Holographic Description of Black Hole Space-time

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by

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ブラックホール時空の
ホログラフィックな記述について

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Contents

Preface

In the present thesis, I want to summarize the main results of my research and work of the past three years on high energy physics, in particular, on *black hole* as quantum theory of gravity, while adding my motivation and the whole theoretical background in a self-contained form. I hope that the fruit of my research would help other researchers as well as myself study gravity theory in further detail and throw more light on an abyss of this deep and enigmatic black hole physics.

Overview

Black hole in general theory of relativity is of interest and significant not only because it is an unique astronomical object in our universe but because it offers a tool of a survey of the fundamental quantum theory beyond Standard Model. It is especially remarkable that physical quantities of the black hole, such as the mass, angular momentum and charges, seem to obey a thermodynamical relation. Above all, the area of the event horizon of the black hole can be seen as *entropy* in the thermodynamical behaviour [1, 2]. This suggestion by Bekenstein and Hawking may imply a quantum aspect of the fundamental theory of gravity, if this entropy is reproduced from the microscopic counting of some states, just as we once found the statistical viewpoint of particles as quantum mechanics behind the thermodynamics.

String theory is one of those which consistently include quantum interaction of gravitons. A great success is that the Bekenstein-Hawking entropy defined from general relativity can completely be realized in terms of the statistical counting of string states on the so-called D-branes [3], which represent the black hole or black brane in (higher-dimensional) gravity theory while provide a non-purterbative picture of string [4]. Thus, a lot of efforts have been devoted to a full understanding of string theory in the recent decades.

However, we would here like to take the liberty of reconsidering whether string theory is really essencial as quantum theory of gravity, particularly, as the one which reproduces the Bekenstein-Hawking formula from the microscopic standpoint. This is my motivation of research and a main subject of this thesis. In the thesis I want to conclude that *string theory is not necessarily required, at least, for a microscopic reproduction of the black hole entropy.*

The key idea is the AdS/CFT correspondence rather than string theory itself. This is a conjecture saying that gravity theory on anti-de Sitter (AdS) space is the same as conformal field theory (CFT) on the lower dimensional boundary of this AdS [5]. It was first claimed in string context by Maldacena, and is based on the duality between closed string and open string for D-branes. For examples, it is considered that the D3-brane (an object extended in 1+3 dimensions) admits the $AdS_5 \times S^5$ solution in type IIB supergravity while 1+3 dimensional CFT resides on the boundary as $N = 4$ super Yang-Mills gauge theory.

But this concept has actually been known in three dimensional gravity for a long time. By using the AdS_3 geometry in three dimensional gravity with the negative cosmological constant [6], Brown and Henneaux showed that there exists the Virasoro algebra on the AdS_3 boundary [7]. Namely, the Poisson bracket algebra of the Hamiltonian of gravity becomes equivalent to the algebra characterizing two dimensional CFT, irrelevantly to supersymmetry or string theory, and its central charge is expressed by parameters of gravity. Once the Virasoro algebra is confirmed and the central charge is obtained, we can calculate a quantum degeneracy of this CFT_2 by the Cardy formula [8], whether we specify a precise expression of this CFT_2 or not. On the other hand, Banados-Teitelboim-Zanelli (BTZ) black hole is known as one of the AdS_3 geometries [9], and its entropy is given by the Bekenstein-Hawking formula. Then these two expressions of the entropy agree with each other [10]. This was the first example of the AdS_3/CFT_2 correspondence.

The application of this Brown-Henneaux's approach is not limited to three dimensional gravity. Guica, Hartman, Song and Strominger recently applied it to the four dimensional extremal (the most stable state in mass) Kerr black holes and obtained the Virasoro algebra on the AdS geometry of the event horizon [11].[1] The macroscopic and microscopic entropies are still in a complete agreement between the Bekenstein-Hawking and the Cardy formula. The Kerr black hole is observed in our universe, so it is a great step for us to uncover the microscopic interpritation of the real black hole. And in fact, any extremal black holes generally have the AdS behaviour near the horizon, which was mathmatically proven [12, 13]. That is to say, it is not impossible to understand and describe the AdS near horizon geometry of extremal black holes as a certain CFT. From the above reasons, the AdS/CFT is considered to be rather essencial.

There is another universal feature of the extremal black holes, called *attractor mechanism* [14, 15, 16]. In general gravity coupled to scalar fields and gauge fields, the solution of the scalars to equations of motion displays a characteristic behaviour. The scalar solutions have degrees of freedom of integration constants corresponding to the values at the asymptotic infinity. But, whatever value they are given, their values at the horizon are always fixed by the black hole (electric and magnetic) charges. As a result, the entropy, which is generally a function of the horizon values of the scalars in matter-coupled gravity, is given by only charges. This mechanism was first found in a BPS black hole solution of the four dimensional $N = 2$ supergravity, but afterward it was confirmed that it is a unique nature arising from not supersymmetry but extremality of black holes [17, 18, 19].

The horizon is so special and becomes AdS in extremal case, as mentioned above, but the attractor makes us believe that there is a microscopic understanding of black holes *not only on the horizon but also as a whole space-time.* In string context, this perspective is also known as the gauge/gravity duality, which means a wider correspondence, rather than AdS/CFT, between gravity theory in the bulk and gauge theory on the boundary. Gubser, Klebanov and Polyakov and Witten claimed that the classical action of bulk gravity should be regarded as the generating functional of the boundary field theory [20, 21]. In other words, at each position of the radial coordinate of bulk gravity, the lower dimensional (non-conformal) quantum field theory (QFT) is realized on the surface. According to this idea,

[1]Strictly speaking, it is not an exact AdS_3 but a *warped* AdS_3. Their discovery is that the Virasoro algebra is, nevertheless, obtainable.

holography, a change of the energy scale in QFT is related to that of the radial coordinate on the gravity side. The UV or IR region of QFT corresponds to the spatial infinity or horizon in gravity theory. In ref. [22], de Boer, Verlinde and Verlinde showed that the Hamilton-Jacobi equation for the bulk gravity implies the Callan-Symanzik equation for the dual QFT on the surface of the fixed radial coordinate. The scalar fields can be identified with running couplings if the radial coordinate of the bulk can be seen as the cut-off scale for the dual QFT, and flows of their solutions in the bulk are understood as the holographic renormalization group (RG) flow [23, 24, 22, 25]. Also, this was originally motivated by string theory, but should be applicable to any gravity theory.

At this point one may notice that the attractor flow is similar to this RG flow, and as a matter of fact, they are equivalent since the scalar fields become the running couplings in the holographic duality. The mechanism is not merely an interesting behaviour of the black hole solutions but a quite significant clue to imply the existence of the underlying QFT in the whole space-time. Conversely, one can believe in the dual QFT obeying a RG flow equation whenever the attractor works. Our hope is to investigate the quantum aspects of gravity as the boundary field theory which lives in the full black hole space-time, like the Kerr space-time in real world, with the help of this idea of the holographic RG flow.

Organization of my thesis

On the basis of the above background, in my thesis I focus myself on (I) the black hole thermodynamics, especially the entropy and the attractor mechanism, (II) the AdS/CFT correspondence in black holes with the use of Brown-Henneaux's canonical approach, (III) the search of the RG flow of field theory holographically dual to the full black hole space-time based on the Hamilton-Jacobi formalism.

(I) In Chapter 1, we introduce a notion of the black hole entropy and its more general formula proposed by Wald [26], which is useful even in the case where the gravity theory includes any higher derivative correction. In Chapter 2, we illustrate the attractor behaviour of extremal black holes both in a general observation not depending on supersymmetry and in explicit analytic solutions to equations of motion in $N = 2$ supergravity setup.

(II) Chapter 3 is a review of Brown-Henneaux's statement on the AdS_3/CFT_2 correspondence and its application to the four dimensional Kerr black hole. In Chapter 4, we confirm that the AdS_3/CFT_2 correspondence is truely valid even with higher derivative corrections, in particular we use the most difficult theory of gravity to deal with, called Topologically Massive Gravity (TMG) [27].

(III) In Chapter 5, we first review Zamolodchikov's c-theorem which is an universal property of general two dimensional field theories [28], and next demonstrate how it can be observed by the Hamilton-Jacobi equation from the gravity side. In Chapter 6, various three dimensional gravity theories are presented as examples to see the RG flow of the corresponding characteristic QFT_2s. In Chapter 7, all results of this thesis are summarized.

In Appendix A, we show a technique of constructing any black brane solutions in gravity theory with the negative cosmological constant.

Main results of my work

Finally, I list up the major results of my work in the past. This thesis is mainly based on the following references.

In ref. [29], the attractor mechanism for the four dimensional $N = 2$ supergravity black hole solution was analyzed in a system where D0 and D4 branes are wrapping a compact Calabi-Yau threefold. We newly derived exact solutions, which explicitly exhibit the attractor mechanism for extremal non-BPS black holes. Our solutions account for the moduli as general complex fields, while in almost all non-BPS solutions had obtained previously, the moduli fields are restricted to be purely imaginary, thereby assuming no axions. Then our solutions contain an extra parameter corresponding to the asymptotic value of the axions.

In ref. [30], we studied the symmetry realized asymptotically on the two dimensional boundary of AdS_3 geometry in TMG, which consists of the gravitational Chern-Simons (GCS) term as well as the usual Einstein-Hilbert and negative cosmological constant terms. Our analysis was based on the conventional canonical method and proceeded along the line completely parallel to the original Brown and Henneaux's. In spite of the presence of the GCS term, it was confirmed by the canonical method that the boundary theory actually has the conformal symmetry satisfying the left and right moving Virasoro algebras. The central charges of the Virasoro algebras were computed explicitly and were shown to be left-right asymmetric due to the GCS term. It was also argued that the Cardy formula for the BTZ black hole entropy capturing *all* higher derivative corrections agrees with the extended version of Wald's entropy formula. The M5-brane system (or equivalently to D0-D4 system) was illustrated as an application of the present calculation.

In ref. [31], we presented a new exact black hole solution in three dimensional gravity coupled to a single scalar field. This is one of the extended solutions of the BTZ black hole and has in fact AdS_3 geometries both at the spatial infinity and at the event horizon. An explicit derivation of Virasoro algebras for CFT_2 at the two boundaries was shown to be possible à la Brown and Henneaux's calculation. Moreover, if we regard the scalar field as a running coupling in the dual two dimensional field theory, and its flow in the bulk as the holographic RG flow, our black hole should interpolate the two CFT_2 living at the infinity and at the horizon. Following the Hamilton-Jacobi analysis by de Boer, Verlinde and Verlinde, we calculated the central charges c_{UV} and c_{IR} for the CFT_2 on the infinity and the horizon, respectively. We also confirmed that the inequality $c_{\mathrm{IR}} < c_{\mathrm{UV}}$ is satisfied, which is consistent with Zamolodchikov's c-theorem.

In ref. [32], we extended the discussion of the Kerr/CFT correspondence and its recent developments to the more general gauge/gravity correspondence in the full extremal black hole space-time of the bulk by using a technique of the holographic RG flow. It was conjectured that the extremal black hole space-time is holographically dual to the chiral two dimensional field theory. Our example is a typical four dimensional Reissner-Nordstrøm black hole, the M5-brane configuration. In five dimensional supergravity view point this near horizon geometry is $AdS_3 \times S^2$, and three dimensional gravity coupled to moduli fields is effectively obtained after a dimensional reduction on S^2. Constructing the Hamilton-Jacobi equation, we defined the holographic RG flow from the three dimensional gravity. The central charge of the Virasoro algebra was calculable from the conformal anomaly at the point where the beta function defined from gravity side becomes zero. We showed that these flow equations are completely equivalent to not only BPS but also non-BPS attractor flow equations of the muduli fields. The attractor mechanism can be understood equivalently to the fact that the RG flows are fixed at the critical points in the dual QFT.

In ref. [33], we considered the holographic RG flow in three dimensional gravity with the GCS term coupled to some scalar fields. We applied the canonical approach to this higher derivative case and employ the Hamilton-Jacobi formalism to analyze the flow equations of two dimensional field theory. Especially we obtained flow equations of Weyl and gravitational anomalies, and derived c-functions for left and right moving modes. Both of them are monotonically non-increasing along the flow, and the difference between them is determined by the coefficient of the GCS term. This is completely consistent with the c-theorem for parity-violating QFT_2.

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Part I

Black Hole Thermodynamics

In this part of the thesis we show some thermodynamical aspects of black holes, especially the black hole entropy. It is a purpose of later parts to give some understanding of these macroscopic facts of gravity from microscopic view points.

In general relativity it is known that there are thermodynamical behaviours among the physical quantities of black holes, in particalar the area of the event horizon can be seen as the entropy. This Bekenstein-Hawking entropy is generalized to the form for the higher derivative gravity by Wald and other authers [26, 34, 35, 36, 37, 38, 39]. Chaprter 1 is devoted to this review.

In matter-coupled gravity, like supergravity, the black hole entropy given by the Wald formula is expressed by charges and values of scalars at the horizon, but it is known in extremal case that these values are attracted to the ones which are completely determined by the charges. This behaviour, the attractor mechanism, was originally claimed in the black hole solution of $N = 2$ supergravity [14, 15, 16], but has been confirmed in various setups even without supersymmetry [18, 19] and with higher derivative corrections [17]. In Chapter 2, we present a brief explanation for this and show that exact solutions actually behave as expected.

And it is also known that there is another important nature of extremal black holes. It was proven that the near horizon geometry of the extremal black holes always has the AdS_2 factor even if any higher derivative corrections are included in the action [12, 13]. Namely, the most important thing to determine the extremal black hole entropy is not supersymmetry but the above two conditions, the attractor mechanism and the AdS_2 geometry near the horizon. This fact implies the existence of the CFT on the horizon, and will become a clue to uncover microscopic view of black holes.

Anyway, assuming that two conditions are always satisfied, we have a very powerful tool to compute the extremal black hole entropy, called *entropy function formalism* [40]. It is based on the Wald formula, so it enables us to get the thermodynamical entropy of any black hole in any gravity theory. In the last section of this part we introduce it briefly. In the next part we will give some discussions of reproducing the results obtained in this part from the dual side.

1 Black Hole Entropy

1.1 Bekenstein-Hawking Entropy and Extremality

Let us consider a simple Einstein-Maxwell theory in four dimensions

$$\mathcal{I}_{(4)} = \frac{1}{8\pi} \int d^4x \sqrt{-G} \left(\frac{1}{2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \tag{1.1.1}$$

and put an ansatz

$$ds^2 = -e^{2g(r)} dt^2 + e^{2f(r)} dr^2 + r^2 d\Omega_{S^2}^2. \tag{1.1.2}$$

The Newton constant will be recovered below. The gauge field must satisfy

$$\partial_\mu (\sqrt{-G} F^{\mu\nu}) = 0, \qquad \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0, \tag{1.1.3}$$

so we obtain

$$F_{tr} = \frac{e^{g+f}}{r^2} q, \qquad F_{\theta\varphi} = p \sin\theta. \tag{1.1.4}$$

The constants q and p correspond to electric and magnetic charges. Substituting them into the Einstein equation

$$R_{\mu\nu} = -\frac{1}{4} G_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + F_{\mu\rho} F_\nu{}^\rho, \tag{1.1.5}$$

the solution of the metric is

$$e^{2g} = e^{-2f} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \tag{1.1.6}$$

where $Q^2 = q^2 + p^2$. This is the well-known Reissner-Nordstrøm black hole.

It has outer and inner horizons $r_\pm = M \pm \sqrt{M^2 - Q^2}$. The surface gravity

$$\kappa_s = \left[\exp(2g) \frac{\partial g}{\partial r} \right] \Big|_{r_+} = \frac{\sqrt{M^2 - Q^2}}{2M(M + \sqrt{M^2 - Q^2}) - Q^2} \tag{1.1.7}$$

and the area of horizon

$$A = 4\pi r_+^2 = 4\pi \left(M + \sqrt{M^2 - Q^2} \right)^2 \tag{1.1.8}$$

are calculated at $r = r_+$. According to the cosmic censorship hypothesis, we have the mass bound $M \geq |Q|$. In extremal case $M = |Q|$, the surface gravity vanishes and the black hole has the lowest energy.

In the more general Kerr-Newman space-time, we have a relation

$$\delta M = \frac{\kappa_s}{8\pi}\delta A + \mu\delta Q + \Omega\delta J \tag{1.1.9}$$

among the mass M, the charge Q and the angular momentum J. Here Ω is the angular velocity and $\mu = Q/r_+$. This reminds us of the first law of thermodynamics. Therefore, defining the Hawking temperature as $T = \kappa_s/2\pi$, which satisfies the zeroth law because κ_s is constant on $r = r_+$ by its definition,[1] and recovering the dimension by the Newton constant $G_N^{(4)}$,[2] we can identify

$$\mathcal{S} = \frac{A}{4G_N^{(4)}} \tag{1.1.10}$$

as the black hole entropy. The surprising fact that δA is non-negative in any dynamical processes is also known as an analogue of the second law.

There are two comments on this Bekenstein-Hawking formula. The first is a case with higher derivatives. In gravity theory up to second derivative, the entropy is written by the area of the horizon, but in such cases it deviates from that. The more general Wald-Tachikawa formula including these corrections will be mentioned in the next section. The second is a case with scalar fields. In matter-coupled gravity like supergravity, the area also depends on the values of scalars at the horizon, which is a function of the values at the spatial infinity and the charges, namely, $A = A(Q, M, \phi_H) = A(Q, M(\phi_\infty, Q), \phi_H(\phi_\infty, Q))$. However, in extremal case, $M(\phi_\infty, Q)$ and $\phi_H(\phi_\infty, Q)$ become functions of only Q. As a result, the entropy is completely determined by the charges $\mathcal{S} = \mathcal{S}(Q)$. This is the attractor mechanism discussed later.

1.2 Wald-Tachikawa Entropy

In this section we review the general entropy formula for the D-dimensional black hole, which is originally derived by Wald [26] and entended to the case with gravitational Chern-Simons terms by Tachikawa [39]. Their arguments are based on the Noether charge formulation with respect to the diffeomorphism of the theory.

Let us denote $L(\phi)$ as the D-form Lagrangian density, where ϕ means physical fields including the metric. Under the diffeomorphism for the vector field ξ, its variation is

$$\delta_\xi L(\phi) = \pounds_\xi L(\phi) + d\,\Xi_\xi, \tag{1.2.11}$$

where $\pounds_\xi$ is the Lie derivative with respect to ξ. The last term is, in ordinary covariant theories, absent, but arises when the gravitational Chern-Simons term is added to the action in odd dimensions.

On the other hand, the first-order variation of L becomes

$$\delta L = \sum_\phi E_\phi \delta\phi + d\Theta(\phi, \delta\phi), \tag{1.2.12}$$

[1] According to Hawking, the temperature is identified up to the normalization $1/2\pi$.

[2] In the following of this thesis, we will use G_N for the Newton constant in three dimensions. On the other hand, $G_N^{(D)}$ denotes the Newton constant in D-dimensions.

so $E_\phi = 0$ expresses equations of motion. The function Θ is called the symplectic potential. Then we can define a current with respect to the diffeomorphism as

$$J_\xi = \Theta(\phi, \delta_\xi \phi) - \iota_\xi L - \Xi_\xi, \tag{1.2.13}$$

where ι_ξ is the interior product defined by $\iota_\xi \omega = (1/(r-1)!)\xi^\nu \omega_{\nu\mu_2\cdots\mu_r} dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_r}$ for the r-form $\omega = (1/r!)\omega_{\mu_1\cdots\mu_r} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}$. This is conserved up to the equations of motion, i.e., $dJ_\xi \simeq 0$, because of the identity $(d\iota_\xi + \iota_\xi d)\,\omega = \pounds_\xi \omega$ for any r-form ω. It is therefore possible to write down a $(D-2)$-form Q_ξ satisfying

$$J_\xi \simeq dQ_\xi. \tag{1.2.14}$$

In the case of $\Xi_\xi = 0$, the explicit form of the antisymmetric Noether potential $Q_\xi^{\mu\nu}$ is

$$Q_\xi^{\mu\nu} = -\mathcal{L}^{\mu\nu\rho\sigma}\nabla_\rho \xi_\sigma + 2\nabla_\rho \mathcal{L}^{\mu\nu\rho\sigma}\xi_\sigma + (\text{matter}), \tag{1.2.15}$$

where $\mathcal{L}^{\mu\nu\rho\sigma} = \partial\mathcal{L}/\partial R_{\mu\nu\rho\sigma}$. This is of course changed when Ξ_ξ is added, but only the first term of the above is important in the end.

Here let us define Π_ξ by

$$\delta_\xi \Theta = \pounds_\xi \Theta + \Pi_\xi. \tag{1.2.16}$$

Then, from the expression $\delta\delta_\xi L$, we have $d\Pi_\xi \simeq \delta d\,\Xi_\xi$. And it is easy to see that this leads to an equation

$$\Pi_\xi - \delta\,\Xi_\xi \simeq d\Sigma_\xi. \tag{1.2.17}$$

Now we put

$$\mathbf{Q}_\xi = Q_\xi - C_\xi, \tag{1.2.18}$$

where C_ξ satisfies a relation

$$\delta C_\xi = \iota_\xi \Theta + \Sigma_\xi. \tag{1.2.19}$$

According to the paper [26], we can identify the black hole entropy satisfying the first law, $\kappa_s\,\delta\mathcal{S} \simeq \delta M - \Omega\delta J$, as follows:

$$\mathcal{S} = \frac{1}{8G_N^{(D)}\kappa_s}\int_H \mathbf{Q}_\xi, \tag{1.2.20}$$

where the integration is done on the event horizon H. It is the most general formula to compute the black hole entropy. On this integral we can put $\xi_\mu = 0$ and $\nabla_\mu \xi_\nu = \kappa_s \epsilon_{\mu\nu}$, where $\epsilon_{\mu\nu}$ is binormal vector of the horizon normalized by $\epsilon^{\mu\nu}\epsilon_{\mu\nu} = -2$, with the help of a property on the bifurcation surface. The replacement $\xi_\mu = 0$ and $\nabla_\mu \xi_\nu = \kappa_s \epsilon_{\mu\nu}$ are true on the bifurcation surface where the horizontal Killing vector field vanishes, but in general the event horizon of the black holes is not the bifurcation surface. Hence, the analytic continuation from the bifurcation surface is needed, and then we can carry out the above replacement on the integral (1.2.20).

