Exotic hadrons in the QCD sum rule

（QCD和則によるエキゾチックハドロン）

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Abstract

We study the baryon and tetraquark currents systematically in the flavor, color and Lorentz spaces. The tetraquark currents are also studied in both the diquark-antidiquark $(qq)(\bar{q}\bar{q})$ construction and meson-meson $(\bar{q}q)(\bar{q}q)$ construction, which are proved to be equivalent. By using these currents, we perform the QCD sum rule analyses, and study light scalar mesons ($\sigma(600)$, $\kappa(800)$, $a_0(980)$ and $f_0(980)$) with quantum numbers $J^{PC} = 0^{++}$, $Y(2175)$ with $J^{PC} = 1^{--}$, $\pi_1(1400)$, $\pi_1(1600)$ and $\pi_1(2000)$ with $I^G J^{PC} = 1^- 1^{-+}$.

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Preface

The theory of the strong interactions, Quantum Chromodynamics (QCD), originated from the systematics of hadron spectroscopy. The spectroscopy contains meson and baryon states, many of which are well classified by the quark model with quark contents $q\bar{q}$ and qqq. Besides the quark model, QCD allows much richer hadron spectrum such as multiquark states, hadron molecules, hybrid states, and glueballs etc. However the spectrum of QCD seem to saturate at $q\bar{q}$ and qqq. Therefore, we call these spectrum beyond $q\bar{q}$ and qqq exotic hadrons (exotica).

Exotica have been studied more than thirties years. R. L. Jaffe wrote two famous papers about scalar tetraquark states in 1976 [93, 94], whose structure is still not clear yet. In 2003, the pentaquark Θ^+ was observed in several experiments, but then several experiments denied its existence. After five years of intense study, the status of Θ^+ is still controversial [137]. There are many other exotic candidates, such as $\pi_1(1400)$ [10], $D_{sJ}(2317)$ [18], $X(3872)$ [45], and $Y(4260)$ [19], etc. Their properties are difficult to be explained by the conventional picture of $q\bar{q}$ and qqq.

In order to study these exotica, lots of methods have been used. Although we have known a lot about QCD, but still there are many important and essential dynamical aspects that we need to clarify. As a doctor student in RCNP, Osaka University, I spent my latest three years on the study of QCD. I hope I contributed, although the time is not long, and my contribution is rather restricted. Now I am trying to graduate and changing my career in the research, and I am required to write this doctor thesis.

The method we used in this thesis is the QCD sum rule, which has proven to be a powerful and successful non-perturbative method for the past decades [155, 160]. An introduce of QCD sum rule is written in Chapter 1, which contains the SVZ sum rule, and the finite energy sum rule.

This thesis is separated into two parts. In the first part, we classify the interpolating fields (currents) for hadrons in QCD, which are used in the QCD sum rule analysis in the second part. QCD currents can contain quark fields, antiquark fields and gluon fields. The quark and antiquark fields are Dirac spinors, and so currents can also be spinors, such as baryon current

$$\epsilon_{abc} q_1^{aT} C\gamma_5 q_2^b q_3^c \, .$$

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Currents can also be scalars other than matrices, such as the meson current

$$\bar{q}_1^a \gamma_5 q_2^a \,.$$

The notations and conventions we used are written in Chapter 1, where we construct meson currents ($\bar{q}q$), diquark currents (qq) and antidiquark currents ($\bar{q}\bar{q}$). In Chapters 2 and 3, we construct baryon currents and tetraquark currents, respectively. Chapter 4 is the discussion of color structure of multiquark currents.

After classifying current in the first part, we can start to perform the QCD sum rule analysis, which is the second part of this thesis. We have three important criteria:

1. Convergence of Operator Product Expansion (OPE),

2. Positivity of spectral density,

3. Sufficient amount of pole contribution.

We take $ud\bar{s}\bar{s}$ currents as an example and show our QCD sum rule analysis in Chapter 5. This procedure will be used in the following chapters: in Chapter 6, we study light scalar mesons; in Chapter 7, we study $Y(2175)$ as a tetraquark states; in Chapter 8, we study $\pi_1(1400)$, $\pi_1(1600)$ and $\pi_1(2015)$. In Chapter 9, the QCD sum rule is used to study the bottom baryons which contain heavy quarks.

Above I just gave a short introduction to my thesis. In my three years' research, I learned much and had a great deal of fun. I hope the readers would enjoy my thesis.

Notations and Conventions

Notations and conventions used in this thesis mostly follow the book "An Introduction to Quantum Field Theory" written by M. E. Peskin and D. V. Schroeder (Addison-Wesley Publishing Company, 1997) [152].

Quark field $q_A^a(x)$ is a Dirac spinor at location x, and contains a flavor index A and a color index a. For antiquark field, we use $\bar{q}_A^a(x)$. By using the following $\gamma-$matrices:

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we can write the quark field as a combination of left- and right-handed fields:

$$q = q_L + q_R,$$

where

$$q_L = \frac{1 - \gamma_5}{2} q, q_R = \frac{1 + \gamma_5}{2} q.$$

For gluon field, we use $G_{\mu\nu}^n$, which has a color index n. The covariant derivative is

$$D_\mu = \partial_\mu + i g_s \frac{\lambda^n}{2} A_\mu^n,$$

where we take the fix-point gauge

$$A_\mu^n = -\frac{1}{2} x^\nu G_{\mu\nu}^n.$$

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The coupling constant g_s defined here is different from Peskin's book, where $D = \partial - ig_s A$. But it is used in some other QCD sum rule studies [85, 177].

We work under the metric tensor:

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

with Greek indices running over 0, 1, 2, 3.

We use $S_{ABC\ldots}$ to represent a totally symmetric matrix, and $\epsilon_{ABC\ldots}$ to represent a totally antisymmetric matrix. Especially, we use $\epsilon_{\mu\nu\rho\sigma}$ in the four-dimension:

$$\epsilon_{0123} = -1.$$

In order to describe the color structure of QCD, $SU(3)_C$, we use the eight Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

They are traceless, Hermitian, and their normalizations are

$$Tr(\lambda_i \lambda_i) = 2\delta_{ij}.$$

The three discrete symmetries of QCD are

1. Parity (P):

$$\mathbf{P}q(t, \vec{x})\mathbf{P} = \lambda^0 q(t, -\vec{x}) \,;$$

2. Time Reversal (T):

$$\mathbf{T}q(t, \vec{x})\mathbf{T} = -i\lambda^2 \lambda^3 q(-t, \vec{x}) \,;$$

3. Charge Conjugation (C):

$$\mathbf{C}q(t, \vec{x})\mathbf{C} = -C\bar{q}(t, \vec{x}) \,,$$

where the charge-conjugation operator C is defined to be $C = i\gamma^2 \gamma^0$.

Chapter 1

Introduction

1.1 QCD Lagrangian

Quantum Chromodynamics (QCD), the theory of strong interactions among quarks and gluons, is a quantum field theory of a special kind called non-Abelian gauge theory. The gauge invariant QCD Lagrangian is:

$$\mathcal{L} = \bar{\psi}_i \left(i\gamma^\mu (D_\mu)_{ij} - m\delta_{ij} \right) \psi_j - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a \tag{1.1}$$

$$= \bar{\psi}_i \left(i\gamma^\mu \partial_\mu - m \right) \psi_i - g A^a_\mu \bar{\psi}_i \gamma^\mu T^a_{ij} \psi_j - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a \,,$$

where $\psi_i(x)$ $(i = 1, 2, 3)$ is the quark field, the fundamental representation of the $SU(3)$ gauge group; A^a_μ are the gluon fields, the adjoint representation of the $SU(3)$ gauge group; γ_μ are the Dirac matrices, connecting the spinor representation to the vector representation of the Lorentz group; and T^a_{ij} $(a = 1, 2, \cdots, 8)$ are the generators, connecting the fundamental, anti-fundamental and adjoint representations of the $SU(3)$ gauge group. The Gell-Mann matrices λ^a_{ij} provide one such representation for the generators:

$$T^a_{ij} = \frac{\lambda^a_{ij}}{2} \,. \tag{1.2}$$

We emphasize here that the covariant derivative in this thesis is defined to be

$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} + i g A^a_\mu T^a_{ij} \,. \tag{1.3}$$

Although we know QCD Lagrangian very clearly, its non-Abelian nature prevents us to solve it accurately. There are many different kinds of theories, such as Lattice QCD, 1/N expansion and many effective theories. QCD sum rule is one of them. The QCD sum rule has proven to be a very powerful and successful non-perturbative method for the past decades [155, 160]. The idea is to work with gauge invariant operators and operator product expansions of them.

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1.2   Two-point Correlation Function

In both the Lattice QCD and the QCD sum rule we need to study the two-point correlation function:

$$\Pi(x) = \langle \Omega | T\phi(x)\phi(0) | \Omega \rangle , \tag{1.4}$$

where $|\Omega\rangle$ denotes the ground state, and T is the time-ordering operator. Correlation functions contain information about the distribution of points or events, or things across some spacetime. It is used in in astronomy, financial analysis, quantum field theory and statistical mechanics, etc. In the quantum field theory the two-point correlation function can be interpreted as the amplitude for the particle propagation or particle excitation.

In lattice QCD spacetime is represented not as continuous but as a crystalline lattice, vertices connected by lines. Therefore, we use following correlation function:

$$\Pi(L) = \langle \Omega | T\phi(L)\phi(0) | \Omega \rangle , \tag{1.5}$$

where L is not discrete rather than continuous, and we work in the region $L \to$ Large.

While in the QCD sum rule we use the dispersion relation:

$$\Pi(q^2) = \frac{(q^2)^N}{\pi} \int_0^\infty ds \frac{\mathbf{Im}\Pi(s)}{s^N(s - q^2 - i\epsilon)} + \sum_{n=0}^{N-1} (q^2)^n a_n , \tag{1.6}$$

which is derived from the integration shown in Fig. 1.1. Here we need to work in the region $-Q^2 \to \infty$. In this region, we can use a method called operator product expansion to calculate the two-point correlation function.

1.3   OPE

The method of operator product expansion is useful not only in QCD, but also in the more general quantum field theory. Its basic idea is to replace a product of several operators with a single effective vertex, which was first studied by Kenneth G. Wilson [176].

First we assume that there are two operators $\mathcal{O}_1(x)$ and $\mathcal{O}_2(0)$, with a small distance x. As an example, we choose

$$\mathcal{O}_1 = \bar{d}_L \gamma_\mu u_L, \quad \mathcal{O}_2 = \bar{u}_L \gamma_\mu s_L, \tag{1.7}$$

whose product is just the weak interaction vertex. By studying this product, we can study the renormalization of the weak interaction in QCD.

In order to study this product, we define the following Green's function:

$$G(x; y_1, \cdots, y_m) = \langle \mathcal{O}_1(x)\mathcal{O}_2(0)\eta_1(y_1) \cdots \eta_m(y_m) \rangle , \tag{1.8}$$
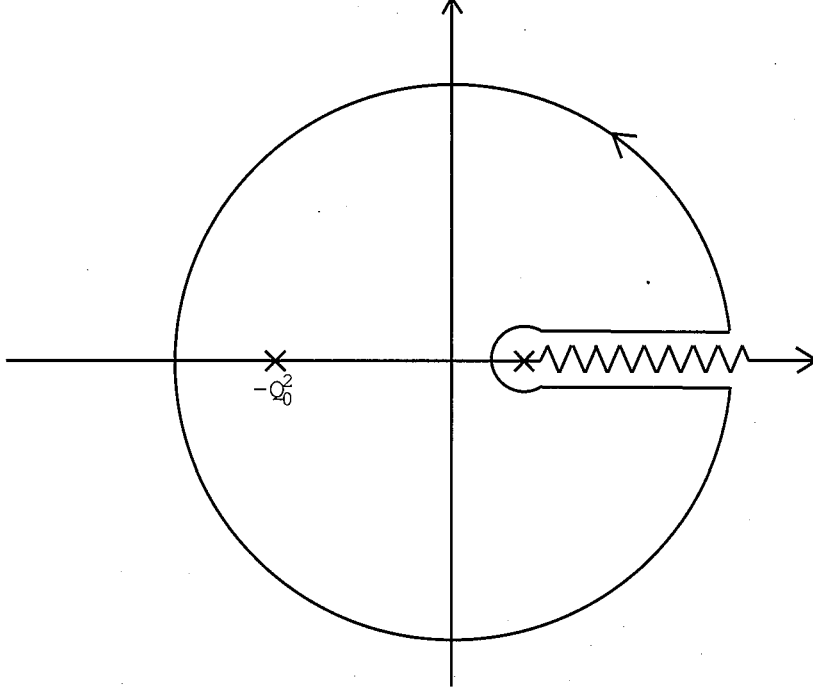
Figure 1.1: Contour of integration involved in the QCD sum rule.

where $\eta_i(y_i)$ are the fields located much farther away, and so irrelevant with the calculation of the product of $\mathcal{O}_1$ and $\mathcal{O}_2$. We assume $x \to 0$, and so the effect of this product can be described as the effect of a local operator placed at 0. It is natural to assume that there is a standard basis of operators, and so the local operator coming from the product is just a linear combination of these basic operators:

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_n C_{12}^n(x)\mathcal{O}_n(x), \tag{1.9}$$

where $C_{12}^n(x)$ are the coefficients depending on the small distance x. The Green function $G(x; y_1, \cdots, y_m)$ can be then expanded:

$$G(x; y_1, \cdots, y_m) = \sum_n C_{12}^n(x)\langle\mathcal{O}_n(x)\eta_1(y_1)\cdots\eta_m(y_m)\rangle. \tag{1.10}$$

To calculate the product of $\mathcal{O}_1(x)$ and $\mathcal{O}_2(0)$, we need to calculate the QCD corrections to the strength of the non-leptonic weak interaction vertex. We just show the final result here:

$$\begin{aligned} \mathcal{J}_M^1 &\equiv [\mathcal{O}_1\mathcal{O}_2]|_M \tag{1.11} \\ &= \mathcal{J}_0^1 + a^{11}\mathcal{J}_0^1 + a^{12}\mathcal{J}_0^2, \end{aligned}$$

where the subscript 0 is used to denote that the operator is located at location 0, and M is the renormalization scale (in this case it is of order m_W^{-1}). The operator $\mathcal{J}_0^2$ is another local operator used in the weak interaction vertex:

$$\mathcal{J}^2 \equiv \mathcal{O}_3 \mathcal{O}_4 , \tag{1.12}$$

where

$$\mathcal{O}_3 = \bar{u}_L \gamma_\mu u_L , \quad \mathcal{O}_4 = \bar{d}_L \gamma_\mu s_L . \tag{1.13}$$

Two coefficients a^{11} and a^{12} are counterterms, which depend on the renormalization scale M:

$$a^{11} = -\frac{g^2}{16\pi^2} \frac{\Gamma(2 - d/2)}{(M^2)^{2-d/2}} , \quad a^{12} = +3 \frac{g^2}{16\pi^2} \frac{\Gamma(2 - d/2)}{(M^2)^{2-d/2}} . \tag{1.14}$$

We can also study the operator product $\mathcal{J}^2$:

$$\begin{aligned} \mathcal{J}_M^2 &\equiv [\mathcal{O}_3 \mathcal{O}_4]|_M \\ &= \mathcal{J}_0^2 + a^{21} \mathcal{J}_0^1 + a^{22} \mathcal{J}_0^2 , \end{aligned} \tag{1.15}$$

where

$$a^{21} = a^{12} , \quad a^{22} = a^{11} . \tag{1.16}$$

So we can obtain the Callan-Symanzik equation, and now the matrix γ linking two operators $\mathcal{J}^1$ and $\mathcal{J}^2$ is

$$\gamma = M \frac{\partial}{\partial M}[-\mathbf{a}] = \frac{g^2}{16\pi^2} \begin{pmatrix} -2 & 6 \\ 6 & -2 \end{pmatrix} . \tag{1.17}$$

The eigen-operators are:

$$\begin{aligned} \mathcal{J}^{1/2} &= \frac{1}{2}(\mathcal{J}^1 - \mathcal{J}^2) = \frac{1}{2}(\bar{d}_L \gamma_\mu u_L \bar{u}_L \gamma_\mu s_L - \bar{u}_L \gamma_\mu u_L \bar{d}_L \gamma_\mu s_L) , \tag{1.18} \\ \mathcal{J}^{3/2} &= \frac{1}{2}(\mathcal{J}^1 + \mathcal{J}^2) = \frac{1}{2}(\bar{d}_L \gamma_\mu u_L \bar{u}_L \gamma_\mu s_L + \bar{u}_L \gamma_\mu u_L \bar{d}_L \gamma_\mu s_L) , \end{aligned}$$

with the eigenvalues:

$$\gamma_{\frac{1}{2}} = -8 \frac{g^2}{16\pi^2} , \quad \gamma_{\frac{3}{2}} = +4 \frac{g^2}{16\pi^2} . \tag{1.19}$$

The first eigen-operator has isospin 1/2, and the second one has isospin 3/2. Indeed these two eigen-operators have also been differentiated in the experiments, and we have the OZI rule that the first process is much faster that the second one [87, 147, 188].

1.4 QCD sum rule

In QCD sum rule, we use the method of operator product expansion, and now the local operators are unit operator I and those constructed from quark and gluon fields, for example:

$$m_q \bar{q}q, \quad G^a_{\mu\nu}G^{\mu\nu}_a, \cdots \qquad (1.20)$$

These operators have non-zero vacuum expectation values due to non-perturbative QCD effects. In the asymptotically free limit, this expansion can be calculated by using the perturbative method. Then we can relate this to the quantities of QCD at the low energy side by using dispersion relations. At the low energy region, the degrees of freedom are hadrons other than quarks and gluons. By relating them, we can obtain their masses and decay widths.

In QCD sum rule, first we consider two-point correlation functions:

$$\Pi(q^2) \equiv i \int d^4x e^{iqx} \langle 0|T\eta(x)\eta^\dagger(0)|0\rangle, \qquad (1.21)$$

where η is an interpolating current, which is written as a combination of quark fields and gluon fields. We can calculate it at the quark-gluon level up to certain order in the expansion, by using the method of perturbative QCD with non-zero quark and gluon condensates, such as $\langle \bar{q}q \rangle$ and $\langle g^2GG \rangle$, etc. The obtained OPE can be matched with a hadronic parametrization at the hadronic level to extract information of hadron properties. At the hadron level, we express the correlation function in the form of the dispersion relation with a spectral function:

$$\Pi(p) = \int_0^\infty \frac{\rho(s)}{s - p^2 - i\varepsilon}ds, \qquad (1.22)$$

where

$$\begin{aligned}
\rho(s) &\equiv \sum_n \delta(s - M_n^2)\langle 0|\eta|n\rangle\langle n|\eta^\dagger|0\rangle \\
&= f_X^2 \delta(s - M_X^2) + \text{higher states}.
\end{aligned} \qquad (1.23)$$

By assuming that these exists a kinematic region where these two aspects both works, we can evaluate many physical observables, such as masses, coupling constants, etc. For the second equation, as usual, we adopt a parametrization of one pole dominance for the ground state X and a continuum contribution. The sum rule analysis is then performed after the Borel transformation of the two expressions of the correlation function, (1.21) and (1.22)

$$\Pi^{(all)}(M_B^2) \equiv \mathcal{B}_{M_B^2}\Pi(p^2) = \int_0^\infty e^{-s/M_B^2}\rho(s)ds. \qquad (1.24)$$

Assuming that the contribution from the continuum states can be approximated well by the spectral density of OPE above a threshold value s_0 (duality), we arrive at the sum rule equation

$$\Pi(M_B^2) \equiv f_X^2 e^{-M_X^2/M_B^2} = \int_0^{s_0} e^{-s/M_B^2} \rho(s) ds. \tag{1.25}$$

The use of the OPE expression for the continuum part ($s > s_0$) of the spectral density $\rho(s)$ which is the basic assumption of the duality greatly simplifies the actual sum rule analyses. Although ambiguities coming from the uncertainties in the continuum contribution exist [127], we shall rely on that assumption as in most of the previous studies. Differentiating Eq. (1.25) with respect to $\frac{1}{M_B^2}$ and dividing it by Eq. (1.25), finally we obtain

$$M_X^2 = \frac{\int_0^{s_0} e^{-s/M_B^2} s \rho(s) ds}{\int_0^{s_0} e^{-s/M_B^2} \rho(s) ds}. \tag{1.26}$$

Another sum rule which is widely used is the so-called finite energy sum rule (FESR). In order to calculate the mass in the FESR, we first define the nth moment by using the spectral function $\rho(s)$ in Eq. (1.23)

$$W(n, s_0) = \int_0^{s_0} \rho(s) s^n ds. \tag{1.27}$$

This integral is used for the phenomenological side, while the integral along the circular contour of radius s_0 on the q^2 complex plain should be performed for the theoretical side.

With the assumption of quark-hadron duality, we obtain

$$W(n, s_0)\Big|_{Hadron} = W(n, s_0)\Big|_{OPE}. \tag{1.28}$$

The mass of the ground state can be obtained as

$$M_Y^2(n, s_0) = \frac{W(n+1, s_0)}{W(n, s_0)}. \tag{1.29}$$

Here we just briefly introduced the basic concept of the QCD sum rule. While a detailed example is given in Chapter 5.

During the studies of multiquark system, we found that the most complicated part is the construction of interpolating current η, which is written as a combination of quark fields and gluon fields, and can couple to the physical states. It has almost all the properties that the physical states have, such as the flavor structure, color structure, and quantum numbers **J**, **P** and **C**, etc. Therefore, to begin the discussion, we first study the basic currents:

1. meson current, which contains one quark field and one antiquark field,

2. diquark current, which contains two quark fields.

We will just study the local fields which do not contain derivatives, while those containing derivatives couple to excited states, which are beyond our studies. The properties of the currents corresponding to these objects can be easily obtained, and so we will just show the results. In the following chapters, we will use these simple objects to construct currents for the baryon and tetraquark which are more complicated.

1.5 Meson

In this section, we study interpolating fields which contain one quark and one antiquark. They couple to meson states, such as π, ρ, etc. Due to the confinement nature of QCD, there is only one choice for its color structure:

$$\mathbf{3_c} \otimes \mathbf{\bar{3}_c} \longrightarrow \mathbf{1_c} \, .$$

The flavor can be either octet ($\bar{q}_A \lambda^N_{AB} q_B$) or singlet ($\bar{q}_A q_A$). In the following, we will just keep $\bar{q}_A q_B$, then the flavor octet and singlet can be constructed by adding λ^N_{AB} and δ_{AB}, respectively. The Lorentz structure can be differentiated by using $\gamma-$matrices, and we can construct five different interpolating fields:

1. Scalar:

$$S = \bar{q}^a_A(x) q^a_B(x) \, . \tag{1.30}$$

It has quantum numbers $J^P = 0^+$.

2. Vector:

$$V_\mu = \bar{q}^a_A(x) \gamma_\mu q^a_B(x) \, . \tag{1.31}$$

It has spin $J = 1$ and parity $P = (-1)^\mu$, where $(-1)^\mu = 1$ for $\mu = 0$, and $(-1)^\mu = -1$ for $\mu = 1, 2, 3$. For simplicity, we write it as $J^P = 1^-$.

3. Tensor:

$$T_{\mu\nu} = \bar{q}^a_A(x) \sigma_{\mu\nu} q^a_B(x) \, . \tag{1.32}$$

It has quantum numbers $J^P = 1^\pm$, and can be separated into two parts: T_{0i} and T_{ij}. T_{0i} has quantum numbers $J^P = 1^-$ and T_{ij} has quantum numbers $J^P = 1^+$.

4. Axial-Vector:

$$A_\mu = \bar{q}^a_A(x) \gamma_\mu \gamma_5 q^a_B(x) \, . \tag{1.33}$$

It has quantum numbers $J^P = 1^+$.

5. Pseudoscalar:

$$P = \bar{q}_A^a(x)\gamma_5 q_B^a(x) \,.$$ (1.34)

It has quantum numbers $J^P = 0^-$.

Besides the tensor current listed above, there is another one:

$$T'_{\mu\nu} = \bar{q}_A^a(x)\sigma_{\mu\nu}\gamma_5 q_B^a(x) \,.$$ (1.35)

By using the equation

$$\sigma_{\mu\nu}\gamma_5 = \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}\sigma^{\rho\sigma} \,,$$

It can be related to $T_{\mu\nu}$. And so, it just has an opposite parity. T'_{0i} has quantum numbers $J^P = 1^+$ and T'_{ij} has quantum numbers $J^P = 1^-$.

1.6 Diquark and Antidiquark

The diquark and antidiquark can not be combined to be color singlet, and so they do not exist by themselves. But it is still useful to study them in order to study baryon and tetraquark currents, which can be constructed by these basic fields together with quark fields and antiquark fields. In this section, we just study diquark currents, and antidiquark currents can be studied similarly.

The diquark field contains two quark spinors, and its color can be either $\bar{3}_c$ ($\epsilon_{abc}q^b q^c$) or 6_c ($S_{ab}^m q^a q^b$). The flavor can also be either $\bar{3}_f$ ($\epsilon^{ABC}q_B q_C$) or 6_f ($S_N^{AB}q_A q_B$), where ϵ^{ABC} is the totally antisymmetric matrix, and S_N^{AB} is the totally symmetric matrix with $N = 1, \cdots, 6$. Together with γ-matrices and the charge-conjugation operator C, we can construct the diquark currents:

1. Scalar:

$$\begin{aligned} S^3 &= \epsilon_{abc}q_A^{bT}(x)C\gamma_5 q_B^c(x) \,, \\ S^6 &= q_A^{aT}(x)C\gamma_5 q_B^b(x) + q_A^{bT}(x)C\gamma_5 q_B^a(x) \,. \end{aligned}$$ (1.36)

The first one S^3 has color $\bar{3}_c$. It has antisymmetric color structure, antisymmetric spin structure and symmetric orbital structure, and so it should have antisymmetric flavor $\bar{3}_f$ due to the Pauli principle. The second one S^6 has color 6_c and so flavor 6_f. They both have quantum numbers $J^P = 0^+$. The spin can be studied more carefully:

$$S = 0 \,, L = 0 \,, J = 0 \,,$$

which can be written as $^1S_0$.

2. Vector:

$$V_\mu^3 = \epsilon_{abc} q_A^{bT}(x) C\gamma_\mu \gamma_5 q_B^c(x) \,, \tag{1.37}$$
$$V_\mu^6 = q_A^{aT}(x) C\gamma_\mu \gamma_5 q_B^b(x) + q_A^{bT}(x) C\gamma_\mu \gamma_5 q_B^a(x) \,.$$

The first one V_μ^3 has color $\mathbf{\bar{3}_c}$ and flavor $\mathbf{\bar{3}_f}$; the second one V_μ^6 has color $\mathbf{6_c}$ and flavor $\mathbf{6_f}$. They both have quantum numbers $J^P = 1^-$ ($^3P_1$).

3. Tensor:

$$T_{\mu\nu}^3 = \epsilon_{abc} q_A^{bT}(x) C\sigma_{\mu\nu} q_B^c(x) \,, \tag{1.38}$$
$$T_{\mu\nu}^6 = q_A^{aT}(x) C\sigma_{\mu\nu} q_B^b(x) + q_A^{bT}(x) C\sigma_{\mu\nu} q_B^a(x) \,.$$

The first one $T_{\mu\nu}^3$ has color $\mathbf{\bar{3}_c}$ and flavor $\mathbf{6_f}$; the second one $T_{\mu\nu}^6$ has color $\mathbf{6_c}$ and flavor $\mathbf{\bar{3}_f}$. They both have quantum numbers $J^P = 1^\pm$. T_{0i}^3 and T_{0i}^6 have quantum numbers $J^P = 1^+$ ($^3S_1$), and T_{ij}^3 and T_{ij}^6 have quantum numbers $J^P = 1^-$ ($^1P_1$).

4. Axial-Vector:

$$A_\mu^3 = \epsilon_{abc} q_A^{bT}(x) C\gamma_\mu q_B^c(x) \,, \tag{1.39}$$
$$A_\mu^6 = q_A^{aT}(x) C\gamma_\mu q_B^b(x) + q_A^{bT}(x) C\gamma_\mu q_B^a(x) \,.$$

The first one A_μ^3 has color $\mathbf{\bar{3}_c}$ and flavor $\mathbf{6_f}$; the second one A_μ^6 has color $\mathbf{6_c}$ and flavor $\mathbf{\bar{3}_f}$. They both have quantum numbers $J^P = 1^+$ ($^3S_1$).

5. Pseudoscalar:

$$P^3 = \epsilon_{abc} q_A^{bT}(x) C q_B^c(x) \,, \tag{1.40}$$
$$P^6 = q_A^{aT}(x) C q_B^b(x) + q_A^{bT}(x) C q_B^a(x) \,.$$

The first one P^3 has color $\mathbf{\bar{3}_c}$ and flavor $\mathbf{\bar{3}_f}$; the second one P^6 has color $\mathbf{6_c}$ and flavor $\mathbf{6_f}$. They both have quantum numbers $J^P = 0^-$ ($^3P_0$).

Again, we emphasize here that there are two other tensor currents:

$$T_{\mu\nu}^{3\prime} = \epsilon_{abc} q_A^{bT}(x) C\sigma_{\mu\nu}\gamma_5 q_B^c(x) \,, \tag{1.41}$$
$$T_{\mu\nu}^{6\prime} = q_A^{aT}(x) C\sigma_{\mu\nu} q_B^b(x) + q_A^{bT}(x) C\sigma_{\mu\nu}\gamma_5 q_B^a(x) \,.$$

which can be related to tensor currents $T_{\mu\nu}^3$ and $T_{\mu\nu}^6$, but have an opposite parity.

Altogether we have ten different kinds of diquark currents which are listed in Table 1.1. By using these diquark currents and adding another quark spinor, we can construct baryon currents; while by adding another antidiquark current, we can construct tetraquark currents. It is also interesting to study the diquark itself [97, 153], which we will not discuss in this thesis.

Table 1.1: Diquark Properties of Single Currents.

(qq)	S_3	V_3	T_6	A_6	P_3	S_6	V_6	T_3	A_3	P_6
Flavor ($\mathbf{f}$)	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	$\bar{3}$	6	6	6	6	6
Color ($\mathbf{c}$)	$\bar{3}$	$\bar{3}$	6	6	$\bar{3}$	6	6	$\bar{3}$	6	$\bar{3}$
Spin ($\mathbf{S}$)	0	0	(0, 1)	1	1	0	0	(0, 1)	1	1
Orbit angular momentum ($\mathbf{L}$)	0	1	(1, 0)	0	1	0	1	(1, 0)	0	1
Total Spin ($\mathbf{J} = \mathbf{S} + \mathbf{L}$)	0	1	1	1	0	0	1	1	1	0

Chapter 2

Baryon Fields

In this chapter, we perform a complete classification of baryon fields written as local products (without derivatives) of three quarks according to the chiral symmetry group $SU(3)_L \otimes SU(3)_R$. The case of flavor $SU(2)$ has been studied in the reference [136]. These baryon fields have been studied long time ago, and are used as interpolators for the study of two-point correlation functions in the QCD sum rule approach and in the lattice QCD [38, 48, 51, 65, 65, 88, 103, 120, 122, 180]. Although the chiral structure of an interpolator does not directly reflect that of the physical state when chiral symmetry is spontaneously broken, the minimal configuration of three quarks provides at least a guide to the simplest expectations for baryons.

We first establish a classification under the ordinary (vector) flavor $SU(3)$ symmetry, and then investigate the properties under the full chiral symmetry group $SU(3)_L \otimes SU(3)_R$. Here, we want to study chiral symmetry together with the flavor symmetry, the reason is that there are situations when it makes sense to consider algebraic aspects of chiral symmetry, i.e. the chiral multiplets of hadrons, as pointed out by Weinberg [173], and studied in many other references [79, 103, 118, 119]. We can also use the chiral representation as a theoretical probe for the internal structure of hadrons. For instance, for a $\bar{q}q$ spin-one meson, the possible chiral representations are $(\mathbf{8}, \mathbf{1})$ and $(\mathbf{3}, \bar{\mathbf{3}})$ and their left-right conjugates for flavor octet mesons. As a matter of fact, for the multiquark hadrons, the allowed chiral representations can be more complicated/higher dimensional with increasing number of quarks and antiquarks. Hence the study of chiral representations may provide some hints to the structure of hadrons, extending possibly beyond the minimal constituent picture [27, 28, 55, 77, 101, 102].

We first establish a classification under the ordinary (vector) flavor $SU(3)$ symmetry, and then investigate the properties under the full chiral symmetry group. The method is based essentially on the tensor method for the $SU(3)$ group representations, while the Fierz method for the Pauli principle associated with the structure in the color, flavor and Lorentz (spin) spaces is utilized when establishing the independent fields. It turns out that for local three-quark fields, the Pauli principle puts a constraint on the structure of

the Lorentz and chiral representations. This leads essentially to the same permutation symmetry structures as in the case of flavor $SU(2)$ symmetry, with the one important difference being the existence of flavor singlets in the present case:

2.1 Flavor Symmetries of Three-Quark Baryon Fields

Local fields for baryons consisting of three quarks can be generally written as

$$B(x) \sim \epsilon_{abc} \left(q_A^{aT}(x) C \Gamma_1 q_B^b(x) \right) \Gamma_2 q_C^c(x) , \tag{2.1}$$

where a, b, c denote the color and A, B, C the flavor indices, $C = i\gamma_2\gamma_0$ is the charge-conjugation operator, $q_A(x) = (u(x), d(x), s(x))$ is the flavor triplet quark field at location x, and the superscript T represents the transpose of the Dirac indices only (the flavor and color $SU(3)$ indices are *not* transposed). The antisymmetric tensor in color space ϵ_{abc}, ensures the baryons' being color singlets. For local fields, the space-time coordinate x does nothing with our studies, and we shall omit it. The matrices $\Gamma_{1,2}$ are Dirac matrices which describe the Lorentz structure. With a suitable choice of $\Gamma_{1,2}$ and taking a combination of indices of A, B and C, the baryon operators are defined so that they form an irreducible representation of the Lorentz and flavor groups, as we shall show in this section.

We employ the tensor formalism for flavor $SU(3)$ *a la* Okubo [78, 129, 145, 146, 158] for the quark field q, although the explicit expressions in terms of *up, down* and *strange* quarks are usually employed in lattice QCD and QCD sum rule studies. We shall see that the tensor formulation simplifies the classification of baryons into flavor multiplets and leads to a straightforward, but lengthy derivation of the Fierz identities and the chiral transformations of baryon operators. This is in contrast to the $N_f = 2$ case where we explicitly included isospin/flavour into the $\Gamma_{1,2}$ matrices and thus produced isospin invariant/covariant objects [136].

2.1.1 Flavor $SU(3)_f$ decomposition for baryons

For the sake of notational completeness, we start with some definitions. The quarks of flavor $SU(3)$ form either the contra-variant (3) or the covariant ($\bar{3}$) fundamental representations. They are distinguished by either upper or lower index as

$$q^A \in q = \begin{pmatrix} u \\ d \\ s \end{pmatrix} , \tag{2.2}$$

$$q_A \in q^\dagger = (u^*, d^*, s^*) .$$

The two conjugate fundamental representations transform under flavor $SU(3)$ transformations as

$$q \rightarrow \exp(i\frac{\vec{\lambda}}{2}\vec{a})q, \tag{2.3}$$

$$q^\dagger \rightarrow q^\dagger \exp(-i\frac{\vec{\lambda}}{2}\vec{a}),$$

where a_N $(N = 1, \cdots, 8)$ are the octet of $SU(3)_F$ group parameters and λ^N are the eight Gell-Mann matrices. Since the latter are Hermitian, we may replace the transposed matrices with the complex conjugate ones. The set of eight $\bar{\lambda}^N = -(\lambda^N)^T = -(\lambda^N)^*$ matrices form the generators of the irreducible $\bar{\mathbf{3}}$ representation.

Now for three quarks, we show flavor $SU(3)$ irreducible decomposition $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}$ explicitly in terms of three quarks. It can be done by making suitable permutation symmetry representations of three-quark products $q_A q_B q_C$.

1. The totally antisymmetric combination which forms the singlet,

$$\Psi_{[ABC]} = \mathcal{N}\left(q_A q_B q_C + q_B q_C q_A + q_C q_A q_B - q_B q_A q_C - q_A q_C q_B - q_C q_B q_A\right). \tag{2.4}$$

The normalization constant here is $\mathcal{N} = 1/\sqrt{6}$. In the quark model this corresponds to $\Lambda(1405)$. In order to represent this totally antisymmetric combination, we can use the totally antisymmetric tensor ϵ^{ABC}. Then the flavor singlet baryon field Λ can be written as:

$$\Lambda \equiv \epsilon^{ABC}\epsilon_{abc}\left(q_A^{aT}C\Gamma_1 q_B^b\right)\Gamma_2 q_C^c. \tag{2.5}$$

2. The totally symmetric combination which forms the decuplet,

$$\Psi_{\{ABC\}} = \mathcal{N}\left(q_A q_B q_C + q_B q_C q_A + q_C q_A q_B + q_B q_A q_C + q_A q_C q_B + q_C q_B q_A\right). \tag{2.6}$$

The normalization constant depends on the set of quarks for baryons. For example, for $q_A, q_B, q_C = u, d, s$, $\mathcal{N} = 1/\sqrt{6}$, while it is $1/6$ for $q_A, q_B, q_C = u, u, u$. In order to represent this totally symmetric flavor structure, we introduce the totally symmetric tensor S_P^{ABC} $(P = 1, \cdots, 10)$. Then the flavor decuplet baryon field Δ can be written as:

$$\Delta^P \equiv S_P^{ABC}\epsilon_{abc}\left(q_A^{aT}C\Gamma_1 q_B^b\right)\Gamma_2 q_C^c. \tag{2.7}$$

The non-zero components of S_P^{ABC} $(= 1)$ are summarized in Table 2.1. The rest of components are just zero, for instance, $S_1^{112} = 0$.

3. The two mixed symmetry tensors of the ρ and λ types are defined by

$$\Psi^\rho_{[A\{B]C\}} = \mathcal{N}\left(2q_A q_B q_C - q_B q_C q_A - q_C q_A q_B - 2q_B q_A q_C + q_A q_C q_B + q_C q_B q_A\right),$$

$$\Psi^\lambda_{\{A[B\}C]} = \mathcal{N}\left(2q_A q_B q_C - q_B q_C q_A - q_C q_A q_B + 2q_B q_A q_C - q_A q_C q_B - q_C q_B q_A\right). \tag{2.8}$$

Table 2.1: Non-Zero Components of $S_P^{ABC}(=1)$

P	1	2	3	4	5	6	7	8	9	10
ABC	111	112	113	122	123	133	222	223	233	333
Baryons	Δ^{++}	Δ^+	Σ^{*+}	Δ^0	Σ^{*0}	Ξ^{*0}	Δ^-	Σ^{*-}	Ξ^{*-}	Ω^-

Here the two symbols in { } are first symmetrized and then the symbols in [ ] are anti-symmetrized. The normalization constant depends again on the number of different kinds of terms. The correspondence of the octet fields of (2.8) and the physical ones can be made first by taking the following combinations

$$N_{8\rho}^N = \epsilon^{ABD}(\boldsymbol{\lambda}^N)_{DC}\Psi^\rho_{[A\{B]C\}}, \qquad (2.9)$$
$$N_{8\lambda}^N = \epsilon^{BCD}(\boldsymbol{\lambda}^N)_{DA}\Psi^\lambda_{\{A[B\}C]},$$

where N is an octet index $N = 1, 2, \cdots, 8$. This kind of "double index" (DC for $N_{8\rho}^N$ and DA for $N_{8\lambda}^N$) notation for the baryon flavor has been used by Christos [47]. In our discussions, we shall use the following form for the flavor octet baryon field

$$N^N \equiv \epsilon^{ABD}(\boldsymbol{\lambda}^N)_{DC}\epsilon_{abc}\left(q_A^{aT}C\Gamma_1 q_B^b\right)\Gamma_2 q_C^c. \qquad (2.10)$$

It is of the ρ type. But after using Fierz transformations to interchange the second and the third quarks, the transformed one contains λ type also, as we shall show in the following. The octet of physical baryon fields are then determined by

$$N^1 \pm iN^2 \sim \Sigma^\mp, \quad N^3 \sim \Sigma^0, \quad N^8 \sim \Lambda, \qquad (2.11)$$
$$N^4 \pm iN^5 \sim \Xi^-, p, \quad N^6 \pm iN^7 \sim \Xi^0, n,$$

or put into the 3×3 baryon matrix

$$\mathfrak{N} = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^8}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda^8}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda^8 \end{pmatrix}. \qquad (2.12)$$

2.1.2 Dirac fields

In this section we investigate independent baryon fields for each Lorentz group representation which is formed by three quarks. The Clebsch-Gordan series for the irreducible decomposition of the direct product of three $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representations of the Lorentz group (the three quark Dirac fields) is

$$\left((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\right)^3 \sim \left((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\right) \oplus \left((1, \frac{1}{2}) \oplus (\frac{1}{2}, 1)\right) \oplus \left((\frac{3}{2}, 0) \oplus (0, \frac{3}{2})\right),$$

$$(2.13)$$

where we have ignored the different multiplicities of the representations on the right-hand side. The three representations $\left((\frac{1}{2},0)\oplus(0,\frac{1}{2})\right)$, $\left((1,\frac{1}{2})\oplus(\frac{1}{2},1)\right)$, $\left((\frac{3}{2},0)\oplus(0,\frac{3}{2})\right)$ describe the Dirac spinor field, the Rarita-Schwinger's vector-spinor field and the antisymmetric-tensor-spinor field, respectively. In order to establish independent fields we employ the Fierz transformations for the color, flavor, and Lorentz (spin) degrees of freedom, which is essentially equivalent to the Pauli principle for three quarks.

The Flavor Singlet Baryon

Let us start with writing down five baryon fields which contain a diquark formed by five sets of Dirac matrices, $1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5$ and $\sigma_{\mu\nu}$,

$$
\begin{aligned}
\Lambda_1 &= \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}Cq_B^b)\gamma_5 q_C^c\,, \\
\Lambda_2 &= \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}C\gamma_5 q_B^b)q_C^c\,, \\
\Lambda_3 &= \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}C\gamma_\mu\gamma_5 q_B^b)\gamma^\mu q_C^c\,, \\
\Lambda_4 &= \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}C\gamma_\mu q_B^b)\gamma^\mu\gamma_5 q_C^c\,, \\
\Lambda_5 &= \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}C\sigma_{\mu\nu} q_B^b)\sigma_{\mu\nu}\gamma_5 q_C^c\,.
\end{aligned}
\tag{2.14}
$$

Among these five fields, we can show that the fourth and fifth ones vanish, $\Lambda_{4,5}=0$. This is due to the Pauli principle between the first two quarks, and can be verified, for instance, by taking the transpose of the diquark component and compare the resulting three-quark field with the original expressions [47]. The Pauli principle can also be used between the first and the third quarks, so we construct the primed fields where the second and the third quarks are interchanged, for instance,

$$
\Lambda_1' = \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}Cq_C^c)\gamma_5 q_B^b\,.
$$

Now expressing Λ_i in terms of the Fierz transformed fields Λ_i', we find the following relations (see Appendix B),

$$
\begin{aligned}
\Lambda_1 &= -\tfrac{1}{4}\Lambda_1' - \tfrac{1}{4}\Lambda_2' - \tfrac{1}{4}\Lambda_3'\,, \\
\Lambda_2 &= -\tfrac{1}{4}\Lambda_1' - \tfrac{1}{4}\Lambda_2' + \tfrac{1}{4}\Lambda_3'\,, \\
\Lambda_3 &= -\Lambda_1' + \Lambda_2' + \tfrac{1}{2}\Lambda_3'\,.
\end{aligned}
$$

On the other hand, by changing the indices B, C and b, c, for instance,

$$
\begin{aligned}
\Lambda_1' &= \epsilon_{acb}\epsilon^{ACB}(q_A^{aT}Cq_B^b)\gamma_5 q_C^c \\
&= \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}Cq_B^b)\gamma_5 q_C^c\,,
\end{aligned}
$$

we see that the primed fields are just the corresponding unprimed ones, $\Lambda_i' = \Lambda_i$. Consequently, we obtain three homogeneous linear equations whose rank is just one, and we find the following solution

$$
\Lambda_3 = 4\Lambda_2 = -4\Lambda_1\,, \Lambda_4 = \Lambda_5 = 0\,.
\tag{2.15}
$$

We see that there is only one non-vanishing independent field, which in the quark model corresponds to the odd-parity $\Lambda(1405)$.

The Flavor Decuplet Baryons

Among the five decuplet baryon fields formed by the five different γ-matrices, only two are non-zero:

$$\Delta_4^P = \epsilon_{abc} S_P^{ABC} (q_A^{aT} C \gamma_\mu q_B^b) \gamma^\mu \gamma_5 q_C^c \,,$$
$$\Delta_5^P = \epsilon_{abc} S_P^{ABC} (q_A^{aT} C \sigma_{\mu\nu} q_B^b) \sigma_{\mu\nu} \gamma_5 q_C^c \,. \tag{2.16}$$

Performing the Fierz transformation and with the relation $\Delta_i^{P\prime} = -\Delta_i^P$ ($\epsilon_{acb} S_P^{ACB} = -\epsilon_{abc} S_P^{ABC}$), we find that there is only a trivial (null) solution to the homogeneous linear equations. Therefore, the Dirac baryon fields (fundamental representation of the Lorentz group) formed by three quarks can not survive the flavor decuplet.

The Flavor Octet Baryon

Let us start once again with five fields, which have three potentially non-zero ones

$$N_1^N = \epsilon_{abc} \epsilon^{ABD} \lambda_{DC}^N (q_A^{aT} C q_B^b) \gamma_5 q_C^c \,,$$
$$N_2^N = \epsilon_{abc} \epsilon^{ABD} \lambda_{DC}^N (q_A^{aT} C \gamma_5 q_B^b) q_C^c \,,$$
$$N_3^N = \epsilon_{abc} \epsilon^{ABD} \lambda_{DC}^N (q_A^{aT} C \gamma_\mu \gamma_5 q_B^b) \gamma^\mu q_C^c \,, \tag{2.17}$$
$$N_4^N = \epsilon_{abc} \epsilon^{ABD} \lambda_{DC}^N (q_A^{aT} C \gamma_\mu q_B^b) \gamma^\mu \gamma_5 q_C^c = 0 \,,$$
$$N_5^N = \epsilon_{abc} \epsilon^{ABD} \lambda_{DC}^N (q_A^{aT} C \sigma_{\mu\nu} q_B^b) \sigma_{\mu\nu} \gamma_5 q_C^c = 0 \,.$$

These octet baryon fields have been studied in Refs [48, 65, 88], where the independent ones are clarified. As before, we perform the Fierz rearrangement to obtain five equations with the primed fields, while $N_4^{N\prime}$ and $N_5^{N\prime}$ are not zero. For the first three equations, $N_{1,2,3}$ on the left hand side should be expressed by the primed fields. To this end, we can use the Jacobi identity

$$\epsilon^{ABD} \lambda_{DC}^N + \epsilon^{BCD} \lambda_{DA}^N + \epsilon^{CAD} \lambda_{DB}^N = 0 \,, \tag{2.18}$$

which can be used to relate the original fields N_i^N and primed ones $N_i^{N\prime}$, for instance,

$$\left(\epsilon^{ABD} \lambda_{DC}^N + \epsilon^{BCD} \lambda_{DA}^N + \epsilon^{CAD} \lambda_{DB}^N \right) (q_A^{aT} C q_B^b) \gamma_5 q_C^c = 0 \,,$$

from which we find

$$N_1^{N\prime} = -\frac{1}{2} N_1^N \,,$$

and the same relations for $N_{2,3}^N$. There are no relations between $N_{4,5}^N$ and $N_{4,5}^{N\prime}$. Altogether, we have five equations. The equations related to N_4^N and N_5^N are also necessary because the corresponding primed ones are not zero. They can be solved to obtain the following solutions:

$$\frac{2}{3} N_4^{N\prime} = N_3^N = N_1^N - N_2^N \,, N_5^{N\prime} = -3(N_1^N + N_2^N) \,, \tag{2.19}$$

which indicates that there are two independent octet fields, for instance, N_1^N and N_2^N. Thus we have shown the same result just as in the two-flavor case [136]. In the following sections we shall show that the difference between the two fields N_1 and N_2 lies in their chiral properties: $N_1^N - N_2^N$ together with Λ belong to $(\bar{\mathbf{3}}, \mathbf{3}) \oplus (\mathbf{3}, \bar{\mathbf{3}})$, and the other $N_1^N + N_2^N$ belongs to $(\mathbf{8}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8})$.

There are two ways to construct the octet baryon fields. One is done already as shown in Eqs. (2.17), whose flavor structure is the same as the ρ type baryon field $N_{8\rho}^N$ in Eqs. (2.9):

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \longrightarrow (\mathbf{3} \otimes \mathbf{3}) \otimes \mathbf{3} \longrightarrow \bar{\mathbf{3}} \otimes \mathbf{3} \longrightarrow \mathbf{8}_\rho \,. \tag{2.20}$$

The other λ type baryon field $N_{8\lambda}^N$ is complicated when used straightforwardly:

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \longrightarrow (\mathbf{3} \otimes \mathbf{3}) \otimes \mathbf{3} \longrightarrow \mathbf{6} \otimes \mathbf{3} \longrightarrow \mathbf{8}_\lambda \,. \tag{2.21}$$

Therefore, we use another way based on

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} \longrightarrow \mathbf{3} \otimes (\mathbf{3} \otimes \mathbf{3}) \longrightarrow \mathbf{3} \otimes \bar{\mathbf{3}} \longrightarrow \mathbf{8}_\rho' \,. \tag{2.22}$$

This contains partly $\mathbf{8}_\lambda$, and it is easily to verify that (2.20) and (2.22) compose a full description of octet baryon which is also fully described by using (2.20) and (2.21). The way $\mathbf{8}_\rho$ leads to octet fields N_i^N, and the other way $\mathbf{8}_\rho'$ leads to other five ones

$$\begin{aligned}
\widetilde{N}_1^N &= \epsilon_{abc}\epsilon^{ACD}\lambda_{DB}^N (q_A^{aT} C q_B^b)\gamma_5 q_C^c \,, \\
\widetilde{N}_2^N &= \epsilon_{abc}\epsilon^{ACD}\lambda_{DB}^N (q_A^{aT} C \gamma_5 q_B^b) q_C^c \,, \\
\widetilde{N}_3^N &= \epsilon_{abc}\epsilon^{ACD}\lambda_{DB}^N (q_A^{aT} C \gamma_\mu \gamma_5 q_B^b)\gamma^\mu q_C^c \,, \\
\widetilde{N}_4^N &= \epsilon_{abc}\epsilon^{ACD}\lambda_{DB}^N (q_A^{aT} C \gamma_\mu q_B^b)\gamma^\mu \gamma_5 q_C^c \,, \\
\widetilde{N}_5^N &= \epsilon_{abc}\epsilon^{ACD}\lambda_{DB}^N (q_A^{aT} C \sigma_{\mu\nu} q_B^b)\sigma_{\mu\nu}\gamma_5 q_C^c \,.
\end{aligned} \tag{2.23}$$

However, these fields can be related to the previous ones by changing the flavor and color indices B, C and b, c:

$$\widetilde{N}_i^N = -N_i^{N'} \,. \tag{2.24}$$

In nearly all the cases, the octet baryon fields from the second way can be related to the ones from the first way. Therefore, we shall omit the discussion of the second octet.

2.1.3 Rarita-Schwinger fields

In this section, we study the properties of Rarita-Schwinger fields, in the form of

$$B_\mu(x) \sim \epsilon_{abc}(q_A^{aT}(x) C \Gamma_1 q_B^b(x))\Gamma_2 q_C^c(x) \,, \tag{2.25}$$

where there are eight possible pairs of Γ_1 and Γ_2,

$$\begin{aligned}
(\Gamma_1, \ \Gamma_2) \ = \ & (\mathbf{1}, \ \gamma_\mu), \ (\gamma_5, \ \gamma_\mu\gamma_5), \ (\gamma_\mu\gamma_5, \ \gamma_5), \ (\gamma^\nu\gamma_5, \ \sigma_{\mu\nu}\gamma_5), \\
& (\gamma_\mu, \ \mathbf{1}), \ (\gamma^\nu, \ \sigma_{\mu\nu}), \ (\sigma_{\mu\nu}, \ \gamma^\nu), \ (\sigma_{\mu\nu}\gamma_5, \ \gamma^\nu\gamma_5).
\end{aligned}$$

The discussion is separated into singlet, decuplet and octet.

The Flavor Singlet Baryon

For flavor singlet fields, there are four apparently non-zero fields

$$\begin{aligned}
\Lambda_{1\mu} &= \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}Cq_B^b)\gamma_\mu q_C^c \,, \\
\Lambda_{2\mu} &= \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}C\gamma_5 q_B^b)\gamma_\mu\gamma_5 q_C^c \,, \\
\Lambda_{3\mu} &= \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}C\gamma_\mu\gamma_5 q_B^b)\gamma_5 q_C^c \,, \\
\Lambda_{4\mu} &= \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}C\gamma^\nu\gamma_5 q_B^b)\sigma_{\mu\nu}\gamma_5 q_C^c \,.
\end{aligned}
\qquad (2.26)$$

As before, the Fierz transformed fields (primed fields) are just the corresponding unprimed ones, $\Lambda'_{i\mu} = \Lambda_{i\mu}$. By performing the Fierz transformation (see Appendix. B), we obtain four equations

$$\begin{aligned}
\Lambda_{1\mu} &= -\tfrac{1}{4}\Lambda'_{1\mu} - \tfrac{1}{4}\Lambda'_{2\mu} + \tfrac{1}{4}\Lambda'_{3\mu} - \tfrac{i}{4}\Lambda'_{4\mu} \,, \\
\Lambda_{2\mu} &= -\tfrac{1}{4}\Lambda'_{1\mu} - \tfrac{1}{4}\Lambda'_{2\mu} - \tfrac{1}{4}\Lambda'_{3\mu} + \tfrac{i}{4}\Lambda'_{4\mu} \,, \\
\Lambda_{3\mu} &= \tfrac{1}{4}\Lambda'_{1\mu} - \tfrac{1}{4}\Lambda'_{2\mu} - \tfrac{1}{4}\Lambda'_{3\mu} - \tfrac{i}{4}\Lambda'_{4\mu} \,, \\
\Lambda_{4\mu} &= \tfrac{3i}{4}\Lambda'_{1\mu} - \tfrac{3i}{4}\Lambda'_{2\mu} + \tfrac{3i}{4}\Lambda'_{3\mu} + \tfrac{1}{4}\Lambda'_{4\mu} \,.
\end{aligned}$$

Thus we find the following solution

$$\Lambda_{1\mu} = -\Lambda_{2\mu} = \Lambda_{3\mu} = -\frac{i}{3}\Lambda_{4\mu} = \gamma_\mu\gamma_5\Lambda_1 \,, \qquad \Lambda_{6\mu} = \Lambda_{7\mu} = \Lambda_{8\mu} = 0. \qquad (2.27)$$

We see that there is only one non-vanishing independent field. However, it has a structure of $\gamma_\mu\Lambda_i$. Therefore, they are all Dirac fields, and there is no flavor singlet fields of the Rarita-Schwinger type.

The Flavor Decuplet Baryon

For flavor decuplet fields, we have four potentially non-zero interpolators

$$\begin{aligned}
\Delta_{5\mu}^P &= \epsilon_{abc}S_P^{ABC}(q_A^{aT}C\gamma_\mu q_B^b)q_C^c \,, \\
\Delta_{6\mu}^P &= \epsilon_{abc}S_P^{ABC}(q_A^{aT}C\gamma^\nu q_B^b)\sigma_{\mu\nu}q_C^c \,, \\
\Delta_{7\mu}^P &= \epsilon_{abc}S_P^{ABC}(q_A^{aT}C\sigma_{\mu\nu}q_B^b)\gamma^\nu q_C^c \,, \\
\Delta_{8\mu}^P &= \epsilon_{abc}S_P^{ABC}(q_A^{aT}C\sigma_{\mu\nu}\gamma_5 q_B^b)\gamma^\nu\gamma_5 q_C^c \,.
\end{aligned}
\qquad (2.28)$$

As before, the Fierz transformed fields can be related to the corresponding unprimed ones, $\Delta_{i\mu}^{P\prime} = -\Delta_{i\mu}^P$. Similarly performing the Fierz transformation to relate $\Delta_{i\mu}^N$ and $\Delta_{i\mu}^{N\prime}$, we obtain the solution

$$\Delta_{5\mu}^P = i\Delta_{6\mu}^P = -i\Delta_{7\mu}^P = i\Delta_{8\mu}^P \,. \qquad (2.29)$$

There are no Dirac decuplet fields. Therefore, we obtain one extra non-vanishing field.

The Flavor Octet Baryon

To study the octet baryon fields, we start with eight baryon fields:

$$
\begin{aligned}
N_{1\mu}^N &= \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N(q_A^{aT}Cq_B^b)\gamma_\mu q_C^c\,, \\
N_{2\mu}^N &= \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N(q_A^{aT}C\gamma_5 q_B^b)\gamma_\mu\gamma_5 q_C^c\,, \\
N_{3\mu}^N &= \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N(q_A^{aT}C\gamma_\mu\gamma_5 q_B^b)\gamma_5 q_C^c\,, \\
N_{4\mu}^N &= \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N(q_A^{aT}C\gamma^\nu\gamma_5 q_B^b)\sigma_{\mu\nu}\gamma_5 q_C^c\,, \\
N_{5\mu}^N &= \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N(q_A^{aT}C\gamma_\mu q_B^b)q_C^c = 0\,, \\
N_{6\mu}^N &= \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N(q_A^{aT}C\gamma^\nu q_B^b)\sigma_{\mu\nu}q_C^c = 0\,, \\
N_{7\mu}^N &= \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N(q_A^{aT}C\sigma_{\mu\nu}q_B^b)\gamma^\nu q_C^c = 0\,, \\
N_{8\mu}^N &= \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N(q_A^{aT}C\sigma_{\mu\nu}\gamma_5 q_B^b)\gamma^\nu\gamma_5 q_C^c = 0\,.
\end{aligned}
\tag{2.30}
$$

There are four zero fields, but the Fierz transformed ones are non-zero. By using the Jacobi identity in Eq. (2.18), we obtain

$$
N_{1\mu}^{N\prime} = -\frac{1}{2}N_{1\mu}^N\,,\ N_{2\mu}^{N\prime} = -\frac{1}{2}N_{2\mu}^N\,,\ N_{3\mu}^{N\prime} = -\frac{1}{2}N_{3\mu}^N\,,\ N_{4\mu}^{N\prime} = -\frac{1}{2}N_{4\mu}^N\,.
$$

Similarly performing the Fierz transformation to relate $N_{i\mu}^N$ and $N_{i\mu}^{N\prime}$, we obtain the solution

$$
\begin{aligned}
N_{4\mu}^N &= -iN_{1\mu}^N + iN_{2\mu}^N - iN_{3\mu}^N\,, \\
N_{5\mu}^{N\prime} &= -\frac{1}{2}N_{1\mu}^N + \frac{1}{2}N_{2\mu}^N - \frac{1}{2}N_{3\mu}^N\,, \\
N_{6\mu}^{N\prime} &= -iN_{1\mu}^N + iN_{2\mu}^N + \frac{i}{2}N_{3\mu}^N\,, \\
N_{7\mu}^{N\prime} &= iN_{1\mu}^N + \frac{i}{2}N_{2\mu}^N + iN_{3\mu}^N\,, \\
N_{8\mu}^{N\prime} &= \frac{i}{2}N_{1\mu}^N + iN_{2\mu}^N - iN_{3\mu}^N\,.
\end{aligned}
$$

Thus we have shown that there are three different kinds of octets. $N_{1\mu}^N$ and $N_{2\mu}^N$ have a structure of $\gamma_\mu N_1^N$ and $\gamma_\mu N_2^N$. Therefore, we only obtain one extra octet baryon field $N_{3\mu}^N$.

2.1.4 Tensor Fields

In this section, we study the baryons fields with two free antisymmetric Lorentz indices: $J_{\mu\nu}$, if $J_{\mu\nu} = -J_{\nu\mu}$, it can have spin 3/2. For the tensor fields, we can form nine three-quark fields where the possible pairs of Γ_1 and Γ_2 are

$$
\begin{aligned}
(\Gamma_1,\ \Gamma_2) = \ & (\gamma_\mu,\ \gamma_\nu\gamma_5) - (\mu \leftrightarrow \nu),\ (\gamma_\mu\gamma_5,\ \gamma_\nu) - (\mu \leftrightarrow \nu), \\
& \epsilon_{\mu\nu\rho\sigma}(\gamma^\rho,\ \gamma^\sigma),\ \epsilon_{\mu\nu\rho\sigma}(\gamma^\rho\gamma_5,\ \gamma^\sigma\gamma_5),\ (1,\ \sigma_{\mu\nu}\gamma_5),\ (\gamma_5,\ \sigma_{\mu\nu}), \\
& (\sigma_{\mu\nu},\ \gamma_5),\ (\sigma_{\mu\nu}\gamma_5,\ 1),\ \epsilon_{\mu\nu\rho\sigma}(\sigma_{\rho l},\sigma_{\sigma l})\,.
\end{aligned}
$$

The discussion is separated into singlet, decuplet and octet.

The Flavor Singlet Baryon

The flavor singlet baryon fields have four potentially non-zero interpolators among nine fields:

$$
\begin{aligned}
\Lambda_{2\mu\nu} &= \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}C\gamma_\mu\gamma_5 q_B^b)\gamma_\nu q_C^c - (\mu \leftrightarrow \nu)\,,\\
\Lambda_{4\mu\nu} &= \epsilon_{abc}\epsilon^{ABC}\epsilon_{\mu\nu\rho\sigma}(q_A^{aT}C\gamma_\rho\gamma_5 q_B^b)\gamma_\sigma\gamma_5 q_C^c\,,\\
\Lambda_{5\mu\nu} &= \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}C q_B^b)\sigma_{\mu\nu}\gamma_5 q_C^c\,,\\
\Lambda_{6\mu\nu} &= \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}C\gamma_5 q_B^b)\sigma_{\mu\nu} q_C^c\,.
\end{aligned}
\tag{2.31}
$$

As before, the Fierz transformed fields are just the corresponding unprimed ones, $\Lambda'_{i\mu\nu} = \Lambda_{i\mu\nu}$. Similarly performing the Fierz transformation to relate $\Lambda_{i\mu\nu}$ and $\Lambda'_{1\mu\nu}$, we obtain the solution:

$$
i\Lambda_{2\mu\nu} = \Lambda_{4\mu\nu} = 2\Lambda_{5\mu\nu} = -2\Lambda_{6\mu\nu}\,,
$$

The Fierz transformation is listed in the Appendix B. There is only one independent field. However, it has a structure of $\sigma_{\mu\nu}\Lambda_i$. Therefore, there are no extra fields.

The Flavor Decuplet Baryon

The flavor decuplet baryon fields have five potentially non-zero interpolators:

$$
\begin{aligned}
\Delta_{1\mu\nu}^P &= \epsilon_{abc}S^{ABC}(q_A^{aT}C\gamma_\mu q_B^b)\gamma_\nu\gamma_5 q_C^c - (\mu \leftrightarrow \nu)\,,\\
\Delta_{3\mu\nu}^P &= \epsilon_{abc}S^{ABC}\epsilon_{\mu\nu\rho\sigma}(q_A^{aT}C\gamma_\rho q_B^b)\gamma_\sigma q_C^c\,,\\
\Delta_{7\mu\nu}^P &= \epsilon_{abc}S^{ABC}(q_A^{aT}C\sigma_{\mu\nu} q_B^b)\gamma_5 q_C^c\,,\\
\Delta_{8\mu\nu}^P &= \epsilon_{abc}S^{ABC}(q_A^{aT}C\sigma_{\mu\nu}\gamma_5 q_B^b)q_C^c\,,\\
\Delta_{9\mu\nu}^P &= \epsilon_{abc}S^{ABC}\epsilon_{\mu\nu\rho\sigma}(q_A^{aT}C\sigma_{\rho l} q_B^b)\sigma_{\sigma l} q_C^c\,.
\end{aligned}
\tag{2.32}
$$

As before, the Fierz transformed fields can be related to the corresponding unprimed ones, $\Delta_{i\mu\mu}^{P\prime} = -\Delta_{i\mu\mu}^P$. Similarly performing the Fierz transformation to relate $\Delta_{i\mu\nu}^P$ and $\Delta_{i\mu\nu}^{P\prime}$, we obtain two independent fields: $\Delta_{1\mu\nu}^P$ and $\Delta_{7\mu\nu}^P$:

$$
\Delta_{3\mu\nu}^P = -i\Delta_{1\mu\nu}^P\,,\ \Delta_{8\mu\nu}^P = i\Delta_{1\mu\nu}^P + \Delta_{7\mu\nu}^P\,,\ \Delta_{9\mu\nu}^P = -i\Delta_{1\mu\nu}^P - 2\Delta_{7\mu\nu}^P\,.
$$

The first one $\Delta_{1\mu\nu}^P$ can be related to the Rarita-Schwinger baryon fields, but the second one $\Delta_{7\mu\nu}^P$ can not. Therefore, we obtain one extra decuplet fields.

The Flavor Octet Baryon

To study the octet baryon fields, we start with nine octet baryon fields

$$
\begin{aligned}
N^N_{1\mu\nu} &= \epsilon_{abc}\epsilon^{ABD}\lambda^N_{DC}(q^{aT}_A C\gamma_\mu q^b_B)\gamma_\nu\gamma_5 q^c_C - (\mu \leftrightarrow \nu) = 0\,, \\
N^N_{2\mu\nu} &= \epsilon_{abc}\epsilon^{ABD}\lambda^N_{DC}(q^{aT}_A C\gamma_\mu\gamma_5 q^b_B)\gamma_\nu q^c_C - (\mu \leftrightarrow \nu)\,, \\
N^N_{3\mu\nu} &= \epsilon_{abc}\epsilon^{ABD}\lambda^N_{DC}\epsilon_{\mu\nu\rho\sigma}(q^{aT}_A C\gamma_\rho q^b_B)\gamma_\sigma q^c_C = 0\,, \\
N^N_{4\mu\nu} &= \epsilon_{abc}\epsilon^{ABD}\lambda^N_{DC}\epsilon_{\mu\nu\rho\sigma}(q^{aT}_A C\gamma_\rho\gamma_5 q^b_B)\gamma_\sigma\gamma_5 q^c_C\,, \\
N^N_{5\mu\nu} &= \epsilon_{abc}\epsilon^{ABD}\lambda^N_{DC}(q^{aT}_A C q^b_B)\sigma_{\mu\nu}\gamma_5 q^c_C\,, \\
N^N_{6\mu\nu} &= \epsilon_{abc}\epsilon^{ABD}\lambda^N_{DC}(q^{aT}_A C\gamma_5 q^b_B)\sigma_{\mu\nu} q^c_C\,, \\
N^N_{7\mu\nu} &= \epsilon_{abc}\epsilon^{ABD}\lambda^N_{DC}(q^{aT}_A C\sigma_{\mu\nu} q^b_B)\gamma_5 q^c_C = 0\,, \\
N^N_{8\mu\nu} &= \epsilon_{abc}\epsilon^{ABD}\lambda^N_{DC}(q^{aT}_A C\sigma_{\mu\nu}\gamma_5 q^b_B)q^c_C = 0\,, \\
N^N_{9\mu\nu} &= \epsilon_{abc}\epsilon^{ABD}\lambda^N_{DC}\epsilon_{\mu\nu\rho\sigma}(q^{aT}_A C\sigma_{\rho l} q^b_B)\sigma_{\sigma l} q^c_C = 0\,.
\end{aligned}
\tag{2.33}
$$

There are five zero fields, but the Fierz transformed ones are non-zero. By using the Jacobi identity in Eq. (2.18), we obtain

$$
N^{N\prime}_{2\mu\nu} = -\frac{1}{2}N^N_{2\mu\nu}\,, \quad N^{N\prime}_{4\mu\nu} = -\frac{1}{2}N^N_{4\mu\nu}\,, \quad N^{N\prime}_{5\mu\nu} = -\frac{1}{2}N^N_{5\mu\nu}\,, \quad N^{N\prime}_{6\mu\nu} = -\frac{1}{2}N^N_{6\mu\nu}\,.
$$

Similarly performing the Fierz transformation to relate $N^N_{i\mu\nu}$ and $N^{N\prime}_{i\mu\nu}$, we find that there are three independent fields $N^N_{2\mu\nu}$, $N^N_{5\mu\nu}$ and $N^N_{6\mu\nu}$. Here are the relations:

$$
\begin{aligned}
N^N_{4\mu\nu} &= -iN^N_{2\mu\nu} - N^N_{5\mu\nu} + N^N_{6\mu\nu}\,, \\
N^{N\prime}_{1\mu\nu} &= -\frac{1}{2}N^N_{2\mu\nu} + iN^N_{5\mu\nu} - iN^N_{6\mu\nu}\,, \\
N^{N\prime}_{3\mu\nu} &= \frac{i}{2}N^N_{2\mu\nu} - \frac{1}{2}N^N_{5\mu\nu} + \frac{1}{2}N^N_{6\mu\nu}\,, \\
N^{N\prime}_{7\mu\nu} &= -\frac{i}{2}N^N_{2\mu\nu} - \frac{1}{2}N^N_{5\mu\nu}\,, \\
N^{N\prime}_{8\mu\nu} &= \frac{i}{2}N^N_{2\mu\nu} - \frac{1}{2}N^N_{6\mu\nu}\,, \\
N^{N\prime}_{9\mu\nu} &= -N^N_{5\mu\nu} - N^N_{6\mu\nu}\,.
\end{aligned}
$$

All these three fields can be related to the Rarita-Schwinger fields. Therefore, there are no extra octet fields.

2.1.5 A short summary of independent baryon fields

Here we shall make a short summary of independent baryon fields for all cases constructed from three quarks. For simplicity, here we suppress the antisymmetric tensor in color space ϵ_{abc}, since it appears in all baryon fields in the same manner. Furthermore, it is convenient to introduce a "tilde-transposed" quark field $\tilde{q}$ as follows

$$
\tilde{q} = q^T C\gamma_5\,.
\tag{2.34}
$$

which differs from the two-flavor definition in Ref. [136] by the absence of the flavor (G-parity) matrix.

As we have shown already, for Dirac fields without Lorentz index, there are one singlet field Λ and two octet fields N_1^N and N_2^N:

$$
\begin{aligned}
\Lambda_1 &= \epsilon^{ABC}(\widetilde{q}_A\gamma_5 q_B)\gamma_5 q_C\,, \\
N_1^N &= \epsilon^{ABD}\lambda_{DC}^N(\widetilde{q}_A\gamma_5 q_B)\gamma_5 q_C\,, \\
N_2^N &= \epsilon^{ABD}\lambda_{DC}^N(\widetilde{q}_A q_B)q_C\,.
\end{aligned}
$$

For the Rarita-Schwinger fields with one Lorentz index, we would consider one singlet, three octet and one decuplet fields:

$$
\begin{aligned}
\Lambda_{1\mu} &= \epsilon^{ABC}(\widetilde{q}_A\gamma_5 q_B)\gamma_\mu q_C\,, \\
N_{1\mu}^N &= \epsilon^{ABD}\lambda_{DC}^N(\widetilde{q}_A\gamma_5 q_B)\gamma_\mu q_C\,, \\
N_{2\mu}^N &= \epsilon^{ABD}\lambda_{DC}^N(\widetilde{q}_A q_B)\gamma_\mu\gamma_5 q_C\,, \\
N_{3\mu}^N &= -\epsilon^{ABD}\lambda_{DC}^N(\widetilde{q}_A\gamma_\mu q_B)\gamma_5 q_C\,, \\
\Delta_{5\mu}^P &= -S_P^{ABC}(\widetilde{q}_A\gamma_\mu\gamma_5 q_B)q_C\,.
\end{aligned}
$$

However, we find that $\Lambda_{1\mu} = \gamma_\mu\gamma_5\Lambda$, $N_{1\mu}^N = \gamma_\mu\gamma_5 N_1^N$ and $N_{2\mu}^N = \gamma_\mu\gamma_5 N_2^N$. So, there are two non-vanishing independent fields: one octet field N_μ^N and one decuplet field Δ_μ. By using the projection operator:

$$
P_{\mu\nu}^{3/2} = \left(g_{\mu\nu} - \frac{1}{4}\gamma_\mu\gamma_\nu\right)\,, \tag{2.35}
$$

they can be written as

$$
\begin{aligned}
N_\mu^N = P_{\mu\nu}^{3/2}N_{3\nu}^N &= -\left(g_{\mu\nu} - \frac{1}{4}\gamma_\mu\gamma_\nu\right)\epsilon^{ABD}\lambda_{DC}^N(\widetilde{q}_A\gamma_\mu q_B)\gamma_5 q_C \\
&= N_{3\mu}^N + \frac{1}{4}\gamma_\mu\gamma_5(N_1^N - N_2^N)\,, \\
\Delta_\mu^P = P_{\mu\nu}^{3/2}\Delta_{5\nu}^P &= -\left(g_{\mu\nu} - \frac{1}{4}\gamma_\mu\gamma_\nu\right)S_P^{ABC}(\widetilde{q}_A\gamma_\mu\gamma_5 q_B)q_C \\
&= \Delta_{5\mu}^P\,.
\end{aligned}
$$

For tensor fields with two antisymmetric Lorentz indices, we would have one singlet, three octet and two decuplet fields:

$$
\begin{aligned}
\Lambda_{1\mu} &= \epsilon^{ABC}(\widetilde{q}_A\gamma_5 q_B)\sigma_{\mu\nu}\gamma_5 q_C\,, \\
N_{3\mu\nu}^N &= -\epsilon^{ABD}\lambda_{DC}^N(\widetilde{q}_A\gamma_\mu q_B)\gamma_\nu q_C + (\mu\leftrightarrow\nu)\,, \\
N_{10\mu\nu}^N &= \epsilon^{ABD}\lambda_{DC}^N(\widetilde{q}_A\gamma_5 q_B)\sigma_{\mu\nu}\gamma_5 q_C\,, \\
N_{11\mu\nu}^N &= \epsilon^{ABD}\lambda_{DC}^N(\widetilde{q}_A q_B)\sigma_{\mu\nu}q_C\,, \\
\Delta_{2\mu\nu}^P &= -S_P^{ABC}(\widetilde{q}_A\gamma_\mu\gamma_5 q_B)\gamma_\nu\gamma_5 q_C + (\mu\leftrightarrow\nu)\,, \\
\Delta_{7\mu\nu}^P &= S_P^{ABC}(\widetilde{q}_A\sigma_{\mu\nu}\gamma_5 q_B)\gamma_5 q_C\,.
\end{aligned}
$$

But in this case, we can show that there is only one non-vanishing field $\Delta_{\mu\nu}$:

$$\begin{aligned}
\Delta^P_{\mu\nu} = \Gamma^{\mu\nu\alpha\beta}\Delta^P_{7\mu\nu} &= \Gamma^{\mu\nu\alpha\beta}S^{ABC}_P(\widetilde{q}_A\sigma_{\mu\nu}\gamma_5 q_B)\gamma_5 q_C \\
&= \Delta^P_{7\mu\nu} - \frac{i}{2}\gamma_\mu\gamma_5\Delta^P_{5\nu} + \frac{i}{2}\gamma_\nu\gamma_5\Delta^P_{5\mu},
\end{aligned}$$

where

$$\Gamma^{\mu\nu\alpha\beta} = \left(g^{\mu\alpha}g^{\nu\beta} - \frac{1}{2}g^{\nu\beta}\gamma^\mu\gamma^\alpha + \frac{1}{2}g^{\mu\beta}\gamma^\nu\gamma^\alpha + \frac{1}{6}\sigma^{\mu\nu}\sigma^{\alpha\beta}\right). \tag{2.36}$$

2.2 Chiral Transformations

In this section, we establish the chiral transformation properties of the baryon fields which we have obtained in the previous section. Technically, this leads to somewhat complicated algebraic results. However, the final result will be understood by making the left- and right-handed decomposition, which we shall perform in the next section.

Let us start with the chiral transformation properties of quarks which are given by the following equations:

$$\begin{aligned}
\mathbf{U(1)_V} &: \quad q \to \exp(i\frac{\lambda^0}{2}a_0)q = q + \delta q, \\[2mm]
\mathbf{SU(3)_V} &: \quad q \to \exp(i\frac{\vec{\lambda}}{2}\cdot\vec{a})q = q + \delta^{\vec{a}}q, \\[2mm]
\mathbf{U(1)_A} &: \quad q \to \exp(i\gamma_5\frac{\lambda^0}{2}b_0)q = q + \delta_5 q, \\[2mm]
\mathbf{SU(3)_A} &: \quad q \to \exp(i\gamma_5\frac{\vec{\lambda}}{2}\cdot\vec{b})q = q + \delta^{\vec{b}}_5 q,
\end{aligned} \tag{2.37}$$

where $\lambda^0 = \sqrt{2/3}\,\mathbf{1}$, $\vec{\lambda}$ are the eight Gell-Mann matrices and $\mathbf{1}$ is a 3×3 unit matrix. Here a^0 is an infinitesimal parameter for the $U(1)_V$ transformation, $\vec{a}$ the octet of $SU(3)_V$ group parameters, b^0 an infinitesimal parameter for the $U(1)_A$ transformation, and $\vec{b}$ the octet of the chiral transformations.

The $U(1)_V$ chiral transformation is trivial which picks up a phase factor proportional to the baryon number. The $U(1)_A$ chiral transformation is slightly less trivial, and the baryon fields are transformed as

$$\begin{aligned}
\delta_5\Lambda &= -i\gamma_5\sqrt{\frac{1}{6}}b^0\Lambda, \\[2mm]
\delta_5(N^N_1 - N^N_2) &= -i\gamma_5\sqrt{\frac{1}{6}}b^0(N^N_1 - N^N_2),
\end{aligned}$$

$$\delta_5(N_1^N + N_2^N) = i\gamma_5\sqrt{\frac{3}{2}}b^0(N_1^N + N_2^N),$$

$$\delta_5 N_\mu^N = i\gamma_5\sqrt{\frac{1}{6}}b^0 N_\mu^N, \qquad (2.38)$$

$$\delta_5\Delta_\mu^P = i\gamma_5\sqrt{\frac{1}{6}}b^0\Delta_\mu^P,$$

$$\delta_5\Delta_{\mu\nu}^P = i\gamma_5\sqrt{\frac{3}{2}}b^0\Delta_{\mu\nu}^P.$$

We note that the combinations of $N_1^N \pm N_2^N$ form different representations.

To study the vector chiral transformation and axial-vector chiral transformation, we first show the following equation which define the d and f coefficients

$$\lambda_{AB}^N\lambda_{BC}^M = (\lambda^N\lambda^M)_{AC} = \frac{1}{2}\{\lambda^N,\lambda^M\}_{AC} + \frac{1}{2}[\lambda^N,\lambda^M]_{AC} \qquad (2.39)$$

$$= \frac{2}{3}\delta^{NM}\delta_{AC} + (d^{NMO} + if^{NMO})\lambda_{AC}^O.$$

Furthermore, the following formulae define the coefficients g_3, g_5 and g_7, which are proved by using *Mathematica*, a software good at matrix calculation:

$$\epsilon^{ADE}\lambda_{DB}^N\lambda_{EC}^M = g_1^{NMO}\epsilon^{ABD}\lambda_{DC}^O + g_2^{NMO}\epsilon^{ACD}\lambda_{DB}^O + g_3^{NMP}S_P^{ABC} + g_4^{NM}\epsilon^{ABC},$$
$$S_Q^{ABD}\lambda_{DC}^M = g_5^{QMO}\epsilon^{ABD}\lambda_{DC}^O + g_6^{QMO}\epsilon^{ACD}\lambda_{DB}^O + g_7^{QMP}S_P^{ABC} + g_8^{QM}\epsilon^{ABC},$$

$$(2.40)$$

where indices $A \sim E$ take values 1, 2 and 3, N, M and O $1, \cdots, 8$, and P and Q $1, \cdots, 10$. The coefficients g_3, g_5 and g_7 are listed in Table 2.2, where we use "0" instead of "10". Other coefficients can be related to d, f, g_3, g_5 and g_7:

$$g_1^{MNO} = -d^{MNO} - \frac{i}{3}f^{MNO},$$

$$g_2^{MNO} = d^{MNO} - \frac{i}{3}f^{MNO},$$

$$g_4^{MN} = -\frac{1}{3}\delta^{MN}, \qquad (2.41)$$

$$g_6^{QMO} = -2g_5^{QMO},$$

$$g_8^{MN} = 0.$$

Let us explain Eqs. (2.40) a bit more. The quantities on the left hand side have three indices A, B and C, and therefore, they are regarded as direct products of three fundamental representations of $SU(3)$: **3** $\otimes$ **3** $\otimes$ **3**. They can be decomposed into irreducible

Table 2.2: g-Coefficients Defined by Eqs. (2.40)

g_3					
	133, 138, 144, 146, 254, 256, 272, 279, 439, 463, 468, 573, 578, 612, 619, 636				$-1/3$
	162, 169, 313, 318, 349, 366, 414, 416, 524, 526, 643, 648, 722, 729, 753, 758				$1/3$
	154, 179, 215, 233, 246, 269, 328, 359, 376, 424, 455, 478, 516, 563, 622, 658, 712, 743, 765				$-i/3$
	125, 156, 172, 238, 244, 262, 323, 426, 473, 514, 539, 545, 568, 629, 653, 675, 719, 736, 748				$i/3$
	183, 686, 818, 835, 849	$-1/\sqrt{3}$	167, 251, 277, 411, 570, 640	-1	342, 364	$-2/3$
	188, 385, 489, 813, 866	$1/\sqrt{3}$	141, 460, 521, 617, 727, 750	1	432, 634	$2/3$
	283, 288, 589, 876	$-i/\sqrt{3}$	177, 421, 470, 511, 560, 627	$-i$	352, 374	$-2i/3$
	786, 823, 828, 859	$i/\sqrt{3}$	151, 241, 267, 650, 717, 740	i	532, 734	$2i/3$
g_5	125, 141, 227, 261, 313, 346, 357, 414, 425, 614, 625	$1/6$	318, 668, 881, 984	$1/2\sqrt{3}$
	663, 716, 727, 813, 846, 857, 927, 943, 961, 057, 064		381, 686, 818, 948	$-1/2\sqrt{3}$
	114, 152, 216, 272, 331, 364, 375, 441, 452, 636, 641	$-1/6$	382, 678, 882, 985	$i/2\sqrt{3}$
	652, 761, 772, 831, 864, 875, 916, 934, 972, 046, 075		328, 687, 828, 958	$-i/2\sqrt{3}$
	115, 124, 217, 226, 332, 347, 365, 424, 451, 615, 642	$i/6$	234, 436	$1/3$
	673, 726, 771, 823, 856, 874, 953, 962, 971, 065, 074		243, 463	$-1/3$
	142, 151, 262, 271, 323, 356, 374, 415, 442, 624, 637	$-i/6$	253, 473, 512, 554, 567	$i/3$
	651, 717, 762, 832, 847, 865, 917, 926, 935, 047, 056		235, 437, 521, 545, 576	$-i/3$
	583	$1/\sqrt{3}$	538	$-1/\sqrt{3}$
g_7	112, 143, 232, 245, 263, 315, 362, 448, 465, 619	$1/3$	214, 333, 346, 412, 513, 518	$2/3$
	636, 665, 714, 768, 815, 844, 916, 945, 046, 069		542, 549, 564, 566, 643, 869, 968	
	434, 939	$-1/3$	838	$-2/3$
	372, 675, 724, 825, 854, 926, 955, 056, 079	$i/3$	422, 523, 552, 574, 653, 978	$2i/3$
	122, 153, 255, 273, 325, 458, 475, 629, 778	$-i/3$	224, 356, 528, 559, 576, 879	$-2i/3$
	131, 211, 341, 417, 640, 867, 960	1	181, 282, 484, 787	$1/\sqrt{3}$
	737	-1	686, 989	$-1/\sqrt{3}$
	221, 351, 877	i	080	$-2/\sqrt{3}$
	427, 650, 970	$-i$		

components by applying the four kinds of operators: ϵ_{ABC}, $\epsilon^{ABD}\lambda^O_{DC}$, $\epsilon^{ACD}\lambda^O_{DB}$ and S_P^{ABC}, which correspond to **1**, **8**, **8** and **10** of $SU(3)$, respectively.

