# 対数的ホッジ構造のモジュライ空間の境界：トーザーの自明性

# On the boundary of the moduli spaces of log Hodge structures: Triviality of the torsor

巴山 竜来

Tatsuki Hayama

# On the boundary of the moduli spaces of log Hodge structures: Triviality of the torsor (対数的ホッジ構造のモジュライ空間の境界：トーサーの自明性)

Tatsuki Hayama

# Contents

# On the boundary of the moduli spaces of log Hodge structures: Triviality of the torsor

## 1. Introduction

Let $\varphi : (\Delta^*)^n \to \Gamma\backslash D$ be a period map arising from a variation of Hodge structures with unipotent monodromies on the $n$-fold product of the punctured disk. Here $\Gamma$ is the image of $\pi_1((\Delta^*)^n) \cong \mathbb{Z}^n$ by the monodromy representation, i.e., $\Gamma$ is a free $\mathbb{Z}$-module. By Schmid's nilpotent orbit theorem [**S**], the behavior of the period map around the origin is approximated by a "nilpotent orbit". Then we add the set of nilpotent orbits to $\Gamma\backslash D$ as boundary points and extend the period map satisfying the following diagram:

$$\begin{tikzcd}
(\Delta)^n \arrow[r] & \Gamma\backslash D \cup \{\text{nilpotent orbits}\} \\
(\Delta^*)^n \arrow[u, phantom, "\cup"] \arrow[r, "\varphi"] & \Gamma\backslash D. \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

Here $\Gamma\backslash D$ is an analytic space and $\varphi$ is an analytic morphism. But, except in some cases, we have no way to endow the upper right one with an analytic structure.

In [**KU**] Kato-Usui endow it with a geometric structure as a "logarithmic manifold" and they treat the above diagram as a diagram in the category of logarithmic manifolds. Moreover they define "polarized logarithmic Hodge structures" (PLH for abbr.) and they show that the upper right one is the moduli space of PLH. Our result is for the geometric structure of the moduli spaces of PLH in the following 2 cases:

(A) $D$ is a Hermitian symmetric space.
(B) Polarized Hodge structures (PH for abbr.) are of weight $w = 3$ and Hodge number $h^{p,q} = 1$ $(p + q = 3, p, q \geq 0)$ and logarithms of the monodromy transformations are of type $N_\alpha$ in [**KU**, §12.3] (or type $\mathrm{II}_1$ in [**GGK**]).

The case (A) corresponds to degenerations of algebraic curves or K3 surfaces. This case is classical and well-known. The case (B) corresponds to degenerations of certain Calabi-Yau three-folds, for instance those occurring in the mirror quintic family. This case is studied recently. For example, Griffiths et al ([**GGK**]) describe "Néron models" for VHS of this type. Usui ([**U**]) shows a logarithmic Torelli theorem for this quintic mirror family.

**Construction of a moduli space of PLH.** To explain our result, we describe Kato-Usui's construction of the moduli space of PLH roughly (§3 for detail). Steps of the construction are given as follows:

**Step 1.** Define the nilpotent cone $\sigma$ and the set $D_\sigma$ of nilpotent orbits.
**Step 2.** Define the toric variety $\mathrm{toric}_\sigma$ and the space $E_\sigma$.
**Step 3.** Define the map $E_\sigma \to \Gamma\backslash D_\sigma$ and endow $\Gamma\backslash D_\sigma$ with a geometric structure by this map.

Firstly, we fix a point $s_0 \in (\Delta^*)^n$ and let $(H_{s_0}, F_{s_0}, \langle\ ,\ \rangle_{s_0})$ be the corresponding polarized Hodge structure (PH for abbr.). The period domain $D$ is a homogeneous space for the real Lie group $G = \mathrm{Aut}\,(H_{s_0,\mathbb{R}}, \langle\ ,\ \rangle_{s_0})$ and also an open $G$-orbit in the flag manifold $\check{D}$ (§2 for detail).

**Step 1**: Take the logarithms $N_1, \ldots, N_n$ of the monodromy transformations and make the cone $\sigma$ in $\mathfrak{g}$ generated by them, which is called a *nilpotent cone*. By Schmid's nilpotent orbit theorem, there exists the limiting Hodge filtration $F_\infty$.

We call the orbit $\exp(\sigma_{\mathbb{C}})F_\infty$ in $\check{D}$ a *nilpotent orbit.* $D_\sigma$ is the set of all nilpotent orbits generated by $\sigma$.

**Step 2**: Take the monoid $\Gamma(\sigma) := \Gamma \cap \exp(\sigma)$. It determines the toric variety $\mathrm{toric}_\sigma := \mathrm{Spec}(\mathbb{C}[\Gamma(\sigma)^\vee])_{\mathrm{an}}$. We define the subspace $E_\sigma$ of $\mathrm{toric}_\sigma \times \check{D}$ in §3.4.

**Step 3**: We define the map $E_\sigma \to \Gamma\backslash D_\sigma$ in §3.5 (here $\Gamma = \Gamma(\sigma)^{\mathrm{gp}}$). By [**KU**, Theorem A], $E_\sigma$ and $\Gamma\backslash D_\sigma$ are logarithmic manifolds. Moreover the map is a $\sigma_{\mathbb{C}}$-torsor in the category of logarithmic manifolds, i.e., there exists a proper and free $\sigma_{\mathbb{C}}$-action on $E_\sigma$ and $E_\sigma \to \Gamma\backslash D_\sigma$ is isomorphic to $E_\sigma \to \sigma_{\mathbb{C}}\backslash E_\sigma$ in the category of logarithmic manifolds.

**Main result.** Our main result is for properties of the torsor $E_\sigma \to \Gamma\backslash D_\sigma$.

- In the case (A), the torsor is trivial. (Theorem 5.6)
- In the case (B), the torsor is non-trivial. (Proposition 5.8)

In the case (A), $\Gamma\backslash D_\sigma$ is just a toroidal partial compactification of $\Gamma\backslash D$ introduced by [**AMRT**]. To show the triviality, we review the theory of bounded symmetric domains in §4. Realization of $D$ as the Siegel domain of the 3rd kind (4.2) is a key of the proof. This induces the triviality of $B(\sigma) \to \mathbb{B}(\sigma)$ (Lemma 5.2). By the triviality of this torsor, we show the triviality of $E_\sigma \to \Gamma\backslash D_\sigma$. We also describe a simple example (Example 5.7).

In the case (B), $D$ is not a Hermitian symmetric domain, i.e., isotropy subgroups are not maximally compact. We fix a point $F_0 \in D$ and take a maximally compact subgroup $K$ as in (5.2). Then $KF_0$ is a compact subvariety of $D$. Existence of such a variety (of positive dimension) is a distinction between the cases where the period domain is Hermitian symmetric and otherwise. The compact subvariety plays an important role in the proof.

**Acknowledgment.** A part of main results is from the author's master thesis. The author is grateful to Professors Sampei Usui, Christian Schnell for their valuable advice and warm encouragement. The author is also grateful to the referee for his careful reading and valuable suggestions and comments on presentations.

## 2. Polarized Hodge structures and period domains

We recall the definition of polarized Hodge structures and of period domains. A *Hodge structure* of weight $w$ and of Hodge type $(h^{p,q})$ is a pair $(H, F)$ consisting of a free $\mathbb{Z}$-module of rank $\sum_{p,q} h^{p,q}$ and of a decreasing filtration on $H_{\mathbb{C}} := H \otimes \mathbb{C}$ satisfying the following conditions:

(H1) $\dim_{\mathbb{C}} F^p = \sum_{r \geq p} h^{r,w-r}$ for all $p$.

(H2) $H_{\mathbb{C}} = \bigoplus_{p+q} H^{p,q}$ $(H^{p,q} := F^p \cap \overline{F^{w-p}})$.

A *polarization* $\langle\ ,\ \rangle$ for a Hodge structure $(H, F)$ of weight $w$ is a non-degenerate bilinear form on $H_{\mathbb{Q}} := H \otimes \mathbb{Q}$, symmetric if $w$ is even and skew-symmetric if $w$ is odd, satisfying the following conditions:

(P1) $\langle F^p, F^q\rangle = 0$ for $p + q > w$.

(P2) The Hermitian form

$$H_{\mathbb{C}} \times H_{\mathbb{C}} \to \mathbb{C}, \quad (x, y) \mapsto \langle C_F(x), \bar{y}\rangle$$

is positive definite.

Here $\langle\ ,\ \rangle$ is regarded as the natural extension to $\mathbb{C}$-bilinear form and $C_F$ is the Weil operator, which is defined by $C_F(x) := (\sqrt{-1})^{p-q}x$ for $x \in H^{p,q}$.

We fix a polarized Hodge structure $(H_0, F_0, \langle\ ,\ \rangle_0)$ of weight $w$ and of Hodge type $(h^{p,q})$. We define the set of all Hodge structures of this type

$$D := \left\{ F \;\middle|\; \begin{array}{r} (H_0, F, \langle\ ,\ \rangle_0) \text{ is a polarized Hodge structure} \\ \text{of weight } w \text{ and of Hodge type } (h^{p,q}) \end{array} \right\}.$$

$D$ is called a *period domain.* Moreover, we have the flag manifold

$$\check{D} := \left\{ F \;\middle|\; \begin{array}{r} (H_0, F, \langle\ ,\ \rangle_0) \text{ satisfy the conditions} \\ \text{(H1), (H2) and (P1)} \end{array} \right\}.$$

$\check{D}$ is called the *compact dual of* $D$. $D$ is contained in $\check{D}$ as an open subset. $D$ and $\check{D}$ are homogeneous spaces under the natural actions of $G$ and $G_{\mathbb{C}}$ respectively, where $G := \operatorname{Aut}(H_{0,\mathbb{R}}, \langle\ ,\ \rangle_0)$ and $G_{\mathbb{C}}$ is the complexification of $G$. $G$ is a classical group such that

$$G \cong \begin{cases} Sp(h, \mathbb{R})/\{\pm 1\} & \text{if } w \text{ is odd,} \\ SO(h_{\mathrm{odd}}, h_{\mathrm{even}}) & \text{if } w \text{ is even,} \end{cases}$$

where $2h = \operatorname{rank} H_0$, $h_{\mathrm{odd}} = \sum_{\mathrm{p:odd}} h^{p,q}$ and $h_{\mathrm{even}} = \sum_{\mathrm{p:even}} h^{p,q}$. The isotropy subgroup of $G$ at $F_0$ is isomorphic to

$$\begin{cases} \prod_{p \le m} U(h^{p,q}) & \text{if } w = 2m+1, \\ \prod_{p<m} U(h^{p,q}) \times SO(h^{m,m}) & \text{if } w = 2m. \end{cases}$$

They are compact subgroups of $G$ but not maximal compact in general. $D$ is a Hermitian symmetric domain if and only if the isotropy subgroup is a maximally compact subgroup, i.e., one of the following is satisfied:

(1) $w = 2m+1$, $h^{p,q} = 0$ unless $p = m+1, m$.
(2) $w = 2m$, $h^{p,q} = 1$ for $p = m+1, m-1$, $h^{m,m}$ is arbitary, $h^{p,q} = 0$ otherwise.
(3) $w = 2m$, $h^{p,q} = 1$ for $p = m+a, m+a-1, m-a, m-a+1$ for some $a \ge 2$, $h^{p,q} = 0$ otherwise.

In the case (1), $D$ is a Hermitian symmetric domain of type III. In the case (2) or (3), an irreducible component of $D$ is a Hermitian symmetric domain of type IV.

## 3. Moduli spaces of polarized log Hodge structures

In this section, we review some basic facts in [**KU**]. Firstly, we introduce $D_\Sigma$, a set of nilpotent orbits associated with a fan $\Sigma$ consisting of nilpotent cones. Secondly, for some subgroup $\Gamma$ in $G_{\mathbb{Z}}$, we endow $\Gamma\backslash D_\Sigma$ with a geometric structure. Finally we see some fundamental properties of $\Gamma\backslash D_\Sigma$, which are among the main results of [**KU**].

**3.1. Nilpotent orbits.** A *nilpotent cone* $\sigma$ is a strongly convex and finitely generated rational polyhedral cone in $\mathfrak{g} := \operatorname{Lie} G$ whose generators are nilpotent and commute with each other. For $A = \mathbb{R}, \mathbb{C}$, we denote by $\sigma_A$ the $A$-linear span of $\sigma$ in $\mathfrak{g}_A$.

Definition 3.1. Let $\sigma = \sum_{j=1}^n \mathbb{R}_{\ge 0} N_j$ be a nilpotent cone and $F \in \check{D}$. $\exp(\sigma_{\mathbb{C}})F \subset \check{D}$ is a *$\sigma$-nilpotent orbit* if it satisfies following conditions:

(1) $\exp(\sum_j iy_j N_j)F \in D$ for all $y_j \gg 0$.
(2) $NF^p \subset F^{p-1}$ for all $p \in \mathbb{Z}$ and for all $N \in \sigma$.

The condition (2) says the map $\mathbb{C}^n \to \check{D}$ given by $(z_j) \mapsto \sum_j \exp(z_j N_j)F$ is horizontal. Let $\Sigma$ be a fan consisting of nilpotent cones. We define the set of nilpotent orbits

$$D_\Sigma := \{(\sigma, Z) |\ \sigma \in \Sigma,\ Z \text{ is a } \sigma\text{-nilpotent orbit}\}.$$

For a nilpotent cone $\sigma$, the set of faces of $\sigma$ is a fan, and we abbreviate $D_{\{\text{faces of } \sigma\}}$ as $D_\sigma$.