In $\Xi_\xi = 0$ case, it is reduced to the form

$$\mathcal{S} = -\frac{1}{8G_N^{(D)}}\int_H d^{D-2}x\sqrt{h}\frac{\partial\mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}, \tag{1.2.21}$$

where $d^{D-2}x\sqrt{h}$ gives the invariant surface element on the horizon. This form says that only the curvature terms in the Lagrangian contribute to the thermodynamical entropy.

Of course it becomes equal to the Bekenstein-Hawking formula (1.1.10) in the case of the Einstein gravity $\mathcal{L} = R$. Furthermore, when we add the gravitational Chern-Simons term in three dimensions

$$\mathcal{L}_{CS} = \frac{\beta}{2}\sqrt{-G}\epsilon^{\mu\nu\rho}\left(\Gamma^{\sigma}_{\mu\lambda}\partial_{\nu}\Gamma^{\lambda}_{\ \rho\sigma} + \frac{2}{3}\Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\nu\theta}\Gamma^{\theta}_{\ \rho\sigma}\right), \tag{1.2.22}$$

we have the correction to the entropy

$$\Delta\mathcal{S} = \frac{\beta}{4G_N}\int_H \epsilon^{\mu}{}_{\nu}\Gamma^{\nu}_{\mu\rho}dx^{\rho}, \tag{1.2.23}$$

by following steps to compute Ξ_ξ, Π_ξ, Σ_ξ and C_ξ. This expression of the correction in three dimensions was also obtained in various setups [36, 37, 38].

This thermodynamical entropy is of course a formal expression, and indeed, we have to insert on-shell solutions into them. It is often difficult to get exact solutions in gravity with higher derivative corrections except for some cases, such as BPS black holes in supergravity or the BTZ black hole in three dimensions. However, what we need is their values at the horizon. An estimation of the entropy may be possible if we know universal behaviours near the horizon of the black holes. For the extremal black holes, they are actually controlled by the attractor mechanism and the AdS near horizon geometry. In such cases, the entropy function formalism is a very powereful tool to obtain the one-shell solutions at the horizon, as will seen later.

2 Attractor Mechanism

2.1 Attractor Behaviors in Extremal Black Holes

Now let us consider the four dimensional action with an arbitrary number of complex scalar fields z^a and $U(1)$ gauge fields A^I_μ,

$$\mathcal{I}_{(4)} = \frac{1}{16\pi G_N^{(4)}} \int d^4x\sqrt{-G}\Big(R - 2G_{a\bar{b}}\partial_\mu z^a \partial^\mu \bar{z}^{\bar{b}} - \frac{1}{4}\mu_{IJ}F^I_{\mu\nu}F^{J\mu\nu} - \frac{i}{4}\nu_{IJ}F^I_{\mu\nu}\tilde{F}^{J\mu\nu}\Big) \quad (2.1.1)$$

The dual field strength $\tilde{F}^{I\mu\nu}$ is defined by $\tilde{F}^{I\mu\nu} = \frac{i}{2}\epsilon^{\mu\nu\rho\sigma}F^I_{\rho\sigma}$, and the couplings $G_{a\bar{b}}$, μ_{IJ} and ν_{IJ} are some functions of z^a.

At this stage we assume that z^a depends only on the radial coordinate r and take an ansatz of the following isotropic form

$$ds^2_{(4)} = -e^{2U(r)}dt^2 + e^{-2U(r)}\left(dr^2 + r^2 d\Omega^2_{S^2}\right). \quad (2.1.2)$$

It is known that the ansatz relating the coefficients of dt^2 and dr^2 satisfies the extremal black hole condition [41]. In order to solve equations of motion for U and z^a coupled to the gauge fields, we put

$$F^I_{tr} = \frac{\hat{q}^I}{2}, \quad F^I_{\theta\varphi} = \frac{p^I}{2}\sin\theta, \quad G_{Itr} = \frac{\hat{p}_I}{2}, \quad G_{I\theta\varphi} = \frac{q_I}{2}\sin\theta, \quad (2.1.3)$$

where the magnetic fields are defined as $G_{I\mu\nu} = \nu_{IJ}F^J_{\mu\nu} - i\mu_{IJ}\tilde{F}^J_{\mu\nu}$. But $\hat{q}^I$ and $\hat{p}_I$ are actually given by the electric and magnetic charges q_I and p^I of the black hole as

$$\begin{aligned} \hat{q}^I &= \frac{e^{2U}}{r^2}[-(\mu^{-1})^{IJ}\nu_{JK}p^K + (\mu^{-1})^{IJ}q_J], \\ \hat{p}_I &= \frac{e^{2U}}{r^2}\left[-\nu_{IJ}(\mu^{-1})^{JK}\nu_{KL}p^L + \nu_{IJ}(\mu^{-1})^{JK}q_K - \mu_{IJ}p^J\right], \end{aligned} \quad (2.1.4)$$

due to equations of motion of the gauge fields.

With the gauge field configurations appearing in (2.1.3), the equations of motion for the action (2.1.1) turn out to be

$$\begin{aligned} &U'' = e^{2U}V_{BH}, \\ &-\left\{U'' - 2(U')^2\right\} + 2G_{a\bar{b}}(z^a)'(\bar{z}^b)' - e^{2U}V_{BH} = 0, \\ &\{G_{a\bar{b}}(\bar{z}^b)'\}' - \partial_a G_{b\bar{c}}(z^b)'(\bar{z}^c)' = e^{2U}\partial_a V_{BH}, \end{aligned} \quad (2.1.5)$$

where $' \equiv d/d(-1/r)$. The function

$$V_{BH}(z,\bar{z},p,q) = \frac{1}{16}(p^I, q_J)\begin{pmatrix}(\nu\mu^{-1}\nu+\mu)_{IK} & -(\nu\mu^{-1})_I{}^L \\ -(\mu^{-1}\nu)^J{}_K & (\mu^{-1})^{JL}\end{pmatrix}\begin{pmatrix}p^K \\ q_L\end{pmatrix} \tag{2.1.6}$$

is often called *black hole potential.*

Incidentally, even if we here consider a system with the Lagrangian

$$\mathcal{L}(U,z,\bar{z}) = (U')^2 + G_{a\bar{b}}(z^a)'(\overline{z}^b)' + e^{2U}V_{BH}(z,\overline{z},p,q), \tag{2.1.7}$$

plus a constraint

$$(U')^2 + G_{a\bar{b}}(z^a)'(\overline{z}^b)' - e^{2U}V_{BH}(z,\overline{z},p,q) = 0, \tag{2.1.8}$$

we can also derive the equations of motion (2.1.5). Thus, this reduced theory is equivalent to the original one (2.1.1) when the metric and scalars depend only on r.

First, let us consider the case in which V_{BH} has a critical point at $\phi^i = \phi_0^i$,

$$\partial_i V_{BH}(\phi_0) = 0, \tag{2.1.9}$$

where ϕ^i stands for the collection of z^a and $\overline{z}^a$. In other words, ϕ_0^i are determined by (q_I, p^I). Second, assume that the second derivative of the potential at the critical point is positive, i.e.,

$$\partial_i\partial_j V_{BH}(\phi_0) > 0. \tag{2.1.10}$$

The statement of the attractor mechanism is that once the above conditions hold, the values of ϕ^i at the horizon are fixed by ϕ_0^i, namely,

$$\phi^i(r=0) = \phi_0^i(q,p). \tag{2.1.11}$$

This has been checked in various ways. A trivial solution under some appropriate V_{BH} is

$$\phi^i(r) = \phi_0^i, \qquad e^{-U(r)} = e^{-U_0(r)} = 1 + \frac{\sqrt{V_0}}{r}, \tag{2.1.12}$$

where V_0 is the extremum of the black hole potential. But in general solutions, the scalars have degrees of freedom of the integration constants, $\phi^i(r=\infty)$. We must, therefore, perturbe this solution by an infinitesimal parameter ϵ, like

$$\phi^i(r) = \phi_0^i + \sum_{n=1}^{\infty}\epsilon^n\phi_n^i(r), \qquad U(r) = U_0(r) + \sum_{n=1}^{\infty}\epsilon^n U_n(r). \tag{2.1.13}$$

When we solve the equations of motion order by order in this ϵ expansion, we will obtain

$$\phi_n^i(r) \sim a_n\left(\frac{1}{1+\frac{\sqrt{V_0}}{r}}\right)^{sn}, \qquad U_n(r) \sim b_n\left(\frac{1}{1+\frac{\sqrt{V_0}}{r}}\right)^{tn}, \tag{2.1.14}$$

where s and t are some positive constants, and a_n and b_n should obey a recursion relation. Aside from the convergence of the sum (2.1.13), it is easy to see that the scalar solutions satisfy (2.1.11). It can be directly checked by the numerical calculation as well. As a result of the attractor mechanism, the Bekenstein-Hawking entropy is found to be

$$\mathcal{S} = \frac{4\pi r^2 e^{-2U}|_{r=0}}{4G_N^{(4)}} = \frac{\pi}{G_N^{(4)}} V_0. \tag{2.1.15}$$

That is, the entropy is determined by the black hole charges in the extremal case.

The above behaviour of extremal black holes is independent of supersymmtry and is observed in many examples. In a specific case, such as $N = 2$ supergravity discussed in the next section, we know exact solutions and they give the attractor. Furthermore, although it is difficult to obtain exact solutions in gravity theory with the higher derivative correction, they still have the above qualitative behaviours.

2.2 Exact Solutions

In this section we present examples of exact solutions in $N = 2$ supergravity embeded into ten dimensional IIA string theory compactified on Calabi-Yau manifold CY_3, or eleven dimensional M-theory on $\mathrm{CY}_3 \times \mathrm{S}^1$. It is interesting because it contains a lot of classes of black hole solutions, BPS or non-BPS.

It is known that four dimensional $N = 2$ supergravity coupled to n_v vector multiplets is described by the so-called *special geometry*. Let us consider complex scalars X^I, $(I = 0, 1, \cdots n_v)$ and the prepotential

$$F(X) = -\frac{1}{6} c_{abc} \frac{X^a X^b X^c}{X^0} \qquad (a = 1, 2 \cdots n_v). \tag{2.2.16}$$

Here c_{abc} denotes the triple intersection number of CY_3. When we define $F_I(X) = \partial F(X)/\partial X^I$, $F_{IJ}(X) = \partial^2 F(X)/\partial X^I \partial X^J$ and $N_{IJ} = 2\mathrm{Im} F_{IJ}$, [1] and choose

$$z^a = \frac{X^a}{X^0}, \qquad N_{IJ} X^I \overline{X}^J = -1, \tag{2.2.17}$$

the action (2.1.1) for the four dimensional supergravity is determined only by the prepotential. Putting $z^0 = 1$ formally and defining the Kähler potential K through

$$e^{-K} = -z^I N_{IJ} \overline{z}^J = |X^0|^{-2}, \tag{2.2.18}$$

we see $G_{a\overline{b}} = \partial_a \partial_{\overline{b}} K$ as the Kähler metric. The couplings ν_{IJ} and μ_{IJ} are also identified in terms of z^a,

$$\nu_{IJ} - i\mu_{IJ} = \overline{F}_{IJ} + i \frac{N_{IK} z^K N_{JL} z^L}{z^M N_{MN} z^N}. \tag{2.2.19}$$

This is realized by compactifying IIA supergravity on CY_3. In type IIA view point, the electric-magnetic charges (q_0, q_a, p^a, p^0) can be seen as (D0,D2,D4,D6)-brane charges. Then a function

$$Z = \frac{e^{K/2}}{2\sqrt{2}} \left(p^I F_I(z) - q_I z^I \right) \tag{2.2.20}$$

[1] We take X^0 to be real and $i\left(F_I(X)\overline{X}^I - X^I \overline{F}_I(\overline{X})\right) = 1$ as a gauge choice. Due to this gauge, we have n_v vector multiplets, and one of the $(n_v + 1)$ gauge fields corresponds to the graviphoton.

gives the graviphoton charge, or the central charge of four dimensional $N = 2$ supergravity theory.

Note that the black hole potential is, in fact, rewritten as

$$V_{BH} = |Z|^2 + G^{a\bar{b}} \mathcal{D}_a Z \overline{\mathcal{D}_{\bar{b}} Z}, \tag{2.2.21}$$

where $\mathcal{D}_a$ is the Kähler covariant derivative, i.e., $\mathcal{D}_a Z = \left(\partial_a + \frac{1}{2}\partial_a K\right) Z$. It is in general known that solutions satisfying first order equations [41, 42]

$$U' = e^U |Z|, \qquad (z^a)' = e^U G^{a\bar{b}} \overline{\mathcal{D}_{\bar{b}} Z} \frac{Z}{|Z|} \tag{2.2.22}$$

express four dimensional BPS black holes which preserve a half of eight supercharges. At the horizon the scalars z^a are attracted to the values specified by $\mathcal{D}_a Z = 0$, so eqs. (2.2.22) exhibit the BPS attractor flow. But there are also, in particular brane configurations, solutions satisfying flow equations

$$U' = e^U |\mathcal{Z}|, \qquad (z^a)' = e^U G^{a\bar{b}} \overline{\mathcal{D}_{\bar{b}} \mathcal{Z}} \frac{\mathcal{Z}}{|\mathcal{Z}|}, \tag{2.2.23}$$

for some function $\mathcal{Z}$. This no longer preserves supersymmetry, but z^a are also attracted to the values specified by $\mathcal{D}_a \mathcal{Z} = 0$ at the horizon. These equations indicate the non-BPS attractor flow [43].

Let us see one example of the exact solution in a simple case, the so-called STU model, whose prepotential is given by

$$F(X) = -\frac{X^1 X^2 X^3}{X^0}. \tag{2.2.24}$$

We would like to solve the equations of motion for the D0-D4 charge configuration (q_0, p^1, p^2, p^3). But we here take $z^a = z = x + iy$ and $p^a = p$ $(a = 1, 2, 3)$ in order to reduce our calculation further.

It is known that the $q_0 < 0$ configuration leads to the BPS black hole. Actually,

$$\begin{aligned} z &= -\frac{1}{2}\frac{\alpha \bar{H} + \beta H_0}{\bar{H}^2 + \alpha\beta} + i\sqrt{\frac{\alpha^2 - \bar{H} H_0}{\bar{H}^2 + \alpha\beta} - \frac{1}{4}\left(\frac{\alpha \bar{H} + \beta H_0}{\bar{H}^2 + \alpha\beta}\right)^2}, \\ e^{-4U} &= (\bar{H}^2 + \alpha\beta)(\alpha^2 - \bar{H} H_0) - \frac{1}{4}(\alpha \bar{H} + \beta H_0)^2, \end{aligned} \tag{2.2.25}$$

where

$$\bar{H} = h + \frac{p}{2r}, \qquad H_0 = h_0 + \frac{q_0}{2r}, \qquad (h_0, q_0 < 0), \qquad \beta q_0 = 3\alpha p, \tag{2.2.26}$$

is a solution to eqs (2.2.22), and the black hole mass is found to be $M = Z|_{r=\infty}$. Even if we choose any h, h_0 and α (or β), the scalars z^a are attracted to the point $x = 0$ and $y = \sqrt{-q_0/p}$. The entropy of the BPS black hole is turned out to be

$$\mathcal{S}_{\rm BPS} = 2\pi\sqrt{-q_0 p^3}. \tag{2.2.27}$$

Here we put $G_N^{(4)} = 1/8$ for later convenience.

On the other hand, the case of $q_0 > 0$ gives the non-BPS black hole of the solution

$$z = \frac{\gamma}{2\bar{H}^2} + i\sqrt{\frac{H_0}{\bar{H}} - \frac{\gamma^2}{4\bar{H}^4}}, \qquad e^{-4U} = \bar{H}^3 H_0 - \frac{\gamma^2}{4}. \tag{2.2.28}$$

For this solution the black hole mass does not saturate $|Z|$, but a short calculation reveals that they satisfy the flow equation (2.2.23) if we define

$$\mathcal{Z} = \frac{e^{K/2}}{2\sqrt{2}} \left(p^a F_a(z) + q_0\right), \tag{2.2.29}$$

in this case. The scalars are attracted to $x = 0$ and $y = \sqrt{q_0/p}$, and this still shows the non-BPS attractor mechanism [44]. Finally, the non-BPS black hole entropy is

$$\mathcal{S}_{\text{non-BPS}} = 2\pi\sqrt{q_0 p^3}. \tag{2.2.30}$$

It was the main result of our paper [29] to find the non-zero axion ($x \neq 0$) solution for the non-BPS black hole.

2.3 Entropy Function

In this chapter we have discussed the attractor mechanism, which is the universal property of the extremal black hole solution. Indeed, there is another nature of the extremal black hole. The authors of refs. [12, 13] proved that the near horizon geometry for any extremal black hole always has a SO(2,1) symmetry, or the AdS_2 factor

$$ds^2_{\text{AdS}_2} = -r^2 dt^2 + \frac{dr^2}{r^2}. \tag{2.3.31}$$

It remains valid in matter-coupled case and even in the presence of higher-derivative corrections, and we can consider that it works together with the attractor mechanism. Conversely, one could say that it is just the definition of the extremal black hole to have two properties that at the horizon the scalars are attracted and the geometry includes the AdS_2 factor.

If so, we know a simple way to calculate any extremal black hole entropy, the entropy function formalism [40, 45, 46, 38]. Sen's proposal is the following. First, consider the solution

$$\begin{aligned} ds^2 &= v_1\left(-r^2 dt^2 + \frac{dr^2}{r^2}\right) + v_2 d\Omega^2_{S^2}, \\ F^I_{tr} &= \frac{e^I}{2}, \qquad F^I_{\theta\phi} = \frac{p^I}{2}\sin\theta, \qquad \phi^a = u^a \end{aligned} \tag{2.3.32}$$

near the horizon. In this ansatz we have assumed the attractor behaviour of the scalars and the AdS_2 geometry of the metric. Here $\vec{v}$, $\vec{e}$, $\vec{p}$ and $\vec{u}$ are constant. Next, introduce electric charges $\vec{q}$ by

$$\frac{\partial f(\vec{v},\vec{e},\vec{p},\vec{u})}{\partial e^I} = q_I, \qquad f(\vec{v},\vec{e},\vec{p},\vec{u}) = \int d\Omega\sqrt{-G}\mathcal{L}. \tag{2.3.33}$$

Then, we can obtain the horizon values of $\vec{v}$, $\vec{e}$ and $\vec{u}$ by algebraically extremizing the *entropy function*

$$\mathcal{E}(\vec{v},\vec{e},\vec{p},\vec{u}) = \frac{1}{8G_N^{(4)}}\left(e^I q_I - f(\vec{v},\vec{e},\vec{p},\vec{u})\right), \tag{2.3.34}$$

because all derivative terms drop out due to the assumption of the attractor mechanism and the AdS_2 geometry. According to ref. [40], the black hole entropy is in fact obtained by the value of this extremum

$$\mathcal{S}(\vec{q},\vec{p}) = \mathcal{E}(\vec{v},\vec{e},\vec{p},\vec{u})|_{\text{extremum}}. \tag{2.3.35}$$

As is clear from (2.3.32), we have to use the above two assumptions on general extremal black holes. Once we admit them, any entropy is obtainable even in case with higher derivative since this formalism is actually based on the Wald formula [40]. The modification is needed in the case which the action includes Chern-Simons terms, and then the definition of the entropy function is slightly changed due to the fact that the Wald-Tachikawa formula becomes valid [38].

Let us return to the example of $N = 2$ supergravity with D0-D4 charges. Although we shall not demonstrate the detail since we will get a similar result in the later chapter, the macroscopic entropy of the BPS and non-BPS black hole is found to be

$$\begin{aligned} \mathcal{S}_{\text{BPS}} &= 2\pi\sqrt{-\frac{q_0 c_R}{6}}, \qquad c_R = 6p^3 + c_{2a}p^a, \\ \mathcal{S}_{\text{non-BPS}} &= 2\pi\sqrt{\frac{q_0 c_L}{6}}, \qquad c_L = 6p^3 + \frac{1}{2}c_{2a}p^a, \end{aligned} \tag{2.3.36}$$

where $p^3 = \frac{1}{6}c_{abc}p^a p^b p^c$ and c_{2a} is the second Chern class number. It is known that each entropy agrees with the microscopic counting for the BPS or non-BPS configuration in M-theory [47, 48]. At least in the non-BPS case, we do not know how to reach this result except for the entropy function formalism because it is almost impossible to solve equations of motion exactly.

Part II

Microscopic Interpretation of Black Hole Entropy

As stated above, the significant properties of the extremal black hole are the AdS behaviour near the horizon and the attractor mechanism. Knowing these and, finally using the entropy function, then we can calculate the black hole entropy, independently of supersymmetry and whether any higher derivative is present or not. From now on, we pay attention to the microscopic description of black hole space-times. It is absolutely necessary to understand and reproduce the thermodynamical entropy we have seen above in terms of the statistical counting of some states in order to disclose the full quantum theory of gravity.

String theory has a lot of examples reconstructing the entropy from microscopic view point. D-brane in the theory enables us to do it. Strominger and Vafa originally found out the microscopic origin of the Bekensten-Hawking entropy [3]. And in the D0-D4 system, or M5 system in M-theory language, the entropy including higher derivative (2.3.36) was also obtained by counting the degeneracy of the states on the brane [47]. The AdS/CFT correspondence is essentially used in these contexts.

However, the information of D-brane or string theory is not always needed to compare to the macroscopic entropy. The key ingredient is the AdS_3/CFT_2 correspondence. In fact, Brown and Henneaux showed the existence of CFT_2 satisfying the Virasoro algebra on AdS_3 boundary of the three dimensional gravity with the negative cosmological constant [7]. Once we derive the Virasoro algebra and get its central charge from gravity side, it is easy to reconstruct the entropy of the BTZ black hole by using the Cardy formula to count the degeneracy of states of the CFT_2 [10]. This idea is recently brought to the four dimensional extremal Kerr black hole [11]. It has not been revealed yet that these Virasoro algebra is of what CFT_2, but the Brown-Henneaux technique relying on the AdS_3/CFT_2 rather than string theory is very attractive so as to explore the quamtum origin of realistic black holes in the sky.

In this part, we first review the Brown-Henneaux formalism in three dimensional gravity and also obtain the similar result in four dimensional Kerr space-time. And next let us confirm that Brown-Henneaux's argument is valid as well in higher derivative gravity, especially Topologically Massive Gravity (TMG) which includes the gravitational Chern-Simons (GCS) term [27]. As a result, if the three dimensional gravity has any higher derivative, one can derive the Virasoro algebras on the AdS_3 boundary and check the agreement between the Wald-Tachikawa formula and the Cardy formula [30]. Finally we mention the dual description of a non-AdS geometry, called warped AdS, in TMG.

