Colored Hook Formula
for a Generalized Young Diagram

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List of Publications.

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K. Nakada, Colored hook formula for a generalized Young diagram, 「組合せ論的表現論とその周辺」研究集会, RIMS, Oct. 25. 2006.

K. Nakada, Colored hook formula for a generalized young diagrams,「組合せ論サマースクール2007」研究集会, Sep. 5. 2007.

K. Nakada, q-Hook formula for a generalized Young diagram, 「組合せ論的表現論の拡がり」研究集会, RIMS, Oct. 26. 2007.

COLORED HOOK FORMULA FOR A GENERALIZED YOUNG DIAGRAM

KENTO NAKADA

ABSTRACT. We prove the colored hook formula for a finite pre-dominant integral weight. As a corollary of this, we get a new proof of the Peterson's hook formula.

1. INTRODUCTION

Let λ be a partition of d, and χ_λ the corresponding irreducible character of the symmetric group $\mathfrak{S}_d$. As is well-known (e.g. [8]), the degree $\chi_\lambda(1)$ of χ_λ is given by the hook formula:

$$\chi_\lambda(1) = \frac{d!}{\prod_{v\in Y_\lambda} h_v}, \tag{1.1}$$

where Y_λ is the Young (or Ferrers) diagram of shape λ, and h_v is the hook-length at a cell v of Y_λ. Since the left hand side of (1.1) is equal to the number $\#\mathrm{STab}(Y_\lambda)$ of standard tableaux of shape λ, the formula (1.1) can be rewritten as:

$$\#\mathrm{STab}(Y_\lambda) = \frac{d!}{\prod_{v\in Y_\lambda} h_v}. \tag{1.2}$$

The purpose of this paper is to prove a generalization of (1.2), the *colored hook formula*, for a generalized Young diagram in the sense of D. Peterson and R. A. Proctor (see [1][5]). We stress that the colored hook formula is new even for a Young diagram.

Let $\Pi = \left\{\alpha_i \,\middle|\, i\in I\right\}$ be the set of simple roots of a Kac-Moody Lie algebra $\mathfrak{g}$, and Φ_+ the set of real positive roots. Then we have the colored hook formula:

$$\sum_{\substack{(\beta_1,\cdots,\beta_l)\in\mathrm{Path}(\lambda)\\ l\geq 0}} \frac{1}{\beta_1}\frac{1}{\beta_1+\beta_2}\cdots\frac{1}{\beta_1+\cdots+\beta_l} = \prod_{\beta\in\mathrm{D}(\lambda)}\left(1+\frac{1}{\beta}\right), \tag{1.3}$$

where λ is a *finite pre-dominant* integral weight of $\mathfrak{g}$, $\mathrm{D}(\lambda)$ is the *diagram of* λ, and $\mathrm{Path}(\lambda)$ is a set of sequences in Φ_+ with certain conditions. See Section 4 and 5 for unexplained notion and furthur details. In Section 2, the reader can see how the colored hook formula looks like in the case of the 2×2 Young diagram.

Taking the lowest degree part of (1.3), we have:

$$\sum_{(\alpha_{i_1},\cdots,\alpha_{i_d})\in \mathrm{MPath}(\lambda)} \frac{1}{\alpha_{i_1}}\frac{1}{\alpha_{i_1}+\alpha_{i_2}}\cdots\frac{1}{\alpha_{i_1}+\cdots+\alpha_{i_d}} = \prod_{\beta\in \mathrm{D}(\lambda)}\frac{1}{\beta}, \tag{1.4}$$

where MPath(λ) is the set of elements of maximal length in Path(λ). Taking the specialization $\alpha_i \to 1$ ($i \in I$) of (1.4), we furthur get:

$$\#\mathrm{MPath}(\lambda)\frac{1}{d!} = \prod_{\beta\in \mathrm{D}(\lambda)}\frac{1}{\mathrm{ht}(\beta)}, \tag{1.5}$$

where ht(β) is the height of β. According to [1], around 1989, D. Peterson proved:

$$\#\mathrm{Red}(w) = \frac{\ell(w)!}{\prod_{\beta\in\Phi(w)}\mathrm{ht}(\beta)} \tag{1.6}$$

for a minuscule element [1][5] w of the Weyl group of $\mathfrak{g}$, where

$$\Phi(w) = \{\beta \in \Phi_+ \,|\, w^{-1}(\beta) < 0\}$$

and #Red(w) is the number of reduced decompostions of w. Peterson's formula (1.6) is equivalent to our reduced formula (1.5).

The colored hook formula (1.3), in the simply-laced case, was conjectured by N. Kawanaka and S. Okamura in their study [9][11] of game-theoretical aspects of Coxeter groups. We also point out that another proof of Peterson's formula (1.6) has been obtained by S. Okamura [10] using a probabilistic argument. Although Okamura's proof was an original motivation behind the colored hook formula (1.3), our proof of (1.3) is entirely algebraic.

We have also succeeded in generalizing the q-hook length formula (R. P. Stanley [2]) to minusucule elements. The proof will be given in a forthcoming paper [12].

2. An Example

Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be a simple system of the root system Φ of type A_3 depicted by the Dynkin diagram in Figure 2.1.

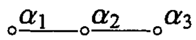

Figure 2.1

Let $\lambda := -\omega_2$, where ω_2 is the fundamental weight corresponding to α_2. Using the standard notation explained in section 3, we put:

$$\mathrm{D}(\lambda) := \left\{\beta \in \Phi_+ \,\middle|\, \langle \lambda, \beta^\vee \rangle = -1\right\}.$$

Then we have:

$$\mathrm{D}(\lambda) = \{\, \alpha_2 \,,\ \alpha_1{+}\alpha_2 \,,\ \alpha_2{+}\alpha_3 \,,\ \alpha_1{+}\alpha_2{+}\alpha_3 \,\},$$

which, in a usual ordering, can be considered as a realization of the 2×2 Young diagram. See Figure 2.2.

$\alpha_1{+}\alpha_2{+}\alpha_3$	$\alpha_2{+}\alpha_3$
$\alpha_1{+}\alpha_2$	α_2

Figure 2.2

Now we consider a directed graph given in Figure 2.3, where, for integral weights μ, ν and $\beta \in \Phi_+$, the arrow $\mu \xrightarrow{\beta} \nu$ means $\langle \mu, \beta^\vee \rangle = -1$ and $\nu = s_\beta(\mu)\,(= \mu + \beta)$, where s_β is the reflection associated with a root β. A sequence of arrows like

$$\lambda = \lambda_0 \xrightarrow{\beta_1} \lambda_1 \xrightarrow{\beta_2} \cdots \xrightarrow{\beta_l} \lambda_l \tag{2.1}$$

is called a *λ-path of length l*, where l is a non-negative integer. Note that the origin of a λ-path is always λ. With each λ-path (2.1), we associate the rational function:

$$\frac{1}{\beta_1}\frac{1}{\beta_1+\beta_2}\cdots\frac{1}{\beta_1+\beta_2+\cdots+\beta_l}. \tag{2.2}$$

For example, with the λ-path

$$\lambda \xrightarrow{\alpha_1+\alpha_2} \lambda+\alpha_1+\alpha_2 \xrightarrow{\alpha_3} \lambda+\alpha_1+\alpha_2+\alpha_3 \xrightarrow{\alpha_2} \lambda+\alpha_1+2\alpha_2+\alpha_3$$

appearing in Figure 2.3, we associate:

$$\frac{1}{\alpha_1{+}\alpha_2}\frac{1}{(\alpha_1{+}\alpha_2){+}\alpha_3}\frac{1}{(\alpha_1{+}\alpha_2){+}\alpha_3{+}\alpha_2} = \frac{1}{\alpha_1{+}\alpha_2}\frac{1}{\alpha_1{+}\alpha_2{+}\alpha_3}\frac{1}{\alpha_1{+}2\alpha_2{+}\alpha_3}.$$

The colored hook formula (1.3), in the present case, asserts that the sum of the rational functions obtained in this way is equal to

$$\prod_{\beta \in \mathrm{D}(\lambda)} \left(1 + \frac{1}{\beta}\right).$$

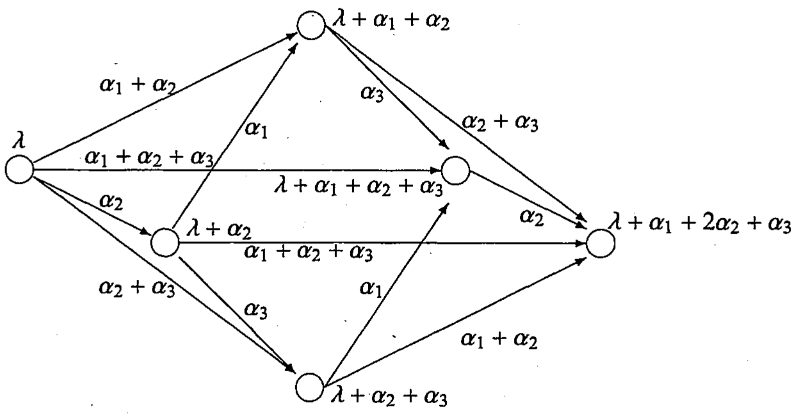

FIGURE 2.3

Thus we have:

$$
\begin{aligned}
&1 + \frac{1}{\alpha_2} + \frac{1}{\alpha_1+\alpha_2} + \frac{1}{\alpha_2+\alpha_3} + \frac{1}{\alpha_1+\alpha_2+\alpha_3} \\
&+ \frac{1}{\alpha_2}\frac{1}{\alpha_1+\alpha_2} + \frac{1}{\alpha_2}\frac{1}{\alpha_2+\alpha_3} + \frac{1}{\alpha_2}\frac{1}{\alpha_1+2\alpha_2+\alpha_3} \\
&+ \frac{1}{\alpha_1+\alpha_2}\frac{1}{\alpha_1+\alpha_2+\alpha_3} + \frac{1}{\alpha_1+\alpha_2}\frac{1}{\alpha_1+2\alpha_2+\alpha_3} \\
&+ \frac{1}{\alpha_2+\alpha_3}\frac{1}{\alpha_1+\alpha_2+\alpha_3} + \frac{1}{\alpha_2+\alpha_3}\frac{1}{\alpha_1+2\alpha_2+\alpha_3} \\
&+ \frac{1}{\alpha_1+\alpha_2+\alpha_3}\frac{1}{\alpha_1+2\alpha_2+\alpha_3} \\
&+ \frac{1}{\alpha_2}\frac{1}{\alpha_1+\alpha_2}\frac{1}{\alpha_1+\alpha_2+\alpha_3} + \frac{1}{\alpha_2}\frac{1}{\alpha_1+\alpha_2}\frac{1}{\alpha_1+2\alpha_2+\alpha_3} \\
&+ \frac{1}{\alpha_2}\frac{1}{\alpha_2+\alpha_3}\frac{1}{\alpha_1+\alpha_2+\alpha_3} + \frac{1}{\alpha_2}\frac{1}{\alpha_2+\alpha_3}\frac{1}{\alpha_1+2\alpha_2+\alpha_3} \\
&+ \frac{1}{\alpha_1+\alpha_2}\frac{1}{\alpha_1+\alpha_2+\alpha_3}\frac{1}{\alpha_1+2\alpha_2+\alpha_3} \\
&+ \frac{1}{\alpha_2+\alpha_3}\frac{1}{\alpha_1+\alpha_2+\alpha_3}\frac{1}{\alpha_1+2\alpha_2+\alpha_3} \\
&+ \frac{1}{\alpha_2}\frac{1}{\alpha_1+\alpha_2}\frac{1}{\alpha_1+\alpha_2+\alpha_3}\frac{1}{\alpha_1+2\alpha_2+\alpha_3} \\
&+ \frac{1}{\alpha_2}\frac{1}{\alpha_2+\alpha_3}\frac{1}{\alpha_1+\alpha_2+\alpha_3}\frac{1}{\alpha_1+2\alpha_2+\alpha_3} \\
&= \left(1+\frac{1}{\alpha_2}\right)\left(1+\frac{1}{\alpha_1+\alpha_2}\right)\left(1+\frac{1}{\alpha_2+\alpha_3}\right)\left(1+\frac{1}{\alpha_1+\alpha_2+\alpha_3}\right).
\end{aligned}
$$

Taking the lowest degree part of this equation, we also get:

$$\frac{1}{\alpha_2}\frac{1}{\alpha_1+\alpha_2}\frac{1}{\alpha_1+\alpha_2+\alpha_3}\frac{1}{\alpha_1+2\alpha_2+\alpha_3}+\frac{1}{\alpha_2}\frac{1}{\alpha_2+\alpha_3}\frac{1}{\alpha_1+\alpha_2+\alpha_3}\frac{1}{\alpha_1+2\alpha_2+\alpha_3}$$
$$=\frac{1}{\alpha_2}\frac{1}{\alpha_1+\alpha_2}\frac{1}{\alpha_2+\alpha_3}\frac{1}{\alpha_1+\alpha_2+\alpha_3}.$$

Note that the left hand side is the sum of the rational functions associated with the λ-paths of maximal length, which are in bijective correspondence with the standard tableaux of the 2×2 Young diagram.

3. Preliminaries

Let $A = (a_{i,j})_{i,j\in I}$ be a (not necessarily symmetrizable) Cartan matrix of a Kac-Moody Lie algebra [3][4]. We denote the set of real numbers by $\mathbb{R}$. Let $\mathfrak{h}$ be an $\mathbb{R}$ -vector space and $\mathfrak{h}^*$ the dual space of $\mathfrak{h}$ and $\langle , \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{R}$ the cannonical bilinear form. We suppose the existence of linearly independent subsets $\Pi := \left\{\alpha_i \middle| i \in I\right\} \subset \mathfrak{h}^*$ and $\Pi^\vee := \left\{\alpha_i^\vee \middle| i \in I\right\} \subset \mathfrak{h}$ such that $\langle \alpha_j, \alpha_i^\vee \rangle = a_{i,j}$. An element $\lambda \in \mathfrak{h}^*$ is said to be an *integral weight* if

$$\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}, \quad i \in I.$$

The set of integral weights is denoted by P. For each $i \in I$, we define $s_i \in GL(\mathfrak{h}^*)$ by:

$$s_i : \lambda \mapsto \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \quad \lambda \in \mathfrak{h}^*.$$

The group W generated by $\left\{s_i \middle| i \in I\right\}$ is called the *Weyl group*, which acts on $\mathfrak{h}$ by:

$$\langle w(\lambda), w(h) \rangle = \langle \lambda, h \rangle, \quad w \in W, \lambda \in \mathfrak{h}^*, h \in \mathfrak{h}.$$

We define the *root system* (resp. *coroot system*) by $\Phi := W\Pi$ (resp. $\Phi^\vee := W\Pi^\vee$). We denote:

$$Q_+ := \bigoplus_{i\in I} \mathbb{N}\alpha_i \ (\subseteq P),$$

where $\mathbb{N}$ is the set of non-negative integers. For integral weights (in particular, roots) λ, μ, we denote $\lambda \leq \mu$ if

$$\mu - \lambda \in Q_+.$$

We denote $\lambda < \mu$ if $\lambda \leq \mu$ and $\lambda \neq \mu$. We denote by Φ_+ and Φ_- the sets of positive and negative roots of Φ, respecively. The *dual* $\beta^\vee \in \Phi^\vee$ of a root $\beta \in \Phi$ is defined so that

$$w(\beta^\vee) = w(\beta)^\vee, \quad w \in W.$$

For each $\beta \in \Phi$, we define $s_\beta \in W$ by:

$$s_\beta(\lambda) = \lambda - \langle \lambda, \beta^\vee \rangle \beta, \quad \lambda \in \mathfrak{h}^*,$$

or, equivalently, by

$$s_\beta(h) = h - \langle \beta, h \rangle \beta^\vee, \quad h \in \mathfrak{h}.$$

We note that $s_{\alpha_i} = s_{-\alpha_i} = s_i$. For each $w \in W$, we define a set $\Phi(w) \, (\subseteq \Phi_+)$ by:

$$\Phi(w) := \left\{\gamma \in \Phi_+ \middle| w^{-1}(\gamma) < 0\right\}.$$

See [4, chap.5] for the following facts.
Let $w = s_{i_1} \cdots s_{i_d}$ be a reduced decomposition of $w \in W$. Then we have

$$\Phi(w) = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \cdots, s_{i_1} \cdots s_{i_{d-1}}(\alpha_{i_d})\}$$

For $w, w' \in W$, we have:

$$\Phi(w) = \Phi(w) \Leftrightarrow w = w',$$

For $\beta, \gamma \in \Phi$, we have:

$$\langle \beta, \gamma^\vee \rangle = 0 \Leftrightarrow \langle \gamma, \beta^\vee \rangle = 0,$$

and

$$\langle \beta, \gamma^\vee \rangle > 0 \Leftrightarrow \langle \gamma, \beta^\vee \rangle > 0.$$

4. Pre-dominant Integral Weights

In this section, we define and study *pre-dominant integral weights*, which play important roles in this paper.

Definition 1. An integral weight λ is *pre-dominant* if

$$\langle \lambda, \beta^\vee \rangle \geq -1 , \quad \beta \in \Phi_+.$$

The set of pre-dominant integral weights is denoted by $P_{\geq -1}$.

Definition 2. For $\lambda \in P_{\geq -1}$, the set $\mathrm{D}(\lambda)$ defined by

$$\mathrm{D}(\lambda) := \left\{ \beta \in \Phi_+ \left| \langle \lambda, \beta^\vee \rangle = -1 \right. \right\}$$

is called the *diagram of* λ. An element of $\mathrm{D}(\lambda)$ is called a *λ-move*. An element of $\mathrm{D}(\lambda) \cap \Pi$ is called a *simple λ-move*. A pre-dominant integral weight λ is said to be *finite* if $\#\mathrm{D}(\lambda) < \infty$.

We note that $\mathrm{D}(\lambda) = \varnothing$ if and only if $\mathrm{D}(\lambda) \cap \Pi = \varnothing$. The terminology "move"is suggested by the game theoretic study of Kawanaka [9].