**3.2. Subgroups in $G_\mathbb{Z}$ which is compatible with a fan.** Let $\Gamma$ be a subgroup of $G_\mathbb{Z}$ and $\Sigma$ a fan of nilpotent cones. We say $\Gamma$ is *compatible* with $\Sigma$ if

$$\mathrm{Ad}(\gamma)(\sigma) \in \Sigma$$

for all $\gamma \in \Gamma$ and for all $\sigma \in \Sigma$. Then $\Gamma$ acts on $D_\Sigma$ if $\Gamma$ is compatible with $\Sigma$. Moreover we say $\Gamma$ is *strongly compatible* with $\Sigma$ if it is compatible with $\Sigma$ and for all $\sigma \in \Sigma$ there exists $\gamma_1, \ldots, \gamma_n \in \Gamma(\sigma) := \Gamma \cap \exp(\sigma)$ such that

$$\sigma = \sum_j \mathbb{R}_{\geq 0} \log(\gamma_j).$$

**3.3. Varieties $\mathrm{toric}_\sigma$ and $\mathrm{torus}_\sigma$.** Let $\Sigma$ be a fan and $\Gamma$ a subgroup of $G_\mathbb{Z}$ which is strongly compatible with $\Sigma$. We have toric varieties associated with the monoid $\Gamma(\sigma)$ such that

$$\begin{aligned} \mathrm{toric}_\sigma &:= \mathrm{Spec}(\mathbb{C}[\Gamma(\sigma)^\vee])_{\mathrm{an}} \cong \mathrm{Hom}\,(\Gamma(\sigma)^\vee, \mathbb{C}), \\ \mathrm{torus}_\sigma &:= \mathrm{Spec}(\mathbb{C}[\Gamma(\sigma)^{\vee\mathrm{gp}}])_{\mathrm{an}} \cong \mathrm{Hom}\,(\Gamma(\sigma)^{\vee\mathrm{gp}}, \mathbb{G}_m) \cong \mathbb{G}_m \otimes \Gamma(\sigma)^{\mathrm{gp}}, \end{aligned}$$

where $\mathbb{C}$ is regarded as a semigroup via multiplication and above homomorphisms are of semigroups. As in [**F**, §2.1], we choose the distinguished point

$$x_\tau : \Gamma(\sigma)^\vee \to \mathbb{C}; \quad u \mapsto \begin{cases} 1 & \text{if } u \in \Gamma(\tau)^\perp, \\ 0 & \text{otherwise,} \end{cases}$$

for a face $\tau$ of $\sigma$. Then $\mathrm{toric}_\sigma$ can be decomposed as

$$\mathrm{toric}_\sigma = \bigsqcup_{\tau \prec \sigma} (\mathrm{torus}_\sigma \cdot x_\tau).$$

For $q \in \mathrm{toric}_\sigma$, there exists $\sigma(q) \prec \sigma$ such that $q \in \mathrm{torus}_\sigma \cdot x_{\sigma(q)}$. By a surjective homomorphism

$$\mathbf{e} : \sigma_\mathbb{C} \to \mathrm{torus}_\sigma \cong \mathbb{G}_m \otimes \Gamma(\sigma)^{\mathrm{gp}};\ w \log(\gamma) \mapsto \exp(2\pi\sqrt{-1}w) \otimes \gamma, \tag{3.1}$$

$q$ can be written as

$$q = \mathbf{e}(z) \cdot x_{\sigma(q)}. \tag{3.2}$$

Here $\ker(\mathbf{e}) = \log(\Gamma(\sigma)^{\mathrm{gp}})$ and $z$ is determined uniquely modulo $\log(\Gamma(\sigma)^{\mathrm{gp}}) + \sigma(q)_\mathbb{C}$.

**3.4. Spaces $\check{E}_\sigma$ and $E_\sigma$.** We define the analytic space $\check{E}_\sigma := \mathrm{toric}_\sigma \times \check{D}$ and endow $\check{E}_\sigma$ with the logarithmic structure $M_{E_\sigma}$ by the inverse image of canonical logarithmic structure on $\mathrm{toric}_\sigma$ (cf. [**K**]). Define the subspace of $\check{E}_\sigma$

$$E_\sigma := \{(q, F) \in \check{E}_\sigma \mid \exp(\sigma(q)_\mathbb{C}) \exp(z) F \text{ is } \sigma(q)\text{-nilpotent orbit}\}$$

where $z$ is an element such that $q = \mathbf{e}(z) \cdot x_{\sigma(q)}$. The set $E_\sigma$ is well-defined. The topology of $E_\sigma$ is the "strong topology" in $\check{E}_\sigma$, which is defined in [**KU**, §3.1], and $\mathcal{O}_{E_\sigma}$(resp. $M_{E_\sigma}$) is the inverse image of $\mathcal{O}_{\check{E}_\sigma}$(resp. $M_{\check{E}_\sigma}$). Then $E_\sigma$ is a logarithmic local ringed space. Note that $E_\sigma$ is *not* an analytic space in general.

**3.5. The structure of $\Gamma\backslash D_\Sigma$.** We define the canonical map

$$\begin{aligned} &E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma, \\ &(q, F) \mapsto (\sigma(q), \exp(\sigma(q)_\mathbb{C}) \exp(z) F) \quad \mathrm{mod}\ \Gamma(\sigma)^{\mathrm{gp}}, \end{aligned}$$

where $q = \mathbf{e}(z) \cdot x_{\sigma(q)}$ as in (3.2). This map is well-defined. We endow $\Gamma\backslash D_\Sigma$ with the strongest topology for which the composite maps $\pi_\sigma : E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \to \Gamma\backslash D_\Sigma$ are continuous for all $\sigma \in \Sigma$. We endow $\Gamma\backslash D_\Sigma$ with $\mathcal{O}_{\Gamma\backslash D_\Sigma}$ (resp. $M_{\Gamma\backslash D_\Sigma}$) as follows:

$$\begin{aligned} &\mathcal{O}_{\Gamma\backslash D_\Sigma}(U)\ (\text{resp. } M_{\Gamma\backslash D_\Sigma}(U)) \\ &:= \{\text{map } f : U \to \mathbb{C} \mid f \circ \pi_\sigma \in \mathcal{O}_{E_\sigma}(\pi_\sigma^{-1}(U))\ (\text{resp. } M_{E_\sigma}(\pi_\sigma^{-1}(U)))\ ({}^\forall \sigma \in \Sigma)\} \end{aligned}$$

for any open set $U$ of $\Gamma\backslash D_\Sigma$. As for $E_\sigma$, note that $\Gamma\backslash D_\Sigma$ is a logarithmic local ringed space but is *not* an analytic space in general. Kato-Usui introduce "logarithmic manifolds" as generalized analytic spaces (cf. [**KU**, §3.5]) and they show the following geometric properties of $\Gamma\backslash D_\Sigma$:

THEOREM 3.2 ([**KU**, Theorem A]). *Let $\Sigma$ be a fan of nilpotent cones and let $\Gamma$ be a subgroup of $G_\mathbb{Z}$ which is strongly compatible with $\Sigma$. Then we have*

*(1) $E_\sigma$ is a logarithmic manifold.*
*(2) If $\Gamma$ is neat (i.e., the subgroup of $\mathbb{G}_m$ generated by all the eigenvalues of all $\gamma \in \Gamma$ is torsion free), $\Gamma\backslash D_\Sigma$ is also a logarithmic manifold.*
*(3) Let $\sigma \in \Sigma$ and define the action of $\sigma_\mathbb{C}$ on $E_\sigma$ over $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ by*

$$a \cdot (q, F) := (\mathbf{e}(a) q, \exp(-a) F) \quad (a \in \sigma_\mathbb{C},\ (q, F) \in E_\sigma).$$

*Then $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ is a $\sigma_\mathbb{C}$-torsor in the category of logarithmic manifold.*
*(4) $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \to \Gamma\backslash D_\Sigma$ is open and locally an isomorphism of logarithmic manifold.*

In [**KU**, §2.4] , Kato-Usui introduce "polarized log Hodge structures" and they show that $\Gamma\backslash D_\Sigma$ is a fine moduli space of polarized log Hodge structures if $\Gamma$ is neat([**KU**, Theorem B]).

## 4. The structure of bounded symmetric domains

In this section we recall some basic facts on Hermitian symmetric domains (for more detail, see [**AMRT**, III] , [**N**, appendix]). We define Satake boundary components, and show that a Hermitian symmetric domain is a family of tube domains parametrized by a vector bundle over a Satake boundary component. This domain is called a Siegel domain of third kind.

**4.1. Satake boundary components.** Let $D$ be a Hermitian symmetric domain. Then Aut $(D)$ is a real Lie group and the identity component $G$ of Aut $(D)$ acts on $D$ transitively. We fix a base point $o \in D$. The isotropy subgroup $K$ at $o$ is a maximally compact subgroup of $G$. Let $s_o$ be a symmetry at $o$ and let

$$\begin{aligned} &\mathfrak{g} := \mathrm{Lie}\,(G), \quad \mathfrak{k} := \mathrm{Lie}\,(K), \\ &\mathfrak{p} := \text{the subspace of } \mathfrak{g} \text{ where } s_o = -\mathrm{Id}. \end{aligned}$$

Then we have a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

$\mathfrak{p}$ is isomorphic to the tangent space to $D$ at $o$. Let $J$ be a complex structure on $\mathfrak{p}$ and let

$$\begin{aligned} \mathfrak{p}_+ &:= \text{the } \sqrt{-1}\text{-eigenspace for } J \text{ in } \mathfrak{p}_{\mathbb{C}}, \\ \mathfrak{p}_- &:= \text{the } -\sqrt{-1}\text{-eigenspace for } J \text{ in } \mathfrak{p}_{\mathbb{C}}. \end{aligned}$$

Here $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are abelian subalgebras of $\mathfrak{g}_{\mathbb{C}}$. Then we have the Harish-Chandra embedding map $D \to \mathfrak{p}_+$ whose image is a bounded domain.

DEFINITION 4.1. A *Satake boundary component* of $D$ is an equivalence class in $\overline{D}$, the topological closure of $D$ in $\mathfrak{p}_+$, under the equivalence relation generated by $x \sim y$ if there exists a holomorphic map

$$\lambda : \{z \in \mathbb{C} |\ |z| < 1\} \longrightarrow \mathfrak{p}_+$$

such that $\mathrm{Im}(\lambda) \subset \overline{D}$ and $x, y \in \mathrm{Im}(\lambda)$.

It is known that Satake boundary components are also Hermitian symmetric domains. Let $S$ be a Satake boundary component. We define

$$\begin{aligned} N(S) &:= \{g \in G |\ gS = S\}, \\ W(S) &:= \text{ the unipotent radical of } N(S), \\ U(S) &:= \text{ the center of } W(S). \end{aligned}$$

These groups have the following properties:

PROPOSITION 4.2. *(1) $N(S)$ acts on $D$ transitively.*
*(2) There exists an abelian Lie subalgebra $\underline{v}(S) \subset \mathfrak{g}$ such that* $\mathrm{Lie}\, W(S) = \underline{v}(S) + \mathrm{Lie}\, U(S)$.
*(3) $W(S)/U(S)$ is an abelian Lie group which is isomorphic to $V(S) := \exp \underline{v}(S)$.*

$S$ is called *rational* if $N(S)$ is defined over $\mathbb{Q}$. If $S$ is rational then $V(S)$ and $U(S)$ are also defined over $\mathbb{Q}$.

**4.2. Siegel domain of third kind.** We define a subspace of $\check{D}$

$$D(S) := U(S)_{\mathbb{C}} \cdot D = \bigcup_{g \in U(S)_{\mathbb{C}}} g \cdot D$$

where $U(S)_{\mathbb{C}} := U(S) \otimes \mathbb{C}$. By Proposition 4.2(1) and the fact that $U(S)$ is a normal subgroup, $N(S)U(S)_{\mathbb{C}}$ acts on $D(S)$ transitively. We choose the base point $o_S$ as in [**AMRT**, §4.2] . The isotropy subgroup $I$ of $G$ at $o_S$ is contained in $N(S)$. Then we have a map

$$\Psi_S : D(S) \cong N(S)U(S)_{\mathbb{C}}/I \to N(S)U(S)_{\mathbb{C}}/N(S) \cong U(S)$$

where the last isomorphism takes imaginary part. By [**AMRT**, §4.2 Theorem 1] , we have an open homogeneous self adjoint cone $C(S) \subset U(S)$ such that $\Psi_S^{-1}(C(S)) = D$.

THEOREM 4.3. *(1) $U(S)_{\mathbb{C}}$ acts freely on $D(S)$. $D(S) \to U(S)_{\mathbb{C}}\backslash D(S)$ is trivial principal homogeneous bundle.*

*(2) $V(S)$ acts freely on $U(S)_{\mathbb{C}}\backslash D(S)$. $V(S)\backslash(U(S)_{\mathbb{C}}\backslash D(S)) \cong S$ and the quotient map $D(S) \to S$ is a complex vector bundle (although $V(S)$ is real). Moreover it is trivial.*

*(3) By (1) and (2), we have a trivialization*

$$D(S) \cong S \times \mathbb{C}^k \times U(S)_{\mathbb{C}}. \tag{4.1}$$

*In this product representation,we have*

$$\Psi_S(x,y,z) = \operatorname{Im} z - h_x(y,y)$$

*where $h_x$ is a real-bilinear quadratic form $\mathbb{C}^k \times \mathbb{C}^k \to U(S)$ depending real-analytically on $x$.*

Thus we have

$$D \cong \left\{ \ (x,y,z) \in S \times \mathbb{C}^{(g-k)k} \times U(S)_{\mathbb{C}} \ \middle| \ \operatorname{Im} z \in C(S) + h_x(y,y) \ \right\}. \tag{4.2}$$

## 5. Main result

**5.1. The case: $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ is trivial.** In this subsection, we assume that $D$ is a period domain and also a Hermitian symmetric domain. The purpose is to show Theorem 5.6. Main theorem for the case where $D$ is upper half plane is described in Example 5.7.