3 Brown-Henneaux's Approach

3.1 Realization of the Virasoro Algebras

Three dimensional gravity with negative cosmological constant

$$\mathcal{I}_{(3)} = \frac{1}{16\pi G_N} \int d^3x \sqrt{-G} \left(R + \frac{2}{\ell^2} \right) \tag{3.1.1}$$

has no degrees of freedom of graviton [6], but nevertheless, it allows the BTZ black hole solution

$$\begin{aligned} ds^2 &= -N^2 dt^2 + N^{-2} dr^2 + r^2 (d\varphi + N^\varphi dt)^2, \qquad (\varphi \sim \varphi + 2\pi) \\ N^2 &= \left(\frac{r}{\ell}\right)^2 + \left(\frac{4G_N j}{r}\right)^2 - 8G_N m \quad , \quad N^\varphi = \frac{4G_N j}{r^2}, \end{aligned} \tag{3.1.2}$$

where m and j are its mass and angular momuntum [9]. This is an excited state from the vacuum solution which is the global AdS geometry

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right) dt^2 + \left(1 + \frac{r^2}{\ell^2}\right)^{-1} dr^2 + r^2 d\varphi^2. \tag{3.1.3}$$

Both solutions are everywhere locally AdS_3 with the radius ℓ from the property of three dimensional gravity.

First of all let us specify the boundary conditions so that field configurations behave as "asymptotically AdS_3". We require that the metric should behave at the spatial infinity $r \longrightarrow \infty$ as

$$\begin{aligned} G_{tt} &= -\frac{r^2}{\ell^2} + \mathcal{O}(1) \quad , \quad G_{tr} = \mathcal{O}(r^{-3}) \quad , \quad G_{t\varphi} = \mathcal{O}(1), \\ G_{rr} &= \frac{\ell^2}{r^2} + \mathcal{O}(r^{-4}) \quad , \quad G_{r\varphi} = \mathcal{O}(r^{-3}) \quad , \quad G_{\varphi\varphi} = r^2 + \mathcal{O}(1), \end{aligned} \tag{3.1.4}$$

which is in accordance with the behavior in (3.1.2) and (3.1.3). The vector field $(\bar{\xi}^0, \bar{\xi}^r, \bar{\xi}^\varphi)$ that transforms the metric while preserving the boundary conditions (3.1.4) are not strongly restricted but are allowed to be a general class of functions. In fact, by using the coordinates of $x^\pm = \frac{t}{\ell} \pm \varphi$, the n-th Fourier component of the vector fields is given by

$$\bar{\xi}^t = \frac{\ell}{2} e^{inx^\pm} \left(1 - \frac{\ell^2 n^2}{2r^2}\right), \quad \bar{\xi}^r = -i\frac{nr}{2} e^{inx^\pm}, \quad \bar{\xi}^\varphi = \pm \frac{1}{2} e^{inx^\pm} \left(1 + \frac{\ell^2 n^2}{2r^2}\right). \tag{3.1.5}$$

In the following we call the above vector fields "Killing vectors". For later use, we assign explicit notations for these Killing vectors:

$$\xi_n^{\pm} \equiv \bar{\xi}^I \partial_I = e^{inx^{\pm}} \left(\partial_{\pm} - \frac{\ell^2 n^2}{2r^2} \partial_{\mp} - \frac{inr}{2} \partial_r \right), \tag{3.1.6}$$

where $\partial_{\pm} = \frac{1}{2}(\ell\partial_t \pm \partial_{\varphi})$. The algebraic structure of the symmetry is encoded in the Killing vector and in fact we can directly compute the commutation relations of these differential operators

$$\left[\xi_m^{\pm}, \xi_n^{\pm}\right] = -i\,(m-n)\,\xi_{m+n}^{\pm}, \qquad \left[\xi_m^{+}, \xi_n^{-}\right] = \mathcal{O}(r^{-4}). \tag{3.1.7}$$

This result clearly shows that the asymptotically AdS_3 spacetime is endowed with the two dimensional conformal symmetry.

In order to evaluate the central extension of the Virasoro algebras, we have to know the asymptotic behaviors of the canonical variables and we introduce the $(2+1)$-dimensional decomposition, often refered to as Arnowitt-Deser-Misner (ADM) decomposition [49], of the three dimensional metric $G_{\mu\nu}$ as

$$ds^2 = G_{\mu\nu}dx^{\mu}dx^{\nu} = -N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \tag{3.1.8}$$

Here $g_{ij}, (i,j = r, \varphi)$ is the two dimensional metric. The lapse and shift functions are denoted by N and N_i, respectively. The action (3.1.1) is rewritten as usual by

$$\mathcal{I}_{(3)} = \frac{1}{16\pi G_N} \int dt d^2x \sqrt{g} N \left(R^{(2)} + \frac{2}{\ell^2} + K^{ij}K_{ij} - K^2 \right), \tag{3.1.9}$$

where $R^{(2)}$ is the scalar curvature made out of g_{ij}, and

$$K_{ij} = \frac{1}{2N}\left(\dot{g}_{ij} - \mathcal{D}_i N_j - \mathcal{D}_j N_i\right), \qquad K = g^{ij}K_{ij}. \tag{3.1.10}$$

The dot over g_{ij} means the t-derivative and $\mathcal{D}_i$ is the covariant derivative with respect to g_{ij}. The momentum variable π^{ij} conjugate to g_{ij} is given by $\pi^{ij} = \sqrt{g}(K^{ij} - g^{ij}K)$ for the theory (3.1.1), and the Hamiltonian density $\mathcal{H}$ is the Legendre transform of the Lagrangian density $\mathcal{L}$, i.e., $\mathcal{H} = \pi^{ij}\dot{g}_{ij} - \mathcal{L}$.

The Hamiltonian consists of an usual combination of constraints together with an appropriate surface term $Q[\xi]$,

$$H[\xi] = \int d^2x \left(\xi^0 \mathcal{H}_0 + \xi^i \mathcal{H}_i\right) + Q[\xi], \tag{3.1.11}$$

where

$$\mathcal{H}_0 = \frac{1}{\sqrt{g}}\left((\pi^{ij})^2 - (\pi_i^i)^2\right) - \sqrt{g}\left(R^{(2)} + \frac{2}{\ell^2}\right), \qquad \mathcal{H}^i = -2\sqrt{g}\mathcal{D}_j\left(\frac{\pi^{ij}}{\sqrt{g}}\right). \tag{3.1.12}$$

The added term $Q[\xi]$ must be determined so that it cancels the surface terms produced by the first term in (3.1.11) under field variation and is a generator of the possible surface deformation [50]. The vector field $(\xi^0, \xi^r, \xi^{\varphi})$ denotes such an allowed surface deformation and is related to the spacetime vector $(\bar{\xi}^0, \bar{\xi}^r, \bar{\xi}^{\varphi})$ via

$$(\xi^0, \xi^r, \xi^{\varphi}) = (N\bar{\xi}^t, \bar{\xi}^r + N^r\bar{\xi}^t, \bar{\xi}^{\varphi} + N^{\varphi}\bar{\xi}^t). \tag{3.1.13}$$

The asymptotic behaviors (3.1.4) are now translated into those of the canonical variables as

$$g_{rr} = \frac{\ell^2}{r^2} + \mathcal{O}(r^{-4}) \quad , \quad g_{r\varphi} = \mathcal{O}(r^{-3}) \quad , \quad g_{\varphi\varphi} = r^2 + \mathcal{O}(1), \tag{3.1.14}$$

$$N = \frac{r}{\ell} + \mathcal{O}(r^{-1}) \quad , \quad N^r = \mathcal{O}(r^{-1}) \quad , \quad N^\varphi = \mathcal{O}(r^{-2}). \tag{3.1.15}$$

The behaviors of the canonical conjugate variables are also derived with the help of (3.1.10), (3.1.14) and (3.1.15):

$$\pi^{rr} = \mathcal{O}(r^{-1}) \quad , \quad \pi^{r\varphi} = \mathcal{O}(r^{-2}) \quad , \quad \pi^{\varphi\varphi} = \mathcal{O}(r^{-5}). \tag{3.1.16}$$

It has been known that conditions (3.1.14), (3.1.15) and (3.1.16) are preserved under the Hamiltonian evolution provided that we impose the Hamiltonian constraints. The generator $Q[\xi]$ in (3.1.11) is found by taking into account the asymptotic behaviors of the canonical variables up to a constant term, which is adjusted so that the charge $Q[\xi]$ vanishes for the globally AdS space.

The algebraic structure of symmetric transformation group is given by the Poisson bracket algebra of Hamiltonian generator $H[\xi]$:

$$\{H[\xi], H[\eta]\}_{\mathrm{P}} = H\big[[\xi,\eta]\big] + K[\xi,\eta], \tag{3.1.17}$$

where $K[\xi,\eta]$ is a possible central extension. While the Dirac bracket $\{Q[\xi], Q[\eta]\}_{\mathrm{D}}$ yields the surface deformation of $Q[\xi]$ with respect to $Q[\eta]$, i.e., $\delta_\eta Q[\xi] = \{Q[\xi], Q[\eta]\}_{\mathrm{D}}$, the charge $Q[\xi]$ forms a conformal group with a central extension $\{Q[\xi], Q[\eta]\}_{\mathrm{D}} = Q\big[[\xi,\eta]\big] + K[\xi,\eta]$. From two expressions we immediately get $\delta_\eta Q[\xi] = Q\big[[\xi,\eta]\big] + K[\xi,\eta]$. If we set the initial condition so that $Q\big[[\xi,\eta]\big] = 0$ for a globally AdS space, the evaluation of the central charge reduces to

$$K\big[\xi,\eta\big] = \delta_\eta Q[\xi]. \tag{3.1.18}$$

In the case of (3.1.1), the explicit form is given by

$$\begin{aligned}&\delta_\eta Q[\xi]\\ &= \int d\varphi \left[\sqrt{g} S^{ijkr}\{\xi^0 \mathcal{D}_k \delta_\eta g_{ij} - \mathcal{D}_k \xi^0 \delta_\eta g_{ij}\} + 2\xi^i \pi^{jr} \delta_\eta g_{ij} + 2\xi_i \delta_\eta \pi^{ir} - \xi^r \pi^{ij} \delta_\eta g_{ij}\right],\end{aligned} \tag{3.1.19}$$

where S^{ijkl} is defined by

$$S^{ijkl} = \frac{1}{2}\left(g^{ik}g^{jl} + g^{il}g^{jk} - 2g^{ij}g^{kl}\right). \tag{3.1.20}$$

In the case of TMG, the derivation of the above equations will be explained in Chapter 4.

Putting the Killing vector (3.1.6) for ξ, we define the Virasoro generators by $L_n^\pm = Q[\xi_n^\pm]/16\pi G_N$. By replacing the Dirac brackets by a commutator ($\{,\}_{\mathrm{D}} \to -i[,]$), the commutation relations become

$$\begin{aligned}&[L_m^+, L_n^+] = (m-n)L_{m+n}^+ + \frac{c_L}{12}m(m^2-1)\delta_{m+n,0},\\ &[L_m^-, L_n^-] = (m-n)L_{m+n}^- + \frac{c_R}{12}m(m^2-1)\delta_{m+n,0},\\ &[L_m^+, L_n^-] = 0,\end{aligned} \tag{3.1.21}$$

and the central charges have been calculated in [7] as

$$c_L = c_R = \frac{3\ell}{2G_N}. \tag{3.1.22}$$

Once we get the central charges, it is straightforward to obtain the BTZ black hole entropy by using the Cardy formula [10]

$$\begin{aligned} \mathcal{S} &= 2\pi\sqrt{\frac{1}{6}c_L L_0^+} + 2\pi\sqrt{\frac{1}{6}c_R L_0^-} \\ &= \frac{\pi}{2G_N}\sqrt{2G_N\ell^2\left(m+\frac{j}{\ell}\right)} + \frac{\pi}{2G_N}\sqrt{2G_N\ell^2\left(m-\frac{j}{\ell}\right)}. \end{aligned} \tag{3.1.23}$$

Here $L_0^\pm$ are related to the mass m and angular momentum j of the black hole by the formulae $L_0^+ + L_0^- = m\ell$ and $L_0^+ - L_0^- = j$.

3.2 The Kerr/CFT Correspondence

The four dimensional Kerr solution is given by

$$ds^2 = -\frac{\Delta}{\rho^2}\left(d\hat{t} - a\sin^2\theta d\hat{\varphi}\right)^2 + \frac{\sin^2\theta}{\rho^2}\left[(\hat{r}^2+a^2)d\hat{\varphi} - ad\hat{t}\right]^2 + \frac{\rho^2}{\Delta}d\hat{r}^2 + \rho^2 d\theta^2,$$

$$\Delta = \hat{r}^2 - 2M\hat{r} + a^2, \qquad \rho^2 = \hat{r}^2 + a^2\cos^2\theta, \tag{3.2.24}$$

where $a = G_N^{(4)}J/M$.[1] In extremal case ($J = M^2/G_N^{(4)}$), the Hawking temperature vanishes and the entropy becomes

$$\mathcal{S} = \frac{2\pi J}{\hbar}. \tag{3.2.25}$$

In that case, the Kerr solution has a $SL(2,R)\times U(1)$ near horizon geometry by taking the Bardeen-Horowitz limit [51]

$$t' = \frac{\lambda\hat{t}}{2M}, \qquad r' = \frac{\lambda M}{\hat{r}-M}, \qquad \varphi' = \hat{\varphi} - \frac{\hat{t}}{2M}, \qquad \lambda\to 0. \tag{3.2.26}$$

Under a further coordinate transformation

$$\begin{aligned} &r' = (\cos t'\sqrt{1+r^2}+r)^{-1}, \qquad t' = r'\sin t\sqrt{1+r^2}, \\ &\varphi' = \varphi + \ln\left(\frac{\cos t' + r\sin t'}{1+\sin t'\sqrt{1+r^2}}\right), \end{aligned} \tag{3.2.27}$$

we finally obtain

$$ds^2 = 2G_N^{(4)}J\Omega^2\left[-(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + \Lambda^2(d\varphi + rdt)^2 + d\theta^2\right], \tag{3.2.28}$$

[1]The usual mass we observe is $M/G_N^{(4)}$ in this unit.

where

$$\Omega^2 = \frac{1+\cos^2\theta}{2}, \qquad \Lambda = \frac{2\sin\theta}{1+\cos^2\theta}. \tag{3.2.29}$$

This geometry can also be seen in warped AdS_3 solution of TMG.

The statement of the Kerr/CFT correspondence is that the isometry $SL(2,R)\times U(1)$ enhances to one copy of the Virasoro algebra on the boundary of $r\to\infty$. Here let us choose

$$\delta G_{\mu\nu} = \left(\begin{array}{ccccc} & t & r & \theta & \varphi \\ t & \mathcal{O}(r^2) & \mathcal{O}(1/r^2) & \mathcal{O}(1/r) & \mathcal{O}(1) \\ r & & \mathcal{O}(1/r^3) & \mathcal{O}(1/r^2) & \mathcal{O}(1/r) \\ \theta & & & \mathcal{O}(1/r) & \mathcal{O}(1/r) \\ \varphi & & & & \mathcal{O}(1) \end{array}\right) \tag{3.2.30}$$

as the boundary condition of the near horizon metric. Then, the allowed Killing vector is turned out to be

$$\xi_n = e^{in\varphi}(\partial_\varphi - inr\partial_r), \tag{3.2.31}$$

supplemented with the time tranlation ∂_t.

It is notable that the Killing vector ξ_n satisfies

$$[\xi_m, \xi_n] = -i\,(m-n)\,\xi_{m+n}, \tag{3.2.32}$$

but that the asymptotic symmetry group contains only one copy of the conformal group unlike the AdS_3 case (3.1.7). But the Killing vector (3.2.31) is enough to derive the entropy, as seen below, because it is considered from the analogy with the BTZ black hole in three dimensions that the extremality of the black hole $M^2/G_N^{(4)} - J = 0$ reduces the Cardy formula. The search for the missing copy is an intriguing task since it directly links with whether the dual theory is really an *ordinary* CFT_2. It is known that the different boundary condition from (3.2.30) leads to one copy of Virasoro algebra times a $U(1)$ current algebra for a warped AdS_3 in TMG [52], but we do not know the boundary condition which also allows the other copy like (3.1.7).

Another comment is that the boundary conditions of δG_{tt} and $\delta G_{\varphi\varphi}$ are the same order of r as the leading term. The covariant charge

$$\begin{aligned}\delta Q[\xi] = -\frac{1}{32\pi G_N^{(4)}}\int \epsilon_{\alpha\beta\mu\nu}\Big[&\xi^\nu\mathcal{D}^\mu\delta G + (\xi_\sigma\mathcal{D}^\nu - \xi^\nu\mathcal{D}_\sigma)\delta G^{\mu\sigma} + \frac{1}{2}\delta G\mathcal{D}^\nu\xi^\mu \\ &- \delta G^{\nu\sigma}\mathcal{D}_\sigma\xi^\mu + \delta G^\nu_\sigma\mathcal{D}^{(\sigma}\xi^{\mu)}\Big]dx^\alpha\wedge dx^\beta,\end{aligned} \tag{3.2.33}$$

instead of the canonical charge (3.1.19) is useful in this case although there are no clear reason why only this charge provides a suitable result [53, 54].

Anyway, inserting (3.2.31) into (3.2.33), one finds the central extention

$$-i(m^3 + 2m)\delta_{m+n}J. \tag{3.2.34}$$

By a shift of zero mode $\hbar L_n^+ = Q[\xi_n] + \frac{3}{2}J\delta_{n,0}$ and a replacement of the commutator ($\{,\}_{\mathrm{D}} \to -\frac{i}{\hbar}[,]$), one copy of the Virasoro algebra

$$[L_m^+, L_n^+] = (m-n)L_{m+n}^+ + \frac{J}{\hbar}m(m^2-1)\delta_{m+n,0} \tag{3.2.35}$$

can be derived. The central charge of the CFT for the extremal Kerr is found to be

$$c_L = \frac{12J}{\hbar}. \qquad (3.2.36)$$

While the conserved charge for the time translation ∂_t is zero, the one for the angular translation ∂_φ ($n = 0$ in (3.2.31)) is now normalized as $J/2\hbar$ according to the claim of the paper [11]. Then, if we can define $\hbar L_0^\pm = (M^2/G_N^{(4)} \pm J)/4$ from the analogy with the BTZ case, the Cardy formula gives the Kerr entropy (3.2.25). Irrespective of whether non-trivial c_R and L_n^- exist or not, the macroscopic black hole entropy can be reproduced from the state counting of *CFT* which obeys at least one copy of the Virasoro algebra.[2]

[2]There are many disccusions on the horizon CFT [55, 56, 57]

4 Topologically Massive Gravity

4.1 Higher Derivative Gravity in Three Dimension

In this chapter we deal with a particular three dimensional gravity which contains the GCS term (1.2.22). This theory is often referred to as Topologically Massive Gravity, which stems from the fact that it has a single, unitary massive graviton with helicity +2 or −2 around the Mikowski background [27]. We here add the negative cosmological constant.[1] Note that there are third derivatives in the GCS, and that it violates parity.

First, we mention the case without the GCS term. The AdS_3/CFT_2 correspondence by Brown and Henneaux was extended to general three dimensional gravity with parity preserving higher derivative terms, i.e., [70]

$$\mathcal{L} = \sqrt{-G}\left[f(R_{\mu\nu}, G_{\mu\nu}) + \frac{2}{\ell_0^2}\right]. \tag{4.1.1}$$

In three dimensions the Riemann tensor is expressed by the Ricci tensor, so parity preserving higher derivatives are included in $f(R_{\mu\nu}, G_{\mu\nu})$. The Einstein equation is modified as

$$\frac{1}{2}G^{\mu\nu}\left(f + \frac{2}{\ell_0^2}\right) + \frac{\partial f}{\partial G_{\mu\nu}} + T^{\mu\nu} = \beta\epsilon^{\rho\sigma(\mu}\mathcal{D}_\rho R_\sigma^{\nu)}, \tag{4.1.2}$$

in this theory. Here we denote

$$T^{\mu\nu} = \frac{1}{2}\left(\mathcal{D}_\rho\mathcal{D}^\mu P^{\rho\nu} + \mathcal{D}_\rho\mathcal{D}^\nu P^{\mu\rho} - \Box P^{\mu\nu} - G^{\mu\nu}\mathcal{D}_\rho\mathcal{D}_\sigma P^{\rho\sigma}\right), \quad P^{\mu\nu} = \frac{\partial f}{\partial R_{\mu\nu}}. \tag{4.1.3}$$

Redefining the field $G_{\mu\nu} \to \Omega^2 G_{\mu\nu}$, where

$$\Omega = \frac{1}{3}G_{\mu\nu}\frac{\partial f}{\partial R_{\mu\nu}} \tag{4.1.4}$$

becomes constant in AdS_3 solutions, we can treat this theory equivalently to the usual Einstein-Hilbert term with the negative cosmological constant [71, 72]. According to the paper [70], even in such a theory, the Virasoro algebras exist on the $r \to \infty$ boundary of the BTZ black hole in spite of a slight shift of the central charges,

$$c_L = c_R = \Omega\frac{3\ell}{2G_N}, \tag{4.1.5}$$

[1] See also references [30, 36, 37, 38, 39, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69] for more topics on TMG.

and the Wald entropy also turn out to agree with the Cardy entropy.

In this parity preserving theory the left and right moving central charges have the same value, but in TMG, parity violating case, this is not true. Taming the GCS term was difficult from the parity preserving case because its form is not affected by the redefinition of $G_{\mu\nu}$. The simple redefinition rule like the parity preserving case is not effective, so we will here verify the AdS_3/CFT_2 by performing the direct application of Brown-Henneaux's canonical formalism to TMG.