Lemma 4.1. *Let* $\lambda \in P_{\geq -1}$ *and* $\beta \in \mathrm{D}(\lambda)$. *Then we have:*

(1) $s_\beta(\lambda) \in P_{\geq -1}$.
(2) $\mathrm{D}(s_\beta(\lambda)) = s_\beta(\mathrm{D}(\lambda) \setminus \Phi(s_\beta))$.

Proof. (1) Let $\gamma \in \Phi_+$, we have:

$$\langle s_\beta(\lambda), \gamma^\vee \rangle = \langle \lambda, s_\beta(\gamma^\vee) \rangle.$$

If $s_\beta(\gamma^\vee) > 0$, then since λ is pre-dominant,

$$\langle \lambda, s_\beta(\gamma^\vee) \rangle \geq -1.$$

If, on the other hand, $s_\beta(\gamma^\vee) < 0$, then

$$\langle \lambda, s_\beta(\gamma^\vee) \rangle = \langle \lambda, \gamma^\vee \rangle - \langle \beta, \gamma^\vee \rangle \langle \lambda, \beta^\vee \rangle = \langle \lambda, \gamma^\vee \rangle + \langle \beta, \gamma^\vee \rangle \geq -1 + 1 \geq -1.$$

This proves part (1).
(2) Let $\gamma \in \mathrm{D}(s_\beta(\lambda))$. Since

$$-1 = \langle s_\beta(\lambda), \gamma^\vee \rangle = \langle \lambda, \gamma^\vee \rangle + \langle \beta, \gamma^\vee \rangle \geq -1 + \langle \beta, \gamma^\vee \rangle,$$

we have $\langle \beta, \gamma^\vee \rangle \leq 0$. Since $\langle \gamma, \beta^\vee \rangle \leq 0$, we have $s_\beta(\gamma) > 0$ and $s_\beta(\gamma) \notin \Phi(s_\beta)$. Since

$$-1 = \langle s_\beta(\lambda), \gamma^\vee \rangle = \langle \lambda, s_\beta(\gamma)^\vee \rangle,$$

we have $s_\beta(\gamma) \in \mathrm{D}(\lambda) \setminus \Phi\left(s_\beta\right)$. Hence, $\gamma \in s_\beta\left(\mathrm{D}(\lambda) \setminus \Phi\left(s_\beta\right)\right)$.
Conversely, let $\gamma \in s_\beta\left(\mathrm{D}(\lambda) \setminus \Phi\left(s_\beta\right)\right)$. Then we have $s_\beta(\gamma) \in \mathrm{D}(\lambda) \setminus \Phi\left(s_\beta\right)$. Since $s_\beta(\gamma) \notin \Phi\left(s_\beta\right)$, we have $\gamma > 0$. Since, moreover,

$$\langle s_\beta(\lambda), \gamma^\vee \rangle = \langle \lambda, s_\beta(\gamma)^\vee \rangle = -1,$$

we have $\gamma \in \mathrm{D}(s_\beta(\lambda))$. This proves part (2). □

Definition 3. Let $\lambda \in P_{\geq -1}$. If $\alpha_i \in \Pi$ satisfies

$$\langle \lambda, (-\alpha_i)^\vee \rangle = -1 \quad (\text{ or, equivalently, } \langle \lambda,\ \alpha_i^\vee \rangle = 1),$$

then $-\alpha_i$ is called a *simple backward* λ*-move.*

Lemma 4.2. *Let* $\lambda \in P_{\geq -1}$ *and* $-\alpha_i$ *a simple backward* λ*-move. Then we have:*

(1) $s_{-\alpha_i}(\lambda) \in P_{\geq -1}$.
(2) $\mathrm{D}(s_{-\alpha_i}(\lambda)) = s_i(\mathrm{D}(\lambda)) \cup \{\alpha_i\}$.

Proof. (1) Let $\gamma \in \Phi_+$, we have:

$$\langle s_{-\alpha_i}(\lambda), \gamma^\vee \rangle = \langle \lambda, s_i(\gamma)^\vee \rangle.$$

If $s_i(\gamma) > 0$, then since λ is pre-dominant, we have:

$$\langle \lambda, s_i(\gamma)^\vee \rangle \geq -1.$$

If, on the other hand, $s_i(\gamma) < 0$, then we have $\gamma = \alpha_i$. Hence,

$$\langle \lambda, s_i(\gamma)^\vee \rangle = -\langle \lambda, \alpha_i^\vee \rangle = -1.$$

This proves part (1).
(2) Since α_i is $s_{-\alpha_i}(\lambda)$-move, we have $\mathrm{D}(s_{\alpha_i}s_{-\alpha_i}(\lambda)) = s_i(\mathrm{D}(s_{-\alpha_i}(\lambda)) \setminus \{\alpha_i\})$ by Lemma 4.1 (2). Hence we get part (2). □

Thus, if $\mu \in P_{\geq -1}$, $-\alpha_i$ is a simple backward μ-move if and only if $\mu = s_{\alpha_i}(\lambda)$ and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$ for some $\lambda \in P_{\geq -1}$.

5. λ-Paths

Definition 4. Let $\lambda \in P_{\geq -1}$. Let l be a nonnegative integer. A sequence of positive roots $\mathcal{B} = (\beta_1, \beta_2, \cdots, \beta_l)$ is said to be a *λ-path* if

$$\beta_p \in \mathrm{D}(s_{\beta_{p-1}} \cdots s_{\beta_1}(\lambda)), \quad 1 \leq p \leq l.$$

We call *l the length of* the λ-path $\mathcal{B}$ and denote it by $\ell(\mathcal{B})$. Note that $\ell(\mathcal{B})$ may be 0. The set of λ-paths is denoted by $\mathrm{Path}(\lambda)$.

Theorem 5.1. *Let $\lambda \in P_{\geq -1}$ and $(\beta_1, \beta_2, \cdots, \beta_l) \in \mathrm{Path}(\lambda)$. Let $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. Then we have:*

$$\langle s_{\beta_l} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = -1, 0 \text{ or } 1.$$

Proof. The statement is trivial for $l = 0$. Since $s_{\beta_l} s_{\beta_{l-1}} \cdots s_{\beta_1}(\lambda) \in P_{\geq -1}$, we have:

$$\langle s_{\beta_l} s_{\beta_{l-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle \geq -1.$$

We also have:

$$\begin{aligned}\langle s_{\beta_l} s_{\beta_{l-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle &= \langle s_i s_{\beta_l} s_i s_i s_{\beta_{l-1}} \cdots s_{\beta_1}(\lambda), s_i(\alpha_i^\vee) \rangle \\ &= -\langle s_{s_i(\beta_l)} s_i s_{\beta_{l-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle.\end{aligned}$$

Hence, it is sufficient to show that $s_{s_i(\beta_l)} s_i s_{\beta_{l-1}} \cdots s_{\beta_1}(\lambda) \in P_{\geq -1}$.
If $\beta_l = \alpha_i$, then we have $s_{s_i(\beta_l)} s_i s_{\beta_{l-1}} \cdots s_{\beta_1}(\lambda) = s_{\beta_{l-1}} \cdots s_{\beta_1}(\lambda) \in P_{\geq -1}$. Hence we may assume $\beta_l \neq \alpha_i$. By induction, we may assume

$$\langle s_{\beta_{l-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = -1, 0, \text{ or } 1.$$

If $\langle s_{\beta_{l-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = -1$, then we have $s_{\alpha_i} s_{\beta_{l-1}} \cdots s_{\beta_1}(\lambda) \in P_{\geq -1}$ by Lemma 4.1 (1).
If $\langle s_{\beta_{l-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = 0$, then we have $s_i s_{\beta_{l-1}} \cdots s_{\beta_1}(\lambda) = s_{\beta_{l-1}} \cdots s_{\beta_1}(\lambda) \in P_{\geq -1}$.
If $\langle s_{\beta_{l-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = 1$, then we have $s_{-\alpha_i} s_{\beta_{l-1}} \cdots s_{\beta_1}(\lambda) \in P_{\geq -1}$ by Lemma 4.2 (1).
Thus, we always have $s_i s_{\beta_{l-1}} \cdots s_{\beta_1}(\lambda) \in P_{\geq -1}$. Since

$$s_i(\beta_l) > 0, \quad \text{and}$$
$$\langle s_i s_{\beta_{l-1}} \cdots s_{\beta_1}(\lambda), s_i(\beta_l)^\vee \rangle = \langle s_{\beta_{l-1}} \cdots s_{\beta_1}(\lambda), \beta_l^\vee \rangle = -1,$$

we have $s_i(\beta_l) \in \mathrm{D}(s_i s_{\beta_{l-1}} \cdots s_{\beta_1}(\lambda))$. Hence, by Lemma 4.1 (1), we have

$$s_{s_i(\beta_l)} s_i s_{\beta_{l-1}} \cdots s_{\beta_1}(\lambda) \in P_{\geq -1}.$$

This proves the Theorem. □

Corollary 5.2. *Let $\lambda \in P_{\geq -1}$ and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. Let $(\beta_1, \cdots, \beta_l) \in \mathrm{Path}(\lambda)$. Then we have:*

$$\langle \beta_k, \alpha_i^\vee \rangle = -2, -1, 0, 1, \text{ or } 2, \quad 1 \leq k \leq l.$$

Proof. By Theorem 5.1, we have:

$$\langle \lambda + \beta_1 + \cdots + \beta_{k-1} + \beta_k, \alpha_i^\vee \rangle = -1, 0, \text{ or } 1.$$
$$\langle \lambda + \beta_1 + \cdots + \beta_{k-1}, \alpha_i^\vee \rangle = -1, 0, \text{ or } 1.$$

Hence we have $\langle \beta_k, \alpha_i^\vee \rangle = -2, -1, 0, 1, \text{ or } 2$. □

Corollary 5.3. *Let $\lambda \in P_{\geq -1}$ and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. Let $(\beta_1, \cdots, \beta_l) \in \mathrm{Path}(\lambda)$. Let $1 \leq k \leq l$.*

(1) *If $\langle \beta_k, \alpha_i^\vee \rangle = 2$, then we have*
$\langle s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = -1$ *and* $\langle s_{\beta_k} s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = 1.$

(2) *If $\langle \beta_k, \alpha_i^\vee \rangle = -2$, then we have*
$\langle s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = 1$ *and* $\langle s_{\beta_k} s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = -1.$

Proof. First, we have

$$-1 \leq \langle s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle \leq 1 \quad \text{and} \quad -1 \leq \langle s_{\beta_k} s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle \leq 1$$

by Theorem 5.1. Since $\langle s_{\beta_k} s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = \langle s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle + \langle \beta_k, \alpha_i^\vee \rangle$, we get part (1) and (2). □

Corollary 5.4. *Let $\lambda \in P_{\geq -1}$, $\beta \in \mathrm{D}(\lambda)$ and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. Then we have:*

$$\langle \beta, \alpha_i^\vee \rangle = 0, 1, \text{ or } 2.$$

Proof. By Theorem 5.1, we have:

$$\langle \lambda + \beta, \alpha_i^\vee \rangle = -1, 0, \text{ or } 1.$$
$$\langle \lambda, \alpha_i^\vee \rangle = -1.$$

Hence we have $\langle \beta, \alpha_i^\vee \rangle = 0, 1, \text{ or } 2.$ □

Lemma 5.5. *Let $\lambda \in P_{\geq -1}$ and $\beta, \gamma \in \mathrm{D}(\lambda)$. Then we have:*

(1) *If $\langle \beta, \gamma^\vee \rangle = 2$, then $\langle \gamma, \beta^\vee \rangle = 1$ or 2*

(2) *If λ is finite and $\langle \beta, \gamma^\vee \rangle = \langle \gamma, \beta^\vee \rangle = 2$, then $\beta = \gamma$*

Proof. (1) Since $\langle \beta, \gamma^\vee \rangle > 0$, we have $\langle \gamma, \beta^\vee \rangle > 0$. Hence it is sufficient to show $\langle \gamma, \beta^\vee \rangle \leq 2$. If $s_\gamma(\beta) > 0$, then, since

$$\langle s_\gamma(\lambda), s_\gamma(\beta)^\vee \rangle = \langle \lambda, \beta^\vee \rangle = -1,$$

a sequence $(\gamma, s_\gamma(\beta))$ is a λ-path. Hence we have $s_{s_\gamma(\beta)} s_\gamma(\lambda) \in P_{\geq -1}$. We note that

$$-1 \leq \langle s_{s_\gamma(\beta)} s_\gamma(\lambda), \beta^\vee \rangle = \langle \lambda + \beta - \gamma, \beta^\vee \rangle = -1 + 2 - \langle \gamma, \beta^\vee \rangle.$$

Hence, we have $\langle \gamma, \beta^\vee \rangle \leq 2$. If, on the oher hand, $s_\gamma(\beta) < 0$, then, since

$$-1 \leq \langle \lambda, (-s_\gamma(\beta))^\vee \rangle = \langle \lambda, -\beta^\vee + \langle \gamma, \beta^\vee \rangle \gamma^\vee \rangle = 1 - \langle \gamma, \beta^\vee \rangle,$$

we have $\langle \gamma, \beta^\vee \rangle \leq 2$. (2) Suppose $\beta \neq \gamma$. We put $\beta_n := (s_\beta s_\gamma)^n(\beta)$ for each integer $n \in \mathbb{Z}$. Then we have:

$$\beta_n = \beta + n(2\beta - 2\gamma),$$

and

$$\beta_n^\vee = \beta^\vee + n(2\beta^\vee - 2\gamma^\vee).$$

Hence

$$\langle \lambda, \beta_n^\vee \rangle = \langle \lambda, \beta^\vee + n(2\beta^\vee - 2\gamma^\vee) \rangle = \langle \lambda, \beta^\vee \rangle + 2n\langle \lambda, \beta^\vee \rangle - 2n\langle \lambda, \gamma^\vee \rangle = -1.$$

Since there exists infinitely many $n \in \mathbb{Z}$ such that $\mathrm{ht}(\beta_n) > 0$, there exists infinitely many $n \in \mathbb{Z}$ such that $\beta_n \in \mathrm{D}(\lambda)$. This contradicts the finiteness of λ. □

Lemma 5.6. *Let* $\beta, \gamma, \beta+\gamma \in \Phi$. *If* $\langle \beta, \gamma^\vee\rangle, \langle \gamma, \beta^\vee\rangle \geq -1$, *then we have:*

(1) $\langle \beta, \gamma^\vee\rangle = \langle \gamma, \beta^\vee\rangle$.

(2) *Put* $n := \langle \beta, \gamma^\vee\rangle = \langle \gamma, \beta^\vee\rangle$. *Then we have* $(\beta+\gamma)^\vee = (\beta^\vee+\gamma^\vee)/(n+2)$.

Proof. We have

$$2 = \langle \beta+\gamma, (\beta+\gamma)^\vee\rangle = \langle \beta, (\beta+\gamma)^\vee\rangle + \langle \gamma, (\beta+\gamma)^\vee\rangle.$$

Since $\langle \beta+\gamma, \beta^\vee\rangle, \langle \beta+\gamma, \gamma^\vee\rangle \geq 1$, we have $\langle \beta, (\beta+\gamma)^\vee\rangle, \langle \gamma, (\beta+\gamma)^\vee\rangle \geq 1$. Hence, we have

$$\langle \beta, (\beta+\gamma)^\vee\rangle = \langle \gamma, (\beta+\gamma)^\vee\rangle = 1.$$

Since $s_{\beta+\gamma}(-\beta) = -\beta + \langle \beta, (\beta+\gamma)^\vee\rangle(\beta+\gamma) = \gamma$, we have:

$$\gamma^\vee = s_{\beta+\gamma}(-\beta^\vee) = -\beta^\vee + \langle \beta+\gamma, \beta^\vee\rangle(\beta+\gamma)^\vee.$$

Hence, we have:

$$\beta^\vee + \gamma^\vee = (\langle \gamma, \beta^\vee\rangle + 2)(\beta+\gamma)^\vee.$$

By the symmetry of β and γ, we have:

$$\beta^\vee + \gamma^\vee = (\langle \beta, \gamma^\vee\rangle + 2)(\beta+\gamma)^\vee.$$

Hence, we get part (1) and (2). □

Lemma 5.7. *Let* $\lambda \in P_{\geq -1}$. *Let* $\beta, \gamma \in \mathrm{D}(\lambda)$. *Suppose that* $\langle \beta, \gamma^\vee\rangle = \langle \gamma, \beta^\vee\rangle = 2$. *Then:*

(1) *We have either* $s_\gamma(\beta) < 0$ *or* $(\gamma, \beta-2\gamma, \gamma) \in \mathrm{Path}(\lambda)$.

(2) *We have* $\beta - \gamma \notin \Phi$ and $\beta+\gamma \notin \Phi$.

Proof. (1) Suppose that $s_\gamma(\beta) > 0$. First, we have:

$$\gamma \in \mathrm{D}(\lambda). \tag{5.1}$$

Next, since $\langle \lambda+\gamma, (\beta-2\gamma)^\vee\rangle = \langle s_\gamma(\lambda), s_\gamma(\beta)^\vee\rangle = \langle \lambda, \beta^\vee\rangle = -1$, we have:

$$\beta - 2\gamma \in \mathrm{D}(\lambda+\gamma). \tag{5.2}$$

Finally, since $\langle \lambda+\gamma+(\beta-2\gamma), \gamma^\vee\rangle = -1 + 2 + (2 - 2\cdot 2) = -1$, we have:

$$\gamma \in \mathrm{D}(\lambda+\gamma+(\beta-2\gamma)). \tag{5.3}$$

By (5.1), (5.2), and (5.3), we have:

$$(\gamma, \beta-2\gamma, \gamma) \in \mathrm{Path}(\lambda).$$

(2) If $\beta-\gamma \in \Phi$, then we have $\langle \beta-\gamma, \gamma^\vee\rangle = \langle \gamma, (\beta-\gamma)^\vee\rangle = 0$. Hence, by Lemma 5.6, we have $\beta^\vee = (1/2)(\beta-\gamma)^\vee + (1/2)\gamma^\vee$. Hence

$$\langle \gamma, \beta^\vee\rangle = \langle \gamma, \frac{1}{2}(\beta-\gamma)^\vee + \frac{1}{2}\gamma^\vee\rangle = \frac{1}{2}\langle \gamma, (\beta-\gamma)^\vee\rangle + \frac{1}{2}\langle \gamma, \gamma^\vee\rangle = 1,$$

which is a contradiction. If, on the other hand, $\beta+\gamma \in \Phi$, then, by Lemma 5.6, we have $(\beta+\gamma)^\vee = (1/4)\beta^\vee + (1/4)\gamma^\vee$. Hence

$$\langle \lambda, (\beta+\gamma)^\vee\rangle = \langle \lambda, \frac{1}{4}\beta^\vee + \frac{1}{4}\gamma^\vee\rangle = \frac{1}{4}\langle \lambda, \beta^\vee\rangle + \frac{1}{4}\langle \lambda, \gamma^\vee\rangle = -\frac{1}{2},$$

which contradicts the fact that λ is an integral weight. □

Definition 5. Let $\beta,\ \gamma \in \Phi$. The root β is said to be *γ-shiftable* if

$$\beta - \gamma \in \Phi \ \text{ or } \ \beta + \gamma \in \Phi.$$

We note that if β is γ-shiftable and $w \in W$, then $w(\beta)$ is $w(\gamma)$-shiftable.