Let $S$ be a Satake rational boundary component of $D$ and $\sigma$ a nilpotent cone included in Lie$(U(s))$. Firstly we show the triviality of the torsor for such a cone. We set

$$B(\sigma) := \exp(\sigma_{\mathbb{C}}) \cdot D \subset \check{D}, \quad \mathbb{B}(\sigma) := \exp(\sigma_{\mathbb{C}})\backslash B(\sigma).$$

Here $B(\sigma), \mathbb{B}(\sigma)$ are defined by Carlson-Cattani-Kaplan ([**CCK**]) from the point of view of mixed Hodge theory.

LEMMA 5.1. *$B(\sigma) \to \mathbb{B}(\sigma)$ is a trivial principal bundle with fiber* $\exp(\sigma_{\mathbb{C}})$.

*Proof.* $\exp(\sigma_{\mathbb{C}})$ is a sub Lie group of $U(S)_{\mathbb{C}}$ and $U(S)_{\mathbb{C}}$ is abelian. By Theorem 4.3(1), $D(S) \to \exp(\sigma_{\mathbb{C}})\backslash D(S)$ is a trivial principal bundle . Furthermore the following diagram is commutative:

$$\begin{array}{ccc} D(S) & \longrightarrow & \exp(\sigma_{\mathbb{C}})\backslash D(S) \\ \cup & & \cup \\ B(\sigma) & \longrightarrow & \mathbb{B}(\sigma) \end{array}$$

Then $B(\sigma) \to \mathbb{B}(\sigma)$ is trivial. □

Now we describe a trivialization of $B(\sigma)$ over $\mathbb{B}(\sigma)$ explicitly. Take a complementary sub Lie group $Z_\sigma$ of $\exp(\sigma_{\mathbb{C}})$ in $U(S)_{\mathbb{C}}$. By (4.1), we have a decomposition of $D(S)$ as $D(S) \cong S \times \mathbb{C}^k \times Z_\sigma \times \exp(\sigma_{\mathbb{C}})$. Here

$$\exp(\sigma_{\mathbb{C}})\backslash D(S) \cong S \times \mathbb{C}^k \times Z_\sigma \tag{5.1}$$

and the decomposition of $D(S)$ makes a trivialization of $D(S)$ over $\exp(\sigma_{\mathbb{C}})\backslash D(S)$. Via (5.1), we have $Y \subset S \times \mathbb{C}^k \times Z_\sigma$ satisfying the following commutative diagram:

$$\begin{array}{ccc} \exp(\sigma_{\mathbb{C}})\backslash D(S) & \cong & S \times \mathbb{C}^k \times Z_\sigma \\ \cup & & \cup \\ \mathbb{B}(\sigma) & \cong & Y \end{array}$$

In fact, $Y$ is the image of $D$ via the projection $D(S) \to S \times \mathbb{C}^k \times Z_\sigma$. Then $B(\sigma) \cong \exp(\sigma_{\mathbb{C}}) \times Y$ is a trivialization of $B(\sigma)$ over $\mathbb{B}(\sigma)$.

Let $\Gamma$ be a subgroup of $G_{\mathbb{Z}}$ which is strongly compatible with $\sigma$. Let us think about a quotient trivial bundle $\Gamma(\sigma)^{\mathrm{gp}}\backslash B(\sigma) \to \mathbb{B}(\sigma)$. Its fiber is the quotient of $\exp(\sigma_{\mathbb{C}})$ by the lattice $\Gamma(\sigma)^{\mathrm{gp}}$. Since $\exp(\sigma_{\mathbb{C}})$ is a unipotent and abelian Lie group, $\sigma_{\mathbb{C}} \cong \exp(\sigma_{\mathbb{C}})$. Via this isomorphism, the lattice action on $\exp(\sigma_{\mathbb{C}})$ is equivalent to the lattice action on $\sigma_{\mathbb{C}}$ by $\log(\Gamma(\sigma)^{\mathrm{gp}})$. Then the fiber is isomorphic to

$$\sigma_{\mathbb{C}}/\log(\Gamma(\sigma)^{\mathrm{gp}}) = \sigma_{\mathbb{C}}/\ker(\mathbf{e}) \cong \mathrm{torus}_\sigma \quad \text{by (3.1)}.$$

By the canonical torus action, $\Gamma(\sigma)^{\mathrm{gp}}\backslash B(\sigma) \to \mathbb{B}(\sigma)$ is also a principal $\mathrm{torus}_\sigma$-bundle whose trivialization is given by

$$\mathrm{torus}_\sigma \times Y \xrightarrow{\sim} \Gamma(\sigma)^{\mathrm{gp}}\backslash B(\sigma); \quad (\mathbf{e}(z), F) \mapsto \exp(z)F \mod \Gamma(\sigma)^{\mathrm{gp}},$$

where we regard $Y$ as a subset of $\check{D}$ via (4.1). By the definition of $E_\sigma$, we have

LEMMA 5.2. *The following diagram is commutative:*

$$\begin{array}{ccc} \Gamma(\sigma)^{\mathrm{gp}}\backslash B(\sigma) & \cong & \mathrm{torus}_\sigma \times Y \\ \cup & & \cup \\ \Gamma(\sigma)^{\mathrm{gp}}\backslash D & \cong & (\mathrm{torus}_\sigma \times Y) \cap E_\sigma \end{array}$$

By the torus embedding $\mathrm{torus}_\sigma \hookrightarrow \mathrm{toric}_\sigma$, we can construct the associated bundle

$$(\Gamma(\sigma)^{\mathrm{gp}}\backslash B(\sigma))_\sigma := (\Gamma(\sigma)^{\mathrm{gp}}\backslash B(\sigma)) \times_{\mathrm{torus}_\sigma} \mathrm{toric}_\sigma,$$

and define

$$(\Gamma(\sigma)^{\mathrm{gp}}\backslash D)_\sigma := \text{ the interior of the closure of } \Gamma(\sigma)^{\mathrm{gp}}\backslash D \text{ in } (\Gamma(\sigma)^{\mathrm{gp}}\backslash B(\sigma))_\sigma.$$

This is a toroidal partial compactification *associated with* $\sigma$.

LEMMA 5.3. *The following diagram is commutative:*

$$\begin{array}{ccc} (\Gamma(\sigma)^{\mathrm{gp}}\backslash B(\sigma))_\sigma & \cong & \mathrm{toric}_\sigma \times Y \\ \cup & & \cup \\ (\Gamma(\sigma)^{\mathrm{gp}}\backslash D)_\sigma & \cong & (\mathrm{toric}_\sigma \times Y) \cap E_\sigma \end{array}$$

*Proof.* For $(q, F) \in \mathrm{toric}_\sigma \times Y$, $(q, F) \in E_\sigma$ if and only if $\exp(\sigma(q)_{\mathbb{C}})\exp(z)F$ is a $\sigma(q)$-nilpotent orbit, where $q = \exp(z)x_{\sigma(q)}$ as in (3.2). Since $D$ is Hermitian symmetric, the horizontal tangent bundle of $D$ coincides with the tangent bundle of $D$. Then the condition of Definition 3.1(2) is trivially satisfied. Let $\{N_i\}$ be a set of rational nilpotent elements generating $\sigma(q)$. $(q, F) \in E_\sigma$ if and only if $\exp(\Sigma_j\, y_j N_j) \exp(z) F \in D$, i.e.,

$$(\mathbf{e}(\Sigma_j\, y_j N_j + z), F) \in (\mathrm{torus}_\sigma \times Y) \cap E_\sigma$$

for all $y_j$ such that $\mathrm{Im}(y_j) \gg 0$. In $\mathrm{toric}_\sigma$,

$$\lim_{\mathrm{Im}(y_j) \to \infty} \mathbf{e}(\Sigma_j\, y_j N_j + z) = \mathbf{e}(z) x_{\sigma(q)} = q.$$

Then $(q, F)$ is in the interior of the closure of $(\mathrm{torus}_\sigma \times Y) \cap E_\sigma$. □

The map $(\mathrm{toric}_\sigma \times Y) \cap E_\sigma \hookrightarrow E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ is bijective. By [**KU**, 8.2.7], $E_\sigma$ is an open set in $\check{E}_\sigma$. Then $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ is a $\sigma_{\mathbb{C}}$-torsor in the category of analytic spaces and

$$(\mathrm{toric}_\sigma \times Y) \cap E_\sigma \cong \sigma_{\mathbb{C}} \backslash E_\sigma \cong \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma.$$

Thus $(\mathrm{toric}_\sigma \times Y) \cap E_\sigma \hookrightarrow E_\sigma$ gives a section of the torsor $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$, i.e. $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ is trivial.

Next we show that $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ is trivial for *all* nilpotent cones $\sigma$. Let $\Gamma = G_{\mathbb{Z}}$. By [**AMRT**, II] , there exists $\Gamma$-admissible collection of fans $\Sigma = \{\Sigma(S)\}_S$ where $\Sigma(S)$ is a fan in $\overline{C(S)}$ for every Satake rational boundary component $S$. Taking logarithm, we identify $\Sigma$ with the collection of fans in $\mathfrak{g}$ which are strongly compatible with $\Gamma$. We show that $\Sigma$ is large enough to cover all nilpotent cones, i.e., $\Sigma$ is complete fan.

Let $U(S)_{\mathbb{Z}} = U(S) \cap \Gamma$. To obtain $U(S)_{\mathbb{Z}} \backslash D_\sigma$, we should confirm the following proposition:

PROPOSITION 5.4. *Generators of $Z_\sigma$ can be taken in $G_{\mathbb{Z}}$.*

*Proof.* $\Gamma(\sigma)^{\mathrm{gp}}$ is saturated in $U(S)_{\mathbb{Z}}$, i.e.,

If $g \in U(S)_{\mathbb{Z}}$ and $g^n \in \Gamma(\sigma)^{\mathrm{gp}}$ for some $n \geq 1$, then $g \in \Gamma(\sigma)^{\mathrm{gp}}$.

Then $U(S)_{\mathbb{Z}}/\Gamma(\sigma)^{\mathrm{gp}}$ is a free module and there exists a subgroup $Z_{\sigma,\mathbb{Z}}$ in $U(S)_{\mathbb{Z}}$ such that $U(S)_{\mathbb{Z}} = \Gamma(\sigma)^{\mathrm{gp}} \oplus Z_{\sigma,\mathbb{Z}}$. Hence we have $U(S)_{\mathbb{C}} = \exp \sigma_{\mathbb{C}} \oplus (Z_{\sigma,\mathbb{Z}} \otimes \mathbb{C})$. □

Gluing $U(S)_{\mathbb{Z}} \backslash D_\sigma = Z_{\sigma,\mathbb{Z}} \backslash (\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma)$ for $\sigma \in \Sigma(S)$, we have $U(S)_{\mathbb{Z}} \backslash D_{\Sigma(S)}$ as a toroidal partial compactification *in the direction* $S$. Then we obtain a compact variety $\Gamma \backslash D_\Sigma$ by [**AMRT**] (Take quotient by $N(S)_{\mathbb{Z}}/U(S)_{\mathbb{Z}}$ and glue neighborhoods of boundaries of $(N(S)_{\mathbb{Z}}/U(S)_{\mathbb{Z}}) \backslash (U(S)_{\mathbb{Z}} \backslash D_{\Sigma(S)})$ with $\Gamma \backslash D$).

On the other hand, we have the following proposition.

PROPOSITION 5.5 ([**KU**, 12.6.4]). *Let $\Gamma$ be a subgroup of $G_{\mathbb{Z}}$ and let $\Sigma$ be a fan which is strongly compatible with $\Gamma$. Assume that $\Gamma \backslash D_\Sigma$ is compact. Then $\Sigma$ is complete.*

The precise definition of complete fan is given in [**KU**]. An important property of a complete fan $\Sigma$ is the following: if there exists $Z \subset \check{D}$ such that $(\sigma, Z)$ is a nilpotent orbit, then $\sigma \in \Sigma$. Now $\Gamma$-admissible collection of fans $\Sigma$ is complete since $\Gamma \backslash D_\Sigma$ is compact. It is to say that a nilpotent cone $\sigma$ has a $\sigma$-nilpotent orbit if

$$\exp(\sigma) \subset \overline{C(S)} \subset U(S)$$

for some Satake boundary component $S$, and $\sigma$ has no $\sigma$-nilpotent orbit ,i.e., $D_\sigma = D$, otherwise. Hence we have

THEOREM 5.6. *Let $\sigma$ be a nilpotent cone in $\mathfrak{g}$. Then $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ is trivial.*

EXAMPLE 5.7. Let $D$ be the upper half plane. $G = SL(2,\mathbb{R})$ acts on $D$ by linear fractional transformation. By the Cayley transformation, $D \cong \Delta := \{\, z \in \mathbb{C} \mid |z| < 1 \,\}$. Take the Satake boundary component $S = \{1\} \in \partial\Delta$. Then

$$N(S) = \left\{ \begin{pmatrix} u & v \\ 0 & u^{-1} \end{pmatrix} \;\middle|\; u \in \mathbb{R} \setminus \{0\},\ v \in \mathbb{R} \right\},$$
$$W(S) = U(S) = \left\{ \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \;\middle|\; v \in \mathbb{R} \right\},$$
$$C(S) = \left\{ \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \;\middle|\; v \in \mathbb{R}_{\geq 0} \right\}$$

(cf. [**N**]). Take the nilpotent $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and the nilpotent cone $\sigma = \mathbb{R}_{\geq 0} N \subset \mathfrak{g}$. Here $\exp(\sigma_{\mathbb{C}}) = U(S)_{\mathbb{C}}$. The compact dual $\check{D}$ and the subspace $B(\sigma) = \exp(\sigma_{\mathbb{C}}) \cdot D \subset \check{D}$ are described as

$$\check{D} = \mathbb{C} \sqcup \{\infty\} \cong \mathbb{P}^1, \quad B(\sigma) = \mathbb{C}.$$