Under the vector chiral transformation, the fields Λ, N_1^N and N_2^N are transformed as

$$
\begin{aligned}
\delta^{\vec{a}}\Lambda &= \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}Cq_B^b)\gamma_5(i\lambda^N_{CD}a^N q_D^c) \\
&+ \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}C(i\lambda^N_{BD}a^N q_D^b))\gamma_5 q_C^c \\
&+ \epsilon_{abc}\epsilon^{ABC}((q_D^{aT}i\lambda^N_{AD}a^N)Cq_B^b)\gamma_5 q_C^c \\
&= ia^N \epsilon_{abc}\epsilon^{ABD}\lambda^N_{DC}(q_A^{aT}Cq_B^b)\gamma_5 q_C^c \\
&- 2ia^N \epsilon_{abc}\epsilon^{ACD}\lambda^N_{DB}(q_A^{aT}Cq_B^b)\gamma_5 q_C^c \\
&= ia^N N_1^N - ia^N N_1^N \\
&= 0,
\end{aligned}
\tag{2.42}
$$

$$
\begin{aligned}
\delta^{\vec{a}}N_1^N &= \epsilon_{abc}\epsilon^{ABD}\lambda^N_{DC}(q_A^{aT}Cq_B^b)\gamma_5(i\lambda^M_{CE}a^M q_E^c) \\
&+ \epsilon_{abc}\epsilon^{ABD}\lambda^N_{DC}(q_A^{aT}C(i\lambda^M_{BE}a^M q_E^b))\gamma_5 q_C^c \\
&+ \epsilon_{abc}\epsilon^{ABD}\lambda^N_{DC}((q_E^{aT}i\lambda^M_{AE}a^M)Cq_B^b)\gamma_5 q_C^c \\
&= ia^M \epsilon_{abc}\epsilon^{ABD}\lambda^N_{DE}\lambda^M_{EC}(q_A^{aT}Cq_B^b)\gamma_5 q_C^c \\
&- ia^M \epsilon_{abc}\epsilon^{EAB}\lambda^N_{DC}\lambda^M_{ED}(q_A^{aT}Cq_B^b)\gamma_5 q_C^c \\
&= \frac{2i}{3}a^N \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}Cq_B^b)\gamma_5 q_C^c + ia^M(d^{NMO}+if^{NMO})\epsilon_{abc}\epsilon^{ABD}\lambda^O_{DC}(q_A^{aT}Cq_B^b)\gamma_5 q_C^c \\
&- \frac{2i}{3}a^N \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}Cq_B^b)\gamma_5 q_C^c - ia^M(d^{MNO}+if^{MNO})\epsilon_{abc}\epsilon^{ABD}\lambda^O_{DC}(q_A^{aT}Cq_B^b)\gamma_5 q_C^c
\end{aligned}
$$

$$= \frac{2i}{3}a^N\Lambda + ia^M(d^{NMO} + if^{NMO})N_1^O$$

$$- \frac{2i}{3}a^N\Lambda - ia^M(d^{MNO} + if^{MNO})N_1^O$$

$$= -2a^M f^{NMO} N_1^O \, , \tag{2.43}$$

and

$$\delta^{\bar{a}} N_2^N = \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N(q_A^{aT}C\gamma_5 q_B^b)(i\lambda_{CE}^M a^M q_E^c)$$

$$+ \; \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N(q_A^{aT}C\gamma_5(i\lambda_{BE}^M a^M q_E^b))q_C^c$$

$$+ \; \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N((q_E^{aT}i\lambda_{AE}^M a^M)C\gamma_5 q_B^b)q_C^c$$

$$= \; ia^M \epsilon_{abc}\epsilon^{ABD}\lambda_{DE}^N\lambda_{EC}^M(q_A^{aT}C\gamma_5 q_B^b)q_C^c$$

$$- \; ia^M \epsilon_{abc}\epsilon^{EAB}\lambda_{DC}^N\lambda_{ED}^M(q_A^{aT}C\gamma_5 q_B^b)q_C^c$$

$$= \; \frac{2i}{3}a^N\epsilon_{abc}\epsilon^{ABC}(q_A^{aT}C\gamma_5 q_B^b)q_C^c + ia^M(d^{NMO} + if^{NMO})\epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^O(q_A^{aT}C\gamma_5 q_B^b)q_C^c$$

$$- \; \frac{2i}{3}a^N\epsilon_{abc}\epsilon^{ABC}(q_A^{aT}C\gamma_5 q_B^b)q_C^c - ia^M(d^{MNO} + if^{MNO})\epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^O(q_A^{aT}C\gamma_5 q_B^b)q_C^c$$

$$= \; -\frac{2i}{3}a^N\Lambda + ia^M(d^{NMO} + if^{NMO})N_2^O$$

$$+ \; \frac{2i}{3}a^N\Lambda - ia^M(d^{MNO} + if^{MNO})N_2^O$$

$$= \; -2a^M f^{NMO} N_2^O \, . \tag{2.44}$$

To study the vector chiral transformation of N_μ^N, we first calculate the transformation of $N_{3\mu}^N$

$$\delta^{\bar{a}} N_{3\mu}^N = \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N(q_A^{aT}C\gamma_\mu\gamma_5 q_B^b)\gamma_5(i\lambda_{CE}^M a^M q_E^c)$$

$$+ \; \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N(q_A^{aT}C\gamma_\mu\gamma_5(i\lambda_{BE}^M a^M q_E^b))\gamma_5 q_C^c$$

$$+ \; \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N((q_E^{aT}i\lambda_{AE}^M a^M)C\gamma_\mu\gamma_5 q_B^b)\gamma_5 q_C^c$$

$$= \; ia^M \epsilon_{abc}\epsilon^{ABD}\lambda_{DE}^N\lambda_{EC}^M(q_A^{aT}C\gamma_\mu\gamma_5 q_B^b)\gamma_5 q_C^c$$

$$+ \; 2ia^M \epsilon_{abc}\epsilon^{AED}\lambda_{DC}^N\lambda_{EB}^M(q_A^{aT}C\gamma_\mu\gamma_5 q_B^b)\gamma_5 q_C^c$$

$$= \; \frac{2i}{3}a^N\epsilon_{abc}\epsilon^{ABC}(q_A^{aT}C\gamma_\mu\gamma_5 q_B^b)\gamma_5 q_C^c$$

$$+ \; ia^M(d^{NMO} + if^{NMO})\epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^O(q_A^{aT}C\gamma_\mu\gamma_5 q_B^b)\gamma_5 q_C^c$$

$$+ \; 2ia^M g_1^{MNO}\epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^O(q_A^{aT}C\gamma_\mu\gamma_5 q_B^b)\gamma_5 q_C^c$$

$$+ \; 2ia^M g_2^{MNO}\epsilon_{abc}\epsilon^{ACD}\lambda_{DB}^O(q_A^{aT}C\gamma_\mu\gamma_5 q_B^b)\gamma_5 q_C^c$$

$$+ \; 2ia^M g_3^{MNP}\epsilon_{abc}S_P^{ABC}(q_A^{aT}C\gamma_\mu\gamma_5 q_B^b)\gamma_5 q_C^c - \frac{2i}{3}a^N\epsilon_{abc}\epsilon^{ABC}(q_A^{aT}C\gamma_\mu\gamma_5 q_B^b)\gamma_5 q_C^c$$

$$= \; \frac{2i}{3}a^N\Lambda_{3\mu} + ia^M(d^{NMO} + if^{NMO})N_{3\mu}^O + 2ia^M g_1^{MNO}N_{3\mu}^O$$

$$+ \ 2ia^M g_2^{MNO} \widetilde{N}_{3\mu}^O + 0 - \frac{2i}{3} a^N \Lambda_{3\mu}$$

$$= \ -2a^M f^{NMO} N_{3\mu}^O .$$

(2.45)

Hence, the vector chiral transformation of N_μ^N is

$$
\begin{aligned}
\delta^{\vec{a}} N_\mu^N &= \delta N_{3\mu}^N + \frac{1}{4}\delta(N_{1\mu}^N - N_{2\mu}^N) \\
&= -2a^M f^{NMO} N_{3\mu}^O - \frac{1}{2} a^M f^{NMO}(N_{1\mu}^O - N_{2\mu}^O) \\
&= -2a^M f^{NMO} N_\mu^N .
\end{aligned}
$$

(2.46)

The chiral transformation of Δ_μ^P is

$$
\begin{aligned}
\delta^{\vec{a}} \Delta_\mu^P &= \epsilon_{abc} S_P^{ABC}(q_A^{aT} C\gamma_\mu q_B^b)(i\lambda_{CE}^M a^M q_E^c) \\
&+ \epsilon_{abc} S_P^{ABC}(q_A^{aT} C\gamma_\mu(i\lambda_{BE}^M a^M q_E^b))q_C^c \\
&+ \epsilon_{abc} S_P^{ABC}((q_E^{aT} i\lambda_{AE}^M a^M)C\gamma_\mu q_B^b)q_C^c \\
&= ia^M \epsilon_{abc} S_P^{ABE} \lambda_{EC}^M(q_A^{aT} C\gamma_\mu q_B^b)q_C^c \\
&+ 2ia^M \epsilon_{abc} S_P^{ACE} \lambda_{EB}^M(q_A^{aT} C\gamma_\mu q_B^b)q_C^c \\
&= -2ia^M g_5^{PMO} \widetilde{N}_{5\mu}^O + ia^M g_7^{PMQ} \Delta_{5\mu}^Q \\
&+ 2ia^M g_5^{PMO} \widetilde{N}_{5\mu}^O + 2ia^M g_7^{PMQ} \Delta_{5\mu}^Q \\
&= 3ia^M g_7^{PMQ} \Delta_\mu^Q .
\end{aligned}
$$

(2.47)

To study the vector chiral transformation of $\Delta_{\mu\nu}^P$, we first calculate the transformation of $\Delta_{7\mu\nu}^P$

$$
\begin{aligned}
\delta^{\vec{a}} \Delta_{7\mu\nu}^P &= \epsilon_{abc} S_P^{ABC}(q_A^{aT} C\sigma_{\mu\nu} q_B^b)\gamma_5(i\lambda_{CE}^M a^M q_E^c) \\
&+ \epsilon_{abc} S_P^{ABC}(q_A^{aT} C\sigma_{\mu\nu}(i\lambda_{BE}^M a^M q_E^b))\gamma_5 q_C^c \\
&+ \epsilon_{abc} S_P^{ABC}((q_E^{aT} i\lambda_{AE}^M a^M)C\sigma_{\mu\nu} q_B^b)\gamma_5 q_C^c \\
&= ia^M \epsilon_{abc} S_P^{ABE} \lambda_{EC}^M(q_A^{aT} C\sigma_{\mu\nu} q_B^b)\gamma_5 q_C^c \\
&+ 2ia^M \epsilon_{abc} S_P^{ACE} \lambda_{EB}^M(q_A^{aT} C\sigma_{\mu\nu} q_B^b)\gamma_5 q_C^c \\
&= -2ia^M g_5^{PMO} \widetilde{N}_{7\mu\nu}^O + ia^M g_7^{PMQ} \Delta_{7\mu\nu}^Q \\
&+ 2ia^M g_5^{PMO} \widetilde{N}_{7\mu\nu}^O + 2ia^M g_7^{PMQ} \Delta_{7\mu\nu}^Q \\
&= 3ia^M g_7^{PMQ} \Delta_{7\mu\nu}^Q .
\end{aligned}
$$

(2.48)

Therefore, the chiral transformation of $\Delta_{\mu\nu}^P$ is

$$
\begin{aligned}
\delta^{\vec{a}} \Delta_{\mu\nu}^P &= \delta\Delta_{7\mu\nu}^P - \frac{i}{2}\gamma_\mu\gamma_5\delta\Delta_{5\nu}^P + \frac{i}{2}\gamma_\nu\gamma_5\delta\Delta_{5\mu}^P \\
&= 3ia^M g_7^{PMQ}(\Delta_{7\mu\nu}^Q - \frac{i}{2}\gamma_\mu\gamma_5\Delta_{5\nu}^P + \frac{i}{2}\gamma_\nu\gamma_5\Delta_{5\mu}^P) \\
&= 3ia^M g_7^{PMQ} \Delta_{\mu\nu}^Q .
\end{aligned}
$$

(2.49)

In summary, under the vector chiral transformation, the baryon fields are transformed as

$$
\begin{aligned}
\delta\Lambda &= 0, \\
\delta^{\vec{a}}N_1^N &= -2a^M f^{NMO}N_1^O, \\
\delta^{\vec{a}}N_2^N &= -2a^M f^{NMO}N_2^O, \\
\delta^{\vec{a}}N_\mu^N &= -2a^M f^{NMO}N_\mu^N, \\
\delta^{\vec{a}}\Delta_\mu^P &= 3ia^M g_7^{PMQ}\Delta_\mu^Q, \\
\delta^{\vec{a}}\Delta_{\mu\nu}^P &= 3ia^M g_7^{PMQ}\Delta_{\mu\nu}^Q,
\end{aligned}
\tag{2.50}
$$

which show nothing but the isospin conservation with the coefficients on the right hand side reflect the isospin charge of the baryons.

Then we go on to study the axial-vector chiral transformation of baryon fields. Under the axial-vector chiral transformation, the field Λ is transformed as

$$
\begin{aligned}
\delta_5^{\vec{b}}\Lambda &= \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}Cq_B^b)\gamma_5(i\gamma_5\lambda_{CD}^N a^N q_D^c) \\
&+ \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}C(i\gamma_5\lambda_{BD}^N a^N q_D^b))\gamma_5 q_C^c \\
&+ \epsilon_{abc}\epsilon^{ABC}((q_D^{aT}i\gamma_5^T\lambda_{AD}^N a^N)Cq_B^b)\gamma_5 q_C^c \\
&= i\gamma_5 a^N \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N(q_A^{aT}Cq_B^b)\gamma_5 q_C^c \\
&- i\gamma_5 a^N \epsilon_{abc}\epsilon^{ACD}\lambda_{DB}^N(q_A^{aT}C\gamma_5 q_B^b)q_C^c \\
&+ i\gamma_5 a^N \epsilon_{abc}\epsilon^{BCD}\lambda_{DA}^N(q_A^{aT}C\gamma_5 q_B^b)q_C^c \\
&= i\gamma_5 a^N N_1^N - \frac{i}{2}\gamma_5 a^N N_2^N - \frac{i}{2}\gamma_5 a^N N_2^N \\
&= i\gamma_5 a^N N_1^N - i\gamma_5 a^N N_2^N .
\end{aligned}
\tag{2.51}
$$

The transformation for N_1^N and N_2^N are

$$
\begin{aligned}
&\delta_5^{\vec{b}}N_1^N \\
&= \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N(q_A^{aT}Cq_B^b)\gamma_5(i\gamma_5\lambda_{CE}^M a^M q_E^c) \\
&+ \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N(q_A^{aT}C(i\gamma_5\lambda_{BE}^M a^M q_E^b))\gamma_5 q_C^c \\
&+ \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N((q_E^{aT}i\gamma_5^T\lambda_{AE}^M a^M)Cq_B^b)\gamma_5 q_C^c \\
&= i\gamma_5 a^M \epsilon_{abc}\epsilon^{ABD}\lambda_{DE}^N\lambda_{EC}^M(q_A^{aT}Cq_B^b)\gamma_5 q_C^c \\
&- i\gamma_5 a^M \epsilon_{abc}\epsilon^{EAB}\lambda_{DC}^N\lambda_{ED}^M(q_A^{aT}C\gamma_5 q_B^b)q_C^c \\
&= \frac{2i}{3}\gamma_5 a^N \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}Cq_B^b)\gamma_5 q_C^c + i\gamma_5 a^M(d^{NMO}+if^{NMO})\epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^O(q_A^{aT}Cq_B^b)\gamma_5 q_C^c \\
&- \frac{2i}{3}\gamma_5 a^N \epsilon_{abc}\epsilon^{ABC}(q_A^{aT}C\gamma_5 q_B^b)q_C^c - i\gamma_5 a^M(d^{MNO}+if^{MNO})\epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^O(q_A^{aT}C\gamma_5 q_B^b)q_C^c \\
&= \frac{2i}{3}\gamma_5 a^N \Lambda + i\gamma_5 a^M(d^{NMO}+if^{NMO})N_1^O
\end{aligned}
$$

$$+ \quad \frac{2i}{3}\gamma_5 a^N \Lambda - i\gamma_5 a^M (d^{MNO} + if^{MNO})N_2^O$$

$$= \quad \frac{4i}{3}\gamma_5 a^N \Lambda + i\gamma_5 a^M (d^{NMO} + if^{NMO})N_1^O - i\gamma_5 a^M (d^{MNO} + if^{MNO})N_2^O , \qquad (2.52)$$

and

$$\delta_5^{\vec{b}} N_2^N$$

$$= \quad \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N (q_A^{aT} C\gamma_5 q_B^b)(i\gamma_5 \lambda_{CE}^M a^M q_E^c)$$

$$+ \quad \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N (q_A^{aT} C\gamma_5 (i\gamma_5 \lambda_{BE}^M a^M q_E^b))q_C^c$$

$$+ \quad \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N ((q_E^{aT} i\gamma_5^T \lambda_{AE}^M a^M)C\gamma_5 q_B^b)q_C^c$$

$$= \quad i\gamma_5 a^M \epsilon_{abc}\epsilon^{ABD}\lambda_{DE}^N \lambda_{EC}^M (q_A^{aT} C\gamma_5 q_B^b)q_C^c$$

$$- \quad i\gamma_5 a^M \epsilon_{abc}\epsilon^{EAB}\lambda_{DC}^N \lambda_{ED}^M (q_A^{aT} C q_B^b)\gamma_5 q_C^c$$

$$= \quad \frac{2i}{3}\gamma_5 a^N \epsilon_{abc}\epsilon^{ABC}(q_A^{aT} C\gamma_5 q_B^b)q_C^c + i\gamma_5 a^M (d^{NMO} + if^{NMO})\epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^O (q_A^{aT} C\gamma_5 q_B^b)q_C^c$$

$$- \quad \frac{2i}{3}\gamma_5 a^N \epsilon_{abc}\epsilon^{ABC}(q_A^{aT} C q_B^b)\gamma_5 q_C^c - i\gamma_5 a^M (d^{MNO} + if^{MNO})\epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^O (q_A^{aT} C q_B^b)\gamma_5 q_C^c$$

$$= \quad -\frac{2i}{3}\gamma_5 a^N \Lambda + i\gamma_5 a^M (d^{NMO} + if^{NMO})N_2^O$$

$$- \quad \frac{2i}{3}\gamma_5 a^N \Lambda - i\gamma_5 a^M (d^{MNO} + if^{MNO})N_1^O$$

$$= \quad -\frac{4i}{3}\gamma_5 a^N \Lambda + i\gamma_5 a^M (d^{NMO} + if^{NMO})N_2^O - i\gamma_5 a^M (d^{MNO} + if^{MNO})N_1^O . \qquad (2.53)$$

To study the vector chiral transformation of N_μ^N, we first calculate the transformation of $N_{3\mu}^N$. Here we need to use the Eq. (2.40), and obtain

$$\delta_5^{\vec{b}} N_{3\mu}^N$$

$$= \quad \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N (q_A^{aT} C\gamma_\mu \gamma_5 q_B^b)\gamma_5 (i\gamma_5 \lambda_{CE}^M a^M q_E^c)$$

$$+ \quad \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N (q_A^{aT} C\gamma_\mu \gamma_5 (i\gamma_5 \lambda_{BE}^M a^M q_E^b))\gamma_5 q_C^c$$

$$+ \quad \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N ((q_E^{aT} i\gamma_5^T \lambda_{AE}^M a^M)C\gamma_\mu \gamma_5 q_B^b)\gamma_5 q_C^c$$

$$= \quad i\gamma_5 a^M \epsilon_{abc}\epsilon^{ABD}\lambda_{DE}^N \lambda_{EC}^M (q_A^{aT} C\gamma_\mu \gamma_5 q_B^b)\gamma_5 q_C^c$$

$$+ \quad 2i\gamma_5 a^M \epsilon_{abc}\epsilon^{AED}\lambda_{DC}^N \lambda_{EB}^M (q_A^{aT} C\gamma_\mu q_B^b)q_C^c$$

$$= \quad \frac{2i}{3}\gamma_5 a^N \epsilon_{abc}\epsilon^{ABC}(q_A^{aT} C\gamma_\mu \gamma_5 q_B^b)\gamma_5 q_C^c$$

$$+ \quad i\gamma_5 a^M (d^{NMO} + if^{NMO})\epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^O (q_A^{aT} C\gamma_\mu \gamma_5 q_B^b)\gamma_5 q_C^c$$

$$+ \quad 2i\gamma_5 a^M g_1^{MNO}\epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^O (q_A^{aT} C\gamma_\mu q_B^b)q_C^c + 2i\gamma_5 a^M g_2^{MNO}\epsilon_{abc}\epsilon^{ACD}\lambda_{DB}^O (q_A^{aT} C\gamma_\mu q_B^b)q_C^c$$

$$+ \quad 2i\gamma_5 a^M g_3^{MNP}\epsilon_{abc}S_P^{ABC}(q_A^{aT} C\gamma_\mu q_B^b)q_C^c - \frac{2i}{3}\gamma_5 a^N \epsilon_{abc}\epsilon^{ABC}(q_A^{aT} C\gamma_\mu q_B^b)q_C^c$$

$$= \quad \frac{2i}{3}\gamma_5 a^N \Lambda_{3\mu} + i\gamma_5 a^M (d^{NMO} + if^{NMO})N_{3\mu}^O + 0$$

$$+ \quad 2i\gamma_5 a^M g_2^{MNO} \widetilde{N}_{5\mu}^O + 2i\gamma_5 a^M g_3^{MNP}\Delta_{5\mu}^P - 0$$

$$= \quad -\frac{2i}{3}\gamma_\mu a^N\Lambda + i\gamma_5 a^M(d^{NMO} + if^{NMO} + g_2^{MNO})N_{3\mu}^O$$

$$- \quad i\gamma_\mu a^M g_2^{MNO}(N_1^O - N_2^O) + 2i\gamma_5 a^M g_3^{MNP}\Delta_\mu^P . \tag{2.54}$$

Therefore, the chiral transformation of N_μ^N is

$$\delta_5^{\vec{b}}N_\mu^N = \delta_5 N_{3\mu}^N + \frac{1}{4}\delta_5(N_{1\mu}^N - N_{2\mu}^N)$$

$$= \delta_5 N_{3\mu}^N + \frac{1}{4}\gamma_\mu\gamma_5\delta_5(N_1^N - N_2^N)$$

$$= \delta_5 N_{3\mu}^N + \frac{1}{4}\gamma_\mu\gamma_5(\frac{8i}{3}\gamma_5 a^N\Lambda + 2i\gamma_5 a^M d^{NMO}(N_1^O - N_2^O))$$

$$= \delta_5 N_{3\mu}^N + \frac{2i}{3}\gamma_\mu a^N\Lambda + \frac{i}{2}\gamma_\mu a^M d^{NMO}(N_1^O - N_2^O)$$

$$= i\gamma_5 a^M(2d^{MNO} - \frac{4i}{3}f^{MNO})(N_{3\mu}^O + \frac{1}{4}\gamma_\mu\gamma_5(N_1^O - N_2^O)) + 2i\gamma_5 a^M g_3^{MNP}\Delta_\mu^P$$

$$= i\gamma_5 a^M(2d^{MNO} - \frac{4i}{3}f^{MNO})N_\mu^O + 2i\gamma_5 a^M g_3^{MNP}\Delta_\mu^P . \tag{2.55}$$

The chiral transformation of Δ_μ^P is

$$\delta_5^{\vec{b}}\Delta_\mu^P = \epsilon_{abc}S_P^{ABC}(q_A^{aT}C\gamma_\mu q_B^b)(i\gamma_5\lambda_{CE}^M a^M q_E^c)$$

$$+ \quad \epsilon_{abc}S_P^{ABC}(q_A^{aT}C\gamma_\mu(i\gamma_5\lambda_{BE}^M a^M q_E^b))q_C^c$$

$$+ \quad \epsilon_{abc}S_P^{ABC}((q_E^{aT}i\gamma_5^T\lambda_{AE}^M a^M)C\gamma_\mu q_B^b)q_C^c$$

$$= \quad i\gamma_5 a^M\epsilon_{abc}S_P^{ABE}\lambda_{EC}^M(q_A^{aT}C\gamma_\mu q_B^b)q_C^c$$

$$+ \quad 2i\gamma_5 a^M\epsilon_{abc}S_P^{ACE}\lambda_{EB}^M(q_A^{aT}C\gamma_\mu\gamma_5 q_B^b)\gamma_5 q_C^c$$

$$= \quad -2i\gamma_5 a^M g_5^{PMO}\widetilde{N}_{5\mu}^O + i\gamma_5 a^M g_7^{PMQ}\Delta_{5\mu}^Q$$

$$+ \quad i\gamma_5 a^M g_5^{PMO}N_{3\mu}^O - 4i\gamma_5 a^M g_5^{PMO}N_{3\mu}^O$$

$$= \quad -4i\gamma_5 a^M g_5^{PMO}N_\mu^O + i\gamma_5 a^M g_7^{PMQ}\Delta_\mu^Q . \tag{2.56}$$

To study the vector chiral transformation of $\Delta_{7\mu\nu}^P$, we first calculate the transformation of $\Delta_{\mu\nu}^P$. Again we need to use the Eq. (2.40), and obtain:

$$\delta_5^{\vec{b}}\Delta_{7\mu\nu}^P = \epsilon_{abc}S_P^{ABC}(q_A^{aT}C\sigma_{\mu\nu}q_B^b)\gamma_5(i\gamma_5\lambda_{CE}^M a^M q_E^c)$$

$$+ \quad \epsilon_{abc}S_P^{ABC}(q_A^{aT}C\sigma_{\mu\nu}(i\gamma_5\lambda_{BE}^M a^M q_E^b))\gamma_5 q_C^c$$

$$+ \quad \epsilon_{abc}S_P^{ABC}((q_E^{aT}i\gamma_5^T\lambda_{AE}^M a^M)C\sigma_{\mu\nu}q_B^b)\gamma_5 q_C^c$$

$$= \quad i\gamma_5 a^M\epsilon_{abc}S_P^{ABE}\lambda_{EC}^M(q_A^{aT}C\sigma_{\mu\nu}q_B^b)\gamma_5 q_C^c$$

$$+ \quad 2i\gamma_5 a^M\epsilon_{abc}S_P^{ACE}\lambda_{EB}^M(q_A^{aT}C\sigma_{\mu\nu}\gamma_5 q_B^b)q_C^c$$

$$
\begin{aligned}
&= -2i\gamma_5 a^M g_5^{PMO}\widetilde{N}_{7\mu\nu}^O + i\gamma_5 a^M g_7^{PMQ}\Delta_{7\mu\nu}^Q \\
&+ 2i\gamma_5 a^M g_5^{PMO}\widetilde{N}_{8\mu\nu}^O + 2i\gamma_5 a^M g_7^{PMQ}\Delta_{8\mu\nu}^Q \\
&= 2\gamma_5 a^M g_5^{PMO}N_{2\mu\nu}^O - i\gamma_5 a^M g_5^{PMO}(N_{5\mu\nu}^O - N_{6\mu\nu}^O) \\
&+ 3i\gamma_5 a^M g_7^{PMQ}\Delta_{7\mu\nu}^Q - 2\gamma_5 a^M g_7^{PMQ}\Delta_{1\mu\nu}^Q \\
&= 2\gamma_5 a^M g_5^{PMO}(\gamma_\nu\gamma_5 N_{3\mu}^O - \gamma_\mu\gamma_5 N_{3\nu}^O) - i\gamma_5 a^M g_5^{PMO}\sigma_{\mu\nu}(N_1^O - N_2^O) \\
&+ 3i\gamma_5 a^M g_7^{PMQ}\Delta_{7\mu\nu}^Q - 2\gamma_5 a^M g_7^{PMQ}(\gamma_\nu\gamma_5\Delta_\mu^Q - \gamma_\mu\gamma_5\Delta_\nu^Q)\,.
\end{aligned}
\tag{2.57}
$$

Therefore, the chiral transformation of $\Delta_{\mu\nu}^P$ is

$$
\begin{aligned}
\delta_5^{\vec{b}}\Delta_{\mu\nu}^P &= \delta_5\Delta_{7\mu\nu}^P - \frac{i}{2}\gamma_\mu\gamma_5\delta_5\Delta_{5\nu}^P + \frac{i}{2}\gamma_\nu\gamma_5\delta_5\Delta_{5\mu}^P \\
&= 2\gamma_5 a^M g_5^{PMO}(\gamma_\nu\gamma_5 N_{3\mu}^O - \gamma_\mu\gamma_5 N_{3\nu}^O) - i\gamma_5 a^M g_5^{PMO}\sigma_{\mu\nu}(N_1^O - N_2^O) \\
&+ 3i\gamma_5 a^M g_7^{PMQ}\Delta_{7\mu\nu}^Q - 2\gamma_5 a^M g_7^{PMQ}(\gamma_\nu\gamma_5\Delta_\mu^Q - \gamma_\mu\gamma_5\Delta_\nu^Q) \\
&- 2\gamma_\mu a^M g_5^{PMO}N_\nu^O + 2\gamma_\nu a^M g_5^{PMO}N_\mu^O \\
&+ \frac{1}{2}\gamma_\mu a^M g_7^{PMQ}\Delta_\nu^Q - \frac{1}{2}\gamma_\nu a^M g_7^{PMQ}\Delta_\mu^Q \\
&= 3i\gamma_5 a^M g_7^{PMQ}\Delta_{\mu\nu}^Q\,.
\end{aligned}
\tag{2.58}
$$

In summary, we show therefore the final result of the axial transformation

$$
\begin{aligned}
\delta_5^{\vec{b}}\Lambda &= i\gamma_5 b^N(N_1^N - N_2^N)\,, \\
\delta_5^{\vec{b}}(N_1^N - N_2^N) &= \frac{8i}{3}\gamma_5 b^N\Lambda + 2i\gamma_5 b^M d^{NMO}(N_1^O - N_2^O)\,, \\
\delta_5^{\vec{b}}(N_1^N + N_2^N) &= -2\gamma_5 b^M f^{NMO}(N_1^O + N_2^O)\,, \\
\delta_5^{\vec{b}}N_\mu^N &= i\gamma_5 b^M(2d^{MNO} - \frac{4i}{3}f^{MNO})N_\mu^O + 2i\gamma_5 b^M g_3^{MNP}\Delta_\mu^P\,, \\
\delta_5^{\vec{b}}\Delta_\mu^P &= -4i\gamma_5 b^M g_5^{PMO}N_\mu^O + i\gamma_5 b^M g_7^{PMQ}\Delta_\mu^Q\,, \\
\delta_5^{\vec{b}}\Delta_{\mu\nu}^P &= 3i\gamma_5 b^M g_7^{PMQ}\Delta_{\mu\nu}^Q\,.
\end{aligned}
\tag{2.59}
$$

2.3 Chiral representations

So far, we have performed classifications without explicitly taking into account the left- and right-handed components of the quark fields. However, it does not require great imagination to see that the chiral properties are also conveniently studied in that language, since chiral symmetry is defined as the symmetries upon each chiral field. Hence, we define the left- and right-handed (chiral or Weyl representation) quark fields as

$$
L \equiv q_L = \frac{1-\gamma_5}{2}q, \quad \text{and} \quad R \equiv q_R = \frac{1+\gamma_5}{2}q\,.
\tag{2.60}
$$

They form the fundamental representations of both the Lorentz group and the chiral group,

$$L: \quad \text{Lorentz}: \ (\tfrac{1}{2}, 0), \quad \text{Chiral}: \ (3, 1),$$

$$R: \quad \text{Lorentz}: \ (0, \tfrac{1}{2}), \quad \text{Chiral}: \ (1, 3).$$

It is convenient first to note that γ-matrices are classified into two categories; chiral-even and chiral-odd classes. The chiral-even γ-matrices survive forming diquarks with identical chiralities, while the chiral-odd ones form diquarks from quarks with opposite chiralities. The chiral-even and -odd γ-matrices are

$$\text{chiral-even:} \quad 1, \gamma_5, \sigma_{\mu\nu},$$
$$\text{chiral-odd:} \quad \gamma_\mu, \gamma_\mu\gamma_5.$$

Therefore, we have six non-vanishing diquarks in the chiral representations,

$$
\left.\begin{array}{l}
L^T C L = -L^T C \gamma_5 L \\
R^T C R = +R^T C \gamma_5 R
\end{array}\right\}
\qquad (0,0) \oplus (0,0), \qquad (\mathbf{\bar{3}},\ \mathbf{1}) \oplus (\mathbf{1},\ \mathbf{\bar{3}}),
$$

$$
\left.\begin{array}{l}
L^T C \gamma_\mu \gamma_5 R = +L^T C \gamma_\mu R \\
R^T C \gamma_\mu \gamma_5 L = -R^T C \gamma_\mu L
\end{array}\right\}
\qquad (\tfrac{1}{2}, \tfrac{1}{2}) \oplus (\tfrac{1}{2}, \tfrac{1}{2}), \qquad (\mathbf{3},\ \mathbf{3}) \oplus (\mathbf{3},\ \mathbf{3}),
$$

$$
\left.\begin{array}{l}
L^T C \sigma_{\mu\nu} L \\
R^T C \sigma_{\mu\nu} R
\end{array}\right\}
\qquad (1,0) \oplus (0,1), \qquad (\mathbf{6},\ \mathbf{1}) \oplus (\mathbf{1},\ \mathbf{6}),
$$

where we have indicated the Lorentz and chiral representations of the diquarks.

For three quarks, we have

$$(L+R)^3 \rightarrow \left\{ \begin{array}{lll} LLL & (\tfrac{1}{2}, 0) \oplus (\tfrac{3}{2}, 0), & (\mathbf{1},\mathbf{1}) \oplus (\mathbf{8},\mathbf{1}) \oplus (\mathbf{8},\mathbf{1}) \oplus (\mathbf{10},\mathbf{1}) \\ LLR & (0, \tfrac{1}{2}) \oplus (1, \tfrac{1}{2}), & (\mathbf{\bar{3}},\mathbf{3}) \oplus (\mathbf{6},\mathbf{3}) \end{array} \right. \qquad (2.61)$$

and together with the terms where L and R are exchanged. Now we discuss the independent fields in terms of the chiral representations.

2.3.1 Chiral properties of Dirac fields

Independent fields of $(LL)L$

The $(LL)L$ must belong to one of the following chiral representations: $(\mathbf{1},\mathbf{1}) \oplus (\mathbf{8},\mathbf{1}) \oplus (\mathbf{8},\mathbf{1}) \oplus (\mathbf{10},\mathbf{1})$. For each chiral representation, there is one flavor representation available.

For $(\mathbf{1},\ \mathbf{1}) \rightarrow \mathbf{1}_f$, there are apparently two non-zero fields

$$
\begin{array}{l}
\Lambda_{L1} = \epsilon_{abc}\epsilon^{ABC}(L_A^{aT} C L_B^b)\gamma_5 L_C^c, \\
\Lambda_{L2} = \epsilon_{abc}\epsilon^{ABC}(L_A^{aT} C \gamma_5 L_B^b) L_C^c, \\
\Lambda_{L3} = \epsilon_{abc}\epsilon^{ABC}(L_A^{aT} C \gamma_\mu \gamma_5 L_B^b)\gamma^\mu L_C^c = 0,
\end{array}
\qquad (2.62)
$$

where Λ_3^L vanishes because $\gamma_\mu\gamma_5$ is chiral-odd

$$L^T C\gamma_\mu\gamma_5 L = 0 \,. \tag{2.63}$$

After performing the Fierz transformation to relate Λ_{Li} and Λ'_{Li} as we have done before, and solving the coupled equations, we find the solution that all such fields vanish.

For $(\mathbf{10},\ \mathbf{1}) \to \mathbf{10_f}$, we would have again two non-zero components:

$$
\begin{aligned}
\Delta_{L4}^P &= \epsilon_{abc}S_P^{ABC}(L_A^{aT}C\gamma_\mu L_B^b)\gamma^\mu\gamma_5 L_C^c \,, \\
\Delta_{L5}^P &= \epsilon_{abc}S_P^{ABC}(L_A^{aT}C\sigma_{\mu\nu} L_B^b)\sigma^{\mu\nu}\gamma_5 L_C^c \,.
\end{aligned}
\tag{2.64}
$$

Performing the Fierz transformation to relate Δ_{Li}^P and $\Delta_{Li}^{P\prime}$, we obtain the solution that all such $(LL)L$ fields vanish.

Finally for $(\mathbf{8},\ \mathbf{1}) \to \mathbf{8_f}$, we may consider once again two non-zero fields to start with

$$
\begin{aligned}
N_{L1}^N &= \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N(L_A^{aT}CL_B^b)\gamma_5 L_C^c \,, \\
N_{L2}^N &= \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N(L_A^{aT}C\gamma_5 L_B^b)L_C^c \,.
\end{aligned}
\tag{2.65}
$$

Applying the Fierz transformation to relate N_{Li}^N and $N_{Li}^{N\prime}$, we obtain the solution

$$N_{L2}^N = N_{L1}^N \,. \tag{2.66}$$

Therefore, there is only one independent $(LL)L$ $\mathbf{8_f}$ field.

Independent $(LL)R$ fields

The chiral representations of $(LL)R$ are $(\bar{\mathbf{3}},\ \mathbf{3}) \oplus (\mathbf{6},\ \mathbf{3})$. We will study them separately in the following.

For $(\bar{\mathbf{3}},\ \mathbf{3}) \to \mathbf{1_f}$, there appears to exist two non-zero components among the five fields,

$$
\begin{aligned}
\Lambda_{M1} &= \epsilon_{abc}\epsilon^{ABC}(L_A^{aT}CL_B^b)\gamma_5 R_C^c \,, \\
\Lambda_{M2} &= \epsilon_{abc}\epsilon^{ABC}(L_A^{aT}C\gamma_5 L_B^b)R_C^c \,, \\
\Lambda_{M3} &= \epsilon_{abc}\epsilon^{ABC}(L_A^{aT}C\gamma_\mu\gamma_5 L_B^b)\gamma^\mu R_C^c = 0 \,, \\
\Lambda_{M4} &= \epsilon_{abc}\epsilon^{ABC}(L_A^{aT}C\gamma_\mu L_B^b)\gamma^\mu\gamma_5 R_C^c = 0 \,, \\
\Lambda_{M5} &= \epsilon_{abc}\epsilon^{ABC}(L_A^{aT}C\sigma_{\mu\nu} L_B^b)\sigma^{\mu\nu}\gamma_5 R_C^c = 0 \,,
\end{aligned}
\tag{2.67}
$$

where M (mixed) indicates that the fields contain both left and right handed quarks. Performing the Fierz transformation to relate Λ_{Mi} and Λ'_{Mi}, we obtain the following relations

$$\Lambda'_{M4} = -\Lambda'_{M3} = -2\Lambda_{M2} = 2\Lambda_{M1} \,. \tag{2.68}$$

We may consider other ten combinations formed by (LR) and (RL) diquarks, $(LR)L$ and $(RL)L$. However, they can be related to the above ones of $(LL)R$ by a rearrangement of indices as well as the Fierz transformation, for instance,

$$\Lambda_{M6} = \epsilon_{abc}\epsilon^{ABC}(L_A^{aT}CR_B^b)\gamma_5 L_C^c = \Lambda'_{M1} \,. \tag{2.69}$$

Therefore, we have only one independent field.

For the chiral representation $(\mathbf{6},\ \mathbf{3}) \to \mathbf{10_f}$, we can write five fields containing diquarks formed by five Dirac matrices. However, we can show that after performing the Fierz transformation all fields vanish. Therefore, this representation can not support three-quark fields.

The baryon fields of chiral representations $(\bar{\mathbf{3}},\ \mathbf{3}) \to \mathbf{8_f}$ can be formed

$$
\begin{aligned}
N_{M1}^{N} &= \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^{N}(L_A^{aT}CL_B^b)\gamma_5 R_C^c \,, \\
N_{M2}^{N} &= \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^{N}(L_A^{aT}C\gamma_5 L_B^b)R_C^c \,, \\
N_{M3}^{N} &= \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^{N}(L_A^{aT}C\gamma_\mu\gamma_5 L_B^b)\gamma^\mu R_C^c = 0 \,, \\
N_{M4}^{N} &= \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^{N}(L_A^{aT}C\gamma_\mu L_B^b)\gamma^\mu\gamma_5 R_C^c = 0 \,, \\
N_{M5}^{N} &= \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^{N}(L_A^{aT}C\sigma_{\mu\nu} L_B^b)\sigma^{\mu\nu}\gamma_5 R_C^c = 0 \,,
\end{aligned}
\tag{2.70}
$$

where we see that there are two non-zero fields. Applying the Fierz transformation, we can verify that there is only one independent field with the following relations

$$
N_{M4}^{N\prime} = -N_{M3}^{N\prime} = -2N_{M2}^{N} = 2N_{M1}^{N} \,.
\tag{2.71}
$$

Another chiral representation $(\mathbf{6},\ \mathbf{3}) \to \mathbf{8_f}$ can be constructed by the combinations similar to (2.70), for instance,

$$
N_{(6,3)1}^{N} = \epsilon_{abc}\epsilon^{ACD}\lambda_{DB}^{N}\{(L_A^{aT}CL_B^b)\gamma_5 R_C^c + (L_B^{aT}CL_A^b)\gamma_5 R_C^c\} \,.
\tag{2.72}
$$

After similar algebra we can verify that all these fields vanish.

2.3.2   Chiral properties of Rarita-Schwinger fields

As previously, we only need to study the properties of $(LL)L$, $(LL)R$, $(LR)L$ and $(RL)L$. Others are similar.

Chiral properties of $(LL)L$

The chiral representations of $(LL)L$ are $(\mathbf{1},\ \mathbf{1}) \oplus (\mathbf{8},\ \mathbf{1}) \oplus (\mathbf{8},\ \mathbf{1}) \oplus (\mathbf{10},\ \mathbf{1})$. We will study them separately in the following.

(1) The chiral representation $(\mathbf{1},\ \mathbf{1})$ has just two non-zero fields:

$$
\begin{aligned}
\Lambda_{L1\mu} &= \epsilon_{abc}\epsilon^{ABC}(L_A^{aT}CL_B^b)\gamma_\mu L_C^c \,, \\
\Lambda_{L2\mu} &= \epsilon_{abc}\epsilon^{ABC}(L_A^{aT}C\gamma_5 L_B^b)\gamma_\mu\gamma_5 L_C^c \,.
\end{aligned}
\tag{2.73}
$$

Similarly performing the Fierz transformation to relate $\Lambda_{Li\mu}$ and $\Lambda'_{Li\mu}$, we obtain the solution that all such kind of fields vanish.

(2) The chiral representation $(\mathbf{10},\ \mathbf{1})$ has two non-zero fields:

$$
\begin{aligned}
\Delta_{L7\mu}^{P} &= \epsilon_{abc}S_P^{ABC}(L_A^{aT}C\sigma_{\mu\nu} L_B^b)\gamma^\nu L_C^c \,, \\
\Delta_{L8\mu}^{P} &= \epsilon_{abc}S_P^{ABC}(L_A^{aT}C\sigma_{\mu\nu}\gamma_5 L_B^b)\gamma^\nu\gamma_5 L_C^c \,.
\end{aligned}
\tag{2.74}
$$

Similarly performing the Fierz transformation to relate $\Delta_{Li\mu}^{P}$ and $\Delta_{Li\mu}^{P'}$, we obtain the solution that all such kind of fields vanish.

(3) The chiral representation $(\mathbf{8},\ \mathbf{1})$ has two non-zero fields:

$$
\begin{aligned}
N_{L1\mu}^{N} &= \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^{N}(L_{A}^{aT}CL_{B}^{b})\gamma_{\mu}L_{C}^{c}\,, \\
N_{L2\mu}^{N} &= \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^{N}(L_{A}^{aT}C\gamma_{5}L_{B}^{b})\gamma_{\mu}\gamma_{5}L_{C}^{c}\,, \\
N_{L3\mu}^{N} &= \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^{N}(L_{A}^{aT}C\gamma_{\mu}\gamma_{5}L_{B}^{b})\gamma_{5}L_{C}^{c} = 0\,, \\
N_{L4\mu}^{N} &= \epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^{N}(L_{A}^{aT}C\gamma^{\nu}\gamma_{5}L_{B}^{b})\sigma_{\mu\nu}\gamma_{5}L_{C}^{c} = 0\,, \\
N_{L5\mu}^{N} &= \epsilon_{abc}\epsilon^{ACD}\lambda_{DB}^{N}(L_{A}^{aT}C\gamma_{\mu}L_{B}^{b})L_{C}^{c} = 0\,, \\
N_{L6\mu}^{N} &= \epsilon_{abc}\epsilon^{ACD}\lambda_{DB}^{N}(L_{A}^{aT}C\gamma^{\nu}L_{B}^{b})\sigma_{\mu\nu}L_{C}^{c} = 0\,, \\
N_{L7\mu}^{N} &= \epsilon_{abc}\epsilon^{ACD}\lambda_{DB}^{N}(L_{A}^{aT}C\sigma_{\mu\nu}L_{B}^{b})\gamma^{\nu}L_{C}^{c} = 0\,, \\
N_{L8\mu}^{N} &= \epsilon_{abc}\epsilon^{ACD}\lambda_{DB}^{N}(L_{A}^{aT}C\sigma_{\mu\nu}\gamma_{5}L_{B}^{b})\gamma^{\nu}\gamma_{5}L_{C}^{c} = 0\,.
\end{aligned}
\tag{2.75}
$$

Similarly performing the Fierz transformation to relate $N_{Li\mu}^{N}$ and $N_{Li\mu}^{N'}$, we obtain the solution

$$
N_{L7\mu}^{N'} = N_{L8\mu}^{N'} = \frac{3i}{2}N_{L2\mu}^{N} = \frac{3i}{2}N_{L1\mu}^{N}\,.
\tag{2.76}
$$

Others are just zero. There is only one non-vanishing octet baryon field.

Chiral properties of $(LL)R$, $(LR)L$ and $(RL)L$

The chiral representations of $(LL)R$, $(LR)L$ and $(RL)L$ are $(\bar{\mathbf{3}},\ \mathbf{3}) \oplus (\mathbf{6},\ \mathbf{3})$. We will study them separately in the following.

(1) The chiral representation $(\bar{\mathbf{3}},\ \mathbf{3}) \to \mathbf{1}_{\mathbf{f}}$ has two non-zero components:

$$
\begin{aligned}
\Lambda_{M1\mu} &= \epsilon_{abc}\epsilon^{ABC}(L_{A}^{aT}CL_{B}^{b})\gamma_{\mu}R_{C}^{c}\,, \\
\Lambda_{M2\mu} &= \epsilon_{abc}\epsilon^{ABC}(L_{A}^{aT}C\gamma_{5}L_{B}^{b})\gamma_{\mu}\gamma_{5}R_{C}^{c}\,, \\
\Lambda_{M3\mu} &= \epsilon_{abc}\epsilon^{ABC}(L_{A}^{aT}C\gamma_{\mu}\gamma_{5}L_{B}^{b})\gamma_{5}R_{C}^{c} = 0\,, \\
\Lambda_{M4\mu} &= \epsilon_{abc}\epsilon^{ABC}(L_{A}^{aT}C\gamma^{\nu}\gamma_{5}L_{B}^{b})\sigma_{\mu\nu}\gamma_{5}R_{C}^{c} = 0\,, \\
\Lambda_{M5\mu} &= \epsilon_{abc}\epsilon^{ABC}(L_{A}^{aT}C\gamma_{\mu}L_{B}^{b})R_{C}^{c} = 0\,, \\
\Lambda_{M6\mu} &= \epsilon_{abc}\epsilon^{ABC}(L_{A}^{aT}C\gamma^{\nu}L_{B}^{b})\sigma_{\mu\nu}R_{C}^{c} = 0\,, \\
\Lambda_{M7\mu} &= \epsilon_{abc}\epsilon^{ABC}(L_{A}^{aT}C\sigma_{\mu\nu}L_{B}^{b})\gamma^{\nu}R_{C}^{c} = 0\,, \\
\Lambda_{M8\mu} &= \epsilon_{abc}\epsilon^{ABC}(L_{A}^{aT}C\sigma_{\mu\nu}\gamma_{5}L_{B}^{b})\gamma^{\nu}\gamma_{5}R_{C}^{c} = 0\,.
\end{aligned}
\tag{2.77}
$$

Similarly performing the Fierz transformation to relate $\Lambda_{Mi\mu}$ and $\Lambda_{Mi\mu}'$, we obtain the solution

$$
3i\Lambda_{M3\mu}' = \Lambda_{M4\mu}' = -3i\Lambda_{M5\mu}' = -\Lambda_{M6\mu}' = \frac{3i}{2}\Lambda_{M1\mu} = -\frac{3i}{2}\Lambda_{M2\mu}\,.
\tag{2.78}
$$

Others are just zero. There is only one non-vanishing field. Others $(LR)L$ and $(RL)L$ can be related to this one.

(2) The chiral representation $(\mathbf{6},\ \mathbf{3}) \to \mathbf{10_f}$ has two non-zero components:

$$
\begin{aligned}
\Delta^P_{M1\mu} &= \epsilon_{abc} S^{ABC} (L^{aT}_A C L^b_B) \gamma_\mu R^c_C = 0\,, \\
\Delta^P_{M2\mu} &= \epsilon_{abc} S^{ABC} (L^{aT}_A C \gamma_5 L^b_B) \gamma_\mu \gamma_5 R^c_C = 0\,, \\
\Delta^P_{M3\mu} &= \epsilon_{abc} S^{ABC} (L^{aT}_A C \gamma_\mu \gamma_5 L^b_B) \gamma_5 R^c_C = 0\,, \\
\Delta^P_{M4\mu} &= \epsilon_{abc} S^{ABC} (L^{aT}_A C \gamma^\nu \gamma_5 L^b_B) \sigma_{\mu\nu} \gamma_5 R^c_C = 0\,, \\
\Delta^P_{M5\mu} &= \epsilon_{abc} S^{ABC} (L^{aT}_A C \gamma_\mu L^b_B) R^c_C = 0\,, \\
\Delta^P_{M6\mu} &= \epsilon_{abc} S^{ABC} (L^{aT}_A C \gamma^\nu L^b_B) \sigma_{\mu\nu} R^c_C = 0\,, \\
\Delta^P_{M7\mu} &= \epsilon_{abc} S^{ABC} (L^{aT}_A C \sigma_{\mu\nu} L^b_B) \gamma^\nu R^c_C\,, \\
\Delta^P_{M8\mu} &= \epsilon_{abc} S^{ABC} (L^{aT}_A C \sigma_{\mu\nu} \gamma_5 L^b_B) \gamma^\nu \gamma_5 R^c_C\,.
\end{aligned}
\tag{2.79}
$$

Others are just zero. Similarly performing the Fierz transformation to relate $\Delta^P_{Mi\mu}$ and $\Delta^{P\prime}_{Mi\mu}$, we obtain the solution

$$
\Delta^{P\prime}_{M3\mu} = i\Delta^{P\prime}_{M4\mu} = -\Delta^{P\prime}_{M5\mu} = -i\Delta^{P\prime}_{M6\mu} = -\frac{i}{2}\Delta^P_{M7\mu} = \frac{i}{2}\Delta^P_{M8\mu}\,.
\tag{2.80}
$$

There is only one non-vanishing field. Others $(LR)L$ and $(RL)L$ can be related to this one.

(3) The chiral representations $(\mathbf{\bar{3}},\ \mathbf{3}) \to \mathbf{8_f}$ has only two non-zero interpolators:

$$
\begin{aligned}
N^N_{M1\mu} &= \epsilon_{abc} \epsilon^{ABD} \lambda^N_{DC} (L^{aT}_A C L^b_B) \gamma_\mu R^c_C\,, \\
N^N_{M2\mu} &= \epsilon_{abc} \epsilon^{ABD} \lambda^N_{DC} (L^{aT}_A C \gamma_5 L^b_B) \gamma_\mu \gamma_5 R^c_C\,, \\
N^N_{M3\mu} &= \epsilon_{abc} \epsilon^{ABD} \lambda^N_{DC} (L^{aT}_A C \gamma_\mu \gamma_5 L^b_B) \gamma_5 R^c_C = 0\,, \\
N^N_{M4\mu} &= \epsilon_{abc} \epsilon^{ABD} \lambda^N_{DC} (L^{aT}_A C \gamma^\nu \gamma_5 L^b_B) \sigma_{\mu\nu} \gamma_5 R^c_C = 0\,, \\
N^N_{M5\mu} &= \epsilon_{abc} \epsilon^{ABD} \lambda^N_{DC} (L^{aT}_A C \gamma_\mu L^b_B) R^c_C = 0\,, \\
N^N_{M6\mu} &= \epsilon_{abc} \epsilon^{ABD} \lambda^N_{DC} (L^{aT}_A C \gamma^\nu L^b_B) \sigma_{\mu\nu} R^c_C = 0\,, \\
N^N_{M7\mu} &= \epsilon_{abc} \epsilon^{ABD} \lambda^N_{DC} (L^{aT}_A C \sigma_{\mu\nu} L^b_B) \gamma^\nu R^c_C = 0\,, \\
N^N_{M8\mu} &= \epsilon_{abc} \epsilon^{ABD} \lambda^N_{DC} (L^{aT}_A C \sigma_{\mu\nu} \gamma_5 L^b_B) \gamma^\nu \gamma_5 R^c_C = 0\,.
\end{aligned}
\tag{2.81}
$$

Similarly performing the Fierz transformation to relate $N^N_{Mi\mu}$ and $N^{N\prime}_{Mi\mu}$, we obtain the solution

$$
3i N^{N\prime}_{M3\mu} = N^{N\prime}_{M4\mu} = -3i N^{N\prime}_{M5\mu} = -N^{N\prime}_{M6\mu} = \frac{3i}{2} N^N_{M1\mu} = -\frac{3i}{2} N^N_{M2\mu}\,.
\tag{2.82}
$$

In order to study the chiral representations $(\mathbf{6},\ \mathbf{3}) \to \mathbf{8_f}$, we need to consider the second

way (see the discussion in the section 2.1.2) which has four non-zero interpolators:

$$
\begin{aligned}
\widetilde{N}^N_{M1\mu} &= \epsilon_{abc}\epsilon^{ACD}\lambda^N_{DB}(L^{aT}_A C L^b_B)\gamma_\mu R^c_C\,,\\
\widetilde{N}^N_{M2\mu} &= \epsilon_{abc}\epsilon^{ACD}\lambda^N_{DB}(L^{aT}_A C\gamma_5 L^b_B)\gamma_\mu\gamma_5 R^c_C\,,\\
\widetilde{N}^N_{M3\mu} &= \epsilon_{abc}\epsilon^{ACD}\lambda^N_{DB}(L^{aT}_A C\gamma_\mu\gamma_5 L^b_B)\gamma_5 R^c_C = 0\,,\\
\widetilde{N}^N_{M4\mu} &= \epsilon_{abc}\epsilon^{ACD}\lambda^N_{DB}(L^{aT}_A C\gamma^\nu\gamma_5 L^b_B)\sigma_{\mu\nu}\gamma_5 R^c_C = 0\,,\\
\widetilde{N}^N_{M5\mu} &= \epsilon_{abc}\epsilon^{ACD}\lambda^N_{DB}(L^{aT}_A C\gamma_\mu L^b_B)R^c_C = 0\,,\\
\widetilde{N}^N_{M6\mu} &= \epsilon_{abc}\epsilon^{ACD}\lambda^N_{DB}(L^{aT}_A C\gamma^\nu L^b_B)\sigma_{\mu\nu}R^c_C = 0\,,\\
\widetilde{N}^N_{M7\mu} &= \epsilon_{abc}\epsilon^{ACD}\lambda^N_{DB}(L^{aT}_A C\sigma_{\mu\nu} L^b_B)\gamma^\nu R^c_C\,,\\
\widetilde{N}^N_{M8\mu} &= \epsilon_{abc}\epsilon^{ACD}\lambda^N_{DB}(L^{aT}_A C\sigma_{\mu\nu}\gamma_5 L^b_B)\gamma^\nu\gamma_5 R^c_C\,.
\end{aligned}
\tag{2.83}
$$

By using the Jacobi identity in Eq. (2.18), we obtain:

$$
\widetilde{N}^N_{M1\mu} = \frac{1}{2}N^N_{M1\mu}\,,\quad \widetilde{N}^N_{M2\mu} = \frac{1}{2}N^N_{M2\mu}\,.
\tag{2.84}
$$

Similarly performing the Fierz transformation to relate $\widetilde{N}^N_{Mi\mu}$ and $\widetilde{N}^{N\prime}_{Mi\mu}$, we obtain the solution

$$
\begin{aligned}
\widetilde{N}^N_{M2\mu} &= -\widetilde{N}^N_{M1\mu} = -\tfrac{1}{2}N^N_{M1\mu}\,,\\
\widetilde{N}^{N\prime}_{M3\mu} &= \tfrac{1}{2}\widetilde{N}^N_{M1\mu} - \tfrac{i}{2}\widetilde{N}^N_{M7\mu}\,,\\
\widetilde{N}^{N\prime}_{M4\mu} &= \tfrac{3i}{2}\widetilde{N}^N_{M1\mu} - \tfrac{1}{2}\widetilde{N}^N_{M7\mu}\,,\\
\widetilde{N}^{N\prime}_{M5\mu} &= -\tfrac{1}{2}\widetilde{N}^N_{M1\mu} + \tfrac{i}{2}\widetilde{N}^N_{M7\mu}\,,\\
\widetilde{N}^{N\prime}_{M6\mu} &= -\tfrac{3i}{2}\widetilde{N}^N_{M1\mu} + \tfrac{1}{2}\widetilde{N}^N_{M7\mu}\,,\\
\widetilde{N}^N_{M8\mu} &= -\widetilde{N}^N_{M7\mu}\,.
\end{aligned}
$$

All together there are two non-vanishing independent fields. Others $(LR)L$ and $(RL)L$ can be related to $(LL)R$. Chiral properties of the tensor fields can be also explored in completely the same manner explained here. Therefore, we do not show this case any more.

2.3.3 A Short Summary for Chiral representations

To summarize this section, we find that possible chiral representations for Dirac spinor baryon fields without Lorentz index are:

$$
\begin{aligned}
\Lambda &= \epsilon_{abc}\epsilon^{ABC}(L^{aT}_A C L^b_B)\gamma_5 R^c_C + \epsilon_{abc}\epsilon^{ABC}(R^{aT}_A C R^b_B)\gamma_5 L^c_C\\
&= \Lambda_{M1} + (L \leftrightarrow R)\,,
\tag{2.85}\\
N^N_1 - N^N_2 &= 2\epsilon_{abc}\epsilon^{ABD}\lambda^N_{DC}(L^{aT}_A C L^b_B)\gamma_5 R^c_C + 2\epsilon_{abc}\epsilon^{ABD}\lambda^N_{DC}(R^{aT}_A C R^b_B)\gamma_5 L^c_C\\
&= 2N^N_{M1} + (L \leftrightarrow R)\,,
\tag{2.86}\\
N^N_1 + N^N_2 &= 2\epsilon_{abc}\epsilon^{ABD}\lambda^N_{DC}(L^{aT}_A C L^b_B)\gamma_5 L^c_C + 2\epsilon_{abc}\epsilon^{ABD}\lambda^N_{DC}(R^{aT}_A C R^b_B)\gamma_5 R^c_C\\
&= 2N^N_{L1} + (L \leftrightarrow R)\,.
\tag{2.87}
\end{aligned}
$$

So we can see that the fields Λ and $N_1^N - N_2^N$ has a type of $LLR \oplus RRL$, and belong to the chiral representation $(\bar{3}, 3) \oplus (3, \bar{3})$; while the field $N_1^N + N_2^N$ has a type of $LLL \oplus RRR$, and belongs to the chiral representation $(8, 1) \oplus (1, 8)$.

We summarize the results here:

$$
\begin{aligned}
N_\mu^N &= 2\epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N(L_A^{aT}C\gamma_\mu\gamma_5 R_B^b)\gamma_5 L_C^c + 2\epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N(R_A^{aT}C\gamma_\mu\gamma_5 L_B^b)\gamma_5 R_C^c \\
&+ \frac{1}{2}\epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N(L_A^{aT}CL_B^b)\gamma_\mu R_C^c + \frac{1}{2}\epsilon_{abc}\epsilon^{ABD}\lambda_{DC}^N(R_A^{aT}CR_B^b)\gamma_\mu L_C^c, \qquad (2.88)
\end{aligned}
$$

$$
\Delta_\mu^P = 2\epsilon_{abc}S_P^{ABC}(L_A^{aT}C\gamma_\mu R_B^b)L_C^c + 2\epsilon_{abc}S_P^{ABC}(R_A^{aT}C\gamma_\mu L_B^b)R_C^c. \qquad (2.89)
$$

So we see that N_μ^N and Δ_μ^P are of the type $LLR \oplus RRL$, and belong to the chiral representation $(6, 3) \oplus (3, 6)$. The (similar) results for $\Delta_{\mu\nu}^P$, which is of the type $LLL \oplus RRR$, and belongs to the chiral representation $(10, 1) \oplus (1, 10)$, are omitted here.

2.4   Axial coupling constants

As a simple application of the present mathematical formalism, we can extract the (diagonal) axial coupling constants g_A for these baryons. All information is contained in Eqs. (2.38) and (2.59), from which one can extract the Abelian $U(1)_A$ axial coupling constant g_A^0 and the non-Abelian $SU(3)_V \times SU(3)_A$ diagonal axial coupling constants, g_A^3 and g_A^8. The latter two can be extracted from the δ_5^{b3} and δ_5^{b8} subset of chiral transformations Eqs. (2.59), respectively.

In general, the diagonal elements of the $SU(3)$ g_A's can be decomposed into so-called F and D components, which are defined by the axial vector current A_μ^a ($a = 0, 1, ...8$)

$$
A_\mu^a = g_A^F \, \mathrm{Tr}\left(\bar{\mathfrak{N}}\gamma_\mu\gamma_5 \left[\frac{\lambda_a}{2}, \mathfrak{N}\right]\right) + g_A^D \, \mathrm{Tr}\left(\bar{\mathfrak{N}}\gamma_\mu\gamma_5 \left\{\frac{\lambda_a}{2}, \mathfrak{N}\right\}\right), \qquad (2.90)
$$

where $\mathfrak{N}$ is the 3×3 baryon octet matrix, Eq. (2.12). Therefore, we have

$$
\begin{aligned}
A_\mu^3 &= (g_A^F + g_A^D)\left(p^+ p - n^+ n\right) \qquad\qquad\qquad\qquad\qquad (2.91) \\
&+ 2g_A^F\left((\Sigma^+)^+\Sigma^+ - (\Sigma^-)^+\Sigma^-\right) \\
&+ (g_A^F - g_A^D)\left((\Xi^0)^+\Xi^0 - (\Xi^-)^+\Xi^-\right),
\end{aligned}
$$

$$
\begin{aligned}
A_\mu^8 &= (\sqrt{3}g_A^F - \frac{g_A^D}{\sqrt{3}})\left(p^+ p + n^+ n\right) \qquad\qquad\qquad\qquad (2.92) \\
&+ \frac{2g_A^D}{\sqrt{3}}\left((\Sigma^+)^+\Sigma^+ + (\Sigma^-)^+\Sigma^-\right) \\
&+ (-\sqrt{3}g_A^F - \frac{g_A^D}{\sqrt{3}})\left((\Xi^0)^+\Xi^0 + (\Xi^-)^+\Xi^-\right) - \frac{2g_A^D}{\sqrt{3}}(\Lambda^8)^+\Lambda^8,
\end{aligned}
$$

where we omit the Lorentz indices. In other words,

$$g_A^3(N) \sim (g_A^F + g_A^D)\mathbf{I_z}, \quad g_A^3(\Sigma) \sim 2g_A^D\mathbf{I_z}, \quad g_A^3(\Xi) \sim (g_A^F - g_A^D)\mathbf{I_z}, \tag{2.93}$$

$$g_A^8(N) \sim \sqrt{3}g_A^F - \frac{g_A^D}{\sqrt{3}}, \quad g_A^8(\Sigma) \sim \frac{2g_A^D}{\sqrt{3}}, \quad g_A^8(\Xi) \sim -\sqrt{3}g_A^F - \frac{g_A^D}{\sqrt{3}}, \quad g_A^8(\Lambda) \sim -\frac{2g_A^D}{\sqrt{3}},$$

for the octet parts. The operator $\mathbf{I_z}$ is the third component of isospin, whereas the $SU(3)$ singlet term g_A^0 contains only the D term and is therefore trivial.

For the decuplet baryons, the $SU(3)$ coupling constants contain only one $SU(3)$ irreducible term because the $SU(3)$ Clebsch-Gordan series for $\overline{10} \otimes 10 \otimes 8$ contains only one singlet. In order to extract the coupling constants, we first rewrite Eqs. (2.38) and (2.59) in the following form, for all the singlet, octet and decuplet baryon fields:

1. The Abelian g_A^0 basically counts the difference between the numbers of left- and right-handed quarks in a baryon of definite/positive chirality (helicity). Several definitions of g_A^0 can be found in the literature. No matter what convention we adopt, we must make sure that it is consistent with the definition of the $SU(3)$ singlet vector current that counts the baryon-, or the quark number. So, either we normalize g_A^0 to the baryon number, or to the quark number. Of course, the difference is just a multiplicative factor (3), but inconsistent definitions will lead to confusion later on when one constructs chirally invariant interactions. At this time we shall adopt the latter (quark number) normalization.

   Because $\lambda_{11}^0 = \lambda_{22}^0 = \lambda_{33}^0$ for g_A^0, the chiral transformations δ_5 are identical for all baryon fields within the same chiral representation, so we may define g_A^0 by

$$\delta_5 B = i\gamma_5\frac{\lambda^0{}_{11}b_0}{2}g_A^0 B = \frac{i\gamma_5 b_0}{\sqrt{6}}g_A^0 B, \tag{2.94}$$

   where B represents the baryon field, such as Λ and $N_1^N - N_2^N$ etc. This convention is based on the quark number, implying that the $SU(3)$ singlet vector charge of a nucleon is three ($+3$).

2. For g_A^3, because $\lambda_{11}^3 = -\lambda_{22}^3$, the chiral transformation δ_5^{b3} is proportional to the isospin value of $\mathbf{I_z}$, which is factored out from the definition of g_A^3

$$\delta_5^{b3}B = i\gamma_5 b_3 g_A^3\mathbf{I_z}B + \cdots, \tag{2.95}$$

   where the ellipsis $\cdots$ on the right-hand side denote the off-diagonal terms.

3. For g_A^8, because $\lambda_{11}^8 = \lambda_{22}^8$, the chiral transformations δ_5^{b8} is the same for the baryon fields belonging to one isospin multiplet. We define it to be

$$\delta_5^{b8}B = i\gamma_5\frac{\lambda_{11}^8 b_8}{2}g_A^8 B + \cdots = \frac{i\gamma_5 b_8}{2\sqrt{3}}g_A^8 B + \cdots. \tag{2.96}$$

Table 2.3: Axial Coupling Constants g_A^0, g_A^3 and g_A^8. In the last column $\alpha = g_A^D/(g_A^F + g_A^D)$.

$SU(3)_L \otimes SU(3)_R$	$SU(3)_F$		g_A^0	g_A^3	g_A^8	α
	1	Λ	-1	–	–	–
$(\bar{3},\ 3) \oplus (3,\ \bar{3})$		N_-	-1	1	-1	
		Σ_-	-1	0	2	
	8	Ξ_-	-1	-1	-1	1
		Λ_-	-1	–	-2	
		N_+	3	1	3	
		Σ_+	3	1	0	
$(8,\ 1) \oplus (1,\ 8)$	**8**	Ξ_+	3	1	-3	0
		Λ_+	3	–	0	
		N_μ	1	5/3	1	
		Σ_μ	1	2/3	2	
	8	Ξ_μ	1	-1/3	-3	3/5
		Λ_μ	1	–	-2	
$(3,\ 6) \oplus (6,\ 3)$		Δ_μ	1	1/3	1	
		Σ_μ^*	1	1/3	0	
	10	Ξ_μ^*	1	1/3	-1	–
		Ω_μ	1	–	-2	
		$\Delta_{\mu\nu}$	3	1	3	
		$\Sigma_{\mu\nu}^*$	3	1	0	
$(10,\ 1) \oplus (1,\ 10)$	**10**	$\Xi_{\mu\nu}^*$	3	1	-3	–
		$\Omega_{\mu\nu}$	3	–	-6	

The resulting axial coupling constants g_A^0, g_A^3 and g_A^8 are shown in Table 2.3, where Λ is the (only) singlet field Λ; then N_-, Σ_-, Ξ_- and Λ_- are the octet fields of the type $N_1^N - N_2^N$; the N_+, Σ_+, Ξ_+ and Λ_+ are the octet fields of the type $N_1^N + N_2^N$; the N_μ, Σ_μ, Ξ_μ and Λ_μ are the octet fields N_μ^N; the Δ_μ, Σ_μ^*, Ξ_μ^* and Ω_μ are the decuplet fields Δ_μ^P; $\Delta_{\mu\nu}$, $\Sigma_{\mu\nu}^*$, $\Xi_{\mu\nu}^*$ and $\Omega_{\mu\nu}$ are the decuplet fields $\Delta_{\mu\nu}^P$.

From the values in Table 2.3, one can compute the F and D couplings easily for the three octet baryon fields $N_1^N - N_2^N$, $N_1^N + N_2^N$, and N_μ^N:

1. $N_1^N - N_2^N$. For λ^3 and λ^8, respectively

$$\frac{i\gamma_5 b_3}{2} \begin{pmatrix} g_A^F + g_A^D \\ 2g_A^F \\ g_A^F - g_A^D \end{pmatrix} = i\gamma_5 b_3 \begin{pmatrix} 1/2 \times 1 \\ 1 \times 0 \\ 1/2 \times (-1) \end{pmatrix},$$

$$\frac{i\gamma_5 b_8}{2} \begin{pmatrix} \sqrt{3}g_A^F - \frac{g_A^D}{\sqrt{3}} \\ \frac{2g_A^D}{\sqrt{3}} \\ -\sqrt{3}g_A^F - \frac{g_A^D}{\sqrt{3}} \\ -\frac{2g_A^D}{\sqrt{3}} \end{pmatrix} = \frac{i\gamma_5 b_8}{2\sqrt{3}} \begin{pmatrix} -1 \\ 2 \\ -1 \\ -2 \end{pmatrix},$$

where the right-hand side of these equations is just g_A^3 and g_A^8, and we show the results explicitly. The solution is $g_A^F = 0$ and $g_A^D = 1$. Therefore, $N_1^N - N_2^N$ only contains D terms.

2. $N_1^N + N_2^N$. For λ^3 and λ^8, respectively

$$\frac{i\gamma_5 b_3}{2} \begin{pmatrix} g_A^F + g_A^D \\ 2g_A^F \\ g_A^F - g_A^D \end{pmatrix} = i\gamma_5 b_3 \begin{pmatrix} 1/2 \times 1 \\ 1 \times 1 \\ 1/2 \times 1 \end{pmatrix},$$

$$\frac{i\gamma_5 b_8}{2} \begin{pmatrix} \sqrt{3}g_A^F - \frac{g_A^D}{\sqrt{3}} \\ \frac{2g_A^D}{\sqrt{3}} \\ -\sqrt{3}g_A^F - \frac{g_A^D}{\sqrt{3}} \\ -\frac{2g_A^D}{\sqrt{3}} \end{pmatrix} = \frac{i\gamma_5 b_8}{2\sqrt{3}} \begin{pmatrix} 3 \\ 0 \\ -3 \\ 0 \end{pmatrix},$$

where the solution is $g_A^F = 1$ and $g_A^D = 0$. Therefore, $N_1^N + N_2^N$ only contains F terms.

3. N_μ^N. For λ^3 and λ^8, respectively

$$\frac{i\gamma_5 b_3}{2} \begin{pmatrix} g_A^F + g_A^D \\ 2g_A^F \\ g_A^F - g_A^D \end{pmatrix} = i\gamma_5 b_3 \begin{pmatrix} 1/2 \times 5/3 \\ 1 \times 2/3 \\ 1/2 \times (-1/3) \end{pmatrix},$$

$$\frac{i\gamma_5 b_8}{2} \begin{pmatrix} \sqrt{3}g_A^F - \frac{g_A^D}{\sqrt{3}} \\ \frac{2g_A^D}{\sqrt{3}} \\ -\sqrt{3}g_A^F - \frac{g_A^D}{\sqrt{3}} \\ -\frac{2g_A^D}{\sqrt{3}} \end{pmatrix} = \frac{i\gamma_5 b_8}{2\sqrt{3}} \begin{pmatrix} 1 \\ 2 \\ -3 \\ -2 \end{pmatrix},$$

where we obtain the solution that $g_A^F = 2/3$ and $g_A^D = 1$. Therefore, N_μ^N contains both F terms and D terms.

The resulting F/D ratio,

$$\alpha = \frac{g_A^D}{g_A^F + g_A^D}, \tag{2.97}$$

is also tabulated in the last column of Table 2.3. Empirically, $\alpha \sim 0.6$, which is fairly close to the $SU(6)$ quark model value. In the present formalism we see that only the $(\mathbf{3}, \mathbf{6}) \oplus (\mathbf{6}, \mathbf{3})$ chiral multiplet/representation reproduces this value. Previous works have shown that this value is physically related to the coupling of the nucleon to the $\Delta(1232)$, as demonstrated in the Adler-Weisberger sum rule [11, 175]. This was also shown algebraically by Weinberg [173]. In both cases, saturation of the pion (axial-vector) induced transition from the nucleon to the $\Delta(1232)$ is essential [58]. In the present study, this is realized by the chiral representation which includes both the nucleon (isospin 1/2) and delta (isospin 3/2) states.

It is also interesting that Table 2.3 shows that $g_A^3(N) = 5/3, g_A^0(N) = 1$ for $(\mathbf{3}, \mathbf{6}) \oplus (\mathbf{6}, \mathbf{3})$, while $g_A^3(N) = 1, g_A^0(N) = -1$ for $(\bar{\mathbf{3}}, \mathbf{3}) \oplus (\mathbf{3}, \bar{\mathbf{3}})$.

The flavor singlet g_A^0 corresponds to the so-called nucleon spin value, as measured in polarized deep-inelastic lepton scattering. A suitable superposition of the two chiral representations may improve the nucleon axial coupling in either the isovector and/or isosinglet sectors. The importance of such mixing for the isovector axial coupling constant has been emphasized by Weinberg since the late 1960-s, Ref. [173].

2.5   Conclusion

In this chapter we have performed a classification of flavor vector and chiral symmetries, and established independence of several types of relativistic $SU(3)$ baryon interpolating fields. The three-quark fields may belong to one of several different Lorentz group representations which fact imposes certain constraints on possible chiral symmetry representations. This is due to the Pauli principle and has been explicitly verified by the method of Fierz transformations.

As the present results reflect essentially the Pauli principle, they can be conveniently summarized by using the permutation symmetry group properties/representations, as shown in Table 2.4. This table "explains" also the previous results for the case of isospin $SU(2)_L \times SU(2)_R$ [136]. In the real world, with spontaneous breaking of chiral symmetry,

Table 2.4: Structure of allowed three-quark baryon fields.

Lorentz	J = Spin	Young diagram for Chiral rep.	Axial $U(1)_A$ charge g_A^0	Chiral $SU(2)$	Chiral $SU(3)$	Flavor $SU(3)$
$(\frac{1}{2},0) \oplus (0,\frac{1}{2})$	1/2	$([21],-) \oplus (-,[21])$ $([1],[11]) \oplus ([11],[1])$	3 -1	$(\frac{1}{2},0) \oplus (0,\frac{1}{2})$	$(8,1) \oplus (1,8)$ $(3,\bar{3}) \oplus (\bar{3},3)$	8 1,8
$(1,\frac{1}{2}) \oplus (\frac{1}{2},1)$	3/2	$([2],[1]) \oplus ([1],[2])$	1	$(\frac{1}{2},1) \oplus (1,\frac{1}{2})$	$(3,6) \oplus (6,3)$	8, 10
$(\frac{3}{2},0) \oplus (0,\frac{3}{2})$	3/2	$([3],-) \oplus (-,[3])$	3	$(\frac{3}{2},0) \oplus (0,\frac{3}{2})$	$(10,1) \oplus (1,10)$	10

physical states of pure chiral (axial) symmetry representation do not occur, but in general they can mix in a state having a definite flavor symmetry. The present results show that the three-quark structures accommodate only a few (sometimes just one) chiral representations, for instance, for the total spin 1/2 field of Dirac spinor, there are two allowed

chiral representations, having the Young diagram structures ([21], −) and ([1],[11]), where − indicates the singlet. The ([21], −) Young diagram corresponds to the $(\frac{1}{2}, 0)$ and $(8, 1)$ representations of $SU(2)$ and $SU(3)$ respectively, whereas the ([1], [11]) Young diagram corresponds to the $(\frac{1}{2}, 0)$ and $(3, \bar{3})$ of $SU(2)$ and $SU(3)$, respectively.

Note that the $N_f = 2$ chiral representations have the same form as those of the Lorentz group. In this way, the Lorentz (spin) and flavor structures are combined into a general structure with total permutation symmetry. As shown in the computation of g_A, in general, various couplings depend on the chiral representations.

We should conclude with a few historical remarks: the two-flavor baryon fields' Fierz identities have been known since the early days of QCD sum rules [88], whereas the three-flavor ones presented here seem to be the first ones. Similarly, the chiral properties of the two-flavor baryon fields' have been known at least since the work of Christos [46,47], but the three-flavor ones have been discussed by Christos and H. Q. Zheng [47,182,183], but not systematically explored.

Chapter 3

Tetraquark Fields

3.1 $ud\bar{s}\bar{s}$ Currents of $J^P = 0^+$

The structure of tetraquark is much more complicated than $\bar{q}q$ mesons and qqq baryons. And so in this section, we fix quark contents to be $ud\bar{s}\bar{s}$. After studying this example, the general tetraquark currents will be studied in the following sections.

Let us consider currents for the tetraquark $ud\bar{s}\bar{s}$ having $J^P = 0^+$. Here again we only consider local currents, and we shall study the diquark-antidiquark currents $((\bar{q}\bar{q})(qq))$ first, while the meson-meson currents $((\bar{q}q)(\bar{q}q))$ will be discussed later. To write a current, Lorentz and color indices are contracted with suitable coefficients $(L_{\mu\nu\rho\sigma}^{abcd})$ to provide necessary quantum numbers,

$$\eta = L_{\mu\nu\rho\sigma}^{abcd}\,\bar{s}_a^\mu \bar{s}_b^\nu u_c^\rho d_d^\sigma\,, \tag{3.1}$$

where the sum over repeated indices (μ, ν, $\cdots$ for Dirac spinor indices, and $a, b, \cdots$ for color indices) is taken.

