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Closed orbits of non-singular Morse-Smale flows on S^3

By Masaaki WADA

The closed orbits of a non-singular Morse-Smale flow ([8], p. 798) on S^3 form an indexed link, that is, a link with the index 0, 1 or 2 attached to each component. Although closed orbits are naturally oriented, we do not consider oriented links since the orientation of a closed orbit of a non-singular Morse-Smale flow can be easily reversed by modifying the flow near the closed orbit.

In this paper, we characterize the set of indexed links which arise as the closed orbits of a non-singular Morse-Smale flow on S^3 in terms of a generator and six operations. The generator is the Hopf link with indices 0 and 2 attached to the components, and the operations are, roughly speaking, split sum, connected sum, and cabling.

Since the author first obtained the result, several papers dealing with the topic have appeared ([6], [7], [9]). Of these, the works of Sasano [7] and Yano [9] were independently done, and are contained in the results in this paper.

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§ 1. Results.

The Hopf link with indices 0 and 2 attached to the components is called the $(0, 2)$-Hopf link. We prove the following:

THEOREM. *Every indexed link which consists of all the closed orbits of a non-singular Morse-Smale flow on S^3 is obtained from $(0, 2)$-Hopf links by applying the following six operations. Conversely, every indexed link obtained from $(0, 2)$-Hopf links by applying the operations is the set of all the closed orbits of some non-singular Morse-Smale flow on S^3.*

OPERATIONS. For given indexed links l_1 and l_2, we define six operations as follows. We denote by $l_1 \cdot l_2$ the split sum of l_1 and l_2, and by $N(k, M)$ the regular neighborhood of k in M. For other terminologies of knot theory, refer to [5].

    I. To make $l_1 \cdot l_2 \cdot u$, where u is an unknot with index 1.

    II. To make $l_1 \cdot (l_2 \setminus k_2) \cdot u$, where k_2 is a component of l_2 of index 0 or 2.

    III. To make $(l_1 \setminus k_1) \cdot (l_2 \setminus k_2) \cdot u$, where k_1 is a component of l_1 of index 0,

and k_2 is a component of l_2 of index 2.

IV. To make $(l_1 \# l_2) \cup m$. The connected sum $l_1 \# l_2$ is obtained by composing a component k_1 of l_1 and a component k_2 of l_2 each of which has index 0 or 2. The index of the composed component $k_1 \# k_2$ is equal to either $\mathrm{ind}(k_1)$ or $\mathrm{ind}(k_2)$. Finally, m is a meridian of $k_1 \# k_2$, and is of index 1.

V. Choose a component k_1 of l_1 of index 0 or 2, and replace $N(k_1, S^3)$ by $D^2 \times S^1$ with three indexed circles in it; $\{0\} \times S^1$, k_2, and k_3. Here, k_2 and k_3 are parallel (p, q)-cables on $\partial N(\{0\} \times S^1, D^2 \times S^1)$. The indices of $\{0\} \times S^1$ and k_2 are either 0 or 2, and one of them is equal to $\mathrm{ind}(k_1)$. The index of k_3 is 1.

VI. Choose a component k_1 of l_1 of index 0 or 2. Replace $N(k_1, S^3)$ by $D^2 \times S^1$ with two indexed circles in it; $\{0\} \times S^1$, and the $(2, q)$-cable k_2 of $\{0\} \times S^1$. The index of $\{0\} \times S^1$ is 1, and $\mathrm{ind}(k_2) = \mathrm{ind}(k_1)$.

We prove the theorem in §3. As an easy consequence of our Lemma 1 in §2 and Corollary 2.5 in [3], we also get the following:

COROLLARY. *Every link which consists of closed orbits of a non-singular Morse-Smale flow on S^3 is a graph link* [4]. *Conversely, given a graph link, we can always construct a non-singular Morse-Smale flow for which each component of the link is a closed orbit.*

§2. Round handle decompositions of S^3.