Lemma 5.8. *Let $\lambda \in P_{\geq -1}$ and $\beta, \gamma \in \mathrm{D}(\lambda)$. Suppose that $\langle \beta, \gamma^\vee \rangle = 2$. Then we have:*

$$\langle \gamma, \beta^\vee \rangle = 1 \ \text{ if and only if } \ \beta \text{ is } \gamma\text{-shiftable.}$$

Proof. Supposing that $\langle \gamma, \beta^\vee \rangle = 1$. Since $\beta - \gamma = s_\beta(-\gamma) \in \Phi, \beta$ is γ-shiftable. Conversely, supposing that β is γ-shiftable. By Lemma 5.5 (1), we have $\langle \gamma, \beta^\vee \rangle = 1$, or 2. If $\langle \gamma, \beta^\vee \rangle = 2$, then, by Lemma 5.7 (2), we have $\beta - \gamma \notin \Phi$ and $\beta + \gamma \notin \Phi$. This contradicts that β is γ-shiftable. Hence, we have $\langle \gamma, \beta^\vee \rangle = 1$. □

Lemma 5.9. *Let $\lambda \in P_{\geq -1}$ and $\beta, \gamma \in \mathrm{D}(\lambda)$. We assume β is γ-shiftable. Then we have:*

(1) $\beta + \gamma \in \mathrm{D}(\lambda)$ if $\langle \beta, \gamma^\vee \rangle = 0$.

(2) $\beta - \gamma \in \mathrm{D}(\lambda)$ if $\langle \beta, \gamma^\vee \rangle = 2$ and $\beta > \gamma$.

Proof. (1) If $\beta - \gamma \in \Phi_+$, then we have $\beta + \gamma = s_\gamma(\beta - \gamma) \in \Phi_+$. Hence, in anyway, we have $\beta + \gamma \in \Phi_+$. Hence, by Lemma 5.6, we have $(\beta + \gamma)^\vee = (1/2)\beta^\vee + (1/2)\gamma^\vee$. Since

$$\langle \lambda,\ (\beta + \gamma)^\vee \rangle = \langle \lambda, (1/2)\beta^\vee + (1/2)\gamma^\vee \rangle = -1,$$

we have $\beta + \gamma \in \mathrm{D}(\lambda)$.

(2) By Lemma 5.8, we have $\langle \gamma, \beta^\vee \rangle = 1$. And, we have $\beta - \gamma = s_\beta(-\gamma) \in \Phi$. Since $\beta > \gamma$, we have $\beta - \gamma \in \Phi_+$. Since

$$\langle \lambda, (\beta - \gamma)^\vee \rangle = \langle \lambda,\ s_\beta(-\gamma)^\vee \rangle = \langle s_\beta(\lambda),\ -\gamma^\vee \rangle = -\langle \lambda + \beta,\ \gamma^\vee \rangle = -1,$$

we have $\beta - \gamma \in \mathrm{D}(\lambda)$. □

Lemma 5.10. *Let $\lambda \in P_{\geq -1}$. Let $(\beta_1, \cdots, \beta_{k-1}, \beta_k, \beta_{k+1}, \beta_{k+2}, \cdots, \beta_l) \in \mathrm{Path}(\lambda)$. If $\langle \beta_k, \beta_{k+1}^\vee \rangle = 0$, then we have: $(\beta_1, \cdots, \beta_{k-1}, \beta_{k+1}, \beta_k, \beta_{k+2}, \cdots, \beta_l) \in \mathrm{Path}(\lambda)$.*

Proof. Since

$$\langle s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda),\ \beta_{k+1}^\vee \rangle = \langle s_{\beta_k} s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda),\ s_{\beta_k}(\beta_{k+1})^\vee \rangle = \langle s_{\beta_k} s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda),\ \beta_{k+1}^\vee \rangle = -1,$$

we have:

$$\beta_{k+1} \in \mathrm{D}(s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda)). \tag{5.4}$$

Since

$$\langle s_{\beta_{k+1}} s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda),\ \beta_k^\vee \rangle = \langle s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda),\ s_{\beta_{k+1}}(\beta_k)^\vee \rangle = \langle s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda),\ \beta_k^\vee \rangle = -1,$$

we have:

$$\beta_k \in \mathrm{D}(s_{\beta_{k+1}} s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda)). \tag{5.5}$$

Since $s_{\beta_k} s_{\beta_{k+1}} = s_{\beta_{k+1}} s_{\beta_k}$, we have:

$$s_{\beta_{k+1}} s_{\beta_k} s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda) = s_{\beta_k} s_{\beta_{k+1}} s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda) \tag{5.6}$$

By (5.4), (5.5), and (5.6), we get $(\beta_1, \cdots, \beta_{k-1}, \beta_{k+1}, \beta_k, \beta_{k+2}, \cdots, \beta_l) \in \mathrm{Path}(\lambda)$. □

6. Hooks

Proposition 6.1. *Let $\beta \in \Phi_+$. Let $\alpha_i \in \Phi\left(s_\beta\right) \cap \Pi$. If $\beta \neq \alpha_i$, then we have:*

$$\Phi\left(s_\beta\right) = \{\alpha_i\} \sqcup s_i\left(\Phi\left(s_{s_i(\beta)}\right)\right) \sqcup \{-s_\beta(\alpha_i)\} \quad \textit{(disjoint union).}$$

Proof. First, we have:

$$\alpha_i \in \Phi\left(s_\beta\right). \tag{6.1}$$

Let $\gamma \in s_i\left(\Phi\left(s_{s_i(\beta)}\right)\right)$. Then $s_i(\gamma) \in \Phi\left(s_{s_i(\beta)}\right)$. Hence, we have $s_i(\gamma) > 0$ and $s_i s_\beta(\gamma) < 0$.
If $\gamma < 0$, then we have $\gamma = -\alpha_i$. Hence, we have $\alpha_i = s_i(\gamma) \in \Phi\left(s_{s_i(\beta)}\right)$. Since $s_i s_\beta s_i(\alpha_i) = s_{s_i(\beta)}(\alpha_i) < 0$, we have $s_i s_\beta(\alpha_i) > 0$. By (6.1), we have $s_\beta(\alpha_i) < 0$. Hence, we have $s_\beta(\alpha_i) = -\alpha_i$. Hence, we get $\beta = \alpha_i$. This contradicts our assumption. Hence, we have $\gamma > 0$.
If $s_\beta(\gamma) > 0$, then we have $s_\beta(\gamma) = \alpha_i$. Hence, we have $\gamma = s_\beta(\alpha_i) < 0$ by (6.1). This contradicts that $\gamma > 0$. Hence, we have $s_\beta(\gamma) < 0$. We have:

$$\gamma \in \Phi\left(s_\beta\right). \tag{6.2}$$

Since $\alpha_i \in \Phi\left(s_\beta\right)$, we have:

$$-s_\beta(\alpha_i) \in \Phi\left(s_\beta\right). \tag{6.3}$$

By (6.1), (6.2), and (6.3), we have:

$$\Phi\left(s_\beta\right) \supseteq \{\alpha_i\} \cup s_i\left(\Phi\left(s_{s_i(\beta)}\right)\right) \cup \{-s_\beta(\alpha_i)\}.$$

If $\alpha_i \in s_i\left(\Phi\left(s_{s_i(\beta)}\right)\right)$, then we have $-\alpha_i = s_i(\alpha_i) \in \Phi\left(s_{s_i(\beta)}\right)$. This is contradiction. Hence, we get:

$$\alpha_i \notin s_i\left(\Phi\left(s_{s_i(\beta)}\right)\right). \tag{6.4}$$

If $\alpha_i = -s_\beta(\alpha_i)$, then $\beta = \alpha_i$. This is contradiction. Hence, we get:

$$\alpha_i \neq -s_\beta(\alpha_i). \tag{6.5}$$

If $-s_\beta(\alpha_i) \in s_i\left(\Phi\left(s_{s_i(\beta)}\right)\right)$, then we have $-s_i s_\beta(\alpha_i) \in \Phi\left(s_{s_i(\beta)}\right)$. Hence, we have $\alpha = s_{s_i(\beta)}(-s_i s_\beta(\alpha_i)) < 0$. This is contradiction. Hence, we get:

$$-s_\beta(\alpha_i) \notin s_i\left(\Phi\left(s_{s_i(\beta)}\right)\right). \tag{6.6}$$

By (6.4), (6.5), and (6.6), we get:

$$\Phi\left(s_\beta\right) \supseteq \{\alpha_i\} \sqcup s_i\left(\Phi\left(s_{s_i(\beta)}\right)\right) \sqcup \{-s_\beta(\alpha_i)\} \quad \text{(disjoint union).} \tag{6.7}$$

Since $s_{s_i(\beta)} = s_i s_\beta s_i$, we have $\ell(s_\beta) = \ell(s_{s_i(\beta)}) - 2, \ell(s_{s_i(\beta)})$, or $\ell(s_{s_i(\beta)}) + 2$. By (6.7), we have:

$$\ell(s_\beta) = \ell(s_{s_i(\beta)}) + 2.$$

This proves the statement. □

Definition 6. Let $\lambda \in P_{\geq -1}$. Let $\beta \in \mathrm{D}(\lambda)$. We define a set $\mathrm{H}_\lambda(\beta)$ by:

$$\mathrm{H}_\lambda(\beta) := \mathrm{D}(\lambda) \cap \Phi\left(s_\beta\right).$$

The set $\mathrm{H}_\lambda(\beta)$ is called the *hook at* β *(in the diagram* $\mathrm{D}(\lambda)$ *)*. The number $\#\mathrm{H}_\lambda(\beta)$ is called the *hooklength at* β *(in the diagram* $\mathrm{D}(\lambda)$ *)*.(See [9])

Lemma 6.2. *Let* $\lambda \in P_{\geq -1}$. *Let* $\alpha_i \in \Pi$ *satisfy* $\langle \lambda, \alpha_i^\vee \rangle = 0$. *Then we have:*

$$s_i(\mathrm{D}(\lambda)) = \mathrm{D}(\lambda).$$

Proof. Let $\beta \in \mathrm{D}(\lambda)$. Since $\beta \neq \alpha_i$, we have $s_i(\beta) > 0$. Since

$$\langle \lambda, s_i(\beta)^\vee \rangle = \langle s_i(\lambda), \beta^\vee \rangle = \langle \lambda, \beta^\vee \rangle = -1,$$

we have $s_i(\beta) \in \mathrm{D}(\lambda)$. Hence, we have $\beta \in s_i(\mathrm{D}(\lambda))$. This proves the statement. □

Lemma 6.3. *Let* $\lambda \in P_{\geq -1}$ *and* $\beta \in \mathrm{D}(\lambda)$. *Let* $\alpha_i \in \Phi\left(s_\beta\right) \cap \Pi$. *If* $\beta \neq \alpha_i$, *then we have either:*

$$\langle \lambda, \alpha_i^\vee \rangle = -1 \text{ and } \langle \beta, \alpha_i^\vee \rangle = 1,$$
$$\langle \lambda, \alpha_i^\vee \rangle = -1 \text{ and } \langle \beta, \alpha_i^\vee \rangle = 2,$$

or

$$\langle \lambda, \alpha_i^\vee \rangle = 0 \text{ and } \langle \beta, \alpha_i^\vee \rangle = 1.$$

Proof. Since $0 > s_\beta(\alpha_i) = \alpha_i - \langle \alpha_i, \beta^\vee \rangle \beta$, we have $\langle \alpha_i, \beta^\vee \rangle, \langle \beta, \alpha_i^\vee \rangle \geq 1$. Since $\beta \neq \alpha_i$, we have $s_i(\beta) > 0$. Since $-1 = \langle \lambda, \beta^\vee \rangle = \langle \lambda, s_i(\beta)^\vee \rangle + \langle \alpha_i, \beta^\vee \rangle \langle \lambda, \alpha_i^\vee \rangle$, we have $\langle \lambda, \alpha_i^\vee \rangle = -1$ or $\langle \lambda, s_i(\beta)^\vee \rangle = -1$. If $\langle \lambda, \alpha_i^\vee \rangle = -1$, then, by Corollary 5.4, we have either $\langle \beta, \alpha_i^\vee \rangle = 1$ or 2.
If, on the other hand, $\langle \lambda, s_i(\beta)^\vee \rangle = -1$, then we have $\langle \lambda, \alpha_i^\vee \rangle = 0$. Since $-s_\beta(\alpha_i) > 0$, we have $-1 \leq \langle \lambda, -s_\beta(\alpha_i)^\vee \rangle = -\langle \beta, \alpha_i^\vee \rangle$. Hence, we have $\langle \beta, \alpha_i^\vee \rangle = 1$. This proves the statement. □

Lemma 6.4. *Let* $\lambda \in P_{\geq -1}$ *and* $\beta \in \mathrm{D}(\lambda)$. *Let* $\alpha_i \in \Phi\left(s_\beta\right) \cap \Pi$. *If* $\beta \neq \alpha_i$, *then we have:*

(1) *If* $\langle \lambda, \alpha_i^\vee \rangle = -1$ and $\langle \beta, \alpha_i^\vee \rangle = 1$, *then we have* $-s_\beta(\alpha_i) \notin \mathrm{D}(\lambda)$.
(2) *If* $\langle \lambda, \alpha_i^\vee \rangle = -1$ and $\langle \beta, \alpha_i^\vee \rangle = 2$, *then we have* $-s_\beta(\alpha_i) \in \mathrm{D}(\lambda)$.
(3) *If* $\langle \lambda, \alpha_i^\vee \rangle = 0$ and $\langle \beta, \alpha_i^\vee \rangle = 1$, *then we have* $-s_\beta(\alpha_i) \in \mathrm{D}(\lambda)$.

Proof. (1) Since

$$\langle \lambda, (-s_\beta(\alpha_i))^\vee \rangle = -\langle s_\beta(\lambda), \alpha_i^\vee \rangle = -(\langle \lambda, \alpha_i^\vee \rangle + \langle \beta, \alpha_i^\vee \rangle) = -(-1+1) = 0,$$

we have $-s_\beta(\alpha_i) \notin \mathrm{D}(\lambda)$.
(2) First, we have $-s_\beta(\alpha_i) > 0$. Since

$$\langle \lambda, (-s_\beta(\alpha_i))^\vee \rangle = -\langle s_\beta(\lambda), \alpha_i^\vee \rangle = -(\langle \lambda, \alpha_i^\vee \rangle + \langle \beta, \alpha_i^\vee \rangle) = -(-1+2) = -1,$$

we have $-s_\beta(\alpha_i) \in \mathrm{D}(\lambda)$.
(3) First, we have $-s_\beta(\alpha_i) > 0$. Since

$$\langle \lambda, (-s_\beta(\alpha_i))^\vee \rangle = -\langle s_\beta(\lambda), \alpha_i^\vee \rangle = -(\langle \lambda, \alpha_i^\vee \rangle + \langle \beta, \alpha_i^\vee \rangle) = -(0+1) = -1,$$

we have $-s_\beta(\alpha_i) \in \mathrm{D}(\lambda)$. □

Lemma 6.5. *Let* $\lambda \in P_{\geq -1}$ *and* $\beta \in \mathrm{D}(\lambda)$. *Let* $\alpha_i \in \Phi\left(s_\beta\right) \cap \Pi$. *If* $\beta \neq \alpha_i$, *then we have:*

(1) *If* $\langle \lambda, \alpha_i^\vee \rangle = -1$ *and* $\langle \beta, \alpha_i^\vee \rangle = 1$, *then* $\mathrm{H}_{s_i(\lambda)}(s_i(\beta)) = s_i(\mathrm{H}_\lambda(\beta) - \{\alpha_i\})$.

(2) *If* $\langle \lambda, \alpha_i^\vee \rangle = -1$ *and* $\langle \beta, \alpha_i^\vee \rangle = 2$, *then* $\mathrm{H}_{s_i(\lambda)}(s_i(\beta)) = s_i\left(\mathrm{H}_\lambda(\beta) - \{\alpha_i, -s_\beta(\alpha_i)\}\right)$.

(3) *If* $\langle \lambda, \alpha_i^\vee \rangle = 0$ *and* $\langle \beta, \alpha_i^\vee \rangle = 1$, *then* $\mathrm{H}_{s_i(\lambda)}(s_i(\beta)) = s_i\left(\mathrm{H}_\lambda(\beta) - \{-s_\beta(\alpha_i)\}\right)$.