For the Dirac spinor space, using possible diquark and antidiquark bilinears [96, 109, 159, 165], there are five independent terms

$$
\begin{aligned}
S_{abcd} &= (\bar{s}_a \gamma_5 C \bar{s}_b^T)(u_c^T C \gamma_5 d_d)\,, \\
V_{abcd} &= (\bar{s}_a \gamma_\mu \gamma_5 C \bar{s}_b^T)(u_c^T C \gamma^\mu \gamma_5 d_d)\,, \\
T_{abcd} &= (\bar{s}_a \sigma_{\mu\nu} C \bar{s}_b^T)(u_c^T C \sigma^{\mu\nu} d_d)\,, \\
A_{abcd} &= (\bar{s}_a \gamma_\mu C \bar{s}_b^T)(u_c^T C \gamma^\mu d_d)\,, \\
P_{abcd} &= (\bar{s}_a C \bar{s}_b^T)(u_c^T C d_d)\,.
\end{aligned}
\tag{3.2}
$$

Here, color indices are not yet specified. For the diquark and antidiquark pair, color structures providing a color-singlet tetraquark are $\mathbf{3} \otimes \bar{\mathbf{3}}$ and $\bar{\mathbf{6}} \otimes \mathbf{6}$, which we will denote by labels $\mathbf{3}$ and $\mathbf{6}$ for short.

Therefore, we have altogether ten terms of products

$$\{S \oplus V \oplus T \oplus A \oplus P\}_{Lorentz} \otimes \{\mathbf{3} \oplus \mathbf{6}\}_{Color}\,. \tag{3.3}$$

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However, half of them drop due to the Pauli principle. For instance

$$P_3 \equiv P_{Lorentz} \otimes \mathbf{3}_{Color} \tag{3.4}$$
$$= \epsilon_{abc}(\bar{s}_b C \bar{s}_c^T)\epsilon_{ab'c'}(u_{b'}^T C d_{c'}) = 0.$$

Eventually, we end up with five independent currents

$$S_6 = (\bar{s}_a \gamma_5 C \bar{s}_b^T)(u_a^T C \gamma_5 d_b), \tag{3.5}$$
$$V_6 = (\bar{s}_a \gamma_\mu \gamma_5 C \bar{s}_b^T)(u_a^T C \gamma^\mu \gamma_5 d_b),$$
$$T_3 = (\bar{s}_a \sigma_{\mu\nu} C \bar{s}_b^T)(u_a^T C \sigma^{\mu\nu} d_b), \tag{3.6}$$
$$A_3 = (\bar{s}_a \gamma_\mu C \bar{s}_b^T)(u_a^T C \gamma^\mu d_b),$$
$$P_6 = (\bar{s}_a C \bar{s}_b^T)(u_a^T C d_b).$$

In the non-relativistic language, these five terms correspond to combinations of diquarks and antidiquarks

$$[(^1S_0)(^1S_0)]_{0^+}, \quad [(^3S_1)(^3S_1)]_{0^+}, \quad [(^1P_1)(^1P_1)]_{0^+},$$
$$[(^3P_0)(^3P_0)]_{0^+}, \quad [(^3P_1)(^3P_1)]_{0^+}. \tag{3.7}$$

Another possible piece of $^3P_2$ is irrelevant, since the five bi-linear forms $q^T \Gamma q$ ($\Gamma = S, V, T, A, P$) can only have spin $j \leq 1$, while the $^3P_2$ diquark has $j = 2$.

Finally we consider the flavor structure. The $\bar{s}\bar{s}$ antidiquark is symmetric in flavor, and hence belongs to the symmetric representation $\bar{\mathbf{6}}_f$. If the other ud diquark belongs to $\mathbf{3}_f$, and so isospin $I = 0$, the diquark and antidiquark will have different flavor symmetry. But they should have the same color and spin symmetries for composing a color-singlet scalar tetraquark. Considering the Pauli principle, they must have different parity, and hence their combination is a negative-parity scalar tetraquark. Accordingly, the other ud diquark also belongs to $\mathbf{6}_f$, and so isospin $I = 1$. Among the irreducible representations of the tetraquark

$$\bar{\mathbf{6}} \otimes \mathbf{6} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{27}, \tag{3.8}$$

$S = +2$ and $I = 1$ states are in the $\mathbf{27}$ representation of $SU(3)_f$, which is the flavor structure of the present tetraquark. As shown in Fig. 3.1, three iso-vector states of the $\mathbf{27}_f$ are $uu\bar{s}\bar{s}$, $1/\sqrt{2}(ud + du)\bar{s}\bar{s}$ and $dd\bar{s}\bar{s}$.

We have constructed five independent currents using diquark and antidiquark combination. Similarly, we can also construct the tetraquark currents using $\bar{q}q$ combination (mesonic construction). Obviously, there are ten combinations of the Dirac (S, V, T, A and P) and color ($\mathbf{1}$ and $\mathbf{8}$) spaces:

$$S_1 = (\bar{s}_a u_a)(\bar{s}_b d_b), \qquad S_8 = (\bar{s}_a \lambda_{ab}^n u_b)(\bar{s}_c \lambda_{cd}^n d_d),$$
$$V_1 = (\bar{s}_a \gamma_\mu u_a)(\bar{s}_b \gamma^\mu d_b), \qquad V_8 = (\bar{s}_a \gamma_\mu \lambda_{ab}^n u_b)(\bar{s}_c \gamma^\mu \lambda_{cd}^n d_d),$$
$$T_1 = (\bar{s}_a \sigma_{\mu\nu} u_a)(\bar{s}_b \sigma^{\mu\nu} d_b), \qquad T_8 = (\bar{s}_a \sigma_{\mu\nu} \lambda_{ab}^n u_b)(\bar{s}_c \sigma^{\mu\nu} \lambda_{cd}^n d_d), \tag{3.9}$$
$$A_1 = (\bar{s}_a \gamma_\mu \gamma_5 u_a)(\bar{s}_b \gamma^\mu \gamma_5 d_b), \qquad A_8 = (\bar{s}_a \gamma_\mu \gamma_5 \lambda_{ab}^n u_b)(\bar{s}_c \gamma^\mu \gamma_5 \lambda_{cd}^n d_d),$$
$$P_1 = (\bar{s}_a \gamma_5 u_a)(\bar{s}_b \gamma_5 d_b), \qquad P_8 = (\bar{s}_a \gamma_5 \lambda_{ab}^n u_b)(\bar{s}_c \gamma_5 \lambda_{cd}^n d_d),$$

$$27_f(6_f \otimes 6_f)$$
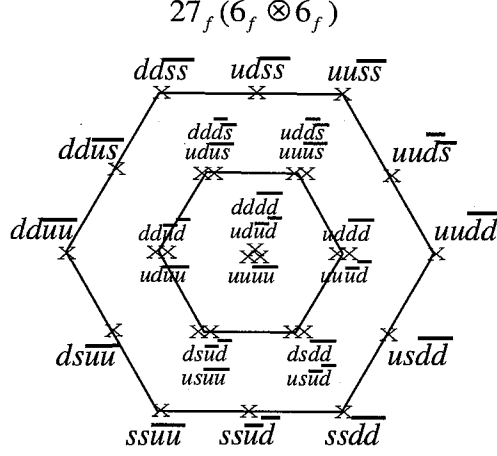
Figure 3.1: $SU(3)$ weight diagram for **27**, where the locations of three tetraquark components of $S = 2$ and $I = 1$ are shown.

where subscripts **1** and **8** denote color singlet and octet representations, respectively. Unlike the diquark construction, all the ten currents in Eq. (3.9) remain finite. However, it is possible to show only five of them (in fact any five of them) are independent. The quark-antiquark pairs in different currents have different properties:

$$
\begin{aligned}
&S_1 : (J^P = 0^+, 8_f, 1_c), &\qquad &S_8 : (J^P = 0^+, 8_f, 8_c), \\
&V_1 : (J^P = 1^-, 8_f, 1_c), &\qquad &V_8 : (J^P = 1^-, 8_f, 8_c), \\
&T_1 : (J^P = 1^+ \& 1^-, 8_f, 1_c), &\qquad &T_8 : (J^P = 1^+ \& 1^-, 8_f, 8_c), \\
&A_1 : (J^P = 1^+, 8_f, 1_c), &\qquad &A_8 : (J^P = 1^+, 8_f, 8_c), \\
&P_1 : (J^P = 0^-, 8_f, 1_c), &\qquad &P_8 : (J^P = 0^-, 8_f, 8_c).
\end{aligned}
$$

In order to establish the five independent currents, first we change their color structures

$$
\begin{aligned}
(\bar{s}_a u_b)(\bar{s}_b d_a) &= \frac{1}{3}(\bar{s}_a u_a)(\bar{s}_b d_b) + \frac{1}{2}(\bar{s}_a u_b)(\bar{s}_c d_d)\lambda_{ab}\lambda_{cd}\,, \\
(\bar{s}_a u_d)(\bar{s}_c d_b)\lambda_{ab}\lambda_{cd} &= \frac{16}{9}(\bar{s}_a u_a)(\bar{s}_b d_b) - \frac{1}{3}(\bar{s}_a u_b)(\bar{s}_c d_d)\lambda_{ab}\lambda_{cd}\,.
\end{aligned}
\tag{3.10}
$$

Then we use the Fierz transformation [131]

$$
\begin{aligned}
&\frac{1}{3}(\bar{s}_a u_a)(\bar{s}_b d_b) + \frac{1}{2}(\bar{s}_a u_b)(\bar{s}_c d_d)\lambda_{ab}\lambda_{cd} \\
&= \quad (\bar{s}_a u_b)(\bar{s}_b d_a)
\end{aligned}
\tag{3.11}
$$

$$= -\frac{1}{4}\{(\bar{s}_a u_a)(\bar{s}_b d_b) + (\bar{s}_a \gamma_\mu u_a)(\bar{s}_b \gamma^\mu d_b) + \frac{1}{2}(\bar{s}_a \sigma_{\mu\nu} u_a)(\bar{s}_b \sigma^{\mu\nu} d_b)$$
$$- (\bar{s}_a \gamma_\mu \gamma_5 u_a)(\bar{s}_b \gamma^\mu \gamma_5 d_b) + (\bar{s}_a \gamma_5 u_a)(\bar{s}_b \gamma_5 d_b)\}\,.$$

We obtain 10 equations in all

$$\frac{1}{3}S_1 + \frac{1}{2}S_8 = -\frac{1}{4}\{S_1 + V_1 + \frac{1}{2}T_1 - A_1 + P_1\},$$

$$\frac{16}{9}S_1 - \frac{1}{3}S_8 = -\frac{1}{4}\{S_8 + V_8 + \frac{1}{2}T_8 - A_8 + P_8\},$$

$$\frac{1}{3}V_1 + \frac{1}{2}V_8 = -\frac{1}{4}\{4S_1 - 2V_1 - 2A_1 - 4P_1\},$$

$$\frac{16}{9}V_1 - \frac{1}{3}V_8 = -\frac{1}{4}\{4S_8 - 2V_8 - 2A_8 - 4P_8\},$$

$$\frac{1}{3}T_1 + \frac{1}{2}T_8 = -\frac{1}{4}\{12S_1 - 2T_1 + 12P_1\}, \tag{3.12}$$

$$\frac{16}{9}T_1 - \frac{1}{3}T_8 = -\frac{1}{4}\{12S_8 - 2T_8 + 12P_8\},$$

$$\frac{1}{3}A_1 + \frac{1}{2}A_8 = -\frac{1}{4}\{-4S_1 - 2V_1 - 2A_1 + 4P_1\},$$

$$\frac{16}{9}A_1 - \frac{1}{3}A_8 = -\frac{1}{4}\{-4S_8 - 2V_8 - 2A_8 + 4P_8\},$$

$$\frac{1}{3}P_1 + \frac{1}{2}P_8 = -\frac{1}{4}\{S_1 - V_1 + \frac{1}{2}T_1 + A_1 + P_1\},$$

$$\frac{16}{9}P_1 - \frac{1}{3}P_8 = -\frac{1}{4}\{S_8 - V_8 + \frac{1}{2}T_8 + A_8 + P_8\}.$$

Solving these linear equations, we find that there are five independent currents. In other words, the rank of the 10×10 coefficient matrix is five. Any five currents among (3.9) are independent and can be expressed by the other five currents. For instance, we have the relations as

$$S_8 = -\frac{7}{6}S_1 - \frac{1}{2}V_1 - \frac{1}{4}T_1 + \frac{1}{2}A_1 - \frac{1}{2}P_1,$$

$$V_8 = -2S_1 + \frac{1}{3}V_1 + A_1 + 2P_1,$$

$$T_8 = -6S_1 + \frac{1}{3}T_1 - 6P_1, \tag{3.13}$$

$$A_8 = 2S_1 + V_1 + \frac{1}{3}A_1 - 2P_1,$$

$$P_8 = -\frac{1}{2}S_1 + \frac{1}{2}V_1 - \frac{1}{4}T_1 - \frac{1}{2}A_1 - \frac{7}{6}P_1.$$

Note that the color octet combinations can be expressed only in terms of color singlet combinations. This point will be discussed in more detail in Chapter 4.

Finally, we establish the relations between the diquark currents and the mesonic currents. For instance, we can verify the relations

$$
\begin{aligned}
S_6 &= -\frac{1}{4}S_1 - \frac{1}{4}V_1 + \frac{1}{8}T_1 - \frac{1}{4}A_1 - \frac{1}{4}P_1\,, \\
V_6 &= S_1 - \frac{1}{2}V_1 + \frac{1}{2}A_1 - P_1\,, \\
T_3 &= 3S_1 + \frac{1}{2}T_1 + 3P_1\,, \\
A_3 &= S_1 + \frac{1}{2}V_1 - \frac{1}{2}A_1 - P_1\,, \\
P_6 &= -\frac{1}{4}S_1 + \frac{1}{4}V_1 + \frac{1}{8}T_1 + \frac{1}{4}A_1 - \frac{1}{4}P_1\,.
\end{aligned}
\tag{3.14}
$$

3.2 Tetraquark fields with $J^P = 0^+$

We have found five independent $ud\bar{s}\bar{s}$ tetraquark currents which have the quantum numbers $J^P = 0^+$, in both the diquark construction and the meson construction. From this section, we will study the tetraquark currents having different quantum numbers. The currents can be constructed by using diquark and antidiquark fields, and they can also be constructed by using quark-antiquark pairs. The same as the $ud\bar{s}\bar{s}$ scalar tetraquark currents, we can find several independent currents.

Following the procedure in the previous section, we can obtain tetraquark currents having other quantum numbers by using the diquark currents and antidiquark currents. The diquark can have the flavor structure $\mathbf{\bar{3}_f}$ and $\mathbf{6_f}$, and the antidiquark can have the flavor structure $\mathbf{3_f}$ and $\mathbf{\bar{6}_f}$. Therefore, there are four combinations and we just need to study three of them:

$$
\mathbf{6}_f \otimes \mathbf{\bar{6}}_f\,, \mathbf{\bar{3}}_f \otimes \mathbf{\bar{6}}_f\,, \mathbf{\bar{3}}_f \otimes \mathbf{3}_f\,,
$$

while $\mathbf{6}_f \otimes \mathbf{3}_f$ can be similarly studied as $\mathbf{\bar{3}}_f \otimes \mathbf{\bar{6}}_f$. For simplicity, we will suppress the symbol $q(x)$, and use the flavor indices instead of it:

$$
q_A^a(x)\,, q_B^a(x) \longrightarrow A_a\,, B_a\,, \text{ and } \bar{q}_X^a(x)\,, \bar{q}_Y^a(x) \longrightarrow \bar{X}_a\,, \bar{Y}_a\,.
$$

The flavor structure of tetraquark is

$$
\begin{aligned}
&\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{\bar{3}} \otimes \mathbf{\bar{3}} \\
={}& (\mathbf{6} \oplus \mathbf{\bar{3}}) \otimes (\mathbf{\bar{6}} \oplus \mathbf{3}) \\
={}& (\mathbf{27} \oplus \mathbf{8} \oplus \mathbf{1}) \oplus (\mathbf{10} \oplus \mathbf{8}) \oplus (\mathbf{\bar{10}} \oplus \mathbf{8}) \oplus (\mathbf{8} \oplus \mathbf{1})\,,
\end{aligned}
\tag{3.15}
$$

In this section, we study scalar currents of $J^P = 0^+$. The diquark and antidiquark can have flavor structure $\mathbf{6}_f \otimes \mathbf{\bar{6}}_f$, then the tetraquark currents have the flavor representations

27_f, 8_f and 1_f; while they can also have the flavor structure $\bar{3}_f \otimes 3_f$, then the tetraquark currents have the flavor representations 8_f and 1_f. The flavor structures $\bar{3}_f \otimes \bar{6}_f$ and $6_f \otimes 3_f$ are not allowed as discussed in the previous section.

3.2.1 $6_f \otimes \bar{6}_f$

In this subsection, we study the tetraquark currents where both the diquark and anti-diquark components have a symmetric flavor structure: 6_f and $\bar{6}_f$, respectively. We can construct five diquark-antidiquark currents:

$$
\begin{aligned}
S_6 &= A_a^T C\gamma_5 B_b (\bar{X}_a \gamma_5 C \bar{Y}_b^T + \bar{X}_b \gamma_5 C \bar{Y}_a^T) \,, \\
V_6 &= A_a^T C\gamma_\mu \gamma_5 B_b (\bar{X}_a \gamma^\mu \gamma_5 C \bar{Y}_b^T + \bar{X}_b \gamma^\mu \gamma_5 C \bar{Y}_a^T) \,, \\
T_3 &= A_a^T C\sigma_{\mu\nu} B_b (\bar{X}_a \sigma^{\mu\nu} C \bar{Y}_b^T - \bar{X}_b \sigma^{\mu\nu} C \bar{Y}_a^T) \,, \\
A_3 &= A_a^T C\gamma_\mu B_b (\bar{X}_a \gamma^\mu C \bar{Y}_b^T - \bar{X}_b \gamma^\mu C \bar{Y}_a^T) \,, \\
P_6 &= A_a^T C B_b (\bar{X}_a C \bar{Y}_b^T + \bar{X}_b C \bar{Y}_a^T) \,,
\end{aligned}
$$

where the subscript is the color representation of the diquark (antidiquark) inside. These five currents are independent. We can also construct ten currents by using quark-antiquark pairs:

$$
\begin{aligned}
S_1 &= (\bar{X}_a A_a)(\bar{Y}_b B_b) + (\bar{X}_a B_a)(\bar{Y}_b A_b) \,, \\
S_8 &= (\bar{X}_a \lambda_{ab} A_b)(\bar{Y}_c \lambda_{cd} B_d) + (\bar{X}_a \lambda_{ab} B_b)(\bar{Y}_c \lambda_{cd} A_d) \,, \\
V_1 &= (\bar{X}_a \gamma_\mu A_a)(\bar{Y}_b \gamma^\mu B_b) + (\bar{X}_a \gamma_\mu B_a)(\bar{Y}_b \gamma^\mu A_b) \,, \\
V_8 &= (\bar{X}_a \gamma_\mu \lambda_{ab} A_b)(\bar{Y}_c \gamma^\mu \lambda_{cd} B_d) + (\bar{X}_a \gamma_\mu \lambda_{ab} B_b)(\bar{Y}_c \gamma^\mu \lambda_{cd} A_d) \,, \\
T_1 &= (\bar{X}_a \sigma_{\mu\nu} A_a)(\bar{Y}_b \sigma^{\mu\nu} B_b) + (\bar{X}_a \sigma_{\mu\nu} B_a)(\bar{Y}_b \sigma^{\mu\nu} A_b) \,, \\
T_8 &= (\bar{X}_a \sigma_{\mu\nu} \lambda_{ab} A_b)(\bar{Y}_c \sigma^{\mu\nu} \lambda_{cd} B_d) + (\bar{X}_a \sigma_{\mu\nu} \lambda_{ab} B_b)(\bar{Y}_c \sigma^{\mu\nu} \lambda_{cd} A_d) \,, \\
A_1 &= (\bar{X}_a \gamma_\mu \gamma_5 A_a)(\bar{Y}_b \gamma^\mu \gamma_5 B_b) + (\bar{X}_a \gamma_\mu \gamma_5 B_a)(\bar{Y}_b \gamma^\mu \gamma_5 A_b) \,, \\
A_8 &= (\bar{X}_a \gamma_\mu \gamma_5 \lambda_{ab} A_b)(\bar{Y}_c \gamma^\mu \gamma_5 \lambda_{cd} B_d) + (\bar{X}_a \gamma_\mu \gamma_5 \lambda_{ab} B_b)(\bar{Y}_c \gamma^\mu \gamma_5 \lambda_{cd} A_d) \,, \\
P_1 &= (\bar{X}_a \gamma_5 A_a)(\bar{Y}_b \gamma_5 B_b) + (\bar{X}_a \gamma_5 B_a)(\bar{Y}_b \gamma_5 A_b) \,, \\
P_8 &= (\bar{X}_a \gamma_5 \lambda_{ab} A_b)(\bar{Y}_c \gamma_5 \lambda_{cd} B_d) + (\bar{X}_a \gamma_5 \lambda_{ab} B_b)(\bar{Y}_c \gamma_5 \lambda_{cd} A_d) \,.
\end{aligned}
$$

Among these ten currents, five are independent, and we can verify following relations:

$$
\begin{aligned}
S_8 &= -\frac{7}{6} S_1 - \frac{1}{2} V_1 - \frac{1}{4} T_1 + \frac{1}{2} A_1 - \frac{1}{2} P_1 \,, \\
V_8 &= -2 S_1 + \frac{1}{3} V_1 + A_1 + 2 P_1 \,, \\
T_8 &= -6 S_1 + \frac{1}{3} T_1 - 6 P_1 \,, \\
A_8 &= 2 S_1 + V_1 + \frac{1}{3} A_1 - 2 P_1 \,,
\end{aligned}
$$

$$P_8 = -\frac{1}{2}S_1 + \frac{1}{2}V_1 - \frac{1}{4}T_1 - \frac{1}{2}A_1 - \frac{7}{6}P_1.$$

The diquark construction and mesonic construction are equivalent, and they can be related to each other:

$$S_6 = -\frac{1}{4}S_1 - \frac{1}{4}V_1 + \frac{1}{8}T_1 - \frac{1}{4}A_1 - \frac{1}{4}P_1,$$

$$V_6 = S_1 - \frac{1}{2}V_1 + \frac{1}{2}A_1 - P_1,$$

$$T_3 = 3S_1 + \frac{1}{2}T_1 + 3P_1,$$

$$A_3 = S_1 + \frac{1}{2}V_1 - \frac{1}{2}A_1 - P_1,$$

$$P_6 = -\frac{1}{4}S_1 + \frac{1}{4}V_1 + \frac{1}{8}T_1 + \frac{1}{4}A_1 - \frac{1}{4}P_1.$$

3.2.2 $\bar{3}_f \otimes 3_f$

In this subsection, we study the tetraquark currents where both the diquark and antidiquark components have a symmetric flavor structure: $\bar{3}_f$ and 3_f, respectively. We can construct five diquark-antidiquark currents:

$$S_3 = A_a^T C\gamma_5 B_b (\bar{X}_a \gamma_5 C\bar{Y}_b^T - \bar{X}_b \gamma_5 C\bar{Y}_a^T),$$

$$V_3 = A_a^T C\gamma_\mu \gamma_5 B_b (\bar{X}_a \gamma^\mu \gamma_5 C\bar{Y}_b^T - \bar{X}_b \gamma^\mu \gamma_5 C\bar{Y}_a^T),$$

$$T_6 = A_a^T C\sigma_{\mu\nu} B_b (\bar{X}_a \sigma^{\mu\nu} C\bar{Y}_b^T + \bar{X}_b \sigma^{\mu\nu} C\bar{Y}_a^T),$$

$$A_6 = A_a^T C\gamma_\mu B_b (\bar{X}_a \gamma^\mu C\bar{Y}_b^T + \bar{X}_b \gamma^\mu C\bar{Y}_a^T),$$

$$P_3 = A_a^T C B_b (\bar{X}_a C\bar{Y}_b^T - \bar{X}_b C\bar{Y}_a^T),$$

which are independent. We can also construct ten currents by using quark-antiquark pairs:

$$S_1 = (\bar{X}_a A_a)(\bar{Y}_b B_b) - (\bar{X}_a B_a)(\bar{Y}_b A_b),$$

$$S_8 = (\bar{X}_a \lambda_{ab} A_b)(\bar{Y}_c \lambda_{cd} B_d) - (\bar{X}_a \lambda_{ab} B_b)(\bar{Y}_c \lambda_{cd} A_d),$$

$$V_1 = (\bar{X}_a \gamma_\mu A_a)(\bar{Y}_b \gamma^\mu B_b) - (\bar{X}_a \gamma_\mu B_a)(\bar{Y}_b \gamma^\mu A_b),$$

$$V_8 = (\bar{X}_a \gamma_\mu \lambda_{ab} A_b)(\bar{Y}_c \gamma^\mu \lambda_{cd} B_d) - (\bar{X}_a \gamma_\mu \lambda_{ab} B_b)(\bar{Y}_c \gamma^\mu \lambda_{cd} A_d),$$

$$T_1 = (\bar{X}_a \sigma_{\mu\nu} A_a)(\bar{Y}_b \sigma^{\mu\nu} B_b) - (\bar{X}_a \sigma_{\mu\nu} B_a)(\bar{Y}_b \sigma^{\mu\nu} A_b),$$

$$T_8 = (\bar{X}_a \sigma_{\mu\nu} \lambda_{ab} A_b)(\bar{Y}_c \sigma^{\mu\nu} \lambda_{cd} B_d) - (\bar{X}_a \sigma_{\mu\nu} \lambda_{ab} B_b)(\bar{Y}_c \sigma^{\mu\nu} \lambda_{cd} A_d),$$

$$A_1 = (\bar{X}_a \gamma_\mu \gamma_5 A_a)(\bar{Y}_b \gamma^\mu \gamma_5 B_b) - (\bar{X}_a \gamma_\mu \gamma_5 B_a)(\bar{Y}_b \gamma^\mu \gamma_5 A_b),$$

$$A_8 = (\bar{X}_a \gamma_\mu \gamma_5 \lambda_{ab} A_b)(\bar{Y}_c \gamma^\mu \gamma_5 \lambda_{cd} B_d) - (\bar{X}_a \gamma_\mu \gamma_5 \lambda_{ab} B_b)(\bar{Y}_c \gamma^\mu \gamma_5 \lambda_{cd} A_d),$$

$$P_1 = (\bar{X}_a \gamma_5 A_a)(\bar{Y}_b \gamma_5 B_b) - (\bar{X}_a \gamma_5 B_a)(\bar{Y}_b \gamma_5 A_b),$$

$$P_8 = (\bar{X}_a \gamma_5 \lambda_{ab} A_b)(\bar{Y}_c \gamma_5 \lambda_{cd} B_d) - (\bar{X}_a \gamma_5 \lambda_{ab} B_b)(\bar{Y}_c \gamma_5 \lambda_{cd} A_d).$$

Among these ten currents, five are independent, and we can verify following relations:

$$S_8 = -\frac{1}{6}S_1 + \frac{1}{2}V_1 + \frac{1}{4}T_1 - \frac{1}{2}A_1 - \frac{1}{2}P_1,$$

$$V_8 = 2S_1 - \frac{5}{3}V_1 - A_1 - 2P_1,$$

$$T_8 = 6S_1 - \frac{5}{3}T_1 + 6P_1,$$

$$A_8 = -2S_1 - V_1 - \frac{5}{3}A_1 + 2P_1,$$

$$P_8 = \frac{1}{2}S_1 - \frac{1}{2}V_1 + \frac{1}{4}T_1 + \frac{1}{2}A_1 - \frac{1}{6}P_1.$$

The diquark construction and mesonic construction are equivalent, and they can be related to each other:

$$S_3 = -\frac{1}{4}S_1 - \frac{1}{4}V_1 + \frac{1}{8}T_1 - \frac{1}{4}A_1 - \frac{1}{4}P_1,$$

$$V_3 = S_1 - \frac{1}{2}V_1 + \frac{1}{2}A_1 - P_1,$$

$$T_6 = 3S_1 + \frac{1}{2}T_1 + 3P_1,$$

$$A_6 = S_1 + \frac{1}{2}V_1 - \frac{1}{2}A_1 - P_1,$$

$$P_3 = -\frac{1}{4}S_1 + \frac{1}{4}V_1 + \frac{1}{8}T_1 + \frac{1}{4}A_1 - \frac{1}{4}P_1.$$

3.3   Tetraquark fields with $J^P = 0^-$

In this section, we study scalar currents of $J^P = 0^-$. The diquark and antidiquark can have flavor structures $\mathbf{6}_f \otimes \bar{\mathbf{6}}_f$, $\bar{\mathbf{3}}_f \otimes \mathbf{3}_f$, $\bar{\mathbf{3}}_f \otimes \bar{\mathbf{6}}_f$ and $\mathbf{6}_f \otimes \mathbf{3}_f$. We will just study the first three of them, since the last one have the similar structure as $\bar{\mathbf{3}}_f \otimes \bar{\mathbf{6}}_f$.

3.3.1 $\mathbf{6}_f \otimes \bar{\mathbf{6}}_f$

In this subsection, we study the tetraquark currents where both the diquark and antidiquark components have a symmetric flavor structure: $\mathbf{6}_f$ and $\bar{\mathbf{6}}_f$, respectively. We can construct three diquark-antidiquark currents:

$$\eta_1 = A_a^T C B_b (\bar{X}_a \gamma_5 C \bar{Y}_b^T + \bar{X}_b \gamma_5 C \bar{Y}_a^T),$$

$$\eta_2 = A_a^T C \gamma_5 B_b (\bar{X}_a C \bar{Y}_b^T + \bar{X}_b C \bar{Y}_a^T),$$

$$\eta_3 = A_a^T C \sigma_{\mu\nu} B_b (\bar{X}_a \sigma^{\mu\nu} \gamma_5 C \bar{Y}_b^T - \bar{X}_b \sigma^{\mu\nu} \gamma_5 C \bar{Y}_a^T),$$

which are independent. We can also construct six currents by using quark-antiquark pairs:

$$
\begin{aligned}
\eta_4 &= (\bar{X}_a A_a)(\bar{Y}_b \gamma_5 B_b) + (\bar{X}_a \gamma_5 A_a)(\bar{Y}_b B_b) \\
&\quad + (\bar{X}_a B_a)(\bar{Y}_b \gamma_5 A_b) + (\bar{X}_a \gamma_5 B_a)(\bar{Y}_b A_b)\,, \\
\eta_5 &= (\bar{X}_a \gamma_\mu A_a)(\bar{Y}_b \gamma^\mu \gamma_5 B_b) + (\bar{X}_a \gamma_\mu \gamma_5 A_a)(\bar{Y}_b \gamma^\mu B_b) \\
&\quad + (\bar{X}_a \gamma_\mu B_a)(\bar{Y}_b \gamma^\mu \gamma_5 A_b) + (\bar{X}_a \gamma_\mu \gamma_5 B_a)(\bar{Y}_b \gamma^\mu A_b)\,, \\
\eta_6 &= (\bar{X}_a \sigma_{\mu\nu} A_a)(\bar{Y}_b \sigma^{\mu\nu} \gamma_5 B_b) + (\bar{X}_a \sigma_{\mu\nu} B_a)(\bar{Y}_b \sigma^{\mu\nu}, \gamma_5 A_b) \\
\eta_7 &= \lambda_{ab}\lambda_{cd}\{(\bar{X}_a A_b)(\bar{Y}_c \gamma_5 B_d) + (\bar{X}_a \gamma_5 A_b)(\bar{Y}_c B_d) \\
&\quad + (\bar{X}_a B_b)(\bar{Y}_c \gamma_5 A_d) + (\bar{X}_a \gamma_5 B_b)(\bar{Y}_c A_d)\}\,, \\
\eta_8 &= \lambda_{ab}\lambda_{cd}\{(\bar{X}_a \gamma_\mu A_b)(\bar{Y}_c \gamma^\mu \gamma_5 B_d) + (\bar{X}_a \gamma_\mu \gamma_5 A_b)(\bar{Y}_c \gamma^\mu B_d) \\
&\quad + (\bar{X}_a \gamma_\mu B_b)(\bar{Y}_c \gamma^\mu \gamma_5 A_d) + (\bar{X}_a \gamma_\mu \gamma_5 B_b)(\bar{Y}_c \gamma^\mu A_d)\}\,, \\
\eta_9 &= \lambda_{ab}\lambda_{cd}\{(\bar{X}_a \sigma_{\mu\nu} A_b)(\bar{Y}_c \sigma^{\mu\nu} \gamma_5 B_d) + (\bar{X}_a \sigma_{\mu\nu} B_b)(\bar{Y}_c \sigma^{\mu\nu} \gamma_5 A_d)\}\,.
\end{aligned}
$$

Among these six currents, three are independent, and we can verify following relations:

$$
\begin{aligned}
\eta_7 &= -\frac{5}{3}\eta_4 - \frac{1}{2}\eta_6\,, \\
\eta_8 &= \frac{4}{3}\eta_5\,, \\
\eta_9 &= -6\eta_4 + \frac{1}{3}\eta_6\,.
\end{aligned}
$$

The diquark construction and mesonic construction are equivalent, and they can be related to each other:

$$
\begin{aligned}
\eta_1 &= -\frac{1}{4}\eta_4 - \frac{1}{4}\eta_5 + \frac{1}{8}\eta_6\,, \\
\eta_2 &= -\frac{1}{4}\eta_4 + \frac{1}{4}\eta_5 + \frac{1}{8}\eta_6\,, \\
\eta_3 &= 3\eta_4 - \frac{1}{2}\eta_6\,.
\end{aligned}
$$

3.3.2 $\bar{\mathbf{3}}_f \otimes \mathbf{3}_f$

In this subsection, we study the tetraquark currents where both the diquark and antidiquark components have a symmetric flavor structure: $\bar{\mathbf{3}}_f$ and $\mathbf{3}_f$, respectively. We can construct three diquark-antidiquark currents:

$$
\begin{aligned}
\eta_1 &= A_a^T C \sigma_{\mu\nu} B_b (\bar{X}_a \sigma^{\mu\nu} \gamma_5 C \bar{Y}_b^T + \bar{X}_b \sigma^{\mu\nu} \gamma_5 C \bar{Y}_a^T)\,, \\
\eta_2 &= A_a^T C B_b (\bar{X}_a \gamma_5 C \bar{Y}_b^T - \bar{X}_b \gamma_5 C \bar{Y}_a^T)\,, \\
\eta_3 &= A_a^T C \gamma_5 B_b (\bar{X}_a C \bar{Y}_b^T - \bar{X}_b C \bar{Y}_a^T)\,,
\end{aligned}
$$

which are independent. We can also construct six currents by using quark-antiquark pairs:

$$
\begin{aligned}
\eta_4 &= (\bar{X}_a A_a)(\bar{Y}_b \gamma_5 B_b) + (\bar{X}_a \gamma_5 A_a)(\bar{Y}_b B_b) \\
&\quad -(\bar{X}_a B_a)(\bar{Y}_b \gamma_5 A_b) - (\bar{X}_a \gamma_5 B_a)(\bar{Y}_b A_b)\,, \\
\eta_5 &= (\bar{X}_a \gamma_\mu A_a)(\bar{Y}_b \gamma^\mu \gamma_5 B_b) + (\bar{X}_a \gamma_\mu \gamma_5 A_a)(\bar{Y}_b \gamma^\mu B_b) \\
&\quad -(\bar{X}_a \gamma_\mu B_a)(\bar{Y}_b \gamma^\mu \gamma_5 A_b) - (\bar{X}_a \gamma_\mu \gamma_5 B_a)(\bar{Y}_b \gamma^\mu A_b)\,, \\
\eta_6 &= (\bar{X}_a \sigma_{\mu\nu} A_a)(\bar{Y}_b \sigma^{\mu\nu} \gamma_5 B_b) - (\bar{X}_a \sigma_{\mu\nu} B_a)(\bar{Y}_b \sigma^{\mu\nu} \gamma_5 A_b)\,, \\
\eta_7 &= \lambda_{ab}\lambda_{cd}\{(\bar{X}_a A_b)(\bar{Y}_c \gamma_5 B_d) + (\bar{X}_a \gamma_5 A_b)(\bar{Y}_c B_d) \\
&\quad -(\bar{X}_a B_b)(\bar{Y}_c \gamma_5 A_d) - (\bar{X}_a \gamma_5 B_b)(\bar{Y}_c A_d)\}\,, \\
\eta_8 &= \lambda_{ab}\lambda_{cd}\{(\bar{X}_a \gamma_\mu A_b)(\bar{Y}_c \gamma^\mu \gamma_5 B_d) + (\bar{X}_a \gamma_\mu \gamma_5 A_b)(\bar{Y}_c \gamma^\mu B_d) \\
&\quad -(\bar{X}_a \gamma_\mu B_b)(\bar{Y}_c \gamma^\mu \gamma_5 A_d) - (\bar{X}_a \gamma_\mu \gamma_5 B_b)(\bar{Y}_c \gamma^\mu A_d)\}\,, \\
\eta_9 &= \lambda_{ab}\lambda_{cd}\{(\bar{X}_a \sigma_{\mu\nu} A_b)(\bar{Y}_c \sigma^{\mu\nu} \gamma_5 B_d) - (\bar{X}_a \sigma_{\mu\nu} B_b)(\bar{Y}_c \sigma^{\mu\nu} \gamma_5 A_d)\}\,.
\end{aligned}
$$

Among these six currents, three are independent, and we can verify following relations:

$$
\begin{aligned}
\eta_7 &= \frac{1}{3}\eta_4 + \frac{1}{2}\eta_6\,, \\
\eta_8 &= -\frac{8}{3}\eta_5\,, \\
\eta_9 &= 6\eta_4 - \frac{5}{3}\eta_6\,.
\end{aligned}
$$

The diquark construction and mesonic construction are equivalent, and they can be related to each other:

$$
\begin{aligned}
\eta_1 &= 3\eta_4 + \frac{1}{2}\eta_6\,, \\
\eta_2 &= -\frac{1}{4}\eta_4 - \frac{1}{4}\eta_5 + \frac{1}{8}\eta_6\,, \\
\eta_3 &= -\frac{1}{4}\eta_4 + \frac{1}{4}\eta_5 + \frac{1}{8}\eta_6\,.
\end{aligned}
$$

3.3.3 $\bar{\mathbf{3}}_f \otimes \bar{\mathbf{6}}_f$

In this subsection, we study the tetraquark currents where the diquark and anti-diquark components have a mixed flavor structure: $\bar{\mathbf{3}}_f$ and $\bar{\mathbf{6}}_f$, respectively. We can construct two diquark-antidiquark currents:

$$
\begin{aligned}
\eta_1 &= A_a^T C\gamma_\mu B_b(\bar{X}_a \gamma^\mu \gamma_5 C\bar{Y}_b^T + \bar{X}_b \gamma^\mu \gamma_5 C\bar{Y}_a^T)\,, \\
\eta_2 &= A_a^T C\gamma_\mu \gamma_5 B_b(\bar{X}_a \gamma^\mu C\bar{Y}_b^T - \bar{X}_b \gamma^\mu C\bar{Y}_a^T)\,,
\end{aligned}
$$

which are independent. We can also construct four currents by using quark-antiquark pairs:

$$
\begin{aligned}
\eta_3 &= (\bar{X}_a A_a)(\bar{Y}_b \gamma_5 B_b) - (\bar{X}_a \gamma_5 A_a)(\bar{Y}_b B_b) \\
&\quad -(\bar{X}_a B_a)(\bar{Y}_b \gamma_5 A_b) + (\bar{X}_a \gamma_5 B_a)(\bar{Y}_b A_b)\,, \\
\eta_4 &= (\bar{X}_a \gamma_\mu A_a)(\bar{Y}_b \gamma^\mu \gamma_5 B_b) - (\bar{X}_a \gamma_\mu \gamma_5 A_a)(\bar{Y}_b \gamma^\mu B_b) \\
&\quad -(\bar{X}_a \gamma_\mu B_a)(\bar{Y}_b \gamma^\mu \gamma_5 A_b) + (\bar{X}_a \gamma_\mu \gamma_5 B_a)(\bar{Y}_b \gamma^\mu A_b)\,, \\
\eta_5 &= \lambda_{ab}\lambda_{cd}\{(\bar{X}_a A_b)(\bar{Y}_c \gamma_5 B_d) - (\bar{X}_a \gamma_5 A_b)(\bar{Y}_c B_d) \\
&\quad -(\bar{X}_a B_b)(\bar{Y}_c \gamma_5 A_d) + (\bar{X}_a \gamma_5 B_b)(\bar{Y}_c A_d)\}\,, \\
\eta_6 &= \lambda_{ab}\lambda_{cd}\{(\bar{X}_a \gamma_\mu A_b)(\bar{Y}_c \gamma^\mu \gamma_5 B_d) - (\bar{X}_a \gamma_\mu \gamma_5 A_b)(\bar{Y}_c \gamma^\mu B_d) \\
&\quad -(\bar{X}_a \gamma_\mu B_b)(\bar{Y}_c \gamma^\mu \gamma_5 A_d) + (\bar{X}_a \gamma_\mu \gamma_5 B_b)(\bar{Y}_c \gamma^\mu A_d)\}\,.
\end{aligned}
$$

Among these four currents, two are independent, and we can verify following relations:

$$
\begin{aligned}
\eta_5 &= -\frac{2}{3}\eta_3 - \eta_4\,, \\
\eta_6 &= -4\eta_3 - \frac{2}{3}\eta_4\,.
\end{aligned}
$$

The diquark construction and mesonic construction are equivalent, and they can be related to each other:

$$
\begin{aligned}
\eta_1 &= \eta_3 - \frac{1}{2}\eta_4\,, \\
\eta_2 &= \eta_3 + \frac{1}{2}\eta_4\,.
\end{aligned}
$$

3.4 Tetraquark fields with $J^P = 1^+$

In this section, we study scalar currents of $J^P = 1^+$. The diquark and antidiquark can have flavor structures $\mathbf{6}_f \otimes \bar{\mathbf{6}}_f$, $\bar{\mathbf{3}}_f \otimes \mathbf{3}_f$, $\bar{\mathbf{3}}_f \otimes \bar{\mathbf{6}}_f$ and $\mathbf{6}_f \otimes \mathbf{3}_f$. We will just study the first three of them, since the last one have the similar structure as $\bar{\mathbf{3}}_f \otimes \bar{\mathbf{6}}_f$.

3.4.1 $\mathbf{6}_f \otimes \bar{\mathbf{6}}_f$

In this subsection, we study the tetraquark currents where both the diquark and antidiquark components have a symmetric flavor structure: $\mathbf{6}_f$ and $\bar{\mathbf{6}}_f$, respectively. We can construct four diquark-antidiquark currents:

$$
\begin{aligned}
\eta_{1\mu} &= A_a^T C B_b (\bar{X}_a \gamma_\mu \gamma_5 C \bar{Y}_b^T + \bar{X}_b \gamma_\mu \gamma_5 C \bar{Y}_a^T)\,, \\
\eta_{2\mu} &= A_a^T C \gamma_\mu \gamma_5 B_b (\bar{X}_a C \bar{Y}_b^T + \bar{X}_b C \bar{Y}_a^T)\,, \\
\eta_{3\mu} &= A_a^T C \gamma^\nu B_b (\bar{X}_a \sigma_{\mu\nu} \gamma_5 C \bar{Y}_b^T - \bar{X}_b \sigma_{\mu\nu} \gamma_5 C \bar{Y}_a^T)\,, \\
\eta_{4\mu} &= A_a^T C \sigma_{\mu\nu} \gamma_5 B_b (\bar{X}_a \gamma^\nu C \bar{Y}_b^T - \bar{X}_b \gamma^\nu C \bar{Y}_a^T)\,,
\end{aligned}
$$

which are independent. We can also construct ten currents by using quark-antiquark pairs:

$$
\begin{aligned}
\eta_{5\mu} &= (\bar{X}_a A_a)(\bar{Y}_b \gamma_\mu \gamma_5 B_b) + (\bar{X}_a \gamma_\mu \gamma_5 A_a)(\bar{Y}_b B_b) \\
&\quad + (\bar{X}_a B_a)(\bar{Y}_b \gamma_\mu \gamma_5 A_b) + (\bar{X}_a \gamma_\mu \gamma_5 B_a)(\bar{Y}_b A_b) \,, \\
\eta_{6\mu} &= (\bar{X}_a \gamma_\mu A_a)(\bar{Y}_b \gamma_5 B_b) + (\bar{X}_a \gamma_5 A_a)(\bar{Y}_b \gamma_\mu B_b) \\
&\quad + (\bar{X}_a \gamma_\mu B_a)(\bar{Y}_b \gamma_5 A_b) + (\bar{X}_a , \gamma_5 B_a)(\bar{Y}_b \gamma_\mu A_b) \\
\eta_{7\mu} &= (\bar{X}_a \gamma^\nu \gamma_5 A_a)(\bar{Y}_b \sigma_{\mu\nu} B_b) + (\bar{X}_a \sigma_{\mu\nu} A_a)(\bar{Y}_b \gamma^\nu \gamma_5 B_b) \\
&\quad + (\bar{X}_a \gamma^\nu \gamma_5 B_a)(\bar{Y}_b \sigma_{\mu\nu} A_b) + (\bar{X}_a \sigma_{\mu\nu} B_a)(\bar{Y}_b \gamma^\nu \gamma_5 A_b) \,, \\
\eta_{8\mu} &= (\bar{X}_a \gamma^\nu A_a)(\bar{Y}_b \sigma_{\mu\nu} \gamma_5 B_b) + (\bar{X}_a \sigma_{\mu\nu} \gamma_5 A_a)(\bar{Y}_b \gamma^\nu B_b) \\
&\quad + (\bar{X}_a \gamma^\nu B_a)(\bar{Y}_b \sigma_{\mu\nu} \gamma_5 A_b) + (\bar{X}_a \sigma_{\mu\nu} \gamma_5 B_a)(\bar{Y}_b \gamma^\nu A_b) \,, \\
\eta_{9\mu} &= \lambda_{ab} \lambda_{cd} \{ (\bar{X}_a A_b)(\bar{Y}_c \gamma_\mu \gamma_5 B_d) + (\bar{X}_a \gamma_\mu \gamma_5 A_b)(\bar{Y}_c B_d) \\
&\quad + (\bar{X}_a B_b)(\bar{Y}_c \gamma_\mu \gamma_5 A_d) + (\bar{X}_a \gamma_\mu \gamma_5 B_b)(\bar{Y}_c A_d) \} \,, \\
\eta_{10\mu} &= \lambda_{ab} \lambda_{cd} \{ (\bar{X}_a \gamma_\mu A_b)(\bar{Y}_c \gamma_5 B_d) + (\bar{X}_a \gamma_5 A_b)(\bar{Y}_c \gamma_\mu B_d) \\
&\quad + (\bar{X}_a \gamma_\mu B_b)(\bar{Y}_c \gamma_5 A_d) + (\bar{X}_a \gamma_5 B_b)(\bar{Y}_c \gamma_\mu A_d) \} \,, \\
\eta_{11\mu} &= \lambda_{ab} \lambda_{cd} \{ (\bar{X}_a \gamma^\nu \gamma_5 A_b)(\bar{Y}_c \sigma_{\mu\nu} B_d) + (\bar{X}_a \sigma_{\mu\nu} A_b)(\bar{Y}_c \gamma^\nu \gamma_5 B_d) \\
&\quad + (\bar{X}_a \gamma^\nu \gamma_5 B_b)(\bar{Y}_c \sigma_{\mu\nu} A_d) + (\bar{X}_a \sigma_{\mu\nu} B_b)(\bar{Y}_c \gamma^\nu \gamma_5 A_d) \} \,, \\
\eta_{12\mu} &= \lambda_{ab} \lambda_{cd} \{ (\bar{X}_a \gamma^\nu A_b)(\bar{Y}_c \sigma_{\mu\nu} \gamma_5 B_d) + (\bar{X}_a \sigma_{\mu\nu} \gamma_5 A_b)(\bar{Y}_c \gamma^\nu B_d) \\
&\quad + (\bar{X}_a \gamma^\nu B_b)(\bar{Y}_c \sigma_{\mu\nu} \gamma_5 A_d) + (\bar{X}_a \sigma_{\mu\nu} \gamma_5 B_b)(\bar{Y}_c \gamma^\nu A_d) \} \,.
\end{aligned}
$$

Among these eight currents, four are independent, and we can verify following relations:

$$
\eta_{9\mu} = -\frac{5}{3}\eta_{5\mu} - i\eta_{8\mu} \,,
$$

$$
\eta_{10\mu} = -\frac{5}{3}\eta_{6\mu} - i\eta_{7\mu} \,,
$$

$$
\eta_{11\mu} = 3i\eta_{6\mu} + \frac{1}{3}\eta_{7\mu} \,,
$$

$$
\eta_{12\mu} = 3i\eta_{5\mu} + \frac{1}{3}\eta_{8\mu} \,.
$$

The diquark construction and mesonic construction are equivalent, and they can be related to each other:

$$
\eta_{1\mu} = -\frac{1}{4}\eta_{5\mu} - \frac{1}{4}\eta_{6\mu} + \frac{i}{4}\eta_{7\mu} + \frac{i}{4}\eta_{8\mu} \,,
$$

$$
\eta_{2\mu} = -\frac{1}{4}\eta_{5\mu} + \frac{1}{4}\eta_{6\mu} - \frac{i}{4}\eta_{7\mu} + \frac{i}{4}\eta_{8\mu} \,,
$$

$$
\eta_{3\mu} = -\frac{3i}{4}\eta_{5\mu} + \frac{3i}{4}\eta_{6\mu} - \frac{1}{4}\eta_{7\mu} + \frac{1}{4}\eta_{8\mu} \,,
$$

$$
\eta_{4\mu} = -\frac{3i}{4}\eta_{5\mu} - \frac{3i}{4}\eta_{6\mu} + \frac{1}{4}\eta_{7\mu} + \frac{1}{4}\eta_{8\mu} \,.
$$

3.4.2 $\bar{3}_f \otimes 3_f$

In this subsection, we study the tetraquark currents where both the diquark and anti-diquark components have a symmetric flavor structure: $\bar{3}_f$ and 3_f, respectively. We can construct four diquark-antidiquark currents:

$$
\begin{aligned}
\eta_{1\mu} &= A_a^T C \gamma^\nu B_b (\bar{X}_a \sigma_{\mu\nu} \gamma_5 C \bar{Y}_b^T + \bar{X}_b \sigma_{\mu\nu} \gamma_5 C \bar{Y}_a^T) \,, \\
\eta_{2\mu} &= A_a^T C \sigma_{\mu\nu} \gamma_5 B_b (\bar{X}_a \gamma^\nu C \bar{Y}_b^T + \bar{X}_b \gamma^\nu C \bar{Y}_a^T) \,, \\
\eta_{3\mu} &= A_a^T C B_b (\bar{X}_a \gamma_\mu \gamma_5 C \bar{Y}_b^T - \bar{X}_b \gamma_\mu \gamma_5 C \bar{Y}_a^T) \,, \\
\eta_{4\mu} &= A_a^T C \gamma_\mu \gamma_5 B_b (\bar{X}_a C \bar{Y}_b^T - \bar{X}_b C \bar{Y}_a^T) \,,
\end{aligned}
$$

which are independent. We can also construct ten currents by using quark-antiquark pairs:

$$
\begin{aligned}
\eta_{5\mu} &= (\bar{X}_a A_a)(\bar{Y}_b \gamma_\mu \gamma_5 B_b) + (\bar{X}_a \gamma_\mu \gamma_5 A_a)(\bar{Y}_b B_b) \\
&\quad -(\bar{X}_a B_a)(\bar{Y}_b \gamma_\mu \gamma_5 A_b) - (\bar{X}_a \gamma_\mu \gamma_5 B_a)(\bar{Y}_b A_b) \,, \\
\eta_{6\mu} &= (\bar{X}_a \gamma_\mu A_a)(\bar{Y}_b \gamma_5 B_b) + (\bar{X}_a \gamma_5 A_a)(\bar{Y}_b \gamma_\mu B_b) \\
&\quad -(\bar{X}_a \gamma_\mu B_a)(\bar{Y}_b \gamma_5 A_b) - (\bar{X}_a \gamma_5 B_a)(\bar{Y}_b \gamma_\mu A_b) \,, \\
\eta_{7\mu} &= (\bar{X}_a \gamma^\nu \gamma_5 A_a)(\bar{Y}_b \sigma_{\mu\nu} B_b) + (\bar{X}_a \sigma_{\mu\nu} A_a)(\bar{Y}_b \gamma^\nu \gamma_5 B_b) \\
&\quad -(\bar{X}_a \gamma^\nu \gamma_5 B_a)(\bar{Y}_b \sigma_{\mu\nu} A_b) - (\bar{X}_a \sigma_{\mu\nu} B_a)(\bar{Y}_b \gamma^\nu \gamma_5 A_b) \,, \\
\eta_{8\mu} &= (\bar{X}_a \gamma^\nu A_a)(\bar{Y}_b \sigma_{\mu\nu} \gamma_5 B_b) + (\bar{X}_a \sigma_{\mu\nu} \gamma_5 A_a)(\bar{Y}_b \gamma^\nu B_b) \\
&\quad -(\bar{X}_a \gamma^\nu B_a)(\bar{Y}_b \sigma_{\mu\nu} \gamma_5 A_b) - (\bar{X}_a \sigma_{\mu\nu} \gamma_5 B_a)(\bar{Y}_b \gamma^\nu A_b) \,, \\
\eta_{9\mu} &= \lambda_{ab} \lambda_{cd} \{ (\bar{X}_a A_b)(\bar{Y}_c \gamma_\mu \gamma_5 B_d) + (\bar{X}_a \gamma_\mu \gamma_5 A_b)(\bar{Y}_c B_d) \\
&\quad -(\bar{X}_a B_b)(\bar{Y}_c \gamma_\mu \gamma_5 A_d) - (\bar{X}_a \gamma_\mu \gamma_5 B_b)(\bar{Y}_c A_d) \} \,, \\
\eta_{10\mu} &= \lambda_{ab} \lambda_{cd} \{ (\bar{X}_a \gamma_\mu A_b)(\bar{Y}_c \gamma_5 B_d) + (\bar{X}_a \gamma_5 A_b)(\bar{Y}_c \gamma_\mu B_d) \\
&\quad -(\bar{X}_a \gamma_\mu B_b)(\bar{Y}_c \gamma_5 A_d) - (\bar{X}_a \gamma_5 B_b)(\bar{Y}_c \gamma_\mu A_d) \} \,, \\
\eta_{11\mu} &= \lambda_{ab} \lambda_{cd} \{ (\bar{X}_a \gamma^\nu \gamma_5 A_b)(\bar{Y}_c \sigma_{\mu\nu} B_d) + (\bar{X}_a \sigma_{\mu\nu} A_b)(\bar{Y}_c \gamma^\nu \gamma_5 B_d) \\
&\quad -(\bar{X}_a \gamma^\nu \gamma_5 B_b)(\bar{Y}_c \sigma_{\mu\nu} A_d) - (\bar{X}_a \sigma_{\mu\nu} B_b)(\bar{Y}_c \gamma^\nu \gamma_5 A_d) \} \,, \\
\eta_{12\mu} &= \lambda_{ab} \lambda_{cd} \{ (\bar{X}_a \gamma^\nu A_b)(\bar{Y}_c \sigma_{\mu\nu} \gamma_5 B_d) + (\bar{X}_a \sigma_{\mu\nu} \gamma_5 A_b)(\bar{Y}_c \gamma^\nu B_d) \\
&\quad -(\bar{X}_a \gamma^\nu B_b)(\bar{Y}_c \sigma_{\mu\nu} \gamma_5 A_d) - (\bar{X}_a \sigma_{\mu\nu} \gamma_5 B_b)(\bar{Y}_c \gamma^\nu A_d) \} \,.
\end{aligned}
$$

Among these eight currents, four are independent, and we can verify following relations:

$$
\begin{aligned}
\eta_{9\mu} &= \frac{1}{3} \eta_{5\mu} + i \eta_{8\mu} \,, \\
\eta_{10\mu} &= \frac{1}{3} \eta_{6\mu} + i \eta_{7\mu} \,, \\
\eta_{11\mu} &= -3i \eta_{6\mu} - \frac{5}{3} \eta_{7\mu} \,, \\
\eta_{12\mu} &= -3i \eta_{5\mu} - \frac{5}{3} \eta_{8\mu} \,.
\end{aligned}
$$

The diquark construction and mesonic construction are equivalent, and they can be related to each other:

$$\eta_{1\mu} = -\frac{3i}{4}\eta_{5\mu} + \frac{3i}{4}\eta_{6\mu} - \frac{1}{4}\eta_{7\mu} + \frac{1}{4}\eta_{8\mu}\,,$$

$$\eta_{2\mu} = -\frac{3i}{4}\eta_{5\mu} - \frac{3i}{4}\eta_{6\mu} + \frac{1}{4}\eta_{7\mu} + \frac{1}{4}\eta_{8\mu}\,,$$

$$\eta_{3\mu} = -\frac{1}{4}\eta_{5\mu} - \frac{1}{4}\eta_{6\mu} + \frac{i}{4}\eta_{7\mu} + \frac{i}{4}\eta_{8\mu}\,,$$

$$\eta_{4\mu} = -\frac{1}{4}\eta_{5\mu} + \frac{1}{4}\eta_{6\mu} - \frac{i}{4}\eta_{7\mu} + \frac{i}{4}\eta_{8\mu}\,.$$

3.4.3 $\bar{3}_f \otimes \bar{6}_f$

In this subsection, we study the tetraquark currents where the diquark and anti-diquark components have a mixed flavor structure: $\bar{3}_f$ and $\bar{6}_f$, respectively. We can construct four diquark-antidiquark currents:

$$\eta_{1\mu} = A_a^T C\gamma_\mu B_b(\bar{X}_a\gamma_5 C\bar{Y}_b^T + \bar{X}_b\gamma_5 C\bar{Y}_a^T)\,,$$

$$\eta_{2\mu} = A_a^T C\sigma_{\mu\nu} B_b(\bar{X}_a\gamma^\nu\gamma_5 C\bar{Y}_b^T + \bar{X}_b\gamma^\nu\gamma_5 C\bar{Y}_a^T)\,,$$

$$\eta_{3\mu} = A_a^T C\gamma_5 B_b(\bar{X}_a\gamma_\mu C\bar{Y}_b^T - \bar{X}_b\gamma_\mu C\bar{Y}_a^T)\,,$$

$$\eta_{4\mu} = A_a^T C\gamma^\nu\gamma_5 B_b(\bar{X}_a\sigma_{\mu\nu} C\bar{Y}_b^T - \bar{X}_b\sigma_{\mu\nu} C\bar{Y}_a^T)\,,$$

which are independent. We can also construct eight currents by using quark-antiquark pairs:

$$\eta_{5\mu} = (\bar{X}_a A_a)(\bar{Y}_b\gamma_\mu\gamma_5 B_b) - (\bar{X}_a\gamma_\mu\gamma_5 A_a)(\bar{Y}_b B_b)$$
$$-(\bar{X}_a B_a)(\bar{Y}_b\gamma_\mu\gamma_5 A_b) + (\bar{X}_a\gamma_\mu\gamma_5 B_a)(\bar{Y}_b A_b)\,,$$

$$\eta_{6\mu} = (\bar{X}_a\gamma_\mu A_a)(\bar{Y}_b\gamma_5 B_b) - (\bar{X}_a\gamma_5 A_a)(\bar{Y}_b\gamma_\mu B_b)$$
$$-(\bar{X}_a\gamma_\mu B_a)(\bar{Y}_b\gamma_5 A_b) + (\bar{X}_a\gamma_5 B_a)(\bar{Y}_b\gamma_\mu A_b)\,,$$

$$\eta_{7\mu} = (\bar{X}_a\gamma^\nu\gamma_5 A_a)(\bar{Y}_b\sigma_{\mu\nu} B_b) - (\bar{X}_a\sigma_{\mu\nu} A_a)(\bar{Y}_b\gamma^\nu\gamma_5 B_b)$$
$$-(\bar{X}_a\gamma^\nu\gamma_5 B_a)(\bar{Y}_b\sigma_{\mu\nu} A_b) + (\bar{X}_a\sigma_{\mu\nu} B_a)(\bar{Y}_b\gamma^\nu\gamma_5 A_b)\,,$$

$$\eta_{8\mu} = (\bar{X}_a\gamma^\nu A_a)(\bar{Y}_b\sigma_{\mu\nu}\gamma_5 B_b) - (\bar{X}_a\sigma_{\mu\nu}\gamma_5 A_a)(\bar{Y}_b\gamma^\nu B_b)$$
$$-(\bar{X}_a\gamma^\nu B_a)(\bar{Y}_b\sigma_{\mu\nu}\gamma_5 A_b) + (\bar{X}_a\sigma_{\mu\nu}\gamma_5 B_a)(\bar{Y}_b\gamma^\nu A_b)\,,$$

$$\eta_{9\mu} = \lambda_{ab}\lambda_{cd}\{(\bar{X}_a A_b)(\bar{Y}_c\gamma_\mu\gamma_5 B_d) - (\bar{X}_a\gamma_\mu\gamma_5 A_b)(\bar{Y}_c B_d)$$
$$-(\bar{X}_a B_b)(\bar{Y}_c\gamma_\mu\gamma_5 A_d)(\bar{X}_a\gamma_\mu\gamma_5 B_b)(\bar{Y}_c A_d)\}\,,$$

$$\eta_{10\mu} = \lambda_{ab}\lambda_{cd}\{(\bar{X}_a\gamma_\mu A_b)(\bar{Y}_c\gamma_5 B_d) - (\bar{X}_a\gamma_5 A_b)(\bar{Y}_c\gamma_\mu B_d)$$
$$-(\bar{X}_a\gamma_\mu B_b)(\bar{Y}_c\gamma_5 A_d) + (\bar{X}_a\gamma_5 B_b)(\bar{Y}_c\gamma_\mu A_d)\}\,,$$

$$\eta_{11\mu} = \lambda_{ab}\lambda_{cd}\{(\bar{X}_a\gamma^\nu\gamma_5 A_b)(\bar{Y}_c\sigma_{\mu\nu} B_d) - (\bar{X}_a\sigma_{\mu\nu} A_b)(\bar{Y}_c\gamma^\nu\gamma_5 B_d)$$

$$-(\bar{X}_a\gamma^\nu\gamma_5 B_b)(\bar{Y}_c\sigma_{\mu\nu}A_d) + (\bar{X}_a\sigma_{\mu\nu}B_b)(\bar{Y}_c\gamma^\nu\gamma_5 A_d)\} \, ,$$
$$\eta_{12\mu} = \lambda_{ab}\lambda_{cd}\{(\bar{X}_a\gamma^\nu A_b)(\bar{Y}_c\sigma_{\mu\nu}\gamma_5 B_d) - (\bar{X}_a\sigma_{\mu\nu}\gamma_5 A_b)(\bar{Y}_c\gamma^\nu B_d)$$
$$-(\bar{X}_a\gamma^\nu B_b)(\bar{Y}_c\sigma_{\mu\nu}\gamma_5 A_d) + (\bar{X}_a\sigma_{\mu\nu}\gamma_5 B_b)(\bar{Y}_c\gamma^\nu A_d)\} \, .$$

Among these eight currents, four are independent, and we can verify following relations:

$$\eta_{9\mu} = -\frac{2}{3}\eta_{5\mu} - \eta_{6\mu} - i\eta_{7\mu} \, ,$$

$$\eta_{10\mu} = -\eta_{5\mu} - \frac{2}{3}\eta_{6\mu} + i\eta_{8\mu} \, ,$$

$$\eta_{11\mu} = 3i\eta_{5\mu} - \frac{2}{3}\eta_{7\mu} - \eta_{8\mu} \, ,$$

$$\eta_{12\mu} = -3i\eta_{6\mu} - \eta_{7\mu} - \frac{2}{3}\eta_{8\mu} \, .$$

The diquark construction and mesonic construction are equivalent, and they can be related to each other:

$$\eta_{1\mu} = \frac{1}{4}\eta_{5\mu} + \frac{1}{4}\eta_{6\mu} - \frac{i}{4}\eta_{7\mu} + \frac{i}{4}\eta_{8\mu} \, ,$$

$$\eta_{2\mu} = \frac{3i}{4}\eta_{5\mu} - \frac{3i}{4}\eta_{6\mu} + \frac{1}{4}\eta_{7\mu} + \frac{1}{4}\eta_{8\mu} \, ,$$

$$\eta_{3\mu} = \frac{1}{4}\eta_{5\mu} - \frac{1}{4}\eta_{6\mu} + \frac{i}{4}\eta_{7\mu} + \frac{i}{4}\eta_{8\mu} \, ,$$

$$\eta_{4\mu} = \frac{3i}{4}\eta_{5\mu} + \frac{3i}{4}\eta_{6\mu} - \frac{1}{4}\eta_{7\mu} + \frac{1}{4}\eta_{8\mu} \, .$$

3.5 Tetraquark fields with $J^P = 1^-$

In this section, we study scalar currents of $J^P = 1^-$. The diquark and antidiquark can have flavor structures $\mathbf{6}_f \otimes \bar{\mathbf{6}}_f$, $\bar{\mathbf{3}}_f \otimes \mathbf{3}_f$, $\bar{\mathbf{3}}_f \otimes \bar{\mathbf{6}}_f$ and $\mathbf{6}_f \otimes \mathbf{3}_f$. We will just study the first three of them, since the last one have the similar structure as $\bar{\mathbf{3}}_f \otimes \bar{\mathbf{6}}_f$.

3.5.1 $\mathbf{6}_f \otimes \bar{\mathbf{6}}_f$

In this subsection, we study the tetraquark currents where both the diquark and antidiquark components have a symmetric flavor structure: $\mathbf{6}_f$ and $\bar{\mathbf{6}}_f$, respectively. We can construct four diquark-antidiquark currents:

$$\eta_{1\mu} = A_a^T C\gamma_5 B_b(\bar{X}_a\gamma_\mu\gamma_5 C\bar{Y}_b^T + \bar{X}_b\gamma_\mu\gamma_5 C\bar{Y}_a^T) \, ,$$
$$\eta_{2\mu} = A_a^T C\gamma_\mu\gamma_5 B_b(\bar{X}_a\gamma_5 C\bar{Y}_b^T + \bar{X}_b\gamma_5 C\bar{Y}_a^T) \, ,$$
$$\eta_{3\mu} = A_a^T C\gamma^\nu B_b(\bar{X}_a\sigma_{\mu\nu}C\bar{Y}_b^T - \bar{X}_b\sigma_{\mu\nu}C\bar{Y}_a^T) \, ,$$
$$\eta_{4\mu} = A_a^T C\sigma_{\mu\nu}B_b(\bar{X}_a\gamma^\nu C\bar{Y}_b^T - \bar{X}_b\gamma^\nu C\bar{Y}_a^T) \, ,$$

which are independent. We can also construct eight currents by using quark-antiquark pairs:

$$
\begin{aligned}
\eta_{5\mu} &= (\bar{X}_a A_a)(\bar{Y}_b \gamma_\mu B_b) + (\bar{X}_a \gamma_\mu A_a)(\bar{Y}_b B_b) \\
&\quad + (\bar{X}_a B_a)(\bar{Y}_b \gamma_\mu A_b) + (\bar{X}_a \gamma_\mu B_a)(\bar{Y}_b A_b)\,, \\
\eta_{6\mu} &= (\bar{X}_a \gamma_\mu \gamma_5 A_a)(\bar{Y}_b \gamma_5 B_b) + (\bar{X}_a \gamma_5 A_a)(\bar{Y}_b \gamma_\mu \gamma_5 B_b) \\
&\quad + (\bar{X}_a \gamma_\mu \gamma_5 B_a)(\bar{Y}_b \gamma_5 A_b) + (\bar{X}_a \gamma_5 B_a)(\bar{Y}_b \gamma_\mu \gamma_5 A_b)\,, \\
\eta_{7\mu} &= (\bar{X}_a \gamma^\nu A_a)(\bar{Y}_b \sigma_{\mu\nu} B_b) + (\bar{X}_a \sigma_{\mu\nu} A_a)(\bar{Y}_b \gamma^\nu B_b) \\
&\quad + (\bar{X}_a \gamma^\nu B_a)(\bar{Y}_b \sigma_{\mu\nu} A_b) + (\bar{X}_a \sigma_{\mu\nu} B_a)(\bar{Y}_b \gamma^\nu A_b)\,, \\
\eta_{8\mu} &= (\bar{X}_a \gamma^\nu \gamma_5 A_a)(\bar{Y}_b \sigma_{\mu\nu} \gamma_5 B_b) + (\bar{X}_a \sigma_{\mu\nu} \gamma_5 A_a)(\bar{Y}_b \gamma^\nu \gamma_5 B_b) \\
&\quad + (\bar{X}_a \gamma^\nu \gamma_5 B_a)(\bar{Y}_b \sigma_{\mu\nu} \gamma_5 A_b) + (\bar{X}_a \sigma_{\mu\nu} \gamma_5 B_a)(\bar{Y}_b \gamma^\nu \gamma_5 A_b)\,, \\
\eta_{9\mu} &= \lambda_{ab}\lambda_{cd}\{(\bar{X}_a A_b)(\bar{Y}_c \gamma_\mu B_d) + (\bar{X}_a \gamma_\mu A_b)(\bar{Y}_c B_d) \\
&\quad + (\bar{X}_a B_b)(\bar{Y}_c \gamma_\mu A_d) + (\bar{X}_a \gamma_\mu B_b)(\bar{Y}_c A_d)\}\,, \\
\eta_{10\mu} &= \lambda_{ab}\lambda_{cd}\{(\bar{X}_a \gamma_\mu \gamma_5 A_b)(\bar{Y}_c \gamma_5 B_d) + (\bar{X}_a \gamma_5 A_b)(\bar{Y}_c \gamma_\mu \gamma_5 B_d) \\
&\quad + (\bar{X}_a \gamma_\mu \gamma_5 B_b)(\bar{Y}_c \gamma_5 A_d) + (\bar{X}_a \gamma_5 B_b)(\bar{Y}_c \gamma_\mu \gamma_5 A_d)\}\,, \\
\eta_{11\mu} &= \lambda_{ab}\lambda_{cd}\{(\bar{X}_a \gamma^\nu A_b)(\bar{Y}_c \sigma_{\mu\nu} B_d) + (\bar{X}_a \sigma_{\mu\nu} A_b)(\bar{Y}_c \gamma^\nu B_d) \\
&\quad + (\bar{X}_a \gamma^\nu B_b)(\bar{Y}_c \sigma_{\mu\nu} A_d) + (\bar{X}_a \sigma_{\mu\nu} B_b)(\bar{Y}_c \gamma^\nu A_d)\}\,, \\
\eta_{12\mu} &= \lambda_{ab}\lambda_{cd}\{(\bar{X}_a \gamma^\nu \gamma_5 A_b)(\bar{Y}_c \sigma_{\mu\nu} \gamma_5 B_d) + (\bar{X}_a \sigma_{\mu\nu} \gamma_5 A_b)(\bar{Y}_c \gamma^\nu \gamma_5 B_d) \\
&\quad + (\bar{X}_a \gamma^\nu \gamma_5 B_b)(\bar{Y}_c \sigma_{\mu\nu} \gamma_5 A_d) + (\bar{X}_a \sigma_{\mu\nu} \gamma_5 B_b)(\bar{Y}_c \gamma^\nu \gamma_5 A_d)\}\,.
\end{aligned}
$$

Among these eight currents, four are independent, and we can verify following relations:

$$
\eta_{9\mu} = -\frac{5}{3}\eta_{5\mu} - i\eta_{8\mu}\,,
$$

$$
\eta_{10\mu} = -\frac{5}{3}\eta_{6\mu} - i\eta_{7\mu}\,,
$$

$$
\eta_{11\mu} = 3i\eta_{6\mu} + \frac{1}{3}\eta_{7\mu}\,,
$$

$$
\eta_{12\mu} = 3i\eta_{5\mu} + \frac{1}{3}\eta_{8\mu}\,.
$$

The diquark construction and mesonic construction are equivalent, and they can be related to each other:

$$
\eta_{1\mu} = -\frac{1}{4}\eta_{5\mu} - \frac{1}{4}\eta_{6\mu} + \frac{i}{4}\eta_{7\mu} + \frac{i}{4}\eta_{8\mu}\,,
$$

$$
\eta_{2\mu} = \frac{1}{4}\eta_{5\mu} - \frac{1}{4}\eta_{6\mu} + \frac{i}{4}\eta_{7\mu} - \frac{i}{4}\eta_{8\mu}\,,
$$

$$
\eta_{3\mu} = \frac{3i}{4}\eta_{5\mu} - \frac{3i}{4}\eta_{6\mu} + \frac{1}{4}\eta_{7\mu} - \frac{1}{4}\eta_{8\mu}\,,
$$

$$
\eta_{4\mu} = -\frac{3i}{4}\eta_{5\mu} - \frac{3i}{4}\eta_{6\mu} + \frac{1}{4}\eta_{7\mu} + \frac{1}{4}\eta_{8\mu}\,.
$$

3.5.2 $\bar{3}_f \otimes 3_f$

In this subsection, we study the tetraquark currents where both the diquark and anti-diquark components have a symmetric flavor structure: $\bar{3}_f$ and 3_f, respectively. We can construct four diquark-antidiquark currents:

$$
\begin{aligned}
\eta_{1\mu} &= A_a^T C \gamma^\nu B_b (\bar{X}_a \sigma_{\mu\nu} C \bar{Y}_b^T + \bar{X}_b \sigma_{\mu\nu} C \bar{Y}_a^T)\,, \\
\eta_{2\mu} &= A_a^T C \sigma_{\mu\nu} B_b (\bar{X}_a \gamma^\nu C \bar{Y}_b^T + \bar{X}_b \gamma^\nu C \bar{Y}_a^T)\,, \\
\eta_{3\mu} &= A_a^T C \gamma_5 B_b (\bar{X}_a \gamma_\mu \gamma_5 C \bar{Y}_b^T - \bar{X}_b \gamma_\mu \gamma_5 C \bar{Y}_a^T)\,, \\
\eta_{4\mu} &= A_a^T C \gamma_\mu \gamma_5 B_b (\bar{X}_a \gamma_5 C \bar{Y}_b^T - \bar{X}_b \gamma_5 C \bar{Y}_a^T)\,,
\end{aligned}
$$

which are independent. We can also construct eight currents by using quark-antiquark pairs:

$$
\begin{aligned}
\eta_{5\mu} &= (\bar{X}_a A_a)(\bar{Y}_b \gamma_\mu B_b) + (\bar{X}_a \gamma_\mu A_a)(\bar{Y}_b B_b) \\
&\quad - (\bar{X}_a B_a)(\bar{Y}_b \gamma_\mu A_b) - (\bar{X}_a \gamma_\mu B_a)(\bar{Y}_b A_b)\,, \\
\eta_{6\mu} &= (\bar{X}_a \gamma_\mu \gamma_5 A_a)(\bar{Y}_b \gamma_5 B_b) + (\bar{X}_a \gamma_5 A_a)(\bar{Y}_b \gamma_\mu \gamma_5 B_b) \\
&\quad - (\bar{X}_a \gamma_\mu \gamma_5 B_a)(\bar{Y}_b \gamma_5 A_b) - (\bar{X}_a \gamma_5 B_a)(\bar{Y}_b \gamma_\mu \gamma_5 A_b)\,, \\
\eta_{7\mu} &= (\bar{X}_a \gamma^\nu A_a)(\bar{Y}_b \sigma_{\mu\nu} B_b) + (\bar{X}_a \sigma_{\mu\nu} A_a)(\bar{Y}_b \gamma^\nu B_b) \\
&\quad - (\bar{X}_a \gamma^\nu B_a)(\bar{Y}_b \sigma_{\mu\nu} A_b) - (\bar{X}_a \sigma_{\mu\nu} B_a)(\bar{Y}_b \gamma^\nu A_b)\,, \\
\eta_{8\mu} &= (\bar{X}_a \gamma^\nu \gamma_5 A_a)(\bar{Y}_b \sigma_{\mu\nu} \gamma_5 B_b) + (\bar{X}_a \sigma_{\mu\nu} \gamma_5 A_a)(\bar{Y}_b \gamma^\nu \gamma_5 B_b) \\
&\quad - (\bar{X}_a \gamma^\nu \gamma_5 B_a)(\bar{Y}_b \sigma_{\mu\nu} \gamma_5 A_b) - (\bar{X}_a \sigma_{\mu\nu} \gamma_5 B_a)(\bar{Y}_b \gamma^\nu \gamma_5 A_b)\,, \\
\eta_{9\mu} &= \lambda_{ab} \lambda_{cd} \{ (\bar{X}_a A_b)(\bar{Y}_c \gamma_\mu B_d) + (\bar{X}_a \gamma_\mu A_b)(\bar{Y}_c B_d) \\
&\quad - (\bar{X}_a B_b)(\bar{Y}_c \gamma_\mu A_d) - (\bar{X}_a \gamma_\mu B_b)(\bar{Y}_c A_d) \}\,, \\
\eta_{10\mu} &= \lambda_{ab} \lambda_{cd} \{ (\bar{X}_a \gamma_\mu \gamma_5 A_b)(\bar{Y}_c \gamma_5 B_d) + (\bar{X}_a \gamma_5 A_b)(\bar{Y}_c \gamma_\mu \gamma_5 B_d) \\
&\quad - (\bar{X}_a \gamma_\mu \gamma_5 B_b)(\bar{Y}_c \gamma_5 A_d) - (\bar{X}_a \gamma_5 B_b)(\bar{Y}_c \gamma_\mu \gamma_5 A_d) \}\,, \\
\eta_{11\mu} &= \lambda_{ab} \lambda_{cd} \{ (\bar{X}_a \gamma^\nu A_b)(\bar{Y}_c \sigma_{\mu\nu} B_d) + (\bar{X}_a \sigma_{\mu\nu} A_b)(\bar{Y}_c \gamma^\nu B_d) \\
&\quad - (\bar{X}_a \gamma^\nu B_b)(\bar{Y}_c \sigma_{\mu\nu} A_d) - (\bar{X}_a \sigma_{\mu\nu} B_b)(\bar{Y}_c \gamma^\nu A_d) \}\,, \\
\eta_{12\mu} &= \lambda_{ab} \lambda_{cd} \{ (\bar{X}_a \gamma^\nu \gamma_5 A_b)(\bar{Y}_c \sigma_{\mu\nu} \gamma_5 B_d) + (\bar{X}_a \sigma_{\mu\nu} \gamma_5 A_b)(\bar{Y}_c \gamma^\nu \gamma_5 B_d) \\
&\quad - (\bar{X}_a \gamma^\nu \gamma_5 B_b)(\bar{Y}_c \sigma_{\mu\nu} \gamma_5 A_d) - (\bar{X}_a \sigma_{\mu\nu} \gamma_5 B_b)(\bar{Y}_c \gamma^\nu \gamma_5 A_d) \}\,.
\end{aligned}
$$

Among these eight currents, four are independent, and we can verify following relations:

$$
\begin{aligned}
\eta_{9\mu} &= \frac{1}{3} \eta_{5\mu} + i \eta_{8\mu}\,, \\
\eta_{10\mu} &= \frac{1}{3} \eta_{6\mu} + i \eta_{7\mu}\,, \\
\eta_{11\mu} &= -3i \eta_{6\mu} - \frac{5}{3} \eta_{7\mu}\,, \\
\eta_{12\mu} &= -3i \eta_{5\mu} - \frac{5}{3} \eta_{8\mu}\,.
\end{aligned}
$$

The diquark construction and mesonic construction are equivalent, and they can be related to each other:

$$\eta_{1\mu} = \frac{3i}{4}\eta_{5\mu} - \frac{3i}{4}\eta_{6\mu} + \frac{1}{4}\eta_{7\mu} - \frac{1}{4}\eta_{8\mu},$$

$$\eta_{2\mu} = -\frac{3i}{4}\eta_{5\mu} - \frac{3i}{4}\eta_{6\mu} + \frac{1}{4}\eta_{7\mu} + \frac{1}{4}\eta_{8\mu},$$

$$\eta_{3\mu} = -\frac{1}{4}\eta_{5\mu} - \frac{1}{4}\eta_{6\mu} + \frac{i}{4}\eta_{7\mu} + \frac{i}{4}\eta_{8\mu},$$

$$\eta_{4\mu} = \frac{1}{4}\eta_{5\mu} - \frac{1}{4}\eta_{6\mu} + \frac{i}{4}\eta_{7\mu} - \frac{i}{4}\eta_{8\mu}.$$

3.5.3 $\bar{3}_f \otimes \bar{6}_f$

In this subsection, we study the tetraquark currents where the diquark and anti-diquark components have a mixed flavor structure: $\bar{3}_f$ and $\bar{6}_f$, respectively. We can construct four diquark-antidiquark currents:

$$\eta_{1\mu} = A_a^T C\gamma_\mu B_b(\bar{X}_a C\bar{Y}_b^T + \bar{X}_b C\bar{Y}_a^T),$$

$$\eta_{2\mu} = A_a^T C\sigma_{\mu\nu}\gamma_5 B_b(\bar{X}_a \gamma^\nu\gamma_5 C\bar{Y}_b^T + \bar{X}_b \gamma^\nu\gamma_5 C\bar{Y}_a^T),$$

$$\eta_{3\mu} = A_a^T C B_b(\bar{X}_a \gamma_\mu C\bar{Y}_b^T - \bar{X}_b \gamma_\mu C\bar{Y}_a^T),$$

$$\eta_{4\mu} = A_a^T C\gamma^\nu\gamma_5 B_b(\bar{X}_a \sigma_{\mu\nu}\gamma_5 C\bar{Y}_b^T - \bar{X}_b \sigma_{\mu\nu}\gamma_5 C\bar{Y}_a^T),$$

which are independent. We can also construct eight currents by using quark-antiquark pairs:

$$\eta_{5\mu} = (\bar{X}_a A_a)(\bar{Y}_b \gamma_\mu B_b) - (\bar{X}_a \gamma_\mu A_a)(\bar{Y}_b B_b)$$
$$-(\bar{X}_a B_a)(\bar{Y}_b \gamma_\mu A_b) + (\bar{X}_a \gamma_\mu B_a)(\bar{Y}_b A_b),$$

$$\eta_{6\mu} = (\bar{X}_a \gamma_\mu\gamma_5 A_a)(\bar{Y}_b \gamma_5 B_b) - (\bar{X}_a \gamma_5 A_a)(\bar{Y}_b \gamma_\mu\gamma_5 B_b)$$
$$-(\bar{X}_a \gamma_\mu\gamma_5 B_a)(\bar{Y}_b \gamma_5 A_b) + (\bar{X}_a \gamma_5 B_a)(\bar{Y}_b \gamma_\mu\gamma_5 A_b),$$

$$\eta_{7\mu} = (\bar{X}_a \gamma^\nu A_a)(\bar{Y}_b \sigma_{\mu\nu} B_b) - (\bar{X}_a \sigma_{\mu\nu} A_a)(\bar{Y}_b \gamma^\nu B_b)$$
$$-(\bar{X}_a \gamma^\nu B_a)(\bar{Y}_b \sigma_{\mu\nu} A_b) + (\bar{X}_a \sigma_{\mu\nu} B_a)(\bar{Y}_b \gamma^\nu A_b),$$

$$\eta_{8\mu} = (\bar{X}_a \gamma^\nu\gamma_5 A_a)(\bar{Y}_b \sigma_{\mu\nu}\gamma_5 B_b) - (\bar{X}_a \sigma_{\mu\nu}\gamma_5 A_a)(\bar{Y}_b \gamma^\nu\gamma_5 B_b)$$
$$-(\bar{X}_a \gamma^\nu\gamma_5 B_a)(\bar{Y}_b \sigma_{\mu\nu}\gamma_5 A_b) + (\bar{X}_a \sigma_{\mu\nu}\gamma_5 B_a)(\bar{Y}_b \gamma^\nu\gamma_5 A_b),$$

$$\eta_{9\mu} = \lambda_{ab}\lambda_{cd}\{(\bar{X}_a A_b)(\bar{Y}_c \gamma_\mu B_d) - (\bar{X}_a \gamma_\mu A_b)(\bar{Y}_c B_d)$$
$$-(\bar{X}_a B_b)(\bar{Y}_c \gamma_\mu A_d) + (\bar{X}_a \gamma_\mu B_b)(\bar{Y}_c A_d)\},$$

$$\eta_{10\mu} = \lambda_{ab}\lambda_{cd}\{(\bar{X}_a \gamma_\mu\gamma_5 A_b)(\bar{Y}_c \gamma_5 B_d) - (\bar{X}_a \gamma_5 A_b)(\bar{Y}_c \gamma_\mu\gamma_5 B_d)$$
$$-(\bar{X}_a \gamma_\mu\gamma_5 B_b)(\bar{Y}_c \gamma_5 A_d) + (\bar{X}_a \gamma_5 B_b)(\bar{Y}_c \gamma_\mu\gamma_5 A_d)\},$$

$$\eta_{11\mu} = \lambda_{ab}\lambda_{cd}\{(\bar{X}_a \gamma^\nu A_b)(\bar{Y}_c \sigma_{\mu\nu} B_d) - (\bar{X}_a \sigma_{\mu\nu} A_b)(\bar{Y}_c \gamma^\nu B_d)$$

$$-(\bar{X}_a\gamma^\nu B_b)(\bar{Y}_c\sigma_{\mu\nu}A_d) + (\bar{X}_a\sigma_{\mu\nu}B_b)(\bar{Y}_c\gamma^\nu A_d)\}\,,$$

$$\eta_{12\mu} = \lambda_{ab}\lambda_{cd}\{(\bar{X}_a\gamma^\nu\gamma_5 A_b)(\bar{Y}_c\sigma_{\mu\nu}\gamma_5 B_d) - (\bar{X}_a\sigma_{\mu\nu}\gamma_5 A_b)(\bar{Y}_c\gamma^\nu\gamma_5 B_d)$$
$$-(\bar{X}_a\gamma^\nu\gamma_5 B_b)(\bar{Y}_c\sigma_{\mu\nu}\gamma_5 A_d) + (\bar{X}_a\sigma_{\mu\nu}\gamma_5 B_b)(\bar{Y}_c\gamma^\nu\gamma_5 A_d)\}\,.$$

Among these eight currents, four are independent, and we can verify following relations:

$$\eta_{9\mu} = -\frac{2}{3}\eta_{5\mu} + \eta_{6\mu} - i\eta_{7\mu}\,,$$

$$\eta_{10\mu} = \eta_{5\mu} - \frac{2}{3}\eta_{6\mu} - i\eta_{8\mu}\,,$$

$$\eta_{11\mu} = 3i\eta_{5\mu} - \frac{2}{3}\eta_{7\mu} - \eta_{8\mu}\,,$$

$$\eta_{12\mu} = 3i\eta_{6\mu} - \eta_{7\mu} - \frac{2}{3}\eta_{8\mu}\,.$$

The diquark construction and mesonic construction are equivalent, and they can be related to each other:

$$\eta_{1\mu} = -\frac{1}{4}\eta_{5\mu} + \frac{1}{4}\eta_{6\mu} + \frac{i}{4}\eta_{7\mu} - \frac{i}{4}\eta_{8\mu}\,,$$

$$\eta_{2\mu} = \frac{3i}{4}\eta_{5\mu} + \frac{3i}{4}\eta_{6\mu} + \frac{1}{4}\eta_{7\mu} + \frac{1}{4}\eta_{8\mu}\,,$$

$$\eta_{3\mu} = \frac{1}{4}\eta_{5\mu} + \frac{1}{4}\eta_{6\mu} + \frac{i}{4}\eta_{7\mu} + \frac{i}{4}\eta_{8\mu}\,,$$

$$\eta_{4\mu} = -\frac{3i}{4}\eta_{5\mu} + \frac{3i}{4}\eta_{6\mu} + \frac{1}{4}\eta_{7\mu} - \frac{1}{4}\eta_{8\mu}\,.$$

3.6 Relations between $(qq)(\bar{q}\bar{q})$ and $(\bar{q}q)(\bar{q}q)$ Structures

3.6.1 General Idea

In the previous sections, we find that there are always some relations between $(qq)(\bar{q}\bar{q})$ and $(\bar{q}q)(\bar{q}q)$ currents. In this section, we will do some detailed study on these relations. The quark field used here is denoted as $q_A^a(x)$ again.