The proof of our theorem is based on round handle decomposition. We have the following by Asimov [1]:

PROPOSITION. *If a manifold M admits a round handle decomposition*

$$\varnothing = M_0 \subset M_1 \subset \cdots \subset M_n = M,$$

there is a non-singular Morse-Smale flow on M such that (1) *the closed orbits of the flow coincide with the cores of round handles, and* (2) *the flow is pointing outward on ∂M_j. Conversely, if M has a non-singular Morse-Smale flow, then M admits a round handle decomposition satisfying* (1) *and* (2).

We will analyze round handle decompositions of S^3 together with the cores. For our purpose, it is more convenient to use the fat round handle decomposition described in [3], §3:

$$\varnothing = M_0 \subset M_1 \subset \cdots \subset M_n = S^3,$$
$$M_i = \bigcup_{j=1}^{i} C_j \qquad (i = 1, 2, \cdots, n).$$

Each C_j has the form

$$C_j = A \times [0, 1] \bigcup_\varphi B_s \oplus B_u \,,$$

where A is a union of components of ∂M_{j-1}, $B_s \oplus B_u$ is the Whitney sum of disk bundles B_s and B_u over S^1, and the image of $\varphi : (\partial B_s) \oplus B_u \to A \times \{1\}$ intersects every component of $A \times \{1\}$. Let us put $\partial_- C_j = A \times \{0\}$, and consider C_j together with $\partial_- C_j$ and the core k_j which is the 0-section of $B_s \oplus B_u$. The component C_j associated to a round 0- or 2-handle is just a solid torus.

LEMMA 1. *The triple* $(C_j, \partial_- C_j, k_j)$ *associated to a round 1-handle is of one of the following types:*

(a) $C \cong T_1 \times [0, 1] \# T_2 \times [0, 1]$ *where* T_1 *and* T_2 *are tori*, $\partial_- C = T_1 \times \{0\} \cup T_2 \times \{0\}$, *and* k *is an unknot in* C.

(b) $C \cong T^2 \times [0, 1] \# D^2 \times S^1$, $\partial_- C = T^2 \times \{0\}$ *or* $T^2 \times \{0\} \cup \partial D^2 \times S^1$, *and* k *is an unknot in* C.

(c) $C \cong V_1 \# V_2$ *where* V_1 *and* V_2 *are solid tori*, $\partial_- C = \partial V_1$, *and* k *is an unknot in* C.

(d) $C \cong F \times S^1$ *where* F *is a disk with two holes*, $\partial_- C$ *is a component or a union of two components of* ∂C, *and* $k = * \times S^1$ *for some point* $*$ *in* $\operatorname{Int} F$.

(e) $C \cong D^2 \times S^1 \setminus \operatorname{Ind} W$ *where* W *is a tubular neighborhood of the* $(2, 1)$-*cable of* $\{0\} \times S^1$ *in* $D^2 \times S^1$, $\partial_- C = \partial W$, *and* $k = \{0\} \times S^1$.

PROOF. We make case-by-case observations of the attaching map

$$\varphi : (\partial B_s) \oplus B_u \longrightarrow A \times \{1\}.$$

The image of φ is an annulus or a union of two annuli, depending on whether the round handle is untwisted or twisted. Since each component of ∂C is a torus (Lemma 3.1, [3]), A is a torus or a union of two tori. We call a component of $\varphi(\partial B_s \oplus \{0\})$ an attaching circle.

Let us first suppose that the round 1-handle is untwisted. Then, it is diffeomorphic to $[-1, 1]_s \times [-1, 1]_u \times S^1$. We denote the two attaching circles by $c_1 = \varphi(\{-1\} \times \{0\} \times S^1)$ and $c_2 = \varphi(\{1\} \times \{0\} \times S^1)$.