Let us add the GCS term (1.2.22) to the three dimensional action. The AdS_3 geometries (3.1.2) and (3.1.3) are also allowed as solutions even in the most general three dimensional gravity with both the terms (4.1.1) and the GCS term (1.2.22), but the defenition of the mass and angular momentum of the BTZ slightly changes like

$$M = \Omega m + \frac{\beta}{\ell^2} j, \qquad J = \Omega j + \beta m. \tag{4.1.6}$$

Here, the *effective* comological constant ℓ in (3.1.2) and (4.1.6) is related to the *bare* one ℓ_0 in (4.1.1) through equations of motion. Note that the contribution of the GCS term is contained in the new definition of the mass and angular momentum [73, 74, 75, 76, 77, 36, 37, 78]. The macroscopic entropy of the BTZ black hole (3.1.2) is given by Wald-Tachikawa formula [26, 37, 38, 39]:

$$\mathcal{S} = \frac{\pi}{2G_N}\left[\left(\Omega + \frac{\beta}{\ell}\right)\sqrt{2G_N\ell^2\left(m + \frac{j}{\ell}\right)} + \left(\Omega - \frac{\beta}{\ell}\right)\sqrt{2G_N\ell^2\left(m - \frac{j}{\ell}\right)}\right]. \tag{4.1.7}$$

The aim of this chapter is the realization of this form from Brown-Henneaux's approach. Since it is straightforward to add to parity preserving higher derivatives, we omit it for a while and consider the action

$$\mathcal{I}_{\text{TMG}} = \frac{1}{16\pi G_N}\int d^3x \left[\sqrt{-G}\left(R + \frac{2}{\ell^2}\right) + \mathcal{L}_{\text{CS}}\right], \tag{4.1.8}$$

where $\mathcal{L}_{\text{CS}}$ is defined by (1.2.22).

4.2 Canonical Formalism in TMG

First of all, we have to decompose the Lagrangian of TMG in the ADM mannar (3.1.8). It is, however, quite difficult because the GCS term contains third derivatives with respect to time. It is known that the canonical formalism of such a system is done by using Ostrogradsky's method [79] in which Lagrange multiplier is introduced. For instance, if there is a Lagrangian $\mathcal{L}(g, \dot{g}, \ddot{g})$, then we define $\mathcal{L}^*(g, \dot{g}, h, \dot{h}, v) = \mathcal{L}(g, h, \dot{h}) + v(\dot{g} - h)$ and construct the Hamiltonian in the usual way. In the case of TMG, it is useful to apply modified version of Ostrogradsky's method as discussed in ref. [80]. In the modified Ostrogradsky method, the extrinsic curvature K_{ij} is dealt with an independent variable. At the same time, Lagrange multiplier v_{ij} should be introduced to give a proper constraint. And furthermore, deviding K_{ij} into a tracepart K and a traceless part H_{ij} like $K_{ij} = H_{ij} + g_{ij}K/2$, one finds that there are in fact no $\dot{K}$ term in the action.

Following these prescriptions, the Lagrangian of the TMG is given by [81, 80]

$$\begin{aligned}\mathcal{L}_{\rm TMG} &= \sqrt{g}N\Big(R^{(2)} + \frac{2}{\ell^2} + K^{ij}K_{ij} - K^2\Big) + v^{ij}(\dot{g}_{ij} - 2NK_{ij} - 2\mathcal{D}_iN_j)\\ &+ \beta\sqrt{g}\epsilon^{mn}\left(\dot{H}_{mk} + \frac{1}{2}K\dot{g}_{mk}\right)H_n{}^k + \beta\sqrt{g}N\Big(2\epsilon^{mn}\mathcal{D}_k\mathcal{D}_nK_m{}^k - A^{kl}K_{kl}\Big)\\ &+ \beta\sqrt{g}N^i\Big\{-2\epsilon^{mn}K_i{}^l\mathcal{D}_nK_{ml} - \epsilon^{mn}\mathcal{D}_k(K_{ni}K_m{}^k) + \frac{1}{2}\epsilon_{ij}\partial^jR^{(2)} + \mathcal{D}_kA_i{}^k\Big\}.\end{aligned} \tag{4.2.9}$$

In the above K_{ij} must be interpreted as $K_{ij} = H_{ij} + g_{ij}K/2$. Here A^{ij} is defined by the following equation:

$$\int d^3x\sqrt{g}\epsilon^{mp}\dot{\gamma}^l{}_{mn}\gamma^n{}_{pl} = -\int d^3x\sqrt{g}A^{ij}\dot{g}_{ij}. \tag{4.2.10}$$

By defining

$$T^{ijk}_{mno} \equiv \frac{1}{2}(\delta^k_m\delta^{(i}_o\delta^{j)}_n + \delta^k_n\delta^{(i}_o\delta^{j)}_m - \delta^k_o\delta^{(i}_m\delta^{j)}_n), \tag{4.2.11}$$

the time derivative of the affine connection is expressed as $\dot{\gamma}^l{}_{mn} = g^{lo}T^{ijk}_{mno}\mathcal{D}_k\dot{g}_{ij}$. After the partial integration in (4.2.10), A^{ij} is explicitly written as

$$\begin{aligned}A^{ij} &= \epsilon^{mp}g^{lo}T^{ijk}_{mno}\mathcal{D}_k\gamma^n{}_{pl}\\ &= \frac{1}{4}\epsilon^{kl}\mathcal{D}_k\gamma^i{}_l{}^j + \frac{1}{4}\epsilon^{il}\mathcal{D}_k\gamma^k{}_l{}^j - \frac{1}{4}\epsilon^{il}\mathcal{D}_k\gamma^j{}_l{}^k + \big(i \leftrightarrow j\big).\end{aligned} \tag{4.2.12}$$

Since A^{ij} depends on the affine connection in an explicit way, it does not behave as a tensor. Canonical variables in this Lagrangian are g_{ij}, $\dot{g}_{ij}$, H_{ij} and $\dot{H}_{ij}$, and Lagrange multipliers are K, N, N^i and v^{ij}. Note that v_{ij}, which is not a tensor, is symmetric under the exchange of indices.

By using this Lagrangian, we can construct the Hamiltonian in the canonical procedure. As usual, momenta conjugate to $\dot{g}_{ij}$ and $\dot{H}_{ij}$ are defined as

$$\begin{aligned}\pi^{ij} &\equiv \frac{\delta\mathcal{L}_{\rm TMG}}{\delta\dot{g}_{ij}} = v^{ij} - \frac{\beta}{2}\sqrt{g}K\epsilon^{k(i}H_k{}^{j)},\\ \Pi^{ij} &\equiv \frac{\delta\mathcal{L}_{\rm TMG}}{\delta\dot{H}_{ij}} = -\beta\sqrt{g}\epsilon^{k(i}H_k{}^{j)}.\end{aligned} \tag{4.2.13}$$

Note that $g^{-\frac{1}{2}}\pi^{ij}$ and $g^{-\frac{1}{2}}\Pi^{ij}$, instead of π^{ij} and Π^{ij}, do behave like tensors. From the second equation, we see that Π^{ij} and H_{ij} are not independent and the system is constrained. Again, such a kind of the constraint should be taken into account by introducing a Lagrange multiplier in the Hamiltonian formalism. Up to total derivative terms, the Hamiltonian of TMG is expressed as

$$\begin{aligned}&\mathcal{H}_{\rm TMG}\\ &= \pi^{ij}\dot{g}_{ij} + \Pi^{ij}\dot{H}_{ij} - \mathcal{L}_{\rm TMG} + f_{ij}\big(\Pi^{ij} - \beta\sqrt{g}\epsilon^{ik}H_k{}^j\big)\\ &\cong \sqrt{g}N\Big\{-R^{(2)} - \frac{2}{\ell^2} - K^{kl}K_{kl} + K^2 - 2\beta\epsilon^{mn}\mathcal{D}_k\mathcal{D}_nK_m{}^k + \big(2g^{-\frac{1}{2}}\pi^{kl} + \beta A^{kl}\big)K_{kl}\Big\}\\ &\quad + \sqrt{g}N^i\Big\{2\beta\epsilon^{mn}K_i{}^l\mathcal{D}_nK_{ml} + \beta\epsilon^{mn}\mathcal{D}_k(K_{ni}K_m{}^k) - \frac{1}{2}\beta\epsilon_{ij}\partial^jR^{(2)} - \mathcal{D}_j\big(2g^{-\frac{1}{2}}\pi_i{}^j + \beta A_i{}^j\big)\\ &\quad - \beta\mathcal{D}^j(K\epsilon_{k(i}H^k_{j)})\Big\} + f_{ij}\big(\Pi^{ij} - \beta\sqrt{g}\epsilon^{ik}H_k{}^j\big),\end{aligned} \tag{4.2.14}$$

where f_{ij} is the new Lagrange multiplier.

In order to derive correct equations of motion, it is necessary to add surface term $Q[\xi]$ to the Hamiltonian. By taking the two dimensional coordinates as (r, φ), the variation of the surface term $\delta Q[\xi]$ is given so as to cancel the total derivative terms in the variation of the Hamiltonian [30]:

$$
\begin{aligned}
&\delta Q[\xi] \\
&= \int d\varphi \Big[\sqrt{g} S^{ijkr} \big(\xi^0 \mathcal{D}_k \delta g_{ij} - \mathcal{D}_k \xi^0 \delta g_{ij} \big) + \xi^i (2\pi^{jr} + \beta g^{\frac{1}{2}} A^{jr}) \delta g_{ij} - \frac{1}{2} \xi^r (2\pi^{ij} + \beta g^{\frac{1}{2}} A^{ij}) \delta g_{ij} \\
&\quad + \xi_i \delta (2\pi^{ir} + \beta g^{\frac{1}{2}} A^{ir}) - 2\beta \sqrt{g} \epsilon^{mr} \mathcal{D}_k \xi^0 g^{kl} \delta K_{ml} + 2\beta \sqrt{g} \epsilon^{mn} \xi^0 \mathcal{D}_n (g^{rl} \delta K_{ml}) \\
&\quad + \frac{1}{2} \beta \sqrt{g} S^{ijkr} \big((\epsilon^{mn} \partial_m \xi_n) \mathcal{D}_k \delta g_{ij} - \mathcal{D}_k (\epsilon^{mn} \partial_m \xi_n) \delta g_{ij} \big) \\
&\quad - 2\beta \sqrt{g} \epsilon^{mr} \xi^i K_i{}^l \delta K_{ml} - \beta \sqrt{g} \epsilon^{mn} \xi^i (\delta K_{ni} K_m{}^r + K_{ni} \delta K_m{}^r) - 2\beta \sqrt{g} \epsilon^{mn} g^{rl} \xi^0 K_{mo} \delta \gamma^o{}_{nl} \\
&\quad - 2\beta \sqrt{g} \epsilon^{mn} g^{kl} g^{op} \big\{ - \mathcal{D}_k \xi^0 K_{mo} T^{ijr}_{nlp} + \xi^0 \mathcal{D}_o K_{ml} T^{ijr}_{knp} + 2\xi^0 \mathcal{D}_n K_{o(l} T^{ijr}_{m)kp} \big\} \delta g_{ij} \\
&\quad + 2\beta \sqrt{g} \epsilon^{mn} \xi^o \big\{ K_o{}^p K_{mk} g^{kl} T^{ijr}_{npl} + g^{qk} g^{lp} (K_{mk} K_{l(o} T^{ijr}_{n)qp} + K_{no} K_{l(k} T^{ijr}_{m)qp}) \big\} \delta g_{ij} \\
&\quad + \frac{1}{2} \beta \sqrt{g} \epsilon^{mn} T^{xyk}_{mlo} g^{op} \big\{ \mathcal{D}_k u_{xy} g^{lq} T^{ijr}_{npq} - u_{xy} \gamma^q{}_{np} g^{ls} T^{ijr}_{kqs} + 2u_{xy} \gamma^l{}_{q(p} g^{qs} T^{ijr}_{n)ks} \big\} \delta g_{ij} \\
&\quad - \frac{1}{2} \beta \sqrt{g} \epsilon^{mn} T^{ijr}_{mlo} u_{ij} g^{op} \delta \gamma^l{}_{np} + \frac{1}{2} \beta \sqrt{g} \epsilon^{mr} \xi_m \delta R^{(2)} \Big].
\end{aligned}
\tag{4.2.15}
$$

The index r which is not contracted represents the radial coordinate. And we here introduced $u_{ij}(\xi) \equiv 2\xi^0 K_{ij} + 2\mathcal{D}_{(i}\xi_{j)}$.

4.3 AdS$_3$/CFT$_2$ in the Most General Higher Derivative Gravity

Now let us evaluate the central charges in TMG. As discussed in Chapter 3, the diffeomorphisms which do not alter the boundary condition are labelled by $\xi_n^{\pm}$ in (3.1.6) whose components are given by (3.1.5). By using those Killing vectors, we can deform the global AdS$_3$ background $G^0_{\mu\nu}$.

A tedious calculation shows that the central extention for $\eta = \xi_n^+$ and $\xi = \xi_m^+$ becomes

$$
\frac{1}{16\pi G_N} \delta_{\eta=\xi_n^+} Q[\xi = \xi_m^+] = -\frac{i}{12} \frac{3\ell}{2G_N} \Big(1 + \frac{\beta}{\ell}\Big) m(m^2-1) \delta_{m,-n}. \tag{4.3.16}
$$

On the other hand, the central extention $\eta = \xi_n^-$ and $\xi = \xi_m^-$ is evaluated as

$$
\frac{1}{16\pi G_N} \delta_{\eta=\xi_n^-} Q[\xi = \xi_m^-] = -\frac{i}{12} \frac{3\ell}{2G_N} \Big(1 - \frac{\beta}{\ell}\Big) m(m^2-1) \delta_{m,-n}. \tag{4.3.17}
$$

Note that the signs of the correction piece to the Brown-Henneaux's result are different from each other. It is possible to check that $\delta_{\eta=\xi_n^+} Q[\xi = \xi_m^-] = 0$ from a similar calculation. If we call ξ_m^+ left mover and ξ_m^- right one, the central charges are written as [36, 38, 37, 30]

$$
c_L = \frac{3\ell}{2G_N} \Big(1 + \frac{\beta}{\ell}\Big), \qquad c_R = \frac{3\ell}{2G_N} \Big(1 - \frac{\beta}{\ell}\Big). \tag{4.3.18}
$$

Thus via canonical formalism of TMG, we have succeeded to realize the Virasoro algebras of left and right movers with the different central charges.

We are now in a position to generalize these results in order to encompass most general cases of higher derivative gravity. In (4.1.5) we have seen that inclusion of all higher derivative corrections other than the GCS term requires us to multiply the central charges of Brown and Henneaux's by the conformal factor Ω [70]. Making use of this simple scaling rule, we get to the following final expression of the central charges for the left and right movers:

$$c_L = \frac{3\ell}{2G_N}\Big(\Omega + \frac{\beta}{\ell}\Big), \qquad c_R = \frac{3\ell}{2G_N}\Big(\Omega - \frac{\beta}{\ell}\Big). \tag{4.3.19}$$

We would like to emphasize that all of the effects due to higher order terms are included in the factors Ω and β. Also note that these central charges are obtained by constructing Virasoro algebras directly.

From the modified definition of the mass and angular momentum of the BTZ black hole (4.1.6), the zero modes L_0^+ and L_0^- of Virasoro algebras for left and right movers are expressed as

$$L_0^\pm = \frac{1}{2}\left(M\ell \pm J\right) = \left(\Omega \pm \frac{\beta}{\ell}\right)\frac{1}{2}\left(m\ell \pm j\right). \tag{4.3.20}$$

Putting these together with (4.3.19), Cardy's formula for counting the states in CFT_2 agrees with the previous Wald-Tachikawa formula for the BTZ (4.1.7). For the BTZ black hole capturing the contributions of all higher derivative corrections, we have thus proved the agreement between the macroscopic entropy and the Cardy entropy of microstate counting. It was the main result of our paper [30] that the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence has been confirmed in the most general three dimensional gravity.

4.4 M5-system Revisited

In this section we again come back to $N = 2$ supergravity. As previously mentioned, the degeneracy of microstates is countable for the BPS and non-BPS black hole with higher derivative correction in the M5 configuration (2.3.36). On the macroscopic side, the entropy function is useful, but the four dimensional supergravity does not give the same answer for non-BPS as the microscopic one since some higher derivative terms are, in fact, missing [45]. It is, however, known that the five dimensional supergravity with eight supercharges does include a complete set of higher derivatives and allows us to get the correct entropy. Although a procedure is similar to the entropy function, we shall here carry out the dimensional reduction of this on S^2 and use the result obtained in the previous section.

The action of five dimensional supergravity coupled to n_v vector multiplets is given by

$$\mathcal{I}_{(5)} = \frac{1}{4\pi^2}\int d^5x\sqrt{-G_{(5)}}(\mathcal{L}_0 + \mathcal{L}_1), \tag{4.4.21}$$

where

$$\mathcal{L}_0 = -2\left(\frac{1}{4}D - \frac{3}{8}R^{(5)} - \frac{1}{2}(v_{MN})^2\right) + \mathcal{N}\left(\frac{1}{2}D + \frac{1}{4}R^{(5)} + 3(v_{MN})^2\right) + 2\mathcal{N}_a v_{MN}F^{aMN}$$
$$+ \mathcal{N}_{ab}\left(\frac{1}{4}F^a_{MN}F^{bMN} + \frac{1}{2}\partial_M M^a \partial^M M^b\right) + \frac{1}{24}c_{abc}A^a_M F^b_{NO}F^c_{PQ}\epsilon^{MNOPQ} \tag{4.4.22}$$

is a two derivative part and

$$\mathcal{L}_1 = \frac{c_{2a}}{24}\Bigg[\frac{1}{8}M^a(C_{MNOP})^2 + \frac{1}{12}M^a D^2 + \frac{1}{6}F^a_{MN}v^{MN}D - \frac{1}{3}M^a C_{MNOP}v^{MN}v^{OP}$$
$$- \frac{1}{2}F^{aMN}C_{MNOP}v^{OP} + \frac{4}{3}M^a\left\{(\mathcal{D}_M v_{NO})^2 + \mathcal{D}_M v_{NO}\mathcal{D}^N v^{OM}\right\}$$
$$+ \frac{8}{3}M^a\left\{v_{MN}\mathcal{D}^N\mathcal{D}_O v^{MO} + \frac{2}{3}v^{MO}v_{ON}R^N_M + \frac{1}{12}(v_{MN})^2 R^{(5)}\right\} - M^a(v_{MN})^2$$
$$- \frac{4}{3}F^{aMN}v_{MO}v^{OP}v_{PN} - \frac{1}{3}F^{aMN}v_{MN}(v_{OP})^2 + 4M^a v_{MN}v^{NO}v_{OP}v^{PM}$$
$$+ \epsilon_{MNOPQ}\Big\{\frac{1}{16}A^{aM}C^{NORS}C^{PQ}_{RS} - \frac{2}{3}M^a v^{MN}v^{OP}\mathcal{D}_R v^{QR}$$
$$+ \frac{2}{3}F^{aMN}v^{OR}\mathcal{D}_R v^{PQ} + F^{aMN}v^{OR}\mathcal{D}^P v^Q_R\Big\}\Bigg] \tag{4.4.23}$$

is a supersymmtric completion of four derivative [82]. Here $\mathcal{N}$, $\mathcal{N}_a$ and $\mathcal{N}_{ab}$ are functions of real scalars M^a:

$$\mathcal{N} = \frac{1}{6}c_{abc}M^a M^b M^c, \qquad \mathcal{N}_a = \frac{1}{2}c_{abc}M^b M^c, \qquad \mathcal{N}_{ab} = c_{abc}M^c, \tag{4.4.24}$$

and C_{MNOP} is the five dimensional Weyl tensor. Auxiliary fields are a two-form v_{MN} and a scalar D. This five dimensional supergravity comes from eleven dimensional supergravity compactified on Calabi-Yau threefold [83]. Note that the Chern-Simons term $\epsilon_{MNOPQ}A^{aM}C^{NORS}C^{PQ}_{RS}$ originates in R^4 terms in M-theory [84, 85] and reduces to the three dimensional GCS term after the dimensional reduction on S^2.