First, we consider the color and flavor structures. The interchange of both color and flavor does not need to be antisymmetric, due to the extra orbital and spin degrees of freedom. Therefore we can not use the Pauli principle such as $q_a^A q_b^B = -q_b^B q_a^A$ within the color and flavor spaces. Altogether there are four types of diquark (qq) and four types of quark-antiquark $(\bar{q}q)$. They are shown in Table 3.1, where the sum over repeated indices $(a, b, \cdots$ for color indices, $A, B, \cdots$ for flavor indices) is taken.

To construct a tetraquark by using $(qq)(\bar{q}\bar{q})$, the color structure is either

$$(\mathbf{3}\otimes\mathbf{3})\otimes(\bar{\mathbf{3}}\otimes\bar{\mathbf{3}}) \to \bar{\mathbf{3}}\otimes\mathbf{3} \to \mathbf{1}\,,$$

Table 3.1: Color and flavor structures of qq and $\bar{q}q$

(Color, Flavor)	$(\bar{3}_c, \bar{3}_f)$	$(\bar{3}_c, 6_f)$	$(6_c, \bar{3}_f)$	$(6_c, 6_f)$
Diquark (qq)	$\epsilon^{abc}\epsilon_{ABC}(q_a^A q_b^B)$	$\epsilon^{abc}(q_a^A q_b^B + q_a^B q_b^A)$	$\epsilon_{ABC}(q_a^A q_b^B + q_b^A q_a^B)$	$(q_a^A q_b^B + q_a^B q_b^A) + (a \leftrightarrow b)$

(Color, Flavor)	$(1_c, 1_f)$	$(1_c, 8_f)$	$(8_c, 1_f)$	$(8_c, 8_f)$
Quark-antiquark $(\bar{q}q)$	$(\bar{q}_a^A q_a^A)$	$\lambda^N_{AB}(\bar{q}_a^A q_a^B)$	$\lambda^{ab}_n(\bar{q}_a^A q_b^A)$	$\lambda^N_{AB}\lambda^{ab}_n(\bar{q}_a^A q_b^B)$

or

$$(3 \otimes 3) \otimes (\bar{3} \otimes \bar{3}) \to 6 \otimes \bar{6} \to 1\,;$$

the flavor structure is

$$(3 \otimes 3) \otimes (\bar{3} \otimes \bar{3}) = (\bar{3} \oplus 6) \otimes (3 \oplus \bar{6}) = 1 \oplus 8 \oplus 8 \oplus 10 \oplus 8 \oplus 10 \oplus 1 \oplus 8 \oplus 27\,.$$

To construct a tetraquark by using $(\bar{q}q)(\bar{q}q)$, the color structure is either

$$(\bar{3} \otimes 3) \otimes (\bar{3} \otimes 3) \to 1 \otimes 1 \to 1\,,$$

or

$$(\bar{3} \otimes 3) \otimes (\bar{3} \otimes 3) \to 8 \otimes \bar{8} \to 1\,,$$

with the same flavor structure as before. In Table 3.2, we show all possible color and flavor structures of tetraquark currents $T_C^{F_1(F_2)}$. Here F_1 denotes the flavor representation of tetraquark; F_2 and C show the intermediate flavor and color representations of either diquark (antidiquark) or quark-antiquark. S^{ABCD} is the totally symmetric matrix. Because we would like to make a scalar tetraquark state, the diquark and antidiquark fields should have the same color, spin and orbital symmetries. Therefore, they must have the same flavor symmetry, which is either symmetric $(6_f \otimes \bar{6}_f)$ or antisymmetric $(\bar{3}_f \otimes 3_f)$.

If the orbital and spin structure between the two quarks (two antiquarks) are symmetric, then the color-flavor structure of diquark (antidiquark) should be anti-symmetric, which means $q_a^A q_b^B = -q_b^B q_a^A$ $(\bar{q}_a^A \bar{q}_b^B = -\bar{q}_b^B \bar{q}_a^A)$. In this case, we can verify

$$T_3^{1(3)} = T_3^{8(3)} = T_3^{8(3,6)} = T_3^{10(3,6)} = T_6^{8(6,3)} = T_6^{10(6,3)} = T_6^{1(6)} = T_6^{8(6)} = T_6^{27(6)} = 0\,,$$
$$(3.16)$$

If the orbital and spin structure between two quarks (two antiquarks) are anti-symmetric, then the color-flavor structure of diquark (antidiquark) should be symmetric, which means $q_a^A q_b^B = q_b^B q_a^A$ $(\bar{q}_a^A \bar{q}_b^B = \bar{q}_b^B \bar{q}_a^A)$. Then we can verify

$$T_6^{1(3)} = T_6^{8(3)} = T_6^{8(3,6)} = T_6^{10(3,6)} = T_3^{8(6,3)} = T_3^{10(6,3)} = T_3^{1(6)} = T_3^{8(6)} = T_3^{27(6)} = 0\,.$$
$$(3.17)$$

Table 3.2: Color and flavor structures of tetraquark currents

$(qq)(\bar{q}\bar{q})$	$(\mathbf{3}\otimes\mathbf{3})\otimes(\bar{\mathbf{3}}\otimes\bar{\mathbf{3}})\to\bar{\mathbf{3}}\otimes\mathbf{3}\to\mathbf{1_c}$	$(\mathbf{3}\otimes\mathbf{3})\otimes(\bar{\mathbf{3}}\otimes\bar{\mathbf{3}})\to\mathbf{6}\otimes\bar{\mathbf{6}}\to\mathbf{1_c}$
$(\mathbf{3}\otimes\mathbf{3})\otimes(\bar{\mathbf{3}}\otimes\bar{\mathbf{3}})$ $\to\bar{\mathbf{3}}\otimes\mathbf{3}\to\mathbf{1_f}$	$\epsilon^{abe}\epsilon^{cde}\epsilon_{ABE}\epsilon_{CDE}(q_a^A q_b^B)(\bar{q}_c^C \bar{q}_d^D)\equiv T_3^{1(3)}$	$\epsilon_{ABE}\epsilon_{CDE}(q_a^A q_b^B + q_a^A q_b^B)(\bar{q}_a^C\bar{q}_b^D + \bar{q}_b^C\bar{q}_a^D)$ $= 2\epsilon_{ABE}\epsilon_{CDE}(q_a^A q_b^B)(\bar{q}_a^C\bar{q}_b^D + \bar{q}_b^C\bar{q}_a^D)\equiv 2T_6^{1(3)}$
$\to\bar{\mathbf{3}}\otimes\mathbf{3}\to\mathbf{8_f}$	$\epsilon^{abe}\epsilon^{cde}\lambda_N^{EF}\epsilon_{ABE}\epsilon_{CDF}(q_a^A q_b^B)(\bar{q}_c^C\bar{q}_d^D)\equiv T_3^{8(3)}$	$\lambda_N^{EF}\epsilon_{ABE}\epsilon_{CDF}(q_a^A q_b^B)(\bar{q}_a^C\bar{q}_b^D + \bar{q}_b^C\bar{q}_a^D)\equiv T_6^{8(3)}$
$\to\bar{\mathbf{3}}\otimes\bar{\mathbf{6}}\to\mathbf{8_f}$	$\epsilon^{abe}\epsilon^{cde}\lambda_N^{DF}\epsilon_{ABE}\epsilon_{CEF}(q_a^A q_b^B)(\bar{q}_c^C\bar{q}_d^D)\equiv T_3^{8(3,6)}$	$\lambda_N^{DF}\epsilon_{ABE}\epsilon_{CEF}(q_a^A q_b^B)(\bar{q}_a^C\bar{q}_b^D + \bar{q}_b^C\bar{q}_a^D)\equiv T_6^{8(3,6)}$
$\to\bar{\mathbf{3}}\otimes\bar{\mathbf{6}}\to\mathbf{10_f}$	$\epsilon^{abe}\epsilon^{cde}S^{CDE}\epsilon_{ABE}(q_a^A q_b^B)(\bar{q}_c^C\bar{q}_d^D)\equiv T_3^{10(3,6)}$	$S^{CDE}\epsilon_{ABE}(q_a^A q_b^B)(\bar{q}_a^C\bar{q}_b^D + \bar{q}_b^C\bar{q}_a^D)\equiv T_6^{10(3,6)}$
$\to\mathbf{6}\otimes\mathbf{3}\to\mathbf{8_f}$	$\epsilon^{abe}\epsilon^{cde}\lambda_N^{BF}\epsilon_{AEF}\epsilon_{CDE}(q_a^A q_b^B)(\bar{q}_c^C\bar{q}_d^D)\equiv T_3^{8(6,3)}$	$\lambda_N^{BF}\epsilon_{AEF}\epsilon_{CDE}(q_a^A q_b^B)(\bar{q}_a^C\bar{q}_b^D + \bar{q}_b^C\bar{q}_a^D)\equiv T_6^{8(6,3)}$
$\to\mathbf{6}\otimes\mathbf{3}\to\mathbf{10_f}$	$\epsilon^{abe}\epsilon^{cde}S^{ABE}\epsilon_{CDE}(q_a^A q_b^B)(\bar{q}_c^C\bar{q}_d^D)\equiv T_3^{10(6,3)}$	$S^{ABE}\epsilon_{CDE}(q_a^A q_b^B)(\bar{q}_a^C\bar{q}_b^D + \bar{q}_b^C\bar{q}_a^D)\equiv T_6^{10(6,3)}$
$\to\mathbf{6}\otimes\bar{\mathbf{6}}\to\mathbf{1_f}$	$\epsilon^{abe}\epsilon^{cde}(q_a^A q_b^B + q_a^B q_b^A)(\bar{q}_c^A\bar{q}_d^B + \bar{q}_c^B\bar{q}_d^A)$ $= 2\epsilon^{abe}\epsilon^{cde}(q_a^A q_b^B)(\bar{q}_c^A\bar{q}_d^B + \bar{q}_c^B\bar{q}_d^A)\equiv 2T_3^{1(6)}$	$(q_a^A q_b^B + q_a^B q_b^A)(\bar{q}_a^A\bar{q}_b^B + \bar{q}_a^B\bar{q}_b^A + (a\leftrightarrow b))$ $= 2(q_a^A q_b^B)(\bar{q}_a^A\bar{q}_b^B + \bar{q}_a^B\bar{q}_b^A + (a\leftrightarrow b))\equiv 2T_6^{1(6)}$
$\to\mathbf{6}\otimes\bar{\mathbf{6}}\to\mathbf{8_f}$	$\lambda_{BC}^N\epsilon^{abe}\epsilon^{cde}(q_a^A q_b^B + q_a^B q_b^A)(\bar{q}_c^A\bar{q}_d^C + \bar{q}_c^C\bar{q}_d^A)\equiv T_3^{8(6)}$	$\lambda_{BC}^N(q_a^A q_b^B + q_a^B q_b^A)(\bar{q}_a^A\bar{q}_b^C + \bar{q}_a^C\bar{q}_b^A + (a\leftrightarrow b))\equiv T_6^{8(6)}$
$\to\mathbf{6}\otimes\bar{\mathbf{6}}\to\mathbf{27_f}$	$S_{ABCD}\epsilon^{abe}\epsilon^{cde}(q_a^A q_b^B)(\bar{q}_c^C\bar{q}_d^D)\equiv T_3^{27(6)}$	$S_{ABCD}(q_a^A q_b^B)(\bar{q}_a^C\bar{q}_b^D + \bar{q}_b^C\bar{q}_a^D)\equiv T_6^{27(6)}$
$(\bar{q}q)(\bar{q}q)$	$(\bar{\mathbf{3}}\otimes\mathbf{3})\otimes(\bar{\mathbf{3}}\otimes\mathbf{3})\to\mathbf{1}\otimes\mathbf{1}\to\mathbf{1_c}$	$(\bar{\mathbf{3}}\otimes\mathbf{3})\otimes(\bar{\mathbf{3}}\otimes\mathbf{3})\to\mathbf{8}\otimes\mathbf{8}\to\mathbf{1_c}$
$(\bar{\mathbf{3}}\otimes\mathbf{3})\otimes(\bar{\mathbf{3}}\otimes\mathbf{3})$ $\to\mathbf{1}\otimes\mathbf{1}\to\mathbf{1_f}$	$(\bar{q}_a^A q_a^A)(\bar{q}_b^B q_b^B)\equiv T_1^{1(1)}$	$(\bar{q}_a^A\lambda_n^{ab}q_b^A)(\bar{q}_c^B\lambda_n^{cd}q_d^B)\equiv T_8^{1(1)}$
$\to\mathbf{1}\otimes\mathbf{8}\to\mathbf{8_f}$	$\lambda_{BC}^N(\bar{q}_a^A q_a^A)(\bar{q}_b^B q_b^C)\equiv T_1^{8(1,\,8)}$	$\lambda_{BC}^N(\bar{q}_a^A\lambda_n^{ab}q_b^A)(\bar{q}_c^B\lambda_n^{cd}q_d^C)\equiv T_8^{8(1,\,8)}$
$\to\mathbf{8}\otimes\mathbf{1}\to\mathbf{8_f}$	$\lambda_{BC}^N(\bar{q}_a^B q_a^C)(\bar{q}_b^A q_b^A)\equiv T_1^{8(8,\,1)}$	$\lambda_{BC}^N(\bar{q}_a^B\lambda_n^{ab}q_b^C)(\bar{q}_c^A\lambda_n^{cd}q_d^A)\equiv T_8^{8(8,\,1)}$
$\to\mathbf{8}\otimes\mathbf{8}\to\mathbf{1_f}$	$(\bar{q}_a^A\lambda_{AB}^N q_a^B)(\bar{q}_b^C\lambda_{CD}^N q_b^D)\equiv T_1^{1(8)}$	$(\bar{q}_a^A\lambda_n^{ab}\lambda_{AB}^N q_b^B)(\bar{q}_c^C\lambda_n^{cd}\lambda_{CD}^N q_d^D)\equiv T_8^{1(8)}$
$\to\mathbf{8}\otimes\mathbf{8}\to\mathbf{8_f}$	$\lambda_N^{FE}\epsilon_{ACE}\epsilon_{BDF}(\bar{q}_a^A q_a^B)(\bar{q}_b^C q_b^D)\equiv T_1^{8(8)}$	$\lambda_N^{FE}\epsilon_{ACE}\epsilon_{BDF}(\bar{q}_a^A\lambda_n^{ab}q_b^B)(\bar{q}_c^C\lambda_n^{cd}q_d^D)\equiv T_8^{8(8)}$
$\to\mathbf{8}\otimes\mathbf{8}\to\mathbf{8_f'}$	$\lambda_N^{BF}\epsilon_{ACE}\epsilon_{DEF}(\bar{q}_a^A q_a^B)(\bar{q}_b^C q_b^D)\equiv T_1^{8'(8)}$	$\lambda_N^{BF}\epsilon_{ACE}\epsilon_{DEF}(\bar{q}_a^A\lambda_n^{ab}q_b^B)(\bar{q}_c^C\lambda_n^{cd}q_d^D)\equiv T_8^{8'(8)}$
$\to\mathbf{8}\otimes\mathbf{8}\to\mathbf{10_f}$	$\epsilon_{ACE}S_{BDE}(\bar{q}_a^A q_a^B)(\bar{q}_b^C q_b^D)\equiv T_1^{10(8)}$	$\epsilon_{ACE}S_{BDE}(\bar{q}_a^A\lambda_n^{ab}q_b^B)(\bar{q}_c^C\lambda_n^{cd}q_d^D)\equiv T_8^{10(8)}$
$\to\mathbf{8}\otimes\mathbf{8}\to\mathbf{10_f'}$	$\epsilon_{BDE}S_{ACE}(\bar{q}_a^A q_a^B)(\bar{q}_b^C q_b^D)\equiv T_1^{10'(8)}$	$\epsilon_{BDE}S_{ACE}(\bar{q}_a^A\lambda_n^{ab}q_b^B)(\bar{q}_c^C\lambda_n^{cd}q_d^D)\equiv T_8^{10'(8)}$
$\to\mathbf{8}\otimes\mathbf{8}\to\mathbf{27_f}$	$S_{ABCD}(\bar{q}_a^A q_a^B)(\bar{q}_b^C q_b^D)\equiv T_1^{27(8)}$	$S_{ABCD}(\bar{q}_a^A\lambda_n^{ab}q_b^B)(\bar{q}_c^C\lambda_n^{cd}q_d^D)\equiv T_8^{27(8)}$

3.6.2 Tetraquark Transformations

Now let us discuss the Fierz rearrangement in order to relate $(qq)(\bar{q}\bar{q})$ and $(\bar{q}q)(\bar{q}q)$ structures. First we perform it in the color and flavor spaces. To do this, it is convenient to consider the interchange of color indices:

$$(q_a^A q_b^B \bar{q}_a^C \bar{q}_b^D) = \frac{1}{3}(q_a^A q_b^B \bar{q}_b^C \bar{q}_a^D) + \frac{1}{2}\lambda_n^{ab}\lambda_n^{cd}(q_a^A q_c^B \bar{q}_d^C \bar{q}_b^D),$$

$$\lambda_n^{ab}\lambda_n^{cd}(q_a^A q_c^B \bar{q}_b^C \bar{q}_d^D) = \frac{16}{9}(q_a^A q_b^B \bar{q}_b^C \bar{q}_a^D) - \frac{1}{3}\lambda_n^{ab}\lambda_n^{cd}(q_a^A q_c^B \bar{q}_d^C \bar{q}_b^D). \tag{3.18}$$

which are obtained by using

$$\delta_{ad}\delta_{bc} = \frac{1}{3}\delta_{ab}\delta_{cd} + \frac{1}{2}\lambda_n^{ab}\lambda_n^{cd},$$

$$\lambda_n^{ad}\lambda_n^{cb} = \frac{16}{9}\delta_{ab}\delta_{cd} - \frac{1}{3}\lambda_n^{ab}\lambda_n^{cd}\,. \tag{3.19}$$

We can obtain the same result for flavor structure.

Let us take $T_3^{1(3)}$ as an example, and perform the simultaneous interchange of both color and flavor indices

$$
\begin{aligned}
T_3^{1(3)} &= \epsilon^{abe}\epsilon^{cde}\epsilon_{ABE}\epsilon_{CDE}(q_a^A q_b^B)(\bar{q}_c^C \bar{q}_d^D) \\
&= (q_a^A q_b^B)(\bar{q}_a^A \bar{q}_b^B) - (q_a^A q_b^B)(\bar{q}_b^A \bar{q}_a^B) - (q_a^A q_b^B)(\bar{q}_a^B \bar{q}_b^A) + (q_a^A q_b^B)(\bar{q}_b^B \bar{q}_a^A) \\
&= (q_a^A q_b^B)(\bar{q}_a^A \bar{q}_b^B) - \left(\frac{1}{3}(q_a^A q_b^B)(\bar{q}_a^A \bar{q}_b^B) + \frac{1}{2}\lambda_n^{ab}\lambda_n^{cd}(q_a^A q_c^B)(\bar{q}_b^A \bar{q}_d^B)\right) \\
&\quad -(q_a^A q_b^B)(\bar{q}_a^B \bar{q}_b^A) + \left(\frac{1}{3}(q_a^A q_b^B)(\bar{q}_a^B \bar{q}_b^A) + \frac{1}{2}\lambda_n^{ab}\lambda_n^{cd}(q_a^A q_c^B)(\bar{q}_b^B \bar{q}_d^A)\right) \\
&= \frac{2}{3}(q_a^A q_b^B)(\bar{q}_a^A \bar{q}_b^B) - \frac{1}{2}\lambda_n^{ab}\lambda_n^{cd}(q_a^A q_c^B)(\bar{q}_b^A \bar{q}_d^B) \\
&\quad -\frac{2}{3}\left(\frac{1}{3}(q_a^A q_b^B)(\bar{q}_a^A \bar{q}_b^B) + \frac{1}{2}\lambda_{AB}^N\lambda_{CD}^N(q_a^A q_b^C)(\bar{q}_a^B \bar{q}_b^D)\right) \\
&\quad +\frac{1}{2}\left(\frac{1}{3}\lambda_n^{ab}\lambda_n^{cd}(q_a^A q_c^B)(\bar{q}_b^A \bar{q}_d^B) + \frac{1}{2}\lambda_{AB}^N\lambda_{CD}^N\lambda_n^{ab}\lambda_n^{cd}(q_a^A q_c^C)(\bar{q}_b^B \bar{q}_d^D)\right) \\
&= \frac{4}{9}(q_a^A q_b^B)(\bar{q}_a^A \bar{q}_b^B) - \frac{1}{3}\lambda_n^{ab}\lambda_n^{cd}(q_a^A q_c^B)(\bar{q}_b^A \bar{q}_d^B) - \frac{1}{3}\lambda_{AB}^N\lambda_{CD}^N(q_a^A q_b^C)(\bar{q}_a^B \bar{q}_b^D) \\
&\quad +\frac{1}{4}\lambda_{AB}^N\lambda_{CD}^N\lambda_n^{ab}\lambda_n^{cd}(q_a^A q_c^C)(\bar{q}_b^B \bar{q}_d^D)\,.
\end{aligned}
$$

Because we only consider the color and flavor structures, by changing the ordering of the second quark and third quark, we arrive at the result:

$$
\begin{aligned}
&\sim \frac{4}{9}(\bar{q}_a^A q_a^A)(\bar{q}_b^B q_b^B) - \frac{1}{3}\lambda_n^{ab}\lambda_n^{cd}(\bar{q}_b^A q_a^A)(\bar{q}_d^B q_c^B) - \frac{1}{3}\lambda_{AB}^N\lambda_{CD}^N(\bar{q}_a^B q_a^A)(\bar{q}_b^D q_b^C) \\
&\quad +\frac{1}{4}\lambda_{AB}^N\lambda_{CD}^N\lambda_n^{ab}\lambda_n^{cd}(\bar{q}_b^B q_a^A)(\bar{q}_d^D q_c^C)\,. \\
&= \frac{4}{9}T_1^{1(1)} - \frac{1}{3}T_8^{1(1)} - \frac{1}{3}T_1^{1(8)} + \frac{1}{4}T_8^{1(8)}\,. \tag{3.20}
\end{aligned}
$$

Next we perform the Fierz rearrangement in the Lorentz indices. The formulae is [67, 131]:

$$
\begin{aligned}
(1)_{\alpha\beta}(1)_{\gamma\delta} &= \frac{1}{4}(1)_{\alpha\delta}(1)_{\gamma\beta} + \frac{1}{4}(\gamma_\mu)_{\alpha\delta}(\gamma^\mu)_{\gamma\beta} + \frac{1}{8}(\sigma_{\mu\nu})_{\alpha\delta}(\sigma^{\mu\nu})_{\gamma\beta} \tag{3.21} \\
&\quad -\frac{1}{4}(\gamma_\mu\gamma_5)_{\alpha\delta}(\gamma^\mu\gamma_5)_{\gamma\beta} + \frac{1}{4}(\gamma_5)_{\alpha\delta}(\gamma_5)_{\gamma\beta}\,.
\end{aligned}
$$

By using this equation, we can obtain various relations such as

$$((q_a^A)^T C q_b^B)(\bar{q}_c^C C (\bar{q}_d^D)^T)$$

$$
\begin{aligned}
= \ & -\frac{1}{4}((q_a^A)^T CC(\bar{q}_d^D)^T)(\bar{q}_c^C q_b^B) - \frac{1}{4}((q_a^A)^T C\gamma_\mu C(\bar{q}_d^D)^T)(\bar{q}_c^C \gamma^\mu q_b^B) \\
& -\frac{1}{8}((q_a^A)^T C\sigma_{\mu\nu}C(\bar{q}_d^D)^T)(\bar{q}_c^C \sigma^{\mu\nu} q_b^B) + \frac{1}{4}((q_a^A)^T C\gamma_\mu\gamma_5 C(\bar{q}_d^D)^T)(\bar{q}_c^C \gamma^\mu\gamma_5 q_b^B) \\
& -\frac{1}{4}((q_a^A)^T C\gamma_5 C(\bar{q}_d^D)^T)(\bar{q}_c^C \gamma_5 q_b^B) \\
= \ & -\frac{1}{4}(\bar{q}_d^D q_a^A)(\bar{q}_c^C q_b^B) + \frac{1}{4}(\bar{q}_d^D \gamma_\mu q_a^A)(\bar{q}_c^C \gamma^\mu q_b^B) + \frac{1}{8}(\bar{q}_d^D \sigma_{\mu\nu} q_a^A)(\bar{q}_c^C \sigma^{\mu\nu} q_b^B) \\
& +\frac{1}{4}(\bar{q}_d^D \gamma_\mu\gamma_5 q_a^A)(\bar{q}_c^C \gamma^\mu\gamma_5 q_b^B) - \frac{1}{4}(\bar{q}_d^D \gamma_5 q_a^A)(\bar{q}_c^C \gamma_5 q_b^B) \,.
\end{aligned}
\tag{3.22}
$$

In order to label the Lorentz structure for a scalar tetraquark field, we introduce S, V, T, A and P instead of T:

$$
\begin{aligned}
& S \text{ for } (q^T C\gamma_5 q)(\bar{q}\gamma_5 C\bar{q}^T) \text{ and } (\bar{q}q)(\bar{q}q) \,, \\
& V \text{ for } (q^T C\gamma_\mu\gamma_5 q)(\bar{q}\gamma^\mu\gamma_5 C\bar{q}^T) \text{ and } (\bar{q}\gamma_\mu q)(\bar{q}\gamma^\mu q) \,, \\
& T \text{ for } (q^T C\sigma_{\mu\nu} q)(\bar{q}\sigma^{\mu\nu} C\bar{q}^T) \text{ and } (\bar{q}\sigma_{\mu\nu} q)(\bar{q}\sigma^{\mu\nu} q) \,, \\
& A \text{ for } (q^T C\gamma_\mu q)(\bar{q}\gamma^\mu C\bar{q}^T) \text{ and } (\bar{q}\gamma_\mu\gamma_5 q)(\bar{q}\gamma^\mu\gamma_5 q) \,, \\
& P \text{ for } (q^T Cq)(\bar{q}C\bar{q}^T) \text{ and } (\bar{q}\gamma_5 q)(\bar{q}\gamma_5 q) \,.
\end{aligned}
$$

For example,

$$
\begin{aligned}
S_6^{27(6)} &\equiv S_{ABCD}(q_a^{AT} C\gamma_5 q_b^B)(\bar{q}_a^C \gamma_5 C\bar{q}_b^{DT} + \bar{q}_b^C \gamma_5 C\bar{q}_a^{DT}) \,, \\
V_1^{27(8)} &\equiv S_{ABCD}(\bar{q}_a^A \gamma_\mu q_a^B)(\bar{q}_b^C \gamma^\mu q_b^D) \,.
\end{aligned}
\tag{3.23}
$$

Diquarks belonging to T and A have a symmetric Lorentz structure (see Eq. 3.16)

$$
(C\gamma_\mu)_{\alpha\beta} = (C\gamma_\mu)_{\beta\alpha} \,, (C\sigma_{\mu\nu})_{\alpha\beta} = (C\sigma_{\mu\nu})_{\beta\alpha} \,,
\tag{3.24}
$$

so they have an anti-symmetric color-flavor structure. Therefore, currents having the symmetric color-flavor structure vanish, such as

$$
A_3^{1(3)} = \epsilon^{abe}\epsilon^{cde}\epsilon_{ABE}\epsilon_{CDE}((q_a^A)^T C\gamma_\mu q_b^B)(\bar{q}_c^C \gamma^\mu C(\bar{q}_d^D)^T) = 0 \,.
\tag{3.25}
$$

Similarly, diquarks belonging to S, V and P have an anti-symmetric Lorentz structure (see Eq. 3.17)

$$
(C)_{\alpha\beta} = -(C)_{\beta\alpha} \,, (C\gamma_\mu\gamma_5)_{\alpha\beta} = -(C\gamma_\mu\gamma_5)_{\beta\alpha} \,, (C\gamma_5)_{\alpha\beta} = -(C\gamma_5)_{\beta\alpha} \,,
\tag{3.26}
$$

and so they have a symmetric color-flavor structure.

By now, we have known the flavor, color and Lorentz structures of scalar tetraquark fields, for both $(qq)(\bar{q}\bar{q})$ and $(\bar{q}q)(\bar{q}q)$ structures, and are ready to derive some relations.

3.6.3   Specifying the flavor structure

In order to establish the relations, we need to specify the flavor quantum numbers of the tetraquark currents. As we are considering in this work, let us choose the flavor octet states $(\mathbf{3} \otimes \mathbf{3}) \otimes (\bar{\mathbf{3}} \otimes \bar{\mathbf{3}}) \rightarrow \bar{\mathbf{3}} \otimes \mathbf{3} \rightarrow \mathbf{8_f}$ for the illustration.

In this case, diquarks and antidiquarks have an anti-symmetric flavor structure, and we can verify

$$S_6^{8(3)} = V_6^{8(3)} = T_3^{8(3)} = A_3^{8(3)} = P_6^{8(3)} = 0 \,. \tag{3.27}$$

Therefore, there are five types of $(qq)(\bar{q}\bar{q})$ fields which are non-zero and independent:

$$S_3^{8(3)}, V_3^{8(3)}, T_6^{8(3)}, A_6^{8(3)}, P_3^{8(3)} \,, \tag{3.28}$$

while all ten types remain for the $(\bar{q}q)(\bar{q}q)$ fields:

$$S_1^{8(8)}, V_1^{8(8)}, T_1^{8(8)}, A_1^{8(8)}, P_1^{8(8)}, S_8^{8(8)}, V_8^{8(8)}, T_8^{8(8)}, A_8^{8(8)}, P_8^{8(8)} \,, \tag{3.29}$$

Among these ten $(\bar{q}q)(\bar{q}q)$ fields, only five are independent. We can derive the following five equation by applying the Fierz transformation for the $(\bar{q}q)(\bar{q}q)$ fields:

$$
\begin{aligned}
S_8^{8(8)} &= -\frac{1}{6}S_1^{8(8)} + \frac{1}{2}V_1^{8(8)} + \frac{1}{4}T_1^{8(8)} - \frac{1}{2}A_1^{8(8)} - \frac{1}{2}P_1^{8(8)} \,, \\
V_8^{8(8)} &= 2S_1^{8(8)} - \frac{5}{3}V_1^{8(8)} - A_1^{8(8)} - 2P_1^{8(8)} \,, \\
T_8^{8(8)} &= 6S_1^{8(8)} - \frac{5}{3}T_1^{8(8)} + 6P_1^{8(8)} \,, \\
A_8^{8(8)} &= -2S_1^{8(8)} - V_1^{8(8)} - \frac{5}{3}A_1^{8(8)} + 2P_1^{8(8)} \,, \\
P_8^{8(8)} &= \frac{1}{2}S_1^{8(8)} - \frac{1}{2}V_1^{8(8)} + \frac{1}{4}T_1^{8(8)} + \frac{1}{2}A_1^{8(8)} - \frac{1}{6}P_1^{8(8)} \,.
\end{aligned} \tag{3.30}
$$

Employing the five currents on the left hand sides of Eqs. (3.30) as independent ones, and applying the Fierz transformation, we can establish the following relations among the five $(qq)(\bar{q}\bar{q})$ and five $(\bar{q}q)(\bar{q}q)$ structures:

$$
\begin{aligned}
S_3^{8(3)} &= -\frac{1}{2}S_1^{8(8)} - \frac{1}{2}V_1^{8(8)} + \frac{1}{4}T_1^{8(8)} - \frac{1}{2}A_1^{8(8)} - \frac{1}{2}P_1^{8(8)} \,, \\
V_3^{8(3)} &= 2S_1^{8(8)} - V_1^{8(8)} + A_1^{8(8)} - 2P_1^{8(8)} \,, \\
T_6^{8(3)} &= 6S_1^{8(8)} + T_1^{8(8)} + 6P_1^{8(8)} \,, \\
A_6^{8(3)} &= 2S_1^{8(8)} + V_1^{8(8)} - A_1^{8(8)} - 2P_1^{8(8)} \,, \\
P_3^{8(3)} &= -\frac{1}{2}S_1^{8(8)} + \frac{1}{2}V_1^{8(8)} + \frac{1}{4}T_1^{8(8)} + \frac{1}{2}A_1^{8(8)} - \frac{1}{2}P_1^{8(8)} \,.
\end{aligned} \tag{3.31}
$$

3.6.4 Specifying the color structure

For completeness of mathematical structure, one can specify the color quantum numbers for the currents. For illustration, let us consider the color structure $(\mathbf{3} \otimes \mathbf{3}) \otimes (\bar{\mathbf{3}} \otimes \bar{\mathbf{3}}) \rightarrow \bar{\mathbf{3}} \otimes \mathbf{3} \rightarrow \mathbf{1_c}$. In order to establish the relations between $(qq)(\bar{q}\bar{q})$ and $(\bar{q}q)(\bar{q}q)$ currents, we find that we need two flavor structures: $(\mathbf{3_f} \otimes \mathbf{3_f}) \otimes (\bar{\mathbf{3}}_\mathbf{f} \otimes \bar{\mathbf{3}}_\mathbf{f}) \rightarrow \bar{\mathbf{3}}_\mathbf{f} \otimes \mathbf{3_f} \rightarrow \mathbf{1}_f$ and $(\mathbf{3_f} \otimes \mathbf{3_f}) \otimes (\bar{\mathbf{3}}_\mathbf{f} \otimes \bar{\mathbf{3}}_\mathbf{f}) \rightarrow \mathbf{6_f} \otimes \bar{\mathbf{6}}_\mathbf{f} \rightarrow \mathbf{1}_f$.

In this case, diquarks and antidiquarks have an anti-symmetric color structure. By using the Pauli principle, we can verify

$$S_3^{1(6)} = V_3^{1(6)} = T_3^{1(3)} = A_3^{1(3)} = P_3^{1(6)} = 0. \tag{3.32}$$

Therefore, there are five types of $(qq)(\bar{q}\bar{q})$ fields, which are non-zero and independent:

$$S_3^{1(3)}, V_3^{1(3)}, T_3^{1(6)}, A_3^{1(6)}, P_3^{1(3)}. \tag{3.33}$$

The single $(\bar{q}q)(\bar{q}q)$ fields can not have an anti-symmetric color structure. Therefore, we need to use their combinations. By using Eq. (3.19), $(\bar{q}q)(\bar{q}q)$ fields can be combined to have an anti-symmetric color structure:

$$
\begin{aligned}
(\bar{q}_a^A q_a^A)(\bar{q}_b^B q_b^B) - (\bar{q}_a^A q_b^A)(\bar{q}_b^B q_a^B) &= (\bar{q}_a^A q_a^A)(\bar{q}_b^B q_b^B) - \frac{1}{3}(\bar{q}_a^A q_a^A)(\bar{q}_b^B q_b^B) - \frac{1}{2}\lambda_n^{ab}\lambda_n^{cd}(\bar{q}_a^A q_b^A)(\bar{q}_c^B q_d^B) \\
&= \frac{2}{3}S_1^{1(1)} - \frac{1}{2}S_8^{1(1)} \equiv S_3^{1(1)}, \tag{3.34}
\end{aligned}
$$

Altogether there are ten types of non-vanishing $(\bar{q}q)(\bar{q}q)$ currents:

$$S_3^{1(1)}, V_3^{1(1)}, T_3^{1(1)}, A_3^{1(1)}, P_3^{1(1)}, S_3^{1(8)}, V_3^{1(8)}, T_3^{1(8)}, A_3^{1(8)}, P_3^{1(8)}.$$

Once again, among them only five are independent

$$
\begin{aligned}
S_3^{1(8)} &= -\frac{1}{6}S_3^{1(1)} + \frac{1}{2}V_3^{1(1)} + \frac{1}{4}T_3^{1(1)} - \frac{1}{2}A_3^{1(1)} - \frac{1}{2}P_3^{1(1)}, \\
V_3^{1(8)} &= 2S_3^{1(1)} - \frac{5}{3}V_3^{1(1)} - A_3^{1(1)} - 2P_3^{1(1)}, \\
T_3^{1(8)} &= 6S_3^{1(1)} - \frac{5}{3}T_3^{1(1)} + 6P_3^{1(1)}, \tag{3.35} \\
A_3^{1(8)} &= -2S_3^{1(1)} - V_3^{1(1)} - \frac{5}{3}A_3^{1(1)} + 2P_3^{1(1)}, \\
P_3^{1(8)} &= \frac{1}{2}S_3^{1(1)} - \frac{1}{2}V_3^{1(1)} + \frac{1}{4}T_3^{1(1)} + \frac{1}{2}A_3^{1(1)} - \frac{1}{6}P_3^{1(1)}.
\end{aligned}
$$

The relations between $(qq)(\bar{q}\bar{q})$ and $(\bar{q}q)(\bar{q}q)$ structures are:

$$S_3^{1(3)} = -\frac{1}{2}S_3^{1(1)} - \frac{1}{2}V_3^{1(1)} + \frac{1}{4}T_3^{1(1)} - \frac{1}{2}A_3^{1(1)} - \frac{1}{2}P_3^{1(1)},$$

$$
\begin{aligned}
V_3^{1(3)} &= 2S_3^{1(1)} - V_3^{1(1)} + A_3^{1(1)} - 2P_3^{1(1)}, \\
T_3^{1(6)} &= 6S_3^{1(1)} + T_3^{1(1)} + 6P_3^{1(1)}, \\
A_3^{1(6)} &= 2S_3^{1(1)} + V_3^{1(1)} - A_3^{1(1)} - 2P_3^{1(1)}, \\
P_3^{1(3)} &= -\frac{1}{2}S_3^{1(1)} + \frac{1}{2}V_3^{1(1)} + \frac{1}{4}T_3^{1(1)} + \frac{1}{2}A_3^{1(1)} - \frac{1}{2}P_3^{1(1)}.
\end{aligned}
\tag{3.36}
$$

3.6.5 Specifying the Lorentz structure

Finally, let us consider the case where the Lorentz structure is specified. As an illustration, let us consider a tetraquark current $(q^T C\gamma_5 q)(\bar{q}\gamma_5 C\bar{q}^T)$. Possible color structures are $(3 \otimes 3) \otimes (\bar{3} \otimes \bar{3}) \to \bar{3} \otimes 3 \to 1_c$ and $(3 \otimes 3) \otimes (\bar{3} \otimes \bar{3}) \to 6 \otimes \bar{6} \to 1_c$; and possible flavor structures are $(3 \otimes 3) \otimes (\bar{3} \otimes \bar{3}) \to \bar{3} \otimes 3 \to 1_f$ and $(3 \otimes 3) \otimes (\bar{3} \otimes \bar{3}) \to 6 \otimes \bar{6} \to 1_f$.

By using the Pauli principle, we can verify

$$
S_3^{1(6)} = S_6^{1(3)} = 0.
\tag{3.37}
$$

Therefore, there are two currents which are non-zero and independent:

$$
\begin{aligned}
S_3^{1(3)} &= \epsilon^{abe}\epsilon^{cde}\epsilon_{ABE}\epsilon_{CDE}(q_a^A C\gamma_5 q_b^B)(\bar{q}_c^C \gamma_5 C\bar{q}_d^D), \\
S_6^{1(6)} &= (q_a^A C\gamma_5 q_b^B)(\bar{q}_a^A \gamma_5 C\bar{q}_b^B + \bar{q}_a^B \gamma_5 C\bar{q}_b^A + (a \leftrightarrow b)),
\end{aligned}
$$

Now from the combination of quark and antiquark, possible color structures are $(\bar{3} \otimes 3) \otimes (\bar{3} \otimes 3) \to 1 \otimes 1 \to 1_c$ and $(\bar{3} \otimes 3) \otimes (\bar{3} \otimes 3) \to 8 \otimes 8 \to 1_c$; and possible flavor structures are $(\bar{3} \otimes 3) \otimes (\bar{3} \otimes 3) \to 1 \otimes 1 \to 1_f$ and $(\bar{3} \otimes 3) \otimes (\bar{3} \otimes 3) \to 8 \otimes 8 \to 1_f$. Therefore, there are four non-vanishing currents:

$$
\begin{aligned}
P_1'^{1(1)} &= (q_a^A C\gamma_5 q_b^B)(\bar{q}_a^A \gamma_5 C\bar{q}_b^B), \\
P_8'^{1(1)} &= \lambda_n^{ab}\lambda_n^{cd}(q_a^A C\gamma_5 q_c^B)(\bar{q}_b^A \gamma_5 C\bar{q}_d^B), \\
P_1'^{1(8)} &= \lambda_{AB}^N \lambda_{CD}^N (q_a^A C\gamma_5 q_b^C)(\bar{q}_a^B \gamma_5 C\bar{q}_b^D), \\
P_8'^{1(8)} &= \lambda_{AB}^N \lambda_{CD}^N \lambda_n^{ab}\lambda_n^{cd}(q_a^A C\gamma_5 q_c^C)(\bar{q}_b^B \gamma_5 C\bar{q}_d^D).
\end{aligned}
$$

The Lorentz structure is still specified to be $(q^T C\gamma_5 q)(\bar{q}\gamma_5 C\bar{q}^T)$. However, if we interchange the second quark and third antiquark as done in Eq. (3.20) within the color and flavor spaces structures, They are now "$(\bar{q}q)(\bar{q}q)$" currents. Among them, only two are independent, through the following relations:

$$
\begin{aligned}
P_1'^{1(8)} &= P_8'^{1(1)}, \\
P_8'^{1(8)} &= \frac{32}{9}P_1'^{1(1)} - \frac{4}{3}P_8'^{1(1)}.
\end{aligned}
\tag{3.38}
$$

Finally, relations between the $(qq)(\bar{q}\bar{q})$ and "$(\bar{q}q)(\bar{q}q)$" currents are

$$
\begin{aligned}
S_3^{1(3)} &= \frac{4}{3} P_1'^{1(1)} - P_8'^{1(1)} , \\
S_6^{1(6)} &= \frac{8}{3} P_1'^{1(1)} + P_8'^{1(1)} .
\end{aligned}
\qquad (3.39)
$$

Chapter 4

Color Structure

In the previous chapter, we find that for all the currents constructed by using quark-antiquark pairs, only half of them are independent. Therefore, all the tetraquark currents which contain two color octet quark-antiquark pairs can be written as combinations of the currents which just contain two color singlet quark-antiquark pairs. In this chapter, we will study this, and we will find that every tetraquark current can be written as a combination of the currents which just contain two color singlet quark-antiquark pairs. This can also be proved in the case of pentaquark that every pentaquark current can be written as a combination of the currents which just contain one color singlet quark-antiquark pair and one color singlet three-quark baryon field.

4.1   Tetraquark Fields

Every tetraquark current can be written as a combination of two quark spinors, two antiquark spinors, a Lorentz matrix L (Lorentz space), a color matrix C (color space), a flavor matrix F (flavor space) and some derivatives ∂_μ

$$\eta = L_{\mu\nu\rho\sigma} F^{abcd} C_{ijkl} \bar{q}_a^{\mu i} \bar{q}_b^{\nu j} q_c^{\rho k} q_d^{\sigma l} , \qquad (4.1)$$

where the sum over repeated indices ($\mu, \nu, \cdots$ for Dirac spinor indices, $a, b, \cdots$ for flavor indices, and $i, j, \cdots$ for color indices) is taken. The quark spinor q may contain derivatives and so there is an extra Lorentz index μ.

We want to prove that every tetraquark current can be expressed by two color singlet quark-antiquark pairs $(\bar{q}^i q^i)(\bar{q}^j q^j)$. To do this, we need to perform some transformations in color and Lorentz spaces.

First we simplify the Lorentz indices to make transformations easier. If two derivatives contract with each other, we write them within the quark spinors

$$(\bar{q}_1 \partial_\mu q_2)(\bar{q}_3 \partial^\mu q_4) \Rightarrow (\bar{q}_1 q_2')(\bar{q}_3 q_4') . \qquad (4.2)$$

If γ_μ does not contract with any other one, which means that its Lorentz index remains in the end, we write it with one quark spinor:

$$\gamma_\mu q_1 \quad \rightarrow \quad q_1 \, . \tag{4.3}$$

This can always be done since we can change the position of γ-matrices:

$$\begin{aligned}
\bar{q}_1 \gamma_\rho \gamma_\mu \gamma_\sigma q_2 &= 2 g_{\mu\sigma} \bar{q}_1 \gamma_\rho q_2 - \bar{q}_1 \gamma_\rho \gamma_\sigma (\gamma_\mu q_2) \\
&\Rightarrow c_1 \bar{q}_1 \gamma_\rho q_2 + c_2 \bar{q}_1 \gamma_\rho \gamma_\sigma q_2' \, .
\end{aligned} \tag{4.4}$$

This may produce some extra metric matrixes $g_{\mu\nu}$, which we keep to the end.

If γ_μ contracts with a derivative ∂^μ which is in the same quark-antiquark pair, we can use the same procedure. If γ_μ contracts with another γ^μ which is in the same quark-antiquark pair, we can contract them directly

$$\bar{q}_1 \cdots \gamma_\mu \cdots \gamma^\mu \cdots q_2 = c_3 \bar{q}_1 \cdots q_2 \, . \tag{4.5}$$

If γ_μ contracts with a derivative ∂^μ which is in the other quark-antiquark pair, we need to use the Fierz transformation to put them together

$$\begin{aligned}
&(\bar{q}_1 \cdots_1 \gamma_\mu)(\cdots_2 q_2)(\bar{q}_3 \cdots_3)(\partial^\mu q_4) \\
&\Rightarrow \sum_\Gamma (\bar{q}_1 \cdots_1 \gamma_\mu) \Gamma (\partial^\mu q_4)(\bar{q}_3 \cdots_3) \Gamma (\cdots_2 q_2) \, .
\end{aligned} \tag{4.6}$$

This may produce some extra Γ matrices. After contracting all these Γ matrices, we arrive at following expression

$$\eta = F^{abcd} C_{ijkl} (\bar{q}_a^i \Gamma_{\mu\nu\cdots} q_b^j)(\bar{q}_c^k \Gamma^{\mu\nu\cdots} q_d^l) \, , \tag{4.7}$$

where the matrix $\Gamma_{\mu\nu\cdots}$ can be written as a combination of $\mathbf{1}$, γ_μ, γ_5 and $\sigma_{\mu\nu}$. The previous coefficient $L_{\mu\nu\rho\sigma}$ is written inside with either $\Gamma_{\mu\nu\cdots}$ or $\Gamma^{\mu\nu\cdots}$. By using the Eq. (4.4) again, every tetraquark current can be written as a combination of five currents

$$\begin{aligned}
\eta^S &= F^{abcd} C_{ijkl} (\bar{q}_a^i q_b^j)(\bar{q}_c^k q_d^l) \, , \\
\eta^V &= F^{abcd} C_{ijkl} (\bar{q}_a^i \gamma_\mu q_b^j)(\bar{q}_c^k \gamma^\mu q_d^l) \, , \\
\eta^T &= F^{abcd} C_{ijkl} (\bar{q}_a^i \sigma_{\mu\nu} q_b^j)(\bar{q}_c^k \sigma^{\mu\nu} q_d^l) \, , \\
\eta^A &= F^{abcd} C_{ijkl} (\bar{q}_a^i \gamma_\mu \gamma_5 q_b^j)(\bar{q}_c^k \gamma^\mu \gamma_5 q_d^l) \, , \\
\eta^P &= F^{abcd} C_{ijkl} (\bar{q}_a^i \gamma_5 q_b^j)(\bar{q}_c^k \gamma_5 q_d^l) \, ,
\end{aligned} \tag{4.8}$$

where the quark spinors may contain some Γ matrixes and derivatives. The currents η^A and η^P can be written in the form of η^V and η^S respectively. However, we will find that they are necessary to compose a complete and independent basis. For tetraquark of

different quantum numbers, the amount of independent currents may change, but there are five independent currents at most, which are just these five ones.

There are two kinds of color structures, which are $(\bar{q}^i q^i)(\bar{q}^j q^j)$ and $(\bar{q}^i \lambda^n_{ij} q^j)(\bar{q}^k \lambda^n_{kl} q^l)$. The flavor symmetry of diquark can be both symmetric and antisymmetric. Here we fix it to be symmetric, and the antisymmetric case can be similarly studied. Therefore, every tetraquark current can be written as a combination of following ten currents

$$
\begin{aligned}
\eta^S_1 &= (\bar{q}^i_1 q^i_2)(\bar{q}^k_3 q^k_4) + (q_2 \leftrightarrow q_4)\,, \\
\eta^S_8 &= (\bar{q}^i_1 \lambda^n_{ij} q^j_2)(\bar{q}^k_3 \lambda^n_{kl} q^l_4) + (q_2 \leftrightarrow q_4)\,, \\
\eta^V_1 &= (\bar{q}^i_1 \gamma_\mu q^i_2)(\bar{q}^k_3 \gamma^\mu q^k_4) + (q_2 \leftrightarrow q_4)\,, \\
\eta^V_8 &= (\bar{q}^i_1 \lambda^n_{ij} \gamma_\mu q^j_2)(\bar{q}^k_3 \lambda^n_{kl} \gamma^\mu q^l_4) + (q_2 \leftrightarrow q_4)\,, \\
\eta^T_1 &= (\bar{q}^i_1 \sigma_{\mu\nu} q^i_2)(\bar{q}^k_3 \sigma^{\mu\nu} q^k_4) + (q_2 \leftrightarrow q_4)\,, \\
\eta^T_8 &= (\bar{q}^i_1 \lambda^n_{ij} \sigma_{\mu\nu} q^j_2)(\bar{q}^k_3 \lambda^n_{kl} \sigma^{\mu\nu} q^l_4) + (q_2 \leftrightarrow q_4)\,, \\
\eta^A_1 &= (\bar{q}^i_1 \gamma_\mu \gamma_5 q^i_2)(\bar{q}^k_3 \gamma^\mu \gamma_5 q^k_4) + (q_2 \leftrightarrow q_4)\,, \\
\eta^A_8 &= (\bar{q}^i_1 \lambda^n_{ij} \gamma_\mu \gamma_5 q^j_2)(\bar{q}^k_3 \lambda^n_{kl} \gamma^\mu \gamma_5 q^l_4) + (q_2 \leftrightarrow q_4)\,, \\
\eta^P_1 &= (\bar{q}^i_1 \gamma_5 q^i_2)(\bar{q}^k_3 \gamma^5 q^k_4) + (q_2 \leftrightarrow q_4)\,, \\
\eta^P_8 &= (\bar{q}^i_1 \lambda^n_{ij} \gamma_5 q^j_2)(\bar{q}^k_3 \lambda^n_{kl} \gamma^5 q^l_4) + (q_2 \leftrightarrow q_4)\,,
\end{aligned}
\tag{4.9}
$$

where the numbers 1, 2, 3 and 4 represent quark flavors, and the subscripts 1 and 8 represent color singlet and octet quark-antiquark pairs respectively.

By performing some transformations, we will see that these ten currents are not independent. First we change their color structure

$$
\begin{aligned}
(\bar{q}^i_1 q^j_2)(\bar{q}^j_3 q^i_4) &= \frac{1}{3}(\bar{q}^i_1 q^i_2)(\bar{q}^j_3 q^j_4) + \frac{1}{2}(\bar{q}^i_1 q^j_2)(\bar{q}^k_3 q^l_4)\lambda^n_{ij}\lambda^n_{kl}\,, \\
(\bar{q}^i_1 q^k_2)(\bar{q}^j_3 q^l_4)\lambda^n_{ij}\lambda^n_{kl} &= \frac{16}{9}(\bar{q}^i_1 q^i_2)(\bar{q}^j_3 q^j_4) - \frac{1}{3}(\bar{q}^i_1 q^j_2)(\bar{q}^k_3 q^l_4)\lambda^n_{ij}\lambda^n_{kl}\,.
\end{aligned}
\tag{4.10}
$$

Then we change their Lorentz structure by using the Fierz transformation

$$
\begin{aligned}
\frac{1}{3}\eta^S_1 + \frac{1}{2}\eta^S_8 &= (\bar{q}^i_1 q^j_2)(\bar{q}^j_3 q^i_4) + (q_2 \leftrightarrow q_4) \\
&= -\frac{1}{4}\{\eta^S_1 + \eta^V_1 + \frac{1}{2}\eta^T_1 - \eta^A_1 + \eta^P_1\}\,.
\end{aligned}
\tag{4.11}
$$

We obtain ten equations in all

$$
\begin{aligned}
\frac{1}{3}\eta^S_1 + \frac{1}{2}\eta^S_8 &= -\frac{1}{4}\{\eta^S_1 + \eta^V_1 + \frac{1}{2}\eta^T_1 - \eta^A_1 + \eta^P_1\}\,, \\
\frac{16}{9}\eta^S_1 - \frac{1}{3}\eta^S_8 &= -\frac{1}{4}\{\eta^S_8 + \eta^V_8 + \frac{1}{2}\eta^T_8 - \eta^A_8 + \eta^P_8\}\,,
\end{aligned}
$$

$$\frac{1}{3}\eta_1^V + \frac{1}{2}\eta_8^V = -\frac{1}{4}\{4\eta_1^S - 2\eta_1^V - 2\eta_1^A - 4\eta_1^P\},$$

$$\frac{16}{9}\eta_1^V - \frac{1}{3}\eta_8^V = -\frac{1}{4}\{4\eta_8^S - 2\eta_8^V - 2\eta_8^A - 4\eta_8^P\},$$

$$\frac{1}{3}\eta_1^T + \frac{1}{2}\eta_8^T = -\frac{1}{4}\{12\eta_1^S - 2\eta_1^T + 12\eta_1^P\},$$

$$\frac{16}{9}\eta_1^T - \frac{1}{3}\eta_8^T = -\frac{1}{4}\{12\eta_8^S - 2\eta_8^T + 12\eta_8^P\},$$

$$\frac{1}{3}\eta_1^A + \frac{1}{2}\eta_8^A = -\frac{1}{4}\{-4\eta_1^S - 2\eta_1^V - 2\eta_1^A + 4\eta_1^P\}, \qquad (4.12)$$

$$\frac{16}{9}\eta_1^A - \frac{1}{3}\eta_8^A = -\frac{1}{4}\{-4\eta_8^S - 2\eta_8^V - 2\eta_8^A + 4\eta_8^P\},$$

$$\frac{1}{3}\eta_1^P + \frac{1}{2}\eta_8^P = -\frac{1}{4}\{\eta_1^S - \eta_1^V + \frac{1}{2}\eta_1^T + \eta_1^A + \eta_1^P\},$$

$$\frac{16}{9}\eta_1^P - \frac{1}{3}\eta_8^P = -\frac{1}{4}\{\eta_8^S - \eta_8^V + \frac{1}{2}\eta_8^T + \eta_8^A + \eta_8^P\}.$$

Solving these linear equations, we find that there are five independent currents at most (some of them may disappear). In other words, the rank of this 10×10 coefficient matrix is five at most. Any five currents among (4.8) can express all the ten currents. These five currents can be either the five $\mathbf{1}_C \otimes \mathbf{1}_C$ currents or the five $\mathbf{8}_C \otimes \mathbf{8}_C$ currents.

If the diquark has a antisymmetric flavor structure, the procedure is similar. Therefore, we arrive at our final conclusion that the tetraquark currents can be written as a combination of two color singlet quark-antiquark pairs (they can also be written as a combination of two color octet quark-antiquark pairs):

$$\eta_8^S = -\frac{7}{6}\eta_1^S - \frac{1}{2}\eta_1^V - \frac{1}{4}\eta_1^T + \frac{1}{2}\eta_1^A - \frac{1}{2}\eta_1^P,$$

$$\eta_8^V = -2\eta_1^S + \frac{1}{3}\eta_1^V + \eta_1^A + 2\eta_1^P,$$

$$\eta_8^T = -6\eta_1^S + \frac{1}{3}\eta_1^T - 6\eta_1^P, \qquad (4.13)$$

$$\eta_8^A = 2\eta_1^S + \eta_1^V + \frac{1}{3}\eta_1^A - 2\eta_1^P,$$

$$\eta_8^P = -\frac{1}{2}\eta_1^S + \frac{1}{2}\eta_1^V - \frac{1}{4}\eta_1^T - \frac{1}{2}\eta_1^A - \frac{7}{6}\eta_1^P.$$

To know more about this, we go on to study $(\bar{q}\bar{q})(qq)$ currents. We use the local scalar tetraquark currents as an example. Because the anti-diquark and diquark must have the same color, spin and orbital symmetries, their flavor symmetry must be the same, which is either $\mathbf{3} \otimes \bar{\mathbf{3}}$ or $\bar{\mathbf{6}} \otimes \mathbf{6}$. However, half of them drop due to the Pauli principle. For instance

$$\eta_3^P (\bar{\mathbf{6}}_f(\bar{q}\bar{q}) \otimes \mathbf{6}_f(qq)) \qquad (4.14)$$

$$\equiv \ P_{Lorentz} \otimes \mathbf{3}_{Color} \otimes \mathbf{6}_{Flavor}$$
$$= \ \epsilon_{ijk}(\bar{q}_1^j C \bar{q}_2^{kT})\epsilon_{ij'k'}(q_3^{j'T} C q_4^{k'}) + (q_3 \Leftrightarrow q_4)$$
$$= \ 0 \,.$$

Eventually, we end up with five independent currents

$$\eta_6^S = (\bar{q}_1^i \gamma_5 C \bar{q}_2^{jT})(q_3^{iT} C \gamma_5 q_4^j) + (q_3 \Leftrightarrow q_4) \,,$$
$$\eta_6^V = (\bar{q}_1^i \gamma_\mu \gamma_5 C \bar{q}_2^{jT})(q_3^{iT} C \gamma^\mu \gamma_5 q_4^j) + (q_3 \Leftrightarrow q_4) \,,$$
$$\eta_3^T = (\bar{q}_1^i \sigma_{\mu\nu} C \bar{q}_2^{jT})(q_3^{iT} C \sigma^{\mu\nu} q_4^j) + (q_3 \Leftrightarrow q_4) \,, \qquad (4.15)$$
$$\eta_3^A = (\bar{q}_1^i \gamma_\mu C \bar{q}_2^{jT})(q_3^{iT} C \gamma^\mu q_4^j) + (q_3 \Leftrightarrow q_4) \,,$$
$$\eta_6^P = (\bar{q}_1^i C \bar{q}^{jT})(q_3^{iT} C q_4^j) + (q_3 \Leftrightarrow q_4) \,.$$

The currents η_3^S, η_3^V, η_6^T, η_6^A and η_3^P all disappear. There are ten $(\bar{q}q)(\bar{q}q)$ currents $\eta_1^S \cdots \eta_8^P$, and five of them are independent. By using the Fierz transformation, we can establish the relations between the $(\bar{q}q)(\bar{q}q)$ currents and the $(\bar{q}\bar{q})(qq)$ currents

$$\eta_6^S = -\frac{1}{4}\eta_1^S - \frac{1}{4}\eta_1^V + \frac{1}{8}\eta_1^T - \frac{1}{4}\eta_1^A - \frac{1}{4}\eta_1^P \,,$$
$$\eta_6^V = \eta_1^S - \frac{1}{2}\eta_1^V + \frac{1}{2}\eta_1^A - \eta_1^P \,,$$
$$\eta_3^T = 3\eta_1^S + \frac{1}{2}\eta_1^T + 3\eta_1^P \,, \qquad (4.16)$$
$$\eta_3^A = \eta_1^S + \frac{1}{2}\eta_1^V - \frac{1}{2}\eta_1^A - \eta_1^P \,,$$
$$\eta_6^P = -\frac{1}{4}\eta_1^S + \frac{1}{4}\eta_1^V + \frac{1}{8}\eta_1^T + \frac{1}{4}\eta_1^A - \frac{1}{4}\eta_1^P \,.$$

Now we know the origin of our conclusion. This is due to the Pauli principle. If the hadron contains two quarks and two antiquarks, after fixing the Lorentz and flavor structures, the color representation of two quarks (antiquark) is also fixed to be either $\bar{\mathbf{3}}$ or $\mathbf{6}$ ($\mathbf{6}$ or $\bar{\mathbf{3}}$). However, the color representation of the quark-antiquark pair can be both $\mathbf{1}$ and $\mathbf{8}$. Therefore, the currents constructed by two color singlet quark-antiquark pairs and two color octet pairs are not independent.

4.2 Pentaquark Fields

From the Young tableau, the only one anti-quark inside the pentaquark has two boxes, while it should be accompanied with one quark (one box) in order to construct a color singlet. Thus, by using the Fierz transformation, we can always change every field to a combination of color singlet meson field and color singlet baryon field in the following

way:

$$P(x) = \left(\bar{q}^a(x)\Gamma_{0X}q^b(x)\right)\left(q^{cT}(x)C\Gamma_{0Y}q^d(x)\right)\Gamma_{0Z}q^e(x) \tag{4.17}$$
$$= \sum_i \left(\bar{q}^a(x)\Gamma_{iX}q^a(x)\right)\epsilon_{bcd}\left(q^{bT}(x)C\Gamma_{iY}q^c x)\right)\Gamma_{iZ}q^d(x),$$

where the flavor indices are omitted, due to that we need to change the position of quarks.

If we change one antiquark to two quarks, we obtain pentaquark currents $(\bar{q}q)(qqq)$. There are three ways to compose a color singlet:

1. $(\bar{3}\otimes 3)\otimes(3\otimes 3\otimes 3) \Rightarrow 1\otimes 1 \Rightarrow 1$,

2. $(\bar{3}\otimes 3)\otimes(3\otimes 3\otimes 3) \Rightarrow 8\otimes(\bar{3}\otimes 3) \Rightarrow 8\otimes 8 \Rightarrow 1$,

3. $(\bar{3}\otimes 3)\otimes(3\otimes 3\otimes 3) \Rightarrow 8\otimes(6\otimes 3) \Rightarrow 8\otimes 8 \Rightarrow 1$.

The second way and the third way are equivalent, for the color representation $\mathbf{8}$ (qqq) has a mixed symmetry, and we can choose two quarks which have an antisymmetric color structure $(\bar{q}^p\lambda^n_{pq}q^q)\epsilon_{ijk}(q^iq^j\lambda^n_{kl}q^l)$. Just as we have proved, this can be expressed by $(\bar{q}_lq_l)\epsilon_{ijk}(q^iq^jq^k)$, which is the first way.

This analysis can be applied to the system which contains more quarks. The color quantum number of quark and antiquark is $\mathbf{3}$ and $\bar{\mathbf{3}}$ respectively. In order to compose a color singlet multiquark current, there are two constructions: one is $(\bar{q}q)\cdots(\bar{q}q)$, the other is $(q\cdots q)(\bar{q}\cdots\bar{q})$. The amount of these combinations in different constructions are the same. However, because of the Pauli principle, only one combination in the second construction remains. Therefore, only one combination in the first construction remains, which we can choose to be $(\bar{q}q)_{1C}\cdots(\bar{q}q)_{1C}$.

The tetraquark and pentaquark states are different from the currents. However, due to Pauli principle, we can obtain the same result.

The quark-antiquark pair can have color representations $\mathbf{1}$ $(\bar{q}^iq^i)$ and $\mathbf{8}$ $(\bar{q}^i\lambda_{ij}q^j)$. In the quark model, we can always fix the flavor structure of the diquark, either symmetric $(\bar{\mathbf{3}})$ or antisymmetric $(\mathbf{6})$. Take the symmetric case as an example. Considering the color structure, there are two combinations

1. $T_1^S = \delta_{ik}\delta_{jl}(\bar{q}_1^i\bar{q}_2^jq_3^kq_4^l) + \delta_{ik}\delta_{jl}(\bar{q}_1^i\bar{q}_2^jq_4^kq_3^l)$,

2. $T_8^S = \lambda_{ik}\lambda_{jl}(\bar{q}_1^i\bar{q}_2^jq_3^kq_4^l) + \lambda_{ik}\lambda_{jl}(\bar{q}_1^i\bar{q}_2^jq_4^kq_3^l)$,

By using of Eqs. (4.10), we can verify

$$4T_1^S = 3T_8^S , \tag{4.18}$$

which means that the states having color structures $\mathbf{1}\otimes\mathbf{1}$ and $\mathbf{8}\otimes\mathbf{8}$ are not independent. This relation may be changed if we consider other structures.

We can apply this analysis to pentaquark states, and obtain the same result.

Having done the analysis on the color structure, we can follow these procedures to study the $SU(3)_f$ flavor structure. There are two tetraquark states which are flavor singlets. In the quark-antiquark constructions, they are $(\bar{q}_a q_a)(\bar{q}_b q_b)$ and $(\bar{q}_a \lambda_n^{ab} q_b)(\bar{q}_c \lambda_n^{cd} q_d)$, where a, b, c, d are flavor indices, and λ is the matrix in the flavor space. Using the same method, we find that they are not independent.

In conclusion we have studied the color structure of the tetraquark and the pentaquark states first by using hadronic currents, and then by using group theory in the quark model. We have found that there is only one color structure for tetraquark and pentaquark states, just as for the conventional mesons and baryons.

Chapter 5

QCD Sum Rule Study of $ud\bar{s}\bar{s}$

From this chapter, we will study several tetraquark candidates as well as some bottom baryons by using the method of QCD sum rule. As the first example, we shall study the tetraquark $ud\bar{s}\bar{s}$ with the quantum numbers $J^{PC} = 0^{++}$ in this chapter.

Historically, tetraquark mesons were investigated long ago as an attempt to explain relatively light masses and excess of states in scalar channels [37, 93–95, 174]. Just as in the exotic baryons, it is interesting to consider genuine exotic states in the meson sector whose minimal component is $qq\bar{q}\bar{q}$. Tetraquark states of $ud\bar{s}\bar{s}$ component have been studied as candidates of such exotic states. Since they may be obtained by replacing one of ud diquarks in Θ^+ by an $\bar{s}$ antiquark, similarities between Θ^+ and $ud\bar{s}\bar{s}$ have been discussed, though precise analogy is a dynamical question [108, 123, 186].

In the former studies, the tetraquark $ud\bar{s}\bar{s}$ of $J^P = 1^+$ was investigated in detail, where it was shown that the state has a relatively low mass and a narrow width decaying into K^*K in the flux tube model [105]. The narrow decay width is associated with the fact that KK channel is forbidden due to the conservation of parity and angular momentum, which partly motivated the study of the 1^+ channel.

In principle, it is also possible to study other channels of the $ud\bar{s}\bar{s}$ tetraquarks [33, 52, 105]. From a naive point of view of mass, it is natural to investigate 0^+ scalar states. In contrast to $\bar{q}q$ mesons, the tetraquark does not need orbital excitation to form the quantum number 0^+, but all quarks may occupy the lowest state. In this case, it is shown that the tetraquark should have isospin one $I = 1$. This is the object that we would like to study in this chapter.

In this chapter, we perform QCD sum rule analyses for the scalar ($J^P = 0^+$) and isovector ($I = 1$) exotic tetraquark $ud\bar{s}\bar{s}$. The independent currents of $I = 1$ and $J^P = 0^+$ have been constructed in Section 3.1. We then consider two-point correlation functions first by using a single current of various types. It turns out that many of them do not achieve a good sum rule. Therefore, we attempt linear combinations of two independent currents. This method was first proposed in Ref. [172]. We then find that there are several cases with good Borel stability, indicating the mass of the tetraquark around 1.5 GeV.

We also investigate the reliability of the sum rule not only from the Borel stability but also from the dependence on the threshold value and the amount of the pole contribution in the total sum rule. We also mention the convergence of OPE.

The difficulties to make a good sum rule for exotic particles of high dimensional operators were nicely discussed in a recent work by Kojo et al. [115]. They proposed a sum rule using a linear combination of two-point functions rather than currents in order, for instance, to suppress large contributions from low dimensional terms that are irrelevant to non-perturbative properties of hadrons. They have successfully achieved a good sum rule that satisfy the necessary requirements. In our present study, our strategy is different from theirs, but the consideration along their idea is certainly important in the discussion of the tetraquark also.

5.1   Analysis of Single Diquark Currents

The scalar tetraquark currents have been classified in the previous section 3.1. There are five independent non-vanishing currents:

$$
\begin{aligned}
S_6 &= (\bar{s}_a\gamma_5 C\bar{s}_b^T)(u_a^T C\gamma_5 d_b)\,, \\
V_6 &= (\bar{s}_a\gamma_\mu\gamma_5 C\bar{s}_b^T)(u_a^T C\gamma^\mu\gamma_5 d_b)\,, \\
T_3 &= (\bar{s}_a\sigma_{\mu\nu} C\bar{s}_b^T)(u_a^T C\sigma^{\mu\nu} d_b)\,, \\
A_3 &= (\bar{s}_a\gamma_\mu C\bar{s}_b^T)(u_a^T C\gamma^\mu d_b)\,, \\
P_6 &= (\bar{s}_a C\bar{s}_b^T)(u_a^T C d_b)\,.
\end{aligned}
\tag{5.1}
$$

We can also construct ten currents by using quark-antiquark pairs:

$$
\begin{aligned}
S_1 &= (\bar{X}_a A_a)(\bar{Y}_b B_b) + (\bar{X}_a B_a)(\bar{Y}_b A_b)\,, \\
S_8 &= (\bar{X}_a\lambda_{ab}A_b)(\bar{Y}_c\lambda_{cd}B_d) + (\bar{X}_a\lambda_{ab}B_b)(\bar{Y}_c\lambda_{cd}A_d)\,, \\
V_1 &= (\bar{X}_a\gamma_\mu A_a)(\bar{Y}_b\gamma^\mu B_b) + (\bar{X}_a\gamma_\mu B_a)(\bar{Y}_b\gamma^\mu A_b)\,, \\
V_8 &= (\bar{X}_a\gamma_\mu\lambda_{ab}A_b)(\bar{Y}_c\gamma^\mu\lambda_{cd}B_d) + (\bar{X}_a\gamma_\mu\lambda_{ab}B_b)(\bar{Y}_c\gamma^\mu\lambda_{cd}A_d)\,, \\
T_1 &= (\bar{X}_a\sigma_{\mu\nu}A_a)(\bar{Y}_b\sigma^{\mu\nu}B_b) + (\bar{X}_a\sigma_{\mu\nu}B_a)(\bar{Y}_b\sigma^{\mu\nu}A_b)\,, \\
T_8 &= (\bar{X}_a\sigma_{\mu\nu}\lambda_{ab}A_b)(\bar{Y}_c\sigma^{\mu\nu}\lambda_{cd}B_d) + (\bar{X}_a\sigma_{\mu\nu}\lambda_{ab}B_b)(\bar{Y}_c\sigma^{\mu\nu}\lambda_{cd}A_d)\,, \\
A_1 &= (\bar{X}_a\gamma_\mu\gamma_5 A_a)(\bar{Y}_b\gamma^\mu\gamma_5 B_b) + (\bar{X}_a\gamma_\mu\gamma_5 B_a)(\bar{Y}_b\gamma^\mu\gamma_5 A_b)\,, \\
A_8 &= (\bar{X}_a\gamma_\mu\gamma_5\lambda_{ab}A_b)(\bar{Y}_c\gamma^\mu\gamma_5\lambda_{cd}B_d) + (\bar{X}_a\gamma_\mu\gamma_5\lambda_{ab}B_b)(\bar{Y}_c\gamma^\mu\gamma_5\lambda_{cd}A_d)\,, \\
P_1 &= (\bar{X}_a\gamma_5 A_a)(\bar{Y}_b\gamma_5 B_b) + (\bar{X}_a\gamma_5 B_a)(\bar{Y}_b\gamma_5 A_b)\,, \\
P_8 &= (\bar{X}_a\gamma_5\lambda_{ab}A_b)(\bar{Y}_c\gamma_5\lambda_{cd}B_d) + (\bar{X}_a\gamma_5\lambda_{ab}B_b)(\bar{Y}_c\gamma_5\lambda_{cd}A_d)\,.
\end{aligned}
$$

Among these ten currents, five are independent. By using them as well as their liner combinations, we can perform a QCD sum rule analysis. In this section, we perform a QCD sum rule analysis using the five independent diquark-antidiquark currents, separately.

Let us first outline briefly how we performed the OPE calculation. For illustration, let us take P_6. Then

$$
\begin{aligned}
\Pi(q^2) &\equiv i \int d^4x e^{iqx} \langle 0|TP_6(x)P_6^\dagger(0)|0\rangle \\
&= Tr[C(S_u^{aa'}(x))^T CS_d^{bb'}(x)]Tr[S_s^{a'a}(-x)C(S_s^{b'b}(-x))^T C] \\
&\quad + Tr[C(S_u^{aa'}(x))^T CS_d^{bb'}(x)]Tr[S_s^{b'a}(-x)C(S_s^{a'b}(-x))^T C] \, .
\end{aligned}
\tag{5.2}
$$

For the quark propagator, we use

$$
\begin{aligned}
iS_q^{ab}(x) &\equiv \langle 0|T[q^a(x)\bar{q}^b(0)]|0\rangle \\
&= \frac{i\delta^{ab}}{2\pi^2 x^4}\hat{x} + \frac{i}{32\pi^2}\frac{\lambda_{ab}^n}{2}g_c G_{\mu\nu}^n \frac{1}{x^2}(\sigma^{\mu\nu}\hat{x} + \hat{x}\sigma^{\mu\nu}) - \frac{\delta^{ab}}{12}\langle\bar{q}q\rangle \\
&\quad + \frac{\delta^{ab}x^2}{192}\langle g_c\bar{q}\sigma Gq\rangle - \frac{\delta^{ab}m_q}{4\pi^2 x^2} + \frac{i\delta^{ab}m_q}{48}\langle\bar{q}q\rangle\hat{x} + \frac{i\delta^{ab}m_q^2}{8\pi^2 x^2}\hat{x} \, .
\end{aligned}
\tag{5.3}
$$

The two-point function is then divided into three parts:

1. Terms proportional to δ^{ab} (a, b being color indices), where no soft gluon is emitted. The lowest term of this kind is the continuum term.

2. Terms containing one λ_{ab} (color matrix), where one soft gluon is emitted. The lowest terms of this type contain condensates such as $\langle g\bar{q}\sigma Gq\rangle$ ($q = u$ and d) and $\langle g\bar{s}\sigma Gs\rangle$.

3. Terms containing two λ_{ab}'s, where two soft gluons are emitted. The lowest terms of this type contain the condensate $\langle g^2G^2\rangle$.