Case 1. Suppose that c_1 and c_2 are contained in different components. Then, A is a disjoint union of two tori T_1 and T_2, where $c_j \subset T_j$ $(j=1, 2)$.

Case 1.1. If both c_1 and c_2 are essential, we get type (d), where $\partial_- C$ is a union of two components of ∂C.

Case 1.2. If one of c_1 and c_2 is essential and the other is inessential, we get type (b), where $\partial_- C = T^2 \times \{0\} \cup \partial D^2 \times S^1$.

Case 1.3. Suppose that both c_1 and c_2 are inessential. For $j=1, 2$, let D_j be a 2-disk in $T_j \times \{1\}$ which bounds c_j. We may assume that D_1 contains $\varphi(\{-1\} \times \{-1\} \times S^1)$. Then, D_2 must contain $\varphi(\{1\} \times \{1\} \times S^1)$, for otherwise we

would have a 2-sphere as a component of ∂C. Let D_j' be a properly embedded 2-disk in $T_j \times [0, 1]$ which bounds c_j. Then, C splits along the 2-sphere $D_1' \cup ([-1, 1] \times \{0\} \times S^1) \cup D_2'$ as a connected sum of $T_1 \times [0, 1]$ and $T_2 \times [0, 1]$. This leads to (a).

Case 2. Suppose that c_1 and c_2 are contained in the same component.

Case 2.1. If both c_1 and c_2 are essential, they are parallel circles in $T^2 \times \{1\}$. Let E be a properly embedded annulus in $T^2 \times [0, 1]$ which bounds $c_1 \cup c_2$. Since the surface $E \cup ([-1, 1] \times \{0\} \times S^1)$ embeds in S^3, it can not be a Klein bottle, hence is a torus, and is 2-sided. From this fact, the attaching map φ is determined up to diffeomorphisms of $T^2 \times \{1\}$. We then get type (d), where $\partial_- C$ is a component of ∂C.

Case 2.2. Suppose that one of c_1 and c_2 is essential and the other is inessential. In this case, the attaching map φ is unique up to diffeomorphisms of $T^2 \times \{1\}$, since there is a diffeomorphism of $T^2 \times \{1\}$ to itself which preserves c_1 and c_2 setwisely, preserves an orientation of the essential one, and reverses an orientation of the other. We then get (c).

Case 2.3. Suppose that both c_1 and c_2 are inessential. For $j=1, 2$, let D_j be a 2-disk in $T^2 \times \{1\}$ which bounds c_j. We first assume that D_1 and D_2 are disjoint. Let E be a properly embedded annulus in $T^2 \times [0, 1]$ which bounds $c_1 \cup c_2$. From that $E \cup ([-1, 1] \times \{0\} \times S^1)$ is a 2-sided torus, the attaching map φ is determined up to diffeomorphisms of $T^2 \times \{1\}$. But in this case, a 2-sphere appears as a component of ∂C. Therefore, D_1 and D_2 must intersect. We may assume that D_1 contains D_2. Since $(D_1 \setminus \operatorname{Int} D_2) \cup ([-1, 1] \times \{0\} \times S^1)$ is a 2-sided torus, C is uniquely determined, and is of type (b) where $\partial_- C = T^2 \times \{0\}$.

If the round 1-handle is twisted, B_s and B_u are non-orientable D^1-bundles over S^1. The attaching circle is $c = \varphi(\partial B_s \oplus \{0\})$. If c bounded a 2-disk D in $T^2 \times \{1\}$, then $D \cup (B_s \oplus \{0\})$ would be a project plane embedded in S^3. Hence, c is essential in $T^2 \times \{1\}$. This leads to (e).

These cases cover all the possibilities, and this completes the proof.

In each case, it is easily verified that both C and $C \setminus \operatorname{Int} N(k, C)$ are graph manifolds. This proves the former part of the corollary.

§3. Proof of Theorem.