Proof. (1) By Lemma 4.1(2), Proposition 6.1 and Lemma 6.4(1), we have:

$$\begin{aligned}\mathrm{H}_{s_{\alpha_i}(\lambda)}(s_i(\beta)) &= \mathrm{D}(s_{\alpha_i}(\lambda)) \cap \Phi\left(s_{s_i(\beta)}\right) \\ &= s_i(\mathrm{D}(\lambda) - \{\alpha_i\}) \cap s_i\left(\Phi\left(s_\beta\right) - \{\alpha_i, -s_\beta(\alpha_i)\}\right) \\ &= s_i\left(\mathrm{D}(\lambda) \cap \Phi\left(s_\beta\right) - \{\alpha_i\}\right) = s_i(\mathrm{H}_\lambda(\beta) - \{\alpha_i\}).\end{aligned}$$

(2) By Lemma 4.1(2), Proposition 6.1 and Lemma 6.4(2), we have:

$$\begin{aligned}\mathrm{H}_{s_{\alpha_i}(\lambda)}(s_i(\beta)) &= \mathrm{D}(s_{\alpha_i}(\lambda)) \cap \Phi\left(s_{s_i(\beta)}\right) \\ &= s_i(\mathrm{D}(\lambda) - \{\alpha_i\}) \cap s_i\left(\Phi\left(s_\beta\right) - \{\alpha_i, -s_\beta(\alpha_i)\}\right) \\ &= s_i\left(\mathrm{D}(\lambda) \cap \Phi\left(s_\beta\right) - \{\alpha_i, -s_\beta(\alpha_i)\}\right) = s_i\left(\mathrm{H}_\lambda\left(s_\beta\right) - \{\alpha_i, -s_\beta(\alpha_i)\}\right).\end{aligned}$$

(3) By Lemma 6.2, Proposition 6.1 and Lemma 6.4(3), we have:

$$\begin{aligned}\mathrm{H}_{s_i(\lambda)}(s_i(\beta)) &= \mathrm{D}(s_i(\lambda)) \cap \Phi\left(s_{s_i(\beta)}\right) \\ &= s_i(\mathrm{D}(\lambda)) \cap s_i\left(\Phi\left(s_\beta\right) - \{\alpha_i, -s_\beta(\alpha_i)\}\right) \\ &= s_i\left(\mathrm{D}(\lambda) \cap \Phi\left(s_\beta\right) - \{-s_\beta(\alpha_i)\}\right) = s_i\left(\mathrm{H}_\lambda\left(s_\beta\right) - \{-s_\beta(\alpha_i)\}\right).\end{aligned}$$

□

Lemma 6.6. *Let* $\lambda \in P_{\geq -1}$ *and* $\beta \in \mathrm{D}(\lambda)$. *Let* $\alpha_i \in \Phi\left(s_\beta\right) \cap \Pi$. *If* $\beta \neq \alpha_i$, *then we have:*

$$\#\mathrm{H}_\lambda(\beta) = \#\mathrm{H}_{s_i(\lambda)}(s_i(\beta)) + \langle \beta, \alpha_i^\vee \rangle.$$

Proof. This follows from Lemma 6.3 and Lemma 6.5. □

Lemma 6.7. *Let* $\lambda \in P_{\geq -1}$ *and* $\beta \in \mathrm{D}(\lambda)$. *Let* $\alpha_i \in \Phi\left(s_\beta\right) \cap \Pi$. *If* $\beta \neq \alpha_i$, *then we have:*

$$\mathrm{ht}(\beta) = \mathrm{ht}(s_i(\beta)) + \langle \beta, \alpha_i^\vee \rangle,$$

where $\mathrm{ht}(\beta)$ *is the height of* β.

Proof. It is straightforward to see. □

Theorem 6.8. *Let* $\lambda \in P_{\geq -1}$. *Let* $\beta \in \mathrm{D}(\lambda)$. *Then we have:*

$$\#\mathrm{H}_\lambda(\beta) = \mathrm{ht}(\beta).$$

Proof. This follows from Lemma 6.6, Lemma 6.7 and induction on $\#\Phi\left(s_\beta\right)$.

□

7. Main Theorem and Its Consequences

We now state the main result of this paper.

Theorem 7.1 (Colored Hook Formula). *Let $\lambda \in P_{\geq -1}$ be finite. Then we have:*

$$\sum_{\substack{(\beta_1,\cdots,\beta_l)\in \mathrm{Path}(\lambda)\\ l\geq 0}} \frac{1}{\beta_1}\frac{1}{\beta_1+\beta_2}\cdots\frac{1}{\beta_1+\cdots+\beta_l} = \prod_{\beta\in \mathrm{D}(\lambda)}\left(1+\frac{1}{\beta}\right). \tag{7.1}$$

where both hand sides are considered as rational functions in $\left\{\alpha_i \,\middle|\, i\in I\right\}\subseteq \mathfrak{h}^*$.

We call α_i $(i \in I)$ *color variables*, when, as in Theorem 7.1, we consider them as independent variables. We note that the Weyl group W naturally acts on the rational function field $\mathbb{Q}(\alpha_i|\, i\in I)$ in color variables.

Let $\lambda \in P_{\geq -1}$ be finite. Put $d := \#\mathrm{D}(\lambda)$. We denote the set of λ-paths of length d by $\mathrm{MPath}(\lambda)$.
By Lemma 4.1 and Theorem 6.8, a λ-path $\mathcal{B}$ in $\mathrm{MPath}(\lambda)$ is a sequence of simple roots of length $\#\mathrm{D}(\lambda)$.

Corollary 7.2. *Let $\lambda \in P_{\geq -1}$ be finite. Put $d := \#\mathrm{D}(\lambda)$. Then we have:*

$$\sum_{(\alpha_{i_1},\cdots,\alpha_{i_d})\in \mathrm{MPath}(\lambda)} \frac{1}{\alpha_{i_1}}\frac{1}{\alpha_{i_1}+\alpha_{i_2}}\cdots\frac{1}{\alpha_{i_1}+\cdots+\alpha_{i_d}} = \prod_{\beta\in \mathrm{D}(\lambda)}\frac{1}{\beta}. \tag{7.2}$$

Proof. Let t be an indeterminate. For each color variable α_i $(i \in I)$, we substitute $t\alpha_i$ in (7.1). Comparing the coefficients of the lowest degree t^{-d} of both hand sides, we get (7.2). □

Corollary 7.3. *Let $\lambda \in P_{\geq -1}$ be finite. Put $d := \#\mathrm{D}(\lambda)$. Then we have:*

$$\#\mathrm{MPath}(\lambda) = \frac{d!}{\prod_{\beta\in \mathrm{D}(\lambda)} \mathrm{ht}(\beta)}. \tag{7.3}$$

where $\mathrm{ht}(\beta)$ *denotes the height of* $\beta \in \Phi_+$.

Proof. For each color variable α_i $(i \in I)$, we substitute 1 in (7.2). Then we get (7.3). □

Applying Theorem 6.8 to (7.3), for a finite pre-dominant integral weight λ, we have:

$$\#\mathrm{MPath}(\lambda) = \frac{\#\mathrm{D}(\lambda)!}{\prod_{\beta\in \mathrm{D}(\lambda)} \#\mathrm{H}_\lambda(\beta)}. \tag{7.4}$$

8. Proof of The Main Theorem (First Part)

Definition 7. Let $\lambda \in P_{\geq -1}$ and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. Let $\mathcal{B} = (\beta_1, \cdots, \beta_l) \in \mathrm{Path}(\lambda)$. If β_k is α_i-shiftable for some $1 \leq k \leq l$, we denote the minimum value of such k by p_1. If such k does not exist, we put $p_1 := l + 1$.
We put

$$\mathcal{G}_{\alpha_i}(\mathcal{B}) := (\beta_1, \cdots, \beta_{p_1 - 1})$$

and

$$\mathcal{F}^{(+)}_{\alpha_i}(\mathcal{B}) := (\beta_{p_1}, \cdots, \beta_l),$$

and call them the *ground α_i-floor* and the *α_i-up-stairs of $\mathcal{B}$*, respectively. Thus $\mathcal{B}$ is written as:

$$\mathcal{B} = (\mathcal{G}_{\alpha_i}(\mathcal{B}), \mathcal{F}^{(+)}_{\alpha_i}(\mathcal{B})).$$

Proposition 8.1. *Let $\lambda \in P_{\geq -1}$ be finite and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. Let $\mathcal{B} \in \mathrm{Path}(\lambda)$, and $\mathcal{G}_{\alpha_i}(\mathcal{B}) = (\beta_1, \cdots, \beta_{p_1 - 1})$. Then, for $1 \leq k \leq p_1 - 1$, we have*

$$\langle \beta_k, \alpha_i^\vee \rangle = 0, \textit{unless } \beta_k = \alpha_i.$$

Moreover, an index k such that $\beta_k = \alpha_i$ is unique if it exists.

Proof. Let $1 \leq k \leq p_1 - 1$. By Corollary 5.2, we have:

$$\langle \beta_k, \alpha_i^\vee \rangle = -2, -1, 0, 1, \text{ or } 2.$$

If $\langle \beta_k, \alpha_i^\vee \rangle = 1$ (resp. -1), then $\beta_k - \alpha_i = s_i(\beta_k) \in \Phi_+$ (resp. $\beta_k + \alpha_i = s_i(\beta_k) \in \Phi_+$). Hence β_k is α_i-shiftable. This contradicts the definition of the ground α_i-floor.
If $\langle \beta_k, \alpha_i^\vee \rangle = -2$, then, by Corollary 5.3, we have:

$$\langle s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = 1.$$

By Lemma 4.2, we have $s_{-\alpha_i} s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda) \in P_{\geq -1}$ and $\alpha_i \in \mathrm{D}(s_{-\alpha_i} s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda))$. Since

$$s_i(\beta_k) > 0, \quad \text{and}$$
$$\langle s_{-\alpha_i} s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda), s_i(\beta_k)^\vee \rangle = \langle s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda), \beta_k^\vee \rangle = -1,$$

we have $s_i(\beta_k) \in \mathrm{D}(s_{-\alpha_i} s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda))$. Since

$$\langle s_i(\beta_k), \alpha_i^\vee \rangle = 2 \text{ and } s_i(\beta_k) \text{ is not } \alpha_i\text{-shiftable},$$

we have $\langle \alpha_i, s_i(\beta_k)^\vee \rangle = 2$, by Lemma 5.8. Hence, we have $s_i(\beta_k) = \alpha_i$ by Lemma 5.5 (2). Hence we have $\beta_k = -\alpha_i$. This contradicts the definition of a λ-path.
If $\langle \beta_k, \alpha_i^\vee \rangle = 2$, then, since β_k is not α_i-shiftable we have $\langle \alpha_i, \beta_k^\vee \rangle = 2$ by Lemma 5.8. Hence, we have $\beta_k = \alpha_i$ by Lemma 5.5 (2).
Hence we have either $\langle \beta_k, \alpha_i^\vee \rangle = 0$ or $\beta_k = \alpha_i$. This proves the first statement.
If there exists k_1, k_2 ($k_1 \neq k_2$) such that $\beta_{k_j} = \alpha_i$ ($j = 1, 2$), then

$$\begin{aligned}\langle s_{\beta_{p_1-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle &= \langle \lambda + \beta_1 + \cdots + \beta_{p_1-1}, \alpha_i^\vee \rangle \\ &\geq \langle \lambda, \alpha_i^\vee \rangle + \langle \beta_{k_1}, \alpha_i^\vee \rangle + \langle \beta_{k_2}, \alpha_i^\vee \rangle = -1 + 2 + 2 = 3\end{aligned}$$

This contradicts Theorem 5.1. Hence we get the uniqueness part of the Proposition. □

Lemma 8.2. *Let $\mu \in P_{\geq -1}$ and $\alpha_i \in \mathrm{D}(\mu) \cap \Pi$.*

(1) *Let $\beta \in \mathrm{D}(\mu)$ be α_i-shiftable. Then we have $\beta - \alpha_i \in \mathrm{D}(s_{\alpha_i}(\mu))$.*

(2) *Let $\beta \in \mathrm{D}(s_{\alpha_i}(\mu))$ be α_i-shiftable. Then we have $\beta + \alpha_i \in \mathrm{D}(\mu)$.*

Proof. (1) By Corollary 5.4, we have $\langle \beta, \alpha_i^\vee \rangle = 0, 1,$ or 2.
If $\langle \beta, \alpha_i^\vee \rangle = 0$, then, by Lemma 5.9 (1), we have $\beta + \alpha_i \in \mathrm{D}(\mu)$. Hence, by Lemma 4.1,

$$\beta - \alpha_i = s_i(\beta + \alpha_i) \in \mathrm{D}(s_{\alpha_i}(\mu))$$

If $\langle \beta, \alpha_i^\vee \rangle = 1$, then we have:

$$\beta - \alpha_i = s_i(\beta) \in \mathrm{D}(s_{\alpha_i}(\mu)).$$

If $\langle \beta, \alpha_i^\vee \rangle = 2$, then, by Lemma 5.9 (2), we have $\beta - \alpha_i \in \mathrm{D}(\mu)$. Hence,

$$\beta - \alpha_i = s_i(\beta - \alpha_i) \in \mathrm{D}(s_{\alpha_i}(\mu)).$$

Thus, we always have $\beta - \alpha_i \in \mathrm{D}(s_{\alpha_i}(\mu))$.
(2) By Corollary 5.4, we have $\langle s_i(\beta), \alpha_i^\vee \rangle = 0, 1,$ or 2.
If $\langle s_i(\beta), \alpha_i^\vee \rangle = 0$, then, by Lemma 5.9 (1), we have $s_i(\beta) + \alpha_i \in \mathrm{D}(\mu)$. Hence,

$$\beta + \alpha_i = s_i(\beta) + \alpha_i \in \mathrm{D}(\mu).$$

If $\langle s_i(\beta), \alpha_i^\vee \rangle = 1$, then, we have:

$$\beta + \alpha_i = s_i(\beta) \in \mathrm{D}(\mu).$$

If $\langle s_i(\beta), \alpha_i^\vee \rangle = 2$, then, by Lemma 5.9 (2), we have $s_i(\beta) - \alpha_i \in \mathrm{D}(\mu)$. Hence,

$$\beta + \alpha_i = s_i(\beta) - \alpha_i \in \mathrm{D}(\mu).$$

Thus, we always have $\beta + \alpha_i \in \mathrm{D}(\mu)$. □

For a sequence $\mathcal{B} = (\beta_1, \cdots, \beta_l)$ of roots. We define a rational function $f_{\mathcal{B}}$ by:

$$f_{\mathcal{B}} := \prod_{p=1}^{l} \frac{1}{\sum_{k=1}^{p} \beta_k},$$

if $\sum_{k=1}^{p} \beta_k \neq 0$ for any p.

Proposition 8.3. *Let $\lambda \in P_{\geq -1}$ be finite and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$.*

(1) *If one of the following sequence $\mathcal{B}, \mathcal{B}_j\,(1 \leq j \leq r)$ is a λ-path with indicated ground α_i-floor and α_i-up-stairs, then any one of the other sequences is a λ-path with indicated ground α_i-floor and α_i-up-stairs.*

$$\begin{aligned}
\mathcal{B} &:= (\overbrace{\beta_1, \cdots\cdots\cdots\cdots, \beta_{r-1}}^{\text{ground } \alpha_i\text{-floor}}, \overbrace{\beta_r, \beta_{r+1}, \cdots, \beta_l}^{\alpha_i\text{-up-stairs}})\\
\mathcal{B}_r &:= (\beta_1, \cdots\cdots\cdots\cdots, \beta_{r-1}, \alpha_i, \beta_r - \alpha_i, \beta_{r+1}, \cdots, \beta_l)\\
&\ \ \vdots\\
\mathcal{B}_j &:= (\beta_1, \cdots, \beta_{j-1}, \alpha_i, \beta_j, \cdots, \beta_{r-1}, \beta_r - \alpha_i, \beta_{r+1}, \cdots, \beta_l)\\
&\ \ \vdots\\
\mathcal{B}_1 &:= (\alpha_i, \beta_1, \cdots\cdots\cdots\cdots, \beta_{r-1}, \beta_r - \alpha_i, \beta_{r+1}, \cdots, \beta_l)
\end{aligned}$$

Here $\beta_k\,(1 \leq k \leq r-1)$ are positive roots such that $\langle \beta_k, \alpha_i^\vee \rangle = 0$ and that β_k are not α_i-shiftable.

(2) *Under the same assumption as in* (1), *we have:*

$$f_{\mathcal{B}} + \sum_{k=1}^{r} f_{\mathcal{B}_k} = \left(1 + \frac{1}{\alpha_i}\right) f_{\mathcal{B}}.$$

Proof. (1) First, we prove $\mathcal{B}_r \in \mathrm{Path}(\lambda)$, supposing that $\mathcal{B} \in \mathrm{Path}(\lambda)$. By our assumption, we have:

$$\langle s_{\beta_{r-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = \langle \lambda, \alpha_i^\vee \rangle + \langle \beta_1, \alpha_i^\vee \rangle + \cdots + \langle \beta_{r-1}, \alpha_i^\vee \rangle = -1.$$

Hence,

$$\alpha_i \in \mathrm{D}(s_{\beta_{r-1}} \cdots s_{\beta_1}(\lambda)). \tag{8.1}$$

If the α_i-up-stairs $\mathcal{F}^{(+)}_{\alpha_i}(\mathcal{B}) = (\beta_r, \cdots, \beta_l)$ is empty, then this proves the assertion. So, we may assume that $\mathcal{F}^{(+)}_{\alpha_i}(\mathcal{B})$ is not empty. Since $\mathcal{B} \in \mathrm{Path}(\lambda)$, we have:

$$\beta_r \in \mathrm{D}(s_{\beta_{r-1}} \cdots s_{\beta_1}(\lambda)). \tag{8.2}$$

By our assumption, β_r is α_i-shiftable. Hence, applying Lemma 8.2(1) to (8.1) and (8.2), we have:

$$\beta_r - \alpha_i \in \mathrm{D}(s_{\alpha_i} s_{\beta_{r-1}} \cdots s_{\beta_1}(\lambda)). \tag{8.3}$$

Furthurmore, since $s_{\beta_r - \alpha_i} s_{\alpha_i} s_{\beta_{r-1}} \cdots s_{\beta_1}(\lambda) = \lambda + \beta_1 + \cdots + \beta_r = s_{\beta_r} s_{\beta_{r-1}} \cdots s_{\beta_1}(\lambda)$, we have:

$$(\beta_{r+1}, \cdots, \beta_l) \in \mathrm{Path}(s_{\beta_r - \alpha_i} s_{\alpha_i} s_{\beta_{r-1}} \cdots s_{\beta_1}(\lambda)). \tag{8.4}$$

By (8.1), (8.3) and (8.4), we get $\mathcal{B}_r \in \mathrm{Path}(\lambda)$.