We have performed the OPE calculation for the spectral function up to dimension eight, which is up to the constant (s^0) term of $\rho(s)$. Actual computation is very complicated. We have performed this calculation using *Mathematica* with *FeynCalc* [66]. *Mathematica* programs are available from the authors. The results are

$$
\begin{aligned}
\rho_{S6}(s) &= \frac{s^4}{61440\pi^6} - \frac{m_s^2 s^3}{3072\pi^6} + \left(\frac{m_s^4}{256\pi^6} - \frac{m_s\langle\bar{s}s\rangle}{192\pi^4} - \frac{\langle g^2GG\rangle}{12288\pi^6}\right)s^2 \\
&\quad + \left(-\frac{m_s^3\langle\bar{s}s\rangle}{32\pi^4} + \frac{m_s^2\langle g^2GG\rangle}{4096\pi^6} - \frac{m_s\langle g\bar{s}\sigma Gs\rangle}{64\pi^4} + \frac{\langle\bar{q}q\rangle^2}{24\pi^2} + \frac{\langle\bar{s}s\rangle^2}{24\pi^2}\right)s \\
&\quad - \frac{m_s^2\langle\bar{q}q\rangle^2}{12\pi^2} + \frac{m_s^2\langle\bar{s}s\rangle^2}{48\pi^2} + \frac{\langle\bar{q}q\rangle\langle g\bar{q}\sigma Gq\rangle}{24\pi^2} + \frac{m_s\langle g^2GG\rangle\langle\bar{s}s\rangle}{1536\pi^4} \\
&\quad + \frac{\langle\bar{s}s\rangle\langle g\bar{s}\sigma Gs\rangle}{24\pi^2} - \frac{m_s^4\langle g^2GG\rangle}{2048\pi^6} \, , \\
\rho_{V6}(s) &= \frac{s^4}{15360\pi^6} - \frac{5m_s^2 s^3}{1536\pi^6} + \left(\frac{m_s^4}{64\pi^6} + \frac{m_s\langle\bar{s}s\rangle}{24\pi^4} + \frac{5\langle g^2GG\rangle}{6144\pi^6}\right)s^2
\end{aligned}
\tag{5.4}
$$

$$+(-\frac{m_s^3\langle\bar{s}s\rangle}{8\pi^4}-\frac{11m_s^2\langle g^2GG\rangle}{2048\pi^6}+\frac{m_s\langle g\bar{s}\sigma Gs\rangle}{32\pi^4}-\frac{\langle\bar{q}q\rangle^2}{12\pi^2}-\frac{\langle\bar{s}s\rangle^2}{12\pi^2})s \quad (5.5)$$

$$+\frac{2m_s^2\langle\bar{q}q\rangle^2}{3\pi^2}+\frac{m_s^2\langle\bar{s}s\rangle^2}{12\pi^2}-\frac{\langle\bar{q}q\rangle\langle g\bar{q}\sigma Gq\rangle}{12\pi^2}$$

$$+\frac{7m_s\langle g^2GG\rangle\langle\bar{s}s\rangle}{768\pi^4}-\frac{\langle\bar{s}s\rangle\langle g\bar{s}\sigma Gs\rangle}{12\pi^2}\,,$$

$$\rho_{T3}(s) = \frac{s^4}{5120\pi^6}-\frac{m_s^2 s^3}{128\pi^6}+(\frac{3m_s^4}{64\pi^6}+\frac{m_s\langle\bar{s}s\rangle}{16\pi^4}+\frac{\langle g^2GG\rangle}{1536\pi^6})s^2$$

$$+(-\frac{3m_s^3\langle\bar{s}s\rangle}{8\pi^4}-\frac{m_s^2\langle g^2GG\rangle}{256\pi^6})s \quad (5.6)$$

$$+\frac{m_s^2\langle\bar{q}q\rangle^2}{\pi^2}+\frac{m_s^2\langle\bar{s}s\rangle^2}{4\pi^2}+\frac{m_s\langle g^2GG\rangle\langle\bar{s}s\rangle}{192\pi^4}-\frac{m_s^4\langle g^2GG\rangle}{256\pi^6}\,,$$

$$\rho_{A3}(s) = \frac{s^4}{30720\pi^6}-\frac{m_s^2 s^3}{1024\pi^6}+(\frac{m_s^4}{128\pi^6}+\frac{\langle g^2GG\rangle}{6144\pi^6})s^2$$

$$+(-\frac{m_s^3\langle\bar{s}s\rangle}{16\pi^4}-\frac{3m_s^2\langle g^2GG\rangle}{2048\pi^6}-\frac{m_s\langle g\bar{s}\sigma Gs\rangle}{64\pi^4}+\frac{\langle\bar{q}q\rangle^2}{24\pi^2}+\frac{\langle\bar{s}s\rangle^2}{24\pi^2})s \quad (5.7)$$

$$+\frac{m_s^2\langle\bar{s}s\rangle^2}{24\pi^2}+\frac{\langle\bar{q}q\rangle\langle g\bar{q}\sigma Gq\rangle}{24\pi^2}+\frac{m_s\langle g^2GG\rangle\langle\bar{s}s\rangle}{256\pi^4}+\frac{\langle\bar{s}s\rangle\langle g\bar{s}\sigma Gs\rangle}{24\pi^2}\,,$$

$$\rho_{P6}(s) = \frac{s^4}{61440\pi^6}-\frac{m_s^2 s^3}{1024\pi^6}+(\frac{m_s^4}{256\pi^6}-\frac{m_s\langle\bar{s}s\rangle}{64\pi^4}-\frac{\langle g^2GG\rangle}{12288\pi^6})s^2$$

$$+(-\frac{m_s^3\langle\bar{s}s\rangle}{32\pi^4}+\frac{3m_s^2\langle g^2GG\rangle}{4096\pi^6}+\frac{m_s\langle g\bar{s}\sigma Gs\rangle}{64\pi^4}-\frac{\langle\bar{q}q\rangle^2}{24\pi^2}-\frac{\langle\bar{s}s\rangle^2}{24\pi^2})s \quad (5.8)$$

$$+\frac{m_s^2\langle\bar{q}q\rangle^2}{4\pi^2}+\frac{m_s^2\langle\bar{s}s\rangle^2}{48\pi^2}-\frac{\langle\bar{q}q\rangle\langle g\bar{q}\sigma Gq\rangle}{24\pi^2}-\frac{m_s\langle g^2GG\rangle\langle\bar{s}s\rangle}{512\pi^4}$$

$$-\frac{\langle\bar{s}s\rangle\langle g\bar{s}\sigma Gs\rangle}{24\pi^2}-\frac{m_s^4\langle g^2GG\rangle}{2048\pi^6}\,.$$

In these equations, q represents a u or d quark, and s represents an s quark. $\langle\bar{q}q\rangle$ and $\langle\bar{s}s\rangle$ are dimension $D=3$ quark condensates; $\langle g^2GG\rangle$ is a $D=4$ gluon condensate; $\langle g\bar{q}\sigma Gq\rangle$ and $\langle g\bar{s}\sigma Gs\rangle$ are $D=5$ mixed condensates. As usual we assume the vacuum saturation for higher dimensional operators such as $\langle 0|\bar{q}q\bar{q}q|0\rangle \sim \langle 0|\bar{q}q|0\rangle\langle 0|\bar{q}q|0\rangle$. There is a minus sign in the definition of the mixed condensate $\langle g_s\bar{q}\sigma Gq\rangle$, which is different with some other QCD sum rule calculation. This is just because the definition of coupling constant g_s is different [85,177]. To obtain these results, we keep the terms of order $O(m_q^2)$ in the propagators of a massive quark in the presence of quark and gluon condensates:

$$iS^{ab} \equiv \langle 0|T[q^a(x)q^b(0)]|0\rangle$$

$$= \frac{i\delta^{ab}}{2\pi^2x^4}\hat{x}+\frac{i}{32\pi^2}\frac{\lambda_{ab}^n}{2}g_cG_{\mu\nu}^n\frac{1}{x^2}(\sigma^{\mu\nu}\hat{x}+\hat{x}\sigma^{\mu\nu})-\frac{\delta^{ab}}{12}\langle\bar{q}q\rangle$$

$$+\frac{\delta^{ab}x^2}{192}\langle g_c\bar{q}\sigma Gq\rangle-\frac{m_q\delta^{ab}}{4\pi^2x^2}+\frac{i\delta^{ab}m_q\langle\bar{q}q\rangle}{48}\hat{x}+\frac{i\delta^{ab}m_q^2}{8\pi^2x^2}\hat{x} \quad (5.9)$$

From these expressions, we observe the followings:

- The coefficients of the lowest dimension, or of the leading term in powers of s, have the relations $c_{S6}^{(4)} = c_{P3}^{(4)}$ and $c_{A3}^{(4)} = 1/2c_{V6}^{(4)}$. These are the consequences of chiral symmetry at the perturbative level [83].

- As empirically known, the terms of quark condensates have important contributions to the sum rule.

For numerical calculations, we use the following values of condensates [71, 85, 89, 99, 140, 148, 177, 179]:

$$\langle \bar{q}q \rangle = -(0.240\text{GeV})^3 \,,$$
$$\langle \bar{s}s \rangle = -(0.8 \pm 0.1) \times (0.240\text{GeV})^3 \,,$$
$$\langle g_s^2 GG \rangle = (0.48 \pm 0.14)\text{GeV}^4 \,,$$
$$m_s(1\text{GeV}) = 0.11\text{GeV} \,, \tag{5.10}$$
$$\langle g_s\bar{q}\sigma Gq \rangle = -M_0^2 \times \langle \bar{q}q \rangle \,,$$
$$M_0^2 = (0.8 \pm 0.2)\text{GeV}^2 \,.$$

In Fig. 5.1, we show all five spectral densities $\rho(s)$ as functions of s. From the definition of (1.23) in Chapter 1, the spectral densities should be positive definite quantities. In practical calculations, however, the positivity may not be necessarily realized, if the OPE up to finite terms does not work due to, for instance, bad choice of currents, weak coupling to physical states and so on. In the present analysis, we find that among the five cases, two functions of V_6 and P_6 currents show such a bad behavior. In particular, the P_6 current takes relatively large (in magnitude) negative values in an expectedly important region of $s \sim$ several GeV. Sum rule values then become negative when the threshold value is chosen around $s_0 \sim$ several GeV^2, which is not physically acceptable. The T_3 current changes the sign twice as in the case of V_6 and P_6 currents, from positive to negative and again to positive values. But the sum rule values take positive values for $s_0 \sim$ several GeV^2.

The tetraquark currents S_6 and A_3 are constructed by diquark fields which correspond to $^1S_0$ and $^3S_1$ in the non-relativistic language, where the two quarks can be in the ground state s-orbit. In contrast, the currents V_6 and P_6 correspond to linear combinations of $^3P_1$, and $^3P_0$, respectively, where one of the two quarks is in an excited p-orbit. The T_3 current is a linear combination of $^3S_1$ and $^1P_1$. Therefore, we verify an empirical fact that the sum rule constructed by currents having the s-wave components in the non-relativistic limit works better than those dominated by p-wave components. For completeness, we show the spectral densities with numerical coefficients for the three better cases, A_3, T_3 and S_6

$$\rho_{P6} = 1.69 \times 10^{-8}s^4 - 1.23 \times 10^{-8}s^3 - 2.35 \times 10^{-7}s^2$$
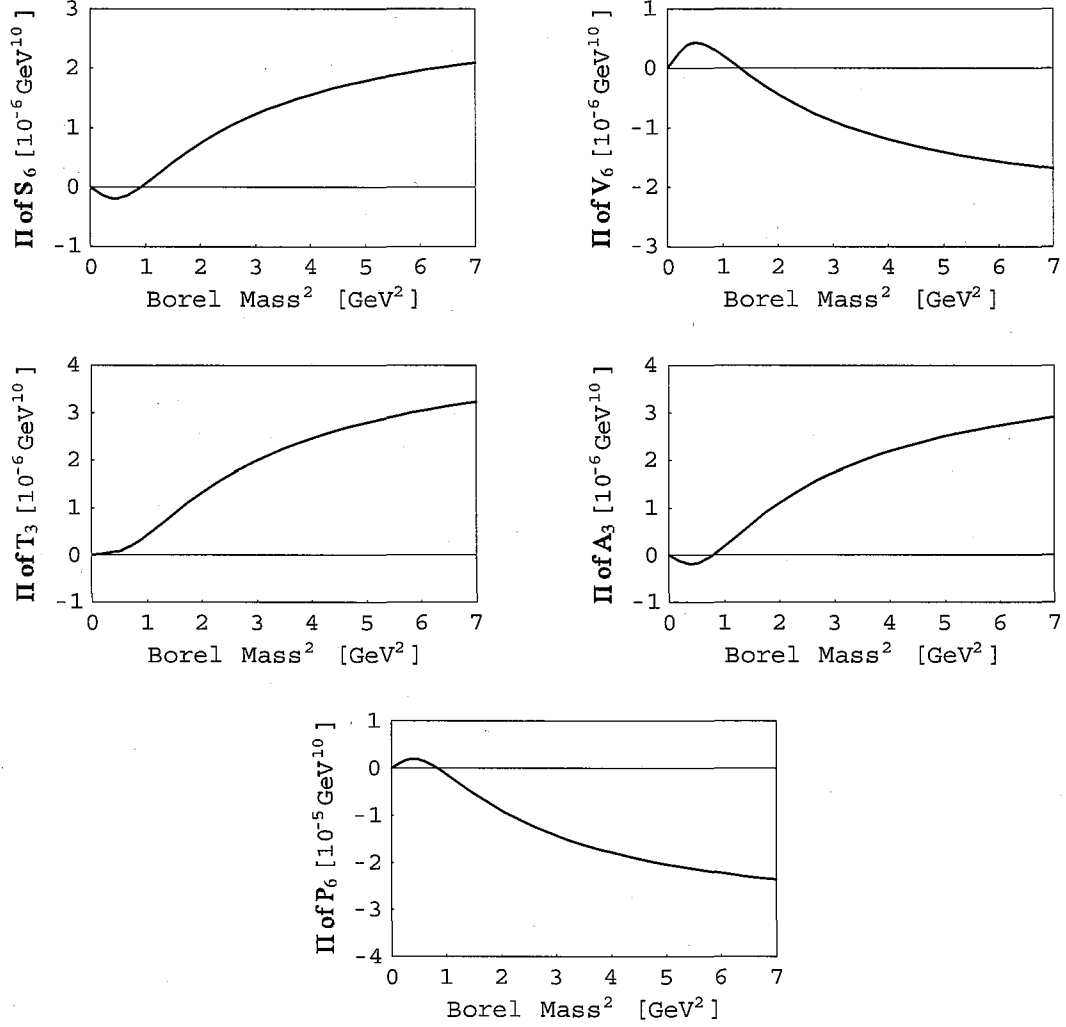
Figure 5.1: Spectral densities ρ_{S6}, ρ_{V6}, ρ_{T3}, ρ_{A3} and ρ_{P6} as functions of s, in units of GeV^8.

$$
\begin{aligned}
&\quad -1.16 \times 10^{-6} s + 1.13 \times 10^{-6}\,, \\
\rho_{A3} &= 3.39 \times 10^{-8} s^4 - 1.23 \times 10^{-8} s^3 + 8.14 \times 10^{-8} s^2 \\
&\quad + 1.17 \times 10^{-6} s - 1.08 \times 10^{-6}\,, \\
\rho_{T3} &= 2.03 \times 10^{-7} s^4 - 9.83 \times 10^{-8} s^3 - 4.53 \times 10^{-7} s^2 \\
&\quad + 3.34 \times 10^{-8} s + 2.41 \times 10^{-7}\,, \\
\rho_{V6} &= 6.77 \times 10^{-8} s^4 - 4.10 \times 10^{-8} s^3 - 1.17 \times 10^{-7} s^2
\end{aligned}
\tag{5.11}
$$

$$-2.35 \times 10^{-6}s + 2.23 \times 10^{-6},$$

$$\rho_{S6} = 1.69 \times 10^{-8}s^4 - 4.10 \times 10^{-9}s^3 + 2.55 \times 10^{-8}s^2$$

$$+1.17 \times 10^{-6}s - 1.08 \times 10^{-6}.$$

From these expressions, we observe that the convergence of the series does not seem very good. Nevertheless, let us proceed further.

As explained in the beginning of this chapter, there are two important parameters remaining in the sum rule analyses; they are the threshold value s_0 [GeV^2] and the Borel mass M_B [GeV]. For a good sum rule, the predicted masses should not depend on these two parameters strongly with sizable pole contribution (Borel window). In Fig. 5.2, we show the masses of the tetraquark as functions of the Borel mass for several threshold values s_0 (Borel curves). We observe that the Borel mass dependence is somewhat strong for the currents S_6 and A_3 in the region $1 < M_B^2 < 2$ GeV^2, which is expected to be a reasonable choice of the Borel mass. For these currents S_6 and A_3, however, we see that the minimum occurs at around 3 GeV^2 when s_0 is varied in the region $M_B^2 \gtrsim 1.5$ GeV^2. (For the current S_6, the mass of $s_0 = 2$ GeV^2 is far above the region shown in the figure.) For this reason, we consider that $s_0 = 3$ GeV^2 is a reasonable choice which we will mainly use for the estimation of the mass of the tetraquark in the following sum rule analyses. At this s_0 value, the mass of the tetraquark turns out to be about 1.6 GeV. For the T_3 current, the Borel stability seems better. The result, however, depends on the threshold value s_0 to some extent. However, it is interesting to see that the mass of the tetraquark is about 1.6 GeV when $s_0 \sim 3$ GeV^2.

From the analysis of the single current of the diquark construction, we expect that the mass of the tetraquark is about 1.6 GeV, although the stability against the variation of both the Borel mass and the threshold parameter is not simultaneously achieved. As we will see, however, a suitable linear combination will improve the stability.

5.2 Analysis of Single Mesonic Currents

In this section, we perform QCD sum rule analysis using the ten mesonic currents, $S_{1,8}$, $V_{1,8}$, $T_{1,8}$, $A_{1,8}$ and $P_{1,8}$, separately. Here we only show two important spectral densities:

$$\rho_{V8}(s) = \frac{s^4}{110592\pi^6} - \frac{19m_s^2 s^3}{55296\pi^6} + \left(\frac{5m_s^4}{2304\pi^6} - \frac{m_s\langle\bar{q}q\rangle}{432\pi^4} + \frac{m_s\langle\bar{s}s\rangle}{432\pi^4} + \frac{17\langle g^2 GG\rangle}{221184\pi^6}\right)s^2$$

$$+\left(\frac{m_s^3\langle\bar{q}q\rangle}{72\pi^4} - \frac{5m_s^3\langle\bar{s}s\rangle}{288\pi^4} - \frac{13m_s^2\langle g^2 GG\rangle}{24576\pi^6} + \frac{m_s\langle g\bar{q}Gq\rangle}{2304\pi^4} - \frac{5m_s\langle g\bar{s}Gs\rangle}{4608\pi^4}\right.$$

$$\left.+\frac{\langle\bar{q}q\rangle^2}{432\pi^2} + \frac{\langle\bar{s}s\rangle^2}{432\pi^2} + \frac{\langle\bar{q}q\rangle\langle\bar{s}s\rangle}{54\pi^2}\right)s + \frac{m_s^2\langle\bar{q}q\rangle^2}{27\pi^2} + \frac{5m_s^2\langle\bar{s}s\rangle^2}{432\pi^2} \qquad (5.12)$$

$$-\frac{m_s\langle\bar{q}q\rangle\langle g^2 GG\rangle}{6912\pi^4} + \frac{5\langle\bar{q}q\rangle\langle g\bar{q}Gq\rangle}{1728\pi^2} + \frac{m_s^3\langle g\bar{q}Gq\rangle}{144\pi^4} - \frac{m_s^2\langle\bar{q}q\rangle\langle\bar{s}s\rangle}{18\pi^2} - \frac{\langle g\bar{q}Gq\rangle\langle\bar{s}s\rangle}{864\pi^2}$$
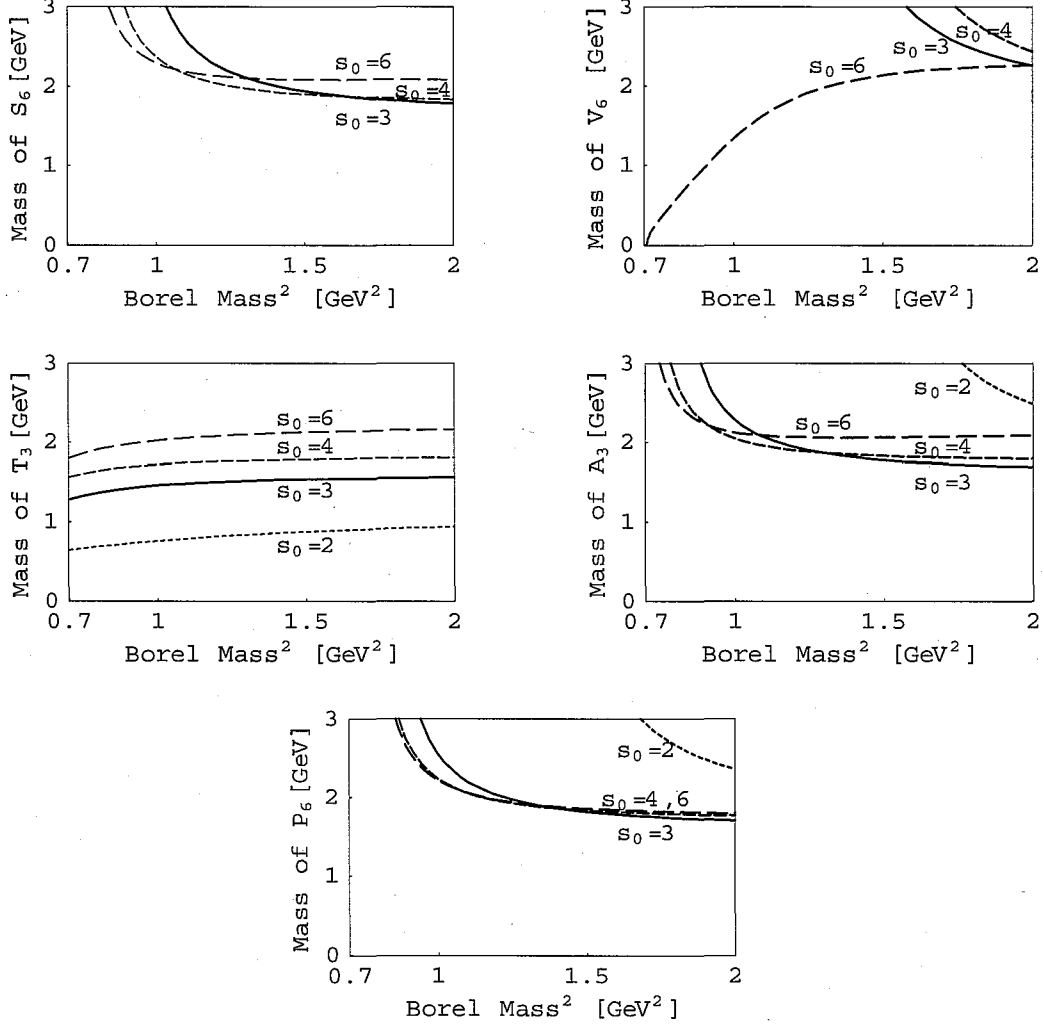
Figure 5.2: Mass of the tetraquark calculated by the three currents S_6, V_6, T_3, A_3 and P_6 as a function of the Borel mass square M_B^2 for several threshold values $s_0 = 2, 3, 4$ and 6 GeV^2.

$$
\begin{aligned}
&+\frac{m_s\langle g^2 GG\rangle\langle\bar{s}s\rangle}{1024\pi^4} - \frac{\langle\bar{q}q\rangle\langle g\bar{s}Gs\rangle}{864\pi^2} + \frac{5\langle\bar{s}s\rangle\langle g\bar{s}Gs\rangle}{1728\pi^2} - \frac{m_s^4\langle g^2 GG\rangle}{9216\pi^6} \ , \\
\rho_{T8}(s) =\ &\frac{s^4}{18432\pi^6} - \frac{5m_s^2 s^3}{2304\pi^6} + \Big(\frac{5m_s^4}{384\pi^6} + \frac{5m_s\langle\bar{s}s\rangle}{288\pi^4} + \frac{31\langle g^2 GG\rangle}{55296\pi^6}\Big)s^2 \\
&+\Big(-\frac{5m_s^3\langle\bar{s}s\rangle}{48\pi^4} - \frac{31m_s^2\langle g^2 GG\rangle}{9216\pi^6}\Big)s + \frac{5m_s^2\langle\bar{q}q\rangle^2}{18\pi^2} + \frac{5m_s^2\langle\bar{s}s\rangle^2}{72\pi^2}
\end{aligned}
\tag{5.13}
$$

$$+\frac{31 m_s \langle g^2 GG \rangle \langle \bar{s}s \rangle}{6912\pi^4} - \frac{13 m_s^4 \langle g^2 GG \rangle}{9216\pi^6}.$$

As shown in Fig. 5.3, we find that two spectral densities for V_8 and T_8 show good behavior: ρ_{T8} is positive definite, while ρ_{V8} takes negative values in the small region $s \leq 0.2 \text{GeV}^2$.

The currents V_1, V_8, P_1 and P_8 are constructed by mesonic fields (either color singlet or color octet) which correspond to $^3S_1$ and $^1S_0$ in the non-relativistic language, where two quark-antiquark pairs can be in the ground state s-orbit. Their spectral densities then show similar behavior to S_6 and A_3 in the previous subsection. In contrast, S_1, S_8, A_1 and A_8 correspond to linear combinations of $^3P_0$ and $^3P_1$, respectively; T_1 and T_8 currents are the combinations of $^3S_1$ and $^1P_1$.

From the above argument, we might expect that six currents, V_1, V_8, P_1, P_8, T_1 and T_8 would work. However, if we test another condition that the quantity

$$f_X^2 e^{-M^2/M_B^2} = \int_0^{s_0} e^{-s/M_B^2} \rho(s) ds, \qquad (5.14)$$

should be positive around $s_0 \sim$ several GeV^2, we found that those by the currents V_1, P_1, P_8 and T_1 take negative values and therefore, they must be abandoned. Now there remain only two better currents V_8 and T_8 in the mesonic construction. This is the reason that we have shown their spectral densities in (5.12) and (5.13). Using the numerical values of various condensates (5.10), we find the spectral densities

$$\begin{aligned}
\rho_{V8} &= 9.41 \times 10^{-9} s^4 - 4.32 \times 10^{-9} s^3 + 4.54 \times 10^{-8} s^2 \\
&\quad +3.52 \times 10^{-7} s - 4.85 \times 10^{-8}, \\
\rho_{T8} &= 5.64 \times 10^{-8} s^4 - 2.73 \times 10^{-8} s^3 + 6.14 \times 10^{-8} s^2 \qquad (5.15) \\
&\quad -4.32 \times 10^{-9} s + 4.89 \times 10^{-8}.
\end{aligned}$$

Once again the convergence of the series does not seem very good, though the coefficient of the constant term of ρ_{V8} (-4.85×10^{-8}) is smaller by about factor ten than that of the first order term of s^1 (3.52×10^{-7}).

In Fig. 5.4, we show the masses of the tetraquark currents V_8 and T_8 as functions of the Borel mass for several threshold values s_0 (Borel curves). As in the case of T_3 current, the Borel stability seems good but the result depends on the threshold value s_0. However, once again, if we take the threshold value at $s_0 \sim 3 \text{ GeV}^2$, the mass of the tetraquark turns out to be reasonable, though the precise values are slightly smaller: the mass of $T_8 \sim 1.5$ GeV and the mass of $V_8 \sim 1.4$ GeV.

5.3 Analysis of Mixed Currents

In order to improve the sum rule, we attempt to make linear combinations of independent currents for both diquark and mesonic currents. Since linear combinations of five currents
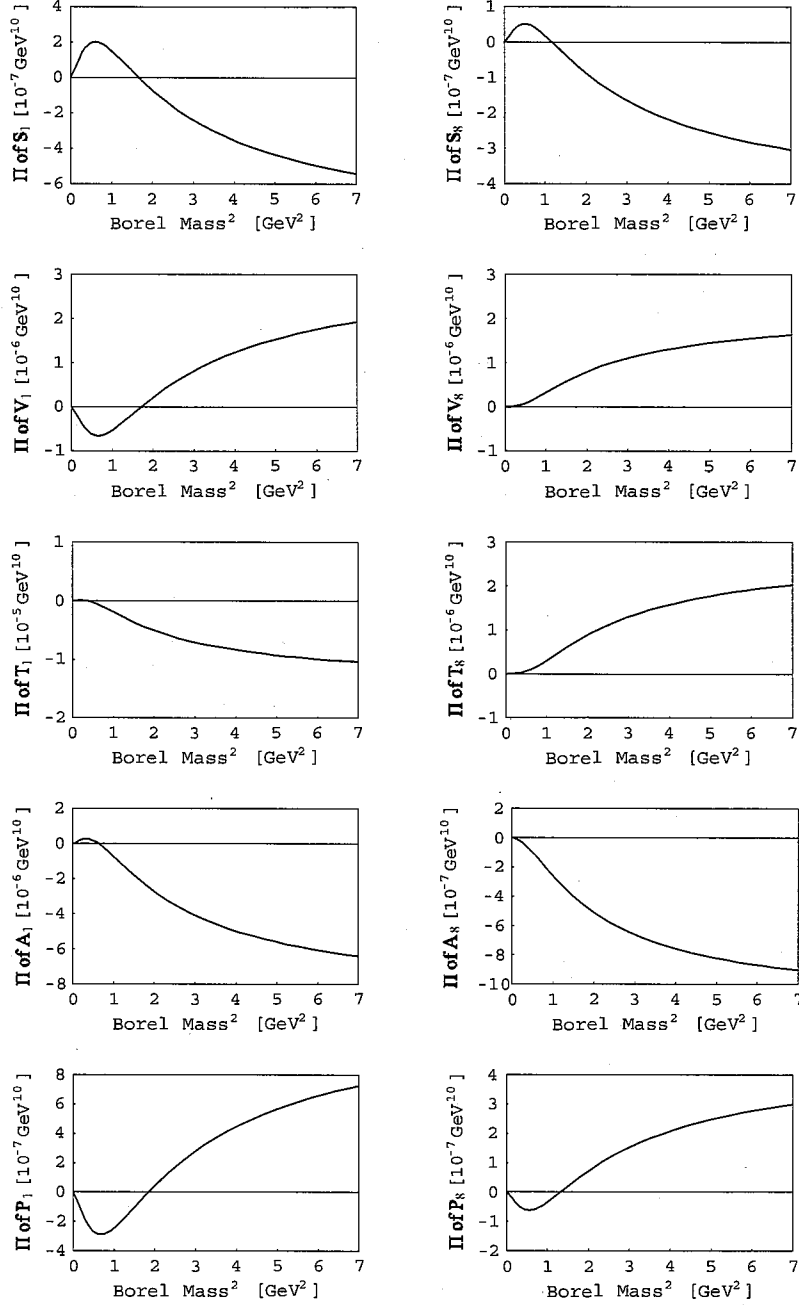
Figure 5.3: Spectral densities ρ_{S1}, ρ_{S8}, ρ_{V1}, ρ_{V8}, ρ_{T1}, ρ_{T8}, ρ_{A1}, ρ_{A8}, ρ_{P1} and ρ_{P8} as functions of s, in units of GeV^8.
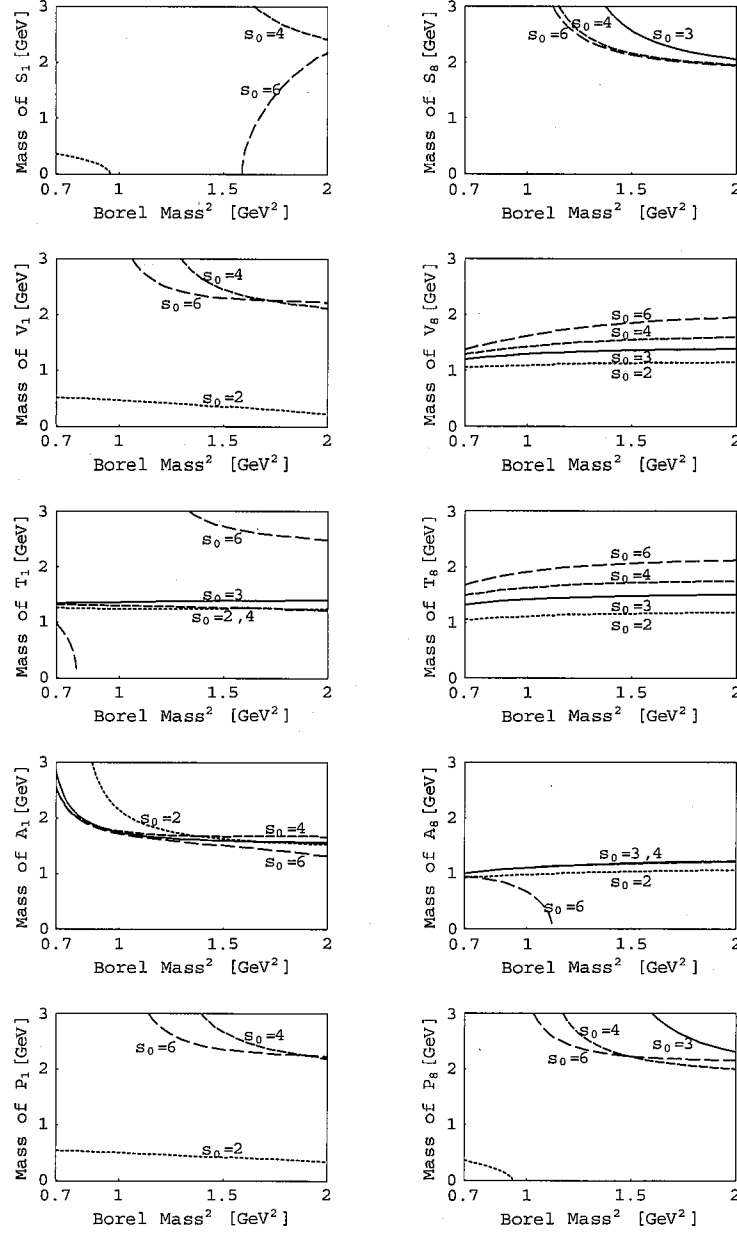
Figure 5.4: Mass of the tetraquark calculated by the currents S_1, S_8, V_1, V_8, T_1, T_8, A_1, A_8, P_1 and P_8, as a function of the Borel mass square M_B^2 for several threshold values $s_0 = 2, 3, 4$ and 6 GeV^2.

contain ten mixing angles, the full consideration with these ten parameters is rather cumbersome. Instead, we make a linear combination of two currents J_1 and J_2 (any two from the independent currents), $\eta = \cos\theta J_1 + \sin\theta J_2$, where θ is a mixing angle. Then the correlation functions are written as

$$\langle \eta\eta^\dagger \rangle = \cos^2\theta\langle J_1 J_1^\dagger \rangle + \sin^2\theta\langle J_2 J_2^\dagger \rangle + \cos\theta\sin\theta\langle J_1 J_2^\dagger \rangle + \cos\theta\sin\theta\langle J_2 J_1^\dagger \rangle. \qquad (5.16)$$

The mixing is chosen with the following requirements:

1. The OPE has a good convergence as going to terms of higher dimensional operators.

2. The spectral density becomes positive quantity for all (or almost all) s values.

3. Pole contribution is sufficiently large.

We have tried various combinations of two currents to realize good sum rules. While doing so, we have realized that the diquark currents are more independent than the mesonic currents. This means that the cross terms of (5.16) have only a minor contribution for diquark currents, while they have a large contribution for mesonic currents.

According to the requirement (1), we would like to make a linear combination such that the highest dimensional (eight) term is suppressed. For diquark currents, we find it convenient to take two combinations:

$$\eta = \cos\theta A_3 + \sin\theta V_6, \qquad (5.17)$$
$$\xi = \cos\theta P_6 + \sin\theta S_6. \qquad (5.18)$$

By choosing $\cot\theta \sim \sqrt{2}$, we find that the term of dimension eight of (5.17) is suppressed, while for $\cot\theta \sim 1$, the term of dimension eight of (5.18) is suppressed. The spectral density of (5.18), however, takes negative values. Therefore, this current should be rejected for the sum rule analysis. In this way we are lead to the current η of (5.17). From now on, we will denote $\eta \to \eta_1$.

For the mesonic case, it turns out that the cross term contributions are large. Accordingly, we attempt a complex angle to improve the sum rule analysis. By choosing $t_1 = 0.91$, $t_2 = -0.41$, we construct a current:

$$\eta_2 = S_1 + (t_1 + it_2)P_1. \qquad (5.19)$$

The numerical spectral densities are:

$$\begin{aligned}
\rho_1 &= 4.5 \times 10^{-8}s^4 - 2.2 \times 10^{-8}s^3 + 2.4 \times 10^{-7}s^2 \qquad (5.20)\\
&\quad -2.0 \times 10^{-8}s + 5.2 \times 10^{-9}, \\
\rho_2 &= 2.1 \times 10^{-8}s^4 - 1.0 \times 10^{-8}s^3 + 4.2 \times 10^{-8}s^2 \\
&\quad -2.2 \times 10^{-8}s + 8.3 \times 10^{-9},
\end{aligned}$$

which may be compared with the spectral densities of the single currents (5.12) and (5.15). It looks that the convergence of the series is improved significantly.

In Fig. 5.5, we show the mass calculated from η_1 and η_2 as functions of the Borel mass square for several threshold values s_0. The Borel stability is improved from the cases of the single currents. Furthermore, the dependence on s_0 is also reduced. When $s_0 \sim 3$ GeV^2, we find the mass calculated from the two currents η_1 and η_2 is about 1.5 GeV.
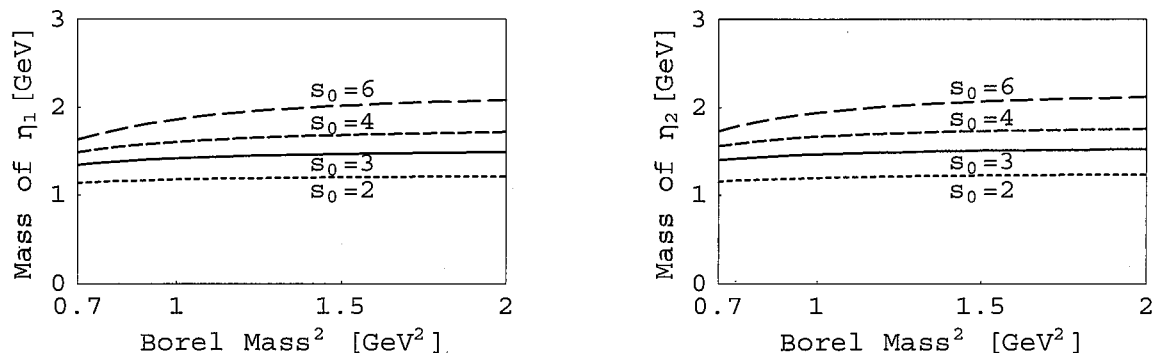
Figure 5.5: Mass of the tetraquark calculated by the mixing currents η_1(Left) and η_2(Right), as a function of the Borel mass square M_B^2 for several threshold values $s_0 = 2, 3, 4$ and 6 GeV^2.

At this point we should also comment on the pole contribution in the sum rule. Generally we expect that the pole contribution should dominate the sum rule, preferably at least more than several tens percent. In the present case, the pole contribution, however, is not always dominant. We have found that it reaches up to 20 percent when we use η_1 and η_2 and the Borel mass is chosen around 1 GeV. As the Borel mass increases, the pole contribution decreases. This would be a general problem for the QCD sum rule for currents of a high dimension, typically for exotic hadrons. Nevertheless, it is interesting to see that a good Borel mass stability has been achieved as shown in Figs. 5.5. In any event, we need further investigations as proposed by Kojo et al [115] to check the stability of the sum rule.

Finally, in order to summarize our analysis, we show in Fig. 5.6 masses of the tetraquark calculated by several reasonable currents used in the present study as functions of the Borel mass square at $s_0 = 3$ GeV^2. They are S_6, A_3 and T_3 for the diquark construction, T_8 and V_8 for the mesonic construction, and η_1 and η_2 for the mixing currents. The plots are extended to a wider region of M_B^2 up to 4 GeV^2. We verify once again a good Borel mass stability for the mixing currents, while some of the single currents show good stability also (T_3, T_8 and V_8). The mass values varies slightly, while we expect the mass of the tetraquark around 1.5 GeV.
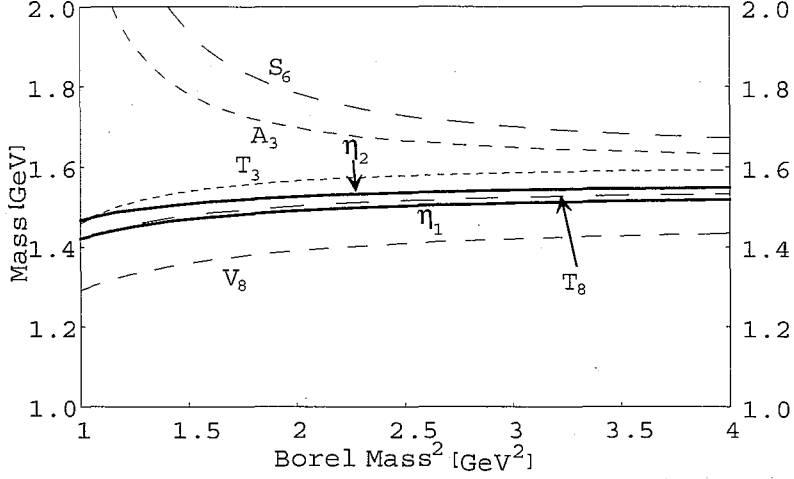
Figure 5.6: Mass of the tetraquark calculated by the currents η_1, η_2, A_3, S_6, T_3, V_8 and T_8, as a function of the Borel mass square M_B^2 in the region $2 < M_B^2 < 4$ GeV^2 for threshold value $s_0 = 3$ GeV^2.

5.4   Conclusion

We have presented a QCD sum rule study of the $uds̄s̄$ tetraquark of $J^P = 0^+$ and $I = 1$, both in the diquark $((\bar{q}\bar{q})(qq))$ and mesonic $((\bar{q}q)(\bar{q}q))$ constructions. We have found that in this channel of tetraquark, there are five independent currents, which is shown both in the diquark and mesonic constructions. For each single current, we have tested the sum rule analysis, but it is found that not all of them provide a good stability.

As an attempt to improve the stability of the sum rule, we have considered linear combinations of independent currents. In order to simplify the analysis, we took a superposition of various combinations of two currents. Among them, we have found two cases that lead to good sum rules, where we investigated s_0 (threshold value) and M_B (Borel mass) dependence, and convergence of OPE. A reasonable choice of the threshold value is taken at $s_0 \sim 3$ GeV^2. A good Borel stability is then achieved in the region $1 \lesssim M_B^2 \lesssim 4$ GeV^2, where the mass of the tetraquark turns out to be around 1.5 GeV.

Despite the seemingly good Borel mass stability, we think that we should investigate the following points more carefully. For instance, estimation of higher dimensional terms of $\mathcal{O}(1/s)$ could be important, as we have found that the pole contribution is around 20% at best. These problems might be related to the high dimensional operators for exotic particles. Another question is the contribution of KK scattering states, since the mass of the tetraquark is around 1.5 GeV, and it can fall apart into the KK states. Such a contribution can be estimated by using the method proposed in Refs. [117, 121]. These will be further investigated in the future work.

Chapter 6

Light Scalar Tetraquark Mesons

The light scalar mesons $\sigma(600)$, $\kappa(800)$, $a_0(980)$ and $f_0(980)$ compose a nonet with the mass below 1 GeV [5,8,9,13–15,17,179]. Almost thirty years ago, Jaffe suggested that they can be tetraquark candidates, which can explain the mass spectrum of the light scalar mesons and also their decay properties [93] (See also Ref. [98] for recent progress).

So far, several different pictures for the scalar mesons have been proposed. In the conventional quark model, they have a $\bar{q}q$ configuration of $^3P_0$ whose masses are expected to be larger than 1 GeV due to the p-wave orbital excitation [50]. Moveover, by a naively counting of the quark mass, the mass ordering should be $m_\sigma \sim m_{a_0} < m_\kappa < m_{f_0}$. They are regarded as chiral partners of the Nambu-Goldstone bosons in chiral models(π, K, η, η') [79], and their masses are expected to be lower than those of the quark model due to their collective nature. Yet another interesting picture is that they are tetraquark states [7,31,32,97,128,134,168,174,181]. In contrast with the $\bar{q}q$ states, their masses are expected to be around 0.6 – 1 GeV with the ordering of $m_\sigma < m_\kappa < m_{a_0,f_0}$, consistent with the recent experimental observations [5,13,14,179]. The lightness of these states is expected to be explained by the strong attractive quark correlation in the scalar and isoscalar channel. There are some lattice studies supporting this [125,162]. Besides their masses, the decay properties are also interesting and important, and are studied in many papers [35,74,75,151,184].

In this chapter, we perform the QCD sum rule analysis for the light scalar mesons. We find once again that there are five independent currents for each scalar tetraquark state. We perform a reliable QCD sum rule by using mixed currents as in the previous chapter, and obtain the masses of the light scalar mesons. The results are consistent with the experiments.

Unlike $\bar{q}q$ and qqq currents, tetraquark currents have complicated structure due to multiquark degrees of freedom. As we will discuss in the next section in detail, there are some independent currents for a given spin with different flavor structures. This is very much different from the ground state baryons, where different flavor representations **8** and **10** correspond to different spins **1/2** and **3/2**, which induce a mass splitting between

$\Delta(1232)$ and $N(939)$.

In this chapter, first we construct the tetraquark currents using diquark and antidiquark fields having the antisymmetric flavor $\bar{\mathbf{3}}_\mathbf{f} \otimes \mathbf{3}_\mathbf{f}$, which is in accordance with the expected light scalar nonet. Furthermore, we construct another set of tetraquark currents by using diquark and antidiquark fields having the symmetric flavor $\mathbf{6}_\mathbf{f} \otimes \bar{\mathbf{6}}_\mathbf{f}$. We do not, however, consider other possibilities such as $\mathbf{6}_\mathbf{f} \otimes \bar{\mathbf{3}}_\mathbf{f}$, since they can not produce tetraquark currents having the scalar quantum numbers (color singlet and $J^P = 0^+$). Then as we have done previously [38], we show that there are five independent currents for both constructions. We will then search linear combinations of the currents that optimize the QCD sum rule and reproduce the results compatible with the expected light scalar mesons. While performing a QCD sum rule analysis, we also find that the results of the two constructions have some similarities. In fact, if we work in the $SU(3)_f$ limit, we obtain identical results for the operator product expansion (OPE).

Since the scalar mesons, especially σ, decays strongly to two pseudoscalar mesons, their effects should be significant for quantitative discussions. The contamination from such two-meson decay should be removed when performing the QCD sum rule analysis, which is however a difficult theoretical problem so far. Nevertheless we consider a phenomenological method by adding another parameter corresponding to a decay width for the QCD sum rule analysis.

6.1   Tetraquark Currents

In order to make a scalar tetraquark current, the diquark and antidiquark fields should have the same color, spin and orbital symmetries. Therefore, they must have the same flavor symmetry, which is either antisymmetric ($\bar{\mathbf{3}}_\mathbf{f} \otimes \mathbf{3}_\mathbf{f}$) or symmetric ($\mathbf{6}_\mathbf{f} \otimes \bar{\mathbf{6}}_\mathbf{f}$). The possible flavor quantum numbers of the tetraquark states are then

$$\begin{aligned}
\bar{\mathbf{3}}_\mathbf{f} \otimes \mathbf{3}_\mathbf{f} &= \mathbf{1}_\mathbf{f} \oplus \mathbf{8}_\mathbf{f}\,, \\
\mathbf{6}_\mathbf{f} \otimes \bar{\mathbf{6}}_\mathbf{f} &= \mathbf{1}_\mathbf{f} \oplus \mathbf{8}_\mathbf{f} \oplus \mathbf{27}_\mathbf{f}\,,
\end{aligned} \tag{6.1}$$

where the corresponding weight diagrams are shown in Fig. 6.1. The scalar nonet $\mathbf{1} + \mathbf{8}$ is therefore included in both representations, independently. For $\bar{\mathbf{3}}_\mathbf{f} \times \mathbf{3}_\mathbf{f} = \mathbf{1}_\mathbf{f} + \mathbf{8}_\mathbf{f}$, κ and a_0 are the members of $\mathbf{8}_\mathbf{f}$ while σ and f_0 can be either in $\mathbf{1}_\mathbf{f}$ or in isospin $I = 0$ component of $\mathbf{8}_\mathbf{f}$. Or, they can also mix and in particular the ideal mixing is achieved by

$$\begin{aligned}
|\sigma\rangle &= \sqrt{\frac{1}{3}}|\mathbf{1}_\mathbf{f}\rangle - \sqrt{\frac{2}{3}}|\mathbf{8}_\mathbf{f}, I = 0\rangle\,, \\
|f_0\rangle &= \sqrt{\frac{2}{3}}|\mathbf{1}_\mathbf{f}\rangle + \sqrt{\frac{1}{3}}|\mathbf{8}_\mathbf{f}, I = 0\rangle\,,
\end{aligned} \tag{6.2}$$

where only isospin symmetry is respected and the currents are classified by the number of strange quarks. We can find another set of linear combinations for the symmetric case.

Hence, denoting light u, d quarks by q, σ currents are constructed as $qq\bar{q}\bar{q}$, κ currents by $qs\bar{q}\bar{q}$ and a_0 and f_0 currents by $qs\bar{q}\bar{s}$. A naive additive quark counting for this construction is consistent with the observed masses, $\sigma(600)$, $\kappa(800)$, $a_0(980)$ and $f_0(980)$. Also, in the QCD sum rule we find that the ideal mixing is needed in order to reproduce the expected mass pattern of σ, κ, a_0 and f_0.

$$\bar{3} \times 3 = 1 + 8 \qquad\qquad 6 \times \bar{6} = 1 + 8 + 27$$
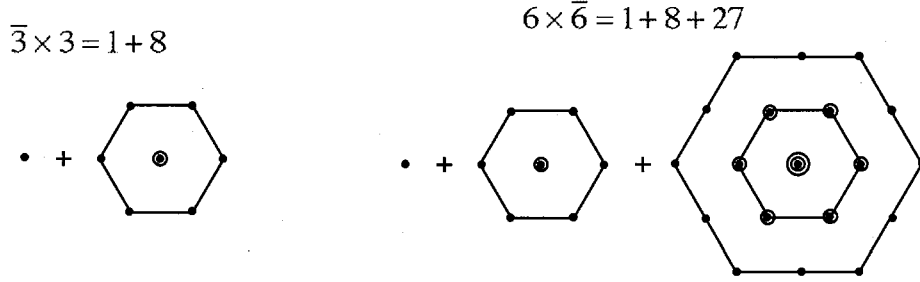
Figure 6.1: SU(3) weight diagrams for tetraquark states of antisymmetric and symmetric diquarks (antidiquarks).

Using the antisymmetric combination for diquark flavor structure, we arrive at the following five independent currents which have been shown in Chapter 4:

$$
\begin{aligned}
S_3^\sigma &= (u_a^T C\gamma_5 d_b)(\bar{u}_a\gamma_5 C\bar{d}_b^T - \bar{u}_b\gamma_5 C\bar{d}_a^T)\,, \\
V_3^\sigma &= (u_a^T C\gamma_\mu\gamma_5 d_b)(\bar{u}_a\gamma^\mu\gamma_5 C\bar{d}_b^T - \bar{u}_b\gamma^\mu\gamma_5 C\bar{d}_a^T)\,, \\
T_6^\sigma &= (u_a^T C\sigma_{\mu\nu} d_b)(\bar{u}_a\sigma^{\mu\nu} C\bar{d}_b^T + \bar{u}_b\sigma^{\mu\nu} C\bar{d}_a^T)\,, \qquad\qquad (6.3)\\
A_6^\sigma &= (u_a^T C\gamma_\mu d_b)(\bar{u}_a\gamma^\mu C\bar{d}_b^T + \bar{u}_b\gamma^\mu C\bar{d}_a^T)\,, \\
P_3^\sigma &= (u_a^T C d_b)(\bar{u}_a C\bar{d}_b^T - \bar{u}_b C\bar{d}_a^T)\,.
\end{aligned}
$$

where the sum over repeated indices ($\mu, \nu, \cdots$ for Dirac, and $a, b, \cdots$ for color indices) is taken. Either plus or minus sign in the second parentheses ensures that the diquarks form the antisymmetric combination in the flavor space. The currents S, V, T, A and P are constructed by scalar, vector, tensor, axial-vector, pseudoscalar diquark and antidiquark fields, respectively. The subscripts 3 and 6 show that the diquarks (antidiquark) are combined into the color representation $\bar{3}_c$ and 6_c (3_c or $\bar{6}_c$), respectively.

We will perform the sum rule analysis using all currents and their various linear combinations. As we have found in the previous chapter, again the results for single currents are not always reliable. In fact, we will find a good sum rule by a linear combination of A_6^σ and V_3^σ

$$\eta_1^\sigma = \cos\theta A_6^\sigma + \sin\theta V_3^\sigma\,, \qquad\qquad (6.4)$$

where θ is the mixing angle. As we will discuss in Sec. 6.3, the best choice of the mixing angle turns out to be $\cot\theta = 1/\sqrt{2}$. The mixed currents for κ, a_0 and f_0 can be found in

the similar way

$$\begin{aligned}
\eta_1^\kappa &= \cos\theta A_6^\kappa + \sin\theta V_3^\kappa, \\
\eta_1^{a_0} &= \cos\theta A_6^{a_0} + \sin\theta V_3^{a_0}, \\
\eta_1^{f_0} &= \cos\theta A_6^{f_0} + \sin\theta V_3^{f_0}.
\end{aligned} \tag{6.5}$$

where the best choices are still $\cot\theta = 1/\sqrt{2}$.

The QCD sum rule results for a_0 and f_0 will give the same results in the QCD sum rule, which is consistent with the experimental masses of a_0 and f_0. For simplicity, we will use the charged a_0 current

$$\begin{aligned}
\eta_1^{a_0} &= \cos\theta A_6^{a_0+} + \sin\theta V_3^{a_0+} \\
&= \cos\theta (u_a^T C\gamma_\mu s_b)(\bar{d}_a\gamma^\mu C\bar{s}_b^T + \bar{d}_b\gamma^\mu C\bar{s}_a^T) + \sin\theta (u_a^T C\gamma_\mu\gamma_5 s_b)(\bar{d}_a\gamma^\mu\gamma_5 C\bar{s}_b^T - \bar{d}_b\gamma^\mu\gamma_5 C\bar{s}_a^T).
\end{aligned} \tag{6.6}$$

We can also construct the tetraquark currents of $J^P = 0^+$ whose diquark and antidiquark have the symmetric flavor structure. We use the same superscripts σ, κ and a_0 because of the same quark contents. There are five independent currents

$$\begin{aligned}
S_6^\sigma &= q_a^T C\gamma_5 q_b(\bar{q}_a\gamma_5 C\bar{q}_b^T + \bar{q}_b\gamma_5 C\bar{q}_a^T), \\
V_6^\sigma &= q_a^T C\gamma_\mu\gamma_5 q_b(\bar{q}_a\gamma^\mu\gamma_5 C\bar{q}_b^T + \bar{q}_b\gamma^\mu\gamma_5 C\bar{q}_a^T), \\
T_3^\sigma &= q_a^T C\sigma_{\mu\nu} q_b(\bar{q}_a\sigma^{\mu\nu} C\bar{q}_b^T - \bar{q}_b\sigma^{\mu\nu} C\bar{q}_a^T), \\
A_3^\sigma &= q_a^T C\gamma_\mu q_b(\bar{q}_a\gamma^\mu C\bar{q}_b^T - \bar{q}_b\gamma^\mu C\bar{q}_a^T), \\
P_6^\sigma &= q_a^T C q_b(\bar{q}_a C\bar{q}_b^T + \bar{q}_b C\bar{q}_a^T).
\end{aligned} \tag{6.7}$$

The quark contents are $\frac{1}{\sqrt{6}}(\{uu\}\{\bar{u}\bar{u}\} - 2\{ud\}\{\bar{u}\bar{d}\} + \{dd\}\{\bar{d}\bar{d}\})$ which compose an isoscalar tetraquark. Either plus or minus sign in the second parentheses ensures that the diquarks form the symmetric combination in the flavor space. We construct the similar mixed currents for κ, a_0 and f_0

$$\begin{aligned}
\eta_2^\sigma &= \cos\theta A_3^\sigma + \sin\theta V_6^\sigma, \\
\eta_2^\kappa &= \cos\theta A_3^\kappa + \sin\theta V_6^\kappa, \\
\eta_2^{a_0} &= \cos\theta A_3^{a_0} + \sin\theta V_6^{a_0}, \\
\eta_2^{f_0} &= \cos\theta A_3^{f_0} + \sin\theta V_6^{f_0},
\end{aligned} \tag{6.8}$$

Here the optimal choice of the mixing angle is $\cot\theta = \sqrt{2}$ for η_2^σ and $\eta_2^{a_0}$, but with a slightly different value for η_2^κ, which is 1.37. This shift is used to keep the spectral density positive, and is due to the nonzero strange quark mass.

The currents η_1 and η_2 have similar structure. We can interchange them under the exchange of $\gamma_\mu \leftrightarrow \gamma_\mu\gamma_5$. We choose the mixing angle $\cot\theta = 1/\sqrt{2}$ for η_1, which corresponds to $\cot\theta = \sqrt{2}$ for η_2.

Concerning linear combinations, we have tested more general cases by using all five currents. However, we could not find significant improvements over the present results of using the two currents.

In Table 1.1, we show the diquark properties of ten single currents. The parity can be obtained by using $P = (-)^L$, which L is the orbital momentum. The structures of tetraquark currents are complicated. The flavor symmetry is not subject to constraints due to the color, spin and orbital symmetries. If the diquark and antidiquark have the antisymmetric flavor, they can have both the antisymmetric color $\bar{3}_c \otimes 3_c$ (S_3^σ, V_3^σ and P_3^σ) and the symmetric color $6_c \otimes \bar{6}_c$ (T_6^σ and A_6^σ); they can have both the antisymmetric spin $0_S \otimes 0_S$ (S_3^σ and V_3^σ) and the symmetric spin $1_S \otimes 1_S$ (A_3^σ and P_3^σ); they can have both positive parity (S_3^σ and A_6^σ) and negative parity (V_3^σ and P_3^σ).

The situation is the same for the color, spin and orbital symmetries. If the diquark and antidiquark have the antisymmetric color $\bar{3}_c \otimes 3_c$, they can have both the antisymmetric flavor (S_3^σ, V_3^σ and P_3^σ) and the symmetric flavor (T_3^σ and A_3^σ); they can have both the antisymmetric spin $0_S \otimes 0_S$ (S_3^σ and V_3^σ) and the symmetric spin $1_S \otimes 1_S$ (A_3^σ and P_3^σ); they can have both positive parity (S_3^σ and A_3^σ) and negative parity (V_3^σ and P_3^σ).

We can also construct $(\bar{q}q)(\bar{q}q)$ currents, and they are equivalent to the $(qq)(\bar{q}\bar{q})$ currents.

6.2 Analysis of Single Currents

In this section, we show the QCD sum rule analysis of κ using single currents S_3^κ, V_3^κ, T_6^κ, A_6^κ and P_3^κ. The results for σ, a_0 and f_0 are quite similar. We have performed the OPE calculation up to dimension eight by using $Mathematica$ with $FeynCalc$ [66]. The results are

$$
\begin{aligned}
\rho_{S3}^\kappa(s) =\ & \frac{s^4}{61440\pi^6} - \frac{m_s^2 s^3}{3072\pi^6} + \left(\frac{\langle g^2 GG\rangle}{6144\pi^6} - \frac{m_s\langle\bar{q}q\rangle}{192\pi^4} + \frac{m_s\langle\bar{s}s\rangle}{384\pi^4}\right)s^2 \\
& + \left(-\frac{m_s^2\langle g^2 GG\rangle}{2048\pi^6} - \frac{m_s\langle g\bar{q}\sigma Gq\rangle}{128\pi^4} + \frac{\langle\bar{q}q\rangle^2}{24\pi^2} + \frac{\langle\bar{q}q\rangle\langle\bar{s}s\rangle}{24\pi^2}\right)s \\
& - \frac{m_s^2\langle\bar{q}q\rangle^2}{12\pi^2} - \frac{m_s\langle g^2 GG\rangle\langle\bar{q}q\rangle}{768\pi^4} + \frac{m_s\langle g^2 GG\rangle\langle\bar{s}s\rangle}{1536\pi^4} \\
& + \frac{\langle\bar{q}q\rangle\langle g\bar{q}\sigma Gq\rangle}{24\pi^2} + \frac{\langle\bar{s}s\rangle\langle g\bar{q}\sigma Gq\rangle}{48\pi^2} + \frac{\langle\bar{q}q\rangle\langle g\bar{s}\sigma Gs\rangle}{48\pi^2},
\end{aligned}
\tag{6.9}
$$

$$
\begin{aligned}
\rho_{V3}^\kappa(s) =\ & \frac{s^4}{15360\pi^6} - \frac{m_s^2 s^3}{768\pi^6} + \left(\frac{\langle g^2 GG\rangle}{3072\pi^6} + \frac{m_s\langle\bar{q}q\rangle}{96\pi^4} + \frac{m_s\langle\bar{s}s\rangle}{96\pi^4}\right)s^2 \\
& + \left(-\frac{m_s^2\langle g^2 GG\rangle}{1024\pi^6} + \frac{m_s\langle g\bar{q}\sigma Gq\rangle}{128\pi^4} - \frac{\langle\bar{q}q\rangle^2}{12\pi^2} - \frac{\langle\bar{q}q\rangle\langle\bar{s}s\rangle}{12\pi^2}\right)s \\
& + \frac{m_s^2\langle\bar{q}q\rangle^2}{6\pi^2} - \frac{m_s\langle g^2 GG\rangle\langle\bar{q}q\rangle}{384\pi^4} + \frac{m_s\langle g^2 GG\rangle\langle\bar{s}s\rangle}{768\pi^4}
\end{aligned}
\tag{6.10}
$$

$$-\frac{\langle\bar{q}q\rangle\langle g\bar{q}\sigma Gq\rangle}{12\pi^2} - \frac{\langle\bar{s}s\rangle\langle g\bar{q}\sigma Gq\rangle}{48\pi^2} - \frac{\langle\bar{q}q\rangle\langle g\bar{s}\sigma Gs\rangle}{16\pi^2},$$

$$\rho^{\kappa}_{T6}(s) = \frac{s^4}{1280\pi^6} - \frac{m_s^2 s^3}{64\pi^6} + \left(\frac{11\langle g^2 GG\rangle}{768\pi^6} + \frac{m_s\langle\bar{s}s\rangle}{8\pi^4}\right)s^2 \quad (6.11)$$

$$-\frac{11m_s^2\langle g^2 GG\rangle}{256\pi^6}s + \frac{11m_s\langle g^2 GG\rangle\langle\bar{s}s\rangle}{192\pi^4},$$

$$\rho^{\kappa}_{A6}(s) = \frac{s^4}{7680\pi^6} - \frac{m_s^2 s^3}{384\pi^6} + \left(\frac{5\langle g^2 GG\rangle}{3072\pi^6} - \frac{m_s\langle\bar{q}q\rangle}{48\pi^4} + \frac{m_s\langle\bar{s}s\rangle}{48\pi^4}\right)s^2$$

$$+\left(-\frac{5m_s^2\langle g^2 GG\rangle}{1024\pi^6} + \frac{m_s\langle g\bar{q}\sigma Gq\rangle}{128\pi^4} + \frac{\langle\bar{q}q\rangle^2}{6\pi^2} + \frac{\langle\bar{q}q\rangle\langle\bar{s}s\rangle}{6\pi^2}\right)s \quad (6.12)$$

$$-\frac{m_s^2\langle\bar{q}q\rangle^2}{3\pi^2} - \frac{m_s\langle g^2 GG\rangle\langle\bar{q}q\rangle}{384\pi^4} + \frac{5m_s\langle g^2 GG\rangle\langle\bar{s}s\rangle}{768\pi^4}$$

$$+\frac{\langle\bar{q}q\rangle\langle g\bar{q}\sigma Gq\rangle}{6\pi^2} - \frac{\langle\bar{s}s\rangle\langle g\bar{q}\sigma Gq\rangle}{48\pi^2} + \frac{3\langle\bar{q}q\rangle\langle g\bar{s}\sigma Gs\rangle}{16\pi^2},$$

$$\rho^{\kappa}_{P3}(s) = \frac{s^4}{61440\pi^6} - \frac{m_s^2 s^3}{3072\pi^6} + \left(\frac{\langle g^2 GG\rangle}{6144\pi^6} + \frac{m_s\langle\bar{q}q\rangle}{192\pi^4} + \frac{m_s\langle\bar{s}s\rangle}{384\pi^4}\right)s^2$$

$$+\left(-\frac{m_s^2\langle g^2 GG\rangle}{2048\pi^6} + \frac{m_s\langle g\bar{q}\sigma Gq\rangle}{128\pi^4} - \frac{\langle\bar{q}q\rangle^2}{24\pi^2} - \frac{\langle\bar{q}q\rangle\langle\bar{s}s\rangle}{24\pi^2}\right)s \quad (6.13)$$

$$+\frac{m_s^2\langle\bar{q}q\rangle^2}{12\pi^2} + \frac{m_s\langle g^2 GG\rangle\langle\bar{q}q\rangle}{768\pi^4} + \frac{m_s\langle g^2 GG\rangle\langle\bar{s}s\rangle}{1536\pi^4} - \frac{\langle\bar{q}q\rangle\langle g\bar{q}\sigma Gq\rangle}{24\pi^2}$$

$$-\frac{\langle\bar{s}s\rangle\langle g\bar{q}\sigma Gq\rangle}{48\pi^2} - \frac{\langle\bar{q}q\rangle\langle g\bar{s}\sigma Gs\rangle}{48\pi^2}.$$

For each single current, we have tested the QCD sum rule analysis, but the result is not good just as in our previous paper [38]. The spectral densities are shown in Fig. 6.2 as functions of the energy square s. Due to the insufficient convergence of the OPE, the positivity of $\rho(s)$ may not be realized. We find that two functions of S_3^{κ} and A_6^{κ} currents show such a bad behavior that $\rho(s)$ becomes negative in the region of $s = 0 \sim 1$ GeV^2, and the QCD sum rule for these two single currents are not reliable.

The convergence of the OPE is another important issue. We show the Borel transformed correlation functions for positive case of V_3^{κ}, T_6^{κ} and P_3^{κ} with numerical coefficients:

$$\Pi_{V3}^{\kappa(all)} = 1.6 \times 10^{-6} M_B^{10} - 1.3 \times 10^{-7} M_B^8 - 3.5 \times 10^{-6} M_B^6$$

$$-2.8 \times 10^{-6} M_B^4 + 2.4 \times 10^{-6} M_B^2,$$

$$\Pi_{T6}^{\kappa(all)} = 2.0 \times 10^{-5} M_B^{10} - 1.5 \times 10^{-6} M_B^8 + 1.1 \times 10^{-5} M_B^6 \quad (6.14)$$

$$-3.3 \times 10^{-7} M_B^4 - 3.9 \times 10^{-7} M_B^2,$$

$$\Pi_{P3}^{\kappa(all)} = 4.1 \times 10^{-7} M_B^{10} - 3.2 \times 10^{-8} M_B^8 - 9.8 \times 10^{-8} M_B^6$$

$$-1.4 \times 10^{-6} M_B^4 + 1.2 \times 10^{-6} M_B^2.$$

From these expressions, we observe that the convergence of the currents V_3^{κ} and P_3^{κ} is not very good at a typical energy scale $M_B \sim 1$ GeV. We have also calculated the pole
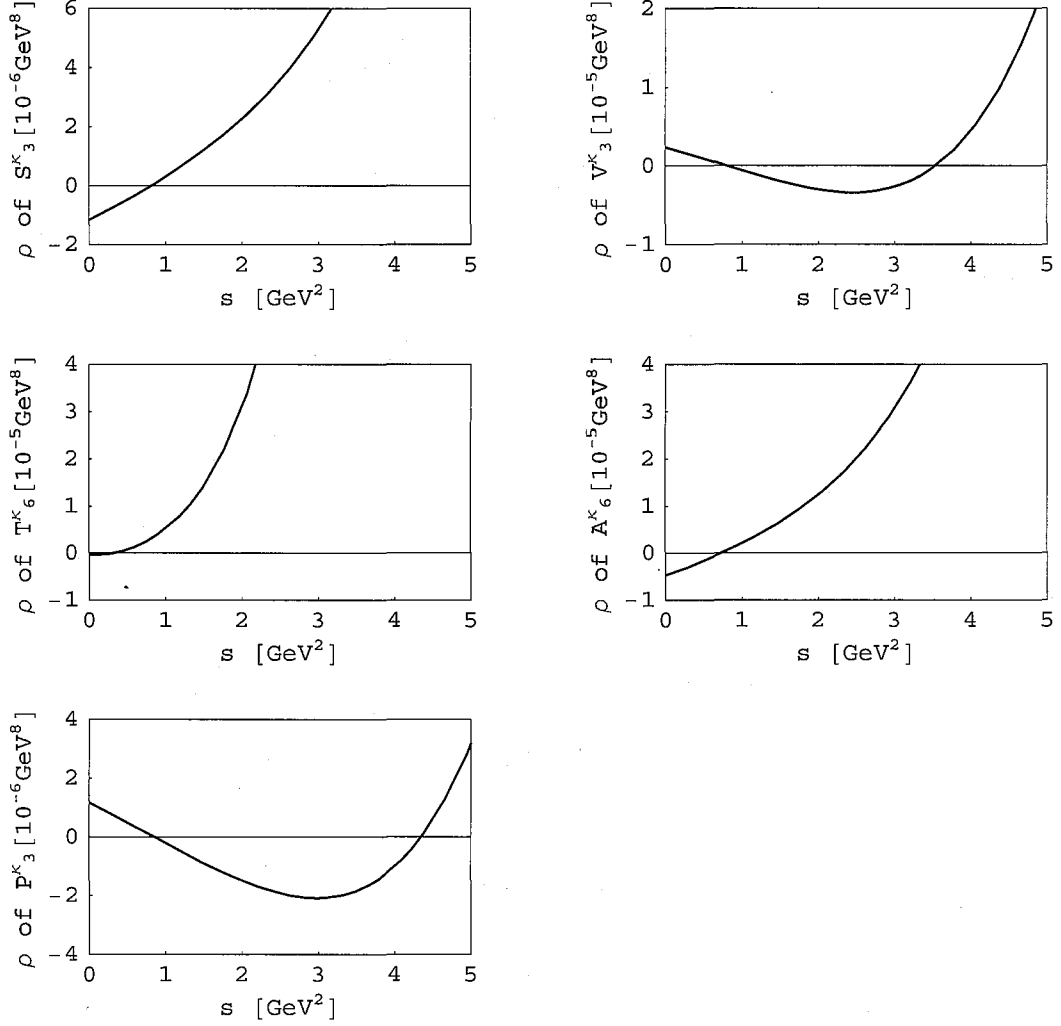
Figure 6.2: Spectral densities $\rho(s)$ for the currents S_3^κ, V_3^κ, T_6^κ, A_6^κ and P_3^κ as functions of s, in units of GeV^8.

contribution which is defined as

$$\text{Pole contribution} \equiv \frac{\int_0^{s_0} e^{-s/M_B^2} \rho(s) ds}{\int_0^\infty e^{-s/M_B^2} \rho(s) ds} , \tag{6.15}$$

However, due to the negative part of the spectral densities, the pole contribution is not well defined. Take the current P_3^κ as an example, when we choose $s_0 = 1 \text{ GeV}^2$ and

$M_B = 0.5$ GeV, the pole contribution is 101% (this is because some parts of the spectral density become negative in the denominator), which is larger than 100%, and does not make sense. The pole contribution is 26% for the current T_6^κ, when we choose $s_0 = 1$ GeV^2 and $M_B = 0.6$ GeV.

Summarizing the QCD sum rule analysis for the single currents, including both the $(qq)(\bar{q}\bar{q})$ currents and $(\bar{q}q)(\bar{q}q)$, we found that T_6^κ gives the best QCD sum rule, which however is not yet good enough for quantitative discussion. In order to improve the sum rule, we move on to study their linear combinations, which are the mixed currents.

6.3   Analysis of Mixed Currents

We have performed the OPE calculation for the mixed currents η_1 and η_2 up to dimension eight, which contains the four-quark condensates. The u and d quark masses are considered in the case of the σ meson, and neglected in other cases.

$$
\rho_1^\sigma(s) = \frac{1}{11520\pi^6}s^4 - \frac{m_u^2 + m_d^2}{288\pi^6}s^3 + \left(\frac{6\sqrt{2}+7}{9216\pi^6}\langle g^2 GG\rangle + \frac{(m_u + m_d)\langle\bar{q}q\rangle}{36\pi^4}\right)s^2 \tag{6.16}
$$
$$
+ \left(-\frac{6\sqrt{2}+7}{1536\pi^6}(m_u^2 + m_d^2)\langle g^2 GG\rangle + \frac{m_u m_d\langle g^2 GG\rangle}{512\pi^6}\right.
$$
$$
\left. - \frac{(m_u^3 + 4m_u^2 m_d + 4m_u m_d^2 + m_d^3)\langle\bar{q}q\rangle}{6\pi^4}\right)s + \frac{(5m_u^2 + 20 m_u m_d + 5m_d^2)\langle\bar{q}q\rangle^2}{9\pi^2}
$$
$$
+ \frac{6\sqrt{2}+1}{1152\pi^4}(m_u + m_d)\langle g^2 GG\rangle\langle\bar{q}q\rangle - \frac{(m_u^2 m_d + m_u m_d^2)\langle\bar{q}\sigma Gq\rangle}{6\pi^4} \ ,
$$

$$
\rho_2^\sigma(s) = \frac{1}{11520\pi^6}s^4 - \frac{m_u^2 + m_d^2}{288\pi^6}s^3 + \left(\frac{6\sqrt{2}+7}{9216\pi^6}\langle g^2 GG\rangle + \frac{(m_u + m_d)\langle\bar{q}q\rangle}{36\pi^4}\right)s^2 \tag{6.17}
$$
$$
+ \left(-\frac{4\sqrt{2}+5}{1024\pi^6}(m_u^2 + m_d^2)\langle g^2 GG\rangle - \frac{m_u m_d\langle g^2 GG\rangle}{768\pi^6}\right.
$$
$$
\left. - \frac{(7m_u^3 + 8m_u^2 m_d + 8m_u m_d^2 + 7m_d^3)\langle\bar{q}q\rangle}{18\pi^4}\right)s + \frac{(25m_u^2 + 40 m_u m_d + 25m_d^2)\langle\bar{q}q\rangle^2}{27\pi^2}
$$
$$
+ \frac{6\sqrt{2}+13}{1152\pi^4}(m_u + m_d)\langle g^2 GG\rangle\langle\bar{q}q\rangle - \frac{(m_u^3 + 2m_u^2 m_d + 2m_u m_d^2 + m_d^3)\langle\bar{q}\sigma Gq\rangle}{18\pi^4} \ ,
$$

$$
\rho_1^\kappa(s) = \frac{1}{11520\pi^6}s^4 - \frac{m_s^2}{572\pi^6}s^3 + \left(\frac{6\sqrt{2}+7}{9216\pi^6}\langle g^2 GG\rangle + \frac{m_s\langle\bar{s}s\rangle}{72\pi^4}\right)s^2
$$
$$
+ \left(-\frac{6\sqrt{2}+7}{3072\pi^6}m_s^2\langle g^2 GG\rangle + \frac{m_s\langle\bar{q}\sigma Gq\rangle}{128\pi^4}\right)s - \frac{m_s\langle g^2 GG\rangle\langle\bar{q}q\rangle}{384\pi^4}
$$
$$
- \frac{\langle\bar{s}s\rangle\langle\bar{q}\sigma Gq\rangle}{48\pi^2} + \frac{\langle\bar{q}q\rangle\langle\bar{s}\sigma Gs\rangle}{48\pi^2} + \frac{6\sqrt{2}+7}{2304\pi^4}m_s\langle g^2 GG\rangle\langle\bar{s}s\rangle \ , \tag{6.18}
$$

$$
\rho_2^\kappa(s) = \frac{1}{11520\pi^6}s^4 - \frac{m_s^2}{572\pi^6}s^3 + \left(\frac{6\sqrt{2}+7}{9216\pi^6}\langle g^2 GG\rangle + \frac{m_s\langle\bar{s}s\rangle}{72\pi^4}\right)s^2
$$

$$+(-\frac{6\sqrt{2}+7}{3072\pi^6}m_s^2\langle g^2GG\rangle - \frac{m_s\langle\bar{q}\sigma Gq\rangle}{128\pi^4})s + \frac{m_s\langle g^2GG\rangle\langle\bar{q}q\rangle}{384\pi^4}$$

$$+\frac{\langle\bar{s}s\rangle\langle\bar{q}\sigma Gq\rangle}{48\pi^2} - \frac{\langle\bar{q}q\rangle\langle\bar{s}\sigma Gs\rangle}{48\pi^2} + \frac{6\sqrt{2}+7}{2304\pi^4}m_s\langle g^2GG\rangle\langle\bar{s}s\rangle, \tag{6.19}$$

$$\rho_1^{a_0}(s) = \frac{1}{11520\pi^6}s^4 - \frac{m_s^2}{288\pi^6}s^3 + (\frac{6\sqrt{2}+7}{9216\pi^6}\langle g^2GG\rangle + \frac{m_s\langle\bar{s}s\rangle}{36\pi^4})s^2$$

$$+(-\frac{6\sqrt{2}+7}{1536\pi^6}m_s^2\langle g^2GG\rangle - \frac{m_s^3\langle\bar{s}s\rangle}{6\pi^4})s - \frac{m_s\langle g^2GG\rangle\langle\bar{q}q\rangle}{192\pi^4}$$

$$+\frac{4m_s^2\langle\bar{q}q\rangle^2}{9\pi^2} + \frac{4m_s^2\langle\bar{s}s\rangle^2}{9\pi^2} + \frac{6\sqrt{2}+7}{1152\pi^4}m_s\langle g^2GG\rangle\langle\bar{s}s\rangle, \tag{6.20}$$

$$\rho_2^{a_0}(s) = \frac{1}{11520\pi^6}s^4 - \frac{m_s^2}{288\pi^6}s^3 + (\frac{6\sqrt{2}+7}{9216\pi^6}\langle g^2GG\rangle + \frac{m_s\langle\bar{s}s\rangle}{36\pi^4})s^2$$

$$+(-\frac{6\sqrt{2}+7}{1536\pi^6}m_s^2\langle g^2GG\rangle - \frac{m_s^3\langle\bar{s}s\rangle}{6\pi^4})s + \frac{m_s\langle g^2GG\rangle\langle\bar{q}q\rangle}{192\pi^4}$$

$$+\frac{4m_s^2\langle\bar{q}q\rangle^2}{9\pi^2} + \frac{4m_s^2\langle\bar{s}s\rangle^2}{9\pi^2} + \frac{6\sqrt{2}+7}{1152\pi^4}m_s\langle g^2GG\rangle\langle\bar{s}s\rangle. \tag{6.21}$$

For σ, terms containing u, d quark masses m_q are small. For instance, the term of $m_q\langle\bar{q}q\rangle$ of dimension four is about ten times smaller than the other term of $\langle g^2GG\rangle$. For κ, a_0 and f_0, the terms containing strangle quark mass are important but those containing u and d quark masses are negligibly small. Therefore, we have ignored them in our sum rule analysis.

To obtain a reliable a QCD sum rule, the mixed currents η_1 and η_2 are chosen with the following requirements:

1. The OPE has a good convergence as going to terms of higher dimensional operators. This can be examined by the following numerical Borel transformed correlation functions, which have a good convergence

$$\Pi_1^{\sigma(all)}(M_B^2) = 2.2\times10^{-6}M_B^{10} - 2.5\times10^{-9}M_B^8 + 1.5\times10^{-6}M_B^6$$
$$-4.4\times10^{-10}M_B^4 - 4.8\times10^{-9}M_B^2,$$

$$\Pi_2^{\sigma(all)}(M_B^2) = 2.2\times10^{-6}M_B^{10} - 2.5\times10^{-9}M_B^8 + 1.5\times10^{-6}M_B^6$$
$$-5.3\times10^{-10}M_B^4 - 1.5\times10^{-8}M_B^2,$$

$$\Pi_1^{\kappa(all)}(M_B^2) = 2.2\times10^{-6}M_B^{10} - 1.7\times10^{-7}M_B^8 + 1.3\times10^{-6}M_B^6$$
$$+7.2\times10^{-8}M_B^4 - 2.3\times10^{-8}M_B^2,$$

$$\Pi_2^{\kappa(all)}(M_B^2) = 2.2\times10^{-6}M_B^{10} - 1.7\times10^{-7}M_B^8 + 1.3\times10^{-6}M_B^6$$
$$-2.8\times10^{-7}M_B^4 + 3.4\times10^{-8}M_B^2,$$

$$\Pi_1^{a_0(all)}(M_B^2) = 2.2\times10^{-6}M_B^{10} - 3.4\times10^{-7}M_B^8 + 8.8\times10^{-7}M_B^6$$
$$-4.1\times10^{-8}M_B^4 + 1.1\times10^{-7}M_B^2,$$

$$\Pi_2^{a_0(all)}(M_B^2) \;=\; 2.2 \times 10^{-6} M_B^{10} - 3.4 \times 10^{-7} M_B^8 + 8.8 \times 10^{-7} M_B^6$$
$$-4.1 \times 10^{-8} M_B^4 + 2.3 \times 10^{-8} M_B^2 \,.$$

It is interesting to observe that the correlation functions of σ have the most rapid convergence, justifying the use of a smaller Borel mass M_B than the other cases of κ, a_0 and f_0.

2. The spectral densities $\rho(s)$ become positive for almost all energy values, as shown in Fig. 6.3. This can be examined for all the mixed currents except η_2^κ. Therefore, we need to change the mixing angle of η_2^κ a little, which is from $\sqrt{2}$ to 1.37.