We denote by r the number of closed orbits of index 1 of the non-singular Morse-Smale flow. We prove the former part of our theorem by induction on r. Associated to the non-singular Morse-Smale flow is a decomposition

$$S^3 = \bigcup_{j=1}^{n} C_j.$$

We denote by l the indexed link which consists of the cores of this decomposition.

First, if there is no closed orbit of index 1, it is easily seen that $n=2$, C_1 is a round 2-handle, and C_2 is a round 0-handle. Hence, l is a $(0, 2)$-Hopf link.

Let us assume $r \geqq 1$ and that every indexed link which consists of all the closed orbits of a non-singular Morse-Smale flow and which has less than r components of index 1 is obtained from $(0, 2)$-Hopf links by applying the operations I-VI. Since $r \geqq 1$, there is a component $C=C_j$ associated to a round 1-handle. We divide the proof into five cases according to the type of C in Lemma 1.

Case (a). Suppose that $C \cong T_1 \times [0, 1] \# T_2 \times [0, 1]$ where T_1 and T_2 are tori. For $j=1, 2$, let N_{j-} and N_{j+} be the components of the complement of C in S^3 which bound $T_j \times \{0\}$ and $T_j \times \{1\}$ respectively. Since the 2-sphere splitting C as a connected sum of $T_1 \times [0, 1]$ and $T_2 \times [0, 1]$ bounds 3-balls on both sides by Schönflies' theorem, we see that $C \cup N_{2-} \cup N_{2+} \cong T^2 \times [0, 1]$. Therefore, $N_{1-} \cup N_{1+} \cong S^3$. This gives a round handle decomposition of S^3. Let l_1 denote the indexed link which consists of the cores of this round handle decomposition. Similarly, we get $N_{2-} \cup N_{2+} \cong S^3$. Let l_2 be the indexed link which consists of the cores of this round handle decomposition. Both l_1 and l_2 have fewer components of index 1 than l. By the assumption of induction, l_1 and l_2 are obtained from $(0, 2)$-Hopf links by applying operations I-VI. We have $l=l_1 \cdot l_2 \cdot u$, where u is an unknot with index 1. Namely, l is obtained from l_1 and l_2 by the operation I.

Case (b). Suppose that $C \cong T^2 \times [0, 1] \# D^2 \times S^1$. Let N_-, N_+, and N_0 be the components of the complement of C in S^3 whose boundaries are $T^2 \times \{0\}$, $T^2 \times \{1\}$, and $\partial D^2 \times S^1$ respectively. In the same manner as in Case (a), we get $N_- \cup N_+ \cong S^3$. Let l_1 denote the indexed link which consists of the cores of this round handle decomposition. We also obtain $C \cup N_- \cup N_+ \cong D^2 \times S^1$. Hence N_0 together with a round i-handle $(D^2 \times S^1, k_2)$ form a round handle decomposition of S^3, where $i=0$ or 2 according as $\partial D^2 \times S^1 \subset \partial_- C$ or not. Let l_2 denote the indexed link which consists of the cores of this decomposition. Then, $l=l_1 \cdot (l_2 \setminus k_2) \cdot u$.

Case (c). Suppose that $C \cong V_1 \# V_2$, where V_1 and V_2 are solid tori. Let N_1 and N_2 be the components of the complement of C in S^3 which bound ∂V_1 and ∂V_2 respectively. Since $C \cup N_2 \cong D^2 \times S^1$, we can construct a 3-sphere by attaching a round 0-handle $(D^2 \times S^1, k_1)$ to N_1. Let l_1 denote the indexed link which consists of the cores of this round handle decomposition. Similarly, let l_2 denote the indexed link which consists of the cores of the round handle decomposition made by attaching N_1 and a round 2-handle $(D^2 \times S^1, k_2)$ to each other. Then,

$$l = (l_1 \setminus k_1) \cdot (l_2 \setminus k_2) \cdot u.$$

So far we have proved the induction step for the cases where there is a component C of type (a), (b), or (c). Now, let us assume that there is no component of type (a), (b), or (c). To proceed further, we have to choose C more carefully.