Next, we prove $\mathcal{B} \in \mathrm{Path}(\lambda)$, supposing that $\mathcal{B}_r \in \mathrm{Path}(\lambda)$. If the α_i-upstairs $\mathcal{F}^{(+)}_{\alpha_i}(\mathcal{B}_r) = (\beta_r - \alpha_i, \beta_{r+1}, \cdots, \beta_l)$ is empty, then the assertion is trivial. So, we may assume that $\mathcal{F}^{(+)}_{\alpha_i}(\mathcal{B}_r)$ is not empty. Then we have:

$$\alpha_i \in \mathrm{D}(s_{\beta_{r-1}} \cdots s_{\beta_1}(\lambda)) \tag{8.5}$$

and

$$\beta_r - \alpha_i \in \mathrm{D}(s_{\alpha_i} s_{\beta_{r-1}} \cdots s_{\beta_1}(\lambda)). \tag{8.6}$$

Applying Lemma 8.2 (2) to (8.5) and (8.6), we have:

$$\beta_r = (\beta_r - \alpha_i) + \alpha_i \in \mathrm{D}(s_{\beta_{r-1}} \cdots s_{\beta_1}(\lambda)). \tag{8.7}$$

Furthurmore, since $s_{\beta_r-\alpha_i} s_{\alpha_i} s_{\beta_{r-1}} \cdots s_{\beta_1}(\lambda) = s_{\beta_r} s_{\beta_{r-1}} \cdots s_{\beta_1}(\lambda)$, we have:

$$(\beta_{r+1}, \cdots, \beta_l) \in \mathrm{Path}(s_{\beta_r} s_{\beta_{r-1}} \cdots s_{\beta_1}(\lambda)). \tag{8.8}$$

By (8.7) and (8.8), we get $\mathcal{B} \in \mathrm{Path}(\lambda)$.
Finally, by Lemma 5.10, for $1 \le j \le r-1$, $\mathcal{B}_{j+1} \in \mathrm{Path}(\lambda)$ is equivalent to $\mathcal{B}_j \in \mathrm{Path}(\lambda)$.
This proves part (1).

(2) For a proof of part (2), it is enough to put $\delta_k := \beta_1 + \cdots + \beta_k$ $(1 \le k \le r-1)$ and apply Lemma 8.4 below. □

Lemma 8.4. *For indeterminates $\alpha, \delta_1, \cdots, \delta_{r-1} (r \ge 2)$, we have:*

$$\begin{aligned} &\frac{1}{\delta_1}\cdots\frac{1}{\delta_{r-2}}\frac{1}{\delta_{r-1}}\frac{1}{\delta_{r-1}+\alpha}\\ &+\frac{1}{\delta_1}\cdots\frac{1}{\delta_{r-2}}\frac{1}{\delta_{r-2}+\alpha}\frac{1}{\delta_{r-1}+\alpha}\\ &+\cdots\cdots\\ &+\frac{1}{\delta_1}\cdots\frac{1}{\delta_k}\frac{1}{\delta_k+\alpha}\cdots\frac{1}{\delta_{r-1}+\alpha} \\ &+\cdots\cdots\\ &+\frac{1}{\delta_1}\frac{1}{\delta_2}\frac{1}{\delta_2+\alpha}\cdots\frac{1}{\delta_{r-1}+\alpha}\\ &+\frac{1}{\delta_1}\frac{1}{\delta_1+\alpha}\frac{1}{\delta_2+\alpha}\cdots\frac{1}{\delta_{r-1}+\alpha}\\ &+\frac{1}{\alpha}\frac{1}{\delta_1+\alpha}\frac{1}{\delta_2+\alpha}\cdots\frac{1}{\delta_{r-1}+\alpha}\\ &= \frac{1}{\delta_1}\frac{1}{\delta_2}\cdots\frac{1}{\delta_{r-2}}\frac{1}{\delta_{r-1}}\frac{1}{\alpha}.\end{aligned}$$

The Lemma can be proved by induction on r. For instance, if $r = 4$, then the proof is given below:

$$\begin{aligned} &\frac{1}{\delta_1}\frac{1}{\delta_2}\frac{1}{\delta_3}\frac{1}{\delta_3+\alpha} +\frac{1}{\delta_1}\frac{1}{\delta_2}\frac{1}{\delta_2+\alpha}\frac{1}{\delta_3+\alpha} +\frac{1}{\delta_1}\frac{1}{\delta_1+\alpha}\frac{1}{\delta_2+\alpha}\frac{1}{\delta_3+\alpha} +\frac{1}{\alpha}\frac{1}{\delta_1+\alpha}\frac{1}{\delta_2+\alpha}\frac{1}{\delta_3+\alpha}\\ &= \frac{1}{\delta_1}\frac{1}{\delta_2}\frac{1}{\delta_3}\frac{1}{\delta_3+\alpha} +\frac{1}{\delta_1}\frac{1}{\delta_2}\frac{1}{\delta_2+\alpha}\frac{1}{\delta_3+\alpha} +\frac{1}{\delta_1}\frac{1}{\alpha}\frac{1}{\delta_2+\alpha}\frac{1}{\delta_3+\alpha}\\ &= \frac{1}{\delta_1}\frac{1}{\delta_2}\frac{1}{\delta_3}\frac{1}{\delta_3+\alpha} +\frac{1}{\delta_1}\frac{1}{\delta_2}\frac{1}{\alpha}\frac{1}{\delta_3+\alpha}\\ &= \frac{1}{\delta_1}\frac{1}{\delta_2}\frac{1}{\delta_3}\frac{1}{\alpha}.\end{aligned}$$

Definition 8. Let $\lambda \in P_{\geq -1}$ and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. The ground α_i-floor $\mathcal{G}_{\alpha_i}(\mathcal{B})$ is said to be *basic* if

$$\alpha_i \notin \mathcal{G}_{\alpha_i}(\mathcal{B}), \text{(namely, if } \alpha_i \text{ does not appear in } \mathcal{G}_{\alpha_i}(\mathcal{B}) \text{).}$$

The set of elements $\mathcal{B}$ of $\mathrm{Path}(\lambda)$ such that $\mathcal{G}_{\alpha_i}(\mathcal{B})$ is basic is denoted by $\mathrm{Path}_{\alpha_i}(\lambda)$.

We note that, for a λ-path $\mathcal{B} = (\beta_1, \cdots, \beta_l)$, the following two conditions are equivalent:

(1) $\mathcal{B} \in \mathrm{Path}_{\alpha_i}(\lambda)$,

(2) if $\beta_q = \alpha_i$, then there exists an index $p < q$ such that β_p is α_i-shiftable.

Proposition 8.5. *Let $\lambda \in P_{\geq -1}$ be finite and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. Then we have:*

$$\sum_{\mathcal{B} \in \mathrm{Path}(\lambda)} f_{\mathcal{B}} = \left(1 + \frac{1}{\alpha_i}\right) \sum_{\mathcal{B} \in \mathrm{Path}_{\alpha_i}(\lambda)} f_{\mathcal{B}}.$$

Proof. For $C, C' \in \mathrm{Path}(\lambda)$, we denote $C \sim C'$, if:

there exists a $\mathcal{B} \in \mathrm{Path}_{\alpha_i}(\lambda)$ such that $C = \mathcal{B}$ or $\mathcal{B}_j$, and $C' = \mathcal{B}$ or $\mathcal{B}_{j'}$,

where $\mathcal{B}_j$ and $\mathcal{B}_{j'}$ are λ-paths indicated in Proposition 8.3 (1). We note that such a $\mathcal{B} \in \mathrm{Path}_{\alpha_i}(\lambda)$ is unique if it exists. The binary relation $\sim$ is an equivalence relation. For each equivalence class $\mathcal{E} \in \mathrm{Path}(\lambda)/\sim$, there exists, by Proposition 8.1 and Proposition 8.3 (1), a unique element $\mathcal{B}$ in $\mathcal{E}$ such that $\mathcal{B} \in \mathrm{Path}_{\alpha_i}(\lambda)$. We denote such $\mathcal{B}$ by $\mathcal{B}_{\mathcal{E}}$. By Proposition 8.3 (2), we have:

$$\begin{aligned}\sum_{\mathcal{B} \in \mathrm{Path}(\lambda)} f_{\mathcal{B}} &= \sum_{\mathcal{E} \in \mathrm{Path}(\lambda)/\sim} \sum_{\mathcal{B} \in \mathcal{E}} f_{\mathcal{B}} = \sum_{\mathcal{E} \in \mathrm{Path}(\lambda)/\sim} \left(1 + \frac{1}{\alpha_i}\right) f_{\mathcal{B}_{\mathcal{E}}} \\ &= \left(1 + \frac{1}{\alpha_i}\right) \sum_{\mathcal{E} \in \mathrm{Path}(\lambda)/\sim} f_{\mathcal{B}_{\mathcal{E}}} = \left(1 + \frac{1}{\alpha_i}\right) \sum_{\mathcal{B} \in \mathrm{Path}_{\alpha_i}(\lambda)} f_{\mathcal{B}}.\end{aligned}$$

□

9. Proof of The Main Theorem (Second Part)

Let $\mu \in P_{\geq -1}$ and $\alpha_i \in \Pi$. We define a set $\mathrm{GD}^{\alpha_i}(\mu)$ by:

$$\mathrm{GD}^{\alpha_i}(\mu) := \left\{\gamma \in \Phi_+ \cup \{-\alpha_i\} \middle| \langle \mu, \gamma^\vee\rangle = -1\right\}.$$

Let $\lambda \in P_{\geq -1}$ and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. Let l be a nonnegative integer. The set of sequences $\mathcal{B} = (\beta_1, \beta_2, \cdots, \beta_l)$ of elements of Φ satisfying the following condition:

$$\beta_p \in \mathrm{GD}^{\alpha_i}(s_{\beta_{p-1}} \cdots s_{\beta_1}(\lambda)), \quad 1 \leq p \leq l,$$

is denoted by $\mathrm{GPath}_{\alpha_i}(\lambda)$.

Lemma 9.1. *Let $\lambda \in P_{\geq -1}$ and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. Let $(\beta_1, \cdots, \beta_{k-1}, \beta_k, \beta_{k+1}, \beta_{k+2}, \cdots, \beta_l) \in \mathrm{GPath}_{\alpha_i}(\lambda)$. If $\langle \beta_k, \beta_{k+1}^\vee\rangle = 0$, then we have:*

$$\mathcal{B}' := (\beta_1, \cdots, \beta_{k-1}, \beta_{k+1}, \beta_k, \beta_{k+2}, \cdots, \beta_l) \in \mathrm{GPath}_{\alpha_i}(\lambda).$$

Proof. We omit the proof that $\mathcal{B}' \in \mathrm{GPath}_{\alpha_i}(\lambda)$, since it is similar to that of Lemma 5.10. □

Lemma 9.2. *Let $\mu \in P_{\geq -1}$ and $\alpha_i \in \Pi$. Suppose that $\langle \mu, \alpha_i^\vee\rangle = -1, 0,$ or 1. Let $\beta \in \mathrm{GD}^{\alpha_i}(\mu)$ be α_i-shiftable.*

(1) *If $\langle s_\beta(\mu), \alpha_i^\vee\rangle = -1$, then we have $\beta + \alpha_i \in \mathrm{GD}^{\alpha_i}(\mu)$.*
(2) *If $\langle s_\beta(\mu), \alpha_i^\vee\rangle = 1$, then we have $\beta - \alpha_i \in \mathrm{GD}^{\alpha_i}(\mu)$.*

Proof. (1) By our assumption, we have $\langle \mu + \beta, \alpha_i^\vee\rangle = -1$. Since β is α_i-shiftable, we have $\beta \neq -\alpha_i$. Hence, we have $\beta \in \mathrm{D}(\mu)$.
If $\langle \mu, \alpha_i^\vee\rangle = -1$, then $\langle \beta, \alpha_i^\vee\rangle = 0$. Since $\beta, \alpha_i \in \mathrm{D}(\mu)$ and β is α_i-shiftable, we have $\beta + \alpha_i \in \mathrm{D}(\mu) \subseteq \mathrm{GD}^{\alpha_i}(\mu)$ by Lemma 5.9 (1).
If $\langle \mu, \alpha_i^\vee\rangle = 0$, then $\langle \beta, \alpha_i^\vee\rangle = -1$. Since $\langle \mu, (\beta + \alpha_i)^\vee\rangle = \langle \mu, s_i(\beta)^\vee\rangle = \langle s_i(\mu), \beta^\vee\rangle = \langle \mu, \beta^\vee\rangle = -1$, we have $\beta + \alpha_i \in \mathrm{GD}^{\alpha_i}(\mu)$
If $\langle \mu, \alpha_i^\vee\rangle = 1$, then $\langle \beta, \alpha_i^\vee\rangle = -2$. Since $\beta \neq \alpha_i$, we have $s_i(\beta) \in \mathrm{D}(s_{-\alpha_i}(\mu))$ by Lemma 4.2 (2). Since $s_i(\beta), \alpha_i \in \mathrm{GD}^{\alpha_i}(s_{-\alpha_i}(\mu)), \langle s_i(\beta), \alpha_i^\vee\rangle = 2$, and $s_i(\beta)$ is α_i-shiftable, we have $s_i(\beta) - \alpha_i \in \mathrm{D}(s_{-\alpha_i}(\mu))$ by Lemma 5.9 (2). Since $s_i(\beta) - \alpha_i \neq \alpha_i$, we have $\beta + \alpha_i \in \mathrm{D}(\mu) \subseteq \mathrm{GD}^{\alpha_i}(\mu)$ by Lemma 4.1 (2).
Thus, we always have $\beta + \alpha_i \in \mathrm{GD}^{\alpha_i}(\mu)$.
(2) By our assumption, we have $\langle \mu + \beta, \alpha_i^\vee\rangle = 1$. Since β is α_i-shiftable, we have $\beta \neq -\alpha_i$. Hence, we have $\beta \in \mathrm{D}(\mu)$.
If $\langle \mu, \alpha_i^\vee\rangle = -1$, then $\langle \beta, \alpha_i^\vee\rangle = 2$. Since $\beta, \alpha_i \in \mathrm{D}(\mu)$ and β is α_i-shiftable, we have $\beta - \alpha_i \in \mathrm{D}(\mu) \subseteq \mathrm{GD}^{\alpha_i}(\mu)$ by Lemma 5.9 (2).
If $\langle \mu, \alpha_i^\vee\rangle = 0$, then $\langle \beta, \alpha_i^\vee\rangle = 1$. Since $\langle \mu, (\beta - \alpha_i)^\vee\rangle = \langle \mu, s_{\alpha_i}(\beta)^\vee\rangle = \langle s_{\alpha_i}(\mu), \beta^\vee\rangle = \langle \mu, \beta^\vee\rangle = -1$, we have $\beta - \alpha_i \in \mathrm{GD}^{\alpha_i}(\mu)$.
If $\langle \mu, \alpha_i^\vee\rangle = 1$, then $\langle \beta, \alpha_i^\vee\rangle = 0$. Since $\beta \neq \alpha_i$, we have $s_i(\beta) \in \mathrm{D}(s_{-\alpha_i}(\mu))$ by Lemma 4.2 (2). Since $s_i(\beta), \alpha_i \in \mathrm{D}(s_{-\alpha_i}(\mu)), \langle s_i(\beta), \alpha_i^\vee\rangle = 0$, and $s_i(\beta)$ is α_i-shiftable, we have $s_i(\beta) + \alpha_i \in \mathrm{D}(s_{-\alpha_i}(\mu))$ by Lemma 5.9 (1). Since $s_i(\beta) + \alpha_i \neq \alpha_i$, we have $\beta - \alpha_i \in \mathrm{D}(\mu) \subseteq \mathrm{GD}^{\alpha_i}(\mu)$ by Lemma 4.1 (2).
Thus, we always have $\beta - \alpha_i \in \mathrm{GD}^{\alpha_i}(\mu)$. □

Lemma 9.3. *Let $\mu \in P_{\geq -1}$ and $\alpha_i \in \Pi$. Let $\beta \in \mathrm{GD}^{\alpha_i}(\mu)$ be α_i-shiftable.*

(1) *If $\langle \mu, \alpha_i^\vee \rangle = 1$, then we have $\beta + \alpha_i \in \mathrm{GD}^{\alpha_i}(s_{-\alpha_i}(\mu))$.*
(2) *If $\langle \mu, \alpha_i^\vee \rangle = -1$, then we have $\beta - \alpha_i \in \mathrm{GD}^{\alpha_i}(s_{\alpha_i}(\mu))$.*