In the following we employ notations used in ref. [59]. Assuming that the five dimensional metric, gauge fields and 2-form auxiliary field v be given by

$$ds^2_{(5)} = \psi^2 G_{\mu\nu}dx^\mu dx^\nu + \chi^2 d\Omega^2_{S^2},$$
$$F^a_{\theta\varphi} = \frac{p^a}{2}\sin\theta, \qquad v_{\theta\varphi} = V\sin\theta, \tag{4.4.25}$$

we obtain the three dimensional supergravity with the curvature squared terms and the GCS term. Here p^a corresponds to the M5-brane charge. In order to realize the Einstein frame in three dimensions, we have to set

$$\psi^{-1} = \frac{\chi^2}{\pi}\left(\frac{3}{4} + \frac{1}{4}\mathcal{N} + \frac{c_{2a}M^a}{288\chi^2} + \frac{c_{2a}M^a V^2}{72\chi^4} - \frac{c_{2a}p^a V}{288\chi^4}\right). \tag{4.4.26}$$

Then the action becomes [59]

$$\mathcal{I}_{(3)} = \int d^3x\sqrt{-G}\left(R + Z(\phi) + A(\phi)R^2 + B(\phi)R_{\mu\nu}R^{\mu\nu}\right) + \int d^3x\mathcal{L}_{\rm CS} + \mathcal{S}', \tag{4.4.27}$$

in the unit of $16\pi G_N = 1$. Here ϕ stands generically for all scalars M^a , V , D and χ, and S' includes their derivative terms. The scalar potential and each coupling are given by

$$Z(\phi) = \psi^3 \frac{\chi^2}{\pi}\left\{\frac{2}{\chi^2}\left(\frac{3}{4}+\frac{\mathcal{N}}{4}\right) - 2\left(\frac{D}{4}-\frac{V^2}{\chi^4}\right) + \mathcal{N}\left(\frac{D}{2}+\frac{6V^2}{\chi^4}\right) + \frac{2\mathcal{N}_a p^a V}{\chi^4} + \frac{\mathcal{N}_{ab}p^a p^b}{8\chi^4}\right.$$
$$\left. + \frac{c_{2a}M^a}{96\chi^4} + \frac{c_{2a}M^a D^2}{288} + \frac{c_{2a}p^a VD}{144\chi^4} - \frac{5c_{2a}M^a V^2}{36\chi^6} - \frac{c_{2a}p^a V}{48\chi^6} + \frac{c_{2a}p^a V^3}{36\chi^8} + \frac{c_{2a}M^a V^4}{6\chi^8}\right\},$$

$$A(\phi) = -\frac{5}{6}\frac{c_{2a}M^a\chi^2}{192\pi\psi} \ , \quad B(\phi) = \frac{8}{3}\frac{c_{2a}M^a\chi^2}{192\pi\psi} \ , \quad \beta = -\frac{c_{2a}p^a}{96\pi}. \tag{4.4.28}$$

The action (4.4.27) enables us to derive an equation of motion for the metric

$$\frac{1}{2}G^{\mu\nu}\{R + AR^2 + B(R_{\mu\nu})^2 + Z\} - R^{\mu\nu} - 2ARR^{\mu\nu} - 2BR^{\mu\rho}R^\nu_\rho + T^{\mu\nu}$$
$$= \beta\epsilon^{\rho\sigma(\mu}\mathcal{D}_\rho R^{\nu)}_\sigma + (\text{derivative terms of } \phi), \tag{4.4.29}$$

and those for scalars

$$\partial_\phi Z + \partial_\phi A R^2 + \partial_\phi B(R_{\mu\nu})^2 = (\text{derivative terms of } \phi). \tag{4.4.30}$$

It is, however, almost impossible for us to find general solutions to these equations. What we can do is to take all ϕ to be constants everywhere and to assume the BTZ black hole which satisfies

$$\frac{1}{2}G^{\mu\nu}\left(R + \frac{2}{\ell^2}\right) - R^{\mu\nu} = 0, \tag{4.4.31}$$

accompanied with the effective cosmological constant ℓ. It corresponds to the black ring solution whose geometry is $\mathrm{AdS}_3 \times \mathrm{S}^2$ in five dimensions. By substituting it, the equations of motion (4.4.29) and (4.4.30) reduce to

$$Z = \frac{2}{\ell^2} + (3A + B)\left(\frac{2}{\ell^2}\right)^2,$$
$$\partial_\phi Z + 3\,(3\,\partial_\phi A + \partial_\phi B)\left(\frac{2}{\ell^2}\right)^2 = 0. \tag{4.4.32}$$

Since c_{2a} indicates the higher derivative corrections in the next order, we can solve five equations (4.4.32) to the first order of c_{2a}. The solutions are

$$M^a = \frac{p^a}{p}\left(1 - \frac{C}{36}\right) \ , \quad V = -\frac{3}{8}p\left(1 + \frac{C}{36}\right) \ , \quad D = \frac{12}{p^2}\left(1 - \frac{C}{18}\right) \ , \quad \chi = \frac{p}{2}\left(1 + \frac{C}{36}\right), \tag{4.4.33}$$

and

$$\ell = \frac{p^3}{4\pi}\left(1 + \frac{37}{288}C\right), \tag{4.4.34}$$

where $p^3 = \frac{1}{6}c_{abc}p^a p^b p^c$ and $C = c_{2a}p^a/p^3$. On the other hand, the conformal factor Ω for this solution is calculated as

$$\begin{aligned}\Omega(\ell) &= 1 + 2AR + \frac{2}{3}BR \\ &\simeq 1 - \frac{C}{288}.\end{aligned} \tag{4.4.35}$$

The assumption of constant scalars admits the BTZ black hole solution. Therefore, Brown-Henneaux's approach explained in the previous sections can be applied to this solution, and we can prove the existence of the CFT_2 satisfying the Virasoro algebra on the AdS_3 boundary. With the use of $16\pi G_N = 1$, $\beta = -c_{2a}p^a/96\pi$ and the formula (4.3.19), the central charges of the left and right movers are given by

$$\begin{aligned}c_L &= 6p^3 + \frac{1}{2}c_{2a}p^a, \\ c_R &= 6p^3 + c_{2a}p^a,\end{aligned} \tag{4.4.36}$$

in agreement with [47, 36, 86, 59]. In four dimensions, these central charges appear in expressions of the entropy for the extremal non-BPS and BPS black holes, respectively [48]. The precise information of the microstates for the CFT at the boundary is veiled in our formalism. Whatever the microstates may be, we can only see the Virasoro algebras and calculate their central charges. But as for the M5-brane system, the explanation for microstates was made clear in ref. [47] from the detailed description of the effective field theory on the brane.

4.5 Warped AdS_3 Space-time

So far, we have concentrated on the AdS_3 solutions in TMG or more general higher derivative theory. The AdS_3 is, however, one of solutions in TMG, and there is the other class of solutions which has $SL(2,R) \times U(1)$ isometry, called the warped AdS_3 [87, 88, 78, 68]. A spacelike warped AdS_3 metric

$$ds^2 = \frac{\ell^2}{\nu^2+3}\left[-\cosh^2\sigma d\tau^2 + d\sigma^2 + \frac{4\nu^2}{\nu^2+3}(du + \sinh\sigma d\tau)^2\right], \tag{4.5.37}$$

where $\nu = \ell/3\beta$, is a solution to (4.1.8), and corresponds to a vacuum, analogous with the global AdS_3. This geometry is actually equivalent to the near horizon behaviour of the extremal Kerr (3.2.28) at fixed polar angle.

Furthermore, a black hole solution

$$ds^2 = -N^2dt^2 + \ell^2R^2(d\varphi + N^\varphi dt)^2 + \frac{\ell^4 dr^2}{4R^2N^2}, \qquad (\varphi \sim \varphi + 2\pi), \tag{4.5.38}$$

where

$$\begin{aligned}R^2 &= \frac{r}{4}\left[3(\nu^2-1)r + (\nu^2+3)(r_+ + r_-) - 4\nu\sqrt{r_+r_-(\nu^2+3)}\right], \\ N^2 &= \frac{\ell^2(\nu^2+3)(r-r_+)(r-r_-)}{4R^2}, \\ N^\varphi &= \frac{2\nu r - \sqrt{r_+r_-(\nu^2+3)}}{2R^2},\end{aligned} \tag{4.5.39}$$

is allowed as an excited state analogous with the BTZ black hole, in a spcelike stretched case ($\nu^2 > 1$). It is obtained by a discrete identification of points on the vacuum warped AdS_3 (4.5.37) such that

$$\mathcal{P} \sim e^{2\pi k \xi}\mathcal{P}, \qquad k = 0, 1, 2\cdots. \tag{4.5.40}$$

The Killing vector ξ is expressed by

$$2\pi\xi = \partial_\varphi = \pi\ell(T_L J - T_R \tilde{J}), \tag{4.5.41}$$

in which we define

$$\begin{aligned} T_L &= \frac{(\nu^2+3)(r_+ - r_-)}{8\pi\ell}, \\ T_R &= \frac{(\nu^2+3)}{8\pi\ell}\left(r_+ + r_- - \frac{\sqrt{(\nu^2+3)r_+ r_-}}{\nu}\right), \end{aligned} \tag{4.5.42}$$

and

$$\begin{aligned} J &= 2\partial_u, \qquad (\in U(1)_L), \\ \tilde{J} &= -2\cos\tau\tanh\sigma\partial_\tau - 2\sin\tau\partial_\sigma - \frac{2\cos\tau}{\cosh\sigma}\partial_u, \qquad (\in SL(2,R)_R). \end{aligned} \tag{4.5.43}$$

It is an analogy with the BTZ case to refer to T_L and T_R as the left and right temperature. But if we can do this, it enables us to obtain the left and right moving central charges

$$c_L = \frac{4\nu\ell}{G_N(\nu^2+3)}, \qquad c_R = \frac{(5\nu^2+3)\ell}{G_N\nu(\nu^2+3)}, \tag{4.5.44}$$

from the Cardy formula for *the ordinary* CFT_2

$$\mathcal{S} = \frac{\pi^2\ell}{3}(c_L T_L + c_R T_R). \tag{4.5.45}$$

This comes from the usual first law of thermodynamics for the CFT_2, $dL_0^+ = T_L dS$ and $dL_0^- = T_R dS$, if it would exist. Because the left hand side of (4.5.45) is computed by the Wald-Tachikawa formula

$$\mathcal{S} = \frac{\pi\ell}{24\nu G_N}\left[(9\nu^2+3)r_+ - (\nu^2+3)r_- - 4\nu\sqrt{(\nu^2+3)r_+ r_-}\right], \tag{4.5.46}$$

one can derive the left and right central charges of the CFT_2 (4.5.44) by inserting (4.5.42) into (4.5.45).

The existence of the CFT_2 with the central charges (4.5.44) is a conjecture on the warped AdS_3 black hole by Anninos, Li, Padi, Song and Strominger [68]. It is an interesting and urgent problem whether we can reconstruct the Virasoro algebras of this CFT_2 and obtain the central charges c_L and c_R by the Brown-Henneaux formalism if it truly exists. Unfortunately, it has not completed yet although one copy of Virasoro times the $U(1)$ current algebra are found under a more relaxed boundary condition than Brown-Henneaux and Kerr/CFT [52]. To identify the dual theory of the warped AdS is not just a topic on lower dimensional gravity, but possibly leads to understanding of a more general correspondence rather than AdS/CFT, such as a *holographic non-relativistic CFT*, since this space-time in higher dimensions is often used for such a dual. Of course it is also notable that the warped AdS as the near horizon geometry of the extremal Kerr directly links to realistic black holes in our universe.

Part III

Holographic RG flow in Black Hole Space-time

In the last part we have discussed the dual description of gravity theory as CFT_2 by the canonical formulation of the Virasoro algebra from the (warped) AdS_3 side. We cannot extract any more information of these CFT_2 through this technique, but, at least in three dimensional gravity or the theories which can be reduced to it, the AdS_3/CFT_2 correspondence is an exact and universal feature of gravity, and very powerful to explain the black hole entropy microscopically. Also, the existence of the dual of the extremal Kerr was partially proved in a similar way.

At this point we want to ask what about gravity theory on the boundary of *arbitrary radius* r. The idea of the generalization of AdS/CFT begin to rise up within us. It is known as the gauge/gravity duality in the context of string theory. Since we have mainly used a cosmological *constant* in the Brown-Henneaux analysis, let us couple non-trivial scalars and consider the gravity with its potential. Then, if one identifies r of gravity solution as the renormalization scale and the scalars as the running couplings in dual quantum field theory (QFT), such a correspondence can be observed explicitly [23, 24, 22, 25]. The flow of the solution for the scalars can be interprited as the renormalization group (RG) flow, and the scalar potential corresponds to *c-function* [28]. Zamolodchikov's c-theorem tells us that this function monotonically decreses with repect to the ronormalization scale, and that at its fixed points the two dimensional field theory becomes conformally invariant and it becomes equal to just the central charge of the Virasoro algebra.

The above idea enables us to study a general framework of the microscopic description of gravity. This concept, the so-called holographic RG flow, is demonstrated by the Hamilton-Jacobi formalism. De Boer-Verlinde-Verlinde said that the Hamilton-Jacobi equation from the gravity side can be identified with the Callan-Symanzik equation of the dual theory [22]. In this part of the thesis, we apply this formalism to scalar-coupled three dimensional gravity and confirm the existence of the dual QFT, which eventually becomes CFT at the fixed points. As an example, we find a black hole solution interpolating between the asymptotic AdS_3 and the horizon AdS_3, which is dual to the RG flow of QFT_2 interpolating between UV CFT_2 and IR CFT_2 [31]. The other model in three dimensional gravity with the GCS term and the scalars is shown to be dual to the RG flow of parity-violating QFT_2 [33].

Moreover, we here notice that the holographic RG flow is just the attractor flow in extremal black hole space-time. In fact, we can explicitly show their connection in the example of the M5 system [32]. Namely, the attractor mechanism for the extremal black holes imply the existence of not only CFT but also QFT which holographically flows in the full black hole space-time. These features give us the possibility to realize a hologram of the black holes, like the Kerr, and understand the quantum aspect of gravity.

5 Hamilton-Jacobi Formalism

5.1 The c-theorem and Anomalies

First of all, we review the c-theorem for generic QFT_2 by Zamolodchikov [28]. For later use, in this section it is generalized to the one for a parity-violating QFT_2 [89]. We will holographically re-obtain these results from three dimensional gravity by the Hamilton-Jacobi formalism.

By using complex coordinates z and $\bar{z}$, the conservation law and the symmetrical property of the energy momentum tensor can be written as

$$\partial\bar{T} + \bar{\partial}\Theta = 0, \quad \bar{\partial}T + \partial\Theta = 0, \tag{5.1.1}$$

where we define $\partial = \partial/\partial z$, $\bar{\partial} = \partial/\partial\bar{z}$, $T = T_{zz}$, $\bar{T} = T_{\bar{z}\bar{z}}$ and $\Theta = T_{z\bar{z}} = T_{\bar{z}z}$. We use these equations to study the scaling properties of two point functions of the energy-momentum tensor. For example, by using the second equation in (5.1.1) we can replace $\bar{\partial}T$ by $-\partial\Theta$. Then we find

$$\begin{aligned}\bar{\partial}\langle T(z,\bar{z})T(0,0)\rangle &= -\partial\langle\Theta(z,\bar{z})T(0,0)\rangle,\\ \bar{\partial}\langle T(z,\bar{z})\Theta(0,0)\rangle &= -\partial\langle\Theta(z,\bar{z})\Theta(0,0)\rangle.\end{aligned} \tag{5.1.2}$$

Note that we can generally write these correlators as

$$\begin{aligned}\langle T(z,\bar{z})T(0,0)\rangle &= \frac{F(z\bar{z})}{z^4}, \quad \langle T(z,\bar{z})\Theta(0,0)\rangle = \frac{G(z\bar{z})}{z^3\bar{z}},\\ \langle\bar{T}(z,\bar{z})\bar{T}(0,0)\rangle &= \frac{\bar{F}(z\bar{z})}{\bar{z}^4}, \quad \langle\bar{T}(z,\bar{z})\Theta(0,0)\rangle = \frac{\bar{G}(z\bar{z})}{\bar{z}^3 z},\\ \langle\Theta(z,\bar{z})\Theta(0,0)\rangle &= \frac{H(z\bar{z})}{z^2\bar{z}^2}.\end{aligned} \tag{5.1.3}$$

Substituting these into (5.1.2), we find

$$\begin{aligned}\mu\frac{d}{d\mu}F(\mu) &= 3G(\mu) - \mu\frac{d}{d\mu}G(\mu),\\ \mu\frac{d}{d\mu}G(\mu) &= G(\mu) + 2H(\mu) - \mu\frac{d}{d\mu}H(\mu),\end{aligned} \tag{5.1.4}$$

where we define a Lorentz invariant scale parameter $\mu = z\bar{z}$.

Combining these two equations, we can prove that a function defined by

$$C_L(\mu) = 2F(\mu) - 4G(\mu) - 6H(\mu) \tag{5.1.5}$$

is monotonically non-increasing along the RG flow, i.e.,

$$\mu\frac{d}{d\mu}\mathcal{C}_L(\mu) = -12H(\mu) \leq 0. \tag{5.1.6}$$

Note that this c-function has its extremum at the fixed points because the trace of the stress tensor $\Theta = T_{z\bar{z}}$ vanishes when the theory has conformal invariance. As is clear from the definition of F, G and H, the value of the c-function at the fixed points is then equal to the central charge of the left-moving Virasoro algebra c_L.

In ordinary case $\bar{F}$ and $\bar{G}$ are equal to F and G, respectively, but in parity-violating case this is not true since in terms of the complex coordinate the parity symmetry means the symmetry exchanging z and $\bar{z}$. Thus one can show that another independent function

$$\mathcal{C}_R(\mu) = 2\bar{F}(\mu) - 4\bar{G}(\mu) - 6H(\mu) \tag{5.1.7}$$

is also monotonically non-increasing,

$$\mu\frac{d}{d\mu}\mathcal{C}_R(\mu) = -12H(t) \leq 0. \tag{5.1.8}$$

Its value at the fixed points is then equal to the central charge of the right-moving Virasoro algebra c_R. Of course, c_L and c_R take the same value in parity-invariant theory.

Nevertheless, it is apparent from (5.1.6) and (5.1.8) that

$$\mu\frac{d}{d\mu}(\mathcal{C}_L(\mu) - \mathcal{C}_R(\mu)) = 0. \tag{5.1.9}$$

The two c-functions, $\mathcal{C}_L(\mu)$ and $\mathcal{C}_R(\mu)$, differ from each other in general and monotonically decrese separately, but their difference is still a constant along the RG flow.

We now briefly discuss the relation of the c-functions to the Weyl and gravitational anomalies [1]. In order to analyse these, here we consider the two dimensional field theory coupled to some curved background.

It is well-known that, at the fixed point of the RG flow, the Weyl anomaly is expressed as

$$\langle T^i{}_i \rangle \;=\; \frac{1}{24\pi}\frac{c_L + c_R}{2}R^{(2)}, \tag{5.1.10}$$

in terms of the sum of the two central charges. It is related to the number of dynamical degrees of freedom of fluctuating fields. Away from the fixed point, the Weyl anomaly (5.1.10) receives corrections proportional to beta functions. But nevertheless, the coefficient of the scalar curvature must be still related to the effective degrees of freedom. Hence, it is natural to regard the coefficient as *the sum of the two c-functions*, $\mathcal{C}_L(\mu) + \mathcal{C}_R(\mu)$, along the RG flow.

Now let us discuss the gravitational anomaly. At the fixed point, it is expressed as a violation of momentum conservation

$$\nabla_i \langle T^{ij} \rangle \;=\; -\frac{c_L - c_R}{96\pi}\epsilon^{jk}\partial_k R^{(2)}. \tag{5.1.11}$$

[1] From here, we move from Euclidean to Minkowski signature.

The gravitational anomaly occurs due to the lack of a regularization which preserves general covariance, and the general covariance of the quantum action is broken by parity-violating one-loop diagrams [90]. This is proportional to the difference between two central charges, and thus we obtained the above expression. Away from the fixed point, in contrast to the Weyl anomaly, eq. (5.1.11) does not receive any corrections. For the same reason as before, it is natural to identify the coefficient of the r.h.s. of (5.1.11) with *the difference of the two c-functions*, $\mathcal{C}_L(\mu) - \mathcal{C}_R(\mu)$, along the RG flow.

5.2 Hamilton-Jacobi Equations in 3D Gravity

A key of the gauge/gravity correspondence is that the radial coordinate of the gravity theory is related to the energy scale of the field theory on the boundary. Then RG flow of the field theory is understood from the gravity side as the variation of boundary values along the radial coordinate. This is the so-called holographic RG flow, and can be well analyzed by using Hamilton-Jacobi formalism [22]. Let us quickly review this formalism below.[2] For a while, we consider three dimensional gravity coupled scalar fields up to two derivative[3]

$$\mathcal{I}_{(3)} = \frac{1}{16\pi G_N}\int d^3x\sqrt{-G}\left(R - V(\phi) - \frac{1}{2}L_{AB}(\phi)\partial_\mu\phi^A\partial^\mu\phi^B\right). \tag{5.2.12}$$

The absence of the GCS term means that the dual two dimensional QFT is parity invariant.

The derivation of Hamilton-Jacobi equation in gravity theory resembles the canonical formalism described in last part of this thesis. The only difference is to take the radial coordinate ρ as *time*. Then, we reparametrize the metric so as to be an Euclidean ADM form[4]

$$ds^2 = N^2 d\rho^2 + g_{ij}\left(dx^i + N^i d\rho\right)\left(dx^j + N^j d\rho\right). \tag{5.2.13}$$

Here x^i parametrizes the two dimensional space-time. As will be seen later, we actually consider gravity solutions in which the scalars do not depend on x^i and the two dimensional metric g_{ij} takes the form $g_{ij} = \eta_{ij}/\mu^2(\rho)$, where $\mu \to 0$ as $\rho \to \infty$ and $\mu \to \infty$ as $\rho \to r_0$. Since μ gives the length scale of the two dimensional field theory, we see that UV region corresponds to the spatial infinity, and IR region does to the horizon.

We insert the ADM decomposition of the metric into the Lagrangian and define momenta π^{ij} and π_A conjugate to g_{ij} and ϕ^A in an usual way. Up to total derivative terms, the Hamiltonian density is then expressed as $\mathcal{H}_E = N\mathcal{H} + N^i\mathcal{P}_i$ in which $\mathcal{H}$ and $\mathcal{P}^i$ are defined by

$$\frac{1}{\sqrt{-g}}\mathcal{H} = \frac{1}{(-g)}\left((\pi^i_i)^2 - (\pi_{ij})^2 - \frac{1}{2}L^{AB}\pi_A\pi_B\right) + V(\phi) - R^{(2)} + \frac{1}{2}L_{AB}\partial_i\phi^A\partial^i\phi^B, \tag{5.2.14}$$

$$\frac{1}{\sqrt{-g}}\mathcal{P}^i = -2\nabla_j\left(\frac{1}{\sqrt{-g}}\pi^{ij}\right) + \frac{1}{\sqrt{-g}}\pi_A\partial^i\phi^A. \tag{5.2.15}$$

[2] See also references [91, 92, 93, 94, 95] for more discussions on the holographic RG.

[3] There are a lot of examples in scalar-coupled three dimensional gravity [96, 97, 98, 99, 100, 101, 102].

[4] Here we employ the same notations for canonical variables, N, N^i and so on, as in the previous part, rather than introducing new ones. Hopefully it might not make any confusion.

Here L^{AB} is the inverse scalar metric of L_{AB}. It is apparent that $\mathcal{H} = \mathcal{P}^i = 0$ since N and N^i are just the Lagrange multipliers.

Now let $\overline{g}_{ij}(x, \rho)$ and $\overline{\phi}^A(x, \rho)$ be the classical solutions of the bulk theory. Then we denote the cut-off scale as ρ_c, and represent boundary values like $\overline{g}_{ij}(x, \rho_c) = g_{ij}(x)$ and $\overline{\phi}^A(x, \rho_c) = \phi^A(x)$. Substituting the classical solutions into the Lagrangian and integrating over the three dimensions, we obtain a functional with respect to g_{ij} and ϕ^A. We denote this functional as $S[g, \phi; \rho_c] = 16\pi G_N \mathcal{I}_{(3)}$. Using the equations of motion, the variation of $S[g, \phi; \rho_c]$ with respect to ρ_c, $g_{ij}(x)$ and $\phi^A(x)$ is given by

$$\delta S[g, \phi; \rho_c] = \frac{\partial S}{\partial \rho_c}\delta\rho_c + \int d^2x \frac{\delta S}{\delta \dot{g}_{ij}(x)}\delta g_{ij}(x) + \int d^2x \frac{\delta S}{\delta \dot{\phi}^A(x)}\delta\phi^A(x). \tag{5.2.16}$$

Combining this relation with $\frac{dS}{d\rho_c} = \int d^2x \mathcal{L}_{\mathrm{E}}$, we find that the classical action is independent of ρ_c,

$$\frac{\partial}{\partial \rho_c} S[g, \phi; \rho_c] = -\int d^2x \left(N\mathcal{H} + N^i\mathcal{P}_i\right) = 0, \tag{5.2.17}$$

and the boundary values of the conjugate variables are

$$\pi^{ij}(x) = \frac{\delta S}{\delta g_{ij}(x)}, \qquad \pi_A(x) = \frac{\delta S}{\delta \phi^A(x)}. \tag{5.2.18}$$

Thus, the Hamilton-Jacobi equation reduces to only two constraints,

$$\mathcal{H}\left(g_{ij}(x), \phi(x), \pi^{ij}(x), \pi_\phi(x)\right) = 0, \qquad \mathcal{P}^i\left(g_{ij}(x), \phi(x), \pi^{ij}(x), \pi_\phi(x)\right) = 0, \tag{5.2.19}$$

with eq. (5.2.18). From the constraint $\mathcal{H} = 0$ one obtains the following equation,

$$\frac{1}{(\sqrt{-g})^2}\left[-\left(g_{ij}\frac{\delta S}{\delta g_{ij}}\right)^2 + \left(\frac{\delta S}{\delta g_{ij}}\right)^2 + \frac{1}{2}L^{AB}\frac{\delta S}{\delta \phi^A}\frac{\delta S}{\delta \phi^B}\right] = V(\phi) - R^{(2)} + \frac{1}{2}L_{AB}\partial_i\phi^A\partial^i\phi^B. \tag{5.2.20}$$

As we will see later, it is possible to derive the conformal anomaly or the Callan-Symanzik equation from this equation. The constraint $\mathcal{P}^i = 0$ implies the invariance under the diffeomorphism of the theory in two dimensional space-time with ρ fixed. However, the two dimensional covariance is broken in the case where GCS term is added into three dimensional gravity, as mentioned later.