Let $\mathfrak{N}$ be the collection of all the submanifolds of S^3 each of which is a union of C_j's including at least one C_j of index 1, and whose boundary is a torus.

ASSERTION 1. *The set $\mathfrak{N}$ contains a solid torus.*

PROOF. Let $C = C_j$ be a component in our decomposition of S^3 which is associated to a round 1-handle. First, assume that C is of type (d) in Lemma 1. Namely, $C \cong F \times S^1$ where F is a disk with two holes. Let ∂_0, ∂_1 and ∂_2 be the boundary components of C. For $j = 0$, 1 and 2, let N_j be the component of the complement of C in S^3 whose boundary is ∂_j. If one of N_0, N_1 and N_2 is not a solid torus, by the solid torus theorem ([5], p. 107), its complement in S^3 is a solid torus which belongs to $\mathfrak{N}$. Hence, we may assume that N_0, N_1 and N_2 are solid tori.

We fix our notation for $\pi_1(C)$ as follows: $\partial_1 \cap (F \times \{1\})$ represents d_1, $\partial_2 \cap (F \times \{1\})$ represents d_2, $\partial_0 \cap (F \times \{1\})$ represents $d_1 d_2$, and $* \times S^1$ represents t. Then, we have

$$\pi_1(C) \cong \langle d_1, d_2, t \mid [d_1, t] = [d_2, t] = 1 \rangle,$$

where $[d, t]$ denotes the commutator of d and t.

Suppose that the meridians of N_0, N_1 and N_2 represent $(d_1 d_2)^{p_0} t^{q_0}$, $d_1^{p_1} t^{q_1}$, and $d_2^{p_2} t^{q_2}$ respectively. Then, $\pi_1(C \cup N_0 \cup N_1 \cup N_2)$ is isomorphic to

$$G = \langle d_1, d_2, t \mid [d_1, t] = [d_2, t] = (d_1 d_2)^{p_0} t^{q_0} = d_1^{p_1} t^{q_1} = d_2^{p_2} t^{q_2} = 1 \rangle.$$

Since $C \cup N_0 \cup N_1 \cup N_2 \cong S^3$, G is trivial. By the Coxeter's theorem ([2], p. 67), the triviality of the group

$$G/\langle t \rangle \cong \langle d_1, d_2 \mid d_1^{p_1} = d_2^{p_2} = (d_1 d_2)^{p_0} = 1 \rangle$$
$$\cong \langle d_1, d_2, d_3 \mid d_1^{p_1} = d_2^{p_2} = d_3^{p_0} = d_1 d_2 d_3 = 1 \rangle$$

implies that at least one of p_1, p_2 and p_0 is equal to 1. We may assume $p_1 = 1$. Since there is a diffeomorphism $f : C \to C$ which preserves ∂_0, ∂_1 and ∂_2 setwisely such that $f_* : \pi_1(C) \to \pi_1(C)$ satisfies $f_*(d_1) = d_1 t^{q_1}$, $f_*(d_2) = d_2$, and $f_*(t) = t$, we may also assume that the meridian of N_1 represents d_1. Then, $C \cup N_1 \cong T^2 \times [0, 1]$. Therefore, $C \cup N_1 \cup N_0$ and $C \cup N_1 \cup N_2$ are solid tori which belong to $\mathfrak{N}$.

The proof for the case where C is of type (e) is similar to the above.

We take a solid torus N in $\mathfrak{N}$ which consists of the least number of C_j's. Let C be the component C_j contained in N which contains ∂N. The component C is associated to a round 1-handle.

Case (d). Suppose that $C \cong F \times S^1$ where F is a disk with two holes. Let us follow the same notation as in the proof of Assertion 1, and assume that $\partial N = \partial_0$. Then, $N = C \cup N_1 \cup N_2$.