3. Pole contribution is sufficiently large. By choosing suitable Borel mass M_B and threshold value s_0, this can be satisfied. The Borel transformed correlation functions are written as power series of the Borel mass M_B. Since the Borel transformation suppresses the contributions from $s > M_B$, smaller values are preferred to suppress the continuum contributions also. However, for smaller M_B convergence of the OPE becomes worse. Therefore, we should find an optimal M_B preferably in a small value region. We have found that the minima of such a region are 0.5 GeV for σ, 0.6 GeV for κ and 0.8 GeV for a_0 and f_0, where the pole contributions reach around 50 % for κ, a_0 and f_0, and is an acceptable amount for σ, as shown in Table 6.1. The pole contribution for the mixed current η_1^κ is improved as compared with the single current T_6^κ.

Table 6.1: Pole contributions of various currents.

	η_1^σ	η_2^σ	η_1^κ	η_2^κ	$\eta_1^{a_0}$	$\eta_2^{a_0}$
M_B (GeV)	0.5	0.5	0.6	0.6	0.8	0.8
$\sqrt{s_0}$ (GeV)	0.7	0.7	1	1	1.3	1.3
Pole (%)	28	21	45	36	40	32

In the $SU(3)_f$ limit, we could find that the differences between ρ_1 and ρ_2 vanish:

$$\rho_1^\sigma(s) - \rho_2^\sigma(s) \;=\; \frac{(m_u^2 + m_d^2)\langle g^2 GG \rangle}{3072\pi^6} s + \frac{5 m_u m_d \langle g^2 GG \rangle}{1536\pi^6} s$$
$$+ \frac{(2m_u^3 - 2m_u^2 m_d - 2m_u m_d^2 + 2m_d^3)\langle \bar{q}q \rangle}{9\pi^4} s$$
$$+ \frac{(-10m_u^2 + 20 m_u m_d - 10 m_d^2)\langle \bar{q}q \rangle^2}{27\pi^2} - \frac{(m_u + m_d)\langle g^2 GG \rangle \langle \bar{q}q \rangle}{96\pi^4}$$
$$+ \frac{(m_u^3 - m_u^2 m_d - m_u m_d^2 + m_d^3)\langle \bar{q}\sigma Gq \rangle}{18\pi^4} \,,$$
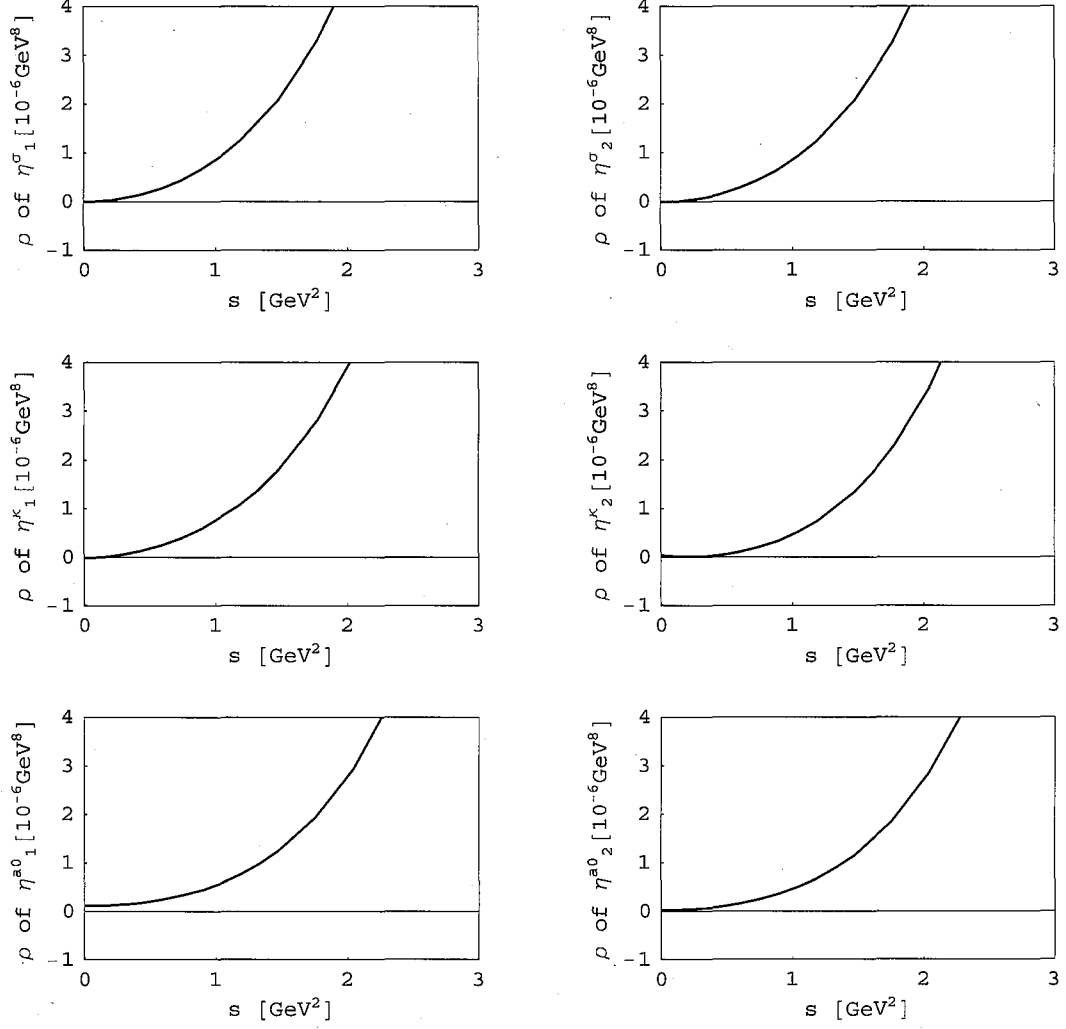
Figure 6.3: Spectral densities $\rho(s)$ for η_1^σ, η_2^σ, η_1^κ, η_2^κ, $\eta_1^{a0,f0}$ and $\eta_2^{a0,f0}$, as functions of s, in units of GeV^8.

$$\rho_1^\kappa(s) - \rho_2^\kappa(s) = \frac{m_s\langle\bar{q}\sigma Gq\rangle}{64\pi^4}s - \frac{m_s\langle g^2GG\rangle\langle\bar{q}q\rangle}{192\pi^4} - \frac{\langle\bar{s}s\rangle\langle\bar{q}\sigma Gq\rangle}{24\pi^2} + \frac{\langle\bar{q}q\rangle\langle\bar{s}\sigma Gs\rangle}{24\pi^2},$$

$$\rho_1^{a0}(s) - \rho_2^{a0}(s) = -\frac{m_s\langle g^2GG\rangle\langle\bar{q}q\rangle}{96\pi^4}. \tag{6.22}$$

From Eqs. (6.16) - (6.21), we find that the gluon condensates are quite important. In the chiral limit where all quark masses vanish, the masses of the scalar mesons are

Table 6.2: Masses of scalar nonet.

Mass (MeV)	$\sigma(600)$	$\kappa(800)$	$a_0(980)$	$f_0(980)$
Experiments (PDG)	$400 \sim 1200$	$841 \pm 30^{+81}_{-73}$	984.7 ± 1.2	980 ± 10
QCD sum rule	600 ± 100	800 ± 100	1000 ± 100	1000 ± 100

dictated only by the gluon condensate. Due to the small u and d quark masses, the mass of the σ is dominated by the gluon condensate. For other masses, however, other condensates with finite value of $m_s \sim 100\text{MeV}$ also play a significant role. As quarks (in particular strange quark) become massive, the degeneracy resolves. We have also tested the case of the SU(3) limit but with the average quark mass, $m_q \sim 50$ MeV, and with average condensates. Then the mass of the scalar mesons turns out to be about 0.8–0.9 GeV.

If the location of a physical state is well separated from the threshold s_0, slight change in s_0 should not affect much on the observables (mass) of the state. Hence we have searched the region where the tetraquark mass varies significantly less than the change in $\sqrt{s_0}$. We have found such regions for s_0 at around 1 GeV^2 from the minimum for σ $s_0(\text{min}) \sim 0.5$ GeV^2, for κ $s_0(\text{min}) \sim 1$ GeV^2 and for a_0 and f_0 $s_0(\text{min}) \sim 1.7$ GeV^2, and up to about 1 GeV^2 higher.

After careful test of the sum rule for a wide range of parameter values of M_B and s_0, we have found reliable sum rules, which are shown in Table 6.2. It is interesting to observe that the masses appear roughly in the order of the number of strange quarks with roughly equal splitting. In Fig. 6.4, the masses of the $\sigma(600)$, $\kappa(800)$, $a_0(980)$ and $f_0(980)$ are shown as functions of the Borel mass M_B. As we see, the mass is very stable in a rather wide region of Borel mass M_B.

The current η_1 has the antisymmetric flavor structure and η_2 has the symmetric flavor structure. By using these currents with different flavor structures, we arrive at similar QCD sum rule results. This suggests that the tetraquarks of different flavor structure may mix with each other, and the tetraquark states can contain diquark and antidiquark having the mixing of the symmetric flavor $6_f \otimes \bar{6}_f$ and the antisymmetric flavor $\bar{3}_f \otimes 3_f$, just like they can have a mixing of different color, spin and orbital symmetries. This is very much different from the ground baryon states, where the different flavor representations 8 and 10 correspond to different spins $1/2$ and $3/2$, which induces a mass splitting between $\Delta(1232)$ and $N(939)$.
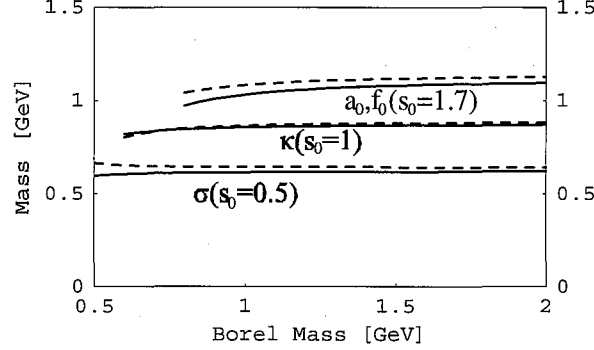
Figure 6.4: Masses of the σ, κ, a_0 and f_0 as tetraquark states calculated by the mixed currents η_1 (solid line) and η_2 (dashed line), as functions of the Borel mass M_B.

6.4 Finite Decay Width

The scalar mesons have large decay widthes, and it is important to consider their effect. In this section, we use a Gaussian distribution for the phenomenal spectral density, instead of δ-function,

$$\rho^{FDW}(\sqrt{s})d\sqrt{s} \equiv \sum_n \langle 0|\eta|n\rangle \langle n|\eta^\dagger|0\rangle \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\sqrt{s}-M_n)^2}{2*\sigma_n^2}\right)d\sqrt{s}$$

$$= \frac{f_X^2}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\sqrt{s}-M_X)^2}{2*\sigma_X^2}\right)d\sqrt{s} + \text{higher states}, \qquad (6.23)$$

where as usual the lowest state denoted by X is isolated from the rest of higher states. The Gaussian width σ_X is related to the Breit-Wigner decay width Γ by $\sigma_X = \Gamma/2.4$.

Again we assume the continuum contribution can be approximated by the spectral density of OPE above a threshold value s_0, and we arrive at the sum rule equation for state having a finite decay width

$$\Pi^{FDW}(M_B^2) \equiv \int_{-\infty}^{+\infty} e^{-s/M_B^2} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\sqrt{s}-M_X)^2}{2\sigma_X^2}\right)d\sqrt{s} = \int_0^{s_0} e^{-s/M_B^2}\rho(s)ds.$$

$$(6.24)$$

For a given Γ, the mass can be obtained by solving the equation

$$\frac{\int_{-\infty}^{+\infty} e^{-s/M_B^2}s\exp\left(-\frac{(\sqrt{s}-M_X)^2}{2\sigma_X^2}\right)d\sqrt{s}}{\int_{-\infty}^{+\infty} e^{-s/M_B^2}\exp\left(-\frac{(\sqrt{s}-M_X)^2}{2\sigma_X^2}\right)d\sqrt{s}}e = \frac{\int_0^{s_0} e^{-s/M_B^2}s\rho(s)ds}{\int_0^{s_0} e^{-s/M_B^2}\rho(s)ds}. \qquad (6.25)$$

In Fig. 6.5, the masses of the $\sigma(600)$, $\kappa(800)$, $a_0(980)$ and $f_0(980)$ are shown as functions of the Borel mass M_B, by setting $\Gamma = 0$, 100, 200 and 400 MeV respectively.

We find that after considering the finite decay width by using the Gaussian distribution, the predicted masses do not change significantly as far as the Borel mass is within a reasonable range, where we can still reproduce the experimental data. However, the question of finite decay width is very important, and we do not consider that our attempt to use the Gaussian form is the final. We need further investigations, which we would like to put as a future important work.
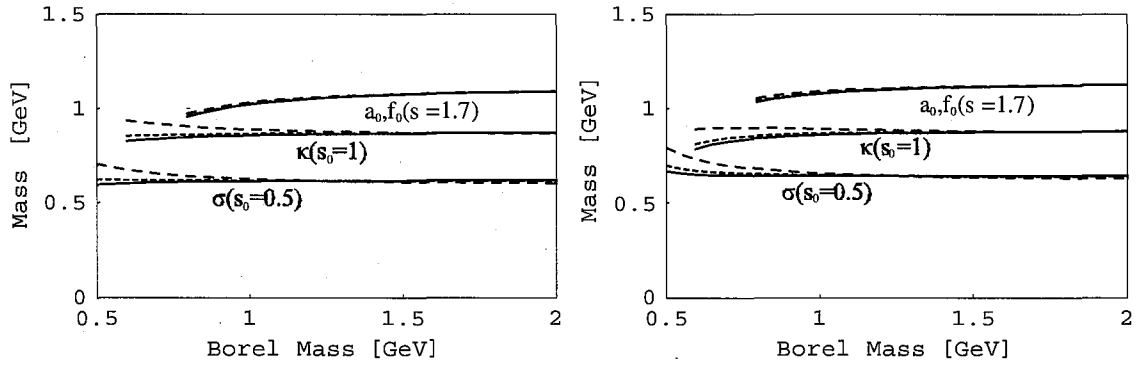
Figure 6.5: Masses of the σ, κ, a_0 and f_0 as tetraquark states calculated by the mixed currents η_1 (left) and η_2 (right), as functions of the Borel mass M_B. For σ and κ, the solid, short-dashed and long-dashed curves are obtained by setting $\Gamma = 0$, 200 and 400 MeV respectively. For a_0 and f_0, the solid, short-dashed and long-dashed curves are obtained by setting $\Gamma = 0$, 100 and 200 MeV respectively.

6.5 Conventional $\bar{q}q$ Mesons

For comparison, we have also performed the QCD sum rule analysis using the $\bar{q}q$ current within the present framework. The QCD sum rule analyses of conventional $\bar{q}q$ mesons have been performed in Ref. [59, 64, 113, 154]. The sum rules using the current $j = \bar{q}_1 q_2$ are

$$f_{(q_1 q_2)}^2 e^{-\frac{m_{(q_1 q_2)}^2}{M_B^2}} = \int_0^{s_0} e^{-s/M_B^2} \frac{3}{8\pi^2} s \left(1 + \frac{17}{3}\frac{\alpha_s}{\pi}\right) ds + \frac{3}{2}\left(m_1\langle \bar{q}_2 q_2\rangle + m_2\langle \bar{q}_1 q_1\rangle\right)$$

$$+\frac{1}{8\pi}\langle \frac{g^2}{4\pi}G^2\rangle - \frac{1}{2M_B^2}\left(m_1\langle g\bar{q}_2\sigma G q_2\rangle + m_2\langle g\bar{q}_1\sigma G q_1\rangle\right) \qquad (6.26)$$

$$-\frac{16\pi}{3M_B^2}\frac{g_s}{4\pi}\langle \bar{q}_1 q_1\rangle\langle \bar{q}_2 q_2\rangle - \frac{16\pi}{27M_B^2}\frac{g_s}{4\pi}\left(\langle \bar{q}_1 q_1\rangle^2 + \langle \bar{q}_2 q_2\rangle^2\right).$$

In Fig. 6.6 we show the mass of the $\bar{q}q$ mesons as functions of Borel mass when the threshold value $s_0 = 2.5$ GeV^2. The masses of σ and a_0 are predicted to be around 1.2

GeV, while the masses of κ and f_0 are larger due to the *strange* quark content. Here again we have tested other values of M_B and s_0, and confirmed that the result shown is optimal. These results are consistent with the previous work [59, 64, 113, 154].
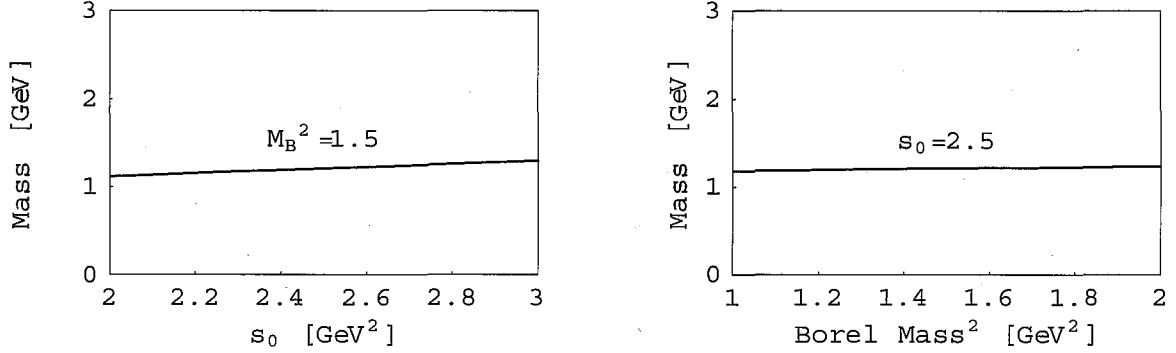
Figure 6.6: Masses of the conventional $\bar{q}q$ mesons calculated by the current $j = \bar{q}_1 q_2$, as functions of the threshold value s_0 (left) and the Borel mass square M_B^2 (right).

6.6 Conclusion

We have performed the QCD sum rule analysis with tetraquark currents, and found the masses of scalar mesons in the region of 600 – 1000 MeV with the ordering, $m_\sigma < m_\kappa < m_{f_0, a_0}$. We have also used the conventional $\bar{q}q$ currents, and verified their masses around 1.2 GeV. We have tested all possible independent tetraquark currents as well as their linear combinations, and considered the effect of finite decay width. Our conclusions are, therefore, rather robust.

The scalar tetraquark currents can have either the antisymmetric flavor or the symmetric flavor structures. We found that there are five independent currents for each state. We investigated Borel mass M_B and threshold value s_0 dependences, which are quite stable. The convergence of the OPE is also good, the positivity (of spectral density) is maintained, and the pole contribution is sufficient large. Therefore, we have achieved a QCD sum rule which is the best reliable within the present calculation of OPE.

Our calculation supports a tetraquark structure for low-lying scalar mesons. We find that the gluon condensate is quite large in the OPE of the mixed currents, which is related to the question of the origin of the mass generation of hadrons [173]. We obtain similar results by using the currents having both the antisymmetric flavor structure and the symmetric flavor structure. This suggests that the tetraquark can have a mixing of different flavor symmetries, as well as different color, spin and orbital symmetries. There is a mass splitting due to the different flavor, color, spin and orbital structures. If this

mass spitting is large enough to be observed in experiments, the tetraquark spectrum would become much more complicated; If the mass splitting is too small to be observed in experiments, a broad decay width would be observed. Such a tetraquark structure will open an alternative path toward the understanding of exotic multiquark dynamics which one does not experience in the conventional hadrons.

Chapter 7

The $Y(2175)$ State

Recently Babar Collaboration observed a resonance $Y(2175)$ near the threshold in the process $e^+e^- \rightarrow \phi f_0(980)$ via initial-state radiation [21–23]. It has the quantum numbers $J^{PC} = 1^{--}$. The Breit-Wigner mass is $M = 2.175 \pm 0.010 \pm 0.015$ GeV, and width is $\Gamma = 0.058 \pm 0.016 \pm 0.020$ GeV. It has been also confirmed by BES collaboration in the process $J/\psi \rightarrow \eta \phi f_0(980)$. A fit with a Breit-Wigner function gives the peak mass and width of $M = 2.186 \pm 0.010 \pm 0.006$ GeV and $\Gamma = 0.065 \pm 0.023 \pm 0.017$ GeV [6].

There are many suggestions to interpret this resonance. Ding and Yan interpreted it as a strangeonium hybrid and studied its decay properties in the flux-tube model and the constituent gluon model. Furthermore, for testing $s\bar{s}g$ scenario, they suggested searching decay modes such as $Y(2175) \rightarrow K_1(1400)K \rightarrow \pi K^*(892)K$, $Y(2175) \rightarrow K_1(1270)K \rightarrow \rho KK$ and $Y(2175) \rightarrow K_1(1270)K \rightarrow \pi K_0^*(1430)$ [56]. In Ref. [57], the authors explored $Y(2175)$ as a 2^3D_1 $s\bar{s}$ meson, and calculated its decay modes by using both the $^3P_0$ model and the flux-tube model. They suggested experimental search of the decay modes KK, K^*K^*, $K(1460)K$ and $h_1(1380)\eta$. The characteristic decay modes of $Y(2175)$ as either a hybrid state or an $s\bar{s}$ state are quite different, which may be used to distinguish the hybrid and $s\bar{s}$ schemes. Wang studied $Y(2175)$ as a tetraquark state $ss\bar{s}\bar{s}$ by using QCD sum rule and suggested that there may be some tetraquark components in the state $Y(2175)$ [169]. In a recent article [187], Zhu reviewed $Y(2175)$ and indicated that the possibility of $Y(2175)$ arising from S-wave threshold effects can not be excluded. Napsuciale, Oset, Sasaki and Vaquera-Araujo studied the reaction $e^+e^- \rightarrow \phi\pi\pi$ for pions in an isoscalar S-wave channel which is dominated by the loop mechanism. By selecting the $\phi f_0(980)$ contribution as a function of the e^+e^- energy, they also reproduced the experimental data except for the narrow peak [138]. Bystritskiy, Volkov, Kuraev, Bartos and Secansky calculated the total probability and the differential cross section of the process $e^+e^- \rightarrow \phi f_0(980)$ by using the local NJL model [34]. Anikin, Pire and Teryaev studied the reaction $\gamma^*\gamma \rightarrow \rho\rho$, and calculated the mass of the isotensor exotic meson [16]. In Ref. [76], the authors performed a QCD sum rule study for 1^{--} hybrid meson, and the mass is predicted to be $2.3 - 2.4$, $2.3 - 2.5$, and $2.5 - 2.6$ GeV for $q\bar{q}g$, $q\bar{s}g$, and $s\bar{s}g$,

respectively.

In this chapter, we revisit the possibility of $Y(2175)$ as an tetraquark state $ss\bar{s}\bar{s}$. The currents of $J^{PC} = 1^-$ have been constructed in Chapter 4, and we can select the currents with charge-conjugation parity negative among them. We find that there are two independent currents. They can have a structure of diquark-antidiquark $(ss)(\bar{s}\bar{s})$, or have a structure of meson-meson $(\bar{s}s)(\bar{s}s)$. We show that they are equivalent, as we have verified many times. Then by using these two independent currents, we also perform a QCD sum rule analysis. We calculate the OPE up to the dimension 12, which contains the $\langle \bar{q}q \rangle^4$ condensates. In these two respects, our study differs from the previous one of Ref. [169].

7.1   Interpolating Currents

In this section, we construct currents for the state $Y(2175)$ of $J^{PC} = 1^{--}$. From the decay pattern $Y(2175) \rightarrow \phi(1020)f_0(980)$, we expect that there is a large $ss\bar{s}\bar{s}$ component in $Y(2175)$ since both ϕ and f_0 have a large $\bar{s}s$ component. We may add further quark and antiquark pairs, but the simplest choice would be $ss\bar{s}\bar{s}$. We will discuss later how this simplest quark content will be compatible with the above decay pattern when considering the possible structure of $\phi(1020)$ and $f_0(980)$.

Let us now briefly see the flavor structure of the current. In the diquark-antidiquark construction $(ss)(\bar{s}\bar{s})$ where ss and $\bar{s}\bar{s}$ pairs have a symmetric flavor structure, the flavor decomposition goes as

$$\mathbf{6_f} \otimes \mathbf{\bar{6}_f} = \mathbf{1_f} \oplus \mathbf{8_f} \oplus \mathbf{27_f}. \tag{7.1}$$

Therefore, the $(ss)(\bar{s}\bar{s})$ state is a mixing of $1_f, 8_f$ and 27_f multiplets in the ideal mixing scheme.

Now we find that there are two non-vanishing currents for each state with the quantum number $J^{PC} = 1^{--}$. For the state $ss\bar{s}\bar{s}$:

$$\eta_{1\mu} = (s_a^T C\gamma_5 s_b)(\bar{s}_a \gamma_\mu \gamma_5 C \bar{s}_b^T) - (s_a^T C\gamma_\mu \gamma_5 s_b)(\bar{s}_a \gamma_5 C \bar{s}_b^T), \tag{7.2}$$

$$\eta_{2\mu} = (s_a^T C\gamma^\nu s_b)(\bar{s}_a \sigma_{\mu\nu} C \bar{s}_b^T) - (s_a^T C\sigma_{\mu\nu} s_b)(\bar{s}_a \gamma^\nu C \bar{s}_b^T), \tag{7.3}$$

where the sum over repeated indices (μ for Dirac spinor indices, and a, b for color indices) is taken. $C = i\gamma_2\gamma_0$ is the Dirac field charge conjugation operator, and the superscript T represents the transpose of the Dirac indices only.

Besides the diquark-antidiquark currents, we can also construct the tetraquark currents by using quark-antiquark $(\bar{s}s)$ pairs. We find that there are four non-vanishing currents:

$$\eta_{3\mu} = (\bar{s}_a s_a)(\bar{s}_b \gamma_\mu s_b),$$

$$\eta_{4\mu} = (\bar{s}_a \gamma^\nu \gamma_5 s_a)(\bar{s}_b \sigma_{\mu\nu} \gamma_5 s_b),$$

$$\eta_{5\mu} = \lambda_{ab}\lambda_{cd}(\bar{s}_a s_b)(\bar{s}_c \gamma_\mu s_d),$$

$$\eta_{6\mu} = \lambda_{ab}\lambda_{cd}(\bar{s}_a \gamma^\nu \gamma_5 s_b)(\bar{s}_c \sigma_{\mu\nu}\gamma_5 s_d).$$

In Ref. [169], the author used $\eta_{5\mu}$ to perform QCD sum rule analysis, which is a mixing of $\eta_{1\mu}$ and $\eta_{2\mu}$. We can verify the following relations by using the Fierz transformation:

$$\eta_{5\mu} = -\frac{5}{3}\eta_{3\mu} - i\eta_{4\mu}, \quad \eta_{6\mu} = 3i\eta_{3\mu} + \frac{1}{3}\eta_{4\mu}. \tag{7.4}$$

Therefore, among the four $(\bar{q}q)(\bar{q}q)$ currents, two are independent. We can also verify the relations between $(ss)(\bar{s}\bar{s})$ currents and $(\bar{s}s)(\bar{s}s)$ currents, by using the Fierz transformation:

$$\eta_{1\mu} = -\eta_{3\mu} + i\eta_{4\mu}, \quad \eta_{2\mu} = 3i\eta_{3\mu} - \eta_{4\mu}. \tag{7.5}$$

Therefore, these two constructions are equivalent, and we will use $\eta_{1\mu}$ and $\eta_{2\mu}$ for QCD sum rule analysis.

7.2 QCD sum rule Analysis

For the currents $\eta_{1\mu}$ and $\eta_{2\mu}$, we have calculated the OPE up to dimension twelve, which contains the $\langle \bar{q}q \rangle^4$ condensate:

$$
\begin{aligned}
\Pi_1(M_B^2) = & \int_{16m_s^2}^{s_0} \left[\frac{s^4}{18432\pi^6} - \frac{m_s^2 s^3}{256\pi^6} + \left(-\frac{\langle g^2 GG \rangle}{18432\pi^6} + \frac{m_s \langle \bar{s}s \rangle}{48\pi^4} \right)s^2 \right. \\
& + \left(\frac{\langle \bar{s}s \rangle^2}{18\pi^2} - \frac{m_s \langle g\bar{s}\sigma Gs \rangle}{48\pi^4} + \frac{17 m_s^2 \langle g^2 GG \rangle}{9216\pi^6} \right)s \\
& + \left. \left(\frac{\langle \bar{s}s \rangle \langle g\bar{s}\sigma Gs \rangle}{12\pi^2} - \frac{m_s \langle g^2 GG \rangle \langle \bar{s}s \rangle}{128\pi^4} - \frac{29 m_s^2 \langle \bar{s}s \rangle^2}{12\pi^2} \right) \right] e^{-s/M_B^2} ds \\
& + \left(\frac{5\langle g^2 GG \rangle \langle \bar{s}s \rangle^2}{864\pi^2} + \frac{\langle g\bar{s}\sigma Gs \rangle^2}{48\pi^2} + \frac{20 m_s \langle \bar{s}s \rangle^3}{9} - \frac{5 m_s \langle g^2 GG \rangle \langle g\bar{s}\sigma Gs \rangle}{2304\pi^4} \right. \\
& - \left. \frac{3 m_s^2 \langle \bar{s}s \rangle \langle g\bar{s}\sigma Gs \rangle}{2\pi^2} \right) + \frac{1}{M_B^2}\left(-\frac{32 g^2 \langle \bar{s}s \rangle^4}{81} - \frac{\langle g^2 GG \rangle \langle \bar{s}s \rangle \langle g\bar{s}\sigma Gs \rangle}{576\pi^2} \right. \\
& - \left. \frac{10 m_s \langle \bar{s}s \rangle^2 \langle g\bar{s}\sigma Gs \rangle}{9} + \frac{m_s^2 \langle g^2 GG \rangle \langle \bar{s}s \rangle^2}{576\pi^2} + \frac{m_s^2 \langle g\bar{s}\sigma Gs \rangle^2}{12\pi^2} \right),
\end{aligned}
\tag{7.6}
$$

$$
\begin{aligned}
\Pi_2(M_B^2) = & \int_{16m_s^2}^{s_0} \left[\frac{s^4}{12288\pi^6} - \frac{3 m_s^2 s^3}{512\pi^6} + \left(\frac{\langle g^2 GG \rangle}{18432\pi^6} + \frac{m_s \langle \bar{s}s \rangle}{32\pi^4} \right)s^2 \right. \\
& + \left. \left(\frac{\langle \bar{s}s \rangle^2}{12\pi^2} - \frac{m_s \langle g\bar{s}\sigma Gs \rangle}{32\pi^4} + \frac{35 m_s^2 \langle g^2 GG \rangle}{9216\pi^6} \right)s \right.
\end{aligned}
$$

$$+ \left(\frac{\langle \bar{s}s \rangle \langle g\bar{s}\sigma Gs \rangle}{8\pi^2} - \frac{3m_s \langle g^2 GG \rangle \langle \bar{s}s \rangle}{128\pi^4} - \frac{29m_s^2 \langle \bar{s}s \rangle^2}{8\pi^2} \right) \Bigg] e^{-s/M_B^2} ds$$

$$+ \left(\frac{5\langle g^2 GG \rangle \langle \bar{s}s \rangle^2}{288\pi^2} + \frac{\langle g\bar{s}\sigma Gs \rangle^2}{32\pi^2} + \frac{10m_s \langle \bar{s}s \rangle^3}{3} - \frac{5m_s \langle g^2 GG \rangle \langle g\bar{s}\sigma Gs \rangle}{768\pi^4} \right.$$

$$\left. - \frac{9m_s^2 \langle \bar{s}s \rangle \langle g\bar{s}\sigma Gs \rangle}{4\pi^2} \right) + \frac{1}{M_B^2} \left(- \frac{16g^2 \langle \bar{s}s \rangle^4}{27} - \frac{\langle g^2 GG \rangle \langle \bar{s}s \rangle \langle g\bar{s}\sigma Gs \rangle}{192\pi^2} \right.$$

$$\left. - \frac{5m_s \langle \bar{s}s \rangle^2 \langle g\bar{s}\sigma Gs \rangle}{3} - \frac{m_s^2 \langle g^2 GG \rangle \langle \bar{s}s \rangle^2}{576\pi^2} + \frac{m_s^2 \langle g\bar{s}\sigma Gs \rangle^2}{8\pi^2} \right).$$

$$(7.7)$$

We find that there is an approximate relation between the correlation functions of $\eta_{1\mu}$ and $\eta_{2\mu}$:

$$3\Pi_1(M_B^2) \sim 2\Pi_2(M_B^2),$$ (7.8)

which is valid for the continuum, $\langle \bar{s}s \rangle$, and $\langle g_c \bar{q}\sigma Gq \rangle$ terms, etc. So the numerical results by using them are also very similar.

7.3 Numerical Analysis

First we want to study the convergence of the operator product expansion, which is the cornerstone of the reliable QCD sum rule analysis. By taking s_0 to be ∞ and the integral subscript $16m_s^2$ to be zero, we obtain the numerical series of the OPE as a function of M_B:

$$\begin{aligned}
\Pi_1(M_B^2) &= 1.4 \times 10^{-6} M_B^{10} - 3.8 \times 10^{-7} M_B^8 - 6.2 \times 10^{-7} M_B^6 + 4.2 \times 10^{-7} M_B^4 \\
&\quad -1.2 \times 10^{-6} M_B^2 + 4.7 \times 10^{-8} - 1.5 \times 10^{-7} M_B^{-2}, \quad\quad (7.9) \\
\Pi_2(M_B^2) &= 2.0 \times 10^{-6} M_B^{10} - 5.7 \times 10^{-7} M_B^8 - 8.0 \times 10^{-7} M_B^6 + 6.4 \times 10^{-7} M_B^4 \\
&\quad -1.7 \times 10^{-6} M_B^2 + 1.0 \times 10^{-7} - 2.2 \times 10^{-7} M_B^{-2}. \quad\quad (7.10)
\end{aligned}$$

After careful testing of the free parameter Borel mass M_B, we find for $M_B^2 > 2$ GeV^2, which is the region suitable for the study of $Y(2175)$, the Borel mass dependence is weak. Moreover, the convergence of the OPE is satisfied in this region. The correlation function of the current $\eta_{1\mu}$ is shown in Fig. 7.1, when we take $s_0 = 5.7$ GeV^2 (the integral subscript is still $16m_s^2$). We find that in the region of 2 GeV$^2 < M_B^2 < 5$ GeV^2, the perturbative term (the solid line in Fig. 7.1) gives the most important contribution, and the convergence is quite good.

It is important to note that the $Y(2175)$ state is not the lowest state in the 1^{--} channel containing $\bar{s}s$ and that the interpolating currents see only the quantum number of the states. It is possible that the low-lying states $\phi(1020)$ and $\phi(1680)$ also couple to the tetraquark currents $\eta_{1\mu}$ and $\eta_{2\mu}$. If so, their contribution to the spectral density and the resulting correlation function should be positive definite.
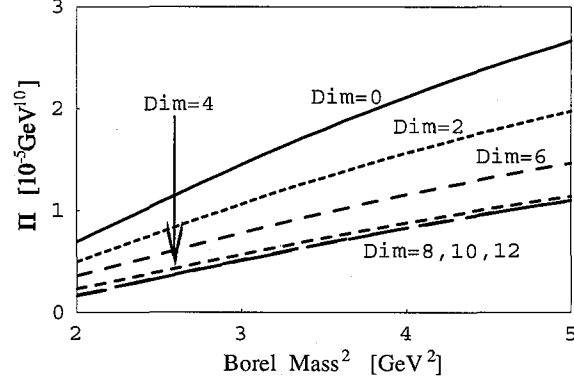
Figure 7.1: Various contribution to the correlation function for the current $\eta_{1\mu}$ as functions of the Borel mass M_B in units of GeV^{10} at $s_0 = 5.7$ GeV^2. The labels indicate the dimension up to which the OPE terms are included.

However, we find that (1) the spectral densities $\rho(s)$ for both currents $\eta_{1\mu}$ and $\eta_{2\mu}$ are negative when $s < 2$ GeV^2; (2) the Borel transformed correlation function $\Pi(M_B^2)$ in Eq. (1.25) is also negative in the region $s_0 < 4.3$ GeV^2 and 1 $GeV^2 < M_B^2 < 4$ GeV^2. As an illustration, we show the correlation function as a function of s_0 in Fig. 7.2. This fact indicates that the $ss\bar{s}\bar{s}$ tetraquark currents couple weakly to the lower states $\phi(1020)$ and $\phi(1680)$ in the present QCD sum rule analysis.

The pole contribution is not large enough for both currents due to a large contribution from $D = 10$ perturbative term $\int_0^{s_0} e^{-s/M_B^2} s^4 ds$, which is a common feature for any multiquark interpolating currents with high dimensions. The mixing of the currents $\eta_{1\mu}$ and $\eta_{2\mu}$ does not improve the rate of the pole contribution. The small pole contribution suggests that the continuum contribution to the spectral density is dominant, which demands a very careful choice of the parameters of the QCD sum rule. In our numerical analysis, we require the extracted mass have a dual minimum dependence on both the Borel parameter M_B and the threshold parameter s_0. In this way, we can find a good working region of M_B and s_0 (Borel window), where the mass of $Y(2175)$ can be determined reliably.

Now the mass is shown as functions of the Borel mass M_B and the threshold value s_0 in Fig. 7.3 and Fig. 7.4. The threshold value is taken to be around $5 \sim 7$ GeV^2, where its square root is around $2.2 \sim 2.7$ GeV. We find that there is a mass minimum around 2.4 GeV for the current $\eta_{1\mu}$, when we take $M_B^2 \sim 4$ GeV^2 and $s_0 \sim 5.7$ GeV^2. While this minimum is around 2.3 GeV for the current $\eta_{2\mu}$, when we take $M_B^2 \sim 4$ GeV^2 and $s_0 \sim 5.4$ GeV^2.

In short summary, we have performed the QCD sum rule analysis for both $\eta_{1\mu}$ and $\eta_{2\mu}$. The obtained results are quite similar. This is due to the similarity of the two correlation
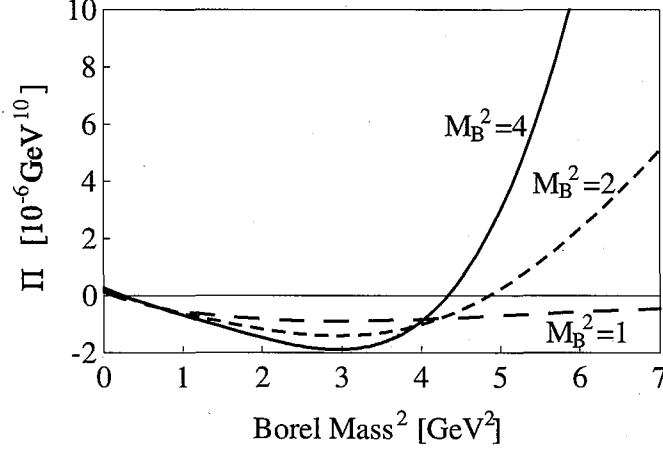
Figure 7.2: The correlation function for the current $\eta_{1\mu}$ as a function of s_0 in units of GeV^{10}. The curves are obtained by setting $M_B^2 = 1$ GeV^2 (long-dashed line), 2 GeV^2 (short-dashed line) and 4 GeV^2 (solid line).

functions as shown in Eq. (7.8). We have also considered their mixing, which also give the similar result. The mass is predicted to be around $2.3 \sim 2.4$ GeV in the QCD sum rule.

7.4 Finite Energy Sum Rule

To test the validity of the results obtained in the SVZ sum rule in the previous section, we use the method of finite energy sum rule (FESR) in this section. For the currents $\eta_{1\mu}$ and $\eta_{2\mu}$, the spectral functions $\rho_1(s)$ and $\rho_2(s)$ can be drawn from Eqs. (7.6) and (7.7). The d = 12 terms which are proportional to $1/(q^2)^2$ do not contribute to the function $W(n, s_0)$ of Eq. (1.27) for $n = 0$, or they have a very small contribution for $n = 1$, when the theoretical side is computed by the integral over the circle of radius s_0 on the complex q^2 plain. Therefore, the spectral densities for $\eta_{1\mu}$ and $\eta_{2\mu}$ take the following form up to dimension 10,

$$
\begin{aligned}
\rho_1(s) = {} & \frac{s^4}{18432\pi^6} - \frac{m_s^2 s^3}{256\pi^6} + \left(-\frac{\langle g^2 GG \rangle}{18432\pi^6} + \frac{m_s \langle \bar{s}s \rangle}{48\pi^4} \right) s^2 \\
& + \left(\frac{\langle \bar{s}s \rangle^2}{18\pi^2} - \frac{m_s \langle g\bar{s}\sigma Gs \rangle}{48\pi^4} + \frac{17 m_s^2 \langle g^2 GG \rangle}{9216\pi^6} \right) s \\
& + \left(\frac{\langle \bar{s}s \rangle \langle g\bar{s}\sigma Gs \rangle}{12\pi^2} - \frac{m_s \langle g^2 GG \rangle \langle \bar{s}s \rangle}{128\pi^4} - \frac{29 m_s^2 \langle \bar{s}s \rangle^2}{12\pi^2} \right)
\end{aligned}
$$
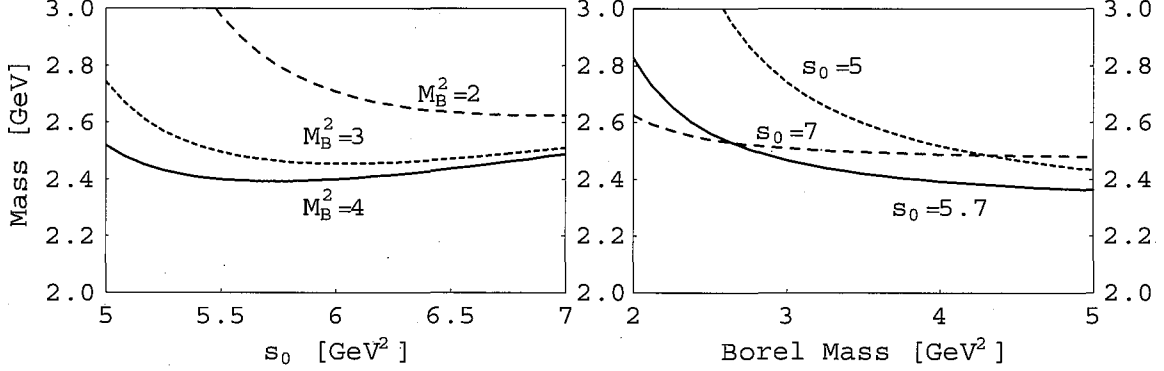
Figure 7.3: The mass of $Y(2175)$ as a function of M_B (Left) and s_0 (Right) in units of GeV for the current $\eta_{1\mu}$.

$$+ \left(\frac{5\langle g^2 GG \rangle \langle \bar{s}s \rangle^2}{864\pi^2} + \frac{\langle g\bar{s}\sigma Gs \rangle^2}{48\pi^2} + \frac{20 m_s \langle \bar{s}s \rangle^3}{9} - \frac{5 m_s \langle g^2 GG \rangle \langle g\bar{s}\sigma Gs \rangle}{2304\pi^4} \right.$$
$$\left. - \frac{3 m_s^2 \langle \bar{s}s \rangle \langle g\bar{s}\sigma Gs \rangle}{2\pi^2} \right) \delta(s) , \tag{7.11}$$

$$\rho_2(s) = \frac{s^4}{12288\pi^6} - \frac{3 m_s^2 s^3}{512\pi^6} + \left(\frac{\langle g^2 GG \rangle}{18432\pi^6} + \frac{m_s \langle \bar{s}s \rangle}{32\pi^4} \right) s^2$$
$$+ \left(\frac{\langle \bar{s}s \rangle^2}{12\pi^2} - \frac{m_s \langle g\bar{s}\sigma Gs \rangle}{32\pi^4} + \frac{35 m_s^2 \langle g^2 GG \rangle}{9216\pi^6} \right) s$$
$$+ \left(\frac{\langle \bar{s}s \rangle \langle g\bar{s}\sigma Gs \rangle}{8\pi^2} - \frac{3 m_s \langle g^2 GG \rangle \langle \bar{s}s \rangle}{128\pi^4} - \frac{29 m_s^2 \langle \bar{s}s \rangle^2}{8\pi^2} \right)$$
$$+ \left(\frac{5 \langle g^2 GG \rangle \langle \bar{s}s \rangle^2}{288\pi^2} + \frac{\langle g\bar{s}\sigma Gs \rangle^2}{32\pi^2} + \frac{10 m_s \langle \bar{s}s \rangle^3}{3} \right.$$
$$\left. - \frac{5 m_s \langle g^2 GG \rangle \langle g\bar{s}\sigma Gs \rangle}{768\pi^4} - \frac{9 m_s^2 \langle \bar{s}s \rangle \langle g\bar{s}\sigma Gs \rangle}{4\pi^2} \right) \delta(s) . \tag{7.12}$$

The mass is shown as a function of the threshold value s_0 in Fig. 7.5, where n is chosen to be 1. We find that there is a mass minimum (stable region). It is around 2.3 GeV for the current $\eta_{1\mu}$ when we take $s_0 \sim 5.2$ GeV^2, while it is around 2.2 GeV for the current $\eta_{2\mu}$ when we take $s_0 \sim 4.8$ GeV^2. For the current $\eta_{1\mu}$, the minimum point occurs at $\sqrt{s_0} = 2.28$ GeV where the mass takes 2.3 GeV, and the threshold value is slightly smaller than the mass, unlike the ordinary expectation that $\sqrt{s_0}$ is larger than the obtained mass. However, the minimum point is on the very shallow minimum curve and the resulting mass is rather insensitive to the change in the $\sqrt{s_0}$ value. Therefore, we can increase $\sqrt{s_0}$ slightly more, for example 2.45 GeV, but the mass still remains at around 2.35 GeV, which is smaller than $\sqrt{s_0}$ now. We interpret this fact as an indication that the state $Y(2175)$ has a narrow decay width which is around 58 MeV.
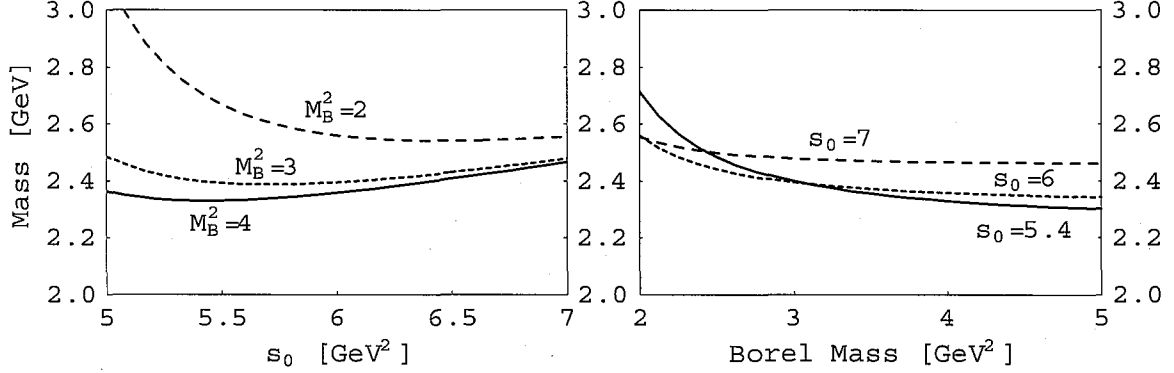
Figure 7.4: The mass of $Y(2175)$ as a function of M_B (Left) and s_0 (Right) in units of GeV for $\eta_{2\mu}$.

7.5 Conclusion

In this chapter we have studied the mass of the state $Y(2175)$ with the quantum numbers $J^{PC} = 1^{--}$ in the QCD sum rule. We have constructed both the diquark-antidiquark currents $(ss)(\bar{s}\bar{s})$ and the meson-meson currents $(\bar{s}s)(\bar{s}s)$. We find that there are two independent currents for both cases and verify the relations between them. Then using the two $(ss)(\bar{s}\bar{s})$ currents, we calculate the OPE up to dimension twelve, which contains the $\langle \bar{s}s \rangle^4$ condensates. The convergence of the OPE turns our to be very good. We find that the OPE's of the two currents are similar, and therefore, the obtained results are also similar. By using both the SVZ sum rule and the finite energy sum rule, we find that there is a mass minimum. For SVZ sum rule, the minimum is in the region $5 < s_0 < 7$ GeV^2 and $2 < M_B^2 < 4$ GeV^2. For finite energy sum rule, the minimum is in the region $4.5 < s_0 < 5.5$ GeV^2. It is about $2.2 \sim 2.4$ GeV. Considering the uncertainty, the state $Y(2175)$ can be accommodated in the QCD sum rule formalism although the central value of the mass is about 100 MeV higher than the experimental value.

We have investigated the coupling of the currents to the lower lying states including $\phi(1020)$ and found that the relevant spectral density becomes negative, implying that the present four-quark currents can not describe those states properly. This fact indicates that the four-quark interpolating currents couple rather weakly to $\phi(1020)$, which is a pure $s\bar{s}$ state.

We can test the tetraquark structure of $Y(2175)$ by considering its decay properties. Naively, the $ss\bar{s}\bar{s}$ tetraquark would fall apart via S-wave into the $\phi(1020)f_0(980)$ pair, and would have a very large width. The experimental width of $Y(2175)$ is only about 60 MeV, which seems too narrow to be a pure tetraquark state. We can discuss the decay of the $Y(2175)$ by borrowing an argument based on a valence quark picture. The
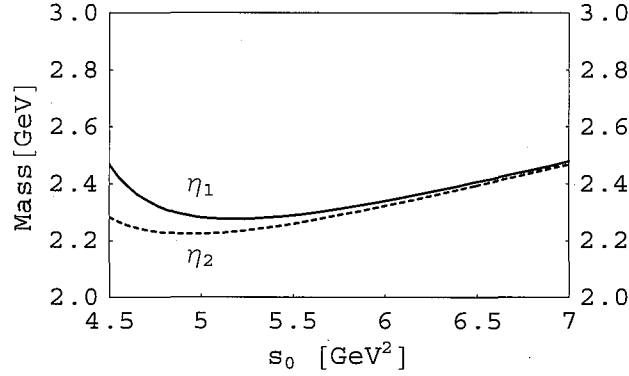
Figure 7.5: The mass of $Y(2175)$ by using the current $\eta_{1\mu}$ (solid line) and $\eta_{2\mu}$ (dashed line) as a function of s_0 in units of GeV.

$(\bar{s}s)(\bar{s}s)$ configuration for $Y(2175)$ can be a combination of $^3S_1$ and $^3P_0$, which may fall apart into two mesons of 1^- and 0^+ in the s-wave. In the QCD sum rule the 1^- $\bar{s}s$ meson is well identified with $\phi(1020)$, while the 0^+ $\bar{s}s$ meson has a mass around 1.5 GeV and is hard to be identified with the observed $f_0(980)$. Therefore, such a fall-apart decay would simply be suppressed due to the kinematical reason. The physical $f_0(980)$ state may be a tetraquark state as discussed in the previous QCD sum rule study [39]. Then the transition $Y(2175) \rightarrow \phi(1020) + f_0(\text{tetraquark})$ should be accompanied by a $\bar{q}q$ creation violating the OZI rule, as well as by an annihilation of one quanta of orbital angular momentum. These facts may once again suppress the decay of $Y(2175) \rightarrow \phi(1020) + f_0(980)$. This fact was studied in the recent paper by Torres, Khemchandani, Geng, Napsuciale and Oset [130]. They studied the $\phi K \bar{K}$ system with the Faddeev equations where the contained $K \bar{K}$ form the $f_0(980)$ resonance. The decay width they calculated is around 18 MeV, not far from the experimental value. The all above evidences would imply that the $Y(2175)$ is a possible candidate of a tetraquark state.

$Y(2175)$ could be a threshold effect, a hybrid state $s\bar{s}G$, a tetraquark, an excited $s\bar{s}$ state or a mixture of all the above possibilities. Because of its non-exotic quantum number, it is not easy to establish its underlying structure. Clearly more experimental and theoretical investigations are required.

One byproduct of the present work is the interesting observation that some type of four-quark interpolating currents may couple weakly to the conventional $q\bar{q}$ ground states. If future work confirms this point, we may have a novel framework to study the excited $q\bar{q}$ mesons using the four-quark interpolating currents, which is not feasible for the traditional $q\bar{q}$ interpolating currents.

Chapter 8

Vector Tetraquark Meson of $I^G J^{PC} = 1^- 1^{-+}$

Hadrons beyond the conventional quark model have been studied for more than thirties years. For example, Jaffe suggested the low-lying scalar mesons as good candidates of tetraquark states composed of strongly correlated diquarks in 1976 [93]. Especially there may exist some low-lying exotic mesons with quantum numbers such as $(J^{PC}) = (1^{-+})$ which $\bar{q}q$ mesons can not access [16,114]. However the hybrid mesons with explicit glue can carry such quantum numbers. The experimental establishment of these states is a direct proof of the glue degree of freedom in the low energy sector of QCD and of fundamental importance.

The mass of the non-strange exotic hybrid meson from lattice QCD simulations includes: 2GeV [135], 1.74 GeV [80], and 1.8 GeV [29]. The mass of its strange partner is 1.92 GeV [80] and 2 GeV [29]. The hybrid meson mass from the constituent glue model is 2 GeV [86] while the value from the flux tube model is around 1.9 GeV [90,150]. The prediction from the QCD sum rule approach is around 1.6 GeV [44,104]. However, Yang obtained a surprisingly low mass around 1.26 GeV for the 1^{-+} hybrid meson using QCD sum rule [178].

Up to now, there are several candidates of the exotic mesons with $I^G(J^{PC}) = 1^-(1^{-+})$ experimentally. They are $\pi_1(1400)$, $\pi_1(1600)$ and $\pi_1(2015)$. Their masses and widths are $(1376 \pm 17, 300 \pm 40)$ MeV, $(1653^{+18}_{-15}, 225^{+45}_{-28})$ MeV, $(2014 \pm 20 \pm 16, 230 \pm 21 \pm 73)$ MeV, respectively [179]. $\pi_1(1400)$ was observed in the reactions $\pi^- p \to \eta \pi^0 n$ [10]; $\bar{p}p \to \pi^0 \pi^0 \eta$ and $\bar{p}n \to \pi^- \pi^0 \eta$ [4]; $\pi^- p \to \eta \pi^- p$ [164]. $\pi_1(1600)$ was observed in the reaction $\pi^- p \to \eta' \pi^- p$ (η' decays to $\eta \pi^+ \pi^-$ with a fraction 44.5%) [92]. Both $\pi_1(1600)$ and $\pi_1(2015)$ were observed in the reactions $\pi^- p \to \omega \pi^- \pi^0 p$ [126] and $\pi^- p \to \eta \pi^+ \pi^- \pi^- p$ [116]. However, a more recent analysis of a higher statistics sample from E852 3π data found no evidence of $\pi_1(1600)$ [61]. All the above observations were from hadron-production experiments.

Recently, the CLAS Collaboration performed a photo-production experiment to search for the 1^{-+} hybrid meson in the speculated 3π final state in the charge exchange reaction

$\gamma p \to \pi^+ \pi^+ \pi^- (n)$ [144]. If $\pi_1(1600)$ was an hybrid state, it was expected to be produced with a strength near or much larger than 10% of the $a_2(1320)$ meson from the theoretical models [12,49,91,163]. However $\pi_1(1600)$ was not observed with the expected strength. In fact its production rate is less than 2% of the $a_2(1320)$ meson. If the $\pi_1(1600)$ signal from the hadron-production experiments is not an artifact, the negative result of the photo-production experiment suggests (1) either theoretical production rates are overestimated significantly or (2) $\pi_1(1600)$ is a meson with a different inner structure instead of a hybrid state.

In fact, the tetraquark states can also carry the exotic quantum numbers $I^G(J^{PC}) = 1^-(1^{-+})$. It is important to note that the gluon inside the hybrid meson can easily split into a pair of $q\bar{q}$. Therefore tetraquarks can always have the same quantum numbers as the hybrid mesons, including the exotic ones. Discovery of hadron candidates with $J^{PC} = 1^{-+}$ does not ensure that it is an exotic hybrid meson. One has to exclude the other possibilities including tetraquarks based on its mass, decay width and decay patterns etc. This argument holds for all these claimed candidates of the hybrid meson.

Tetraquark states in general have a richer internal structure than ordinary $q\bar{q}$ states. For instance, a pair of quarks can be in channels which can not be allowed in the ordinary hadrons. The richness of the structure introduces complication in theoretical studies. Therefore, one usually assumed one or a few particular configurations which are motivated by some intuitions.

To study these states, we follow the same method used in previous sections which is based on complete classification of independent currents. By making suitable linear combinations of the independent currents we can perform advanced analysis as compared with the analysis of using only one type of current which limits the potential of the OPE, and sometimes leads to unphysical results.

In this chapter, we first classify the flavor structure of four-quark system with quantum numbers $J^{PC} = 1^{-+}$. We find that there are five iso-vector states. Then we construct tetraquark interpolating currents by using both diquark-antidiquark construction $((qq)(\bar{q}\bar{q}))$ and quark-antiquark pairs $((q\bar{q})(q\bar{q}))$. We verify that they are just different bases and can be related to each other. Therefore they lead to the same results. By using diquark-antidiquark currents, we perform the QCD sum rule analysis, and calculate their masses. Our results suggest that $\pi_1(1400)$ may not be explained by just using tetraquark structure, and $\pi_1(1600)$ and $\pi_1(2015)$ could be explained by the tetraquark mesons with quark contents $(qq)(\bar{q}\bar{q})$ and $(qs)(\bar{q}\bar{s})$ respectively. The diquark and antidiquark inside have a mixed flavor structure $(\bar{\mathbf{3}} \otimes \bar{\mathbf{6}}) \oplus (\mathbf{6} \otimes \mathbf{3})$.

This chapter is organized as follows. In Sec. 8.1, we construct the tetraquark currents. The tetraquark currents constructed by using both diquark (qq) and antidiquark $(\bar{q}\bar{q})$ are shown in Sec 8.1.1. The tetraquark currents constructed by using quark-antiquark $(\bar{q}q)$ pairs are shown in Sec 8.1.2. In Sec. 8.2, we perform a QCD sum rule analysis by using these currents, and calculate their OPEs. In Sec. 8.3, the numerical result is obtained for their masses. In Sec. 8.4, we use finite energy sum rule to calculate their masses again.

We discuss the decay patterns of these 1^{-+} tetraquark states in Sec. 8.5. In Sec. 8.6, we follow the same approach to study the isoscalar vector tetraquark states. Sec. 8.7 is a summary.

8.1 Tetraquark Currents

In order to construct proper tetraquark currents, let us start with the consideration of the charge-conjugation symmetry. The charge-conjugation transformation changes diquarks into antidiquarks, while it maintains their flavor structures. If a tetraquark state has a definite charge-conjugation parity, either positive or negative, the internal diquark (qq) and antidiquark ($\bar{q}\bar{q}$) must have the same flavor symmetry, which is either symmetric flavor structure $\mathbf{6_f} \otimes \mathbf{\bar{6}_f}$ (**S**) or antisymmetric flavor structure $\mathbf{\bar{3}_f} \otimes \mathbf{3_f}$ (**A**), and can not have mixed flavor symmetry neither $\mathbf{\bar{3}_f} \otimes \mathbf{\bar{6}_f}$ nor $\mathbf{6_f} \otimes \mathbf{3_f}$ (**M**). However, combinations of $\mathbf{\bar{3}_f} \otimes \mathbf{\bar{6}_f}$ and $\mathbf{6_f} \otimes \mathbf{3_f}$ can have a definite charge-conjugation parity. Therefore, in order to study the tetraquark state of $I^G J^{PC} = 1^- 1^{-+}$, we need to consider the following structures of currents

$$qq\bar{q}\bar{q}(\mathbf{S}), qs\bar{q}\bar{s}(\mathbf{S}) \sim \mathbf{6_f} \otimes \mathbf{\bar{6}_f} \quad (\mathbf{S}),$$
$$qs\bar{q}\bar{s}(\mathbf{A}) \sim \mathbf{\bar{3}_f} \otimes \mathbf{3_f} \quad (\mathbf{A}),$$
$$qq\bar{q}\bar{q}(\mathbf{M}), qs\bar{q}\bar{s}(\mathbf{M}) \sim (\mathbf{\bar{3}_f} \otimes \mathbf{\bar{6}_f}) \oplus (\mathbf{6_f} \otimes \mathbf{3_f}) \quad (\mathbf{M}),$$

where q represents an *up* or *down* quark, and s represents a *strange* quark. The flavor structures are shown in Fig. 8.1 in terms of $SU(3)$ weight diagrams. The quark contents indicated at vertices follow the ideal mixing scheme for inner vertices where the mixing is allowed. In the $SU(3)$ limit, the quark contents are suitable combinations of the ones shown in this figures. However, the *strange* quark has a significantly larger mass than *up* and *down* quarks (current quark mass), and so, the ideal mixing is expected to work well for hadrons except for pseudoscalar mesons. The flavor structure in the ideal mixing is also simpler than that in the $SU(3)$ limit. Therefore, we will use the ideal mixing in our QCD sum rule studies.

In the following subsections, we first construct currents by using diquark (qq) and antidiquark ($\bar{q}\bar{q}$) currents as well as quark-antiquark ($\bar{q}q$) pairs, and then we show the currents with explicit quark contents. The tensor currents $\eta_{\mu\nu}$ ($\eta_{\mu\nu} = -\eta_{\nu\mu}$) can also have $I^G J^{PC} = 1^- 1^{-+}$. By using tensor currents, we obtain the similar results, which will be shown in our future work.

8.1.1 $(qq)(\bar{q}\bar{q})$ Currents

We attempt to construct the tetraquark currents using diquark (qq) and antidiquark ($\bar{q}\bar{q}$) currents. For each state having the symmetric flavor structure $\mathbf{6_f} \otimes \mathbf{\bar{6}_f}$ (**S**), there are two

$(qq) \otimes (\overline{qq}) = 6_f \otimes \overline{6}_f$ (S)

27$_f$

$dd\overline{ss}$ $ud\overline{ss}$ $uu\overline{ss}$

$dd\overline{us}$ $\begin{matrix} dd\overline{ds} \\ ud\overline{us} \end{matrix}$ $\begin{matrix} ud\overline{ds} \\ uu\overline{us} \end{matrix}$ $uu\overline{ds}$

$dd\overline{uu}$ $dd\overline{dd}$ $\begin{matrix} dd\overline{dd} \\ ud\overline{ud} \\ uu\overline{uu} \end{matrix}$ $u\overline{ddd}$ $uu\overline{dd}$

$ds\overline{uu}$ $\begin{matrix} ds\overline{ud} \\ us\overline{uu} \end{matrix}$ $\begin{matrix} ds\overline{dd} \\ us\overline{ud} \end{matrix}$ $us\overline{dd}$

$ss\overline{uu}$ $ss\overline{ud}$ $ss\overline{dd}$

8$_f$

$ds\overline{ss}$ $us\overline{ss}$

$ds\overline{us}$ $\begin{matrix} ds\overline{ds} \\ us\overline{us} \end{matrix}$ $us\overline{ds}$

$ss\overline{us}$ $ss\overline{ds}$

1$_f$

$\times$

$ss\overline{ss}$

$(qq) \otimes (\overline{qq}) = \overline{3}_f \otimes 3_f$ (A)   **8**$_f$

$ud\overline{us}$ $ud\overline{ds}$

$ds\overline{us}$ $\begin{matrix} ds\overline{ds} \\ us\overline{us} \end{matrix}$ $us\overline{ds}$

$ds\overline{ud}$ $us\overline{ud}$

1$_f$

$\times$

$ud\overline{ud}$

$(qq) \otimes (\overline{qq}) = \overline{3}_f \otimes \overline{6}_f$ (M)

8$_f$

$us\overline{ss}$ $ds\overline{ss}$

$us\overline{ds}$ $\begin{matrix} us\overline{us} \\ \times\times \\ ds\overline{ds} \end{matrix}$ $ds\overline{us}$

$\overline{ds}(u\overline{u}-d\overline{d})$ $\overline{us}(u\overline{u}-d\overline{d})$

$\overline{10}_f$

$ud\overline{ss}$

$ud\overline{ds}$ $ud\overline{us}$

$ud\overline{dd}$ $ud\overline{ud}$ $ud\overline{uu}$

$us\overline{dd}$ $\overline{ds}(u\overline{u}+d\overline{d})$ $\overline{us}(u\overline{u}+d\overline{d})$ $ds\overline{uu}$

Figure 8.1: Weight diagrams for $6_f \otimes \overline{6}_f(S)$ (top panel), $\overline{3}_f \otimes 3_f(A)$ (middle panel), and $\overline{3}_f \otimes \overline{6}_f(M)$ (bottom panel). The weight diagram for $6_f \otimes 3_f(M)$ is the charge-conjugation transformation of the bottom one.

$(qq)(\bar{q}\bar{q})$ currents of $J^{PC} = 1^{-+}$, which are independent

$$\psi_{1\mu}^{S} = q_{1a}^{T}C\gamma_5 q_{2b}(\bar{q}_{3a}\gamma_\mu\gamma_5 C\bar{q}_{4b}^{T} + \bar{q}_{3b}\gamma_\mu\gamma_5 C\bar{q}_{4a}^{T}) + q_{1a}^{T}C\gamma_\mu\gamma_5 q_{2b}(\bar{q}_{3a}\gamma_5 C\bar{q}_{4b}^{T} + \bar{q}_{3b}\gamma_5 C\bar{q}_{4a}^{T}) \quad (8.1)$$

$$\psi_{2\mu}^{S} = q_{1a}^{T}C\gamma^{\nu} q_{2b}(\bar{q}_{3a}\sigma_{\mu\nu} C\bar{q}_{4b}^{T} - \bar{q}_{3b}\sigma_{\mu\nu} C\bar{q}_{4a}^{T}) + q_{1a}^{T}C\sigma_{\mu\nu} q_{2b}(\bar{q}_{3a}\gamma^{\nu} C\bar{q}_{4b}^{T} - \bar{q}_{3b}\gamma^{\nu} C\bar{q}_{4a}^{T}) \,,$$

where the sum over repeated indices (μ, ν, $\cdots$ for Dirac spinor indices, and $a, b, \cdots$ for color indices) is taken. C is the charge-conjugation matrix, q_1 and q_2 represent quarks, and q_3 and q_4 represent antiquarks. For the antisymmetry flavor structure $\bar{\mathbf{3}}_{\mathbf{f}} \otimes \mathbf{3}_{\mathbf{f}}$ (**A**), we also find that there are two independent $(qq)(\bar{q}\bar{q})$ currents,

$$\psi_{1\mu}^{A} = q_{1a}^{T}C\gamma_5 q_{2b}(\bar{q}_{3a}\gamma_\mu\gamma_5 C\bar{q}_{4b}^{T} - \bar{q}_{3b}\gamma_\mu\gamma_5 C\bar{q}_{4a}^{T}) + q_{1a}^{T}C\gamma_\mu\gamma_5 q_{2b}(\bar{q}_{3a}\gamma_5 C\bar{q}_{4b}^{T} - \bar{q}_{3b}\gamma_5 C\bar{q}_{4a}^{T}) \quad (8.2)$$

$$\psi_{2\mu}^{A} = q_{1a}^{T}C\gamma^{\nu} q_{2b}(\bar{q}_{3a}\sigma_{\mu\nu} C\bar{q}_{4b}^{T} + \bar{q}_{3b}\sigma_{\mu\nu} C\bar{q}_{4a}^{T}) + q_{1a}^{T}C\sigma_{\mu\nu} q_{2b}(\bar{q}_{3a}\gamma^{\nu} C\bar{q}_{4b}^{T} + \bar{q}_{3b}\gamma^{\nu} C\bar{q}_{4a}^{T}) \,,$$

For each state containing diquark and antidiquark having either the flavor structure $\bar{\mathbf{3}}_{\mathbf{f}} \otimes \bar{\mathbf{6}}_{\mathbf{f}}$ or $\mathbf{6}_{\mathbf{f}} \otimes \mathbf{3}_{\mathbf{f}}$, there are no currents of quantum numbers $J^{PC} = 1^{-+}$. However, their combinations $(\bar{\mathbf{3}}_{\mathbf{f}} \otimes \bar{\mathbf{6}}_{\mathbf{f}}) \oplus (\mathbf{6}_{\mathbf{f}} \otimes \mathbf{3}_{\mathbf{f}})$ can have the quantum numbers $J^{PC} = 1^{-+}$. We first define the currents $\psi_{i\mu}^{ML}$ which belong to the flavor representation $\bar{\mathbf{3}}_{\mathbf{f}} \otimes \bar{\mathbf{6}}_{\mathbf{f}}$, and the currents $\psi_{i\mu}^{MR}$ which belong to the flavor representation $\mathbf{6}_{\mathbf{f}} \otimes \mathbf{3}_{\mathbf{f}}$ separately. We find the following four independent currents:

$$\psi_{1\mu}^{ML} = q_{1a}^{T}C\gamma_\mu q_{2b}(\bar{q}_{3a} C\bar{q}_{4b}^{T} + \bar{q}_{3b} C\bar{q}_{4a}^{T}) \,,$$

$$\psi_{2\mu}^{ML} = q_{1a}^{T}C\sigma_{\mu\nu}\gamma_5 q_{2b}(\bar{q}_{3a}\gamma^{\nu}\gamma_5 C\bar{q}_{4b}^{T} + \bar{q}_{3b}\gamma^{\nu}\gamma_5 C\bar{q}_{4a}^{T}) \,,$$

$$\psi_{3\mu}^{ML} = q_{1a}^{T}C q_{2b}(\bar{q}_{3a}\gamma_\mu C\bar{q}_{4b}^{T} - \bar{q}_{3b}\gamma_\mu C\bar{q}_{4a}^{T}) \,,$$

$$\psi_{4\mu}^{ML} = q_{1a}^{T}C\gamma^{\nu}\gamma_5 q_{2b}(\bar{q}_{3a}\sigma_{\mu\nu}\gamma_5 C\bar{q}_{4b}^{T} - \bar{q}_{3b}\sigma_{\mu\nu}\gamma_5 C\bar{q}_{4a}^{T}) \,,$$

$$\psi_{1\mu}^{MR} = q_{1a}^{T}C q_{2b}(\bar{q}_{3a}\gamma_\mu C\bar{q}_{4b}^{T} + \bar{q}_{3b}\gamma_\mu C\bar{q}_{4a}^{T}) \,,$$

$$\psi_{2\mu}^{MR} = q_{1a}^{T}C\gamma^{\nu}\gamma_5 q_{2b}(\bar{q}_{3a}\sigma_{\mu\nu}\gamma_5 C\bar{q}_{4b}^{T} + \bar{q}_{3b}\sigma_{\mu\nu}\gamma_5 C\bar{q}_{4a}^{T}) \,,$$

$$\psi_{3\mu}^{MR} = q_{1a}^{T}C\gamma_\mu q_{2b}(\bar{q}_{3a} C\bar{q}_{4b}^{T} - \bar{q}_{3b} C\bar{q}_{4a}^{T}) \,,$$

$$\psi_{4\mu}^{MR} = q_{1a}^{T}C\sigma_{\mu\nu}\gamma_5 q_{2b}(\bar{q}_{3a}\gamma^{\nu}\gamma_5 C\bar{q}_{4b}^{T} - \bar{q}_{3b}\gamma^{\nu}\gamma_5 C\bar{q}_{4a}^{T}) \,.$$

They all have quantum numbers $J^{P} = 1^{-}$ but no good charge-conjugation parity. However, their mixing can have a definite charge-conjugation parity,

$$\psi_{i\mu}^{M} = \psi_{i\mu}^{ML} \pm \psi_{i\mu}^{MR} \,, \tag{8.3}$$

where the $+$ and $-$ combinations correspond to the charge-conjugation parity positive and negative, respectively. In the present work, we only consider the positive one.

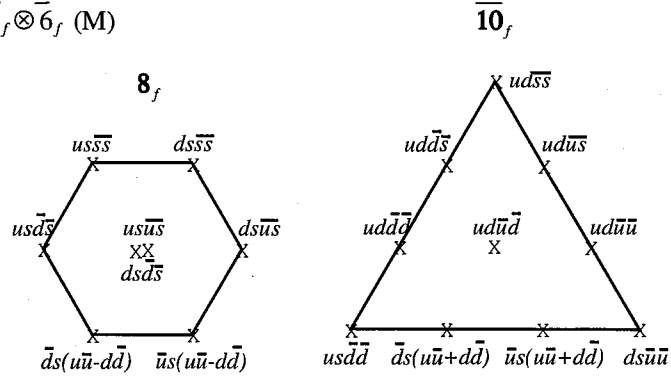## 8.1.2 $(\bar{q}q)(\bar{q}q)$ Currents

In this appendix, we attempt to construct the tetraquark currents using quark-antiquark $(\bar{q}q)$ pairs. For each state containing diquark and antidiquark having the symmetric flavor $\mathbf{6_f} \otimes \mathbf{6_f}$, there are four $(\bar{q}q)(\bar{q}q)$ currents:

$$
\begin{aligned}
\xi_{1\mu}^S &= (\bar{q}_{3a}\gamma_\mu\gamma_5 q_{1a})(\bar{q}_{4b}\gamma_5 q_{2b}) + (\bar{q}_{3a}\gamma_5 q_{1a})(\bar{q}_{4b}\gamma_\mu\gamma_5 q_{2b}) \\
&\quad + (\bar{q}_{3a}\gamma_\mu\gamma_5 q_{2a})(\bar{q}_{4b}\gamma_5 q_{1b}) + (\bar{q}_{3a}\gamma_5 q_{2a})(\bar{q}_{4b}\gamma_\mu\gamma_5 q_{1b}) \, , \\
\xi_{2\mu}^S &= (\bar{q}_{3a}\gamma^\nu q_{1a})(\bar{q}_{4b}\sigma_{\mu\nu} q_{2b}) + (\bar{q}_{3a}\sigma_{\mu\nu} q_{1a})(\bar{q}_{4b}\gamma^\nu q_{2b}) \\
&\quad + (\bar{q}_{3a}\gamma^\nu q_{2a})(\bar{q}_{4b}\sigma_{\mu\nu} q_{1b}) + (\bar{q}_{3a}\sigma_{\mu\nu} q_{2a})(\bar{q}_{4b}\gamma^\nu q_{1b}) \, , \\
\xi_{3\mu}^S &= \lambda_{ab}\lambda_{cd}\{ (\bar{q}_{3a}\gamma_\mu\gamma_5 q_{1b})(\bar{q}_{4c}\gamma_5 q_{2d}) + (\bar{q}_{3a}\gamma_5 q_{1b})(\bar{q}_{4c}\gamma_\mu\gamma_5 q_{2d}) \\
&\quad + (\bar{q}_{3a}\gamma_\mu\gamma_5 q_{2b})(\bar{q}_{4c}\gamma_5 q_{1d}) + (\bar{q}_{3a}\gamma_5 q_{2b})(\bar{q}_{4c}\gamma_\mu\gamma_5 q_{1d}) \} \, , \\
\xi_{4\mu}^S &= \lambda_{ab}\lambda_{cd}\{ (\bar{q}_{3a}\gamma^\nu q_{1b})(\bar{q}_{4c}\sigma_{\mu\nu} q_{2d}) + (\bar{q}_{3a}\sigma_{\mu\nu} q_{1b})(\bar{q}_{4c}\gamma^\nu q_{2d}) \\
&\quad + (\bar{q}_{3a}\gamma^\nu q_{2b})(\bar{q}_{4c}\sigma_{\mu\nu} q_{1d}) + (\bar{q}_{3a}\sigma_{\mu\nu} q_{2b})(\bar{q}_{4c}\gamma^\nu q_{1d}) \} \, .
\end{aligned}
$$

Among these currents, only two are independent. We can verify the following relations

$$
\begin{aligned}
\xi_{3\mu}^S &= -\frac{5}{3}\xi_{1\mu}^S - i\xi_{2\mu}^S \, , \\
\xi_{4\mu}^S &= 3i\xi_{1\mu}^S + \frac{1}{3}\xi_{2\mu}^S \, .
\end{aligned}
$$

Moreover, they are equivalent to the $(qq)(\bar{q}\bar{q})$ currents

$$
\begin{aligned}
\psi_{1\mu}^S &= -\frac{1}{2}\xi_{1\mu}^S + \frac{i}{2}\xi_{2\mu}^S \, , \\
\psi_{2\mu}^S &= -\frac{3i}{2}\xi_{1\mu}^S + \frac{1}{2}\xi_{2\mu}^S \, .
\end{aligned}
$$

For each state containing diquark and antidiquark having the antisymmetric flavor $\mathbf{\bar{3}_f} \otimes \mathbf{3_f}$, there are also four $(\bar{q}q)(\bar{q}q)$ currents which are non-zero:

$$
\begin{aligned}
\xi_{1\mu}^A &= (\bar{q}_{3a}\gamma_\mu\gamma_5 q_{1a})(\bar{q}_{4b}\gamma_5 q_{2b}) + (\bar{q}_{3a}\gamma_5 q_{1a})(\bar{q}_{4b}\gamma_\mu\gamma_5 q_{2b}) \\
&\quad - (\bar{q}_{3a}\gamma_\mu\gamma_5 q_{2a})(\bar{q}_{4b}\gamma_5 q_{1b}) - (\bar{q}_{3a}\gamma_5 q_{2a})(\bar{q}_{4b}\gamma_\mu\gamma_5 q_{1b}) \, , \\
\xi_{2\mu}^A &= (\bar{q}_{3a}\gamma^\nu q_{1a})(\bar{q}_{4b}\sigma_{\mu\nu} q_{2b}) + (\bar{q}_{3a}\sigma_{\mu\nu} q_{1a})(\bar{q}_{4b}\gamma^\nu q_{2b}) \\
&\quad - (\bar{q}_{3a}\gamma^\nu q_{2a})(\bar{q}_{4b}\sigma_{\mu\nu} q_{1b}) - (\bar{q}_{3a}\sigma_{\mu\nu} q_{2a})(\bar{q}_{4b}\gamma^\nu q_{1b}) \, , \\
\xi_{3\mu}^A &= \lambda_{ab}\lambda_{cd}\{ (\bar{q}_{3a}\gamma_\mu\gamma_5 q_{1b})(\bar{q}_{4c}\gamma_5 q_{2d}) + (\bar{q}_{3a}\gamma_5 q_{1b})(\bar{q}_{4c}\gamma_\mu\gamma_5 q_{2d}) \\
&\quad - (\bar{q}_{3a}\gamma_\mu\gamma_5 q_{2b})(\bar{q}_{4c}\gamma_5 q_{1d}) - (\bar{q}_{3a}\gamma_5 q_{2b})(\bar{q}_{4c}\gamma_\mu\gamma_5 q_{1d}) \} \, , \\
\xi_{4\mu}^A &= \lambda_{ab}\lambda_{cd}\{ (\bar{q}_{3a}\gamma^\nu q_{1b})(\bar{q}_{4c}\sigma_{\mu\nu} q_{2d}) + (\bar{q}_{3a}\sigma_{\mu\nu} q_{1b})(\bar{q}_{4c}\gamma^\nu q_{2d}) \\
&\quad - (\bar{q}_{3a}\gamma^\nu q_{2b})(\bar{q}_{4c}\sigma_{\mu\nu} q_{1d}) - (\bar{q}_{3a}\sigma_{\mu\nu} q_{2b})(\bar{q}_{4c}\gamma^\nu q_{1d}) \} \, ,
\end{aligned}
$$

where once again only two are independent

$$\xi_{3\mu}^A = \frac{1}{3}\xi_{1\mu}^A + i\xi_{2\mu}^A,$$

$$\xi_{4\mu}^A = -3i\xi_{1\mu}^A - \frac{5}{3}\xi_{2\mu}^A.$$

They are equivalent to the $(qq)(\bar{q}\bar{q})$ currents

$$\psi_{1\mu}^A = -\frac{1}{2}\xi_{1\mu}^A + \frac{i}{2}\xi_{2\mu}^A,$$

$$\psi_{2\mu}^A = -\frac{3i}{2}\xi_{1\mu}^A + \frac{1}{2}\xi_{2\mu}^A.$$

For the currents which have a mixed flavor symmetry, we just show the $(\bar{q}q)(\bar{q}q)$ currents which belong to the flavor representation $\mathbf{\bar{3}_f} \otimes \mathbf{\bar{6}_f}$. Those belonging to the flavor representation $\mathbf{6_f} \otimes \mathbf{3_f}$ can be obtained similarly.

$$\begin{aligned}
\xi_{1\mu}^M &= (\bar{q}_{3a}q_{1a})(\bar{q}_{4b}\gamma_\mu q_{2b}) - (\bar{q}_{3a}\gamma_\mu q_{1a})(\bar{q}_{4b}q_{2b}) \\
&\quad -(\bar{q}_{3a}q_{2a})(\bar{q}_{4b}\gamma_\mu q_{1b}) + (\bar{q}_{3a}\gamma_\mu q_{2a})(\bar{q}_{4b}q_{1b}), \\
\xi_{2\mu}^M &= (\bar{q}_{3a}\gamma^\mu\gamma_5 q_{1a})(\bar{q}_{4b}\gamma_5 q_{2b}) - (\bar{q}_{3a}\gamma_5 q_{1a})(\bar{q}_{4b}\gamma^\mu\gamma_5 q_{2b}) \\
&\quad -(\bar{q}_{3a}\gamma^\mu\gamma_5 q_{2a})(\bar{q}_{4b}\gamma_5 q_{1b}) + (\bar{q}_{3a}\gamma_5 q_{2a})(\bar{q}_{4b}\gamma^\mu\gamma_5 q_{1b}), \\
\xi_{3\mu}^M &= (\bar{q}_{3a}\gamma^\nu q_{1a})(\bar{q}_{4b}\sigma_{\mu\nu} q_{2b}) - (\bar{q}_{3a}\sigma_{\mu\nu} q_{1a})(\bar{q}_{4b}\gamma^\nu q_{2b}) \\
&\quad -(\bar{q}_{3a}\gamma^\nu q_{2a})(\bar{q}_{4b}\sigma_{\mu\nu} q_{1b}) + (\bar{q}_{3a}\sigma_{\mu\nu} q_{2a})(\bar{q}_{4b}\gamma^\nu q_{1b}), \\
\xi_{4\mu}^M &= (\bar{q}_{3a}\gamma^\nu\gamma_5 q_{1a})(\bar{q}_{4b}\sigma_{\mu\nu}\gamma_5 q_{2b}) - (\bar{q}_{3a}\sigma_{\mu\nu}\gamma_5 q_{1a})(\bar{q}_{4b}\gamma^\nu\gamma_5 q_{2b}) \\
&\quad -(\bar{q}_{3a}\gamma^\nu\gamma_5 q_{2a})(\bar{q}_{4b}\sigma_{\mu\nu}\gamma_5 q_{1b}) + (\bar{q}_{3a}\sigma_{\mu\nu}\gamma_5 q_{2a})(\bar{q}_{4b}\gamma^\nu\gamma_5 q_{1b}).
\end{aligned}$$

There are also four currents which have a color $\mathbf{8_c} \otimes \mathbf{8_c}$ structure, and they can be written as a linear combination of the currents with color structure $\mathbf{1_c} \otimes \mathbf{1_c}$. The relations between $\phi_{i\mu}^{ML}$ and $\xi_{i\mu}^M$ are:

$$\begin{aligned}
\psi_{1\mu}^{ML} &= -\frac{1}{4}\xi_{1\mu}^M + \frac{1}{4}\xi_{2\mu}^M + \frac{i}{4}\xi_{3\mu}^M - \frac{i}{4}\xi_{4\mu}^M, \\
\psi_{2\mu}^{ML} &= \frac{3i}{4}\xi_{1\mu}^M + \frac{3i}{4}\xi_{2\mu}^M + \frac{1}{4}\xi_{3\mu}^M + \frac{1}{4}\xi_{4\mu}^M, \\
\psi_{3\mu}^{ML} &= \frac{1}{4}\xi_{1\mu}^M + \frac{1}{4}\xi_{2\mu}^M + \frac{i}{4}\xi_{3\mu}^M + \frac{i}{4}\xi_{4\mu}^M, \\
\psi_{4\mu}^{ML} &= -\frac{3i}{4}\xi_{1\mu}^M + \frac{3i}{4}\xi_{2\mu}^M + \frac{1}{4}\xi_{3\mu}^M - \frac{1}{4}\xi_{4\mu}^M.
\end{aligned}$$

We can obtain similar results for $\psi_{i\mu}^{MR}$, which we do not show here any more.