5.3 Holographic RG Flow and c-function

Now let us solve the Hamilton-Jacobi equation (5.2.20). First, since the bulk action diverges by taking $\rho_c \to \infty$, it is necessary to subtract such UV divergence. For this purpose we divide the functional $S[g, \phi]$ into the local counter-term and the non-local part $\Gamma[g, \phi]$, which is the generating functional with respect to the external sources $g_{ij}(x)$ and $\phi^A(x)$. Next we assign a weight w to each variable such that $w = 0$ for $g_{ij}(x)$, $\phi^A(x)$ and $\Gamma[g, \phi]$ and $w = 1$ for ∂_i. From these assignment and an equation $\delta\Gamma = \int d^2x(\delta g_{ij}(x)\delta\Gamma/\delta g_{ij}(x) + \delta\phi^A(x)\delta\Gamma/\delta\phi^A(x))$, quantities $R^{(2)}$, $\delta\Gamma/\delta g_{ij}(x)$ and $\delta\Gamma/\delta\phi^A(x)$ turn out to be $w = 2$.

An integrand of the local counter-term with $w = 0$ is written by a function of only the scalar field, $W(\phi)$, and hence the classical action $S[g, \phi]$ is expressed as[5]

$$S[g,\phi] = \int d^2x\sqrt{-g}\left\{W(\phi) + \cdots\right\} + 16\pi G_N \Gamma[g,\phi]. \tag{5.3.21}$$

The dots represent integrands of local counter-terms with $2 < w$. Substituting this into (5.2.20) and comparing the terms with $w = 0$, we obtain

$$V(\phi) = -\frac{1}{2}W(\phi)^2 + \frac{1}{2}L^{AB}\frac{\partial W(\phi)}{\partial\phi^A}\frac{\partial W(\phi)}{\partial\phi^B}. \tag{5.3.22}$$

From the terms with $w = 2$ in eq. (5.2.20), we obtain the following relation,

$$\langle\, T^i_i(x)\,\rangle = \frac{1}{8\pi G_N}\frac{1}{W(\phi)}R^{(2)} + \beta^A(\phi)\frac{1}{\sqrt{-g}}\frac{\delta\Gamma}{\delta\phi^A(x)} - \frac{1}{16\pi G_N}\frac{1}{W(\phi)}L_{AB}\partial_i\phi^A\partial^i\phi^B. \tag{5.3.23}$$

Here the energy momentum tensor is defined as

$$\langle\, T^{ij}(x)\,\rangle = \frac{2}{\sqrt{-g}}\frac{\delta\Gamma[g,\phi]}{\delta g_{ij}(x)}, \tag{5.3.24}$$

and $\beta^A(\phi)$ is given by

$$\beta^A(\phi) = \frac{2L^{AB}}{W(\phi)}\frac{\partial W(\phi)}{\partial\phi^B}. \tag{5.3.25}$$

The notation $\beta^A(\phi)$ is adopted here, since it can actually be interpreted as the beta function for the dual field theory if the parameter μ in $g_{ij} = \eta_{ij}/\mu^2$ and the scalars ϕ^A are regarded as the scale and the running couplings, respectively, of the two dimensional theory of x^i-space at a fixed ρ-slice. We will see that it is really possible to identify $\beta^A(\phi)$ with the beta function for specific gravity solutions presented in the next chapter.

Now we assume that the scalar field $\phi^A(x)$ is homogeneous on the two dimensional surface. Then the third term in the right hand side of eq. (5.3.23) becomes zero. Furthermore, the second term vanishes at the points $\rho = \rho_*$ where $\beta^A(\phi)$ vanishes. Therefore we obtain

$$\langle\, T^i_i(x)\,\rangle\Big|_{\rho=\rho_*} = \frac{1}{24\pi}\frac{3}{G_N W(\phi)}R^{(2)}\Big|_{\rho=\rho_*}. \tag{5.3.26}$$

The critical point of the beta function indicates that the two dimensional theory is conformally invariant, and the above equation corresponds to the conformal anomaly for the CFT_2. We can read off these central charges as $\frac{3}{G_N W(\phi)}|_{\rho=\rho_*} = c$. The left and right moving central charges agree with each other in this case. As a remark, it is found that (5.2.15) leads to $\nabla_j\langle T^{ij}\rangle = 0$. This guarantees the absence of the gravitational anomaly.

From the calculation of the central charges for the CFT_2 (5.3.26), we can think of a function at any value of ρ,

$$C(\phi) = \frac{3}{G_N W(\phi)}. \tag{5.3.27}$$

[5]It is possible to consider integrands of local counter-terms with $w = 2$, such as $\Phi(\phi)R^{(2)}$ and $M_{AB}(\phi)\partial_i\phi^A\partial^i\phi^B$, but these can be absorbed into the non-local term Γ for the present case [92].

This is the so-called c-function for the dual field theory. If the function $W(\phi)$ is non-negative and L_{AB} is positive-definite, which is satisfied in explicit examples, it is clear that

$$\mu\frac{dC(\phi)}{d\mu} = \beta^A(\phi)\frac{dC}{d\phi^A} = -\frac{3}{2G_N W(\phi)}\beta^A(\phi)L_{AB}\beta^B(\phi) \leq 0. \tag{5.3.28}$$

The equality is satisfied only at $\rho = \rho_*$ where the dual theory becomes conformally invariant. The monotonicity of this function (5.3.27) is consistent with the c-theorem [28].

It is more striking that the relation (5.3.23) obtained from the Hamilton-Jacobi equation implies the Callan-Symanzik equation for the two dimensional field theory. Let us assume that $\Gamma[g,\phi]$ is the generating functional of the correlation function in which ϕ^A appears as an external field for a scaling operator $\mathcal{O}_A(x)$. Then n point function in the background of g_{ij} and ϕ^A is given by

$$\langle \mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_n}(x_n)\rangle_{g,\phi} = \frac{1}{\sqrt{-g}}\frac{\delta}{\delta\phi^{A_1}(x_1)}\cdots\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\phi^{A_n}(x_n)}\Gamma[g,\phi], \tag{5.3.29}$$

and ordinary n point function $\langle \mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_{A_n}(x_n)\rangle$ is obtained by setting $g_{ij} = \frac{1}{\mu^2}\eta_{ij}$ and $\phi^A = \phi^A(\rho)$ in the above equation.

Acting n functional derivatives $\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\phi^{A_1}(x_1)}\cdots\frac{1}{\sqrt{-g}}\frac{\delta}{\delta\phi^{A_n}(x_n)}$ on eq. (5.3.23), we obtain

$$\begin{aligned}&\left[-2g_{ij}(x)\frac{\delta}{\delta g_{ij}(x)} + \beta^A(\phi(x))\frac{\delta}{\delta\phi^A(x)}\right]\langle \mathcal{O}_{A_1}(x_1)\mathcal{O}_{A_2}(x_2)\cdots\mathcal{O}_{A_n}(x_n)\rangle_{g,\phi}\\ &+\sum_{k=1}^{n}\delta(x-x_k)\frac{\partial\beta^B(\phi)}{\partial\phi^{A_k}}(x)\langle \mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_B(x_k)\cdots\mathcal{O}_{A_1}(x_n)\rangle_{g,\phi} = (\text{two derivative terms}).\end{aligned} \tag{5.3.30}$$

Therefore, by integrating this equation over two dimensional coordinate x^i and setting $g_{ij} = \frac{1}{\mu^2}\eta_{ij}$ and $\phi = \phi(\rho)$, it becomes

$$\begin{aligned}&\left(2\mu\frac{\partial}{\partial\mu} + \beta^A(\phi)\frac{\partial}{\partial\phi^A}\right)\langle \mathcal{O}_{A_1}(x_1)\mathcal{O}_{A_2}(x_2)\cdots\mathcal{O}_{A_n}(x_n)\rangle\\ &-\sum_{k=1}^{n}\gamma^B_{A_k}(\phi)\langle \mathcal{O}_{A_1}(x_1)\cdots\mathcal{O}_B(x_2)\cdots\mathcal{O}_{A_n}(x_n)\rangle = 0,\end{aligned} \tag{5.3.31}$$

where $\gamma^B_A(\phi) = -\partial\beta^B(\phi)/\partial\phi^A$ is the anomalous dimension. This is just the Callan-Symanzik equation of the two dimensional field theory.

In conclusion, with the help of the Hamilton-Jacobi formalism, we can derive the central charges for CFT_2 from the conformal anomaly, and they are certainly connected by the c-function defined from the three dimensional gravity theory. And what we have done here is to solve equations of motion canonically. Hence, it is remarkable that the equations of motion are solved if we choose the potential $V(\phi)$ so as to satisfy (5.3.22). In the next chapter, we use suitable $W(\phi)$s and consider gravity solutions dual to QFT_2 flowing along holographic RG.

6 Holographic Duals of Various Gravity Theories

6.1 The CFT_2-interpolating Black Hole

As the first example, let us start with a case coupled to a single scalar field ϕ in the action (5.2.12), putting $L_{AB} = 1$. The scalar potential $V(\phi)$ is chosen such as

$$V(\phi) = \frac{1}{8a^4\ell^2}\left(-16 - 4\phi^2 - \phi^4 + 32e^{-a^2+\frac{\phi^2}{4}} - 16e^{-2a^2+\frac{\phi^2}{2}} + 4\phi^2 e^{-2a^2+\frac{\phi^2}{2}}\right), \qquad (6.1.1)$$

where a is a dimensionless parameter [30]. A shape of the potential is illustrated in Fig. 6.1. Extrema of the potential are realized at $\phi = 0$ and $\phi = \pm 2a$, and the values of the potential energy become negative, $V(0) = -2/L^2$ and $V(\pm 2a) = -2/\ell^2$. Here we defined

$$L \equiv \frac{a^2\ell}{1 - e^{-a^2}}, \qquad (6.1.2)$$

which satisfies $\ell < L$ when $0 < a$. Note that there are constant scalar solutions, $\phi(r) = 0$ and $\phi(r) = 2a$. In these cases, the geometries reduce to extremal BTZ black holes which asymptotically become AdS_3 with the radius L and ℓ. Therefore it is expected that a solution which interpolates between $\phi = 0$ and $\phi = 2a$ will generate the CFT_2-interpolating black hole.

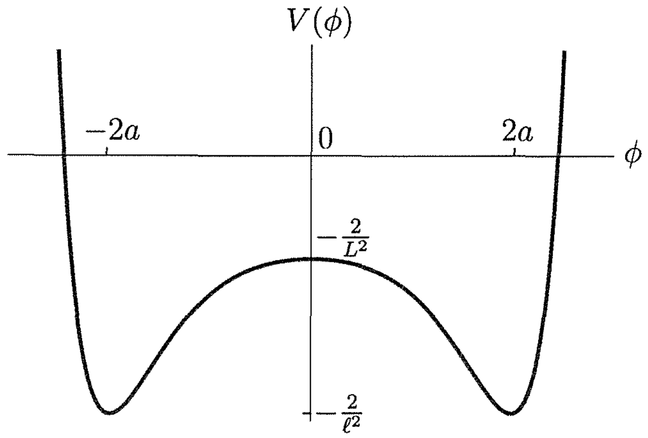

Figure 6.1: Shape of the potential $V(\phi)$.

In fact, it is possible to solve equations of motion under an ansatz

$$ds^2 = -e^{2f(r)}dt^2 + e^{2h(r)}dr^2 + r^2\left(d\varphi + e^{g(r)}\frac{dt}{\ell}\right)^2, \tag{6.1.3}$$

and obtain a solution for $f(r)$, $g(r)$, $h(r)$ and $\phi(r)$ which interpolate between $\phi(\infty) = 0$ and $\phi(r_0) = 2a$,

$$\begin{aligned}
e^{2f(r)} &= \frac{r^2}{a^4\ell^2}\left(e^{-a^2r_0^2/r^2} - e^{-a^2}\right)^2, \\
e^{2h(r)} &= \frac{a^4\ell^2}{r^2}\left[1 - e^{a^2\left(r_0^2/r^2-1\right)}\right]^{-2}, \\
e^{g(r)} &= \frac{1}{a^2}\left(1 - e^{-a^2r_0^2/r^2}\right), \\
\phi(r) &= 2a\frac{r_0}{r}.
\end{aligned} \tag{6.1.4}$$

Here we focus our attention on the region $0 \leq \phi \leq 2a$, that is, $r_0 \leq r$, and φ is the angular coordinate with the periodicity 2π. Notice that when $a = 0$, the solution becomes the extremal BTZ black hole.

The thermodynamical properties of the black hole are evaluated at the horizon $r = r_0$. The temperature is obtained by the inverse of the periodicity of the Euclidean time,

$$T = \frac{1}{2\pi}e^{-h}\frac{de^f}{dr}\bigg|_{r=r_0} = 0. \tag{6.1.5}$$

Therefore the solution corresponds to an extremal black hole. The Bekenstein-Hawking entropy is estimated by the area of the horizon A as

$$\mathcal{S}_{\mathrm{BH}} = \frac{A}{4G_N} = \frac{\pi r_0}{2G_N}. \tag{6.1.6}$$

In the following, we see that this solution represents the extremal black hole which enables us to have two AdS_3 geometries at the spatial infinity and the horizon. First let us investigate behaviors of the geometry around $r = \infty$ by taking the limit of $r_0 \ll r$. Then the solution (6.1.4) approaches

$$\begin{aligned}
ds^2 &\sim -\frac{r^2}{L^2}\left(1 - \frac{2Lr_0^2}{\ell r^2}\right)dt^2 + \frac{L^2}{r^2}\left(1 + \frac{2e^{-a^2}Lr_0^2}{\ell r^2}\right)dr^2 + r^2\left(d\varphi + \frac{r_0^2}{\ell r^2}dt\right)^2 \\
&\sim -\frac{r^2}{L^2}dt^2 + \frac{L^2}{r^2}dr^2 + r^2d\varphi^2,
\end{aligned} \tag{6.1.7}$$

where L is defined in eq. (6.1.2). This is just the AdS_3 geometry whose radius is L.

On the other hand, in order to find the near-horizon geometry of (6.1.3), we have to put

$$t = e^{a^2}\frac{t'}{\epsilon}, \qquad r = r_0 + \epsilon r', \qquad \varphi = \varphi' - \frac{e^{a^2}-1}{a^2\ell}\frac{t'}{\epsilon}, \tag{6.1.8}$$

and then in the limit $\epsilon \to 0$ we obtain

$$ds^2 \sim -\frac{4}{\ell^2}r'^2dt'^2 + \frac{\ell^2}{4}\frac{dr'^2}{r'^2} + r_0^2\left(d\varphi' - \frac{2}{r_0\ell}r'dt'\right)^2. \tag{6.1.9}$$

In fact, we can confirm that this is the same as the near horizon limit of the extremal BTZ black hole whose radius is ℓ. This near horizon behavior follows from the extremality of the black hole [51].

When the radial coordinate r is not around the spatial infinity $r = \infty$ or around the horizon $r = r_0$, the space-time is not AdS$_3$. Therefore our solution interpolates two AdS$_3$ geometries at the spatial infinity and the horizon. It was the main results of our paper [31] to find such an AdS$_3$-interpolating black hole and to make clear that its holographic interpritation is possible as seen below.

We give coordinate transformations which make the solution (6.1.3) with (6.1.4) into the Euclidean ADM form (5.2.13). Actually this can be done as follows,

$$\begin{aligned} ds^2 &= e^{2h(r)}dr^2 + \frac{r^2}{a^4\ell^2}\left[d\theta^2 - 2(e^{-a^2r_0^2/r^2} - e^{-a^2})dtd\theta\right] \\ &= N^2d\rho^2 + \frac{1}{\mu^2}\left[-(dx^+ + N^+d\rho)^2 + (dx^- + N^-d\rho)^2\right]. \end{aligned} \tag{6.1.10}$$

In the first line, we defined $\theta \equiv a^2\ell\varphi + (1 - e^{-a^2})t$. And in the second line we made the coordinate transformation of

$$\rho = r, \qquad x^\pm = \frac{1}{4}\left(\frac{\theta}{e^{-a^2r_0^2/r^2} - e^{-a^2}} - 2t\right) \mp \theta. \tag{6.1.11}$$

The two dimensional coordinate is denoted by $x^i = x^\pm$. The functions μ, N and N^i are written in terms of $(\rho, x^\pm)$ as

$$\mu^2 = \frac{a^4\ell^2}{\rho^2(e^{-a^2r_0^2/\rho^2} - e^{-a^2})}, \tag{6.1.12}$$

$$\begin{aligned} N^2 &= e^{2h(\rho)} = \frac{a^4\ell^2}{\rho^2}\left[1 - e^{a^2(r_0^2/\rho^2-1)}\right]^{-2}, \\ N^+ &= N^- = \frac{a^2r_0^2e^{-a^2r_0^2/\rho^2}}{4\rho^3(e^{-a^2r_0^2/\rho^2} - e^{-a^2})^2}(x^- - x^+). \end{aligned} \tag{6.1.13}$$

In (6.1.10) the coefficient μ can be regarded as the scale of the two dimensional theory of (x^+, x^-)-space at a certain ρ-slice. Since $\rho \to r_0$ means $\mu \to \infty$, let us call it the IR-region. On the other hand, $\rho \to \infty$ or $\mu \to 0$ indicates the UV-region. Also notice that the scalar field becomes $\phi = \phi(\rho)$.

If t and φ are tuned, we are able to choose arbitrary (x^+, x^-) at any radius ρ except for $\rho = r_0$. But we have to be careful of the transformation (6.1.11) at the horizon. For example, from the near horizon limit (6.1.8),

$$x^+ = \frac{e^{a^2}}{4}\left(\frac{\ell r_0}{2}\frac{\varphi'}{r'} - 2t'\right)\frac{1}{\epsilon} - a^2\ell\varphi' \tag{6.1.14}$$

is naively seen to be divergent as $\sim \mathcal{O}(1/\epsilon)$. However, since t', r' and φ' can be chosen arbitrary after rewriting (6.1.8), x^+ can be kept finite and taken arbitrary if the inside of the brackets of (6.1.14) is fine-tuned as $\sim \mathcal{O}(\epsilon)$ by the appropriate t', r' and φ'.

Now we are ready to perform the holographic RG flow described in the last chapter. The potential given by eq. (6.1.1) can be devided by the function $W(\phi)$ which is expressed by

$$W(\phi) = \frac{2}{a^2\ell}\left(\frac{\phi^2}{4} + 1 - e^{-a^2 + \frac{\phi^2}{4}}\right). \tag{6.1.15}$$

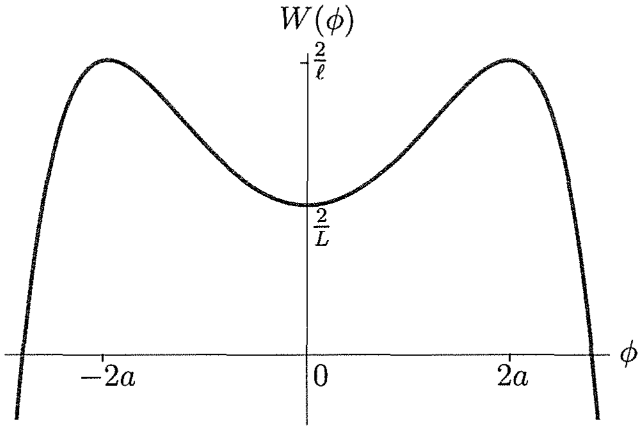

Figure 6.2: Shape of the function $W(\phi)$.

In the range of $0 \leq \phi \leq 2a$ (or $r_0 \leq \rho$), this function monotonically increases as ϕ does. (See Fig. 6.2.)

Namely, the function

$$\beta(\phi) = \frac{2}{W(\phi)} \frac{\partial W(\phi)}{\partial \phi} = \frac{\phi(1 - e^{-a^2+\phi^2/4})}{\frac{\phi^2}{4} + 1 - e^{-a^2+\phi^2/4}} \tag{6.1.16}$$

is non-negative. Moreover, one finds

$$\beta(\phi) = \mu \frac{d\phi}{d\mu} \tag{6.1.17}$$

from the solution of the scalar and (6.1.12). The parameter μ and the scalar ϕ are regarded as the scale and the running coupling, respectively, of the QFT_2 of x^i-slice at a fixed ρ-slice, so it is really possible to identify $\beta(\phi)$ with the beta function.

The Hamilton-Jacobi analysis enables us to claim the holographic flow of RG for the QFT_2. The theory becomes conformally invariant at the fixed points of the beta function, $\rho = \infty$ and $\rho = r_0$. The central charges of the Virasoro algebra are estimated by the critical values of the c-function (5.3.27) with (6.3.46). For the UV CFT ($\rho = \infty$), it gives

$$c_{UV} = \frac{3L}{2G_N}. \tag{6.1.18}$$

This is always larger than the central charge of the IR CFT ($\rho = r_0$),

$$c_{IR} = \frac{3\ell}{2G_N}, \tag{6.1.19}$$

because of $L > \ell$.