ASSERTION 2. *Either N_1 or N_2 is a solid torus.*

PROOF. If both N_1 and N_2 are not solid torus, they are non-trivial knot complements. Hence, ∂_1 and ∂_2 are incompressible in N_1 and N_2 respectively. Since ∂_1 and ∂_2 are also incompressible in C, every incompressible surface in C is incompressible in $N = C \cup N_1 \cup N_2$. Especially, ∂_0 is incompressible in N. This contradicts the fact that N is a solid torus.

Then, there are two possibilities.

Case (d.1). Suppose that one of N_1 and N_2 is a solid torus, and the other is not.

We may assume that N_2 is a solid torus. Since N consists of the least number of C_j's, N_2 is not in $\mathfrak{N}$. Hence, N_2 is either a round 0-handle or a round 2-handle depending on whether $\partial_2 \subset \partial_- C$ or not.

Let α be an essential arc properly embedded in $F \times \{1\}$ whose end points lie in ∂_2. The properly embedded annulus $E = \alpha \times S^1$ cuts C into two components P_0 and P_1 which contain ∂_0 and ∂_1 respectively. The two components of ∂E, a_1 and a_2, are meridians of N_2, since otherwise $C \cup N_2$ would be a Seifert fibered space over an annulus, and each component of the boundary would be incompressible. Then, $(C \cup N_2) \cup N_1$ would not be a solid torus. Let D_1 and D_2 be disjoint meridian disks of N_2 bounding a_1 and a_2 respectively. We also assume for $j = 1, 2$ that $D_j \cap k'$ is a point, where k' is the core of N_2. Two disks D_1 and D_2 together split N_2 into two 3-balls B_0 and B_1. Since $\partial(N_0 \cup P_0 \cup B_0) = E \cup D_1 \cup D_2$ is a 2-sphere, by the Schönflies' theorem, $N_0 \cup P_0 \cup B_0$ is a 3-ball.

We can make a 3-sphere by attaching the standard ball pair (D^3, D^1) to $(N_0 \cup P_0 \cup B_0, k' \cap B_0)$ by a diffeomorphism $\varphi : (\partial D^3, \partial D^1) \to (E \cup D_1 \cup D_2, k' \cap (D_1 \cup D_2))$. Then, $P_0 \cup B_0 \cup D^3$ is a solid torus, and $(k' \cap B_0) \cup D^1$ is its core. Regard this solid torus as a round 0-handle or a round 2-handle according as $\partial_0 \subset \partial_- C$ or not. Then, we obtain a round handle decomposition for S^3. Let l_1 denote the indexed link which consists of the cores of this round handle decomposition.

Since $N_1 \cup P_1 \cup B_1$ is also a 3-ball, we can obtain an indexed link l_2 in the same way as the above. We see that l is obtained from l_1 and l_2 by the operation IV.

Case (d.2). Suppose that both N_1 and N_2 are solid tori. Since N_j $(j=1, 2)$

contains fewer C_j's than N, N_j does not contain a piece of index 1. Hence, N_j is a round 0-handle or a round 2-handle according as $\partial_j \subset \partial_- C$ or not. Let us assume that the meridians of N_1 and N_2 represent $d_1^{p_1} t^{q_1}$ and $d_2^{p_2} t^{q_2}$ respectively. Then, $\pi_1(N)$ is isomorphic to

$$G = \langle d_1, d_2, t \mid [d_1, t]=[d_2, t]=d_1^{p_1} t^{q_1}=d_2^{p_2} t^{q_2}=1 \rangle.$$