Proof. (1) Since β is α_i-shiftable, we have $\beta \neq -\alpha_i$. Hence, we have $\beta \in \mathrm{D}(\mu)$. Since $s_i(\beta), \alpha_i \in \mathrm{D}(s_{-\alpha_i}(\mu))$, we have $\langle s_i(\beta), \alpha_i^\vee \rangle = 0, 1$, or 2 by Lemma 4.2 (2) and Corollary 5.4.
If $\langle s_i(\beta), \alpha_i^\vee \rangle = 0$, then we have $\beta + \alpha_i = s_i(\beta) + \alpha_i \in \mathrm{D}(s_{-\alpha_i}(\mu)) \subseteq \mathrm{GD}^{\alpha_i}(s_{-\alpha_i}(\mu))$ by Lemma 5.9(1).
If $\langle s_i(\beta), \alpha_i^\vee \rangle = 1$, then since $\langle s_{-\alpha_i}(\mu), (\beta+\alpha_i)^\vee \rangle = \langle s_{-\alpha_i}(\mu), s_i(\beta)^\vee \rangle = \langle \mu, \beta^\vee \rangle = -1$, we have $\beta+\alpha_i \in \mathrm{D}(s_{-\alpha_i}(\mu)) \subseteq \mathrm{GD}^{\alpha_i}(s_{-\alpha_i}(\mu))$.
If $\langle s_i(\beta), \alpha_i^\vee \rangle = 2$, then we have $\beta + \alpha_i = s_i(\beta) - \alpha_i \in \mathrm{D}(s_{-\alpha_i}(\mu)) \subseteq \mathrm{GD}^{\alpha_i}(s_{-\alpha_i}(\mu))$ by Lemma 5.9(2). Thus, we always have $\beta + \alpha_i \in \mathrm{GD}^{\alpha_i}(s_{-\alpha_i}(\mu))$.
(2) Since β is α_i-shiftable, we have $\beta \neq -\alpha_i$. Hence, we have $\beta \in \mathrm{D}(\mu)$. Since $\beta, \alpha_i \in \mathrm{D}(\mu)$, we have $\langle \beta, \alpha_i^\vee \rangle = 0, 1$, or 2 by Corollary 5.4.
If $\langle \beta, \alpha_i^\vee \rangle = 0$, then we have $\beta + \alpha_i \in \mathrm{D}(\mu)$ by Lemma 5.9 (1). Since $\beta + \alpha_i \neq \alpha_i$, we have $\beta - \alpha_i = s_i(\beta + \alpha_i) \in \mathrm{D}(s_{\alpha_i}(\mu)) \subseteq \mathrm{GD}^{\alpha_i}(s_{\alpha_i}(\mu))$ by Lemma 4.1 (2).
If $\langle \beta, \alpha_i^\vee \rangle = 1$, then since $\langle s_{\alpha_i}(\mu), (\beta - \alpha_i)^\vee \rangle = \langle s_{\alpha_i}(\mu), s_i(\beta)^\vee \rangle = \langle \mu, \beta^\vee \rangle = -1$, we have $\beta - \alpha_i \in \mathrm{GD}^{\alpha_i}(s_{\alpha_i}(\mu))$.
If $\langle \beta, \alpha_i^\vee \rangle = 2$, then we have $\beta - \alpha_i \in \mathrm{D}(\mu)$ by Lemma 5.9 (2). Hence, we have $\beta - \alpha_i = s_i(\beta - \alpha_i) \in \mathrm{D}(s_{\alpha_i}(\mu)) \subseteq \mathrm{GD}^{\alpha_i}(s_{\alpha_i}(\mu))$ by Lemma 4.1 (2).
Thus, we always have $\beta - \alpha_i \in \mathrm{GD}^{\alpha_i}(s_{\alpha_i}(\mu))$. □

Lemma 9.4. *Let $\lambda \in P_{\geq -1}$ and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. If $\mathcal{B} = (\beta_1, \cdots, \beta_l) \in \mathrm{GPath}_{\alpha_i}(\lambda)$, then there exists a $(\beta'_1, \cdots, \beta'_{l'}) \in \mathrm{Path}(\lambda)$ such that*

$$s_{\beta_l} \cdots s_{\beta_1}(\lambda) = s_{\beta'_{l'}} \cdots s_{\beta'_1}(\lambda).$$

Proof. If $-\alpha_i \notin \mathcal{B}$, then there is nothing to prove. If $-\alpha_i \in \mathcal{B}$, then let k be the smallest index such that $\beta_k = -\alpha_i$. Applying Lemma 9.1 to $\mathcal{B}$ repeatedly, we get an element:

$$(\beta_1, \cdots, \beta_p, -\alpha_i, \beta_{p+1}, \cdots, \beta_{k-1}, \beta_{k+1}, \cdots, \beta_l) \in \mathrm{GPath}_{\alpha_i}(\lambda) \text{ such that } \langle \beta_p, (-\alpha_i)^\vee \rangle \neq 0.$$

Since $-\alpha_i \in \mathrm{GD}^{\alpha_i}(s_{\beta_p} \cdots s_{\beta_1}(\lambda))$, we have $\langle s_{\beta_p} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = 1$. Since $(\beta_1, \cdots, \beta_{p-1}) \in \mathrm{Path}(\lambda)$, we have $\langle s_{\beta_{p-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = -1, 0$, or 1 by Theorem 5.1. Hence, we have $\langle \beta_p, \alpha_i^\vee \rangle = 1$, or 2.
If $\langle \beta_p, \alpha_i^\vee \rangle = 1$, then β_p is α_i-shiftable. Hence, by Lemma 9.2 (2), we have $(\beta_p - \alpha_i, \alpha_i) \in \mathrm{Path}(s_{\beta_{p-1}} \cdots s_{\beta_1}(\lambda))$. We have:

$$(\beta_1, \cdots, \beta_{p-1}, \beta_p - \alpha_i, \beta_{p+1}, \cdots, \beta_{k-1}, \beta_{k+1}, \cdots, \beta_l) \in \mathrm{GPath}_{\alpha_i}(\lambda).$$

If, on the other hand, $\langle \beta_p, \alpha_i^\vee \rangle = 2$, then we have $\beta_p, \alpha_i \in \mathrm{D}(s_{\beta_{p-1}} \cdots s_{\beta_1}(\lambda))$ by Corollary 5.3(2). By Lemma 5.5 (1), we have $\langle \alpha_i, \beta_p^\vee \rangle = 1$, or 2.
If $\langle \alpha_i, \beta_p^\vee \rangle = 1$, then β_p is α_i-shiftable. Hence, by Lemma 9.2 (2), we have $(\beta_p - \alpha_i, \alpha_i) \in \mathrm{Path}(s_{\beta_{p-1}} \cdots s_{\beta_1}(\lambda))$. We have:

$$(\beta_1, \cdots, \beta_{p-1}, \beta_p - \alpha_i, \beta_{p+1}, \cdots, \beta_{k-1}, \beta_{k+1}, \cdots, \beta_l) \in \mathrm{GPath}_{\alpha_i}(\lambda).$$

If $\langle \alpha_i, \beta_p^\vee \rangle = 2$, then we have $\beta_p = \alpha_i$ or $(\alpha_i, \beta_p - 2\alpha_i, \alpha_i) \in \mathrm{Path}(s_{\beta_{p-1}} \cdots s_{\beta_1}(\lambda))$ by Lemma 5.7(1).
If $\beta_p = \alpha_i$, then we have:

$$(\beta_1, \cdots, \beta_{p-1}, \beta_{p+1}, \cdots, \beta_{k-1}, \beta_{k+1}, \cdots, \beta_l) \in \mathrm{GPath}_{\alpha_i}(\lambda).$$

If, on the other hand, $(\alpha_i, \beta_p - 2\alpha_i, \alpha_i) \in \mathrm{Path}(s_{\beta_{p-1}} \cdots s_{\beta_1}(\lambda))$, then we have:

$$(\beta_1, \cdots, \beta_{p-1}, \alpha_i, \beta_p - 2\alpha_i, \beta_{p+1}, \cdots, \beta_{k-1}, \beta_{k+1}, \cdots, \beta_l) \in \mathrm{GPath}_{\alpha_i}(\lambda).$$

Applying the above argument repeatedly, we finally get an element

$$\mathcal{B}' = (\beta_1', \cdots, \beta_{l'}') \in \mathrm{GPath}_{\alpha_i}(\lambda)$$

which contains no $-\alpha_i$. Thus, we get $\mathcal{B}' \in \mathrm{Path}(\lambda)$ such that $s_{\beta_l} \cdots s_{\beta_1}(\lambda) = s_{\beta_{l'}'} \cdots s_{\beta_1'}(\lambda)$. □

Let $\mathcal{B} = (\beta_1, \cdots, \beta_l) \in \mathrm{GPath}_{\alpha_i}(\lambda)$. For each $1 \le k \le l$, we have $\sum_{p=1}^{k} \beta_p > 0$ by the above Lemma. Hence, we can always define $f_{\mathcal{B}}$ for $\mathcal{B} \in \mathrm{GPath}_{\alpha_i}(\lambda)$.

Theorem 9.5. *Let $\lambda \in P_{\ge -1}$ and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. Let $(\beta_1, \beta_2, \cdots, \beta_l) \in \mathrm{GPath}_{\alpha_i}(\lambda)$. Then we have:*

$$\langle s_{\beta_l} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = -1, 0 \text{ or } 1.$$

Proof. This follows from Theorem 5.1 and Lemma 9.4. □

Corollary 9.6, Corollary 9.7, and Corollary 9.8 below are generalization of Corollary 5.2, Corollary 5.3, and Corollary 5.4, respectively. Since proofs are entirely similar to those of the corresponding ones, we omit them.

Corollary 9.6. *Let $\lambda \in P_{\ge -1}$ and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. Let $(\beta_1, \cdots, \beta_l) \in \mathrm{GPath}_{\alpha_i}(\lambda)$. Then we have:*

$$\langle \beta_k, \alpha_i^\vee \rangle = -2, -1, 0, 1, \text{ or } 2, \quad 1 \le k \le l.$$

Corollary 9.7. *Let $\lambda \in P_{\ge -1}$ and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. Let $(\beta_1, \cdots, \beta_l) \in \mathrm{GPath}_{\alpha_i}(\lambda)$. Let $1 \le k \le l$.*

(1) *If $\langle \beta_k, \alpha_i^\vee \rangle = 2$, then we have*
$\langle s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = -1$ *and* $\langle s_{\beta_k} s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = 1$.
(2) *If $\langle \beta_k, \alpha_i^\vee \rangle = -2$, then we have*
$\langle s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = 1$ *and* $\langle s_{\beta_k} s_{\beta_{k-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = -1$.

Corollary 9.8. *Let $\lambda \in P_{\ge -1}$ and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. Let $\beta \in \mathrm{GD}^{\alpha_i}(\lambda)$. Then we have:*

$$\langle \beta, \alpha_i^\vee \rangle = 0, 1, \text{ or } 2.$$

Let $\lambda \in P_{\ge -1}$ and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. We define a set $s_i(\mathrm{Path}(s_{\alpha_i}(\lambda)))$ by:

$$s_i(\mathrm{Path}(s_{\alpha_i}(\lambda))) := \left\{ (s_i(\gamma_1), \cdots, s_i(\gamma_l)) \,\middle|\, (\gamma_1, \cdots, \gamma_l) \in \mathrm{Path}(s_{\alpha_i}(\lambda)) \right\}.$$

Lemma 9.9. *Let $\lambda \in P_{\geq -1}$ and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$.*

(1) *We have:*

$$\mathrm{Path}(\lambda) = \left\{\mathcal{B} \in \mathrm{GPath}_{\alpha_i}(\lambda) \,\middle|\, -\alpha_i \notin \mathcal{B}\right\}.$$

(2) *For $\mathcal{B}=(\beta_1,\cdots,\beta_l) \in \mathrm{GPath}_{\alpha_i}(\lambda)$, the following two conditions* (a),(b) *are equivalent:*

(a) *$\mathcal{B} \in \mathrm{Path}_{\alpha_i}(\lambda)$,*

(b) *$-\alpha_i \notin \mathcal{B}$, and if $\beta_q = \alpha_i$, then there exists an index $p<q$ such that β_p is α_i-shiftable.*

(3) *We have:*

$$s_i\,(\mathrm{Path}(s_{\alpha_i}(\lambda))) = \left\{\mathcal{B} \in \mathrm{GPath}_{\alpha_i}(\lambda) \,\middle|\, \alpha_i \notin \mathcal{B}\right\}.$$

(4) *Let $(\beta_1,\cdots,\beta_l) \in s_i\,(\mathrm{Path}(s_{\alpha_i}(\lambda)))$. Suppose that λ is finite. If $\beta_q = -\alpha_i$, then there exists an index $p<q$ such that β_p is α_i-shiftable.*

Proof. (1) It is straightforward to see.

(2) This follows from definition 8.

(3) It is straightforward to see.

(4) Let $\mathcal{B} = (\beta_1,\cdots,\beta_l) \in s_i\,(\mathrm{Path}(s_{\alpha_i}(\lambda)))$. Then we have $s_i(\mathcal{B}) = (s_i(\beta_1),\cdots,s_i(\beta_l)) \in \mathrm{Path}(s_{\alpha_i}(\lambda))$. Let $\beta_q = -\alpha_i$. Then we have $s_i(\beta_q) = \alpha_i$. We have:

$$\langle s_{\alpha_i}(\lambda),\, \alpha_i^\vee\rangle = 1, \tag{9.1}$$

and

$$\langle s_{\alpha_i} s_{s_i(\beta_{q-1})}\cdots s_{s_i(\beta_1)} s_{\alpha_i}\lambda,\, \alpha_i^\vee\rangle = \langle s_{s_i(\beta_q)} s_{s_i(\beta_{q-1})}\cdots s_{s_i(\beta_1)} s_{\alpha_i}\lambda,\, \alpha_i^\vee\rangle = 1.$$

Hence, we have:

$$\langle s_{s_i(\beta_{q-1})}\cdots s_{s_i(\beta_1)} s_{\alpha_i}\lambda,\, \alpha_i^\vee\rangle = -1. \tag{9.2}$$

By (9.1) and (9.2), we have, for some $1 \leq p \leq q-1$,

(9.3)

$$\langle s_{s_i(\beta_{p-1})}\cdots s_{s_i(\beta_1)} s_{\alpha_i}\lambda,\, \alpha_i^\vee\rangle = 1 \text{ and } \langle s_{s_i(\beta_p)} s_{s_i(\beta_{p-1})}\cdots s_{s_i(\beta_1)} s_{\alpha_i}\lambda,\, \alpha_i^\vee\rangle = -1,$$

or

(9.4)

$$\langle s_{s_i(\beta_{p-1})}\cdots s_{s_i(\beta_1)} s_{\alpha_i}\lambda,\, \alpha_i^\vee\rangle = 1 \text{ and } \langle s_{s_i(\beta_p)} s_{s_i(\beta_{p-1})}\cdots s_{s_i(\beta_1)} s_{\alpha_i}\lambda,\, \alpha_i^\vee\rangle = 0.$$

If (9.3) holds, then, since λ is finite, $s_i(\beta_p)$ is α_i-shiftable by Lemma 5.5(2) and Lemma 5.8. If, on the other hand, (9.4) holds, it is trivial that $s_i(\beta_p)$ is α_i-shiftable.

Thus, $s_i(\beta_p)$ is always α_i-shiftable. And so is β_p. This proves part (4). □

Definition 9. Let $\lambda \in P_{\geq -1}$ and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. Let $\mathcal{B} = (\beta_1,\cdots,\beta_l) \in \mathrm{GPath}_{\alpha_i}(\lambda)$.We suppose that there exists exactly m indices $1 \leq p_1 < \cdots < p_m \leq l$ such that β_{p_q} is α_i-shiftable($1 \leq q \leq m$). We put $p_{m+1} := l+1$. The subsequence $\mathcal{G}_{\alpha_i}(\mathcal{B}) := (\beta_1,\cdots,\beta_{p_1-1})$ of $\mathcal{B}$ is called the *ground α_i-floor of $\mathcal{B}$*. For $1 \leq q \leq m$, the subsequence $\mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B}) := (\beta_{p_q},\cdots,\beta_{p_{q+1}-1})$ of $\mathcal{B}$ is called the *q-th α_i-floor of $\mathcal{B}$*. The subsequence $\mathcal{F}^{(+)}_{\alpha_i}(\mathcal{B}) := (\mathcal{F}^{(1)}_{\alpha_i}(\mathcal{B}), \cdots, \mathcal{F}^{(m)}_{\alpha_i}(\mathcal{B}))$ is called the *α_i-up-stairs*. Thus $\mathcal{B}$ is written as:

$$\mathcal{B} = (\mathcal{G}_{\alpha_i}(\mathcal{B}), \mathcal{F}^{(+)}_{\alpha_i}(\mathcal{B})) = (\mathcal{G}_{\alpha_i}(\mathcal{B}), \mathcal{F}^{(1)}_{\alpha_i}(\mathcal{B}), \cdots, \mathcal{F}^{(m)}_{\alpha_i}(\mathcal{B})).$$

Proposition 9.10. *Let $\lambda \in P_{\geq -1}$ be finite and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. Let $\mathcal{B} = (\beta_1, \cdots, \beta_l) \in \mathrm{GPath}_{\alpha_i}(\lambda)$. We suppose that there exists exactly m indices $1 \leq p_1 < \cdots < p_m \leq l$ such that β_{p_q} is α_i-shiftable $(1 \leq q \leq m)$. Let $1 \leq q \leq m$. Let $\mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B}) = (\beta_{p_q}, \beta_{p_q+1}, \cdots, \beta_{p_{q+1}-1})$.*

(1) *Then, for $p_q+1 \leq k \leq p_{q+1}-1$, we have:*

$$\langle \beta_k, \alpha_i^\vee \rangle = 0, \quad \textit{unless}\ \beta_k = \alpha_i \text{ or } -\alpha_i.$$

(2) *If $-\alpha_i \notin \mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B})$, then*
an index $k\,(p_q+1 \leq k \leq p_{q+1}-1)$ such that $\beta_k = \alpha_i$ is unique if it exists.

(3) *If $\alpha_i \notin \mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B})$, then*
an index $k\,(p_q+1 \leq k \leq p_{q+1}-1)$ such that $\beta_k = -\alpha_i$ is unique if it exists.

Proof. We omit the proof, since it is similar to that of Proposition 8.1. □

Let $\lambda \in P_{\geq -1}$ be finite and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. The set of $\mathcal{B} = (\beta_1, \cdots, \beta_l) \in \mathrm{GPath}_{\alpha_i}(\lambda)$ satisfying the following two conditions:

(1) if $\beta_q = \alpha_i$ or $-\alpha_i$, then there exists an index $p < q$ such that β_p is α_i-shiftable.

(2) $\alpha_i \notin \mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B})$ or $-\alpha_i \notin \mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B})$, for each q.

is denoted by $\mathrm{QPath}_{\alpha_i}(\lambda)$.
By Lemma 9.9 and Proposition 9.10, we have $\mathrm{Path}_{\alpha_i}(\lambda) \subseteq \mathrm{QPath}_{\alpha_i}(\lambda)$ and $s_i\,(\mathrm{Path}(s_{\alpha_i}(\lambda))) \subseteq \mathrm{QPath}_{\alpha_i}(\lambda)$.