These results can also be realized by the Brown-Henneaux approach since at the critical points the solution has exact AdS_3 isometries. It is, however, obvious that the concept of the holographic RG is more general in that it constructs the beta function or c-function dual to gravity solution in the whole space-time.

Finally, let us compare to the black hole entropy. The ADM mass and angular momentum of our black hole at $\rho = \infty$ and $\rho = r_0$ appear as charges with respect to the time translation and angular translation. At UV ($\rho = \infty$), they are

$$M_{UV}L = J_{UV} = \frac{r_0^2}{4G_N\ell}, \tag{6.1.20}$$

and at IR ($\rho = r_0$)

$$M_{IR}\ell = J_{IR} = \frac{r_0^2}{4G_N\ell}. \tag{6.1.21}$$

In both cases, they must satisfy the extremality of the black hole $L_0^{-UV} = (M_{UV}L - J_{UV})/2 = 0$ and $L_0^{-IR} = (M_{IR}\ell - J_{IR})/2 = 0$. As a result, one finds that the Cardy formula for the IR CFT_2

$$\mathcal{S}_{IR} = 2\pi\sqrt{\frac{c_{IR}L_0^{+IR}}{6}} = \frac{\pi r_0}{2G_N} \tag{6.1.22}$$

completely agrees with the Bekenstein-Hawking formula (6.1.6), but is smaller than the one for the UV CFT_2

$$\mathcal{S}_{UV} = 2\pi\sqrt{\frac{c_{UV}L_0^{+UV}}{6}} = \frac{\pi r_0}{2G_N}\sqrt{\frac{L}{\ell}}. \tag{6.1.23}$$

6.2 The Kerr/CFT Correspondence in 4D Reissner-Nordstrøm Black Hole

In this section we come back to the discussion of the Kerr/CFT correspondence, but now would like to consider the dual description not only on the horizon but also in the full black hole space-time. Soon after the authors of ref. [11] originally proposed that the near horizon geometry of the extremal four dimensional Kerr can be mapped to the Virasoro algebra of CFT, their approach was also applied to the four dimensional Reissner-Nordstrøm black hole in refs. [103, 104]. By regarding the $U(1)$ gauge field as the one coming from the KK reduction of fifth dimension, one only has to treat the near horizon geometry of five dimensional rotating black holes. And moreover, if S^2 is compact and can be reducible by the KK reduction, one may define the holographic RG flow dual to the effective three dimensional gravity in the mannar previously described. In other words, the dual QFT exists on each boundary of the four dimensional Reissner-Nordstrøm black hole space-time. In this section, we again use the M5 system in the five dimensional supergravity in order to confirm this argument.

The five dimensional on-shell action up to two derivative [105]

$$\begin{aligned}\mathcal{I}_{(5)} =& \frac{1}{4\pi^2}\int d^5x\sqrt{-G_{(5)}}\Bigg[R^{(5)} - \frac{1}{2}\left(\frac{2\mathcal{N}_a\mathcal{N}_b}{3\mathcal{N}^2} - \frac{\mathcal{N}_{ab}}{\mathcal{N}}\right)\partial_M M^a\partial^M M^b \\ &- \frac{\mathcal{N}^{2/3}}{2}G_{ab}F^a_{MN}F^{bMN}\Bigg] + \frac{1}{4\pi^2}\int\frac{1}{6}c_{abc}A^a\wedge F^b\wedge F^c,\end{aligned} \tag{6.2.24}$$

is obtained after solving equations of motion of auxiliary fields D and v_{MN} in (4.4.22). [1]

[1] Strictly speaking, one has to solve D, redefine the moduli like $M^a \to \mathcal{N}^{-1/3}M^a$ and finally solve v_{MN}.

The functions $\mathcal{N}$, $\mathcal{N}_a$ and $\mathcal{N}_{ab}$ were given in (4.4.24) and

$$G_{ab} = \frac{1}{2}\left(\frac{\mathcal{N}_a\mathcal{N}_b}{\mathcal{N}^2} - \frac{\mathcal{N}_{ab}}{\mathcal{N}}\right). \tag{6.2.25}$$

Because we can fix the value of $\mathcal{N}$ which can be seen as a total volume of the CY_3, a constraint $\mathcal{N} = 1$ is often imposed due to the decoupling of hypermultiplets.

Here let us carry out a compactification on S^1. Decomposing the five dimensional metric and $U(1)$ gauge field like

$$\begin{aligned} ds^2_{(5)} &= e^{-s}(g^{(4)}_{mn}dx^m dx^n) + e^{2s}(dy + A_m dx^m)^2, \\ A^a &= A^a_m dx^m - a^a(dy + A_m dx^m), \end{aligned} \tag{6.2.26}$$

we have the four dimensional supergravity action (2.1.1) with $G_N^{(4)} = 1/8$ which is described by the prepotential (2.2.16). The indices $m, n, \cdots$ label the four dimensional space-time coordinate and $I, J, \cdots = (0, a)$. A new gauge field strength $F^0_{mn} = 2\partial_{[m}A_{n]}$ comes from KK $U(1)$ part. The complex scalar field z^a is given by a combination

$$z^a = a^a + iM^a e^s \mathcal{N}^{-1/3}, \tag{6.2.27}$$

and $G_{a\bar{b}}$, ν_{IJ} and μ_{IJ} are then expressed as

$$\begin{aligned} \nu_{IJ} &= \begin{pmatrix} -\frac{1}{3}c_{cde}a^c a^d a^e & \frac{1}{2}c_{acd}a^c a^d \\ \frac{1}{2}c_{bcd}a^c a^d & -c_{abc}a^c \end{pmatrix}, \qquad G_{a\bar{b}} = \frac{e^{-2s}\mathcal{N}^{2/3}}{2}G_{ab}, \\ \mu_{IJ} &= \begin{pmatrix} e^{3s} + 2\mathcal{N}^{2/3}e^s G_{cd}a^c a^d & -2\mathcal{N}^{2/3}e^s G_{ac}a^c \\ -2\mathcal{N}^{2/3}e^s G_{bc}a^c & 2\mathcal{N}^{2/3}e^s G_{ab} \end{pmatrix}. \end{aligned} \tag{6.2.28}$$

Now let us return to the five dimensional action

$$\mathcal{I}_{(5)} = \frac{1}{4\pi^2}\int d^5x\sqrt{-G_{(5)}}\left[R^{(5)} - G_{ab}\partial_M M^a \partial^M M^b - \frac{\mathcal{N}^{2/3}}{2}G_{ab}F^a_{MN}F^{bMN}\right] + \mathcal{I}_{CS}. \tag{6.2.29}$$

This is slightly different from (6.2.24). But it is useful for the Hamilton-Jacobi formalism since the coefficient matrix of the kinetic term of M^a has the inverse matrix in contrast to (6.2.24). Of course solutions to equations of motion are not affected if we recall $\mathcal{N} = 1$. As is noted in ref. [32], the final results are the same even if we assume to start with the action (6.2.24). Technically, to follow these procedures is the most significant point in finding the beta function and the c-function of the dual theory.

From now on we want to consider the M5-brane configuration, or D0-D4-brane configration in four dimensional view point. Since we have already known that this is the static spherically symmetric black hole in four dimensions, the dimensional reduction of the five dimensional action (6.2.29) on S^2

$$ds^2_{(5)} = e^{4\omega}(G^{(3)}_{\mu\nu}dx^\mu dx^\nu) + e^{-2\omega}d\Omega^2_{\mathrm{S}^2}, \qquad F^a_{\theta\varphi} = \frac{p^a}{2}\sin\theta \tag{6.2.30}$$

gives the effective three dimensional gravity. The Lagrangian is the following:

$$\mathcal{I}_{(3)} = \frac{1}{\pi}\int d^3x\sqrt{-G_{(3)}}\left[R^{(3)} - 6(\partial\omega)^2 - G_{ab}\partial_\mu M^a \partial^\mu M^b - V(\phi)\right], \tag{6.2.31}$$

where the potential term takes the form

$$V(\phi) = -2e^{6\omega} + \frac{e^{8\omega}}{4}\mathcal{N}^{2/3}G_{ab}p^a p^b. \tag{6.2.32}$$

Here ϕ^A denotes the three dimensional scalars $\phi^A = (\omega,\ M^a)$.

Incidentally, the solutions of the M5-brane configuration, or the five dimensional black ring has been already known. For the BPS black hole with zero axions, the metric and the moduli M^a are found by

$$ds^2_{(5)} = e^{-s}\left[-e^{2U}dt^2 + e^{-2U}(dr^2 + r^2 d\Omega^2_{S^2})\right] + e^{2s}(dy + Jdt)^2, \tag{6.2.33}$$

together with

$$e^{-4U} = H^3(-H_0), \qquad e^{2s} = \frac{-H_0}{H}, \qquad J = \frac{1}{H_0}, \tag{6.2.34}$$

and

$$M^a = \frac{H^a}{H}. \tag{6.2.35}$$

They are obtained by harmonic functions

$$H = \left(\frac{1}{6}c_{abc}H^a H^b H^c\right)^{1/3}, \qquad H^a = h^a + \frac{p^a}{2r}, \qquad H_0 = h_0 + \frac{q_0}{2r}. \tag{6.2.36}$$

In the BPS case, h_0 and q_0 are both negative. On the other hand, the four dimensional solution $z^a = iH^a(-H_0)^{1/2}H^{-3/2}$ and $e^{2U} = H^{-3/2}(-H_0)^{-1/2}$ are read off from eqs. (6.2.27) and (6.2.26).

Eq. (6.2.30) suggests that the M5 solution can be seen as the asymptotically flat BTZ black hole in three dimensions,

$$ds^2_{(3)} = -r^4\frac{H^3}{-H_0}dt^2 + r^4H^6dr^2 + r^4H^3(-H_0)\left(dy + \frac{1}{H_0}dt\right)^2, \tag{6.2.37}$$

plus the scalars (6.2.35) and

$$e^{-\omega} = e^{-U-s/2}r = rH. \tag{6.2.38}$$

This black hole is extremal, and in the holographic view point it corresponds to the QFT_2 in which only left movers are excited.[2] In the extremal non-BPS black hole, only right movers are excited.

In fact, after taking the near horizon limit, we have

$$ds^2_{(3)} \sim -\frac{p^3}{4(-q_0)}r^2dt^2 + \frac{(p^3)^2}{64}\frac{dr^2}{r^2} + \frac{-q_0p^3}{16}\left(dy - \frac{2}{-q_0}rdt\right)^2, \tag{6.2.39}$$

where $p^3 = \frac{1}{6}c_{abc}p^ap^bp^c$. This space-time has the isometry of AdS_3 with a radius $\ell = p^3/4$. As was shown in ref. [31], one can derive the Virasoro algebra for CFT_2 on the horizon through the Brown-Henneaux-like computation with the use of the boundary condition founded by ref. [11]. The central charge of the Virasoro algebra is expected to be $c =$

[2] At the conformal fixed point corresponding to the horizon, it can be checked that the right moving Virasoro charge $\overline{L}_0$ vanishes.

$3\ell/2G_N$. Because in our notation the three dimensional Newton constant is $G_N = 1/16$, one can obtain a well-known result without higher derivative corrections $c = 6p^3$ [47]. But here, we will not follow this calculation and will concentrate on c-function for the dual QFT_2 on each boundary.

In order to reparametrize the metric (6.2.37) for later convenience, let us introduce new coordinates

$$\rho = r, \qquad x^{\pm} = \frac{1}{4}(-H_0 y + 2t) \mp y. \tag{6.2.40}$$

Then, the metric (6.2.37) can be transformed into the ADM form (6.1.10) where

$$N = \rho^2 H^3, \qquad N^{\pm} = -\frac{q_0}{16\rho^2}(x^- - x^+), \qquad \mu^2 = \frac{1}{\rho^4 H^3}. \tag{6.2.41}$$

It is the same as before that the limit $\rho \to \infty$ and $\rho \to 0$ lead to $\mu \to 0$ and $\mu \to \infty$, respectively.

Now that we have written down the three dimensional action (5.2.12) with $L_{\omega\omega} = 12$, $L_{ab} = 2G_{ab}$ and $G_N = 1/16$, we can follow the Hamilton-Jacobi procedure. The inverse matrix L^{AB} is

$$L^{\omega\omega} = \frac{1}{12}, \qquad L^{ab} = \frac{1}{2}G^{ab} = \frac{1}{2}M^a M^b - \mathcal{N}\mathcal{N}^{ab}, \tag{6.2.42}$$

where $\mathcal{N}^{ab}$ is defined through $\mathcal{N}_{ab}\mathcal{N}^{bc} = \delta_a^c$.[3] One can finally obtain

$$W(\phi) = 4e^{3\omega} - \frac{1}{2}e^{4\omega}\mathcal{N}^{-2/3}\mathcal{N}_a p^a, \tag{6.2.43}$$

as a solution to eq. (5.3.22), using the inverse matrix (6.2.42). Note that at the horizon ($\rho = 0$) this function gives the value $W(\phi)|_{\rho=0} = 8/p^3$ from the exact solution (6.2.35) and (6.2.38). The central charge $c = 6p^3$ is, therefore, recovered from the Weyl anomaly of the CFT_2 at the horizon, as seen in (5.3.26).

It is a very remarkable thing that the holographic RG flow (5.3.25) with (6.2.43) is actually equivalent to the BPS attractor flow (2.2.22). The attractor, which is known as a characteristic behaviour in the extremal black hole solution, implies that the dual QFT flowing along RG flow really exists.[4] The horizon is of course special and can be described by CFT satisfying the Virasoro algebra. These statement is valid even in non-BPS case, and then the first order equation (2.2.23) with (2.2.29) can also be regarded as the RG flow. It was the main result of my paper [32] to make sure the correspondence between the attractor flow and the holographic RG flow.

We have holographically described the Reissnor-Nordstrøm black hole in the bulk. The similar issue on the Kerr black hole need further study, since it is difficult to write down the Hamilton-Jacobi equation for the Kerr solution.

6.3 Left-Right Asymmetric Holographic RG Flow

As a next example, we add the scalar-coupled gravity (5.2.12) to the GCS term (1.2.22). Its gravity solutions are holographically dual to left-right asymmetric RG flow of parity-violating QFT_2.

[3] In the case of the action (6.2.24) we cannot define the inverse matrix L^{AB} like (6.2.42).

[4] The relation between the attractor and the c-function is also discussed in refs. [106, 107].

By substituting the ansatz

$$ds^2 = dr^2 + e^{-2f(r)}\eta_{ij}dx^i dx^j, \quad \phi^I = \phi^I(r), \tag{6.3.44}$$

into equations of motion, one notices that first order equations

$$\dot{f} = \frac{1}{2}W(\phi), \qquad \dot{\phi}^A = L^{AB}(\phi)\frac{\partial W(\phi)}{\partial \phi^B} \tag{6.3.45}$$

are really the solutions when the potential term is written as (5.3.22). Since they are just the first order differential equations, we can always find a solution for any appropriate $W(\phi)$. For instance, we consider only one scalar field $\phi \equiv \phi^1$ with the metric $L_{11}(\phi) = 1$, and choose $W(\phi)$ like

$$W(\phi) = \frac{2}{\ell}\left(\sin\phi + \alpha\right), \tag{6.3.46}$$

where ℓ is some positive constant and $\alpha > 1$. This is monotonically non-decreasing in the region $-\pi/2 \leq \phi \leq \pi/2$. Now the solution is given by

$$\begin{aligned} f &= \frac{1}{2}\log(\cos\phi) - \alpha\tanh^{-1}\left(\tan\frac{\phi}{2}\right) + b, \\ \phi &= 2\arctan\left(\tanh\left(\frac{r-a}{\ell}\right)\right), \end{aligned} \tag{6.3.47}$$

in the region $-\infty < r < \infty$. Namely, the scalar field is represented by a kink solution.

As formulated in TMG, we decompose the action in ADM mannar by using the Ostrogradsky method. With the change of sign from treating r as time, the total action can then be written as

$$\begin{aligned} &S\left[g_{ij}, H_{ij}, K, \phi^A, \pi^{ij}, \Pi^{ij}, \pi_A, N, N^i, f_{ij}; r_0\right] \\ &= \int d^2x \int^{r_c} dr\left[\pi^{ij}\dot{g}_{ij} + \Pi^{ij}\dot{H}_{ij} + \pi_A\dot{\phi}^A - \left(N\mathcal{H} + N_i\mathcal{P}^i\right) + f_{ij}\left(\beta\sqrt{-g}\epsilon^{k(i}H^{j)}_{\ k} - \Pi^{ij}\right)\right], \end{aligned}$$

where r-integration has been cut off at $r = r_c$. Here we introduced the Hamiltonian and momentum as follows:

$$\begin{aligned} \frac{1}{\sqrt{-g}}\mathcal{H} :=& -R^{(2)} + V(\phi) + H^{kl}H_{kl} - \frac{1}{2}K^2 - \frac{1}{2(-g)}L^{AB}\pi_A\pi_B + \frac{1}{2}L_{AB}\partial_i\phi^A\partial^i\phi^B \\ &- 2\beta\epsilon^{mn}\nabla_k\nabla_n H_m^{\ k} + \left(\frac{2}{\sqrt{-g}}\pi^{kl} + \beta A^{kl}\right)\left(H_{kl} + \frac{1}{2}g_{kl}K\right), \end{aligned} \tag{6.3.48}$$

$$\begin{aligned} \frac{1}{\sqrt{-g}}\mathcal{P}^i :=& -2\beta\epsilon^{mn}K^{ik}\nabla_n K_{mk} - \beta\epsilon^{mn}\nabla_k\left(K_n^{\ i}K_m^{\ k}\right) + \beta\nabla_j\left(K\epsilon^{k(i}H^{j)}_{\ k}\right) \\ &- \frac{1}{2}\beta\epsilon^{ij}\partial_j R^{(2)} - \nabla_j\left(\frac{2}{\sqrt{-g}}\pi^{ij} + \beta A^{ij}\right) + \frac{1}{\sqrt{-g}}\pi_A\partial^i\phi^A. \end{aligned} \tag{6.3.49}$$

We find N, N_i, f_{ij} and K are Lagrange multipliers in the action. Path integrations over them lead to the following constraints:

$$\mathcal{H} = 0, \qquad \mathcal{P}^i = 0, \tag{6.3.50}$$

$$\Pi^{ij} = \beta\sqrt{-g}\epsilon^{k(i}H^{j)}_{\ k}, \tag{6.3.51}$$

$$K = \left(\frac{1}{\sqrt{-g}}\pi^{ij} + \frac{\beta}{2}A^{ij}\right)g_{ij}. \tag{6.3.52}$$

Since we put the Lagrange multiplier f_{ij}, the Hamilton-Jacobi equation acompanied by the addtional source $H_{ij}(x)$ can be written down similarly to the previous case. Again, one finds

$$\frac{\delta S}{\delta g_{ij}(x)} = \pi^{ij}(x), \quad \frac{\delta S}{\delta \phi^A(x)} = \pi_A(x), \quad \frac{\delta S}{\delta H_{ij}(x)} = \Pi^{ij}(x), \quad \frac{\delta S}{\partial r_c} = 0. \tag{6.3.53}$$

In order to interpret the constraints $\mathcal{H} = 0$ and $\mathcal{P}^i = 0$ as RG flow equations, one should assign the weight 0 and 2 to H_{ij} and $\delta\Gamma/\delta H_{ij}$, respectively and compare the term weight by weight. Hereafter, let us consider solutions which satisfy $H_{ij} = 0$ for simplicity, although it is an interesting task to relax this assumption.

Under the assumption, we find that the Hamiltonian constraint reduces to

$$-\frac{1}{2g}L^{AB}\frac{\delta S}{\delta\phi^A}\frac{\delta S}{\delta\phi^B} - \frac{1}{2}\left\{\left(\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g_{kl}} + \frac{\beta}{2}A^{kl}\right)g_{kl}\right\}^2 = -R^{(2)} + V(\phi) + \frac{1}{2}L_{AB}\partial_i\phi^A\partial^i\phi^B. \tag{6.3.54}$$

Here use has been made of (6.3.52). Since the scalar fields are regarded as coupling constants of the dual field theory, we set ϕ to be constant on the two-dimensional surface. By noting that A^{ij} has weight 2, the $w = 2$ flow equation turns out to be

$$g_{ij}\left(\frac{2}{\sqrt{-g}}\frac{\delta\Gamma}{\delta g_{ij}} + \frac{\beta}{16\pi G_{\rm N}}A^{ij}\right) - L^{AB}\frac{2}{W}\frac{\partial W}{\partial\phi^A}\left(\frac{1}{\sqrt{-g}}\frac{\delta\Gamma}{\delta\phi^B}\right) = \frac{1}{16\pi G_N}\frac{2}{W}R^{(2)}. \tag{6.3.55}$$

Putting (5.3.25) back into (6.3.55), we can regard (6.3.55) as the Weyl anomaly equation of the dual field theory on the curved backgrounds:

$$\langle T^i{}_i\rangle = \frac{1}{24\pi}\frac{3}{G_N W}R^{(2)} + \beta^A(\phi)\frac{1}{\sqrt{-g}}\frac{\delta\Gamma}{\delta\phi^A}. \tag{6.3.56}$$

Here the energy-momentum tensor T^{ij} is modified by adding the so-called Bardeen-Zumino term $\beta A^{ij}/16\pi G_N$, which makes it covariant [108], to the ordinary piece (5.3.24). At the fixed points where $\beta^A(\phi) = 0$ in the second term of the r.h.s of (6.3.56), we can read off the sum of the central charges of the left- and right-movers from the coefficients of the scalar curvature. Away from the fixed points, it is legitimate to define the sum of c-functions for left- and right-movers in the dual field theory as follows:

$$\mathcal{C}_L(\phi) + \mathcal{C}_R(\phi) = \frac{6}{G_N W(\phi)}. \tag{6.3.57}$$

On the other hand, $\mathcal{P}^i = 0$ means that the classical bulk action S is, in this case, not invariant under two-dimensional diffeomorphisms. More explicitly, by extracting $w = 3$ terms of both sides and set ϕ to be constant on the two-dimensional surface, we obtain

$$\nabla_i\langle T^{ij}\rangle = -\frac{3\beta}{G_N}\frac{1}{96\pi}\epsilon^{jk}\partial_k R^{(2)}, \tag{6.3.58}$$

which implies that, in the two-dimensional dual field theory, there is a gravitational anomaly proportional to the Chern-Simons coupling β. Hence, we can *define the difference between two c-functions for left-and right-mover* as

$$\mathcal{C}_L(\phi) - \mathcal{C}_R(\phi) = \frac{3\beta}{G_N}. \tag{6.3.59}$$

This is compatible with the result in the generalized c-theorem (5.1.9). Combining (6.3.57) and (6.3.59), we finally obtain the holographic expression of the left-right asymmetric c-functions:

$$\mathcal{C}_L(\phi) = \frac{3}{G_N}\left(\frac{1}{W(\phi)} + \frac{\beta}{2}\right), \quad \mathcal{C}_R(\phi) = \frac{3}{G_N}\left(\frac{1}{W(\phi)} - \frac{\beta}{2}\right). \tag{6.3.60}$$

It was the main result of our paper [33] to identify these c-functions in comparison to the generalized c-theorem.