$\exp(\sigma_{\mathbb{C}}) \backslash B_\sigma = \mathbb{B}(\sigma)$ is a point and $B(\sigma) \to \mathbb{B}(\sigma)$ is a trivial principal bundle over $\mathbb{B}(\sigma)$. Take $F_0 \in B(\sigma)$. We have a trivialization

$$B(\sigma) = \exp(\sigma_{\mathbb{C}}) \cdot F_0 \cong \exp(\sigma_{\mathbb{C}}) \times \{F_0\}.$$

For $\Gamma = SL(2,\mathbb{Z})$,

$$\Gamma(\sigma)^{\mathrm{gp}} = \left\{ \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \;\middle|\; v \in \mathbb{Z} \right\} = \exp(\mathbb{Z}N)$$

is a lattice of $\exp(\sigma_{\mathbb{C}})$ and $\mathbb{G}_m \cong \exp(\sigma_{\mathbb{C}})/\Gamma(\sigma)^{\mathrm{gp}}$. Then we have the trivial principal $\mathbb{G}_m$-bundle $\Gamma(\sigma)^{\mathrm{gp}} \backslash B(\sigma) \to \mathbb{B}(\sigma)$. By the torus embedding $\mathbb{G}_m \hookrightarrow \mathbb{C}$, we have the trivial associated bundle

$$(\Gamma(\sigma)^{\mathrm{gp}} \backslash B(\sigma)) \times_{\mathbb{G}_m} \mathbb{C} \cong \mathbb{C} \times \{F_0\}.$$

And we have

$$E_\sigma = \left\{ (q, F) \in \mathbb{C} \times B(\sigma) \;\middle|\; \begin{array}{l} \exp((2\pi i)^{-1} \log(q) N) F \in D \text{ if } q \neq 0 \\ \text{and } F \in B(\sigma) \text{ if } q = 0 \end{array} \right\}$$

The map

$$(\mathbb{C} \times \{F_0\}) \cap E_\sigma \hookrightarrow E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma;$$
$$(q, F_0) \to \begin{cases} (\{0\}, \exp((2\pi i)^{-1} \log(q) N) F_0) \mod \Gamma(\sigma)^{\mathrm{gp}} & \text{if } q \neq 0, \\ (\sigma, \mathbb{C}) & \text{if } q = 0. \end{cases}$$

is an isomorphism by Lemma 5.3 and we have the following commutative diagram:

$$\begin{tikzcd} (\mathbb{C} \times \{F_0\}) \cap E_\sigma \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\wr\|"] & E_\sigma \arrow[dl] \\ \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma. & \end{tikzcd}$$

Then the torsor $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$ has a section, i.e., the torsor is trivial.

**5.2. The case: $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ is non-trivial.** Let $w = 3$, and $h^{p,q} = 1$ $(p+q=3, p,q \geq 0)$. Let $H_0$ be a free module of rank 4, $\langle\ ,\ \rangle_0$ a non-degenerate alternating bilinear form on $H_0$. In this case $D \cong Sp(2,\mathbb{R})/(U(1)\times U(1))$. Then $D$ is not a Hermitian symmetric space. Take $e_1,\dots,e_4$ as a symplectic basis for $(H_0, \langle\ ,\ \rangle_0)$, i.e.,

$$(\langle e_i, e_j\rangle_0)_{i,j} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Define $N \in \mathfrak{g}$ as follows:

$$N(e_3) = e_1, \quad N(e_j) = 0\ (j \neq 3).$$

PROPOSITION 5.8. *Let $\sigma = \mathbb{R}_{\geq 0}N$. Then $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ is non-trivial.*

*Proof.* Define

$$(u_1,\dots,u_4) := \frac{1}{\sqrt{2}}(e_1,\dots,e_4)\begin{pmatrix} I & I \\ iI & -iI \end{pmatrix}, \text{ i.e.,}$$

$$(\langle u_i, u_j\rangle_0)_{i,j} = \begin{pmatrix} 0 & iI \\ -iI & 0 \end{pmatrix}, \quad (\langle u_i, \overline{u_j}\rangle_0)_{i,j} = \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}.$$

Take $F_w, F_\infty \in D$ $(w \in \mathbb{C})$ as follows:

$$F_w^3 = \mathrm{span}_{\mathbb{C}}\{wu_1 + u_2\}, \quad F_w^2 = \mathrm{span}_{\mathbb{C}}\{wu_1+u_2, u_3 - wu_4\},$$
$$F_\infty^3 = \mathrm{span}_{\mathbb{C}}\{u_1\}, \quad F_\infty^2 = \mathrm{span}_{\mathbb{C}}\{u_1, u_4\}.$$

We have the maximally compact subgroup of $G$ at $F_0$

$$\text{(5.2)} \qquad \begin{aligned} K &= \{\ g \in G \mid \langle C_{F_0}(gv), \overline{gw}\rangle_0 = \langle C_{F_0}(v), \overline{w}\rangle_0 \quad \text{for } v,w \in H_{\mathbb{C}}\ \} \\ &= \left\{ \begin{pmatrix} X & 0 \\ 0 & \overline{X} \end{pmatrix} \;\middle|\; X \in U(2) \right\}, \end{aligned}$$

where matrices in the second equation are expressed with respect to the basis $(u_1,\dots,u_4)$. The $K$-orbit of $F_0$ is given by

$$K \cdot F_0 = K_{\mathbb{C}} \cdot F_0 = \{\ F_w \mid w \in \mathbb{C}\ \} \sqcup F_\infty \cong \mathbb{P}^1.$$

We assume that $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ is trivial. Let $\varphi$ be a section of $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$. We define a holomorphic morphism $\Phi : D \to \mathbb{C}$ such that

$$\Phi : D \xrightarrow{\text{quot.}} \Gamma(\sigma)^{\mathrm{gp}}\backslash D \hookrightarrow \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \xrightarrow{\varphi} E_\sigma \xrightarrow{\text{proj.}} \mathrm{toric}_\sigma \cong \mathbb{C}.$$

Since $KF_0 \cong \mathbb{P}^1$, $\Phi|_{KF_0}$ is constant.

On the other hand, $(\sigma, \exp(\sigma_{\mathbb{C}})F_0)$ is a nilpotent orbit (it is easy to check the condition of Definition 3.1). Then

$$\text{(5.3)} \qquad \lim_{x\to\infty} \Phi(\exp(ixN)F_0) = 0.$$

Define $N' \in \mathfrak{g}$ as follows:

$$N'(u_3) = u_1, \quad N'(u_j) = 0\ (j \neq 3).$$

Then we have

$$\text{(5.4)} \qquad \exp(ixN)F_0 = \exp\left(\frac{x}{2+x}N'\right)F_0,$$

$$\text{(5.5)} \qquad F_\infty = \exp\left(\frac{x}{2+x}N'\right)F_\infty$$

for $x \in \mathbb{R} \setminus \{-2\}$ and

$$\exp(zN')KF_0 \subset D \quad \text{for } |z| < 1. \tag{5.6}$$

Then $\Phi|_{\exp(zN')KF_0}$ is constant for each $|z| < 1$, again because $\exp(zN')KF_0 \cong \mathbb{P}^1$. Finally we have

$$\begin{aligned} \Phi(\exp(ixN)F_0) &= \Phi\left(\exp\left(\tfrac{x}{2+x}N'\right)F_0\right) && \text{(by (5.4))} \\ &= \Phi\left(\exp\left(\tfrac{x}{2+x}N'\right)F_\infty\right) && \text{(by (5.6) and } |\tfrac{x}{2+x}| < 1) \\ &= \Phi(F_\infty) && \text{(by (5.5))} \end{aligned}$$

for $x > -1$. This contradicts the condition (5.3), since $\Phi(F_\infty) \in \mathrm{torus}_\sigma$ if $F_\infty \in D$. $\square$

# Bibliography


[AMRT] A. Ash, D. Mumford, M. Rapoport and Y. S. Tai, *Smooth compactification of locally symmetric varieties,* Math. Sci. Press, Brookline, 1975.

[CCK] J. A. Carlson, E. Cattani and A. Kaplan, *Mixed Hodge structures and compactification of Siegel's space*, in Journées de géométrie algébrique d'Angers 1979 (A. Beauville, ed.), Sijthohoff & Nordhoff, 1980, 77–105.

[F] W.Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, **131**. Princeton University Press, Princeton, NJ, 1993.

[GGK] M. Green, P. Griffiths and M. Kerr, *Néron models and boundary components for degenerations of Hodge structures of mirror quintic type*, in *Curves and Abelian Varieties* (V. Alexeev, Ed.), Contemp. Math **465** (2007), AMS, 71–145.

[K] K.Kato, *Logarithmic structures of Fontaine-Illusie*, Algebraic analysis, geometry, and number theory, Johns Hopkins Univ. Press, Baltimore, 1989, 191–224.

[KU] K. Kato and S. Usui, *Classifying space of degenerating polarized Hodge structures*, Annals of Mathematics Studies, **169**. Princeton University Press, Princeton, NJ, 2009.

[N] Y. Namikawa, *Toroidal compactification of Siegel spaces*, Lecture Notes in Math **812**, Springer-Verlag, 1980.

[S] W. Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. **22** (1973), 211–319.

[U] S. Usui, *Generic Torelli theorem for quintic-mirror family*, Proc. Japan Acad. Ser. A Math. Sci. **84** (2008), no. 8, 143–146.

# Appendix. Néron models of Green-Griffiths-Kerr and log Néron models

## 1. introduction

Let $J \to \Delta^*$ be a family of intermediate Jacobians arising from a variation of polarized Hodge structure (VHS for abbr.) of weight $-1$ with a unipotent monodromy on the punctured disk. By Carlson ([**C**]), the intermediate Jacobians are isomorphic to the extension groups of the Hodge structures in the category of mixed Hodge structures (MHS for abbr.). Then a section of $J \to \Delta^*$ gives a VMHS. A VMHS satisfying the *admissibility condition* (cf. [**P**]) is called an admissible VMHS (AVMHS for abbr.) and a section which gives an AVMHS is called an admissible normal function (ANF for abbr.).

For the VHS, Green-Griffiths-Kerr ([**GGK1**]) introduce the family $J^{\mathrm{GGK}} \to \Delta$ satisfying the following conditions:

- The family restricted to $\Delta^*$ is $J \to \Delta^*$.
- The fiber over 0 is a complex Lie group.
- Any ANF is a section of $J^{\mathrm{GGK}} \to \Delta$.
- $J^{\mathrm{GGK}}$ is a Hausdorff space.

$J^{\mathrm{GGK}}$ is called the *Néron model.* Here $J^{\mathrm{GGK}}$ is just a topological space. In [**GGK1**], they propose that "One may "do geometry"" on the Néron models.

On the other hand, Kato-Nakayama-Usui construct Néron models by their log mixed Hodge theory. To explain their work, we describe $J \to \Delta^*$ by another formulation. Let $\Delta^* \to \Gamma\backslash D$ be the period map arising from the VHS. Then the family of intermediate Jacobian is obtained as the fiber product

$$\begin{tikzcd} J \arrow[r] \arrow[d] & \Gamma'\backslash D' \arrow[d, "\mathrm{Gr}^W_{-1}"] \\ \Delta^* \arrow[r] & \Gamma\backslash D \end{tikzcd}$$

where $D'$ and $\Gamma'$ are for the MHS corresponding to the intermediate Jacobians.

Kato-Nakayama-Usui ([**KNU1**]) extend the above diagram. Firstly, by Kato-Usui ([**KU**]), the period map can be extended to

$$\begin{array}{ccc} \Delta & \longrightarrow & \Gamma\backslash D_\Sigma \\ \cup & & \cup \\ \Delta^* & \longrightarrow & \Gamma\backslash D \end{array}$$

where $\Sigma$ is the fan of nilpotent cones arising from the monodromy of the VHS. Here a boundary point of $\Gamma\backslash D_\Sigma$ is a *nilpotent orbit*, which approximate the period map by Schmid ([**Sc**]). The main theorem of [**KU**] says $\Gamma\backslash D_\Sigma$ is a logarithmic manifold and is a moduli space of log (pure) Hodge structures.

Secondly, ANF is written as

$$\Delta^* \to \Gamma'\backslash D'.$$

A fan $\Sigma'$ including all nilpotent cones arising from the monodromy of any ANF over $\Delta^*$ is given by [**KNU1**]. Then, by [**KNU2**], this map can be extended to

$$\begin{array}{ccc} \Delta & \longrightarrow & \Gamma'\backslash D'_{\Sigma'} \, . \\ \cup & & \cup \\ \Delta^* & \longrightarrow & \Gamma'\backslash D' \end{array}$$

Similarly in the pure case ([**KU**]), a boundary point of $\Gamma'\backslash D'_{\Sigma'}$ is a *nilpotent orbit*, which approximate the ANF by Pearlstein ([**P**]). The main Theorem of [**KNU2**] says $\Gamma'\backslash D'_{\Sigma'}$ is a logarithmic manifold and is a moduli space of log mixed Hodge structures.

And finally, they define the *log Néron model* $J^{\mathrm{KNU}}$ as the fiber product

$$\begin{tikzcd} J^{\mathrm{KNU}} \arrow[r] \arrow[d] & \Gamma'\backslash D'_{\Sigma'} \arrow[d, "\mathrm{Gr}^W_{-1}"] \\ \Delta \arrow[r] & \Gamma\backslash D_{\Sigma} \end{tikzcd}$$

in the category of logarithmic manifolds. We remark that $J^{\mathrm{KNU}}$ is not only a topological space but also has a geometric structure as a *logarithmic manifold.*

However, [**KNU1**] does not show the relationship between $J^{\mathrm{GGK}}$ and $J^{\mathrm{KNU}}$; [**KNU1**, §12.2] says the relationship does not seem to be known between this $J^{\mathrm{KNU}}$ and the Néron model constructed by Green-Griffiths-Kerr. Our main result answers this problem.