Definition 10. Let $\lambda \in P_{\geq -1}$ and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. Let $\mathcal{B} = (\beta_1, \cdots, \beta_l) \in \mathrm{QPath}_{\alpha_i}(\lambda)$. We suppose that there exists exactly m indices $1 \leq p_1 < \cdots < p_m \leq l$ such that β_{p_q} is α_i-shiftable$(1 \leq q \leq m)$. Let $1 \leq q \leq m$.
The q-th α_i-floor $\mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B})$ is said to be *basic* if

$$\alpha_i \notin \mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B}) \text{ and } -\alpha_i \notin \mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B}).$$

The q-th α_i-floor $\mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B})$ is said to be *positive* if

$$\alpha_i \in \mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B}) \text{ and } -\alpha_i \notin \mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B}).$$

The q-th α_i-floor $\mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B})$ is said to be *negative* if

$$\alpha_i \notin \mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B}) \text{ and } -\alpha_i \in \mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B}).$$

We note that each q-th floor $\mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B})$ is either positive, basic, or negative by Proposition 9.10.

Proposition 9.11. *Let $\lambda \in P_{\geq -1}$ be finite and $\alpha_i \in \mathrm{D}(\lambda)\cap\Pi$. Let $\mathcal{B} = (\beta_1, \cdots, \beta_l) \in \mathrm{QPath}_{\alpha_i}(\lambda)$. We suppose that there exists exactly m indices $1 \leq p_1 < \cdots < p_m \leq l$ such that β_{p_q} is α_i-shiftable$(1 \leq q \leq m)$. For $1 \leq q \leq m$, we define $\tau_q(\mathcal{B}) := (\beta'_1, \cdots, \beta'_l)$ by:*

$$\beta'_k := \begin{cases} \beta_k & \text{if } k < p_q, \\ \beta_k + \alpha_i & \text{if } \mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B}) \textit{ is positive}, \text{ and } k = p_q, \\ -\alpha_i & \text{if } \mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B}) \textit{ is positive}, p_q < k < p_{q+1}, \text{ and } \beta_k = \alpha_i, \\ \beta_k & \text{if } \mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B}) \textit{ is positive}, p_q < k < p_{q+1}, \text{ and } \langle \beta_k, \alpha_i^\vee \rangle = 0, \\ \beta_k + \alpha_i & \text{if } \mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B}) \textit{ is positive}, \text{ and } k = p_{q+1}, \\ \beta_k & \text{if } \mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B}) \textit{ is basic}, \text{ and } p_q \leq k \leq p_{q+1}, \\ \beta_k - \alpha_i & \text{if } \mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B}) \textit{ is negative}, \text{ and } k = p_q, \\ \alpha_i & \text{if } \mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B}) \textit{ is negative}, p_q < k < p_{q+1}, \text{ and } \beta_k = -\alpha_i, \\ \beta_k & \text{if } \mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B}) \textit{ is negative}, p_q < k < p_{q+1}, \text{ and } \langle \beta_k, \alpha_i^\vee \rangle = 0, \\ \beta_k - \alpha_i & \text{if } \mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B}) \textit{ is negative}, \text{ and } k = p_{q+1}, \\ \beta_k & \text{if } p_{q+1} < k. \end{cases}$$

Then we have:

(1) $\tau_q(\mathcal{B}) \in \mathrm{QPath}_{\alpha_i}(\lambda)$.
(2) *The subsequence $(\beta'_{p_q}, \beta'_{p_q+1}, \cdots, \beta'_{p_{q+1}-1})$ is the q-th α_i-floor of $\tau_q(\mathcal{B})$.*
(3) $\tau_q\tau_q(\mathcal{B}) = \mathcal{B}$.

Proof. (1) and (2) If $\mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B})$ is basic, then there is nothing to prove. So, we may assume that $\mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B})$ is positive or negative. Put $\epsilon := 1$ (resp. -1) if $\mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B})$ is positive (resp. negative). Let k_0 be the unique index in $\{p_q+1, \cdots, p_{q+1}-1\}$ such that $\beta_{k_0} = \epsilon\alpha_i$. (See Proposition 9.10)
Since $\epsilon\alpha_i \in \mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B})$, by Corollary 9.7, we have

$$(9.5) \qquad \langle s_{\beta_{k_0-1}} \cdots s_{\beta_{p_q+1}} s_{\beta_{p_q}} s_{\beta_{p_q-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = -\epsilon,$$

and

$$(9.6) \qquad \langle s_{\epsilon\alpha_i} s_{\beta_{k_0-1}} \cdots s_{\beta_{p_q+1}} s_{\beta_{p_q}} s_{\beta_{p_q-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = \epsilon.$$

By Proposition 9.10 (2) and (3), we have:

$$(9.7) \qquad \langle \beta_k, \epsilon\alpha_i^\vee \rangle = 0, \quad p_q < k < k_0,$$

and

$$(9.8) \qquad \langle \beta_k, \epsilon\alpha_i^\vee \rangle = 0, \quad k_0 < k < p_{q+1}.$$

By (9.5) and (9.7), we have:

$$(9.9) \qquad \langle s_{\beta_{p_q}} s_{\beta_{p_q-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee \rangle = -\epsilon.$$

Applying Lemma 9.2 (1) to (9.9), we have:

(9.10)
$$\beta_{p_q} + \epsilon\alpha_i \in \mathrm{GD}^{\alpha_i}(s_{\beta_{p_q-1}} \cdots s_{\beta_1}(\lambda))$$
$$\beta_{p_q} + \epsilon\alpha_i \text{ is } \alpha_i\text{-shiftable.}$$

By (9.10) and induction on k (from $k = p_q+1$ to k_0-1), we have:

$$\begin{aligned}
&\langle s_{\beta_{k-1}} \cdots s_{\beta_{p_q+1}} s_{\beta_{p_q}+\epsilon\alpha_i} s_{\beta_{p_q-1}} \cdots s_{\beta_1}(\lambda), \beta_k^\vee\rangle \\
&= \langle \beta_{k-1} + \cdots + \beta_{p_q+1} + (\beta_{p_q} + \epsilon\alpha_i) + \beta_{p_q-1} + \cdots + \beta_1 + \lambda, \beta_k^\vee\rangle \\
&= \langle (\beta_{k-1} + \cdots + \beta_{p_q+1} + \beta_{p_q} + \beta_{p_q-1} + \cdots + \beta_1 + \lambda) + \epsilon\alpha_i, \beta_k^\vee\rangle \\
&= \langle s_{\beta_{k-1}} \cdots s_{\beta_{p_q+1}} s_{\beta_{p_q}} s_{\beta_{p_q-1}} \cdots s_{\beta_1}(\lambda), \beta_k^\vee\rangle + \epsilon\langle\alpha_i, \beta_k^\vee\rangle \\
&= -1 + \epsilon \cdot 0 = -1.
\end{aligned}$$

Hence, we have:

(9.11)
$$\beta_k \in \mathrm{GD}^{\alpha_i}(s_{\beta_{k-1}} \cdots s_{\beta_{p_q+1}} s_{\beta_{p_q}+\epsilon\alpha_i} s_{\beta_{p_q-1}} \cdots s_{\beta_1}(\lambda)), \quad p_q + 1 \le k \le k_0 - 1.$$
$$\beta_k \text{ is not } \alpha_i\text{-shiftable.}$$

By (9.10) and (9.11), we have:

$$\begin{aligned}
&\langle s_{\beta_{k_0-1}} \cdots s_{\beta_{p_q+1}} s_{\beta_{p_q}+\epsilon\alpha_i} s_{\beta_{p_q-1}} \cdots s_{\beta_1}(\lambda), (-\epsilon\alpha_i)^\vee\rangle \\
&= \langle \beta_{k_0-1} + \cdots + \beta_{p_q+1} + (\beta_{p_q} + \epsilon\alpha_i) + \beta_{p_q-1} + \cdots + \beta_1 + \lambda, (-\epsilon\alpha_i)^\vee\rangle \\
&= \langle (\beta_{k_0-1} + \cdots + \beta_{p_q+1} + \beta_{p_q} + \beta_{p_q-1} + \cdots + \beta_1 + \lambda) + \epsilon\alpha_i, (-\epsilon\alpha_i)^\vee\rangle \\
&= -\epsilon\langle s_{\beta_{k_0-1}} \cdots s_{\beta_{p_q+1}} s_{\beta_{p_q}} s_{\beta_{p_q-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee\rangle - \langle\alpha_i, \alpha_i^\vee\rangle \\
&= -\epsilon \cdot (-\epsilon) - 2 = -1.
\end{aligned}$$

Hence, we have:

(9.12)
$$-\epsilon\alpha_i \in \mathrm{GD}^{\alpha_i}(s_{\beta_{k_0-1}} \cdots s_{\beta_{p_q+1}} s_{\beta_{p_q}+\epsilon\alpha_i} s_{\beta_{p_q-1}} \cdots s_{\beta_1}(\lambda)).$$
$$-\epsilon\alpha_i \text{ is not } \alpha_i\text{-shiftable.}$$

By (9.10), (9.11), (9.12) and induction on k (from $k = k_0+1$ to $p_{q+1}-1$), we have:

$$\begin{aligned}
&\langle s_{\beta_{k-1}} \cdots s_{\beta_{k_0+1}} s_{-\epsilon\alpha_i} s_{\beta_{k_0-1}} \cdots s_{\beta_{p_q+1}} s_{\beta_{p_q}+\epsilon\alpha_i} s_{\beta_{p_q-1}} \cdots s_{\beta_1}(\lambda), \beta_k^\vee\rangle \\
&= \langle \beta_{k-1} + \cdots + \beta_{k_0+1} + (-\epsilon\alpha_i) + \beta_{k_0-1} + \cdots + \beta_{p_q+1} + (\beta_{p_q} + \epsilon\alpha_i) + \beta_{p_q-1} + \cdots + \beta_1 + \lambda, \beta_k^\vee\rangle \\
&= \langle (\beta_{k-1} + \cdots + \beta_{k_0+1} + \epsilon\alpha_i + \beta_{k_0-1} + \cdots + \beta_{p_q+1} + \beta_{p_q} + \beta_{p_q-1} + \cdots + \beta_1 + \lambda) - \epsilon\alpha_i, \beta_k^\vee\rangle \\
&= \langle s_{\beta_{k-1}} \cdots s_{\beta_{k_0+1}} s_{\epsilon\alpha_i} s_{\beta_{k_0-1}} \cdots s_{\beta_{p_q+1}} s_{\beta_{p_q}} s_{\beta_{p_q-1}} \cdots s_{\beta_1}(\lambda), \beta_k^\vee\rangle - \epsilon\langle\alpha_i, \beta_k^\vee\rangle \\
&= -1 - \epsilon \cdot 0 = -1,
\end{aligned}$$

Hence, we have:

(9.13)
$$\beta_k \in \mathrm{GD}^{\alpha_i}(s_{\beta_{k-1}} \cdots s_{\beta_{k_0+1}} s_{-\epsilon\alpha_i} s_{\beta_{k_0-1}} \cdots s_{\beta_{p_q+1}} s_{\beta_{p_q}+\epsilon\alpha_i} s_{\beta_{p_q-1}} \cdots s_{\beta_1}(\lambda)), \quad k_0+1 \le k \le p_{q+1}-1.$$

By (9.6) and (9.8), we have:

(9.14)
$$\langle s_{\beta_{p_{q+1}-1}} \cdots s_{\beta_{k_0+1}} s_{\epsilon\alpha_i} s_{\beta_{k_0-1}} \cdots s_{\beta_{p_q+1}} s_{\beta_{p_q}} s_{\beta_{p_q-1}} \cdots s_{\beta_1}(\lambda), \alpha_i^\vee\rangle = \epsilon.$$

Applying Lemma 9.3 (1) to (9.14), we have:

(9.15)

$\beta_{p_{q+1}} + \epsilon\alpha_i \in \mathrm{GD}^{\alpha_i}(s_{\beta_{p_{q+1}-1}} \cdots s_{\beta_{k_0+1}} s_{-\epsilon\alpha_i} s_{\beta_{k_0-1}} \cdots s_{\beta_{p_q+1}} s_{\beta_{p_q}+\epsilon\alpha_i} s_{\beta_{p_q-1}} \cdots s_{\beta_1}(\lambda))$.

$\beta_{q+1} + \epsilon\alpha_i$ is α_i-shiftable.

Furthuremore, we have:

$$(9.16)\quad \begin{aligned} & s_{\beta_{p_{q+1}}} s_{\beta_{p_{q+1}-1}} \cdots s_{\beta_{k_0+1}} s_{\epsilon\alpha_i} s_{\beta_{k_0-1}} \cdots s_{\beta_{p_q+1}} s_{\beta_{p_q}} s_{\beta_{p_q-1}} \cdots s_{\beta_1}(\lambda) \\ &= s_{\beta_{p_{q+1}}+\epsilon\alpha_i} s_{\beta_{p_{q+1}-1}} \cdots s_{\beta_{k_0+1}} s_{-\epsilon\alpha_i} s_{\beta_{k_0-1}} \cdots s_{\beta_{p_q+1}} s_{\beta_{p_q}+\epsilon\alpha_i} s_{\beta_{p_q-1}} \cdots s_{\beta_1}(\lambda). \end{aligned}$$

By (9.10),(9.11),(9.12), (9.13),(9.15) and (9.16),We have $\tau_q(\mathcal{B}) \in \mathrm{GPath}_{\alpha_i}(\lambda)$. Since there exists a unique index k ($p_q < k < p_{q+1}$) such that $\beta_k = -\alpha_i$ and $\langle \beta_p, \alpha_i^\vee \rangle = 0$ ($p \neq k$), we have:

$$\tau_q(\mathcal{B}) \in \mathrm{QPath}_{\alpha_i}(\lambda).$$
$$(\beta'_{p_q}, \beta'_{p_q+1}, \cdots, \beta'_{p_{q+1}-1}) = \mathcal{F}^{(q)}_{\alpha_i}(\tau_q(\mathcal{B})).$$

(3) This follows from definition of τ_q. □

Corollary 9.12. *Let $\lambda \in P_{\geq -1}$ be finite and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. Let $\mathcal{B} = (\beta_1, \cdots, \beta_l) \in \mathrm{QPath}_{\alpha_i}(\lambda)$. We suppose that there exists exactly m indices $1 \leq p_1 < \cdots < p_m \leq l$ such that β_{p_q} is α_i-shiftable($1 \leq q \leq m$). Then, for $1 \leq q \leq m$, we have:*

1. *If $\mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B})$ is not negative, then $\mathcal{F}^{(q)}_{\alpha_i}(\tau_q(\mathcal{B}))$ is not positive.*
2. *If $\mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B})$ is not positive, then $\mathcal{F}^{(q)}_{\alpha_i}(\tau_q(\mathcal{B}))$ is not negative.*

Proof. This follows from Proposition 9.11. □

Let $\lambda \in P_{\geq -1}$ be finite and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. Let $\mathcal{B} = (\beta_1, \cdots, \beta_l) \in \mathrm{QPath}_{\alpha_i}(\lambda)$. We suppose that there exists exactly m indices $1 \leq p_1 < \cdots < p_m \leq l$ such that β_{p_q} is α_i-shiftable ($1 \leq q \leq m$). We define transformations $\tau(\mathcal{B})$ and $\hat{\tau}(\mathcal{B})$ of $\mathcal{B}$ by:

$$\tau(\mathcal{B}) := \tau_q \cdots \tau_1(\mathcal{B}),$$

and

$$\hat{\tau}(\mathcal{B}) := \tau_1 \cdots \tau_q(\mathcal{B}).$$

Remark 1. Let $\lambda \in P_{\geq -1}$ be finite and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. Let $\mathcal{B} = (\beta_1, \cdots, \beta_l) \in \mathrm{QPath}_{\alpha_i}(\lambda)$.We suppose that there exists exactly m indices $1 \leq p_1 < \cdots < p_m \leq l$ such that β_{p_q} is α_i-shiftable($1 \leq q \leq m$). Then, for $1 \leq q_1, q_2 \leq m$, we have:

$$\tau_{q_1}\tau_{q_2} = \tau_{q_2}\tau_{q_1}.$$

Thus, we actually have $\tau = \hat{\tau}$.