For an appropriate potential such as the example (6.3.46), these c-functions are both monotonically non-increasing toward the infrared direction. In that example, the beta function is evaluated as

$$\beta(\phi) = \frac{2\cos\phi}{\sin\phi + \alpha}. \tag{6.3.61}$$

At a fixed point $\phi = \pi/2$ ($r = \infty$), three dimensional space-time becomes AdS_3,[5] and then we find that $W(\phi) = 2/\ell$ and (6.3.60) lead to the central charges

$$c_L = \frac{3}{G_N}\left(\frac{\ell}{2} + \frac{\beta}{2}\right), \quad c_R = \frac{3}{G_N}\left(\frac{\ell}{2} - \frac{\beta}{2}\right) \tag{6.3.62}$$

which we previously obtained by Brown-Henneaux's method [30].

In conclusion, we know that the Hamilton-Jacobi equation in three dimensional gravity with the scalar fields and the GCS term reproduces the RG flow for left-right asymmteric QFT_2. Both left and right c-functions monotonically decrese with respect to the renormalization scale, but their difference is always constant, which are just parallel to Zamolodchikov's c-theorem [28, 72].

6.4 Gravity Dual of the Minimal CFT Model ?

At this stage, one may notice that second order equations of motion in gravity theory is reducible to the first order ones if the scalar potential is chosen as (5.3.22). By using the Hamilton-Jacobi formalism, we can realize the RG flow of field theory under a suitable scalar function $W(\phi)$. One cannot, in a real sense, confirm the consistency of this correspondence any more, just as gravity theory on the background of $AdS_5 \times S^5$ was claimed to be dual to $N = 4$ super Yang-Mills gauge theory. But the beta function, the c-function and the anomalous dimension are at least fitted completely in our framework.

To see this, we try to construct the gravity dual of perturbed minimal CFT model. Let us consider the CFT_2 of the unitary discrete series $\mathcal{M}_p$ with the central charge [109]

$$c_p = 1 - \frac{6}{p(p+1)}, \qquad p = 2, 3, \cdots. \tag{6.4.63}$$

When we act the purterbation of the operator $\mathcal{O}_{13}$ with the scaling dimension

$$x = 2 - \frac{4}{p+1} = 2 - \epsilon \tag{6.4.64}$$

[5] Of course, there is another fixed point at $r = -\infty$.

upon this $\mathcal{M}_p$, it is known that the beta function for its coupling ϕ is, in a large p case, expressed by

$$\beta(\phi) = \epsilon\phi + \frac{2}{\sqrt{3}}\left(1 - \frac{3\epsilon}{4}\right)\phi^2 + \mathcal{O}(\phi^3). \tag{6.4.65}$$

Hence, there are the other fixed point $\phi = \phi^* = -\sqrt{3}\epsilon/2$. Since the c-function is related to the beta function via $\beta(\phi) = -\frac{2}{3}\frac{\partial \mathcal{C}}{\partial \phi}$, the value of $\mathcal{C}(\phi)$ at $\phi = \phi^*$ becomes

$$\mathcal{C}(\phi^*) = c_p - \frac{12}{p^3} + \mathcal{O}(p^{-4}) \simeq c_{p-1}. \tag{6.4.66}$$

In other words, it is concluded that $\mathcal{M}_p$ flows to $\mathcal{M}_{p-1}$ by the $\mathcal{O}_{13}$-purterbation.

The above setup is the most familiar and well analysed as a model to explicitly describe the c-theorem, or the RG flow of the field theory interpolating between CFTs [110, 111]. Our aim is to present the three dimensional gravity theory which is consistent with these qualitative behaviours of two dimensional QFT. Various realization may be possible, but we here use the AdS$_3$-interpolating black hole solution.

First, we take the action

$$\mathcal{I}_{(3)} = \frac{1}{16\pi G_N}\int d^3x\sqrt{-G}\left[R - \frac{1}{2}L(\phi)(\partial\phi)^2 + \frac{1}{2}W(\phi)^2 - \frac{1}{2L(\phi)}(\partial_\phi W(\phi))^2,\right] \tag{6.4.67}$$

and suppose an ansatz

$$ds^2 = -\frac{r^2}{\ell^2}e^{2g(r)}dt^2 + e^{2h(r)}dr^2 + r^2\left(d\varphi + e^{g(r)}\frac{dt}{\ell}\right)^2. \tag{6.4.68}$$

If one defines

$$\mu^2 = \frac{a^2\ell^2}{r^2}e^{-g}, \tag{6.4.69}$$

the equations of motion can be written as first order:

$$\frac{d\log\mu}{dr} = -\frac{e^h}{2}W, \tag{6.4.70}$$

$$\mu\frac{d\phi}{d\mu} = \frac{2\partial_\phi W}{LW}, \tag{6.4.71}$$

$$g' = \frac{2r_0^2}{\ell r^2}e^h. \tag{6.4.72}$$

Eqs. (6.4.69), (6.4.70) and (6.4.72) enable us to put

$$e^g = \frac{1}{a^2}\left(e^{-a^2r_0^2/r^2} - e^{-a^2}\right), \qquad e^h = \frac{a^2\ell}{r\left(1 - e^{-a^2+a^2r_0^2/r^2}\right)}, \tag{6.4.73}$$

$$W(\phi(r)) = \frac{1}{a^2\ell}\left(1 + a^2\frac{r_0^2}{r^2} - e^{-a^2+a^2r_0^2/r^2}\right). \tag{6.4.74}$$

Since we know the beta function (6.4.65) from the field theory analysis, the scalar ϕ in gravity theory must behave as

$$\phi(r) = \frac{-\mu^\epsilon}{1 + \frac{2}{\sqrt{3\epsilon}}\left(1 - \frac{3\epsilon}{4}\right)\mu^\epsilon}, \qquad \mu^2 = \frac{a^4\ell^2}{r^2\left(e^{-a^2r_0^2/r^2} - e^{-a^2}\right)}. \tag{6.4.75}$$

That is, ϕ becomes zero at $r = \infty$ and ϕ^* at $r = r_0$. Next, let us consider to tune the scalar function $L(\phi)$ to satisfy (6.4.71). After a short calculation, one finds that it should be of the form

$$L(\phi(r)) = \frac{4a^2r_0^2}{W^2r^4}\frac{e^{-2h}}{\beta(\phi)^2}. \tag{6.4.76}$$

Finally, we only have to choose ℓ/G_N and a such that the central charges satisfy

$$\begin{aligned} c_{UV} &= \frac{3\ell}{2G_N}\frac{a^2}{1 - e^{-a^2}} = c_p, \\ c_{IR} &= \frac{3\ell}{2G_N} = c_{p-1}. \end{aligned} \tag{6.4.77}$$

Though it is impossible to write down the explicit dependence of $W(\phi)$ and $L(\phi)$ on ϕ, we know the solution with respect to the radial coordinate r. The $\mathcal{O}_{13}$-perturbed RG flow interpolating between $\mathcal{M}_p$ and $\mathcal{M}_{p-1}$ has been projected to the AdS_3-interpolating black hole in three dimensional gravity by the Hamilton-Jacobi formalism.

However, we have no idea to confirm this correspondence further, at least in our framework. For example, it is known that the $\mathcal{O}_{13}$-perturbed minimal CFT model still has an infinite numbers of conserved charges and can be mapped to an integrable QFT, such as the sine-Gordon theory [112]. In order to construct the infinite conserved charges, it is necessary to use the information that the CFT at the critical point is really the minimal CFT model. We have not acquired the knowledge that even the AdS_3 gravity is dual to *what* CFT_2, except for a proposal by Witten that it may be a specific extremal CFT called the monter theory [113]. Nevertheless, if such a duality between AdS_3 and CFT_2 would confirm in the other analysis, it might be straightforward to define the holographic RG flow as seen here.

7 Toward a Construction of the Full Black Hole Hologram

7.1 Conclusion

Throughout this thesis, we discussed in detail the possibility of the holographic description of the black holes in the whole space-time. In other words, it was confirmed that the information of quantum gravity may be extracted from understanding of field theory on the boundary.

The macroscopic behaviors of the black holes, such as the attractor mechanism and the AdS near horizon geometry, are very important in (matter-coupled) gravity theory, as seen in Part I. We gave the naive observation and the explicit examples for the exact solutions about the attractor mechanism in Chapter 2. If they are true independently of supersymmetry, the black hole entropy in any gravity theory can be computed by using the entropy function formalism, which is based on the Wald formula presented in Chapter 1, even without the full solution to the equations of motion. Two properties of the extremal black holes, the attractor mechanism and the AdS_2 near horizon geometry, are not merely unique features of solutions of gravity, but a quite crucial hint to tell us that the dual field theory lies behind such thermodynamical behaviours.

Obviously, the horizon is a special point, and the AdS geometry there implies that the dual CFT is possible to describe the gravity theory. This mapping to CFT can be done by Brown-Henneaux's approach as noted in Part II. Especially in three dimensions, it was shown that the Virasoro algebra for CFT_2 is surely realized on the AdS_3 boundary even if any higher derivative correction is included in the action. In Chapter 4, it was shown that the GCS term was difficult to handle, but we could carry out the canonical method on it. As a result, the Wald-Tachikawa formula for the BTZ black hole completely agrees with the Cardy formula for the CFT_2. In addtion, taking into account that the Bekenstein-Hawking formula for the extremal Kerr is also reproduced by the Cardy formula, we have to conclude that the CFT, in a sense, carries the quantum information of gravity.

Furthermore, we discussed in Part III the wider dual framework between gravity and boundary field theory rather than the AdS/CFT correspondence. If the radial coordinate r and the scalar fields of gravity play the role of the scale and the running couplings of QFT, respectively, the Hamilton-Jacobi equation in gravity side is re-interprited as the RG flow of field theory on each r-boundary. The attractor flow is equivalent to this holographic RG flow and indicates that the dual QFT certainly exists on each boundary of the black hole space-time. In some examples in Chapter 6, we established the solutions in matter-coupled gravity and their holographic RG flows defined consistently to the generalized c-theorem.

If all consideration given above is true, our universe subjected to general relativity also has the dual description as field theory satisfying a certain RG flow equation. However, there is a limitation in our approach used here, such as Brown-Henneaux's canonical method or the Hamilton-Jacobi formalism. It is not possible to identify anymore our holographic duals defined by those procedures with a certain specific QFT, just as $N = 4$ super Yang-Mills gauge theory is considered to be dual to type IIB supergravity on the D3-brane background. Although the gravity dual of the minimal CFT model was presented as an example here, this should not be realistic since there are many kinds of CFT_2 satisfying the Virasoro algebra. The brane concept in string theory is now illuminating in the sense that it helps us consider both the gravitational source as black branes and gauge fields as the open string. But if we assumed the duality between bulk gravity and boundary field theory while not relying on the brane concept, the specification of the duality would be very difficult.

The monster theory is the sole candidate known as a specific CFT_2 dual to AdS_3 gravity. Witten conjectured that if and only if it is left-right holomorphically factorizable, the number of its primary operators creating the BTZ black holes exactly corresponds to the Bekenstein-Hawking formula [113]. It is so interesting, and the validity of this conjecture has been attempted but not concluded yet [114]. A nature of the chiral gravity in three dimensions may support it to some extent [60]. In order for the massive gravitons around AdS_3 in TMG (4.1.8) to keep unitarity (to have non-negative mass), the coupling of the GCS term must be $\beta = -\ell$ and then the dual CFT_2 becomes chiral, or $c_L = 0$ in (4.3.18). This suggestion seems to constrain left-right factorizability of the CFT_2, but has not been confirmed as well [61, 62, 63, 64, 65, 66, 67].

The attractor mechanism still has a problem to solve. In general, one has the multiple-centered black hole universe in Einstein-Maxwell system, where all of the gravitational attraction and the electromagnetic repulsion cancel out. In this case the attractor flow should split towards their respective centers [115]. How can one interpret this split flow if the attractor means the RG flow of the dual QFT? It is conjectured that this implies the decay of bounded branes (for example M5-branes) in string context [116]. But it is too hard to write the Hamilton-Jacobi equation in such solutions because we cannot perform the ADM decomposition with respect to "r". If this could be done, the split flow might be understood as a decomposition of running couplings, or a spontanous symmetry breaking of boundary field theory. A correct explanation for the split attractor flow is also closely related to the development of string theory itself, or gives more insights into the connection of (super)gravity to topological string theory [117].

To begin with, there is a fundamental difficulty in finding the underlying description of black holes in real world. We do not know even a complete proof on whether the dual of the extremal Kerr is really an ordinary CFT_2 because another set of Virasoro algebra is missing. It is necessary to further analyse the dual boundary theory which underlies the warped AdS_3 geometry in the near horizon of the Kerr or in TMG. These problems are left open for the future work.

Black holes are observed a lot in our universe. According to the paper [11], GRS 1915+105 in our Milky Way galaxy, with the mass $M \sim 14M_\odot$ obeying a near extremality $J/(G_N^{(4)}M^2) > 0.98$ [118], is dual to the CFT with the central charge $c \sim 2\times10^{79}$. Whether their dual is the minimal model or the monster, or still string theory, if we identify it with such a particular typical QFT in a real sense, this will be just a completion of quantum gravity, equal to the successful string theory. We hope to shed more light on this nature of quantum gravity hidden behind the literally black event horizon.

A AdS Space-time

A.1 Black Brane Solution of AdS Space-time

The AdS$_d$ geometry is realized on the hypersurface satisfying

$$(X^{-1})^2 + (X^0)^2 - (X^1)^2 \cdots - (X^{d-1})^2 = \ell^2 \tag{A.1.1}$$

in $\mathbb{R}^{2,d-1}$ space-time. When carrying out the coordinate transformation

$$\begin{aligned} X^{-1} &= \sqrt{\ell^2 + r^2} \cos\left(\frac{t}{\ell}\right), \\ X^0 &= \sqrt{\ell^2 + r^2} \sin\left(\frac{t}{\ell}\right), \\ X^i &= r\Omega^i_{d-1}, \qquad (i = d-1), \end{aligned} \tag{A.1.2}$$

in the region $0 \leq r \leq \infty$, one obtains the line element

$$\begin{aligned} ds^2 &= -(dX^{-1})^2 - (dX^0)^2 + (dX^1)^2 \cdots + (dX^{d-1})^2, \\ &= \frac{dr^2}{1 + \frac{r^2}{\ell^2}} - \left(1 + \frac{r^2}{\ell^2}\right) dt^2 + r^2 d\Omega^2_{d-2}. \end{aligned} \tag{A.1.3}$$

This is the globally AdS$_d$ solution to the d-dimensional gravity with the negative cosmological constant $d(d-1)/\ell^2$.

For instance, in three dimensions, putting

$$\begin{aligned} X^{-1} &= \sqrt{L^2 + r^2} \cos\left(\frac{t}{\ell}\right), \\ X^0 &= \sqrt{L^2 + r^2} \sin\left(\frac{t}{\ell}\right), \\ X^1 &= r\cos\varphi, \\ X^2 &= r\sin\varphi, \end{aligned} \tag{A.1.4}$$

we obtain (3.1.3). But when we transform the coordinate in $r > r_+$ region like

$$\begin{aligned}
X^{-1} &= \sqrt{B(r)}\sinh\left(\frac{r_+}{\ell^2}t - \frac{r_-}{\ell}\varphi\right), \\
X^{0} &= \sqrt{A(r)}\cosh\left(-\frac{r_-}{\ell^2}t + \frac{r_+}{\ell}\varphi\right), \\
X^{1} &= \sqrt{A(r)}\sinh\left(-\frac{r_-}{\ell^2}t + \frac{r_+}{\ell}\varphi\right), \\
X^{2} &= \sqrt{B(r)}\cosh\left(\frac{r_+}{\ell^2}t - \frac{r_-}{\ell}\varphi\right),
\end{aligned} \tag{A.1.5}$$

where

$$A(r) = \ell^2\frac{r^2 - r_-^2}{r_+^2 - r_-^2}, \qquad B(r) = \ell^2\frac{r^2 - r_+^2}{r_+^2 - r_-^2}, \tag{A.1.6}$$

the BTZ black hole solution (3.1.2) appears. This means an identification of points in the globally AdS_3, then it still has the isometry of locally AdS_3. The transformation for the extremal BTZ is also noted in ref. [119]

It is easy to make the AdS_4 solution, but now consider the following parametrization:

$$\begin{aligned}
X^{i} &= X^{i}_{\mathrm{AdS}_3}\cosh\left(\frac{x}{\ell}\right), \qquad (i = -1, 0, 1, 2), \\
X^{3} &= \ell\sinh\left(\frac{x}{\ell}\right).
\end{aligned} \tag{A.1.7}$$

In fact, this is a solution to Einstein gravity with the negative cosmological constant $6/\ell^2$. If we choose $X^i_{\mathrm{AdS}_3}$ as the BTZ black hole, the line element becomes

$$\begin{aligned}
ds^2 &= \cosh^2\left(\frac{x}{\ell}\right)\left[-N^2dt^2 + N^{-2}dr^2 + r^2(d\varphi + N^\varphi dt)^2\right] + dx^2. \\
N^2 &= \left(\frac{r}{\ell}\right)^2 + \left(\frac{4G_N j}{r}\right)^2 - 8G_N m \quad , \quad N^\varphi = \frac{4G_N j}{r^2},
\end{aligned} \tag{A.1.8}$$

This is everywhere locally AdS_4 since the Weyl tensor vanishes, and it is always regular except for $r = 0$. Especially there is no singularity in any x, which is clear from $R^{\mu\nu}R_{\mu\nu} = 36/\ell^4$ and $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = 24/\ell^4$. The singularity $r = 0$ of the BTZ is extended along x direction, so this is a four dimensional black string solution with an infinite mass. The Emparan-Horowitz-Myers black string is a generalization of this solution [120].

Similarly, one notices that d-dimensional Einstein gravity with the negative cosmological constant $(d-1)(d-2)/\ell^2$, in general, allows the locally AdS_d solution

$$\begin{aligned}
X^{i} &= X^{i}_{\mathrm{AdS}_3}\cosh\left(\frac{x_1}{\ell}\right)\cdots\cosh\left(\frac{x_{d-3}}{\ell}\right), \qquad (i = -1, 0, 1, 2), \\
X^{k+2} &= \ell\sinh\left(\frac{x_k}{\ell}\right)\cosh\left(\frac{x_{k+1}}{\ell}\right)\cdots\cosh\left(\frac{x_{d-3}}{\ell}\right), \qquad (1 \le k \le d-4), \\
X^{d-1} &= \ell\sinh\left(\frac{x_{d-3}}{\ell}\right).
\end{aligned} \tag{A.1.9}$$

When $X^i_{\mathrm{AdS}_3}$ is the parametrization of the BTZ black hole, this becomes the black $(d-3)$-brane, but there is also no singularity except for $r = 0$.

Inspired by these results, one may notice that in five dimensional gravity with $12/\ell^2$ the form

$$\left(ds_{(5)}^{[12/\ell^2]}\right)^2 = \cosh^2\left(\frac{x}{\ell}\right)\left(ds_{(4)}^{[6/\ell^2]}\right)^2 + dx^2 \tag{A.1.10}$$

is also a solution to the Einstein equation, where $\left(ds_{(4)}^{[6/\ell^2]}\right)^2$ is the four dimensional space-time with $\Lambda = 6/\ell^2$, for example the AdS-Schwarzchild or the AdS-Kerr. In the former case

$$\begin{aligned}\left(ds_{(4)}^{[6/\ell^2]}\right)^2 &= -e^{2f(r)}dt^2 + e^{-2f(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right),\\ e^{2f(r)} &= \frac{r^2}{\ell^2} + 1 - \frac{2M}{r},\end{aligned} \tag{A.1.11}$$

the scalar invariants are computed as follows: $R^{\mu\nu}R_{\mu\nu} = 40/\ell^4$, $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = 40/\ell^4 + 48M^2/(r^6\cosh^4(x/\ell))$. The solution is not locally AdS_5 except for $r = \infty$ any more, but still a regular black string in contrast to the Chamblin-Hawking-Reall black string [121] which is based on the Randall-Sundrum-like warped space-time [122].

The method of adding more cosh-form warp factors is applicable to any higher dimensional gravity. The instability of the black brane solutions is not discussed here, and in fact they have the infinite mass [123]. However, apart from the instability, it is in general easy for us to construct the following solution in d-dimensional gravity with $\Lambda = (d-1)(d-2)/\ell^2$:

$$\begin{aligned}&\left(ds_{(d)}^{[(d-1)(d-2)/\ell^2]}\right)^2\\ &=\cosh^2\left(\frac{x_{d-3}}{\ell}\right)\left(ds_{(d-1)}^{[(d-2)(d-3)/\ell^2]}\right)^2 + dx_{d-3}^2\\ &=\cosh^2\left(\frac{x_{d-3}}{\ell}\right)\left\{\cosh^2\left(\frac{x_{d-2}}{\ell}\right)\left(ds_{(d-2)}^{[(d-3)(d-4)/\ell^2]}\right)^2 + dx_{d-2}^2\right\} + dx_{d-3}^2\\ &=\cdots\\ &=\cosh^2\left(\frac{x_{d-3}}{\ell}\right)\left[\cosh^2\left(\frac{x_{d-2}}{\ell}\right)\left\{\cdots\left(\cosh^2\left(\frac{x_1}{\ell}\right)\left(ds_{(3)}^{[2/\ell^2]}\right)^2 + dx_1^2\right)\cdots\right\} + dx_{d-2}^2\right] + dx_{d-3}^2.\end{aligned} \tag{A.1.12}$$

The dimension of the black brane depends on where one inserts the asymptotically AdS black hole solution (the BTZ, the AdS-Schwarzchild and so on[1]) into the lower dimensional space-time [125].

[1]The general (A)dS-Kerr solution in arbitrary dimensions was obtained in the Kerr-Schild form [124].

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