THEOREM 1.1 (Theorem 5.1). *$J^{\mathrm{GGK}}$ is homeomorphic to $J^{\mathrm{KNU}}$.*

We explain a key of the proof. By using the liftings in (4.1) and in (4.6), we construct the bijective map between them (Proposition 4.4). In §5, we show that this map is homeomorphism. The diagram (3.5) and the admissibility condition (2.5) or (2.9) play important roles in the proof.

**Acknowledgment.** The author is grateful to Professors Kazuya Kato, Chikara Nakayama, Gregory Pearlstein, Christian Schnell and Sampei Usui for their valuable advice and warm encouragement.

## 2. Preliminary

In this section, we recall the definitions of Néron models of [**GGK1**] and of [**KNU2**]. Let $(\mathcal{H}_{\mathbb{Z}}, \mathcal{F}, \nabla)$ be a variation of polarized Hodge structure of weight $-1$ over the punctured disc $\Delta^*$, where $\mathcal{H}_{\mathbb{Z}}$ is a local system, $\mathcal{F}$ is a filtration of locally free sheaf $\mathcal{H} := \mathcal{H}_{\mathbb{Z}} \otimes \mathcal{O}_{\Delta^*}$ and $\nabla$ is a Gauss-Manin connection. We assume that the monodromy transformation $T$ is unipotent.

**2.1. Families of intermediate Jacobians.** Let $(H, F)$ be the total space of the vector bundle corresponding to the VHS $(\mathcal{H}, \mathcal{F})$. The intermediate Jacobian over $s \in \Delta^*$ is defined as

$$J_s := F_s\backslash H_s/\mathcal{H}_{\mathbb{Z};s}$$

where the subscript $s$ means the fiber (or the stalk) over $s$ and the polarization gives the inclusion map $\mathcal{H}_{\mathbb{Z};s} \hookrightarrow F_s\backslash H_s$. By Carlson ([**C**]), we have the isomorphism

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), H_s) \cong J_s \tag{2.1}$$

where $\mathbb{Z}(0)$ is the Tate's Hodge structure.

We describe the family of intermediate Jacobians $J \to \Delta^*$ by using the MHS in (2.1). Fix a refference point $s_0 \in \Delta^*$. For the PHS $H_{s_0} = (H_\mathbb{Z}, F_{s_0}, \langle\ ,\ \rangle)$ over $s_0$, take a MHS $H'$ which represents an extension class in $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), H_{s_0})$. Let $D$ (resp. $D'$) be the period domain for the type of $H_{s_0}$ (resp. $H'$), defined in [**G**] (resp. [**U**]). Set the monodromy group $\Gamma := \{T^n \in \mathrm{Aut}(H_\mathbb{Z}) \mid n \in \mathbb{Z}\}$ and the period map $\phi : \Delta^* \to \Gamma\backslash D$ arising from the VHS. Then the family of intermediate Jacobian is obtained by the following cartesian diagram:

$$\begin{tikzcd} J \arrow[r] \arrow[d] & \Gamma'\backslash D' \arrow[d, "\mathrm{Gr}^W_{-1}"] \\ \Delta^* \arrow[r, "\phi"] & \Gamma\backslash D, \end{tikzcd}$$

where $\Gamma' := \{T' \in \mathrm{Aut}(H'_\mathbb{Z}) \mid T'|_{\mathrm{Aut}(H_\mathbb{Z})} \in \Gamma\}$.

We review some properties about the period domains $D$ and $D'$. Put $\check{D}$ (resp. $\check{D}'$) the compact dual of $D$ (resp. $D'$), defined in [**G**] (resp. [**U**]). [**G**, §4] (resp. [**U**, §2]) shows the following properties for the pure case (resp. for some kind of mixed case including the case of $D'$):

PROPOSITION 2.1. *Put* $G_A := \mathrm{Aut}(H_A, \langle\ ,\ \rangle)$ *(resp.* $G'_A := \mathrm{Aut}(H'_A, \langle\ ,\ \rangle_\bullet)$*) for* $A = \mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{C}$. *Then*

*(1)* $G_\mathbb{R}$ *(resp.* $G'_\mathbb{R}$*) acts on* $D$ *(resp.* $D'$*) transitively.*
*(2)* $G_\mathbb{C}$ *(resp.* $G'_\mathbb{C}$*) acts on* $\check{D}$ *(resp.* $\check{D}'$*) transitively.*
*(3) Any subgroup of* $G_\mathbb{Z}$ *(resp.* $G'_\mathbb{Z}$*) acts on* $D$ *(resp.* $D'$*) properly discontinuously.*

Since $H'$ is a extension of $H_{s_0}$ by $\mathbb{Z}(0)$, we have the exact sequence of $\mathbb{Z}$-module

$$0 \to H_\mathbb{Z} \xrightarrow{i} H'_\mathbb{Z} \xrightarrow{j} \mathbb{Z} \to 0.$$

We fix $e \in H'_\mathbb{Z}$ such that $j(e) = 1$. Then

$$H'_\mathbb{Z} \cong H_\mathbb{Z} \oplus \mathbb{Z}e. \tag{2.2}$$

Set

$$\mathfrak{h} := \{X \in \mathrm{End}(H'_\mathbb{C}) \mid X|_{\mathrm{End}(H_\mathbb{C})} = 0,\ X(e) \in H_\mathbb{C}\}.$$

PROPOSITION 2.2 ([**U**] Theorem 2.16). $\mathrm{Gr}^{W'}_{-1} : \check{D}' \to \check{D}$ *is a fiber bundle, whose fiber is* $\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{b})$. *Here* $\mathfrak{b}$ *is the Lie algebra of an isotropy subgroup of* $G_\mathbb{C}$.

**2.2. Normal functions and the identity components.** Firstly we define the normal functions for the VHS according to [**GGK1**, §II.A]. Define the following shaves over $\Delta^*$:

$$\mathcal{J} := \mathcal{F}^0\backslash\mathcal{H}/\mathcal{H}_\mathbb{Z},$$
$$\mathcal{J}_\nabla := \left\{ \nu \in \mathcal{J} \;\middle|\; \begin{array}{l} \nabla\tilde{\nu} \in \mathcal{F}^{-1} \otimes \Omega^1 \\ \text{for any local lifting } \tilde{\nu} \end{array} \right\}.$$

Since the monodromy is unipotent, we have the Deligne extension $(\mathcal{H}_e, \mathcal{F}_e)$. Define the following sheaves over $\Delta$:

$$\mathcal{J}_e := \mathcal{F}^0_e\backslash\mathcal{H}_e/j_*\mathcal{H}_\mathbb{Z}, \quad \mathcal{J}_{e,\nabla} := \mathcal{J}_e \cap j_*\mathcal{J}_\nabla$$

where $j : \Delta^* \hookrightarrow \Delta$. A section of $\mathcal{J}_{e,\nabla}$ is called a *normal function* (NF for abbr.).

Secondly we define a space including values of NF according to [**GGK1**, §II.A]. Let $(H_e, F_e)$ be the total space of the vector bundles corresponding to $(\mathcal{H}_e, \mathcal{F}_e)$. Since these vector bundles are trivial, we have a trivialization

$$F_e^n \cong \Delta \times F_{e;0}^n.$$

Since $(F_{e;0}, W(N))$ is a MHS ([**Sc**]), we have the Deligne decomposition $H_{e;0} = \bigoplus_{p,q} I^{p,q}$. This decomposition induces

$$F_{e;0}^0 \backslash H_{e;0} \cong \bigoplus_{p<0} I^{p,q} =: V, \quad F_e^0 \backslash H_e \cong \Delta \times V. \tag{2.3}$$

Define

$$J^Z := F_e^0 \backslash H_e / \sim$$

where

$$(s,x) \sim (s',x') \iff s = s',\ x - x' \in j_* \mathcal{H}_{\mathbb{Z};s}$$

for $(s,x),(s',x') \in \Delta \times V \cong F_e^0 \backslash H_e$. $J^Z$ is called the Zucker space.

$J^Z$ include values of NF. But $J^Z$ is *not* a Hausdorff space generally (cf. [**GGK1**, II.B.8]). Then [**GGK1**] defines the subspace of $J^Z$ so that it is a Hausdorff space including values of NF.

Define

$$W := \{(s,x) \in \Delta \times V \mid x \in \operatorname{Ker}(N) \text{ if } s = 0\}. \tag{2.4}$$

We identify $W$ as the subspace of $F_e^0 \backslash H_e$ by (2.3)

DEFINITION 2.3 ([**GGK1**] II.A.9). Define

$$J^{\mathrm{GGK},0} := W / \sim .$$

Here the topology on $J^{\mathrm{GGK},0}$ is induced from the *strong topology* of $W$ in $\Delta \times V$ ([**KU**, §3.1.1]). $J^{\mathrm{GGK},0}$ is called the *identity component of the Néron model.*

The identity componet has the following property:

PROPOSITION 2.4 ([**GGK1**] II.A.9). *For a NF $\nu$, $\nu(0) \in J_0^{\mathrm{GGK},0}$.*

REMARK 2.5. In [**GGK1**], the definition of the topology on $J^{\mathrm{GGK},0}$ seems to be unclear (Remark after [**GGK1**, Theorem II.A.9] says "This topology is modeled on the "strong topology" in [**KU**]"). In this paper, we use the strong topology on $W \subset \Delta \times V$. Saito ([**Sa**]) shows the Hausdorff property in the case of the ordinary topology.

**2.3. Admissible normal functions and Néron models.** [**GGK1**, §II.B] defines the sheaf

$$\tilde{\mathcal{J}}_\nabla := \left\{ \nu \in j_* \mathcal{J}_\nabla \;\middle|\; \begin{array}{l} \tilde{\nu} \text{ has a logarithmic growth as a section of } \check{\mathcal{F}}_e^0, \\ (T-I)\tilde{\nu} \in (T-I)\mathcal{H}_{\mathbb{Q}} \cap \mathcal{H}_{\mathbb{Z}} \text{ for any local lifting } \tilde{\nu}. \end{array} \right\}, \tag{2.5}$$

where we denote by $(T-I)\tilde{\nu}$ the analytic continuation around the origin 0 of $\tilde{\nu}$. A section of $\tilde{\mathcal{J}}_{e,\nabla}$ is called an *admissible normal function* (for abbr. ANF). By the definition, we have the following exact sequence:

$$0 \to \mathcal{J}_{e,\nabla} \xrightarrow{i} \tilde{\mathcal{J}}_{e,\nabla} \xrightarrow{j} G_0 \to 0. \tag{2.6}$$

Here $G_0$ is the skyscraper sheaf supported at 0, whose stalk is

$$G := \frac{(T-I)H_{\mathbb{Q}} \cap H_{\mathbb{Z}}}{(T-I)H_{\mathbb{Z}}}.$$

Define

$$J_s^{\mathrm{GGK}} := \frac{J_s^{\mathrm{GGK},0} \times \tilde{\mathcal{J}}_{e,\nabla;s}}{\{(\nu(s), [i(\nu)]_s) \mid \nu \in \mathcal{J}_{e,\nabla}\}}$$

where $[i(\nu)]_s$ is the germ at $s \in \Delta$. Since $\tilde{\mathcal{J}}_{e,\nabla;s}$ is divisible abelian group and $G$ is a finite group, the exact sequence of the stalks of (2.6) is split ([**GGK1**, II.a.11]). Then

$$J_s^{\mathrm{GGK}} \cong \begin{cases} J_s^{\mathrm{GGK},0} & \text{if } s \neq 0, \\ J_s^{\mathrm{GGK},0} \times G & \text{if } s = 0. \end{cases}$$

DEFINITION 2.6 ([**GGK1**] II.B.9). Define

$$J^{\mathrm{GGK}} := \bigsqcup_{s \in \Delta} J_s^{\mathrm{GGK}}.$$

Here the topology on $J^{\mathrm{GGK}}$ is defined by the open sets

$$S(\nu) := \{((s,x), [\nu]_s) \in J^{\mathrm{GGK}} \mid (s,x) \in S\} \tag{2.7}$$

where $S$ is an open set of $J^{\mathrm{GGK},0}$ and $\nu$ is an ANF. $J^{\mathrm{KNU}}$ is called the *Néron model* (of Green-Griffiths-Kerr).

EXAMPLE 2.7 (Classical case). Let $\bar{f} : \bar{E} \to \Delta$ be a degenerating family of elliptic curves of Kodaira-type $I_n$. For the restriction $f : E \to \Delta^*$, put the local system $\mathcal{H}_{\mathbb{Z}} := R^1 f_* \mathbb{Z}$ and the filtration $\mathcal{F}^p = R^1 f_*(\Omega^{\geq p}_{E/\Delta^*})$. Then $(\mathcal{H}_{\mathbb{Z}}, \mathcal{F})$ is a VHS over $\Delta^*$ with an unipotent monodromy. In this case,

$$J_0^{\mathrm{GGK},0} \cong \mathbb{G}_m, \quad G \cong \mathbb{Z} \backslash n\mathbb{Z}$$

twisting $(\mathcal{H}_{\mathbb{Z}}, \mathcal{F})$ into the VHS of weight $-1$. [**N**] constructs Néron models in this case by using toroidal embeddings.

**2.4. A nonclassical example.** We give an example where the Néron model is not an analytic space. [**GGK2**, §III.A] or [**KNU1**, §9] deals with a special situation of this example.