8.1.3   Iso-Vector Currents

For the study of the present exotic tetraquark state, we need to construct iso-vector ($I = 1$) currents. There are two isospin triplets belonging to the flavor representation $\mathbf{6}_f \otimes \bar{\mathbf{6}}_f$, one isospin triplet belonging to the flavor representation $\bar{\mathbf{3}}_f \otimes \mathbf{3}_f$, and two isospin triplets belonging to the flavor representation $(\bar{\mathbf{3}}_f \otimes \bar{\mathbf{6}}_f) \oplus (\mathbf{6}_f \otimes \mathbf{3}_f)$ (Fig. 8.1). For each state, there are several independent currents. We list them in the following.

1. For the two isospin triplets belonging to $\mathbf{6}_f \otimes \bar{\mathbf{6}}_f$ (**S**):

$$
\begin{cases}
\eta_{1\mu}^S \equiv \psi_{1\mu}^S(qq\bar{q}\bar{q}) \sim & u_a^T C\gamma_5 d_b(\bar{u}_a\gamma_\mu\gamma_5 C\bar{d}_b^T + \bar{u}_b\gamma_\mu\gamma_5 C\bar{d}_a^T) \\
& +u_a^T C\gamma_\mu\gamma_5 d_b(\bar{u}_a\gamma_5 C\bar{d}_b^T + \bar{u}_b\gamma_5 C\bar{d}_a^T)\,, \\
\eta_{2\mu}^S \equiv \psi_{2\mu}^S(qq\bar{q}\bar{q}) \sim & u_a^T C\gamma^\nu d_b(\bar{u}_a\sigma_{\mu\nu} C\bar{d}_b^T - \bar{u}_b\sigma_{\mu\nu} C\bar{d}_a^T) \\
& +u_a^T C\sigma_{\mu\nu} d_b(\bar{u}_a\gamma^\nu C\bar{d}_b^T - \bar{u}_b\gamma^\nu C\bar{d}_a^T)\,,
\end{cases}
$$

$$
\begin{cases}
\eta_{3\mu}^S \equiv \psi_{1\mu}^S(qs\bar{q}\bar{s}) \sim & u_a^T C\gamma_5 s_b(\bar{u}_a\gamma_\mu\gamma_5 C\bar{s}_b^T + \bar{u}_b\gamma_\mu\gamma_5 C\bar{s}_a^T) \\
& +u_a^T C\gamma_\mu\gamma_5 s_b(\bar{u}_a\gamma_5 C\bar{s}_b^T + \bar{u}_b\gamma_5 C\bar{s}_a^T)\,, \\
\eta_{4\mu}^S \equiv \psi_{2\mu}^S(qs\bar{q}\bar{s}) \sim & u_a^T C\gamma^\nu s_b(\bar{u}_a\sigma_{\mu\nu} C\bar{s}_b^T - \bar{u}_b\sigma_{\mu\nu} C\bar{s}_a^T) \\
& +u_a^T C\sigma_{\mu\nu} s_b(\bar{u}_a\gamma^\nu C\bar{s}_b^T - \bar{u}_b\gamma^\nu C\bar{s}_a^T)\,.
\end{cases}
$$

where $\eta_{1\mu}^S$ and $\eta_{2\mu}^S$ are the two independent currents containing only light flavors, and $\eta_{3\mu}^S$ and $\eta_{4\mu}^S$ are the two independent ones containing one $s\bar{s}$ quark pair.

2. For the isospin triplet belonging to $\bar{\mathbf{3}}_f \otimes \mathbf{3}_f$ (**A**):

$$
\begin{cases}
\eta_{1\mu}^A \equiv \psi_{1\mu}^A(qs\bar{q}\bar{s}) \sim & u_a^T C\gamma_5 s_b(\bar{u}_a\gamma_\mu\gamma_5 C\bar{s}_b^T - \bar{u}_b\gamma_\mu\gamma_5 C\bar{s}_a^T) \\
& +u_a^T C\gamma_\mu\gamma_5 s_b(\bar{u}_a\gamma_5 C\bar{s}_b^T - \bar{u}_b\gamma_5 C\bar{s}_a^T)\,, \\
\eta_{2\mu}^A \equiv \psi_{2\mu}^A(qs\bar{q}\bar{s}) \sim & u_a^T C\gamma^\nu s_b(\bar{u}_a\sigma_{\mu\nu} C\bar{s}_b^T + \bar{u}_b\sigma_{\mu\nu} C\bar{s}_a^T) \\
& +u_a^T C\sigma_{\mu\nu} s_b(\bar{u}_a\gamma^\nu C\bar{s}_b^T + \bar{u}_b\gamma^\nu C\bar{s}_a^T)\,,
\end{cases}
$$

where $\eta_{1\mu}^A$ and $\eta_{2\mu}^A$ are the two independent currents.

3. For the two isospin triplets belonging to $(\bar{\mathbf{3}}_f \otimes \bar{\mathbf{6}}_f) \oplus (\mathbf{6}_f \otimes \mathbf{3}_f)$ (**M**):

$$
\begin{cases}
\eta_{1\mu}^M \equiv \psi_{1\mu}^M(qq\bar{q}\bar{q}) \sim & u_a^T C\gamma_\mu d_b(\bar{u}_a C\bar{d}_b^T + \bar{u}_b C\bar{d}_a^T) \\
& +u_a^T C d_b(\bar{u}_a\gamma_\mu C\bar{d}_b^T + \bar{u}_b\gamma_\mu C\bar{d}_a^T)\,, \\
\eta_{2\mu}^M \equiv \psi_{2\mu}^M(qq\bar{q}\bar{q}) \sim & u_a^T C\sigma_{\mu\nu}\gamma_5 d_b(\bar{u}_a\gamma^\nu\gamma_5 C\bar{d}_b^T + \bar{u}_b\gamma^\nu\gamma_5 C\bar{d}_a^T) \\
& +u_a^T C\gamma^\nu\gamma_5 d_b(\bar{u}_a\sigma_{\mu\nu}\gamma_5 C\bar{d}_b^T + \bar{u}_b\sigma_{\mu\nu}\gamma_5 C\bar{d}_a^T)\,, \\
\eta_{3\mu}^M \equiv \psi_{3\mu}^M(qq\bar{q}\bar{q}) \sim & u_a^T C d_b(\bar{u}_a\gamma_\mu C\bar{d}_b^T - \bar{u}_b\gamma_\mu C\bar{d}_a^T) \\
& +u_a^T C\gamma_\mu d_b(\bar{u}_a C\bar{d}_b^T - \bar{u}_b C\bar{d}_a^T)\,, \\
\eta_{4\mu}^M \equiv \psi_{4\mu}^M(qq\bar{q}\bar{q}) \sim & u_a^T C\gamma^\nu\gamma_5 d_b(\bar{u}_a\sigma_{\mu\nu}\gamma_5 C\bar{d}_b^T - \bar{u}_b\sigma_{\mu\nu}\gamma_5 C\bar{d}_a^T) \\
& +u_a^T C\sigma_{\mu\nu}\gamma_5 d_b(\bar{u}_a\gamma^\nu\gamma_5 C\bar{d}_b^T - \bar{u}_b\gamma^\nu\gamma_5 C\bar{d}_a^T)\,,
\end{cases}
$$

$$
\left\{
\begin{aligned}
\eta_{5\mu}^M &\equiv \psi_{1\mu}^M(qs\bar{q}\bar{s}) \sim u_a^T C\gamma_\mu s_b(\bar{u}_a C\bar{s}_b^T + \bar{u}_b C\bar{s}_a^T) \\
&\quad + u_a^T C s_b(\bar{u}_a \gamma_\mu C\bar{s}_b^T + \bar{u}_b \gamma_\mu C\bar{s}_a^T), \\
\eta_{6\mu}^M &\equiv \psi_{2\mu}^M(qs\bar{q}\bar{s}) \sim u_a^T C\sigma_{\mu\nu}\gamma_5 s_b(\bar{u}_a \gamma^\nu \gamma_5 C\bar{s}_b^T + \bar{u}_b \gamma^\nu \gamma_5 C\bar{s}_a^T) \\
&\quad + u_a^T C\gamma^\nu \gamma_5 s_b(\bar{u}_a \sigma_{\mu\nu}\gamma_5 C\bar{s}_b^T + \bar{u}_b \sigma_{\mu\nu}\gamma_5 C\bar{s}_a^T), \\
\eta_{7\mu}^M &\equiv \psi_{3\mu}^M(qs\bar{q}\bar{s}) \sim u_a^T C s_b(\bar{u}_a \gamma_\mu C\bar{s}_b^T - \bar{u}_b \gamma_\mu C\bar{s}_a^T) \\
&\quad + u_a^T C\gamma_\mu s_b(\bar{u}_a C\bar{s}_b^T - \bar{u}_b C\bar{s}_a^T), \\
\eta_{8\mu}^M &\equiv \psi_{4\mu}^M(qs\bar{q}\bar{s}) \sim u_a^T C\gamma^\nu \gamma_5 s_b(\bar{u}_a \sigma_{\mu\nu}\gamma_5 C\bar{s}_b^T - \bar{u}_b \sigma_{\mu\nu}\gamma_5 C\bar{s}_a^T) \\
&\quad + u_a^T C\sigma_{\mu\nu}\gamma_5 s_b(\bar{u}_a \gamma^\nu \gamma_5 C\bar{s}_b^T - \bar{u}_b \gamma^\nu \gamma_5 C\bar{s}_a^T),
\end{aligned}
\right.
$$

where $\eta_{1,2,3,4}^M$ are the four independent currents containing only light flavors, and $\eta_{1,2,3,4}^M$ are the four independent ones containing one $s\bar{s}$ quark pair.

We use $\sim$ to make clear that the quark contents here are not exactly correct. For instance, in the current $\eta_{1\mu}^A$, the state $us\bar{u}\bar{s}$ does not have isospin one. The correct quark contents should be $(us\bar{u}\bar{s} - ds\bar{d}\bar{s})$. However, in the following QCD sum rule analysis, we shall not include the mass of up and $down$ quarks and choose the same value for $\langle \bar{u}u \rangle$ and $\langle \bar{d}d \rangle$. Therefore, the QCD sum rule results for η_1^A with quark contents $us\bar{u}\bar{s}$ and $(us\bar{u}\bar{s} - ds\bar{d}\bar{s})$ are the same.

8.2 SVZ sum rule

We have performed the OPE calculation up to dimension twelve:

$$
\begin{aligned}
\Pi_1^A(M_B^2) &= \int_{s_<}^{s_0} \left[\frac{1}{36848\pi^6}s^4 - \frac{17m_s^2}{15360\pi^6}s^3 + \left(\frac{\langle g_s^2 GG \rangle}{18432\pi^6} + \frac{m_s\langle \bar{q}q \rangle}{192\pi^4} + \frac{m_s\langle \bar{s}s \rangle}{96\pi^4} \right)s^2 \right. \\
&\quad + \left(-\frac{\langle \bar{q}q \rangle^2}{72\pi^2} - \frac{\langle \bar{s}s \rangle^2}{72\pi^2} - \frac{\langle \bar{q}q \rangle\langle \bar{s}s \rangle}{18\pi^2} + \frac{m_s\langle g_s\bar{q}\sigma Gq \rangle}{96\pi^4} + \frac{m_s\langle g_s\bar{s}\sigma Gs \rangle}{192\pi^4} \right. \\
&\quad \left. - \frac{m_s^2\langle g_s^2 GG \rangle}{4608\pi^6} \right)s - \frac{\langle \bar{q}q \rangle\langle g_s\bar{q}\sigma Gq \rangle}{48\pi^2} - \frac{\langle \bar{s}s \rangle\langle g_s\bar{s}\sigma Gs \rangle}{48\pi^2} - \frac{\langle \bar{q}q \rangle\langle g_s\bar{s}\sigma Gs \rangle}{24\pi^2} \\
&\quad - \frac{\langle \bar{s}s \rangle\langle g_s\bar{q}\sigma Gq \rangle}{24\pi^2} + \frac{m_s\langle g_s^2 GG \rangle\langle \bar{q}q \rangle}{256\pi^4} - \frac{m_s^2\langle \bar{q}q \rangle^2}{12\pi^2} + \frac{m_s^2\langle \bar{s}s \rangle^2}{48\pi^2} \\
&\quad \left. + \frac{m_s^2\langle \bar{q}q \rangle\langle \bar{s}s \rangle}{4\pi^2} \right] e^{-s/M_B^2} ds + \left(-\frac{\langle g_s\bar{q}\sigma Gq \rangle^2}{192\pi^2} - \frac{\langle g_s\bar{s}\sigma Gs \rangle^2}{192\pi^2} \right. \\
&\quad - \frac{\langle g_s\bar{q}\sigma Gq \rangle\langle g_s\bar{s}\sigma Gs \rangle}{48\pi^2} - \frac{5\langle g_s^2 GG \rangle\langle \bar{q}q \rangle\langle \bar{s}s \rangle}{864\pi^2} + \frac{m_s\langle \bar{q}q \rangle^2\langle \bar{s}s \rangle}{3} - \frac{2m_s\langle \bar{q}q \rangle\langle \bar{s}s \rangle^2}{9} \\
&\quad \left. + \frac{5m_s\langle g_s^2 GG \rangle\langle g_s\bar{q}\sigma Gq \rangle}{4608\pi^4} + \frac{m_s^2\langle \bar{q}q \rangle\langle g_s\bar{s}\sigma Gs \rangle}{12\pi^2} + \frac{m_s^2\langle \bar{s}s \rangle\langle g_s\bar{q}\sigma Gq \rangle}{8\pi^2} \right) \\
&\quad + \frac{1}{M_B^2}\left(-\frac{16g_s^2\langle \bar{q}q \rangle^2\langle \bar{s}s \rangle^2}{81} + \frac{\langle g_s^2 GG \rangle\langle \bar{q}q \rangle\langle g_s\bar{s}\sigma Gs \rangle}{1152\pi^2} + \frac{\langle g_s^2 GG \rangle\langle \bar{s}s \rangle\langle g_s\bar{q}\sigma Gq \rangle}{1152\pi^2} \right. \\
&\quad \left. - \frac{m_s\langle \bar{q}q \rangle^2\langle g_s\bar{s}\sigma Gs \rangle}{9} - \frac{m_s\langle \bar{s}s \rangle^2\langle g_s\bar{q}\sigma Gq \rangle}{18} - \frac{5m_s\langle \bar{q}q \rangle\langle \bar{s}s \rangle\langle g_s\bar{q}\sigma Gq \rangle}{18} \right.
\end{aligned}
$$

$$- \frac{m_s \langle \bar{q}q \rangle \langle \bar{s}s \rangle \langle g_s \bar{s}\sigma Gs \rangle}{18} + \frac{m_s^2 \langle g_s \bar{q}\sigma Gq \rangle^2}{48\pi^2} + \frac{m_s^2 \langle g_s \bar{q}\sigma Gq \rangle \langle g_s \bar{s}\sigma Gs \rangle}{48\pi^2} \Bigg) . \quad (8.4)$$

$$\Pi_1^M(M_B^2) = \int_0^{s_0} \left[\frac{1}{18432\pi^6} s^4 - \frac{\langle g_s^2 GG \rangle}{18432\pi^6} s^2 + \frac{\langle \bar{q}q \rangle^2}{18\pi^2} s + \frac{\langle \bar{q}q \rangle \langle g_s \bar{q}\sigma Gq \rangle}{12\pi^2} \right] e^{-s/M_B^2} ds$$
$$+ \left(\frac{\langle g_s \bar{q}\sigma Gq \rangle^2}{48\pi^2} - \frac{5\langle g_s^2 GG \rangle \langle \bar{q}q \rangle^2}{864\pi^2} \right)$$
$$+ \frac{1}{M_B^2} \left(- \frac{32 g_s^2 \langle \bar{q}q \rangle^4}{81} + \frac{\langle g_s^2 GG \rangle \langle \bar{q}q \rangle \langle g_s \bar{q}\sigma Gq \rangle}{576\pi^2} \right). \quad (8.5)$$

$$\Pi_2^M(M_B^2) = \int_0^{s_0} \left[\frac{1}{6144\pi^6} s^4 + \frac{11\langle g_s^2 GG \rangle}{18432\pi^6} s^2 + \frac{\langle \bar{q}q \rangle^2}{6\pi^2} s + \frac{\langle \bar{q}q \rangle \langle g_s \bar{q}\sigma Gq \rangle}{4\pi^2} \right] e^{-s/M_B^2} ds$$
$$+ \left(\frac{\langle g_s \bar{q}\sigma Gq \rangle^2}{16\pi^2} + \frac{5\langle g_s^2 GG \rangle \langle \bar{q}q \rangle^2}{864\pi^2} \right)$$
$$+ \frac{1}{M_B^2} \left(- \frac{32 g_s^2 \langle \bar{q}q \rangle^4}{27} - \frac{\langle g_s^2 GG \rangle \langle \bar{q}q \rangle \langle g_s \bar{q}\sigma Gq \rangle}{576\pi^2} \right). \quad (8.6)$$

$$\Pi_3^M(M_B^2) = \int_0^{s_0} \left[\frac{1}{36864\pi^6} s^4 + \frac{\langle g_s^2 GG \rangle}{18432\pi^6} s^2 + \frac{\langle \bar{q}q \rangle^2}{36\pi^2} s + \frac{\langle \bar{q}q \rangle \langle g_s \bar{q}\sigma Gq \rangle}{24\pi^2} \right] e^{-s/M_B^2} ds$$
$$+ \left(\frac{\langle g_s \bar{q}\sigma Gq \rangle^2}{96\pi^2} + \frac{5\langle g_s^2 GG \rangle \langle \bar{q}q \rangle^2}{864\pi^2} \right)$$
$$+ \frac{1}{M_B^2} \left(- \frac{16 g_s^2 \langle \bar{q}q \rangle^4}{81} - \frac{\langle g_s^2 GG \rangle \langle \bar{q}q \rangle \langle g_s \bar{q}\sigma Gq \rangle}{576\pi^2} \right). \quad (8.7)$$

$$\Pi_4^M(M_B^2) = \int_0^{s_0} \left[\frac{1}{12288\pi^6} s^4 + \frac{\langle g_s^2 GG \rangle}{18432\pi^6} s^2 + \frac{\langle \bar{q}q \rangle^2}{12\pi^2} s + \frac{\langle \bar{q}q \rangle \langle g_s \bar{q}\sigma Gq \rangle}{8\pi^2} \right] e^{-s/M_B^2} ds$$
$$+ \left(\frac{\langle g_s \bar{q}\sigma Gq \rangle^2}{32\pi^2} - \frac{5\langle g_s^2 GG \rangle \langle \bar{q}q \rangle^2}{864\pi^2} \right)$$
$$+ \frac{1}{M_B^2} \left(- \frac{16 g_s^2 \langle \bar{q}q \rangle^4}{27} + \frac{\langle g_s^2 GG \rangle \langle \bar{q}q \rangle \langle g_s \bar{q}\sigma Gq \rangle}{576\pi^2} \right). \quad (8.8)$$

$$\Pi_5^M(M_B^2) = \int_{4m_s^2}^{s_0} \left[\frac{1}{18432\pi^6} s^4 - \frac{17 m_s^2}{7680\pi^6} s^3 + \left(- \frac{\langle g_s^2 GG \rangle}{18432\pi^6} - \frac{m_s \langle \bar{q}q \rangle}{96\pi^4} + \frac{m_s \langle \bar{s}s \rangle}{48\pi^4} \right) s^2 \right.$$
$$+ \left(- \frac{\langle \bar{q}q \rangle^2}{36\pi^2} + \frac{\langle \bar{q}q \rangle \langle \bar{s}s \rangle}{9\pi^2} - \frac{\langle \bar{s}s \rangle^2}{36\pi^2} - \frac{m_s \langle g_s \bar{q}\sigma Gq \rangle}{48\pi^4} + \frac{m_s \langle g_s \bar{s}\sigma Gs \rangle}{96\pi^4} \right.$$
$$\left. + \frac{m_s^2 \langle g_s^2 GG \rangle}{4608\pi^6} \right) s - \frac{\langle \bar{q}q \rangle \langle g_s \bar{q}\sigma Gq \rangle}{24\pi^2} + \frac{\langle \bar{q}q \rangle \langle g_s \bar{s}\sigma Gs \rangle}{12\pi^2} + \frac{\langle \bar{s}s \rangle \langle g_s \bar{q}\sigma Gq \rangle}{12\pi^2}$$

$$- \frac{\langle \bar{s}s \rangle \langle g_s \bar{s}\sigma Gs \rangle}{24\pi^2} + \frac{m_s \langle g_s^2 GG \rangle \langle \bar{q}q \rangle}{256\pi^4} - \frac{m_s^2 \langle \bar{q}q \rangle^2}{6\pi^2} - \frac{m_s^2 \langle \bar{q}q \rangle \langle \bar{s}s \rangle}{2\pi^2}$$

$$+ \frac{m_s^2 \langle \bar{s}s \rangle^2}{24\pi^2} \Big] e^{-s/M_B^2} ds + \Big(- \frac{\langle g_s \bar{q}\sigma Gq \rangle^2}{96\pi^2} + \frac{\langle g_s \bar{q}\sigma Gq \rangle \langle g_s \bar{s}\sigma Gs \rangle}{24\pi^2}$$

$$- \frac{\langle g_s \bar{s}\sigma Gs \rangle^2}{96\pi^2} - \frac{5\langle g_s^2 GG \rangle \langle \bar{q}q \rangle \langle \bar{s}s \rangle}{864\pi^2} + \frac{2m_s \langle \bar{q}q \rangle^2 \langle \bar{s}s \rangle}{3} + \frac{4m_s \langle \bar{q}q \rangle \langle \bar{s}s \rangle^2}{9}$$

$$+ \frac{5m_s \langle g_s^2 GG \rangle \langle g_s \bar{q}\sigma Gq \rangle}{4608\pi^4} - \frac{m_s^2 \langle \bar{s}s \rangle \langle g_s \bar{q}\sigma Gq \rangle}{4\pi^2} - \frac{m_s^2 \langle \bar{q}q \rangle \langle g_s \bar{s}\sigma Gs \rangle}{6\pi^2} \Big)$$

$$+ \frac{1}{M_B^2} \Big(- \frac{32 g_s^2 \langle \bar{q}q \rangle^2 \langle \bar{s}s \rangle^2}{81} + \frac{\langle g_s^2 GG \rangle \langle \bar{q}q \rangle \langle g_s \bar{s}\sigma Gs \rangle}{1152\pi^2} + \frac{\langle g_s^2 GG \rangle \langle \bar{s}s \rangle \langle g_s \bar{q}\sigma Gq \rangle}{1152\pi^2}$$

$$- \frac{2m_s \langle \bar{q}q \rangle^2 \langle g_s \bar{s}\sigma Gs \rangle}{9} - \frac{5m_s \langle \bar{q}q \rangle \langle \bar{s}s \rangle \langle g_s \bar{q}\sigma Gq \rangle}{9} + \frac{m_s \langle \bar{q}q \rangle \langle \bar{s}s \rangle \langle g_s \bar{s}\sigma Gs \rangle}{9}$$

$$+ \frac{m_s \langle \bar{s}s \rangle^2 \langle g_s \bar{q}\sigma Gq \rangle}{9} + \frac{m_s^2 \langle g_s \bar{q}\sigma Gq \rangle^2}{24\pi^2} - \frac{m_s^2 \langle g_s \bar{q}\sigma Gq \rangle \langle g_s \bar{s}\sigma Gs \rangle}{24\pi^2} \Big). \tag{8.9}$$

$$\Pi_6^M(M_B^2) = \int_{4m_s^2}^{s_0} \Big[\frac{1}{6144\pi^6} s^4 - \frac{17m_s^2}{2560\pi^6} s^3 + \Big(\frac{11\langle g_s^2 GG \rangle}{18432\pi^6} - \frac{m_s \langle \bar{q}q \rangle}{32\pi^4} + \frac{m_s \langle \bar{s}s \rangle}{16\pi^4} \Big) s^2$$

$$+ \Big(- \frac{\langle \bar{q}q \rangle^2}{12\pi^2} + \frac{\langle \bar{q}q \rangle \langle \bar{s}s \rangle}{3\pi^2} - \frac{\langle \bar{s}s \rangle^2}{12\pi^2} - \frac{m_s \langle g_s \bar{q}\sigma Gq \rangle}{16\pi^4} + \frac{m_s \langle g_s \bar{s}\sigma Gs \rangle}{32\pi^4}$$

$$- \frac{109 m_s^2 \langle g_s^2 GG \rangle}{18432\pi^6} \Big) s - \frac{\langle \bar{q}q \rangle \langle g_s \bar{q}\sigma Gq \rangle}{8\pi^2} + \frac{\langle \bar{q}q \rangle \langle g_s \bar{s}\sigma Gs \rangle}{4\pi^2} + \frac{\langle \bar{s}s \rangle \langle g_s \bar{q}\sigma Gq \rangle}{4\pi^2}$$

$$- \frac{\langle \bar{s}s \rangle \langle g_s \bar{s}\sigma Gs \rangle}{8\pi^2} - \frac{3m_s \langle g_s^2 GG \rangle \langle \bar{q}q \rangle}{128\pi^4} + \frac{5m_s \langle g_s^2 GG \rangle \langle \bar{s}s \rangle}{256\pi^4} - \frac{m_s^2 \langle \bar{q}q \rangle^2}{2\pi^2}$$

$$- \frac{3m_s^2 \langle \bar{q}q \rangle \langle \bar{s}s \rangle}{2\pi^2} + \frac{m_s^2 \langle \bar{s}s \rangle^2}{8\pi^2} \Big] e^{-s/M_B^2} ds + \Big(- \frac{\langle g_s \bar{q}\sigma Gq \rangle^2}{32\pi^2} + \frac{\langle g_s \bar{q}\sigma Gq \rangle \langle g_s \bar{s}\sigma Gs \rangle}{8\pi^2}$$

$$- \frac{\langle g_s \bar{s}\sigma Gs \rangle^2}{32\pi^2} - \frac{25\langle g_s^2 GG \rangle \langle \bar{q}q \rangle^2}{1728\pi^2} + \frac{5\langle g_s^2 GG \rangle \langle \bar{q}q \rangle \langle \bar{s}s \rangle}{144\pi^2} - \frac{25\langle g_s^2 GG \rangle \langle \bar{s}s \rangle^2}{1728\pi^2}$$

$$- \frac{5m_s \langle g_s^2 GG \rangle \langle g_s \bar{q}\sigma Gq \rangle}{768\pi^4} + \frac{25m_s \langle g_s^2 GG \rangle \langle g_s \bar{s}\sigma Gs \rangle}{4608\pi^4} + 2m_s \langle \bar{q}q \rangle^2 \langle \bar{s}s \rangle$$

$$+ \frac{4m_s \langle \bar{q}q \rangle \langle \bar{s}s \rangle^2}{3} - \frac{m_s^2 \langle \bar{q}q \rangle \langle g_s \bar{s}\sigma Gs \rangle}{2\pi^2} - \frac{3m_s^2 \langle \bar{s}s \rangle \langle g_s \bar{q}\sigma Gq \rangle}{4\pi^2} \Big)$$

$$+ \frac{1}{M_B^2} \Big(- \frac{32 g_s^2 \langle \bar{q}q \rangle^2 \langle \bar{s}s \rangle^2}{27} + \frac{5\langle g_s^2 GG \rangle \langle \bar{q}q \rangle \langle g_s \bar{q}\sigma Gq \rangle}{1152\pi^2} - \frac{\langle g_s^2 GG \rangle \langle \bar{q}q \rangle \langle g_s \bar{s}\sigma Gs \rangle}{192\pi^2}$$

$$- \frac{\langle g_s^2 GG \rangle \langle \bar{s}s \rangle \langle g_s \bar{q}\sigma Gq \rangle}{192\pi^2} + \frac{5\langle g_s^2 GG \rangle \langle \bar{s}s \rangle \langle g_s \bar{s}\sigma Gs \rangle}{1152\pi^2} - \frac{2m_s \langle \bar{q}q \rangle^2 \langle g_s \bar{s}\sigma Gs \rangle}{3}$$

$$- \frac{5m_s \langle \bar{q}q \rangle \langle \bar{s}s \rangle \langle g_s \bar{q}\sigma Gq \rangle}{3} + \frac{m_s \langle \bar{q}q \rangle \langle \bar{s}s \rangle \langle g_s \bar{s}\sigma Gs \rangle}{3} + \frac{m_s \langle \bar{s}s \rangle^2 \langle g_s \bar{q}\sigma Gq \rangle}{3}$$

$$- \frac{5m_s^2 \langle g_s^2 GG \rangle \langle \bar{s}s \rangle^2}{1152\pi^2} + \frac{m_s^2 \langle g_s \bar{q}\sigma Gq \rangle^2}{8\pi^2} - \frac{m_s^2 \langle g_s \bar{q}\sigma Gq \rangle \langle g_s \bar{s}\sigma Gs \rangle}{8\pi^2} \Big). \tag{8.10}$$

$$\Pi_7^M(M_B^2) = \int_{4m_s^2}^{s_0} \left[\frac{1}{36864\pi^6}s^4 - \frac{17m_s^2}{15360\pi^6}s^3 + \left(\frac{\langle g_s^2 GG \rangle}{18432\pi^6} - \frac{m_s\langle \bar{q}q \rangle}{192\pi^4} + \frac{m_s\langle \bar{s}s \rangle}{96\pi^4} \right)s^2 \right.$$

$$+ \left(-\frac{\langle \bar{q}q \rangle^2}{72\pi^2} + \frac{\langle \bar{q}q \rangle\langle \bar{s}s \rangle}{18\pi^2} - \frac{\langle \bar{s}s \rangle^2}{72\pi^2} - \frac{m_s\langle g_s\bar{s}\sigma Gs \rangle}{96\pi^4} + \frac{m_s\langle g_s\bar{s}\sigma Gs \rangle}{192\pi^4} \right.$$

$$\left. -\frac{m_s^2\langle g_s^2 GG \rangle}{4608\pi^6} \right)s - \frac{\langle \bar{q}q \rangle\langle g_s\bar{q}\sigma Gq \rangle}{48\pi^2} + \frac{\langle \bar{q}q \rangle\langle g_s\bar{s}\sigma Gs \rangle}{24\pi^2} + \frac{\langle \bar{s}s \rangle\langle g_s\bar{q}\sigma Gq \rangle}{24\pi^2}$$

$$-\frac{\langle \bar{s}s \rangle\langle g_s\bar{s}\sigma Gs \rangle}{48\pi^2} - \frac{m_s\langle g_s^2 GG \rangle\langle \bar{q}q \rangle}{256\pi^4} - \frac{m_s^2\langle \bar{q}q \rangle^2}{12\pi^2} - \frac{m_s^2\langle \bar{q}q \rangle\langle \bar{s}s \rangle}{4\pi^2}$$

$$\left. +\frac{m_s^2\langle \bar{s}s \rangle^2}{48\pi^2} \right] e^{-s/M_B^2}ds + \left(-\frac{\langle g_s\bar{q}\sigma Gq \rangle^2}{192\pi^2} + \frac{\langle g_s\bar{q}\sigma Gq \rangle\langle g_s\bar{s}\sigma Gs \rangle}{48\pi^2} \right.$$

$$-\frac{\langle g_s\bar{s}\sigma Gs \rangle^2}{192\pi^2} + \frac{5\langle g_s^2 GG \rangle\langle \bar{q}q \rangle\langle \bar{s}s \rangle}{864\pi^2} + \frac{m_s\langle \bar{q}q \rangle^2\langle \bar{s}s \rangle}{3} + \frac{2m_s\langle \bar{q}q \rangle\langle \bar{s}s \rangle^2}{9}$$

$$\left. -\frac{5m_s\langle g_s^2 GG \rangle\langle g_s\bar{q}\sigma Gq \rangle}{4608\pi^4} - \frac{m_s^2\langle \bar{s}s \rangle\langle g_s\bar{q}\sigma Gq \rangle}{8\pi^2} - \frac{m_s^2\langle \bar{q}q \rangle\langle g_s\bar{s}\sigma Gs \rangle}{12\pi^2} \right)$$

$$+\frac{1}{M_B^2} \left(-\frac{16g_s^2\langle \bar{q}q \rangle^2\langle \bar{s}s \rangle^2}{81} - \frac{\langle g_s^2 GG \rangle\langle \bar{q}q \rangle\langle g_s\bar{s}\sigma Gs \rangle}{1152\pi^2} - \frac{\langle g_s^2 GG \rangle\langle \bar{s}s \rangle\langle g_s\bar{q}\sigma Gq \rangle}{1152\pi^2} \right.$$

$$-\frac{m_s\langle \bar{q}q \rangle^2\langle g_s\bar{s}\sigma Gs \rangle}{9} - \frac{5m_s\langle \bar{q}q \rangle\langle \bar{s}s \rangle\langle g_s\bar{q}\sigma Gq \rangle}{18} + \frac{m_s\langle \bar{q}q \rangle\langle \bar{s}s \rangle\langle g_s\bar{s}\sigma Gs \rangle}{18}$$

$$\left. +\frac{m_s\langle \bar{s}s \rangle^2\langle g_s\bar{q}\sigma Gq \rangle}{18} + \frac{m_s^2\langle g_s\bar{q}\sigma Gq \rangle^2}{48\pi^2} - \frac{m_s^2\langle g_s\bar{q}\sigma Gq \rangle\langle g_s\bar{s}\sigma Gs \rangle}{48\pi^2} \right). \quad (8.11)$$

$$\Pi_8^M(M_B^2) = \int_{4m_s^2}^{s_0} \left[\frac{1}{12288\pi^6}s^4 - \frac{17m_s^2}{5120\pi^6}s^3 + \left(\frac{\langle g_s^2 GG \rangle}{18432\pi^6} - \frac{m_s\langle \bar{q}q \rangle}{64\pi^4} + \frac{m_s\langle \bar{s}s \rangle}{32\pi^4} \right)s^2 \right.$$

$$+ \left(-\frac{\langle \bar{q}q \rangle^2}{24\pi^2} + \frac{\langle \bar{q}q \rangle\langle \bar{s}s \rangle}{6\pi^2} - \frac{\langle \bar{s}s \rangle^2}{24\pi^2} - \frac{m_s\langle g_s\bar{q}\sigma Gq \rangle}{32\pi^4} + \frac{m_s\langle g_s\bar{s}\sigma Gs \rangle}{64\pi^4} \right.$$

$$\left. -\frac{17m_s^2\langle g_s^2 GG \rangle}{18432\pi^6} \right)s - \frac{\langle \bar{q}q \rangle\langle g_s\bar{q}\sigma Gq \rangle}{16\pi^2} + \frac{\langle \bar{q}q \rangle\langle g_s\bar{s}\sigma Gs \rangle}{8\pi^2} + \frac{\langle \bar{s}s \rangle\langle g_s\bar{q}\sigma Gq \rangle}{8\pi^2}$$

$$-\frac{\langle \bar{s}s \rangle\langle g_s\bar{s}\sigma Gs \rangle}{16\pi^2} + \frac{m_s\langle g_s^2 GG \rangle\langle \bar{s}s \rangle}{256\pi^4} - \frac{m_s^2\langle \bar{q}q \rangle^2}{4\pi^2} - \frac{3m_s^2\langle \bar{q}q \rangle\langle \bar{s}s \rangle}{4\pi^2}$$

$$\left. +\frac{m_s^2\langle \bar{s}s \rangle^2}{16\pi^2} \right] e^{-s/M_B^2}ds + \left(-\frac{\langle g_s\bar{q}\sigma Gq \rangle^2}{64\pi^2} + \frac{\langle g_s\bar{q}\sigma Gq \rangle\langle g_s\bar{s}\sigma Gs \rangle}{16\pi^2} \right.$$

$$-\frac{\langle g_s\bar{s}\sigma Gs \rangle^2}{64\pi^2} - \frac{5\langle g_s^2 GG \rangle\langle \bar{q}q \rangle^2}{1728\pi^2} - \frac{5\langle g_s^2 GG \rangle\langle \bar{s}s \rangle^2}{1728\pi^2} + \frac{5m_s\langle g_s^2 GG \rangle\langle g_s\bar{s}\sigma Gs \rangle}{4608\pi^4}$$

$$\left. +m_s\langle \bar{q}q \rangle^2\langle \bar{s}s \rangle + \frac{2m_s\langle \bar{q}q \rangle\langle \bar{s}s^2 \rangle}{3} - \frac{3m_s^2\langle \bar{s}s \rangle\langle g_s\bar{q}\sigma Gq \rangle}{8\pi^2} - \frac{m_s^2\langle \bar{q}q \rangle\langle g_s\bar{s}\sigma Gs \rangle}{4\pi^2} \right)$$

$$+\frac{1}{M_B^2} \left(-\frac{16g_s^2\langle \bar{q}q \rangle^2\langle \bar{s}s \rangle^2}{27} + \frac{\langle g_s^2 GG \rangle\langle \bar{q}q \rangle\langle g_s\bar{q}\sigma Gq \rangle}{1152\pi^2} + \frac{\langle g_s^2 GG \rangle\langle \bar{s}s \rangle\langle g_s\bar{s}\sigma Gs \rangle}{1152\pi^2} \right.$$

$$-\frac{m_s\langle \bar{q}q \rangle^2\langle g_s\bar{s}\sigma Gs \rangle}{3} - \frac{5m_s\langle \bar{q}q \rangle\langle \bar{s}s \rangle\langle g_s\bar{q}\sigma Gq \rangle}{6} + \frac{m_s\langle \bar{q}q \rangle\langle \bar{s}s \rangle\langle g_s\bar{s}\sigma Gs \rangle}{6}$$

$$+\frac{m_s\langle\bar{s}s\rangle^2\langle g_s\bar{q}\sigma Gq\rangle}{6} - \frac{m_s^2\langle g_s^2GG\rangle\langle\bar{s}s\rangle^2}{1152\pi^2} + \frac{m_s^2\langle g_s\bar{q}\sigma Gq\rangle^2}{16\pi^2}$$
$$-\frac{m_s^2\langle g_s\bar{q}\sigma Gq\rangle\langle g_s\bar{s}\sigma Gs\rangle}{16\pi^2}\Bigg)\,. \tag{8.12}$$

8.3 Numerical Analysis

For the currents which belong to the flavor representations $\mathbf{6_f}\otimes\mathbf{\bar{6}_f}$ (**S**) and $\mathbf{\bar{3}_f}\otimes\mathbf{3_f}$ (**A**), the spectral densities turn out to be negative in the energy region 1 GeV $\sim$ 2 GeV as shown in Fig. 8.2. The spectral densities of these currents become positive in the region $s > 4$ GeV^2. They may couple to the state $\pi_1(2015)$. However, after performing the sum rule calculation, we find that the mass obtained from the currents $\eta_{i\mu}^A$ and $\eta_{i\mu}^S$ is larger than 2.5 GeV, for instance, we show the mass calculated from the current $\eta_{1\mu}^A$ in Fig. 8.4. The curves are obtained by setting $M_B^2 = 2$ GeV^2 (solid line), 3 GeV^2 (short-dashed line) and 4 GeV^2 (long-dashed line). The left curves (disconnected from the right part) are obtained from a negative Borel transformed correlation function, and have no physical meaning. Therefore, our QCD sum rule analysis does not support $\pi_1(1400)$, $\pi_1(1600)$ and $\pi_1(2015)$ as tetraquark states with a flavor structure either $\mathbf{6_f}\otimes\mathbf{\bar{6}_f}$ or $\mathbf{\bar{3}_f}\otimes\mathbf{3_f}$.
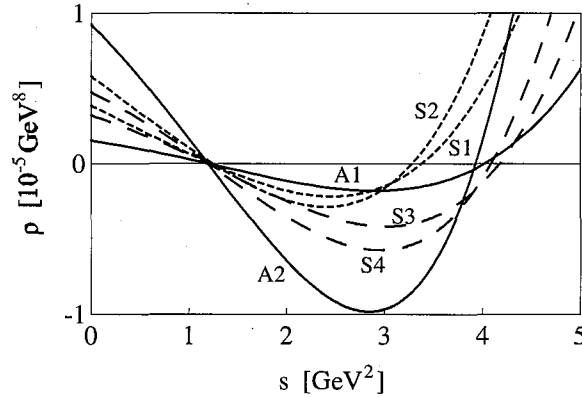
Figure 8.2: Spectral densities for the current $\eta_{1\mu}^A$, $\eta_{2\mu}^A$ (solid lines), $\eta_{1\mu}^S$, $\eta_{2\mu}^S$ (short-dashed lines), $\eta_{3\mu}^S$ and $\eta_{4\mu}^S$ (long-dashed lines). The labels besides the lines indicate the flavor symmetry (S or A) and suffix i of the current $\eta_{i\mu}^{S,A}$ ($i = 1, 2, 3, 4$).

When using the currents $\eta_{i\mu}^M$, the spectral densities are positive as shown in Fig. 8.3. And so we shall use these currents to perform a QCD sum rule analysis. First we need to study the convergence of the OPE. The Borel transformed correlation function of the current $\eta_{5\mu}^M$ is shown in Fig. 8.5, when we take $s_0 = 4$ GeV^2. Besides the first term, which is the continuum piece, the D=6 and D=8 terms give large contributions.
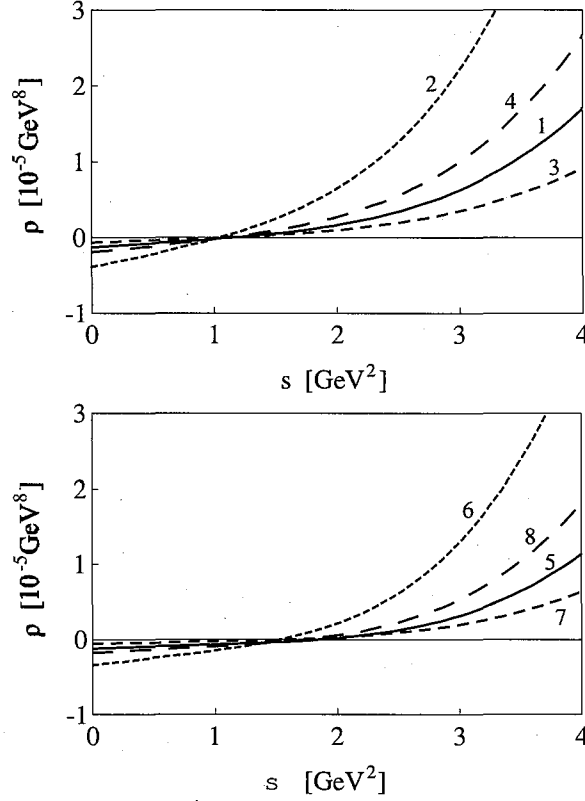
Figure 8.3: Spectral densities for the current $\eta_{i\mu}^M$. The spectral densities for the currents with the quark contents $qq\bar{q}\bar{q}$ are shown in the left hand side, and those with the quark contents $qs\bar{q}\bar{s}$ are shown in the right hand side. The labels besides the lines indicate the suffix i of the current $\eta_{i\mu}^M$ $(i = 1, \cdots, 8)$.

The D=6 terms contain $\langle \bar{q}q \rangle^2$ and the D=8 terms contain $\langle \bar{q}q \rangle \langle g_c \bar{q} \sigma G q \rangle$, which are the important condensates. We find that the convergence is very good in the region of $2 \text{ GeV}^2 < M_B^2 < 5\text{GeV}^2$. Therefore, in this region, OPEs are reliable.

The mass is calculated by using Eq. (1.26), and results are obtained as functions of Borel mass M_B and threshold value s_0. In Figs. 8.6, 8.7, 8.8 and 8.9, we show the mass calculated from currents $\eta_{1\mu}^M$, $\eta_{2\mu}^M$, $\eta_{3\mu}^M$ and $\eta_{4\mu}^M$, whose quark contents are $qq\bar{q}\bar{q}$. Although these four independent currents look much different, we find that they give a similar result. From figures at LHS, we find that the dependence on Borel mass is weak. From figures at RHS where the mass is shown as functions of s_0, we find that there is a mass minimum for all curves where the stability is the best. It is 1.7 GeV, 1.6 GeV, 1.6 GeV and 1.7 GeV for four independent currents, respectively. We find that sometimes the threshold values become smaller than the mass obtained in the mass minimum region.
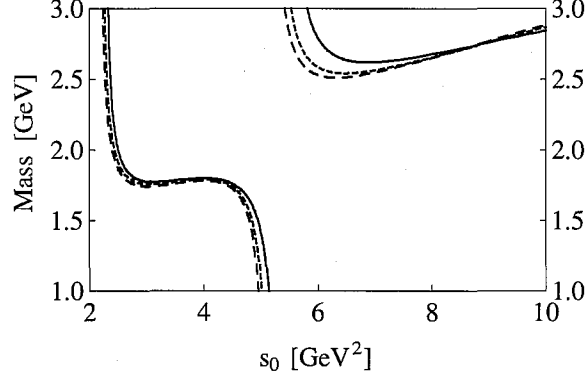
Figure 8.4: The mass calculated by using the current $\eta^A_{1\mu}$, as functions of s_0 in units of GeV. The curves are obtained by setting $M^2_B = 2\ \mathrm{GeV^2}$ (solid line), $3\ \mathrm{GeV^2}$ (short-dashed line) and $4\ \mathrm{GeV^2}$ (long-dashed line). The left curves (disconnected from the right part) are obtained from a negative correlation function, and have no physical meaning.
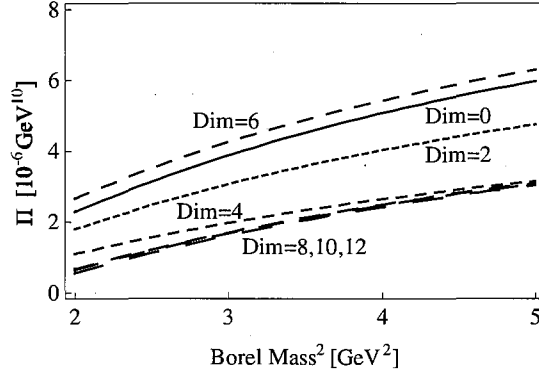
Figure 8.5: Various contribution to the correlation function for the current $\eta^M_{5\mu}$ as functions of the Borel mass M_B in units of $\mathrm{GeV^{1}0}$ at $s_0 = 4\ \mathrm{GeV^2}$. The labels indicate the dimension up to which the OPE terms are included.

This is due to the negative part of the spectral densities. We also met this in the study of $Y(2175)$. See Ref [42] for details.

In Figs. 8.10, 8.11, 8.12 and 8.13, we show the mass calculated from currents $\eta^M_{5\mu}$, $\eta^M_{6\mu}$, $\eta^M_{7\mu}$ and $\eta^M_{8\mu}$, whose quark contents are $qs\bar{q}\bar{s}$. The results are similar as previous four currents. But now the mass obtained is about 0.4 GeV larger than the previous ones. The minimum occurs at 2.1 GeV, 2.0 GeV, 1.9 GeV and 2.0 GeV, respectively.

In a short summary, we have performed a QCD sum rule analysis for $qq\bar{q}\bar{q}$ and $qs\bar{q}\bar{s}$.

Figure 8.6: The mass of the state $qq\bar{q}\bar{q}$ calculated by using the current $\eta^M_{1\mu}$, as functions of M^2_B (Left) and s_0 (Right) in units of GeV.
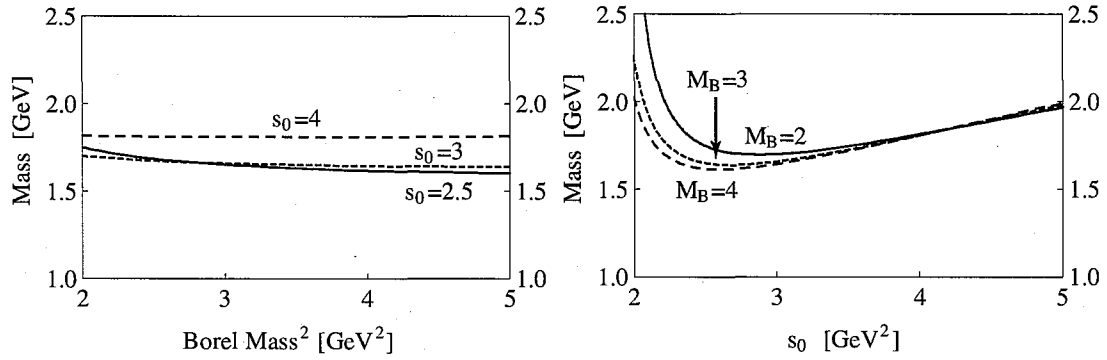
Figure 8.7: The mass of the state $qq\bar{q}\bar{q}$ calculated by using the current $\eta^M_{2\mu}$, as functions of M^2_B (Left) and s_0 (Right) in units of GeV.

The mass obtained is around 1.6 GeV and 2.0 GeV, respectively. There are four independent currents for each case, which give a similar results. Their mixing would lead to a similar result, too. Compared with the experimental data, they can be used to interpret the states $\pi_1(1600)$ and $\pi_1(2015)$ of $I^G J^{PC} = 1^- 1^{-+}$. These analyses are very similar to our previous paper [42], where we studied the state $Y(2175)$ by using vector tetraquark currents which have quantum numbers $J^{PC} = 1^{--}$ and quark contents $ss\bar{s}\bar{s}$.

The pole contribution

$$\frac{\int_{s_<}^{s_0} e^{-s/M^2_B} \rho(s) ds}{\int_{s_<}^{\infty} e^{-s/M^2_B} \rho(s) ds} \tag{8.13}$$

is not large enough for all currents due to the high dimension nature of tetraquark currents. Another reason is that these currents have a large coupling to the continuum,

Figure 8.8: The mass of the state $qq\bar{q}\bar{q}$ calculated by using the current $\eta_{3\mu}^M$, as functions of M_B^2 (Left) and s_0 (Right) in units of GeV.
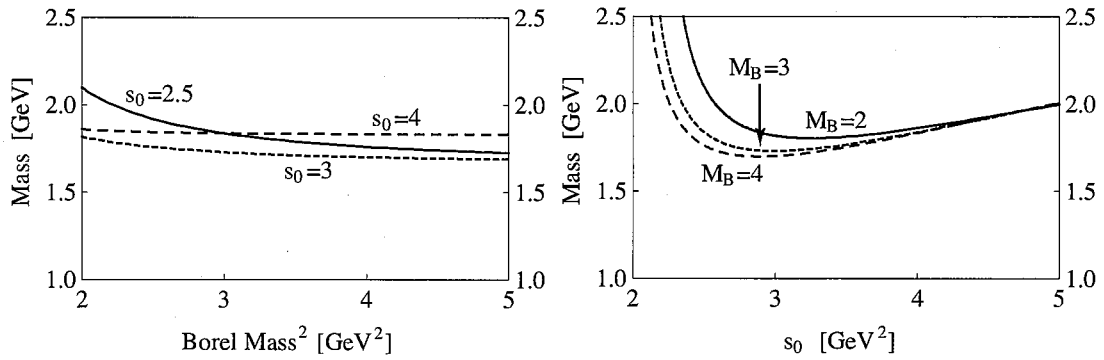
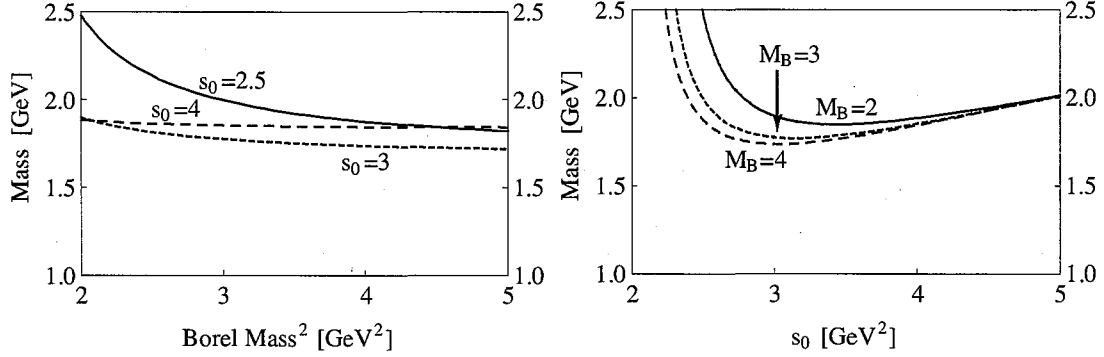Figure 8.9: The mass of the state $qq\bar{q}\bar{q}$ calculated by using the current $\eta_{4\mu}^M$, as functions of M_B^2 (Left) and s_0 (Right) in units of GeV.

which is difficult to be removed. Therefore, we arrive at a stable mass, but with a small pole. To make our analysis more reliable, we go on to use the finite energy sum rule in the following section.

8.4 Finite Energy Sum Rule

The spectral functions $\rho_i^M(s)$ can be drawn from the Borel transformed correlation functions shown in section 8.2. The $Dim = 12$ terms which are proportional to $1/(q^2)^2$ do not contribute to the function $W(n, s_0)$ of Eq. (1.27) for $n = 0$, or they have a very small contribution for $n = 1$, when the theoretical side is computed by the integral over the circle of radius s_0 on the complex q^2 plain.
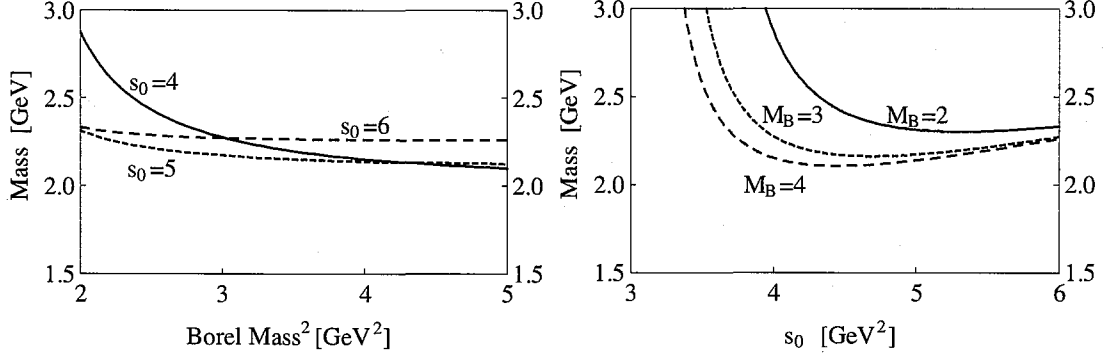
Figure 8.10: The mass of the state $qs\bar{q}\bar{s}$ calculated by using the current $\eta^M_{5\mu}$, as functions of M^2_B (Left) and s_0 (Right) in units of GeV.
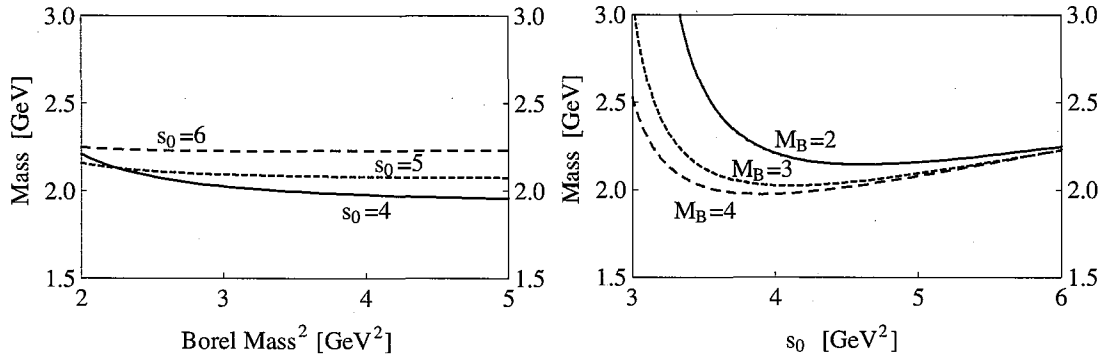
Figure 8.11: The mass of the state $qs\bar{q}\bar{s}$ calculated by using the current $\eta^M_{6\mu}$, as functions of M^2_B (Left) and s_0 (Right) in units of GeV.

The mass is shown as a function of the threshold value s_0 in Fig. 8.14, where n is chosen to be 1. We find that there is a mass minimum around which the result is stable under the change in s_0. It is around 1.6 GeV for currents η^M_1, η^M_2, η^M_3 and η^M_4, whose quark contents are $qq\bar{q}\bar{q}$, while it is around 2.0 GeV for currents η^M_5, η^M_6, η^M_7 and η^M_8, whose quark contents are $qs\bar{q}\bar{s}$. In a short summary, we arrive at the same results as the previous SVZ QCD sum rule.

8.5   Decay Patterns of the 1^{-+} Tetraquark States

In the Section 8.1.2, we have verified that $(qq)(\bar{q}\bar{q})$ construction and $(\bar{q}q)(\bar{q}q)$ construction are equivalent, and from the second one we can obtain some decay information. The four
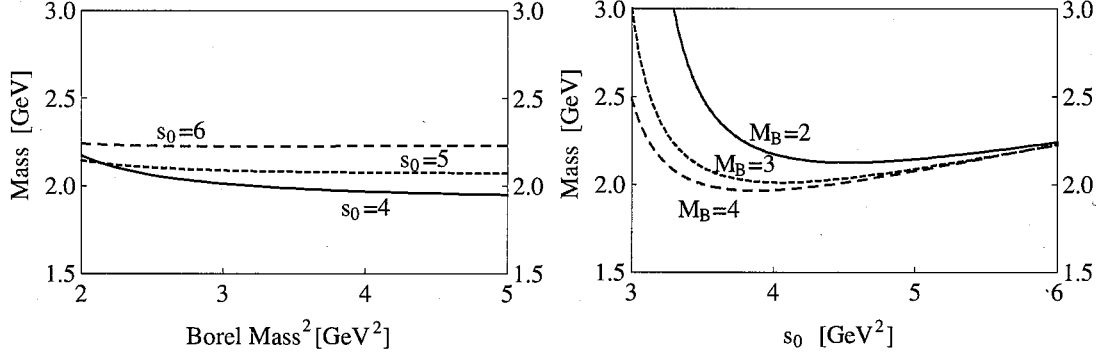
Figure 8.12: The mass of the state $qs\bar{q}\bar{s}$ calculated by using the current $\eta_{7\mu}^{M}$, as functions of M_{B}^{2} (Left) and s_{0} (Right) in units of GeV.
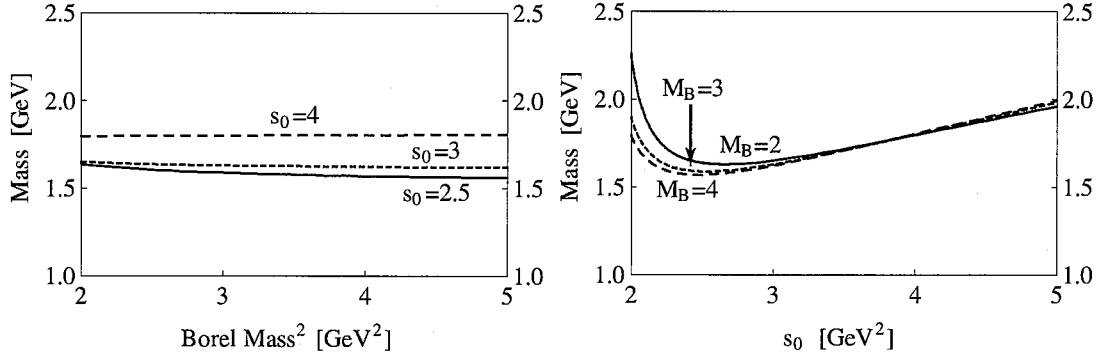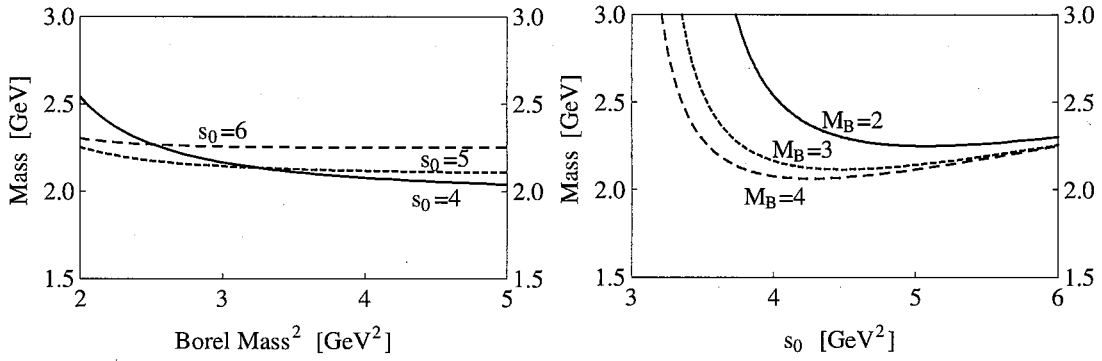

Figure 8.13: The mass of the state $qs\bar{q}\bar{s}$ calculated by using the current $\eta_{8\mu}^{M}$, as functions of M_{B}^{2} (Left) and s_{0} (Right) in units of GeV.

independent $(\bar{q}q)(\bar{q}q)$ currents $\xi_{i\mu}^{M}$ lead to the same mass, and therefore, we shall study the decay patterns from all these currents. We can obtain the S-wave decay patterns straightforwardly:

1. The current $\xi_{1\mu}^{M}$ naively falls apart to one scalar meson and one vector meson:

$$\xi_{1\mu}^{M} \quad : \quad \pi_{1}(1600) \rightarrow 0^{+}\left(\sigma(600), f_{0}(980)\cdots\right) + 1^{-}\left(\rho(770), \omega(782)\cdots\right), \quad (8.14)$$
$$\pi_{1}(2000) \rightarrow 0^{+}\left(\sigma(600), \kappa(800)\cdots\right) + 1^{-}\left(\rho(770), K^{*}(892)\cdots\right).$$

2. The current $\xi_{2\mu}^{M}$ naively falls apart to one axial-vector meson and one pseudoscalar meson:

$$\xi_{2\mu}^{M} \quad : \quad \pi_{1}(1600) \rightarrow 1^{+}\left(a_{1}(1260), b_{1}(1235)\cdots\right) + 0^{-}\left(\pi(135)\cdots\right), \quad (8.15)$$
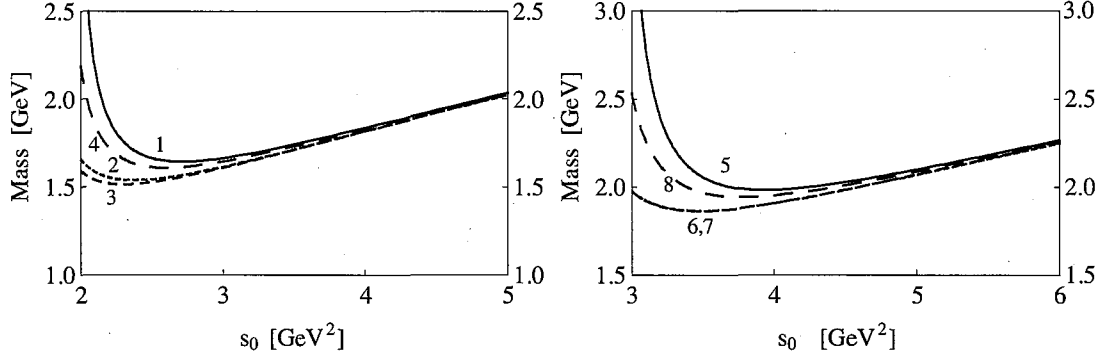
Figure 8.14: The mass calculated using the finite energy sum rule. The mass for the currents $\eta^M_{1\mu}$, $\eta^M_{2\mu}$, $\eta^M_{3\mu}$ and $\eta^M_{4\mu}$ is shown in the left hand side, and The mass for the currents $\eta^M_{5\mu}$, $\eta^M_{6\mu}$, $\eta^M_{7\mu}$ and $\eta^M_{8\mu}$ are shown in the right hand side. The labels besides the lines indicate the suffix i of the current $\eta^M_{i\mu}$ ($i = 1, \cdots, 8$).

$$\pi_1(2000) \rightarrow 1^+ \left(a_1(1260), K_1(1270), \cdots\right) + 0^- \left(\pi(135), K(498) \cdots\right).$$

3. The current $\xi^M_{3\mu}$ naively falls apart to one vector meson and one axial-vector meson:

$$\xi^M_{3\mu} : \quad \pi_1(1600) \rightarrow 1^- \left(\rho(770), \omega(782) \cdots\right) + 1^+ \left(a_1(1260), b_1(1235) \cdots\right), \quad (8.16)$$
$$\pi_1(2000) \rightarrow 1^- \left(\rho(770), K^*(892) \cdots\right) + 1^+ \left(a_1(1260), K_1(1270) \cdots\right).$$

4. The current $\xi^M_{4\mu}$ naively falls apart to one axial-vector meson and one vector meson:

$$\xi^M_{4\mu} : \quad \pi_1(1600) \rightarrow 1^+ \left(a_1(1260), b_1(1235) \cdots\right) + 1^- \left(\rho(770), \omega(782) \cdots\right), \quad (8.17)$$
$$\pi_1(2000) \rightarrow 1^+ \left(a_1(1260), K_1(1270) \cdots\right) + 1^- \left(\rho(770), K^*(892) \cdots\right).$$

$\pi_1(2000)$ contains one $\bar{s}s$ pair, so its final states should also contain one $\bar{s}s$ pair, and its decay patterns are more complicated than $\pi_1(1600)$. We see that the decay modes (8.16) and (8.17) are kinematically forbidden (or strongly suppressed) due to energy conservation. The decay modes (8.14) are difficult to be observed in the experiments due to the large decay width of scalar mesons (σ and κ). Moreover, the scalar mesons below 1 GeV are sometimes interpreted as tetraquark states, and if so, these decay modes should be suppressed due to the extra $\bar{q}q$ pair [41]. Therefore, the decay modes (8.15) are preferred. The π_1 meson first decays to one axial-vector meson and one pseudoscalar meson. Then the axial-vector meson decays into two or more pseudoscalar mesons. However, the second step is a P-wave decay. Considering the conservation of G parity, the decay mode $a_1(1260)\pi$ is forbidden. One possible decay pattern is that $\pi_1(1600)$ first decays to $b_1(1235)\pi$, and then decays to $\omega\pi\pi$.

We can also check the P-wave decay patterns besides S-wave decay patterns. We find that the current $\xi^M_{2\mu}$ leads to a decay mode of two P-wave pseudoscalar mesons by naively relating $\bar{q}\gamma_\mu\gamma_5 q$ and $\partial_\mu\pi$

$$\pi_1(1600) \rightarrow 0^- (\pi, \eta, \eta' \cdots) + 0^- (\pi, \eta, \eta' \cdots) , \tag{8.18}$$
$$\pi_1(2000) \rightarrow 0^- (\pi, \eta, \eta' \cdots) + 0^- (\pi, \eta, \eta' \cdots) .$$

Considering the conservation of G parity, decay modes $\pi\pi$ and $\eta\eta$ etc. are forbidden, and possible decay modes are $\pi\eta$ and $\pi\eta'$ etc. Summarizing the decay patterns, there are two possible decay modes: P-wave many body decay, such as $\omega\pi\pi$, and P-wave two body decay, such as $\pi\eta$ and $\pi\eta'$. This is partly consistent with the experiments which observe $\pi_1(1600)$ and $\pi_1(2015)$ in the decay modes $\pi\eta'$, $\omega\pi\pi$ and $\eta\pi\pi\pi$. However, the experiment has not observe them in the final state $\pi\eta$. Certainly it is desired to study these decay patterns to obtain more information on the structure of the π_1s mesons.

8.6 The $I^G J^{PC} = 0^+ 1^{-+}$ Tetraquark State

The tetraquark currents with the quantum numbers $J^{PC} = 1^{-+}$ have been constructed in the previous section. Now we need construct the isoscalar ones. The flavor structures are shown in Fig. 8.1 in terms of $SU(3)$ weight diagrams. The ideal mixing scheme is used since it is expected to work well for hadrons except for the pseudoscalar mesons. In order to have a definite charge-conjugation parity, the diquark and antidiquark inside can have the same flavor symmetry, which is either symmetric $\mathbf{6_f} \otimes \mathbf{\bar{6}_f}$ (**S**) or antisymmetric $\mathbf{\bar{3}_f} \otimes \mathbf{3_f}$ (**A**). Another option is the combination of $\mathbf{\bar{3}_f} \otimes \mathbf{\bar{6}_f}$ and $\mathbf{6_f} \otimes \mathbf{3_f}$ (**M**), which can also have a definite charge-conjugation parity.

From Fig. 8.1, we find that there are three isospin singlets belonging to the flavor representation $\mathbf{6}_f \otimes \mathbf{\bar{6}}_f$, two isospin singlets belonging to the flavor representation $\mathbf{\bar{3}}_f \otimes \mathbf{3}_f$, and one isospin singlet belonging to the flavor representation $(\mathbf{\bar{3}}_f \otimes \mathbf{\bar{6}}_f) \oplus (\mathbf{6}_f \otimes \mathbf{3}_f)$:

$$qq\bar{q}\bar{q}(\mathbf{S}), qs\bar{q}\bar{s}(\mathbf{S}), ss\bar{s}\bar{s}(\mathbf{S}) \sim \mathbf{6_f} \otimes \mathbf{\bar{6}_f} \quad (\mathbf{S}),$$
$$qq\bar{q}\bar{q}(\mathbf{A}), qs\bar{q}\bar{s}(\mathbf{A}) \sim \mathbf{\bar{3}_f} \otimes \mathbf{3_f} \quad (\mathbf{A}), \tag{8.19}$$
$$qs\bar{q}\bar{s}(\mathbf{M}) \sim (\mathbf{\bar{3}_f} \otimes \mathbf{\bar{6}_f}) \oplus (\mathbf{6_f} \otimes \mathbf{3_f}) \; (\mathbf{M}),$$

where q represents an *up* or *down* quark, and s represents a *strange* quark. For each state, there are several independent currents. We list them in the following.

1. For the three isospin singlets of $\mathbf{6}_f \otimes \mathbf{\bar{6}}_f$ (**S**):

$$\begin{aligned}
\zeta^S_{1\mu} &\sim u_a^T C\gamma_5 d_b(\bar{u}_a\gamma_\mu\gamma_5 C\bar{d}_b^T + \bar{u}_b\gamma_\mu\gamma_5 C\bar{d}_a^T) \\
&\quad + u_a^T C\gamma_\mu\gamma_5 d_b(\bar{u}_a\gamma_5 C\bar{d}_b^T + \bar{u}_b\gamma_5 C\bar{d}_a^T) , \\
\zeta^S_{2\mu} &\sim u_a^T C\gamma^\nu d_b(\bar{u}_a\sigma_{\mu\nu} C\bar{d}_b^T - \bar{u}_b\sigma_{\mu\nu} C\bar{d}_a^T) \\
&\quad + u_a^T C\sigma_{\mu\nu} d_b(\bar{u}_a\gamma^\nu C\bar{d}_b^T - \bar{u}_b\gamma^\nu C\bar{d}_a^T) ,
\end{aligned} \tag{8.20}$$

$$
\begin{aligned}
\zeta_{3\mu}^S &\sim u_a^T C\gamma_5 s_b(\bar{u}_a\gamma_\mu\gamma_5 C\bar{s}_b^T + \bar{u}_b\gamma_\mu\gamma_5 C\bar{s}_a^T) \\
&+u_a^T C\gamma_\mu\gamma_5 s_b(\bar{u}_a\gamma_5 C\bar{s}_b^T + \bar{u}_b\gamma_5 C\bar{s}_a^T)\,, \\
\zeta_{4\mu}^S &\sim u_a^T C\gamma^\nu s_b(\bar{u}_a\sigma_{\mu\nu} C\bar{s}_b^T - \bar{u}_b\sigma_{\mu\nu} C\bar{s}_a^T) \\
&+u_a^T C\sigma_{\mu\nu} s_b(\bar{u}_a\gamma^\nu C\bar{s}_b^T - \bar{u}_b\gamma^\nu C\bar{s}_a^T)\,.
\end{aligned}
\tag{8.21}
$$

$$
\begin{aligned}
\zeta_{5\mu}^S &\sim s_a^T C\gamma_5 s_b(\bar{s}_a\gamma_\mu\gamma_5 C\bar{s}_b^T + \bar{s}_b\gamma_\mu\gamma_5 C\bar{s}_a^T) \\
&+s_a^T C\gamma_\mu\gamma_5 s_b(\bar{s}_a\gamma_5 C\bar{s}_b^T + \bar{s}_b\gamma_5 C\bar{s}_a^T)\,, \\
\zeta_{6\mu}^S &\sim s_a^T C\gamma^\nu s_b(\bar{s}_a\sigma_{\mu\nu} C\bar{s}_b^T - \bar{s}_b\sigma_{\mu\nu} C\bar{s}_a^T) \\
&+s_a^T C\sigma_{\mu\nu} s_b(\bar{s}_a\gamma^\nu C\bar{s}_b^T - \bar{s}_b\gamma^\nu C\bar{s}_a^T)\,.
\end{aligned}
\tag{8.22}
$$

where $\zeta_{1\mu}^S$ and $\zeta_{2\mu}^S$ are the two independent currents containing only light flavors; $\zeta_{3\mu}^S$ and $\zeta_{4\mu}^S$ are the two independent ones containing one $s\bar{s}$ pair; $\zeta_{5\mu}^S$ and $\zeta_{6\mu}^S$ are the two independent ones containing two $s\bar{s}$ pairs.

2. For the two isospin singlets of $\bar{3}_f \otimes 3_f$ (**A**):

$$
\begin{aligned}
\zeta_{1\mu}^A &\sim u_a^T C\gamma_5 d_b(\bar{u}_a\gamma_\mu\gamma_5 C\bar{d}_b^T - \bar{u}_b\gamma_\mu\gamma_5 C\bar{d}_a^T) \\
&+u_a^T C\gamma_\mu\gamma_5 d_b(\bar{u}_a\gamma_5 C\bar{d}_b^T - \bar{u}_b\gamma_5 C\bar{d}_a^T)\,, \\
\zeta_{2\mu}^A &\sim u_a^T C\gamma^\nu d_b(\bar{u}_a\sigma_{\mu\nu} C\bar{d}_b^T + \bar{u}_b\sigma_{\mu\nu} C\bar{d}_a^T) \\
&+u_a^T C\sigma_{\mu\nu} d_b(\bar{u}_a\gamma^\nu C\bar{d}_b^T + \bar{u}_b\gamma^\nu C\bar{d}_a^T)\,,
\end{aligned}
\tag{8.23}
$$

$$
\begin{aligned}
\zeta_{3\mu}^A &\sim u_a^T C\gamma_5 s_b(\bar{u}_a\gamma_\mu\gamma_5 C\bar{s}_b^T - \bar{u}_b\gamma_\mu\gamma_5 C\bar{s}_a^T) \\
&+u_a^T C\gamma_\mu\gamma_5 s_b(\bar{u}_a\gamma_5 C\bar{s}_b^T - \bar{u}_b\gamma_5 C\bar{s}_a^T)\,, \\
\zeta_{4\mu}^A &\sim u_a^T C\gamma^\nu s_b(\bar{u}_a\sigma_{\mu\nu} C\bar{s}_b^T + \bar{u}_b\sigma_{\mu\nu} C\bar{s}_a^T) \\
&+u_a^T C\sigma_{\mu\nu} s_b(\bar{u}_a\gamma^\nu C\bar{s}_b^T + \bar{u}_b\gamma^\nu C\bar{s}_a^T)\,,
\end{aligned}
\tag{8.24}
$$

where $\zeta_{1\mu}^A$ and $\zeta_{2\mu}^A$ are the two independent currents containing only light flavors; $\zeta_{3\mu}^A$ and $\zeta_{4\mu}^A$ are the two independent ones containing one $s\bar{s}$ pair.

3. For the isospin singlet of $(\bar{3}_f \otimes \bar{6}_f) \oplus (6_f \otimes 3_f)$ (**M**):

$$
\begin{aligned}
\zeta_{1\mu}^M &\sim u_a^T C\gamma_\mu s_b(\bar{u}_a C\bar{s}_b^T + \bar{u}_b C\bar{s}_a^T) \\
&+u_a^T C s_b(\bar{u}_a\gamma_\mu C\bar{s}_b^T + \bar{u}_b\gamma_\mu C\bar{s}_a^T)\,, \\
\zeta_{2\mu}^M &\sim u_a^T C\sigma_{\mu\nu}\gamma_5 s_b(\bar{u}_a\gamma^\nu\gamma_5 C\bar{s}_b^T + \bar{u}_b\gamma^\nu\gamma_5 C\bar{s}_a^T) \\
&+u_a^T C\gamma^\nu\gamma_5 s_b(\bar{u}_a\sigma_{\mu\nu}\gamma_5 C\bar{s}_b^T + \bar{u}_b\sigma_{\mu\nu}\gamma_5 C\bar{s}_a^T)\,, \\
\zeta_{3\mu}^M &\sim u_a^T C s_b(\bar{u}_a\gamma_\mu C\bar{s}_b^T - \bar{u}_b\gamma_\mu C\bar{s}_a^T) \\
&+u_a^T C\gamma_\mu s_b(\bar{u}_a C\bar{s}_b^T - \bar{u}_b C\bar{s}_a^T)\,, \\
\zeta_{4\mu}^M &\sim u_a^T C\gamma^\nu\gamma_5 s_b(\bar{u}_a\sigma_{\mu\nu}\gamma_5 C\bar{s}_b^T - \bar{u}_b\sigma_{\mu\nu}\gamma_5 C\bar{s}_a^T) \\
&+u_a^T C\sigma_{\mu\nu}\gamma_5 s_b(\bar{u}_a\gamma^\nu\gamma_5 C\bar{s}_b^T - \bar{u}_b\gamma^\nu\gamma_5 C\bar{s}_a^T)\,,
\end{aligned}
\tag{8.25}
$$

where $\zeta_{i\mu}^M$ are the four independent ones containing one $s\bar{s}$ pair. The above structure has some implications on their decay patterns.

The expressions of Eqs. (8.20)-(8.25) are not exactly correct, since they do not have a definite isospin. For instance, the current $\zeta_{3\mu}^A$ should contain $(us\bar{u}\bar{s} + ds d\bar{s})$ in order to

have $I = 0$. However, in the following QCD sum rule analysis, we find that there is no difference between these two cases in the limit that the masses and condensates of the *up* and *down* quarks are the same. Actually we also ignore a small quark mass effect $(m_u \sim m_d \lesssim 10 \text{ MeV})$.

By using these tetraquark currents, we have performed the OPE calculation up to dimension 12. Values for various condensates and m_s follow the references [71, 85, 89, 99, 140, 148, 177, 179]. There are altogether 14 currents. It turns out that some of them lead to the same results of OPEs as the previous ones in previous sections [43]:

$$\zeta^S_{1,2,3,4\mu} \sim \eta^S_{1,2,3,4\mu},$$
$$\zeta^A_{3,4\mu} \sim \eta^A_{1,2\mu},$$
$$\zeta^M_{1,2,3,4\mu} \sim \eta^M_{5,6,7,8\mu},$$

Therefore, we just need calculate the OPEs of $\zeta^S_{5,6\mu}$ and $\zeta^A_{1,2\mu}$. The full OPE expressions are too lengthy and are omitted here.