Since N is a solid torus, G is isomorphic to $\mathbf{Z}$, and

$$G/\langle t \rangle \cong \langle d_1, d_2 \mid d_1^{p_1}=d_2^{p_2}=1 \rangle \cong \mathbf{Z}/p_1 * \mathbf{Z}/p_2$$

is a factor group of $\mathbf{Z}$. Hence, either p_1 or p_2 is equal to 1. We may assume $p_2=1$. We may also assume that the meridian of N_2 represents d_2, and hence $C \cup N_2 \cong T^2 \times [0, 1]$. Therefore, N is as an indexed link equivalent to $D^2 \times S^1$ with three indexed circles $\{0\} \times S^1$, k_2 and k_3, where k_2 and k_3 are parallel (p, q)-cables on $\partial N(\{0\} \times S^1)$. It is now easily seen that l is obtained from an indexed link which has fewer components of index 1 than l by applying the operation V.

Case (e). In this case, we put $P=C \setminus \text{Int } N(k, C)$. Then, $P \cong F \times S^1$, where F is a disk with two holes. Let N_2 be the complement of C in N. We denote the components of ∂P by ∂_0, ∂_1 and ∂_2, so that $\partial_0=\partial N$, $\partial_1=\partial N(k, C)$, and $\partial_2=\partial N_2$. We may assume that the meridian of $N(k, C)$ represents $d_1^2 t$. If N_2 is not a solid torus, the same argument as in Case (d.1) shows that $* \times S^1$ represents d_1. But, this contradicts the fact that the meridian of $N(k, C)$ represents $d_1^2 t$. Hence, N_2 is a solid torus. Let $d_2^p t^q$ be represented by the meridian of N_2. We can show as in Case (d.2) that $p=1$. We obtain $P \cup N_2 \cong T^2 \times [0, 1]$. Therefore, $C \cup N_2$ is equivalent to $D^2 \times S^1$ with two indexed circles $\{0\} \times S^1$ of index 1, and k_2, the $(2, q)$-cable around $\{0\} \times S^1$ of index i, where $i=0$ or 2 according as $\partial_0 \subset \partial_- C$ or not. We can construct a round handle decomposition of S^3 by replacing $C \cup N_2$ by a round i-handle $(D^2 \times S^1, k_1)$. Let l_1 denote the indexed link which consists of the cores of this round handle decomposition. Then, l is obtained from l_1 by the operation VI.

It only remains to prove the latter part of Theorem. Obviously, $(0, 2)$-Hopf link is the set of cores of a round handle decomposition of S^3.

Suppose that we have two round handle decomposition of S^3,

$$S^3 = \bigcup_{j=1}^{s} C_j$$

whose cores form l_1, and

$$S^3 = \bigcup_{j=1}^{t} C_j'$$

whose cores form l_2. First, suppose that l is obtained from l_1 and l_2 by the operation I. Replace C_1 by $C_1 \cup (\partial C_1 \times [0, 1])$, and C_1' by $C_1' \cup (\partial C_1' \times [0, 1])$. Then,

perform the connected sum operation using $\partial C_1 \times [0, 1]$ and $\partial C_1' \times [0, 1]$. We obtain a decomposition

$$S^3 = C_1 \cup C_1' \cup (\partial C_1 \times [0, 1] \# \partial C_1' \times [0, 1]) \cup \left(\bigcup_{j=2}^{s} C_j \right) \cup \left(\bigcup_{j=2}^{t} C_j' \right).$$

If we regard $(\partial C_1 \times [0, 1] \# \partial C_1' \times [0, 1])$ in this decomposition as C of type (a) in Lemma 1, we have a round handle decomposition of S^3. The set of cores of this round handle decomposition is $l = l_1 \cdot l_2 \cdot u$. A similar argument applies to l obtained by the operation II, III, or IV by using C of type (b), (c) or (d) respectively. For l obtained from l_1 by applying V or VI, it is easy to construct a round handle decomposition of S^3 which has l as the set of cores, since there is a round handle decomposition of $D^2 \times S^1$ whose cores are $\{0\} \times S^1$, k_2 and k_3 defined in V, or $\{0\} \times S^1$ and k_2 defined in VI.

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