Let $Y$ be a singular $K3$ surface, i.e., $\rho(Y) = 20$, $\bar{f} : \bar{E} \to \Delta$ a degenerating family of elliptic curves of Kodaira-type $I_n$. By the Shioda-Inose correspondence ([**SI**]), for a transcendental basis $\{t_1, t_2\}$ of $H^2(Y)$ the intersection form is represented as

$$(t_i \cdot t_j)_{i,j} = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$$

where $a, b, c \in \mathbb{Z}$, $a, c > 0$ and $b^2 - 4ac < 0$. We assume that $a = m$ (square free positive integer), $b = 0$ and $c = 1$. Take a symplectic basis $\{\alpha, \beta\}$ of $H^1(E_s)$ for $s \neq 0$ such that the monodromy action is

$$\alpha \mapsto \alpha + n\beta, \quad \beta \mapsto \beta.$$

Set

$$e_1 = t_1 \times \alpha, \quad e_2 = t_2 \times \alpha, \quad e_3 = \frac{t_1}{2m} \times \beta, \quad e_4 = \frac{t_2}{2} \times \beta$$

in $H^3(Y \times E_s, \mathbb{Q})$. Then the intersection form is represented as

$$(e_i \cdot e_j)_{i,j} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Put $g := f \circ \mathrm{pr}_2 : Y \times E \to \Delta^*$. We have the local system $\mathcal{H}_{\mathbb{Z}} \subset R^3 g_* \mathbb{Q}$ such that $\mathcal{H}_{\mathbb{Z},s} = \sum_i \mathbb{Z} e_i$ and the filtration $\mathcal{F}^p$ induced from $R^3 g_*(\Omega^{\geq p}_{Y \times E/\Delta^*})$. Then $(\mathcal{H}_{\mathbb{Z}}, \mathcal{F})$ is a VHS and a fiber $(\mathcal{H}_{\mathbb{Z},s}, F_s)$ is a PHS of weight $-1$ where $h^{1,-2} = h^{0,-1} = h^{-1,0} = h^{-2,1} = 1$, twisting it into the VHS of weight $-1$. The monodromy transformation is represented as

$$T = \begin{pmatrix} I_2 & 0 \\ \begin{matrix} 2mn & 0 \\ 0 & 2n \end{matrix} & I_2 \end{pmatrix}.$$

By [**KU**, §12.3], the limiting MHS is described by the following diamond:

$$\begin{tikzcd} \underset{\bullet}{(1,-1)} \arrow[d, "N"] & \underset{\bullet}{(-1,1)} \arrow[d, "N"] \\ \underset{\bullet}{(0,-2)} & \underset{\bullet}{(-2,0)} \end{tikzcd}$$

Then

$$J_0^{\mathrm{GGK},0} \cong I^{-2,0}/j_* \mathcal{H}_{\mathbb{Z};0}, \quad G \cong \mathbb{Z}/2mn\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z}.$$

In this case the dimension of $J_0^{\mathrm{GGK},0}$ is smaller than the dimension of a general fiber and $J^{\mathrm{Z}}$ is not a Hausdorff space (cf. [**KNU1**, §9]).

**2.5. Moduli spaces of log Hodge structures and log Néron models.** Let $\mathfrak{g}_A$ (resp. $\mathfrak{g}'_A$) be the Lie algebra of $G_A$ (resp. $G'_A$) for $A = \mathbb{R}, \mathbb{C}$. Set the nilpotent cone $\sigma = \mathbb{R}_{\geq 0} N$ ($N = \log(T)$) in $\mathfrak{g}_{\mathbb{R}}$, the fan $\Sigma := \{\{0\}, \sigma\}$ and the set

$$D_\Sigma = \{(\sigma, Z) \mid \sigma \in \Sigma,\ Z = \exp(\sigma_{\mathbb{C}})F \text{ is a } \sigma\text{-nilpotent orbit}\}. \tag{2.8}$$

By [**KU**], the period map $\phi : \Delta^* \to \Gamma\backslash D$ extends to the *log* period map $\phi : \Delta \to \Gamma\backslash D_\Sigma$.

In [**KNU1**], the following fan of $\mathfrak{g}'_{\mathbb{R}}$ is defined:

(2.9)

$$\Sigma' := \left\{ \mathbb{R}_{\geq 0} N' \;\middle|\; \begin{array}{l} N' \in \operatorname{End} H_{\mathbb{Q}}, N'|_{\operatorname{End} H_{\mathbb{Q}}} = N, \\ N'(e) = N(a) \text{ for some } a \in H_{\mathbb{Q}} \text{ such that } (T - I)a \in H_{\mathbb{Z}} \end{array} \right\}.$$

PROPOSITION 2.8. *Take $\sigma' = \mathbb{R}_{\geq 0} N' \in \Sigma'$. Then*

$$\exp(N') \in \Gamma', \quad \operatorname{Ad}(\gamma)\sigma' \in \Sigma'$$

*for $\gamma \in \Gamma'$. Therefore $\Gamma'$ is strongly compatible with $\Sigma'$.*

*Proof.* $N'$ and $\Gamma'$ are represented as

$$N' = \begin{pmatrix} N & Na \\ 0 & 0 \end{pmatrix}, \quad \Gamma' = \left\{ \begin{pmatrix} T^n & b \\ 0 & 1 \end{pmatrix} \;\middle|\; b \in H_{\mathbb{Z}},\ n \in \mathbb{Z} \right\}$$

with respect to the decomposition (2.2). Since $(T - I)a \in H_{\mathbb{Z}}$

$$\exp(N') = \begin{pmatrix} T & (T - I)a \\ 0 & 1 \end{pmatrix} \in \Gamma'.$$

For $\gamma = \begin{pmatrix} T^n & b \\ 0 & 1 \end{pmatrix} \in \Gamma'$, we have

$$\mathrm{Ad}\,(\gamma)N' = \begin{pmatrix} N & N(T^n a - b) \\ 0 & 0 \end{pmatrix}. \tag{2.10}$$

Since $(T-I)(T^n a - b) \in H_{\mathbb{Z}}$, $\mathrm{Ad}\,(\gamma)N' \in \Sigma'$. □

Similarly in (2.8), $D'_{\Sigma'}$ is defined as the set of nilpotent orbits ([**KNU2**, §2.1.3]). By above proposition, we can define the action

$$\Gamma' \times D'_{\Sigma'} \to D'_{\Sigma'};\ (\gamma, (\sigma', Z)) \mapsto (\mathrm{Ad}\,(\gamma)\sigma', \gamma Z)$$

and the orbit space $\Gamma'\backslash D'_{\Sigma'}$.

The geometric structure on $\Gamma'\backslash D'_{\Sigma'}$ is defined in [**KNU2**, §2.2.2], similarly in the pure case ([**KU**]). For $\sigma' \in \Sigma'$, set the monoid

$$\Gamma'(\sigma') := \Gamma' \cap \exp(\sigma')$$

and the toric variety

$$\mathrm{toric}_{\sigma'} := \mathrm{Spec}\,(\mathbb{C}[\Gamma'(\sigma')^\vee])_{\mathrm{an}} \cong \mathbb{C}.$$

Moreover we define the analytic space

$$\check{E}'_{\sigma'} := \mathrm{toric}_{\sigma'} \times \check{D}'$$

and the subspace

$$E'_{\sigma'} = \left\{ (s, F) \in \check{E}'_{\sigma'} \;\middle|\; \begin{array}{l} \exp(l(s)N')F \in D' \text{ if } s \neq 0, \\ \exp(\sigma'_{\mathbb{C}})F \text{ is a nilpotent orbit if } s = 0 \end{array} \right\}$$

where $l(s)$ is a branch of $(2\pi i)^{-1}\log(s)$. The topology on $E'_{\sigma'}$ is the strong topology in $\check{E}'_{\sigma'}$. Then we have the map

$$E'_{\sigma'} \xrightarrow{p'_1} \Gamma'(\sigma')^{\mathrm{gp}}\backslash D'_{\sigma'} \xrightarrow{p'_2} \Gamma'\backslash D'_{\Sigma'};\quad (s, F) \mapsto \begin{cases} (0, \exp(l(s)N')F) & \text{if } s \neq 0, \\ (\sigma', \exp(\sigma'_{\mathbb{C}})F) & \text{if } s = 0. \end{cases}$$

The geometric structure on $\Gamma'\backslash D'_{\Sigma'}$ is induced from $E'_{\sigma'}$ locally through this map. Moreover Kato-Nakayama-Usui announce the following theorem:

THEOREM 2.9 ([**KNU2**] Main Theorem). *Similarly in the pure case ([**KU**, Main Theorem]), the following is hold*

*(1)* $E'_{\sigma'}, \Gamma'(\sigma')^{\mathrm{gp}}\backslash D'_{\sigma'}$ *and* $\Gamma'\backslash D'_{\Sigma'}$ *are logarithmic manifolds.*
*(2)* $E'_{\sigma'} \to \Gamma'(\sigma')^{\mathrm{gp}}\backslash D'_{\sigma'}$ *is a* $\sigma'_{\mathbb{C}}$*-torsor.*
*(3)* $\Gamma'(\sigma')^{\mathrm{gp}}\backslash D'_{\sigma'} \to \Gamma'\backslash D'_{\Sigma'}$ *is a locally isomorphism.*
*(4)* $\Gamma'\backslash D'_{\Sigma'}$ *is a moduli space of log mixed Hodge structures.*

DEFINITION 2.10 ([**KNU1**] §7). Define the fiber product

$$\begin{array}{ccc} J^{\mathrm{KNU}} & \longrightarrow & \Gamma'\backslash D'_{\Sigma'} \\ \downarrow & & \downarrow {\scriptstyle \mathrm{Gr}^W_{-1}} \\ \Delta & \xrightarrow{\phi} & \Gamma\backslash D_{\Sigma} \end{array}$$

in the category $\mathcal{B}(\log)$ ([**KU**, (3.2.4)]). $J^{\mathrm{KNU}}$ is called the *log Néron model*.

We describe the topology on $J^{\mathrm{KNU}}$. Now we define the following diagram:

$$\begin{tikzcd} K_{\sigma'} \arrow[r] \arrow[d] & E'_{\sigma'} \arrow[d, "p_1"] \\ J_{\sigma'} \arrow[r] \arrow[d] & \Gamma'(\sigma')^{\mathrm{gp}}\backslash D'_{\sigma'} \arrow[d, "p_2"] \\ J^{\mathrm{KNU}} \arrow[r] & \Gamma'\backslash D'_{\Sigma'}, \end{tikzcd}$$

where $K_{\sigma'}$ and $J_{\sigma'}$ are the fiber products in $\mathcal{B}(\log)$. Here the topology on $K_{\sigma'}$ is the strong topology in $\Delta \times \check{E}'_{\sigma'}$. The topological structures of $J_{\sigma'}$ (resp. $J^{\mathrm{KNU}}$) is induced from $K_{\sigma'}$ through the morphism $K_{\sigma'} \to J_{\sigma'}$ (resp. $K_{\sigma'} \to J^{\mathrm{KNU}}$).

## 3. The relation between $E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ and $E'_{\sigma'} \to \Gamma'(\sigma')^{\mathrm{gp}}\backslash D'_{\sigma'}$

The results in this section can be verified easily following after **[KNU2]**, but for our later use we write them here in details. In the following section, we regard $E_\sigma$ (resp. $E'_{\sigma'}$) as a topological space whose topology is the strong topology in $\check{E}_\sigma$ (resp. $\check{E}'_{\sigma'}$).

**3.1. $\sigma_{\mathbb{C}}$-action on $E_\sigma$ and $\sigma'_{\mathbb{C}}$-action on $E'_{\sigma'}$.** For $\sigma = \mathbb{R}_{\geq 0}N \in \Sigma$ put

$$\mathrm{torus}_\sigma := \mathrm{Spec}\,(\mathbb{C}[\Gamma(\sigma)^{\vee\mathrm{gp}}])_{\mathrm{an}} \cong \mathbb{G}_m.$$

Then we have the surjective map

$$\sigma_{\mathbb{C}} \to \mathrm{torus}_\sigma; \quad wN \mapsto \exp(2\pi\sqrt{-1}w),$$

which induces the action

$$\sigma_{\mathbb{C}} \times \mathrm{toric}_\sigma \to \mathrm{toric}_\sigma; \quad (wN, s) \mapsto \exp(2\pi\sqrt{-1}w)s.$$

For $\sigma' = \mathbb{R}_{\geq 0}N' \in \Sigma'$, $\sigma'_{\mathbb{C}}$-action on $\mathrm{toric}_{\sigma'}$ is defined similarly.

By the correspondence $N \leftrightarrow N'$ (resp. $\exp(N) \leftrightarrow \exp(N')$), we have $\sigma_{\mathbb{C}} \cong \sigma'_{\mathbb{C}}$ (resp. $\mathrm{toric}_\sigma \cong \mathrm{toric}_{\sigma'}$) and the following commutative diagram:

$$\begin{tikzcd} \sigma'_{\mathbb{C}} \times \mathrm{toric}_{\sigma'} \arrow[r] \arrow[d, "\wr"] & \mathrm{toric}_{\sigma'} \arrow[d, "\wr"] \\ \sigma_{\mathbb{C}} \times \mathrm{toric}_\sigma \arrow[r] & \mathrm{toric}_\sigma \end{tikzcd} \tag{3.1}$$

Moreover we define the $\sigma_{\mathbb{C}}$-action

$$\sigma_{\mathbb{C}} \times E_\sigma \to E_\sigma; \quad (wN, (s, F)) \mapsto (\exp(2\pi\sqrt{-1}w)s, \exp(-wN)F).$$

$\sigma'_{\mathbb{C}}$-action on $E'_{\sigma'}$ is defined similarly. Put

$$\mathrm{Gr}^W_{-1} : \check{E}'_{\sigma'} \to \check{E}_\sigma; \quad (s, F) \mapsto (s, \mathrm{Gr}^W_{-1}(F)).$$

Then the diagram (3.1) induces the following commutative diagram:

$$\begin{tikzcd} \sigma'_{\mathbb{C}} \times E'_{\sigma'} \arrow[r] \arrow[d] & E'_\sigma \arrow[d] & \subset & \check{E}'_{\sigma'} \arrow[d, "\mathrm{Gr}^W_{-1}"] \\ \sigma_{\mathbb{C}} \times E_\sigma \arrow[r] & E_\sigma & \subset & \check{E}_\sigma. \end{tikzcd} \tag{3.2}$$

**3.2. The torsor property of $E'_{\sigma'} \to \Gamma'(\sigma')^{\mathrm{gp}}\backslash D'_{\sigma'}$.**

LEMMA 3.1. *The action of $\sigma'_{\mathbb{C}}$ on $E'_{\sigma'}$ is proper and free.*

*Proof.* Since the lower horizontal action in (3.2) is proper and free ([**KU**, (7.2.9)]), the upper horizontal action in (3.2) is free.