In our previous paper [43] we have found that the OPEs of the currents $\zeta^S_{i\mu}$'s and $\zeta^A_{i\mu}$'s lead to unphysical results where the spectral densities $\rho(s)$ become negative in the region of $2 \text{ GeV}^2 \lesssim s \lesssim 4 \text{ GeV}^2$. We find this to be the case also for the isoscalar currents. Therefore, our QCD sum rule analysis does not support a tetraquark state which has a flavor structure either $\mathbf{6_f} \otimes \mathbf{\bar{6}_f}$ or $\mathbf{\bar{3}_f} \otimes \mathbf{3_f}$ and a mass less than 2 GeV.

We shall discuss only the currents of the mixed flavor symmetry. For the isoscalar case, there is only one set of four independent currents as given in Eqs. (8.25), unlike the isovector case which have two sets. The spectral densities calculated by the mixed currents $\zeta^M_{i\mu}$ are shown in Fig. 8.15, which are positive for a wide range of s. The convergence of OPE is very good in the region of $2 \text{ GeV}^2 < M_B^2 < 5\text{GeV}^2$ as in our previous study [43]. In general, the pole contribution should be large enough in the SVZ sum rule. However, the pole contributions of multiquark states are rather small due to the large continuum contribution. Therefore a careful choice of the threshold parameter is important in order to subtract the continuum contribution. At this moment we do not have a complete solution to this problem, while we can perform a sum rule analysis phenomenologically. Besides the SVZ sum rule, we will also use the finite energy sum rule. As we shall discuss in the following, the remarkable stability in both the SVZ sum rule and the finite energy sum rule indicates the signal of the physical state of the present exotic channel with a very similar mass.

When using the SVZ sum rule, the mass is obtained as functions of Borel mass M_B and threshold value s_0. As an example, we show the mass calculated from currents $\zeta^M_{2\mu}$ in Fig. 8.16. The Borel mass dependence is weak, as shown in the upper figure; the s_0 dependence has a minimum where the stability is the best, as shown in the bottom figure. The minimum is around 2.0 GeV, which we choose to be our prediction. The other three independent currents $\zeta^M_{1\mu}$, $\zeta^M_{3\mu}$ and $\eta^M_{4\mu}$ lead to similar results, which are around 2.1 GeV, 1.9 GeV and 2.0 GeV respectively.
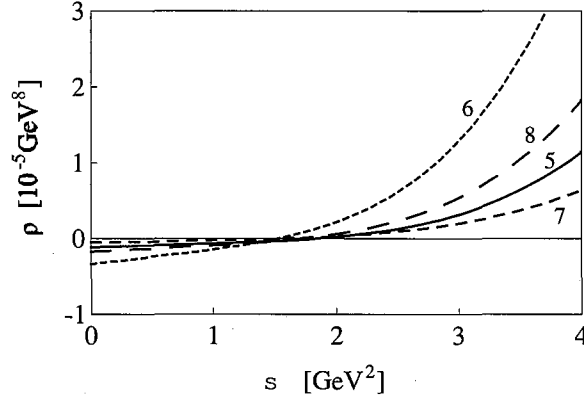
Figure 8.15: Spectral densities for the currents $\eta_{i\mu}^M$. The labels besides the lines indicate the suffix i of the currents $\zeta_{i\mu}^M$ $(i = 1, \cdots, 4)$.
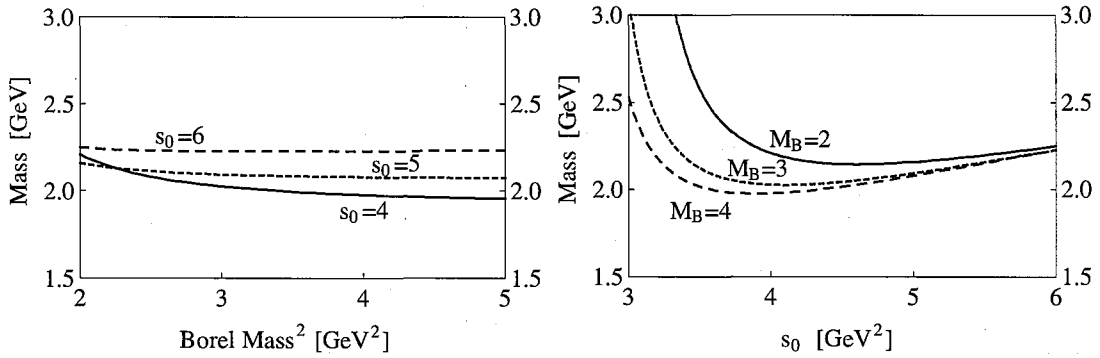
Figure 8.16: The mass of the state $qs\bar{q}\bar{s}$ calculated by using the current $\zeta_{2\mu}^M$, as functions of M_B^2 (upper) and s_0 (bottom) in units of GeV.

When using the finite energy sum rule, the mass is obtained as a function of the threshold value s_0, which is shown in Fig. 8.17. There is also a mass minimum around 2.1 GeV, 1.9 GeV, 1.9 GeV and 2.0 GeV for currents $\zeta_{1\mu}^M$, $\zeta_{2\mu}^M$, $\zeta_{3\mu}^M$ and $\zeta_{4\mu}^M$ respectively. In a short summary, we have performed a QCD sum rule analysis for $qs\bar{q}\bar{s}$. The mass obtained is around 2.0 GeV. We label this state $\sigma_1(2000)$.

Now let us discuss its decay properties as expected from a naive fall-apart process. This has a direct relevance to the experimental observations. As shown in Eqs. (8.25) the currents contain one $s\bar{s}$ pair. Therefore, we expect that the final states should also contain one $s\bar{s}$ pair. In order to spell out the possible spin of decaying particles and their orbital angular momentum, we need perform a Fierz rearrangement to change $(qq)(\bar{q}\bar{q})$
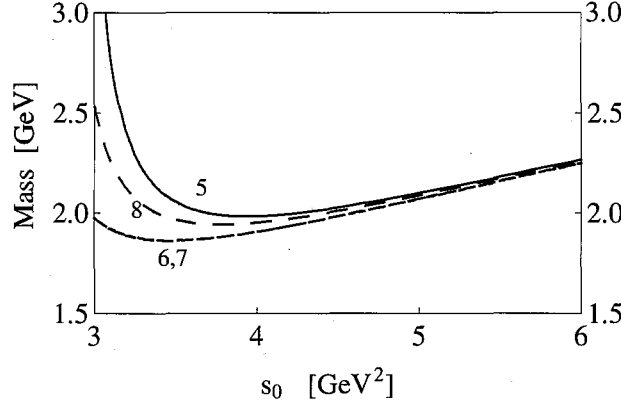
Figure 8.17: The mass calculated using the finite energy sum rule. The mass for the currents $\zeta_{1\mu}^M$, $\zeta_{2\mu}^M$, $\zeta_{3\mu}^M$ and $\zeta_{4\mu}^M$ are shown. The labels besides the lines indicate the suffix i of the current $\zeta_{i\mu}^M$ $(i = 1, \cdots, 4)$.

currents to $(\bar{q}q)(\bar{q}q)$ ones. For illustration, we use one of the four independent $(\bar{q}q)(\bar{q}q)$ currents [43]:

$$\xi_{2\mu}'^M = (\bar{s}_a\gamma^\mu\gamma_5 s_a)(\bar{u}_b\gamma_5 u_b) - (\bar{s}_a\gamma_5 s_a)(\bar{u}_b\gamma^\mu\gamma_5 u_b)$$
$$+ \cdots . \tag{8.26}$$

All terms of this current have the structure $(\bar{q}_a\gamma^\mu\gamma_5 q_a)(\bar{q}_b\gamma_5 q_b)$. Therefore, the expected decay patterns are: (1) 1^+ and 0^- particles with relative angular momentum $L = 0$, and (2) 0^- and 0^- particles with $L = 1$.

For the S-wave decay, we expect the following two-body decay patterns

$$\sigma_1(I^G J^{PC} = 0^+1^{-+}) \rightarrow a_1(1260)\eta, a_1\eta', \cdots ,$$
$$b_1(1235)\eta, b_1\eta' \cdots . \tag{8.27}$$

If we consider, however, the G parity conservation, the fist line is forbidden and the second line is the only one allowed. These modes can be observed in the final states $\omega\pi\eta$ and $\omega\pi\eta'$.

For the P-wave decay, we expect (with the G parity conservation):

$$\sigma_1(I^G J^{PC} = 0^+1^{-+}) \rightarrow KK, \eta\eta, \eta\eta', \eta'\eta' \cdots . \tag{8.28}$$

We can also estimate the (partial) decay width through the comparison with the observed $\pi_1(2015)$ [126], which has $\Gamma_{\text{tot}} \sim 230$ MeV. Assuming that the decay of $\pi_1(2015)$ solely goes through S-wave $b_1\pi$ and that of $\sigma_1(2000)$ through $b_1\eta$, we expect $\Gamma_{\sigma_1 \rightarrow b_1\eta} \sim 160$ MeV, as they are proportional to the S-wave phase space. For the P-wave decay there

is an information $\pi_1(2015) \to \eta'\pi$, which corresponds to $\sigma_1(2000) \to \eta'\eta$ (Because both $\pi_1(1600)$ and $\pi_1(2015)$ have been observed in the final states $\pi\eta'$ other than $\pi\eta$, we choose $\eta\eta'$ to be the final states of $\sigma_1(2000)$ other than KK and $\eta\eta$). Assuming once again that this is the unique decay mode, we expect that the decay width is approximately 130 MeV. If the decay occurs 50% through $b_1\pi$ ($b_1\eta$) and 50% through $\eta'\pi$ ($\eta'\eta$), we expect that $\Gamma_{\sigma_1} \sim 150$ MeV.

In summary, we have performed the QCD sum rule analysis of the exotic tetraquark states with $I^G J^{PC} = 0^+ 1^{-+}$. We test all possible flavor structures in the diquark-antidiquark $(qq)(\bar{q}\bar{q})$ construction, $\mathbf{6} \otimes \bar{\mathbf{6}}$, $\bar{\mathbf{3}} \otimes \mathbf{3}$ and $(\bar{\mathbf{3}} \otimes \bar{\mathbf{6}}) \oplus (\mathbf{6} \otimes \mathbf{3})$. We find that the former two cases can not result in a meaningful sum rule since the spectral functions become negative. On the other hand, the mixed currents of the flavor structure $(\bar{\mathbf{3}} \otimes \bar{\mathbf{6}}) \oplus (\mathbf{6} \otimes \mathbf{3})$ allows a positive and convergent OPE with which we can perform a QCD sum rule analysis. There is only one choice with the quark content $qs\bar{q}\bar{s}$, which have four independent currents. We have then performed both the SVZ sum rule and the finite energy sum rule. The resulting mass is around 2.0 GeV. The possible decay modes are S-wave $b_1(1235)\eta$ and $b_1(1235)\eta'$, and P-wave KK, $\eta\eta$, $\eta\eta'$ and $\eta'\eta'$, etc. The decay width is around 150 MeV through a rough estimation.

8.7   Conclusion

In this chapter we have performed the QCD sum rule analysis of the exotic tetraquark states with $I^G J^{PC} = 1^- 1^{-+}$. The tetraquark currents have rich internal structure. There are several independent currents for a given set of quantum numbers. We have classified the complete set of independent currents and constructed the currents in the form of either $(qq)(\bar{q}\bar{q})$ or $(\bar{q}q)(\bar{q}q)$. As expected, they are shown to be equivalent by having the complete set of independent currents. Physically, this seems to make it difficult to draw interpretation of the internal structure such as diquark (qq) dominated or meson $(\bar{q}q)$ dominated ones. Using the complete set of the currents, one can perform an optimal analysis of the QCD sum rule.

Somewhat complicated feature arises from the flavor structure. We have tested all possibilities for the isovector $I = 1$ states. In the $SU(3)$ limit, there are three cases of, in the diquark $(qq)(\bar{q}\bar{q})$ construction, $\mathbf{6} \otimes \bar{\mathbf{6}}$, $\bar{\mathbf{3}} \otimes \mathbf{3}$ and $(\bar{\mathbf{3}} \otimes \bar{\mathbf{6}}) \oplus (\mathbf{6} \otimes \mathbf{3})$. We find that the former two cases can not result in meaningful sum rule since the spectral functions become negative. On the other hand, the mixed case $(\bar{\mathbf{3}} \otimes \bar{\mathbf{6}}) \oplus (\mathbf{6} \otimes \mathbf{3})$ allows positive OPE with which we can perform the QCD sum rule analysis. Actual currents have been constructed in the limit of the ideal mixing where the currents are classified by the number of the strange quarks. Hence the quark contents are either $qq\bar{q}\bar{q}$ or $qs\bar{q}\bar{s}$.

We have then performed the SVZ and finite energy sum rules. The resulting masses are around 1.6 GeV for $qq\bar{q}\bar{q}$, and around 2.0 GeV for $qs\bar{q}\bar{s}$. The four independent currents lead to the same mass and couple to a single state as shown above. Hence

one of our main conclusions is that the higher energy states $\pi_1(1600)$ and $\pi_1(2015)$ are well compatible with the tetraquark picture in the present QCD sum rule analysis. On the other hand, any combination of the independent currents does not seem to couple sufficiently to the lower mass state $\pi_1(1400)$, which was, however, described as a hybrid state by K. C. Yang in Ref. [178]. He obtained a low mass around 1.26 GeV by using the renormalization-improved QCD sum rules. The $\pi_1(1400)$ state seems somewhat special, as the experiments show the similarity between $\pi_1(1600)$ and $\pi_1(2015)$ as well as the difference between $\pi_1(1400)$ and the above two states, which we have discussed in the introduction.

We have also studied their decay patterns and found that these states can be searched for in the decay mode of the axial-vector and pseudoscalar meson pair such as $b_1(1235)\pi$, which is sometimes considered as the characteristic decay mode of the hybrid mesons. The P-wave modes $\pi\eta, \pi\eta'$ are also quite important.

It is also interesting to study the partners of π_1s. Especially, we can study the one with quark contents $ud\bar{s}\bar{s}$, which is at the top of the flavor representation $\overline{10}$ (see Fig. 8.1). It has a mass around 2.0 GeV, and the decay modes are $K^+(\bar{s}u)K^0(\bar{s}d)$ (P-wave) and KKK (P-wave) etc. BESIII will start taking data very soon. The search/identification of exotic mesons is one of its important physical goals. Hopefully the dedicated experimental programs on the exotic mesons at BESIII and JLAB in the coming years will shed light on their existence, and then their internal structure. More work on theoretical side is also needed. We will go on to study other tetraquark candidates.

Chapter 9

Bottom Baryons

Recently CDF Collaboration observed four bottom baryons $\Sigma_b^{\pm}$ and $\Sigma_b^{*\pm}$ [1, 72]. D0 Collaboration announced the observation of Ξ_b [3], which was confirmed by CDF collaboration later [2, 124]. Very recently, Babar Collaboration reported the observation of Ω_c^* with the mass splitting $m_{\Omega_c^*} - m_{\Omega_c} = 70.8 \pm 1.0 \pm 1.1$ MeV [20]. We collect the masses of these recently observed bottom baryons in Table 9.1.

The heavy hadron containing a single heavy quark is particularly interesting. The light degrees of freedom (quarks and gluons) circle around the nearly static heavy quark. Such a system behaves as the QCD analogue of the familiar hydrogen bounded by electromagnetic interaction. The heavy quark expansion provides a systematic tool for heavy hadrons. When the heavy quark mass $m_Q \rightarrow \infty$, the angular momentum of the light degree of freedom is a good quantum number. Therefore heavy hadrons form doublets. For example, Ω_b and Ω_b^* will be degenerate in the heavy quark limit. Their mass splitting is caused by the chromo-magnetic interaction at the order $\mathcal{O}(1/m_Q)$, which can be taken into account systematically in the framework of heavy quark effective field theory (HQET).

In the past two decades, various phenomenological models have been used to study heavy baryon masses [24, 36, 62, 100, 132, 156]. Capstick and Isgur studied the heavy baryon system in a relativized quark potential model [36]. Roncaglia et al. predicted the masses of baryons containing one or two heavy quarks using the Feynman-Hellmann theorem and semiempirical mass formulas [156]. Jenkins studied heavy baryon masses using a combined expansion of $1/m_Q$ and $1/N_c$ [100]. Mathur et al. predicted the masses of charmed and bottom baryons from lattice QCD [132]. Ebert et al. calculated the masses of heavy baryons with the light-diquark approximation [62]. Using the relativistic Faddeev approach, Gerasyuta and Ivanov calculated the masses of the S-wave charmed baryons [69]. Later, Gerasyuta and Matskevich studied the charmed $(\mathbf{70}, 1^-)$ baryon multiplet using the same approach [70]. Stimulated by recent experimental progress, there have been several theoretical papers on the masses of Σ_b, Σ_b^* and Ξ_b using the hyperfine interaction in the quark model [106, 107, 110–112, 157]. Recently the strong

Table 9.1: The masses of bottom baryons recently observed by CDF and D0 collaborations.

	mass (MeV)	Experiment
Σ_b^+	$5808^{+2.0}_{-2.3}$(stat.) $\pm$ 1.7(syst.)	
Σ_b^-	$5816^{+1.0}_{-1.0}$(stat.) $\pm$ 1.7(syst.)	CDF [1,72]
Σ_b^{*+}	$5829^{+1.6}_{-1.8}$(stat.) $\pm$ 1.7(syst.)	
Σ_b^{*-}	$5837^{+2.1}_{-1.9}$(stat.) $\pm$ 1.7(syst.)	
Ξ_b^-	5774 ± 11(stat.) $\pm$ 15(syst.)	D0 [3]
	5793 ± 2.5(stat.) $\pm$ 1.7(syst.)	CDF [2,124]

decays of heavy baryons were investigated systematically using $^3P_0$ model in Ref. [40].

QCD sum rule (QSR) has been applied to study heavy baryon masses previously [24–26, 53, 54, 60, 73, 84, 141, 161, 166, 167, 170, 185]. The mass sum rules of $\Lambda_{c,b}$ and $\Sigma_{c,b}$ were obtained in full QCD in Refs. [24, 25, 141]. The mass sum rules of Σ_Q and Λ_Q in the leading order of the heavy quark effective theory (HQET) have been discussed in Refs. [26, 73, 161]. Dai et al. calculated the $1/m_Q$ correction to the mass sum rules of Λ_Q and $\Sigma_Q^{(*)}$ in HQET [53, 54]. Later the mass sum rules of Λ_Q and $\Sigma_Q^{(*)}$ were reanalyzed in Ref. [166]. The mass sum rules of orbitally excited heavy baryons in the leading order of HQET were discussed in Refs. [84, 185] while the $1/m_Q$ correction was considered in Ref. [167]. Recently Wang studied the mass sum rule of $\Omega_{c,b}^*$ [171] while Durães and Nielsen studied the mass sum rule of $\Xi_{c,b}$ using full QCD Lagrangian [60].

In order to extract the chromo-magnetic splitting between the bottom baryon doublets reliably, we derive the mass sum rules up to the order of $1/m_Q$ in the heavy quark effective field theory in this work. We perform a systematic study of the masses of Ξ_b, Ξ_b', Ξ_b^*, Ω_b and Ω_b^* through the inclusion of the strange quark mass correction. The resulting chromo-magnetic mass splitting agrees well with the available experimental data. As a cross-check, we reproduce the mass sum rules of Λ_b, Σ_b and Σ_b^* which have been derived in literature previously. As a byproduct, we extend the same formalism to the case of charmed baryons while keeping in mind that the heavy quark expansion does not work well for the charmed hadrons.

9.1 QCD sum rules for heavy baryons

We first introduce our notations for the heavy baryons. Inside a heavy baryon there are one heavy quark and two light quarks (u, d or s). It belongs to either the symmetric $\mathbf{6_F}$ or antisymmetric $\mathbf{3_F}$ flavor representation (see Fig. 9.1). For the S-wave heavy baryons,

the total flavor-spin wave function of the two light quarks must be symmetric since their color wave function is antisymmetric. Hence the spin of the two light quarks is either $S = 1$ for $\mathbf{6_F}$ or $S = 0$ for $\mathbf{\bar{3}_F}$. The angular momentum and parity of the S-wave heavy baryons are $J^P = \frac{1}{2}^+$ or $\frac{3}{2}^+$ for $\mathbf{6_F}$ and $J^P = \frac{1}{2}^+$ for $\mathbf{\bar{3}_F}$. The names of S-wave heavy baryons are listed in Fig. 9.1, where we use $*$ to denote $\frac{3}{2}^+$ baryons and the $\prime$ to denote the $J^P = \frac{1}{2}^+$ baryons in the $\mathbf{6_F}$ representation. In this work, we use B to denote the heavy baryons with $\frac{1}{2}^+$ in $\mathbf{\bar{3}_F}$ and B' and B^* to denote those states with $\frac{1}{2}^+$ and $\frac{3}{2}^+$ in $\mathbf{6_F}$.
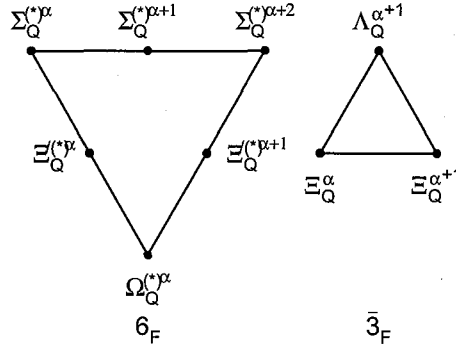
Figure 9.1: The SU(3) flavor multiplets of heavy baryons. Here α, $\alpha + 1$, $\alpha + 2$ denote the charges of heavy baryons.

We will study heavy baryon masses in HQET using QCD sum rule approach. HQET plays an important role in the investigation of the heavy hadron properties [143]. In the limit of $m_Q \to \infty$, the heavy quark field $Q(x)$ in full QCD can be decomposed into its small and large components

$$Q(x) = e^{-im_Q v \cdot x}[H_v(x) + h_v(x)], \tag{9.1}$$

where v^μ is the velocity of the heavy baryon. Accordingly the heavy quark field $h_v(x)$ reads

$$h_v(x) = e^{im_Q v \cdot x} \frac{1 + \not{v}}{2} Q(x), \tag{9.2}$$

$$H_v(x) = e^{im_Q v \cdot x} \frac{1 - \not{v}}{2} Q(x). \tag{9.3}$$

The Lagrangian in HQET reads

$$\mathcal{L}_{HQET} = \bar{h}_v iv \cdot D h_v + \frac{1}{2m_Q} \bar{h}_v (iD_\perp)^2 h_v - C_{mag} \frac{g}{4m_Q} \bar{h}_v \sigma_{\mu\nu} G^{\mu\nu} h_v. \tag{9.4}$$

The second and third term in the above Lagrangian corresponds to the kinetic and chromo-magnetic corrections at the order of $1/m_Q$. Here $D_\perp^\mu = D^\mu - v^\mu v \cdot D$ and $D^\mu = \partial^\mu + igA^\mu$. $C_{mag}(\mu)$ is renormalization coefficient $C_{mag}(\mu) = (\alpha_s(m_Q)/\alpha_s(\mu))^{3/\beta_0}[1 + \frac{13\alpha_s}{6\pi}]$, where $\beta_0 = 11 - 2n_f/3$ and n_f is the number of quark flavors [143].

In order to derive the mass sum rules of B, B' and B^*, we use the following interpolating currents for the heavy baryons with $J^P = \frac{1}{2}^+$ in $\mathbf{6_F}$,

$$J_{B'}(x) = \epsilon_{abc}[q_1^{aT}(x)C\gamma_\mu q_2^b(x)]\gamma_t^\mu \gamma_5 h_v^c(x), \tag{9.5}$$

$$\bar{J}_{B'}(x) = -\epsilon_{abc}\bar{h}_v^c(x)\gamma_5 \gamma_t^\mu [\bar{q}_2^b(x)\gamma_\mu C\bar{q}_1^{aT}(x)]. \tag{9.6}$$

For the heavy baryons with $J^P = \frac{3}{2}^+$ in $\mathbf{6_F}$,

$$J_{B^*}^\mu(x) = \epsilon_{abc}[q_1^{aT}(x)C\gamma_\nu q_2^b(x)] \times \left(-g_t^{\mu\nu} + \frac{1}{3}\gamma_t^\mu \gamma_t^\nu\right)h_v^c(x), \tag{9.7}$$

$$\bar{J}_{B^*}^\mu(x) = \epsilon_{abc}\bar{h}_v^c(x)\left(-g_t^{\mu\nu} + \frac{1}{3}\gamma_t^\nu \gamma_t^\mu\right) \times [\bar{q}_2^b(x)\gamma_\nu C\bar{q}_1^{aT}(x)]. \tag{9.8}$$

For the heavy baryons with $J^P = \frac{1}{2}^+$ in $\mathbf{\bar{3}_F}$

$$J_B(x) = \epsilon_{abc}[q_1^{aT}(x)C\gamma_5 q_2^b(x)]h_v^c(x), \tag{9.9}$$

$$\bar{J}_B(x) = -\epsilon_{abc}\bar{h}_v^c(x)[\bar{q}_2^b(x)\gamma_5 C\bar{q}_1^{aT}(x)]. \tag{9.10}$$

Here a, b and c are color indices, $q_i(x)$ denotes up, down and strange quark fields. T is the transpose matrix and C is the charge conjugate matrix. $g_t^{\mu\nu} = g^{\mu\nu} - v^\mu v^\nu$, $\gamma_t^\mu = \gamma^\mu - \not{v}v^\mu$.

The overlapping amplitudes of the interpolating currents with B, B' and B^* are defined as

$$\langle 0|J_B|B\rangle = f_B u_B, \tag{9.11}$$

$$\langle 0|J_{B'}|B'\rangle = f_{B'}u_{B'}, \tag{9.12}$$

$$\langle 0|J_{B^*}^\mu|B^*\rangle = \frac{1}{\sqrt{3}}f_{B^*}u_{B^*}^\mu, \tag{9.13}$$

where $u_{B^*}^\mu$ is the Rarita-Schwinger spinor in HQET. $f_{B'} = f_{B^*}$ due to heavy quark symmetry.

The binding energy $\bar{\Lambda}_i$ is defined as the mass difference between the heavy baryon and heavy quark when $m_Q \to \infty$. In order to extract $\bar{\Lambda}_i$, we consider the following correlation function

$$i\int d^4x \, e^{iq\cdot x}\langle 0|T\{J_{B^{(\prime)}}(x)\bar{J}_{B^{(\prime)}}(0)\}|0\rangle = \frac{1 + \not{v}}{2}\Pi_{B^{(\prime)}}(\omega), \tag{9.14}$$

with $\omega = v \cdot q$.

The dispersion relation for $\Pi(\omega)$ is

$$\Pi(\omega) = \int \frac{\rho(\omega')}{\omega' - \omega - i\epsilon} \, d\omega', \tag{9.15}$$

where $\rho(\omega)$ denotes the spectral density in the limit of $m_Q \to \infty$. At the phenomenological level,

$$\Pi(\omega) = \frac{f_i^2}{\bar{\Lambda}_i - \omega} + \text{continuum}. \tag{9.16}$$

Making the Borel transformation with variable ω, we obtain

$$f_i^2 e^{-\bar{\Lambda}_i/T} = \int_0^{\omega_0} \rho(\omega) e^{-\omega/T} d\omega, \tag{9.17}$$

where T is the Borel mass obtained by using Borel transformation. We have invoked the quark-hadron duality assumption and approximated the continuum above ω_0 with the perturbative contribution at the quark-gluon level. The mass sum rules of B, B' and B^* are

$$
\begin{aligned}
f_B^2 e^{-\bar{\Lambda}_B/T} = & \\
\int_0^{\omega_B} & \Bigg[\frac{\omega^5}{20\pi^4} - \frac{(m_{q_1}^2 + m_{q_2}^2 - m_{q_1} m_{q_2})\omega^3}{4\pi^4} \\
& + \frac{\langle g^2 GG \rangle \omega}{128\pi^4} + \frac{m_{q_2}\langle \bar{q}_2 q_2 \rangle + m_{q_1}\langle \bar{q}_1 q_1 \rangle}{4\pi^2}\omega \\
& - \frac{2m_{q_2}\langle \bar{q}_1 q_1 \rangle + 2m_{q_1}\langle \bar{q}_2 q_2 \rangle}{4\pi^2} \Bigg] e^{-\omega/T} d\omega \\
& - \frac{m_{q_1}\langle g_c \bar{q}_2 \sigma G q_2 \rangle + m_{q_2}\langle g_c \bar{q}_1 \sigma G q_1 \rangle}{32\pi^2} \\
& + \frac{m_{q_1}\langle g_c \bar{q}_1 \sigma G q_1 \rangle + m_{q_2}\langle g_c \bar{q}_2 \sigma G q_2 \rangle}{12 \cdot 32\pi^2} + \frac{\langle \bar{q}_1 q_1 \rangle \langle \bar{q}_2 q_2 \rangle}{6} \\
& + \frac{\langle \bar{q}_1 q_1 \rangle \langle g_c \bar{q}_2 \sigma G q_2 \rangle + \langle \bar{q}_2 q_2 \rangle \langle g_c \bar{q}_1 \sigma G q_1 \rangle}{96T^2},
\end{aligned}
\tag{9.18}
$$

$$
\begin{aligned}
f_{B'}^2 e^{-\bar{\Lambda}_{B'}/T} = & \\
\int_0^{\omega_{B'}} & \Bigg[\frac{3\omega^5}{20\pi^4} + \frac{(3m_{q_1} m_{q_2} - 3m_{q_1}^2 - 3m_{q_2}^2)\omega^3}{4\pi^4} \\
& - \frac{\langle g^2 GG \rangle \omega}{128\pi^4} - \frac{6m_{q_1}\langle \bar{q}_2 q_2 \rangle + 6m_{q_2}\langle \bar{q}_1 q_1 \rangle}{4\pi^2}\omega \\
& + \frac{3m_{q_1}\langle \bar{q}_1 q_1 \rangle + 3m_{q_2}\langle \bar{q}_2 q_2 \rangle}{4\pi^2}\omega \Bigg] e^{-\omega/T} d\omega
\end{aligned}
$$

$$+\frac{\langle \bar{q}_1 q_1 \rangle \langle \bar{q}_2 q_2 \rangle}{2} - \frac{3 m_{q_1} \langle g_c \bar{q}_2 \sigma G q_2 \rangle + 3 m_{q_2} \langle g_c \bar{q}_1 \sigma G q_1 \rangle}{32 \pi^2}$$

$$+\frac{5 m_{q_1} \langle g_c \bar{q}_1 \sigma G q_1 \rangle + 5 m_{q_2} \langle g_c \bar{q}_2 \sigma G q_2 \rangle}{128 \pi^2}$$

$$+\frac{\langle \bar{q}_2 q_2 \rangle \langle g_c \bar{q}_1 \sigma G q_1 \rangle + \langle \bar{q}_1 q_1 \rangle \langle g_c \bar{q}_2 \sigma G q_2 \rangle}{32 T^2}. \tag{9.19}$$

The mass sum rule of B^* is same as that of B' at the leading order of HQET. In the above equations, $\langle \bar{q}_i q_i \rangle$ is the quark condensates, $\langle g^2 GG \rangle$ is the gluon condensate and $\langle g\bar{q}_i \sigma G q_i \rangle$ is the quark-gluon mixed condensate. The above sum rules have been derived in the massless light quark limit in Refs. [26, 53, 54, 73, 161]. Up and down quark mass correction is tiny for heavy baryons Λ_b, Σ_b and Σ_b^*. In this work we have included the finite quark mass correction which is important for heavy baryons Ξ_b, Ξ_b', Ξ_b^*, Ω_b and Ω_b^*.

The binding energy $\bar{\Lambda}_i$ can be extracted using the following formula

$$\bar{\Lambda}_i = \frac{T^2}{\mathbb{R}_i} \cdot \frac{d\mathbb{R}_i}{dT}, \tag{9.20}$$

where $\mathbb{R}_i$ denotes the right-hand part in the above sum rules.

9.2   The $1/m_Q$ correction

In order to calculate the $1/m_Q$ correction, we insert the heavy baryon eigen-state of the Hamiltonian up to the order $\mathcal{O}(1/m_Q)$ into the correlation function

$$i \int d^4 x e^{iq \cdot x} \langle 0|T[J_i(x) \bar{J}_i(0)]|0 \rangle. \tag{9.21}$$

Its pole contribution is

$$\begin{aligned}
\Pi(\omega) &= \frac{(f + \delta f)^2}{(\bar{\Lambda} + \delta m) - \omega} \\
&= \frac{f^2}{\bar{\Lambda} - \omega} - \frac{f^2 \delta m}{(\bar{\Lambda} - \omega)^2} + \frac{2 f \delta f}{\bar{\Lambda} - \omega},
\end{aligned} \tag{9.22}$$

where both δm and δf are $\mathcal{O}(1/m_Q)$.

We consider the three-point correlation function

$$\frac{1 + \not{v}}{2} \delta^O \Pi(\omega, \omega')$$

$$= i^2 \int d^4 z d^4 y e^{ip \cdot z} e^{ip' \cdot y} \langle 0|T[J_i(z) O(x) \bar{J}(y)]|0 \rangle, \tag{9.23}$$

where operators $O = \mathcal{K}$ and $\mathcal{S}$ correspond to the kinetic energy and chromo-magnetic interaction in Eq. (9.4). The double dispersion relation for $\delta^O \Pi(\omega, \omega')$ reads

$$\delta^O \Pi(\omega, \omega') = \int_0^\infty ds \int_0^\infty ds' \frac{\rho^O(s, s')}{(s - \omega)(s' - \omega')}.$$ (9.24)

At the hadronic level,

$$\delta^{\mathcal{K}} \Pi(\omega, \omega') = \frac{f^2 \mathcal{K}_i}{(\bar{\Lambda} - \omega)(\bar{\Lambda} - \omega')} + \cdots,$$ (9.25)

$$\delta^{\mathcal{S}} \Pi(\omega, \omega') = \frac{f^2 \mathcal{S}_i}{(\bar{\Lambda} - \omega)(\bar{\Lambda} - \omega')} + \cdots$$ (9.26)

with

$$\mathcal{K}_i = \frac{1}{2m_Q} \langle B_i | \bar{h}_v (iD_\perp)^2 h_v | B_i \rangle,$$ (9.27)

$$\mathcal{S}_i = -\frac{1}{4m_Q} \langle B_i | \bar{h}_v g \sigma_\mu G^{\mu\nu} h_v | B_i \rangle.$$ (9.28)

After setting $\omega = \omega'$ in Eqs. (9.25) and (9.26) and comparing them with Eq. (9.22), we can extract δm

$$\delta m_i = -(\mathcal{K}_i + C_{mag} \mathcal{S}_i).$$ (9.29)

Here the renormalization coefficient C_{mag} for bottom baryons is $C_{mag} \approx 0.8$ [185].

We calculate the diagrams listed in Fig. 9.2 to derive $\delta^O \Pi(\omega, \omega')$. After invoking double Borel transformation to Eq. 9.24, we obtain the spectral density $\rho^O(s, s')$. Then we redefine the integration variable

$$s_+ = \frac{s + s'}{2},$$ (9.30)

$$s_- = \frac{s - s'}{2}.$$ (9.31)

Now the integral in Eq. (9.24) is changed as

$$\int_0^\infty ds \int_0^\infty ds' \ldots = 2 \int_0^\infty ds_+ \int_{-s_+}^{+s_+} ds_- \ldots.$$ (9.32)

In the subtraction of the continuum contribution, quark hadron duality is assumed for the integration variable s_+ [30, 142].
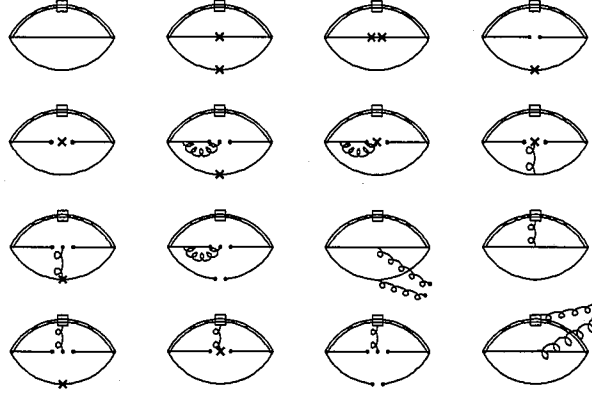
Figure 9.2: The diagrams for the $1/m_Q$ corrections. Here the current quark mass correction is denoted by the cross. The first eleven diagrams correspond to the kinetic corrections and the last five diagrams are chromo-magnetic corrections. White squares denote the operators of $1/m_Q$.

For $B(\frac{1}{2}^+)$ in $\mathbf{\bar{3}_F}$, the $1/m_Q$ correction comes from the kinetic term only.

$$
\begin{aligned}
\mathcal{K}_B = & \\
& -\frac{e^{\bar{\Lambda}_B/T}}{m_Q f_B^2}\Big\{ \int_0^{\omega_B} \Big[\frac{54\omega^7}{7!\pi^4} - \frac{9\omega^5}{5!\pi^4}(m_{q_1}^2 + m_{q_2}^2 - m_{q_1}m_{q_2}) \\
& + \frac{3\langle g^2 GG\rangle \omega^3}{128\cdot 3!\pi^4} + \frac{3\omega^3}{4\cdot 3!\pi^2}\Big(m_{q_1}\langle \bar{q}_1 q_1\rangle + m_{q_2}\langle \bar{q}_2 q_2\rangle \\
& - 2m_{q_2}\langle \bar{q}_1 q_1\rangle - 2m_{q_1}\langle \bar{q}_2 q_2\rangle \Big) \\
& - \frac{3\omega}{128\pi^2}\Big(m_{q_1}\langle g_c\bar{q}_1\sigma G q_1\rangle + m_{q_2}\langle g_c\bar{q}_2\sigma G q_2\rangle \Big) \\
& + \frac{3\omega}{32\pi^2}\Big(m_{q_1}\langle g_c\bar{q}_2\sigma G q_2\rangle + m_{q_2}\langle g_c\bar{q}_1\sigma G q_1\rangle \Big) \Big] e^{-\omega/T}\mathrm{d}\omega \\
& - \frac{1}{32}\Big[\langle \bar{q}_1 q_1\rangle\langle g_c\bar{q}_2\sigma G q_2\rangle + \langle \bar{q}_2 q_2\rangle\langle g_c\bar{q}_1\sigma G q_1\rangle \Big] \Big\},
\end{aligned}
\tag{9.33}
$$

$$
\mathcal{S}_B = 0.
\tag{9.34}
$$

Here $\mathcal{S}_B = 0$ is consistent with the simple expectation in the constituent quark model that the chromo-magnetic interaction $\langle S_Q \cdot j_l\rangle = 0$ since $j_l = 0$ for $B(\frac{1}{2}^+)$ in $\mathbf{\bar{3}_F}$.

For $B'(\frac{1}{2}^+)$ in $\mathbf{6_F}$, the $1/m_Q$ corrections are

$$
\begin{aligned}
\mathcal{K}_{B'} = {} & -\frac{e^{\bar{\Lambda}_{B'}/T}}{m_Q f_{B'}^2} \Bigg\{ \int_0^{\omega_{B'}} \Bigg[\frac{18 \cdot 11\omega^7}{7!\pi^4} - \frac{9\omega^5}{5!\pi^4}(4m_{q_1}^2 + 4m_{q_2}^2 \\
& - 3m_{q_1}m_{q_2}) - \frac{\langle g^2 GG \rangle \omega^3}{128 \cdot 3!\pi^4} + \frac{3\omega^3}{4 \cdot 3!\pi^2}\Big(5m_{q_1}\langle \bar{q}_1 q_1 \rangle \\
& + 5m_{q_2}\langle \bar{q}_2 q_2 \rangle - 6m_{q_2}\langle \bar{q}_1 q_1 \rangle - 6m_{q_1}\langle \bar{q}_2 q_2 \rangle \Big) \\
& + \frac{11\omega}{128 \cdot 4\pi^2}\Big(m_{q_1}\langle g_c \bar{q}_1 \sigma G q_1 \rangle + m_{q_2}\langle g_c \bar{q}_2 \sigma G q_2 \rangle \Big) \Bigg] e^{-\omega/T} \mathrm{d}\omega \\
& - \frac{3}{32}\Big[\langle \bar{q}_1 q_1 \rangle \langle g_c \bar{q}_2 \sigma G q_2 \rangle + \langle \bar{q}_2 q_2 \rangle \langle g_c \bar{q}_1 \sigma G q_1 \rangle \Big] \Bigg\}.
\end{aligned}
\tag{9.35}
$$

$$
\begin{aligned}
\mathcal{S}_{B'} = {} & \frac{e^{\bar{\Lambda}_{B'}/T}}{m_Q f_{B'}^2} \Bigg\{ \int_0^{\omega_{B'}} \Bigg[\frac{2g_c^2 \omega^7}{105\pi^6} + \frac{\langle g^2 GG \rangle \omega^3}{16 \cdot 3!\pi^4} \\
& - \frac{\omega}{32\pi^2}\Big(m_{q_1}\langle g_c \bar{q}_1 \sigma G q_1 \rangle + m_{q_2}\langle g_c \bar{q}_2 \sigma G q_2 \rangle \\
& - 2m_{q_2}\langle g_c \bar{q}_1 \sigma G q_1 \rangle - 2m_{q_1}\langle g_c \bar{q}_2 \sigma G q_2 \rangle \Big) \Bigg] e^{-\omega/T} \mathrm{d}\omega \\
& - \frac{1}{48}\Big[\langle \bar{q}_1 q_1 \rangle \langle g_c \bar{q}_2 \sigma G q_2 \rangle + \langle \bar{q}_2 q_2 \rangle \langle g_c \bar{q}_1 \sigma G q_1 \rangle \Big] \Bigg\}.
\end{aligned}
\tag{9.36}
$$

Through explicit calculation, we obtain

$$
\mathcal{K}_{B^*} = \mathcal{K}_{B'} ,
\tag{9.37}
$$

$$
\mathcal{S}_{B^*} = -\mathcal{S}_{B'}/2 ,
\tag{9.38}
$$

$$
m_{B^*} - m_{B'} = \frac{3}{2}\mathcal{S}_{B'},
\tag{9.39}
$$

which are consistent with the heavy quark symmetry.

9.3 Results and discussion

In our numerical analysis, we use the previous values (5.10) as well as [68, 71, 89, 99, 148, 177, 179]:

$$
m_c = 1.25 \pm 0.09 \text{ GeV}, \ \ m_b = 4.8 \text{ GeV}.
$$
$$
\alpha_s(m_c) = 0.328, \ \ \alpha_s(m_b) = 0.189.
$$

The values of the u, d, s and charm quark masses correspond to the $\overline{MS}$ scheme at a scale $\mu \approx 2$ GeV and $\mu = \overline{m}_c$ respectively [179]. The b quark mass is obtained from the Upsilon $1S$ mass [81, 82, 179].

Since the energy gap between the S-wave heavy baryons and their radial/orbital excitations is around 500 MeV, the continuum contribution can be subtracted quite cleanly. We require that the high-order power corrections be less than 30% of the perturbative term to ensure the convergence of the operator product expansion. This condition yields the minimum value for the working region of the Borel parameter. In this work, we choose the working region as $0.4 < T < 0.6$ GeV.

In Fig. 9.3-9.5, we give the dependence of $\bar{\Lambda}$, $\mathcal{K}_i$, $\mathcal{S}_i$ and mass splitting $m_{B_b^*} - m_{B_b'}$ on T and ω_c for Σ_b, Ξ_b', Ω_b. The variation of a sum rule with both T and ω_i contributes to the errors of the extracted value, together with the truncation of the operator product expansion and the uncertainty of vacuum condensate values. We collect the extracted $\bar{\Lambda}$, $\mathcal{K}_i$, $\mathcal{S}_i$ and mass splitting $m_{B_c^*} - m_{B_c'}$ in Table 9.2.

The masses of bottom baryons from the present work are presented in Table 9.3. It's well known that the heavy quark expansion does not work very well for the charmed baryons since the charm quark is not heavy enough to ensure the good convergence of $1/m_Q$ expansion. For example, the chromo-magnetic splitting between Ω_c^* and Ω_c from our work is around 133 MeV, which is much larger than the experimental value 67.4 MeV. However, we still choose to present the masses of S-wave charmed baryons also in Table 9.3 simply for the sake of comparison with experimental data.

Table 9.2: The central values in this table are extracted at $T = 0.5$ GeV, $\omega_i = 1.3$ GeV for $\Sigma_b^{(*)}$, $\omega_i = 1.4$ GeV for $\Xi_b'^{(*)}$, $\omega_i = 1.55$ GeV for $\Omega_b^{(*)}$, $\omega_i = 1.1$ GeV for Λ_b and $\omega_i = 1.25$ GeV for Ξ_b (in MeV).

	Σ_b	Ξ_b'	Ω_b^0	Λ_b	Ξ_b
Λ	950^{+78}_{-74}	1042^{+76}_{-74}	1169 ± 74	773^{+68}_{-59}	908^{+72}_{-67}
δm	59^{+4}_{-2}	60^{+6}_{-4}	67^{+7}_{-3}	65^{+2}_{-1}	72 ± 1
mass splitting	$m_{\Sigma_b^*} - m_{\Sigma_b}$	$m_{\Xi_b^*} - m_{\Xi_b'}$	$m_{\Omega_b^*} - m_{\Omega_b}$	-	-
this work	26 ± 1	26 ± 1	28^{+8}_{-2}	-	-
experiment [1, 72]	21	-	-	-	-

In our calculation, we adopt the phenomenological spectral function by the classical and simple ansatz of a single resonance pole plus the perturbative continuum. The systematic uncertainty of hadron parameters obtained with such an approximation was discussed recently in Ref. [127]. We have not considered the next-to-leading order α_s corrections, which may also result in large contribution and uncertainty as indicated by the study of the α_s corrections in the light-quark baryon system in Ref. [149].

Table 9.3: Masses of the heavy baryons from the present work and other approaches and the comparison with experimental data (in MeV).

Baryon	$I(J^P)$	Ours	Ref. [36]	Ref. [156]	Ref. [100]	Ref. [132]	Ref. [62]	Ref. [166,171]	EXP [3,20,72,124,179]
Σ_c	$1(\frac{1}{2}^+)$	2411^{+93}_{-81}	2440	2453		2452	2439	2470	2454.02(0.18)
Ξ'_c	$\frac{1}{2}(\frac{1}{2}^+)$	2508^{+97}_{-91}		2580	2580.8	2599	2578		2575.7(3.1)
Ω_c	$0(\frac{1}{2}^+)$	2657^{+102}_{-99}		2710		2678	2698		2697.5(2.6)
Σ^*_c	$1(\frac{3}{2}^+)$	2534^{+96}_{-81}	2495	2520		2538	2518	2590	2518.4(0.6)
Ξ^*_c	$\frac{1}{2}(\frac{3}{2}^+)$	2634^{+102}_{-94}		2650		2680	2654		2646.6(1.4)
Ω^*_c	$0(\frac{3}{2}^+)$	2790^{+109}_{-105}		2770	2760.5	2752	2768	2790	$\sim$ 2768
Λ_c	$0(\frac{1}{2}^+)$	2271^{+67}_{-49}	2265	2285		2290	2297		2286.46(0.14)
Ξ_c	$\frac{1}{2}(\frac{1}{2}^+)$	2432^{+79}_{-68}		2468		2473	2481		2467.9(0.4)
Σ_b	$1(\frac{1}{2}^+)$	5809^{+82}_{-76}	5795	5820	5824.2	5847	5805	5790	5808
Ξ'_b	$\frac{1}{2}(\frac{1}{2}^+)$	5903^{+81}_{-79}		5950	5950.9	5936	5937		
Ω_b	$0(\frac{1}{2}^+)$	6036 ± 81		6060	6068.7	6040	6065		
Σ^*_b	$1(\frac{3}{2}^+)$	5835^{+82}_{-77}	5805	5850	5840.0	5871	5834	5820	5829
Ξ^*_b	$\frac{1}{2}(\frac{3}{2}^+)$	5929^{+83}_{-79}		5980	5966.1	5959	5963		
Ω^*_b	$0(\frac{3}{2}^+)$	6063^{+83}_{-82}		6090	6083.2	6060	6088	6000	
Λ_b	$0(\frac{1}{2}^+)$	5637^{+68}_{-56}	5585	5620		5672	5622		5624(9)
Ξ_b	$\frac{1}{2}(\frac{1}{2}^+)$	5780^{+73}_{-68}		5810	5805.7	5788	5812		5774,5793

9.4 Conclusion

In short summary, inspired by recent experimental observation of charmed and bottom baryons [1–3,20,72,124], we have investigated the masses of heavy baryons systematically using the QCD sum rule approach in HQET. The chromo-magnetic splitting of the bottom baryon doublet from the present work agrees well with the recent experimental data. Recently $\Xi_b^{(*)}$ was observed by CDF collaboration [1,72]. Our results are also consistent with their experimental value. Our prediction of the masses of Ξ'_b, Ξ^*_b, Ω_b and Ω^*_b can be tested through the future discovery of these interesting states at Tevatron at Fermi Lab.
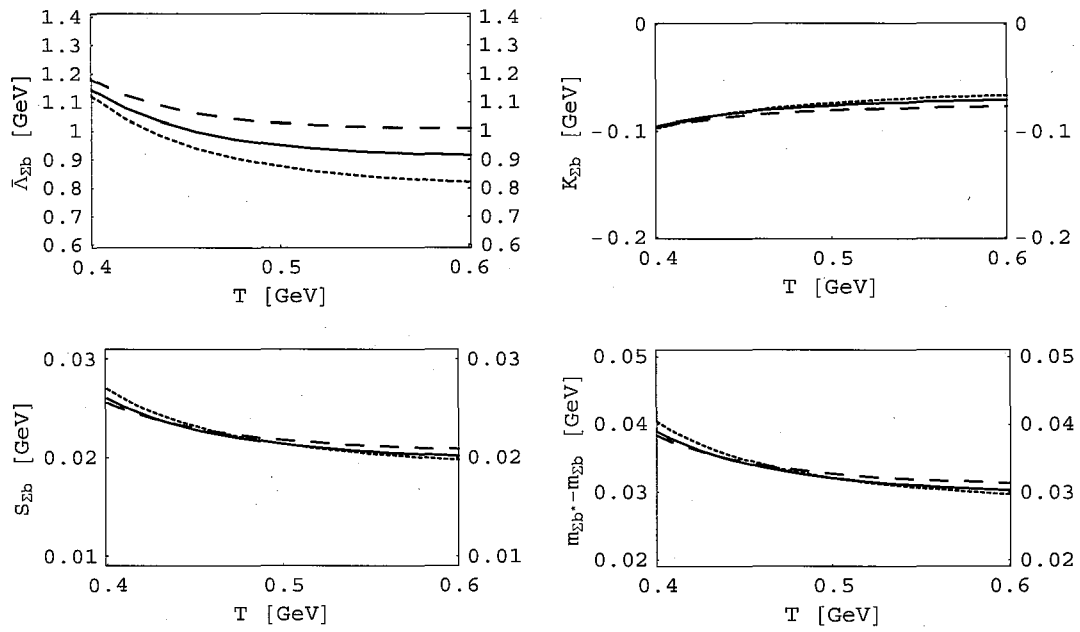
Figure 9.3: The dependences of $\bar{\Lambda}_{\Sigma_b}$, $\mathcal{K}_{\Sigma_b}$, $\mathcal{S}_{\Sigma_b}$, and the mass splitting $m_{\Sigma_b^*} - m_{\Sigma_b}$ on T. Here the dotted, solid and dashed line corresponds to the threshold value $\omega_{\Sigma_b} = 1.2, 1.3, 1.4$ GeV respectively.
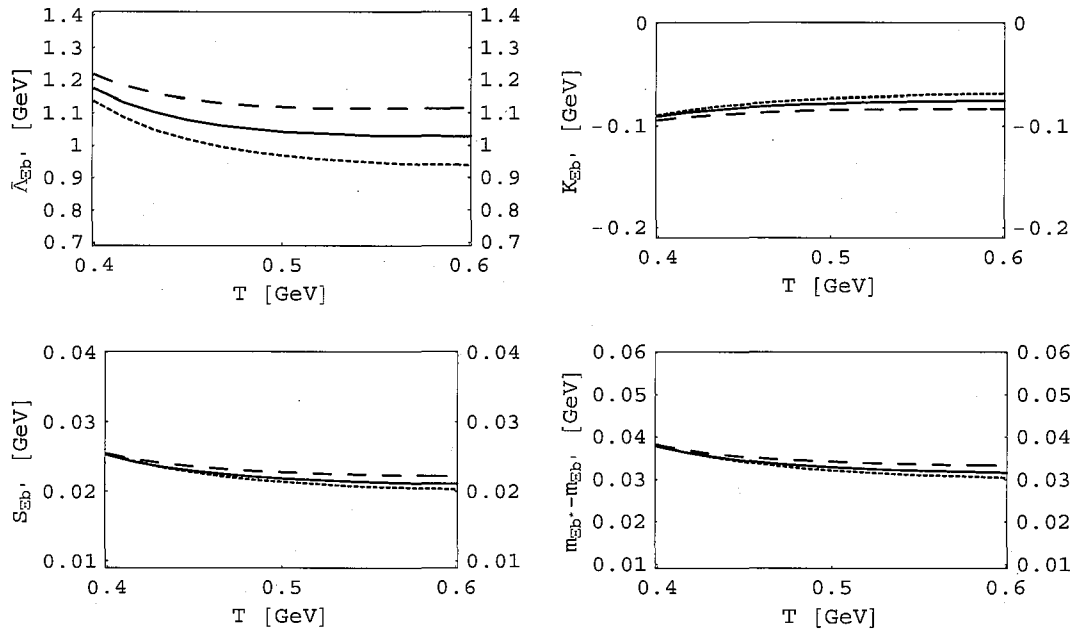
Figure 9.4: The dependences of $\bar{\Lambda}_{\Xi_b'}$, $\mathcal{K}_{\Xi_b'}$, $\mathcal{S}_{\Xi_b'}$, and the mass splitting $m_{\Xi_b^*} - m_{\Xi_b'}$ on T. The dotted, solid and dashed line corresponds to $\omega_{\Xi_b'} = 1.3, 1.4, 1.5$ GeV respectively.

Figure 9.5: The dependences of $\bar{\Lambda}_{\Omega_b}$, $\mathcal{K}_{\Omega_b}$, $\mathcal{S}_{\Omega_b}$, and the mass splitting $m_{\Omega_b^*} - m_{\Omega_b}$ on T. The dotted, solid and dashed line corresponds to $\omega_{\Omega_b} = 1.45, 1.55, 1.65$ GeV respectively.

Chapter 10

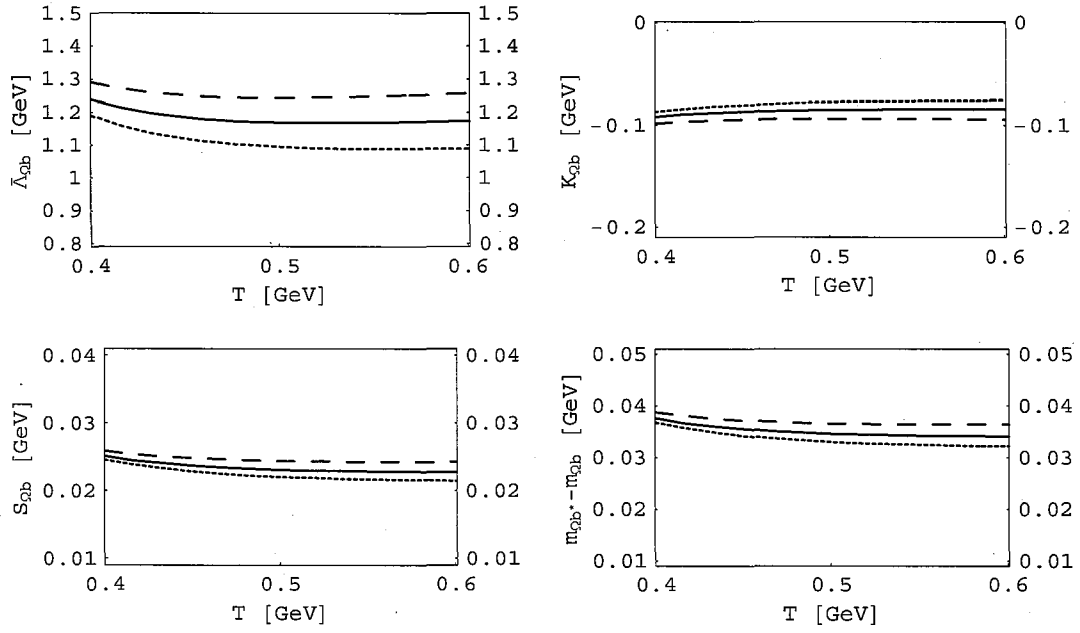# Summary and Outlook

Using the method of QCD sum rule, we have systematically studied many exotic hadrons:

1. light scalar mesons: $\sigma(600)$, $\kappa(800)$, $a_0(980)$ and $f_0(980)$. They have quantum numbers $J^{PC} = 0^{++}$. In the conventional quark model, it is difficult to explain many of their properties by using the $\bar{q}q$ structure. By using the QCD sum rule, we find it is more convenient to interpret them as tetraquark states, while the $\bar{q}q$ scalar meson have a mass around 1.2 GeV, which is considerably heavier.

2. $Y(2175)$. It has quantum numbers $J^{PC} = 1^{--}$, and was observed near the threshold in the process $e^+e^- \rightarrow \phi f_0(980)$ via initial-state radiation. By using the QCD sum rule, we find it can be interpreted as a $ss\bar{s}\bar{s}$ tetraquark state.

3. $\pi_1(1400)$, $\pi_1(1600)$ and $\pi_1(2000)$. They have quantum numbers $I^G J^{PC} = 1^- 1^{-+}$, which $\bar{q}q$ mesons can not access. By using QCD sum rule, we find that $\pi_1(1600)$ and $\pi_1(2000)$ can be interpreted as tetraquark states with quark contents $qq\bar{q}\bar{q}$ and $qs\bar{q}\bar{s}$, respectively. While $\pi_1(1400)$ may be interpreted as a hybrid state.

To study these hadrons, first we do a systematical study on the independent currents, which may couple to these states. This is the first part of our thesis, containing the classification of baryon currents and tetraquark currents. We find this step is very important because there are always more than one currents for each exotic hadrons, and it is important to choose the right one in order to perform a reliable QCD sum rule. Then, in the second part of our thesis, we do this by using all independent currents, and also by using their linear combinations out to two. For the case of light scalar mesons, this largely improves our discussions. While for the cases of $Y(2175)$ and π_1s, all the independent currents lead to the similar results. So does their mixing. We find that this may be due to the similar chiral properties of these different single currents. At last, we do a systematical study on bottom baryons.

During our studies, we find that there are still many things not clear, and our QCD sum rule analysis needs some improvements. We would like to note on the following points:

1. There is a large contribution from the continuum for some exotic hadrons. For example, for the case of $\sigma(600)$, it has a mass larger than the two-pion threshold, and the two-pion contribution should be very large. This is also a difficult question for many other theories when used to study exotic hadrons.

2. The pole contribution is not large enough sometimes. When we study the exotic hadrons, we always meet this problem. This is also related with the first point: the large continuum contribution makes the pole rather small.

3. The relation between currents and states are not so clear. There may be more than one currents coupling to the same state, and one current may also couple to many different states. The current contains quark and gluon fields which are the basic objects of QCD. However, at the low energy region, the degrees of freedom of QCD are hadron states other than quarks and gluons. For exotic hadrons it is difficult to relate these states with the underlying quarks and gluons. Therefore, unclear relation between currents and states is reasonable, while at the same time very interesting.

4. In our studies, the mixed angle is determined by using a try and error process. We just find that the certain mixed angle leads to a good result. However, there may be some intrinsic limitations on this mixed angle, which is a interesting subject.

5. The internal structure of exotic hadrons is interesting. In our QCD sum rule analysis, we find that the diquark-antidiquark $((qq)(\bar{q}\bar{q}))$ construction and meson-meson $((\bar{q}q)(\bar{q}q))$ construction are equivalent by using the Fierz transformation. However, they can be different, and be studied by using other theories.

To end this thesis, we would like to note that we still have many things to study about exotic hadrons. It is important and interesting to study these exotic objects in order to know the non-perturbative nature of QCD for hadron physics.

Appendix A

Calculation of OPE Using Mathematica

A.1 Calculation of OPE Using Mathematica

In this appendix, we introduce the calculation of operator product expansion (OPE) using Mathematica. First we need to install *Mathematica* and a Mathematica package named *FeynCalc*. It can be downloaded at http://www.feyncalc.org.

Take the current P_1 as an example

$$\eta = (\bar{s}_a \gamma_5 u_a)(\bar{s}_b \gamma_5 d_b) \,. \tag{A.1}$$

What we want to calculate is the correlation functions

$$\Pi(q^2) \equiv i \int d^4x e^{iqx} \langle 0|T\eta(x)\eta^\dagger(0)|0\rangle \,. \tag{A.2}$$

Substituting Eq. (A.1) into Eq. (A.2), and contracting quark fields

$$
\begin{aligned}
\langle 0|T\eta(x)\eta^\dagger(0)|0\rangle &= \mathrm{Tr}[iS_s^{a'a}(-x)\gamma_5 iS_u^{aa'}(x)\gamma_5] \times \mathrm{Tr}[iS_s^{b'b}(-x)\gamma_5 iS_d^{bb'}(x)\gamma_5] \\
&\quad -\mathrm{Tr}[iS_s^{b'a}(-x)\gamma_5 iS_u^{aa'}(x)\gamma_5 iS_s^{a'b}(-x)\gamma_5 iS_d^{bb'}(x)\gamma_5] \,, \tag{A.3}
\end{aligned}
$$

where

$$
\begin{aligned}
iS_q^{ab}(x) &\equiv \langle 0|T[q^a(x)\bar{q}^b(0)]|0\rangle \\
&= \frac{i\delta^{ab}}{2\pi^2 x^4}\hat{x} + \frac{i}{32\pi^2}\frac{\lambda_{ab}^n}{2}\mathrm{g}_c\mathrm{G}_{\mu\nu}^n\frac{1}{x^2}(\sigma^{\mu\nu}\hat{x} + \hat{x}\sigma^{\mu\nu}) - \frac{\delta^{ab}}{12}\langle\bar{q}q\rangle + \frac{\delta^{ab}x^2}{192}\langle g_c\bar{q}\sigma Gq\rangle \\
&\quad - \frac{\delta^{ab}m_q}{4\pi^2 x^2} + \frac{i\delta^{ab}m_q}{48}\langle\bar{q}q\rangle\hat{x} + \frac{i\delta^{ab}m_q^2}{8\pi^2 x^2}\hat{x} \,. \tag{A.4}
\end{aligned}
$$

Then we need to substitute the quark propagator Eq. (A.4) into Eq. (A.3). We divide it into three parts:

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1. δ^{ab} part. Gluon part is emitted, and we only consider color matrix δ^{ab}. The lowest term is the continuum term.

2. Two λ_{ab} part. We only consider gluon part in the two quark propagators. We only need to consider color matrix λ. The lowest condensate is $\langle g^2 G^2 \rangle$.

3. One λ_{ab} part. We consider gluon part in one quark propagator, and non-gluon part in the other quark propagator. The lowest condensates are $\langle g\bar{q}\sigma Gq \rangle$ and $\langle g\bar{s}\sigma Gs \rangle$.

A.1.1 δ^{ab} Part

In quark propagator, a lot of terms have color structure δ^{ab}. These parts can be computed together and lead to the continuum contribution and condensates

$$O_4 = m_s \langle \bar{q}q \rangle, O_6 = \langle \bar{q}q \rangle^2 , \; etc. \tag{A.5}$$

We need some definitions:

$$
\begin{aligned}
iS^+(a,b) &= \frac{i\delta^{ab}}{2\pi^2 x^4}\hat{x} - \frac{\delta^{ab}}{12}\langle \bar{q}q \rangle + \frac{\delta^{ab}x^2}{192}\langle g_c\bar{q}\sigma Gq \rangle , \\
iSS^+(a,b) &= \frac{i\delta^{ab}}{2\pi^2 x^4}\hat{x} - \frac{\delta^{ab}}{12}\langle \bar{s}s \rangle + \frac{\delta^{ab}x^2}{192}\langle g_c\bar{s}\sigma Gs \rangle - \frac{\delta^{ab}m_s}{4\pi^2 x^2} + \frac{i\delta^{ab}m_s}{48}\langle \bar{s}s \rangle\hat{x} , \\
iCSC^+(a,b) &= \frac{i\delta^{ab}}{2\pi^2 x^4}\hat{x} + \frac{\delta^{ab}}{12}\langle \bar{q}q \rangle - \frac{\delta^{ab}x^2}{192}\langle g_c\bar{q}\sigma Gq \rangle , \\
iCSSC^+(a,b) &= \frac{i\delta^{ab}}{2\pi^2 x^4}\hat{x} + \frac{\delta^{ab}}{12}\langle \bar{s}s \rangle - \frac{\delta^{ab}x^2}{192}\langle g_c\bar{s}\sigma Gs \rangle + \frac{\delta^{ab}m_s}{4\pi^2 x^2} + \frac{i\delta^{ab}m_s}{48}\langle \bar{s}s \rangle\hat{x} .
\end{aligned}
$$

Here $iS^+(a,b)$ represents $iS^{ab}_{u,d}(x)$, $iSS^+(a,b)$ represents $iS^{ab}_s(x)$, and $iCSSC^+(a,b)$ represents

$$C \times (iS^{ab}_{u,d}(x))^T \times C , \tag{A.6}$$

where C is the charge-conjugation operator. We can also define $iS^-(a,b)$ to represent $iS^{ab}_{u,d}(-x)$.

δ^{ab} part can be written explicitly in Mathematica,

$$
\begin{aligned}
&\text{Quark Part} = \\
&\text{Tr}[iSS^-(a2,a1)\gamma_5 iS^+(a1,a2)\gamma_5] \times \text{Tr}[iSS^-(b2,b1)\gamma_5 iS^+(b1,b2)\gamma_5] \\
&-\text{Tr}[iSS^-(b2,a1)\gamma_5 iS^+(a1,a2)\gamma_5 iSS^-(a2,b1)\gamma_5 iS^+(b1,b2)\gamma_5] .
\end{aligned} \tag{A.7}
$$

Use Mathematica to compute it, and sum color indices,

$$\sum_{a1=1}^{3}\sum_{a2=1}^{3}\sum_{b1=1}^{3}\sum_{b2=1}^{3} \% . \tag{A.8}$$

After using functions "DiracSimplify" and "Expand", finally we obtain the results of δ^{ab} part. There are a lot of terms, and we only choose necessary ones that have a lower dimension.

A.1.2 Two λ Part

When writing propagators in the previous subsection, the gluon part is

$$\frac{i}{32\pi^2}\frac{\lambda_{ab}^n}{2}g_c\mathrm{G}_{\mu\nu}^n\frac{1}{x^2}(\sigma^{\mu\nu}\hat{x}+\hat{x}\sigma^{\mu\nu})=-\frac{i}{16\pi^2x^2}\frac{\lambda_{ab}^n}{2}\gamma_\beta\gamma_5\times g_c\mathrm{G}_{\mu\nu}^n\epsilon^{\mu\nu\sigma\beta}x_\sigma\,. \tag{A.9}$$

For computing two λ part, the definition of λ matrices is needed, and also some more definitions:

$$\mathrm{iSG}^+(a,b,\beta)\;=\;-\frac{i}{16\pi^2x^2}\frac{\lambda_{ab}^n}{2}\gamma_\beta\gamma_5$$
$$\dots$$

Here we have taken $g_c\mathrm{G}_{\mu\nu}^n\epsilon^{\mu\nu\sigma\beta}x_\sigma$ out.

Then We write $\langle 0|T\eta(x)\eta^\dagger(0)|0\rangle$ with two λ explicitly in Mathematica. Because every two propagators together can contribute a two-gluon condensate $\langle g^2G^2\rangle$, every term in the previous subsection is separated into six terms.

> Gluon Part $=$
>
> $\mathrm{Tr}[i\mathrm{SSG}^-(a2,a1,\beta1)\gamma_5 i\mathrm{SG}^+(a1,a2,\beta2)\gamma_5]\times\mathrm{Tr}[i\mathrm{SS}^-(b2,b1)\gamma_5 i\mathrm{S}^+(b1,b2)\gamma_5]$
>
> $+\mathrm{Tr}[i\mathrm{SSG}^-(a2,a1,\beta1)\gamma_5 i\mathrm{S}^+(a1,a2)\gamma_5]\times\mathrm{Tr}[i\mathrm{SSG}^-(b2,b1,\beta2)\gamma_5 i\mathrm{S}^+(b1,b2)\gamma_5]$
>
> $\mathrm{Tr}[i\mathrm{SSG}^-(a2,a1,\beta1)\gamma_5 i\mathrm{S}^+(a1,a2,\beta2)\gamma_5]\times\mathrm{Tr}[i\mathrm{SS}^-(b2,b1)\gamma_5 i\mathrm{SG}^+(b1,b2)\gamma_5]$
>
> $\mathrm{Tr}[i\mathrm{SS}^-(a2,a1,\beta1)\gamma_5 i\mathrm{SG}^+(a1,a2,\beta2)\gamma_5]\times\mathrm{Tr}[i\mathrm{SSG}^-(b2,b1)\gamma_5 i\mathrm{S}^+(b1,b2)\gamma_5]$
>
> $\mathrm{Tr}[i\mathrm{SS}^-(a2,a1,\beta1)\gamma_5 i\mathrm{SG}^+(a1,a2,\beta2)\gamma_5]\times\mathrm{Tr}[i\mathrm{SS}^-(b2,b1)\gamma_5 i\mathrm{SG}^+(b1,b2)\gamma_5]$
>
> $\mathrm{Tr}[i\mathrm{SS}^-(a2,a1,\beta1)\gamma_5 i\mathrm{S}^+(a1,a2,\beta2)\gamma_5]\times\mathrm{Tr}[i\mathrm{SSG}^-(b2,b1)\gamma_5 i\mathrm{SG}^+(b1,b2)\gamma_5]\,.$

We should add the parts which we have taken away (together two $g_c\mathrm{G}_{\mu\nu}^n\epsilon^{\mu\nu\sigma\beta}x_\sigma$). It is

$$g_c\mathrm{G}_{\mu1\nu1}^n\epsilon^{\mu1\nu1\sigma1\beta1}x_{\sigma1}\times g_c\mathrm{G}_{\mu2\nu2}^m\epsilon^{\mu2\nu2\sigma2\beta2}x_{\sigma2}$$
$$=\;\frac{\langle g_c^2\mathrm{G}^2\rangle}{24}\delta^{nm}(x_{\beta1}x_{\beta2}-g_{\beta1\beta2}x^2)\,, \tag{A.10}$$

Here we have already used the condition $n=m$ (δ^{nm}) when writing the Gluon Part.

Use Mathematica to compute (A.10)$\times$(A.10), then use the function "Contract" to reduce redundant indices, do summing in color space, use the functions "DiracSimplify" and "Expand" to simplify them, finally we get the results of step 2.

A.1.3   One λ Part

One gluon and a quark-antiquark pair can form a mixed condensate:

$$\langle g_c\bar{q}\sigma Gq\rangle = \langle g_c\bar{q}_a\sigma_{\mu\nu}\frac{\lambda^n_{ab}}{2}G^{n\mu\nu}q_b\rangle\,. \tag{A.11}$$

In this part we need to change one quark propagator into $q^a\bar{q}^b$. For the other propagator, we will choose the gluon part $\frac{i}{32\pi^2}\frac{\lambda^n_{ab}}{2}g_cG^n_{\mu\nu}\frac{1}{x^2}(\sigma^{\mu\nu}\hat{x} + \hat{x}\sigma^{\mu\nu})$. More definitions are needed. Pay attention that the definition in step 3 is inconsistent with step 2, so we need to compute them separately,

$$\begin{aligned}
iSG^+(a,b) &= -\frac{i}{16\pi^2x^2}\frac{\lambda^n_{ab}}{2}\gamma_{\beta3}\gamma_5 x_{\alpha3}\epsilon^{\alpha3\beta3\alpha4\beta4}\,,\\
iSQG^+(a,b) &= -\frac{1}{192}\frac{\lambda^n_{ab}}{2}\sigma_{\alpha4\beta4}\langle g_c\bar{q}\sigma Gq\rangle
\end{aligned}$$

$$\cdots$$

In these definitions, we have substituted the $g_cG^n_{\mu\nu}$ part of $iSG^+(a,b)$ (which contributes a gluon) into $iSQG^+(a,b)$ (which contributes a $\bar{q}q$ or $\bar{s}s$ pair).

We write the $\langle 0|T\eta(x)\eta^\dagger(0)|0\rangle$ of one λ part explicitly in Mathematica. Every term in the previous subsection is separated into two parts again (so twenty four terms in all):

> Quark-Gluon Part =
> $\mathrm{Tr}[iSSQG^-(a2,a1)\gamma_5 iSG^+(a1,a2)\gamma_5] \times \mathrm{Tr}[iSS^-(b2,b1)\gamma_5 iS^+(b1,b2)\gamma_5]$
> $+\mathrm{Tr}[iSSG^-(a2,a1)\gamma_5 iSQG^+(a1,a2)\gamma_5] \times \mathrm{Tr}[iSS^-(b2,b1)\gamma_5 iS^+(b1,b2)\gamma_5]$
> $+\cdots\,.$

Use Mathematica to compute (A.12), use the function "Contract" to reduce redundant indices, do summing in color space, use the functions "DiracSimplify" and "Expand" to simplify them, finally we get the results of step 3.

A.2   Fourier Transformation and Borel Transformation

After step 1, 2 and 3, we can sum 3 parts together, and get the final $\langle 0|T\eta(x)\eta^+(0)|0\rangle$. To do the Fourier Transformation, we use the formulae:

$$\frac{1}{x^{2n}} \rightarrow \frac{-1 \times n\pi^2 2^{4-2n}p^{2*n-4}}{(n-1)!(n-2)!}ln(-p^2) \text{ For } n \geq 2\,, \tag{A.12}$$

This can also be done by Mathematica easily.

Important Borel transforms include:

$$B[(\frac{1}{p^2 - \alpha})^\beta] = (-1)^\beta (M)^{2-2\beta} \frac{1}{(\beta - 1)!} e^{-\alpha/M^2} \,,$$

$$B[(p^2)^m ln(\frac{1}{-p^2})] = m!(M^2)^{m+1} \,,$$

$$B[(p^2)^m] = 0, \text{m a non-negative integer} \,, \qquad\qquad (A.13)$$

$$B[f(p^2)] = \frac{1}{\pi} \int_0^\infty ds \mathrm{Im} f(s) e^{-s/M^2} \,.$$

Appendix B

Fierz Transformation

In this appendix, we list the Fierz transformations used in our calculation. Here we would like to show only the change in the structure of Lorentz indices of direct products of two Dirac matrices under the Fierz rearrangement. Therefore, in the following equations, we do not include the minus sign which arises from the exchange of quark fields. The formulae go for the three cases corresponding to the Dirac, Rarita-Schwinger and tensor fields when applied to three-quark fields.

1. Products of two Dirac matrices without Lorentz indices:

$$
\begin{pmatrix}
\mathbf{1} \otimes \gamma_5 \\
\gamma_\mu \otimes \gamma^\mu \gamma_5 \\
\sigma_{\mu\nu} \otimes \sigma^{\mu\nu} \gamma_5 \\
\gamma_\mu \gamma_5 \otimes \gamma^\mu \\
\gamma_5 \otimes \mathbf{1}
\end{pmatrix}_{ab,cd}
=
\begin{pmatrix}
\frac{1}{4} & -\frac{1}{4} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} \\
-1 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \\
3 & 0 & -\frac{1}{2} & 0 & 3 \\
1 & -\frac{1}{2} & 0 & -\frac{1}{2} & -1 \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{8} & -\frac{1}{4} & \frac{1}{4}
\end{pmatrix}
\begin{pmatrix}
\mathbf{1} \otimes \gamma_5 \\
\gamma_\mu \otimes \gamma^\mu \gamma_5 \\
\sigma_{\mu\nu} \otimes \sigma^{\mu\nu} \gamma_5 \\
\gamma_\mu \gamma_5 \otimes \gamma^\mu \\
\gamma_5 \otimes \mathbf{1}
\end{pmatrix}_{ad,bc}
\tag{B.1}
$$

2. Products of two Dirac matrices with one Lorentz index:

$$
\begin{pmatrix}
\mathbf{1} \otimes \gamma^\mu \\
\gamma^\mu \otimes \mathbf{1} \\
\gamma_5 \otimes \gamma_\mu \gamma_5 \\
\gamma_\mu \gamma_5 \otimes \gamma_5 \\
\gamma^\nu \otimes \sigma_{\mu\nu} \\
\sigma_{\mu\nu} \otimes \gamma^\nu \\
\gamma^\nu \gamma_5 \otimes \sigma_{\mu\nu} \gamma_5 \\
\sigma_{\mu\nu} \gamma_5 \otimes \gamma^\nu \gamma_5
\end{pmatrix}_{ab,cd}
=
\tag{B.2}
$$

$$
\begin{pmatrix}
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{i}{4} & \frac{i}{4} & \frac{i}{4} & \frac{i}{4} \\
\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{i}{4} & -\frac{i}{4} & \frac{i}{4} & \frac{i}{4} \\
\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{i}{4} & \frac{i}{4} & \frac{i}{4} & -\frac{i}{4} \\
-\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{i}{4} & \frac{i}{4} & \frac{i}{4} & -\frac{i}{4} \\
\frac{3i}{4} & -\frac{3i}{4} & -\frac{3i}{4} & -\frac{3i}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\
-\frac{3i}{4} & \frac{3i}{4} & -\frac{3i}{4} & -\frac{3i}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\
-\frac{3i}{4} & -\frac{3i}{4} & \frac{3i}{4} & -\frac{3i}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\
-\frac{3i}{4} & -\frac{3i}{4} & -\frac{3i}{4} & \frac{3i}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4}
\end{pmatrix}
\begin{pmatrix}
\mathbf{1} \otimes \gamma^\mu \\
\gamma^\mu \otimes \mathbf{1} \\
\gamma_5 \otimes \gamma_\mu \gamma_5 \\
\gamma_\mu \gamma_5 \otimes \gamma_5 \\
\gamma^\nu \otimes \sigma_{\mu\nu} \\
\sigma_{\mu\nu} \otimes \gamma^\nu \\
\gamma^\nu \gamma_5 \otimes \sigma_{\mu\nu} \gamma_5 \\
\sigma_{\mu\nu} \gamma_5 \otimes \gamma^\nu \gamma_5
\end{pmatrix}_{ad,bc}
$$

3. Products of two Dirac matrices with two anti-symmetric Lorentz indices:

$$
\begin{pmatrix}
\mathbf{1} \otimes \sigma_{\mu\nu} \gamma_5 \\
\gamma_5 \otimes \sigma_{\mu\nu} \\
\sigma_{\mu\nu} \otimes \gamma_5 \\
\sigma_{\mu\nu} \gamma_5 \otimes \mathbf{1} \\
\epsilon_{\mu\nu\rho\sigma} \sigma_{\rho l} \otimes \sigma_{\sigma l} \\
\gamma_\mu \otimes \gamma_\nu \gamma_5 - (\mu \leftrightarrow \nu) \\
\gamma_\mu \gamma_5 \otimes \gamma_\nu - (\mu \leftrightarrow \nu) \\
\epsilon_{\mu\nu\rho\sigma} \gamma_\rho \otimes \gamma_\sigma \\
\epsilon_{\mu\nu\rho\sigma} \gamma_\rho \gamma_5 \otimes \gamma_\sigma \gamma_5
\end{pmatrix}_{ab,cd}
=
\tag{B.3}
$$

$$
\begin{pmatrix}
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{i}{4} & -\frac{i}{4} & \frac{1}{4} & -\frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{i}{4} & \frac{i}{4} & -\frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{i}{4} & \frac{i}{4} & \frac{1}{4} & -\frac{1}{4} \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{i}{4} & -\frac{i}{4} & -\frac{1}{4} & \frac{1}{4} \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-\frac{i}{2} & \frac{i}{2} & \frac{i}{2} & -\frac{i}{2} & 0 & 0 & 0 & \frac{i}{2} & \frac{i}{2} \\
\frac{i}{2} & -\frac{i}{2} & -\frac{i}{2} & \frac{i}{2} & 0 & 0 & 0 & \frac{i}{2} & \frac{i}{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & -\frac{i}{2} & -\frac{i}{2} & 0 & 0 \\
-\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & -\frac{i}{2} & -\frac{i}{2} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\mathbf{1} \otimes \sigma_{\mu\nu} \gamma_5 \\
\gamma_5 \otimes \sigma_{\mu\nu} \\
\sigma_{\mu\nu} \otimes \gamma_5 \\
\sigma_{\mu\nu} \gamma_5 \otimes \mathbf{1} \\
\epsilon_{\mu\nu\rho\sigma} \sigma_{\rho l} \otimes \sigma_{\sigma l} \\
\gamma_\mu \otimes \gamma_\nu \gamma_5 - (\mu \leftrightarrow \nu) \\
\gamma_\mu \gamma_5 \otimes \gamma_\nu - (\mu \leftrightarrow \nu) \\
\epsilon_{\mu\nu\rho\sigma} \gamma_\rho \otimes \gamma_\sigma \\
\epsilon_{\mu\nu\rho\sigma} \gamma_\rho \gamma_5 \otimes \gamma_\sigma \gamma_5
\end{pmatrix}_{ad,bc}
$$

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