Lemma 9.13. *Let $\lambda \in P_{\geq -1}$ be finite and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$.*

1. *Let $\mathcal{B} \in \mathrm{Path}_{\alpha_i}(\lambda)$. Then $\tau(\mathcal{B}) \in s_i(\mathrm{Path}(s_{\alpha_i}(\lambda)))$.*
2. *Let $\mathcal{B} \in s_i(\mathrm{Path}(s_{\alpha_i}(\lambda)))$. Then $\hat{\tau}(\mathcal{B}) \in \mathrm{Path}_{\alpha_i}(\lambda)$.*
3. *Moreover, the transformation τ gives a bijection from $\mathrm{Path}_{\alpha_i}(\lambda)$ to $s_i(\mathrm{Path}(s_{\alpha_i}(\lambda)))$.*

Proof. (1) Let $\mathcal{B} = (\beta_1, \cdots, \beta_l) \in \mathrm{Path}_{\alpha_i}(\lambda)$. We suppose that there exists exactly m indices $1 \le p_1 < \cdots < p_m \le l$ such that β_{p_q} is α_i-shiftable $(1 \le q \le m)$. By Lemma 9.9 (2), each q-th floor $\mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B})$ of $\mathcal{B}$ is not negative. By Corollary 9.12 (1), each q-th floor $\mathcal{F}^{(q)}_{\alpha_i}(\tau(\mathcal{B}))$ of $\tau(\mathcal{B})$ is not positive. By Lemma 9.9 (3) and (4), we have $\tau(\mathcal{B}) \in s_i(\mathrm{Path}(s_{\alpha_i}(\lambda)))$.
(2) Let $\mathcal{B} = (\beta_1, \cdots, \beta_l) \in s_i(\mathrm{Path}(s_{\alpha_i}(\lambda)))$. We suppose that there exists exactly m indices $1 \le p_1 < \cdots < p_m \le l$ such that β_{p_q} is α_i-shiftable $(1 \le q \le m)$. By Lemma 9.9 (3) and (4), each q-th floor $\mathcal{F}^{(q)}_{\alpha_i}(\mathcal{B})$ of $\mathcal{B}$ is not positive. By Corollary 9.12 (2), each q-th floor $\mathcal{F}^{(q)}_{\alpha_i}(\hat{\tau}(\mathcal{B}))$ of $\hat{\tau}(\mathcal{B})$ is not negative. By Lemma 9.9 (2), we have $\hat{\tau}(\mathcal{B}) \in \mathrm{Path}_{\alpha_i}(\lambda)$.
(3) By Proposition 9.11 (3), Part (1) and (2), $\hat{\tau}\tau$ is the identity transformation over $\mathrm{Path}_{\alpha_i}(\lambda)$ and $\tau\hat{\tau}$ is the identity transformation over $s_i(\mathrm{Path}(s_{\alpha_i}(\lambda)))$. Hence, the bijectivity is obvious. □

Proposition 9.14. *Let $\lambda \in P_{\ge -1}$ be finite and $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. Let $\gamma_1, \cdots, \gamma_l$, $\gamma'_1, \cdots, \gamma'_{l'}$ be elements of $\Phi_+ \cup \{-\alpha_i\}$. Let $r \ge 2$. Let $\epsilon = 1$ or -1.*

(1) *If one of the following sequence $\mathcal{B}_j$ $(1 \le j \le r-1)$ is an element of $\mathrm{QPath}_{\alpha_i}(\lambda)$ with indicated decomposition, then any one of the other sequences is an element of $\mathrm{QPath}_{\alpha_i}(\lambda)$ with indicated decomposition.*

$$\begin{array}{llll}
 & \text{lower floor} & q\text{-th } \alpha_i\text{-floor} & \text{upper floor} \\
\mathcal{B}_{r-1} := (& \overbrace{\gamma_1, \cdots, \gamma_l,} & \overbrace{\beta_0, \beta_1, \cdots\cdots\cdots, \beta_{r-3}, \beta_{r-2}, \epsilon\alpha_i,} & \overbrace{\gamma'_1, \cdots, \gamma'_{l'}}) \\
\mathcal{B}_{r-2} := (& \gamma_1, \cdots, \gamma_l, & \beta_0, \beta_1, \cdots\cdots\cdots, \beta_{r-3}, \epsilon\alpha_i, \beta_{r-2}, & \gamma'_1, \cdots, \gamma'_{l'}) \\
\vdots & & & \\
\mathcal{B}_j := (& \gamma_1, \cdots, \gamma_l, & \beta_0, \beta_1, \cdots, \beta_{j-1}, \epsilon\alpha_i, \beta_j, \beta_{j+1}, \cdots, \beta_{r-2}, & \gamma'_1, \cdots, \gamma'_{l'}) \\
\vdots & & & \\
\mathcal{B}_2 := (& \gamma_1, \cdots, \gamma_l, & \beta_0, \beta_1, \epsilon\alpha_i, \beta_2, \cdots\cdots\cdots, \beta_{r-2}, & \gamma'_1, \cdots, \gamma'_{l'}) \\
\mathcal{B}_1 := (& \gamma_1, \cdots, \gamma_l, & \beta_0, \epsilon\alpha_i, \beta_1, \beta_2, \cdots\cdots\cdots, \beta_{r-2}, & \gamma'_1, \cdots, \gamma'_{l'})
\end{array}$$

Here β_k $(1 \le k \le r-2)$ are elements of $\Phi_+ \cup \{-\alpha_i\}$ such that $\langle \beta_k, \alpha_i^\vee \rangle = 0$ and that β_k are not α_i-shiftable.

(2) *Under the same assumption as in* (1) *we have:*

$$\sum_{j=1}^{r-1} f_{\mathcal{B}_j} = \sum_{j=1}^{r-1} f_{\tau_q(\mathcal{B}_j)}.$$

Proof. (1) Note that $\epsilon\alpha_i$ never skip α_i-shiftable terms. This follows from Lemma 9.1.
(2) For a proof of Part (2), it is enough to put $\alpha := \epsilon\alpha_i$, $\delta_k := \sum_{p=1}^{l} \gamma_p + \sum_{q=0}^{k-1} \beta_q$ $(1 \le k \le r-1)$, and apply Lemma 9.15 below. □

Lemma 9.15. *For indeterminates* $\alpha, \delta_1, \cdots, \delta_{r-1}$ $(r \geq 2)$, *we have:*

$$\begin{aligned}
&\frac{1}{\delta_1}\cdots\frac{1}{\delta_{r-2}}\frac{1}{\delta_{r-1}}\frac{1}{\delta_{r-1}+\alpha} && \frac{1}{\delta_1+\alpha}\cdots\frac{1}{\delta_{r-2}+\alpha}\frac{1}{\delta_{r-1}+\alpha}\frac{1}{\delta_{r-1}}\\
&+\frac{1}{\delta_1}\cdots\frac{1}{\delta_{r-2}}\frac{1}{\delta_{r-2}+\alpha}\frac{1}{\delta_{r-1}+\alpha} && +\frac{1}{\delta_1+\alpha}\cdots\frac{1}{\delta_{r-2}+\alpha}\frac{1}{\delta_{r-2}}\frac{1}{\delta_{r-1}}\\
&+\cdots\cdots && +\cdots\cdots\\
&+\frac{1}{\delta_1}\cdots\frac{1}{\delta_k}\frac{1}{\delta_k+\alpha}\cdots\frac{1}{\delta_{r-1}+\alpha} \quad = && +\frac{1}{\delta_1+\alpha}\cdots\frac{1}{\delta_k+\alpha}\frac{1}{\delta_k}\cdots\frac{1}{\delta_{r-1}}\\
&+\cdots\cdots && +\cdots\cdots\\
&+\frac{1}{\delta_1}\frac{1}{\delta_2}\frac{1}{\delta_2+\alpha}\cdots\frac{1}{\delta_{r-1}+\alpha} && +\frac{1}{\delta_1+\alpha}\frac{1}{\delta_2+\alpha}\frac{1}{\delta_2}\cdots\cdots\cdots\frac{1}{\delta_{r-1}}\\
&+\frac{1}{\delta_1}\frac{1}{\delta_1+\alpha}\frac{1}{\delta_2+\alpha}\cdots\frac{1}{\delta_{r-1}+\alpha} && +\frac{1}{\delta_1+\alpha}\frac{1}{\delta_1}\frac{1}{\delta_2}\cdots\cdots\cdots\frac{1}{\delta_{r-1}}.
\end{aligned}$$

Proof. This follows from Lemma 8.4. □

Proposition 9.16. *Let* $\lambda \in P_{\geq -1}$ *be finite and* $\alpha_i \in \mathrm{D}(\lambda) \cap \Pi$. *Then we have:*

$$\sum_{\mathcal{B}\in \mathrm{Path}_{\alpha_i}(\lambda)} f_{\mathcal{B}} = \sum_{\mathcal{B}\in s_i(\mathrm{Path}(s_{\alpha_i}(\lambda)))} f_{\mathcal{B}}.$$

Proof. Applying Proposition 9.14 to $\mathrm{Path}_{\alpha_i}(\lambda)$ repeatedly, we get:

$$\sum_{\mathcal{B}\in \mathrm{Path}_{\alpha_i}(\lambda)} f_{\mathcal{B}} = \sum_{\mathcal{B}\in \mathrm{Path}_{\alpha_i}(\lambda)} f_{\tau(\mathcal{B})},$$

by Proposition 9.10. By Lemma 9.13, we get:

$$\sum_{\mathcal{B}\in \mathrm{Path}_{\alpha_i}(\lambda)} f_{\tau(\mathcal{B})} = \sum_{C\in s_i(\mathrm{Path}(s_{\alpha_i}(\lambda)))} f_C.$$

This proves the statement. □

By Lemma 4.1(2), Proposition 8.5, Proposition 9.16, and induction, we have:

$$\begin{aligned}
\sum_{\mathcal{B}\in \mathrm{Path}(\lambda)} f_{\mathcal{B}} &= \left(1+\frac{1}{\alpha_i}\right)\sum_{\mathcal{B}\in \mathrm{Path}_{\alpha_i}(\lambda)} f_{\mathcal{B}} = \left(1+\frac{1}{\alpha_i}\right)\sum_{\mathcal{B}\in s_i(\mathrm{Path}(s_{\alpha_i}(\lambda)))} f_{\mathcal{B}}\\
&= \left(1+\frac{1}{\alpha_i}\right)s_i\left(\sum_{C\in \mathrm{Path}(s_{\alpha_i}(\lambda))} f_C\right) = \left(1+\frac{1}{\alpha_i}\right)s_i\left(\prod_{\gamma\in \mathrm{D}(s_{\alpha_i}(\lambda))}\left(1+\frac{1}{\gamma}\right)\right)\\
&= \left(1+\frac{1}{\alpha_i}\right)\prod_{\beta\in s_i(\mathrm{D}(s_{\alpha_i}(\lambda)))}\left(1+\frac{1}{\beta}\right) = \left(1+\frac{1}{\alpha_i}\right)\prod_{\beta\in \mathrm{D}(\lambda)\setminus\{\alpha_i\}}\left(1+\frac{1}{\beta}\right)\\
&= \prod_{\beta\in D(\lambda)}\left(1+\frac{1}{\beta}\right).
\end{aligned}$$

This completes the proof of the colored hook formula (7.1).

10. Minuscule Elements

Definition 11. Let Λ be a dominant integral weight. Following D. Peterson (see[1][5]), we define $w \in W$ to be Λ-*minuscule* if

$$\langle s_{i_{p+1}} \cdots s_{i_d}(\Lambda), \alpha_{i_p}^\vee \rangle = 1, \quad 1 \le p \le d \tag{10.1}$$

for some reduced decomposition $w = s_{i_1} \cdots s_{i_d}$. If $w \in W$ is Λ-minuscule for some dominant integral weight Λ, then we say that w is *minuscule*.

Proposition 10.1. *For a pair* (Λ, w) *of a dominant integral weight* Λ *and a* Λ*-minuscule element* w*, we put* $\lambda := w(\Lambda)$*. Then,* λ *is a finite pre-dominant integral weight. Furthurmore, the correspondence* $(\Lambda, w) \mapsto \lambda$ *is bijective.*

Proof. First, we prove that $\lambda = w(\Lambda)$ is a finite pre-dominant integral weight, supposing that w is a Λ-minuscule element for a dominant integral weight Λ. Let $\beta \in \Phi_+$. If $\langle \lambda, \beta^\vee \rangle \le -1$, then we have:

$$-1 \ge \langle \lambda, \beta^\vee \rangle = \langle w(\Lambda), \beta^\vee \rangle = \langle \Lambda, w^{-1}(\beta)^\vee \rangle.$$

Since Λ is dominant, $w^{-1}(\beta)$ is a negative root. Hence, we have $\beta \in \Phi(w)$. Let $w = s_{i_1} \cdots s_{i_d}$ be a reduced decomposition of w. Then we have $\Phi(w) = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \cdots, s_{i_1} \cdots s_{i_{d-1}}(\alpha_{i_d})\}$. Hence we have $\beta = s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p})$ for some $1 \le p \le d$. We have:

$$\begin{aligned}\langle \lambda, \beta^\vee \rangle &= \langle s_{i_1} \cdots s_{i_d}(\Lambda), \ s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p})^\vee \rangle \\ &= \langle s_{i_p} s_{i_{p+1}} \cdots s_{i_d}(\Lambda), \ \alpha_{i_p}^\vee \rangle \\ &= -\langle s_{i_{p+1}} \cdots s_{i_d}(\Lambda), \ \alpha_{i_p}^\vee \rangle \\ &= -1\end{aligned}$$

for some $1 \le p \le d$. Hence, for each $\beta \in \Phi_+$, we have $\langle \lambda, \beta^\vee \rangle \ge -1$. Furthurmore, we have also proved that $\langle \lambda, \beta^\vee \rangle = -1$ if and only if $\beta \in \Phi(w)$. Since $\#\Phi(w) = d$ is finite, λ is a finite pre-dominant integral weight.

Next, we prove the bijectivity. Let λ be a finite pre-dominant integral weight. Let $(\alpha_{i_1}, \cdots, \alpha_{i_d}) \in \mathrm{MPath}(\lambda)$. Put $w := s_{i_1} \cdots s_{i_d}$ and $\Lambda := w^{-1}(\lambda) = s_{i_d} \cdots s_{i_1}(\lambda)$. Since the λ-path $(\alpha_{i_1}, \cdots, \alpha_{i_d})$ is of maximal length, we have $\mathrm{D}(\Lambda) = \varnothing$. Hence Λ is a dominant integral weight. Applying Lemma 4.2 (2) to $\mathrm{D}(\Lambda) = \varnothing$ repeatedly, we get:

$$\{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \cdots, s_{i_1} \cdots s_{i_{d-1}}(\alpha_{i_d})\} = \mathrm{D}(\lambda) \subseteq \Phi_+. \tag{10.2}$$

If $s_{i_1} \cdots s_{i_d}$ is not a reduced decomposition of w, then there exists an index p ($1 < p \le d$) such that $\ell(s_{i_1} \cdots s_{i_{p-1}}) > \ell(s_{i_1} \cdots s_{i_{p-1}} s_{i_p})$, where ℓ denotes the length function. Hence, we have $s_{i_1} \cdots s_{i_{p-1}}(\alpha_{i_p}) \in \Phi_-$. This contradicts (10.2). Hence, $s_{i_1} \cdots s_{i_d}$ is a reduced decomposition of w. Since

$$\begin{aligned}\langle s_{i_{p+1}} \cdots s_{i_d}(\Lambda), \ \alpha_{i_p}^\vee \rangle &= \langle s_{i_{p+1}} \cdots s_{i_d} s_{i_d} \cdots s_{i_1}(\lambda), \ \alpha_{i_p}^\vee \rangle \\ &= \langle s_{i_p} s_{i_{p-1}} \cdots s_{i_1}(\lambda), \ \alpha_{i_p}^\vee \rangle \\ &= -\langle s_{i_{p-1}} \cdots s_{i_1}(\lambda), \ \alpha_{i_p}^\vee \rangle \\ &= -(-1) = 1\end{aligned}$$

for $1 \leq p \leq d$, $w = s_{i_1} \cdots s_{i_d}$ is Λ-minuscule. We also have:

$$\Phi(w) = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \cdots, s_{i_1} \cdots s_{i_{d-1}}(\alpha_{i_d})\}. \tag{10.3}$$

By (10.2) and (10.3), we have $\Phi(w) = \mathrm{D}(\lambda)$. Hence, such w is uniquely determined from λ.

The correspondence $\lambda \mapsto (\Lambda(= w^{-1}(\lambda)), w)$ thus obtained is clearly the inverse of the previous map $(\Lambda, w) \mapsto \lambda$. Hence we get the bijectivity. □

For $w \in W$, we denote the set of reduced decompositions $(s_{i_1}, s_{i_2}, \cdots, s_{i_d})$ of w by $\mathrm{Red}(w)$. As a corollary of the proof of Proposition 10.1, we get:

Corollary 10.2. *Let Λ be a dominant integral weight. Let $w \in W$ be Λ-minuscule. Let $(s_{i_1}, s_{i_2}, \cdots, s_{i_d}) \in \mathrm{Red}(w)$. Then, we have:*

$$\langle s_{i_{p+1}} \cdots s_{i_d}(\Lambda), \alpha_{i_p}^\vee \rangle = 1, \quad 1 \leq p \leq d.$$

This corollary show that the definition of Λ-minuscule elements is independent from a choice of reduced decompositions. This is also proved by J. R. Stembridge in [7].

Proposition 10.3. *Let (Λ, w) be a pair of a dominant integral weight Λ and a Λ-minuscule element w. Let $\lambda := w(\Lambda)$ be the corresponding finite pre-dominant integral weight. Put $d := \#\mathrm{D}(\lambda)$. Let $(\alpha_{i_1}, \cdots, \alpha_{i_d}) \in \mathrm{MPath}(\lambda)$. Then we have $(s_{i_1}, \cdots, s_{i_d}) \in \mathrm{Red}(w)$. Furthurmore, the correspondence $(\alpha_{i_1}, \cdots, \alpha_{i_d}) \mapsto (s_{i_1}, \cdots, s_{i_d})$ from $\mathrm{MPath}(\lambda)$ to $\mathrm{Red}(w)$ is bijective.*

Proof. If $(\alpha_{i_1}, \cdots, \alpha_{i_d}) \in \mathrm{MPath}(\lambda)$, then, by the proof of Proposition 10.1, we have $(s_{i_1}, \cdots, s_{i_d}) \in \mathrm{Red}(w)$.

To prove the bijectivity, let $(s_{i_1}, \cdots, s_{i_l}) \in \mathrm{Red}(w)$. By the proof of Proposition 10.1, we have $\Phi(w) = \mathrm{D}(\lambda)$. Hence we have $l = d$. Since

$$\begin{aligned}\langle s_{i_{p-1}} \cdots s_{i_1}(\lambda), \alpha_{i_p}^\vee \rangle &= \langle s_{i_{p-1}} \cdots s_{i_1} s_{i_1} \cdots s_{i_d}(\Lambda), \alpha_{i_p}^\vee \rangle \\ &= \langle s_{i_p} s_{i_{p+1}} \cdots s_{i_d}(\Lambda), \alpha_{i_p}^\vee \rangle \\ &= -\langle s_{i_{p+1}} \cdots s_{i_d}(\Lambda), \alpha_{i_p}^\vee \rangle \\ &= -1\end{aligned}$$

for $1 \leq p \leq d$. Hence, we get $(\alpha_{i_1}, \cdots, \alpha_{i_d}) \in \mathrm{MPath}(\lambda)$. This proves the bijectivity. □

By Corollary 7.3, Proposition 10.1 and Proposition 10.3, we get a proof of Peterson's hook formula (1.6).

Remark 2. Minuscule elements are classified by R. A. Proctor [5][6] (in simply-laced case) and J. R. Stembridge [7] (in non-simply-laced case).

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