The $\sigma'_{\mathbb{C}}$-action is proper if and only if the following condition is satisfied.

- For $x', y' \in E'_{\sigma'}$, sequences $\{x'_\lambda\}$ in $E'_{\sigma'}$ and $\{h'_\lambda\}$ in $\sigma'_{\mathbb{C}}$ such that $x'_\lambda \to x'$ and $h'_\lambda x'_\lambda \to y'$, there exists $h' \in \sigma'_{\mathbb{C}}$ such that $h'_\lambda \to h'$.

We show the above condition is hold. Take $x', y', \{x'_\lambda\}, \{h'_\lambda\}$ as above and set

$$x := \mathrm{Gr}^W_{-1}(x'), \quad y = \mathrm{Gr}^W_{-1}(y'), \quad h_\lambda := h'_\lambda|_{\mathrm{End}\, H_{\mathbb{Q}}}.$$

Since $\sigma_{\mathbb{C}}$-action is proper ([**KU**, (7.2.2)]), there exists $h \in \sigma_{\mathbb{C}}$ such that $h_\lambda \to h$. By the isomorphism $\sigma \cong \sigma'$, there exists $h' \in \sigma'_{\mathbb{C}}$ such that $h = h'|_{\mathrm{End}\, H_{\mathbb{Q}}}$ and $h'_\lambda \to h'$. □

LEMMA 3.2 ([**KU**] Lemma 7.3.3). *Let $H$ be a topological group, $X$ a topological space, and assume we have a action $H \times X \to X$, which is proper and free. Assume moreover the following condition is satisfied.*

- *For $x \in X$, there exists an topological space $S$, a morphism $\iota : S \to X$ and an open neighborhood $U$ of $1$ in $H$ such that $U \times S \to X$; $(h, s) \mapsto h\iota(s)$ induces an isomorphism onto an open set of $X$.*

*Then $X \to H\backslash X$ is an $H$-torsor.*

PROPOSITION 3.3 ([**KNU2**] Theorem A.(2)). *The action of $\sigma'_{\mathbb{C}}$ on $E'_{\sigma'}$ satisfies the condition of Lemma 3.2. Then $E'_{\sigma'} \to \Gamma'(\sigma')^{\mathrm{gp}}\backslash D'_{\sigma'}$ is a $\sigma'_{\mathbb{C}}$-torsor.*

*Proof.* Since $\sigma'(s)_{\mathbb{C}} \hookrightarrow T_{\check{D}'}(F)$ for $(s, F) \in E'_{\sigma'}$ (in this case $\sigma'(s) = \sigma'$ if $s = 0$, $\sigma'(s) = 0$ otherwise), the proof is same as the pure case ([**KU**, (7.3.5)]). □

Since $p_1 : E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ (resp. $p'_1 : E'_{\sigma'} \to \Gamma'(\sigma')^{\mathrm{gp}}\backslash D'_{\sigma'}$) is $\sigma_{\mathbb{C}}$-torsor ($\sigma'_{\mathbb{C}}$-torsor), the diagram (3.2) induces the following property:

COROLLARY 3.4. *The commutative diagram*

$$\begin{tikzcd} E'_{\sigma'} \arrow[r] \arrow[d, "\mathrm{Gr}^W_{-1}"'] & \Gamma'(\sigma')^{\mathrm{gp}}\backslash D'_{\sigma'} \arrow[d] \\ E_\sigma \arrow[r] & \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma \end{tikzcd} \tag{3.3}$$

*is cartesian.*

**3.3. Limiting Hodge filtrations and liftings of the period map.** Let $s$ be a coordinate of $\Delta$. Take the map

$$\hat{\phi} : \Delta^* \to \check{D}; \quad s \mapsto \exp(-l(s)N)\tilde{\phi}(s) \tag{3.4}$$

where $\tilde{\phi}$ is a local lifting of the period map $\phi$. we call $\hat{\phi}$ an *untwisted period map*. This map is extended over $\Delta$ ([**Sc**]). $\hat{\phi}$ gives a lifting

$$\Delta \to E_\sigma; \quad s \mapsto (s, \hat{\phi}(s))$$

of $\phi$. Then we have the following diagram:

(3.5)
$$\begin{tikzcd}
\check{E}'_{\sigma'} \arrow[d, "\mathrm{Gr}^W_{-1}"'] & \supset & E'_{\sigma'} \arrow[r] \arrow[d] & \Gamma'(\sigma')^{\mathrm{gp}}\backslash D'_{\sigma'} \arrow[r] \arrow[d] & \Gamma'\backslash D'_{\Sigma'} \arrow[d] \\
\check{E}_{\sigma} & \supset & E_{\sigma} \arrow[r] & \Gamma(\sigma)^{\mathrm{gp}}\backslash D_{\sigma} \arrow[r, equal] & \Gamma\backslash D_{\Sigma} \\
& & \Delta \arrow[u, "(id,\hat{\phi})"] \arrow[ur, "\phi"'] & &
\end{tikzcd}$$

for $\sigma' \in \Sigma'$ such that $\sigma' \neq \{0\}$.

For $(s, F) \in \check{E}'_{\sigma'}$ such that $\mathrm{Gr}^W_{-1}(F) = F_{\hat{\phi}(s)}$, we have the exact sequence

$$0 \to F^p_{\hat{\phi}(s)} \to F^p \to \mathbb{C} \to 0$$

if $p < 0$, $F^p_{\hat{\phi}(s)} \cong F^p$ otherwise. Then

$$F^p = \begin{cases} \mathbb{C}(z,1) + F^p_{\hat{\phi}(s)} & \text{if } p < 0, \\ F^p_{\hat{\phi}(s)} & \text{if } p \geq 0 \end{cases} \tag{3.6}$$

where $(z, 1) \in H'_{\mathbb{C}}$ is represented with respect to the decomposition (2.2).

PROPOSITION 3.5. *By the admissibility condition (2.9), $\sigma' = \mathbb{R}_{\geq 0}N' \in \Sigma'$ can be written by*

$$N' = \mathbb{R}_{\geq 0}\begin{pmatrix} N & Na \\ 0 & 0 \end{pmatrix}$$

*for some $a \in H_{\mathbb{Q}}$. Then*

$$(s, F) \in E'_{\sigma'} \iff \begin{cases} z \in V & \text{if } s \neq 0, \\ z + a \in V \cap \mathrm{Ker}\,(N) & \text{if } s = 0 \end{cases}$$

*where $z \in H_{\mathbb{C}}$ is in (3.6) and $V$ is in (2.3).*

*Proof.* The transversality condition of nilpotent orbits says that

$$(0, F) \in E'_{\sigma'} \iff N(z + a) \in F^{-1}_{\infty}$$

where we denote the limiting Hodge filtration $F_{\hat{\phi}(0)}$ by $F_\infty$ Since $(F_\infty, W(N))$ is MHS and $N$ is $(-1,-1)$-morphism, $N(z + a) \in F^{-1}_\infty$ if $z + a \in F^0_\infty + \mathrm{Ker}\,(N)$. □

## 4. A bijection

In this section, we define a bijective map between $J^{\mathrm{KNU}}$ and $J^{\mathrm{GGK}}$ as a set. In the following section, we fix a coordinate $s$ of $\Delta$.

### 4.1. A map from $J^{\mathrm{GGK}}$ to $J^{\mathrm{KNU}}$.

Take an ANF $\nu$, i.e., AVMHS

$$\nu : \Delta^* \to \Gamma'\backslash D'.$$

Take a local lifting $\tilde{\nu}$ of $\nu$, which gives a local lifting $\tilde{\phi} := \mathrm{Gr}^W_{-1}(\tilde{\nu})$ of $\phi$. Let $\hat{\nu} : \Delta \to \check{D}'$ (resp. $\hat{\phi} : \Delta \to \check{D}$) be the untwisted period map associated with $\tilde{\nu}$

(resp. $\tilde{\phi}$). Then we have the commutative diagram

(4.1)
$$\begin{tikzcd} & \check{D}' \arrow[d, "\mathrm{Gr}^W_{-1}"] \\ \Delta \arrow[ur, "\hat{\nu}"] \arrow[r, "\hat{\phi}"'] & \check{D}, \end{tikzcd}$$

i.e., $\hat{\nu}$ is a lifting of $\hat{\phi}$. We denote the limiting Hodge filtration $\hat{\nu}(0)$ by $F_{\tilde{\nu}}$.

Fix $F_{\tilde{\nu}}$ as a reference point of $\check{D}'$. By Proposition 2.2, the vertical morphism of the above diagram is the fiber bundle whose fiber is $\mathfrak{h}/(\mathfrak{h}\cap\mathfrak{b})$. Here

$$\mathfrak{h}\cap\mathfrak{b} = \{X \in \mathfrak{h} \mid X(e) \in F^0_{\hat{\phi}(0)}\}.$$

Then

$$\mathfrak{h}/(\mathfrak{h}\cap\mathfrak{b}) \cong V; \quad X_v \leftrightarrow v \tag{4.2}$$

where $X_v \in \mathfrak{h}$ such that $X_v(e) = v$. For a boundary point $((0,v),[\nu]_0) \in J^{\mathrm{GGK}}$, we define

$$\alpha((0,v),[\nu]_0) := (0,(\sigma', \exp(\sigma'_{\mathbb{C}})\exp(X_{-v})F_{\tilde{\nu}}))$$

where $\sigma'$ is the monodromy cone arising from $\tilde{\nu}$. By the admissibility condition (2.5), this monodromy cone $\sigma'$ is in $\Sigma'$. By Proposition 3.5, $\alpha((0,v),[\nu]_0)$ is in $J^{\mathrm{GGK}}$.

Lemma 4.1. *$\alpha((0,v),[\nu]_0)$ is well-defined.*

*Proof.* We show that $\alpha((0,v),[\nu]_0)$ does not depend on the choice of $((0,v),[\nu]_0)$ and $\tilde{\nu}$.

Take $((0,v'),[\nu']_0)$ such that $((0,v),[\nu]_0) \sim ((0,v'),[\nu']_0)$. For a local lifting $\tilde{\nu}$ (resp. $\tilde{\nu}'$) of $\nu$ (resp. $\nu'$), the logarithm of the monodromy of $\tilde{\nu}$ (resp. $\tilde{\nu}'$) is described by

$$\begin{pmatrix} N & Na \\ 0 & 0 \end{pmatrix} \quad \left(\text{resp. } \begin{pmatrix} N & Na' \\ 0 & 0 \end{pmatrix}\right)$$

for some $a \in H_{\mathbb{Q}}$ (resp. $a' \in H_{\mathbb{Q}}$). Put $\mu := \nu' - \nu$. Then $\tilde{\mu} := \tilde{\nu}' - \tilde{\nu}$ is a local lifting, of which the logarithm of the monodromy is

$$\begin{pmatrix} N & N(a'-a) \\ 0 & 0 \end{pmatrix}. \tag{4.3}$$

Since $\mu$ is NF, $a' - a \in H_{\mathbb{Z}}$. By (2.10), we can assume $a' - a = 0$ replacing $\tilde{\nu}'$ by $\gamma\tilde{\nu}'$ for some $\gamma \in \Gamma'$. By the definition, $\hat{\mu}(0) = v - v' \in J^{\mathrm{GGK},0}$. Then

$$\begin{aligned}(\sigma', \exp(\sigma'_{\mathbb{C}})\exp(X_{-v'})F_{\tilde{\nu}'}) &= (\sigma', \exp(\sigma'_{\mathbb{C}})\exp(X_{-v'})\exp(X_{v'-v})F_{\tilde{\nu}'-\tilde{\mu}}) \\ &= (\sigma', \exp(\sigma'_{\mathbb{C}})\exp(X_{-v})F_{\tilde{\nu}}).\end{aligned}$$

Moreover, take another lifting $\gamma\tilde{\nu}$ for $\gamma \in \Gamma'$. The monodromy cone arising from $\gamma\tilde{\nu}$ is $\mathrm{Ad}(\gamma)\sigma'$. The limiting Hodge filtration is $\gamma F_{\tilde{\nu}}$. Since $v \in \operatorname{Ker} N$,

$$\exp(X_{-v})\gamma = \gamma\exp(X_{-v}).$$

Then

$$(\mathrm{Ad}(\gamma)\sigma'_{\mathbb{C}}, \exp(\mathrm{Ad}(\gamma)\sigma'_{\mathbb{C}})\exp(X_{-v})\gamma F_{\tilde{\nu}}) = \gamma(\sigma', \exp(\sigma'_{\mathbb{C}})\exp(X_{-v})F_{\tilde{\nu}}).$$

□

Therefore $\alpha$ defines a map

$$\alpha : J^{\mathrm{GGK}} \to J^{\mathrm{KNU}}$$

where the restriction $\alpha|_J$ is the canonical one.

**4.2. A map from $J^{\mathrm{KNU}}$ to $J^{\mathrm{GGK}}$.** Take a lifting $\tilde{\phi}$ of $\phi$. Let $\hat{\phi} : \Delta \to \check{D}$ be the untwisted period map. By Corollary 3.4, for $(0, (\sigma', Z)) \in J_{\sigma'}$, we have $(0, F) \in E_\sigma$ such that

$$\mathrm{Gr}^W_{-1}(0, F) = (0, F_{\hat{\phi}(0)}), \quad p_1(0, F) = (\sigma', Z) \tag{4.4}$$

uniquely. We denote this filtration by $F_{(\sigma', Z)}$.

LEMMA 4.2. *For $\gamma \in \Gamma'$, $\gamma F_{(\sigma',Z)} = F_{\gamma(\sigma',Z)}$*

*Proof.* By Proposition 3.5,

$$F^p_{(\sigma',Z)} = \begin{cases} \mathbb{C}(x - a, 1) + F^p_{\hat{\phi}(0)} & \text{if } p < 0, \\ F^p_{\hat{\phi}(0)} & \text{if } p \geq 0 \end{cases}$$

where $x \in \mathrm{Ker}\,(N)$. Take $\gamma = \begin{pmatrix} T^n & b \\ 0 & 1 \end{pmatrix} \in \Gamma'$. Then

$$\gamma F^p_{(\sigma',Z)} = \begin{cases} \mathbb{C}(T^n(x - a) + b, 1) + F_{\gamma\hat{\phi}(0)} & \text{if } p < 0, \\ F_{\gamma\hat{\phi}(0)} & \text{if } p \geq 0 \end{cases}.$$

Since $x \in \mathrm{Ker}\,(N)$,

$$T^n(x - a) + b = x - (T^n a - b). \tag{4.5}$$

By (2.10) and Proposition 3.5, $(0, \gamma F_{(\sigma',Z)}) \in E_{\mathrm{Ad}\,(\gamma)\sigma'}$, which satisfy

$$\mathrm{Gr}^W_{-1}(0, \gamma F_{(\sigma',Z)}) = (0, F_{\gamma\hat{\phi}(0)}), \quad p_1(0, \gamma F_{(\sigma',Z)}) = \gamma(\sigma', Z).$$

Then $\gamma F_{(\sigma',Z)} = F_{\gamma(\sigma',Z)}$. □

Since $\check{D}' \to \check{D}$ is a fiber bundle, there exists a lifting of $\hat{\phi}$:

$$\begin{tikzcd} & \check{D}' \arrow[d, "\mathrm{Gr}^W_{-1}"] \\ \Delta \arrow[ur, "\hat{\nu}_{(\sigma',Z)}"] \arrow[r, "\hat{\phi}"'] & \check{D} \end{tikzcd} \tag{4.6}$$

such that $\hat{\nu}_{(\sigma',Z)}(0) = F_{(\sigma',Z)}$, shrinking $\Delta$ if necessarily. Then we have a holomorphic map

$$\Delta^* \to \Gamma' \backslash D'; \quad s \mapsto p_2 \circ p_1(s, \hat{\nu}_{(\sigma',Z)}(s)),$$

which defines AVMHS, i.e., ANF. Denote this ANF by $\nu_{(\sigma',Z)}$. We define

$$\beta(0, (\sigma', Z)) := ((0, 0), [\nu_{(\sigma',Z)}]_0) \in J^{\mathrm{GGK}}.$$

LEMMA 4.3. *$\beta(0, (\sigma', Z))$ is well-defined.*

*Proof.* We show that $\beta(0,(\sigma',Z))$ does not depend on the choice of $\hat{\nu}_{(\sigma',Z)}$ and $(\sigma',Z)$.

Take liftings $\hat{\nu}_{(\sigma',Z)}$ and $\hat{\nu}'_{(\sigma',Z)}$ such that

$$\hat{\nu}_{(\sigma',Z)}(0)=\hat{\nu}'_{(\sigma',Z)}(0)=F_{(\sigma',Z)} \tag{4.7}$$

Then $\mu:=\nu_{(\sigma',Z)}-\nu'_{(\sigma',Z)}$ is a NF. We take a local lifting $\tilde{\mu}$ of $\mu$ by

$$\tilde{\mu}(s)=\exp(l(s)N')\hat{\nu}_{(\sigma',Z)}-\exp(l(s)N')\hat{\nu}'_{(\sigma',Z)}$$

Then the monodromy of $\tilde{\mu}$ is

$$\begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix}.$$

By (4.7), $\hat{\mu}(0)=0\in J^{\mathrm{GGK},0}$. Then

$$((0,0),[\nu_{(\sigma',Z)}]_0)\sim((0,0),[\nu'_{(\sigma',Z)}]_0).$$

Moreover, take $\gamma(\sigma',Z)$ for $\gamma\in\Gamma'$. By Lemma 4.2, $\gamma\hat{\nu}_{(\sigma',Z)}$ gives a lifting $\hat{\nu}_{\gamma(\sigma',Z)}$, which satisfy $\nu_{(\sigma',Z)}=\nu_{\gamma(\sigma',Z)}$. □

Then $\beta$ defines a map

$$\beta: J^{\mathrm{KNU}}\to J^{\mathrm{GGK}}$$

where the restriction map $\beta|_J$ is the canonical one.

PROPOSITION 4.4. *$\alpha=\beta^{-1}$ and $\beta=\alpha^{-1}$, i.e., $J^{\mathrm{GGK}}$ is bijective to $J^{\mathrm{KNU}}$.*

*Proof.* For $((0,v),[\nu]_0)\in J^{\mathrm{GGK}}$, put $(0,(\sigma',Z)):=\alpha((0,v),[\nu]_0)$. Then $F_{(\sigma',Z)}=\exp(X_{-v})F_{\hat{\nu}}$ by taking a lifting suitably. Therefore $\hat{\mu}(0)=v$ for $\mu=\nu-\nu_{(\sigma',Z)}$, which induces

$$((0,v),[\nu]_0)\sim((0,0),[\nu_{(\sigma,Z)}]_0)=\beta(0,(\sigma',Z)).$$

On the other hand, for $(0,(\sigma',Z))\in J^{\mathrm{KNU}}$, put $((0,0),[\nu]_0):=\beta(0,(\sigma',Z))$. Then $F_{\hat{\nu}}=F_{(\sigma',Z)}$ by taking a lifting suitably. Therefore

$$(0,(\sigma',Z))=(0,(\sigma',\exp(\sigma'_{\mathbb{C}})F_{\hat{\nu}}))=\alpha((0,0),[\nu]_0).$$

□

## 5. A homeomorphism

In this section, we describe neighborhoods in $J^{\mathrm{KNU}}$ and show that the bijection constructed in the last section are continuous. In the pure case, neighborhoods of $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ is described in [**KU**, (7.3.5)].

For the period map $\phi$, take an untwisted period map $\hat{\phi}:\Delta\to\check{D}$. Denote the limiting Hodge filtration $\hat{\phi}(0)$ by $F_\infty$. Since $\sigma_{\mathbb{C}}\hookrightarrow T_{\check{D}}(F_\infty)$, we can take a $\mathbb{C}$-subspace $B$ of $\mathfrak{g}_{\mathbb{C}}$ such that $B\oplus\sigma_{\mathbb{C}}\cong T_{\check{D}}(F_\infty)$. Then an open neighborhood of $F_\infty$ in $\check{D}$ is described by

$$\{(\exp(a_1)\exp(a_2)F_\infty)\mid a_1\in U_1,\ a_2\in U_2\}\cong U_1\times U_2$$

where $U_1$ (resp. $U_2$) is a sufficiently small neighborhood of 0 in $\sigma_{\mathbb{C}}$ (resp. $B$). Here we assume that

$$\hat{\phi}(s)\in\{0\}\times U_2\subset U_1\times U_2 \tag{5.1}$$

for $s \in \Delta$, changing the coordinate of $\Delta$ if necessarily. Since $\mathrm{Gr}^W_{-1} : \check{D}' \to \check{D}$ is a fiber bundle, whose fiber is $V$ of (4.2), we have a trivialization

$$(\mathrm{Gr}^W_{-1})^{-1}(U_1 \times U_2) \cong U_1 \times U_2 \times V. \tag{5.2}$$

For $(0,(\sigma',Z)) \in J^{\mathrm{KNU}}$, take the point $(0, F_{(\sigma',Z)}) \in E'_{\sigma'}$ as in (4.4). Since $F_{(\sigma',Z)} \in (\mathrm{Gr}^W_{-1})^{-1}(F_\infty)$, we can assume that $F_{(\sigma',Z)} = (0,0,0)$ in (5.2). By using the trivialization (5.2), an open neighborhood of $(0, F_{(\sigma',Z)})$ in $\check{E}'_{\sigma'}$ can be described by

$$\{(a_0,(a_1,a_2,a_3)) \mid a_0 \in U_0,\ a_1 \in U_1,\ a_2 \in U_2\ a_3 \in U_3\}$$

where $U_0$ (resp. $U_3$) is a sufficiently small neighborhood of 0 in $\mathrm{toric}_\sigma$ (resp. $V$). Set

$$A' = \{(a_0,(0,a_2,a_3)) \mid a_0 \in U_0,\ a_2 \in U_2,\ a_3 \in U_3\}, \quad S' = A' \cap E'_{\sigma'}. \tag{5.3}$$

By the diagram (3.2), the $\sigma'_{\mathbb{C}}$-action defines an open inclusion map

$$U'_1 \times S' \hookrightarrow U'_1 \cdot S' \subset E'_{\sigma'}$$

where $U'_1$ is a small neighborhood of 0 in $\sigma'_{\mathbb{C}}$. By Lemma 3.2, above inclusion map induces

$$\sigma'_{\mathbb{C}} \times S' \hookrightarrow E'_{\sigma'}.$$

Then $p'_1(S')$ (resp. $p'_2 \circ p'_1(S')$) is a open set of $\Gamma'(\sigma')^{\mathrm{gp}}\backslash D'_{\sigma'}$ (resp. $\Gamma'\backslash D'_{\Sigma'}$).

The condition (5.1) and (5.3) induce

$$((p_1)^{-1} \circ \phi(s)) \cap \mathrm{Gr}^W_{-1}(S') = (s, \hat{\phi}(s)). \tag{5.4}$$

Let $\hat{\nu} : U_1 \times U_2 \to \check{D}'$ be the local section which gives the trivialization (5.2). The diagram (3.5), Proposition 3.5 and (5.4) induce

$$(\Delta \times S') \cap K_{\sigma'} = \left\{ (s,(s,\exp(X_v)\hat{\nu}(s))) \;\middle|\; \begin{array}{l} v \in \mathrm{Ker}(N) \cap U_3 \text{ if } s = 0 \\ v \in U_3 \text{ if } s \neq 0 \end{array} \right\}. \tag{5.5}$$

Put $S := W \cap (\Delta \times U_3)$ where $W$ is in (2.4) and $S$ is endowed with the strong topology in $W \cap (\Delta \times U_3)$. Then $S$ is homeomorphic to (5.5). Take a lifting $\hat{\nu}_{(\sigma',Z)}$ as in (4.6). Replacing $\hat{\nu}$ in (5.5) with $\hat{\nu}_{(\sigma',Z)}$, we obtain a neighborhood of $(0, F_{(\sigma',Z)})$ with respect to $\hat{\nu}_{(\sigma',Z)}$. We denote this neighborhood by $S(\hat{\nu}_{(\sigma',Z)})$. $p_1(S(\hat{\nu}_{(\sigma',Z)}))$ (resp. $p_2 \circ p_1(S(\hat{\nu}_{(\sigma',Z)}))$) is a open set of $J_{\sigma'}$ (resp. $J^{\mathrm{KNU}}$) and

$$\beta \circ p_2 \circ p_1(S(\hat{\nu}_{(\sigma',Z)})) = S(\nu_{(\sigma',Z)})$$

where right hand side is a neighborhood of $((0,0),[\nu_{(\sigma',Z)}]_0)$ defined in (2.7). Then we have

THEOREM 5.1. *$J^{\mathrm{GGK}}$ is homeomorphic to $J^{\mathrm{KNU}}$.*

# Bibliography


[C] J. Carlson, *Extensions of mixed Hodge structures*, Journees de Geometrie Algebrique d'Angers, Juillet 1979, 107–127.

[G] P. Griffiths, *Periods of integrals on algebraic manifolds. I. Construction and properties of the modular varieties*, Amer. J. Math. **90** (1968) 568–626.

[GGK1] M. Green, P. Griffiths and M. Kerr, *Néron models and limits of Abel-Jacobi mappings*, to appear in Compositio Math.

[GGK2] M. Green, P. Griffiths and M. Kerr, *Néron models and boundary components for degenerations of Hodge structures of mirror quintic type*, in *Curves and Abelian Varieties* (V. Alexeev, Ed.), Contemp. Math **465** (2007), AMS, 71–145.

[KNU1] K. Kato, C. Nakayama, S. Usui, *Log intermediate Jacobians* , arXiv:0906.3376

[KNU2] K. Kato, C. Nakayama, S. Usui, *Moduli of log mixed Hodge structures*, arXiv:0910.4454

[KU] K. Kato and S. Usui, *Classifying space of degenerating polarized Hodge structures*, Annals of Mathematics Studies, **169**. Princeton University Press, Princeton, NJ, 2009..

[N] I. Nakamura, *Relative compactification of the Néron model and its application*, in *Complex Analysis and Algebraic Geometry*, 207–225, Iwanami Shoten, Tokyo, 1977.

[P] G. Pearlstein, *Degenerations of mixed Hodge structure*, Duke Math. J. **110** (2001), no. 2, 217–251.

[Sa] M. Saito, *Hausdorff property of the Neron models of Green, Griffiths and Kerr*, arXiv:0803.2771

[Sc] W. Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. **22** (1973), 211–319.

[SI] T. Shioda and H. Inose, *On singular K3 surfaces. Complex analysis and algebraic geometry*, Iwanami Shoten, Tokyo, (1977), 119–136.

[U] S. Usui, *Variation of mixed Hodge structure arising from family of logarithmic deformations. II. Classifying space*, Duke Math. J. **51** (1984), no. 4, 851–875.