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A UNIFIED APPROACH TO CONTROL SYSTEM SYNTHESIS VIA LINEAR MATRIX INEQUALITIES

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A UNIFIED APPROACH TO CONTROL SYSTEM SYNTHESIS VIA LINEAR MATRIX INEQUALITIES

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Abstract

Linear matrix inequalities (LMIs in short) provide a lot of algebraic conditions representing control specifications. This dissertation proposes a unified approach to output-feedback linear controller synthesis by using wider variety of LMI-conditions than those ever used before. In contrast to many previous methods to find a particular solution to some individual LMI-based synthesis problem, we define a class of LMIs and give a unified solution to all the synthesis problems for any LMI-condition in the class. The class contains almost all of conventional LMI-conditions for both continuous- and discrete-time systems, such as several root-clustering conditions, H_2-norm conditions, H_∞-norm conditions and positive-real conditions, and so on. Moreover, there are many new multi-objective LMI-conditions belonging to the class.

Though LMIs are convex inequalities if they describe properties of fixed systems, problems to find a controller that makes the closed-loop system satisfy an LMI-condition, which problems we call 'LMI-synthesis problems' here, are no longer convex. To solve such nonconvex problems for any LMIs in the class, we give the unified solution as follows: First, we show a new parametrization of stabilizing output-feedback controllers. The parameter set is finite-dimensional and convex, and therefore appropriate to convex optimization on it. Next, we show a procedure to derive from any LMI-condition in the class a new LMI on the parameter set. The solvability of the new LMI is equivalent to that of the original nonconvex LMI-synthesis problem. Further, all the controllers satisfying the original inequality are parametrized by the parameters satisfying the new LMI.

We show some advantages of the approach: From the theoretical point of view, all the LMI-synthesis problems in the class turn out to be a single optimization problem of different cost functions but of the same kind. This can contribute to further progress of theory with LMIs. Next, our unified solution completely parameterizes the freedom of satisfactory controllers through convex subset of the parameter set. Third, the class contains not only many conventional LMIs but also new multi-objective LMIs,

which have never been used for synthesis before. The unified solution reduces those LMI-synthesis problems to convex optimization problems. With those LMIs and the unified solution to them, more complex specifications such as H_2/H_∞/root-clustering is tractable in computer aided design (CAD) through convex optimization. This new CAD framework includes existing design methods and very flexible in selecting control specifications with the new LMI-conditions presented in this dissertation. Lastly, our solution always gives a full-order controller if the LMI-synthesis problem is solvable at all.

Next, as an application of the above results, this dissertation considers robust performance problems for plants with two types of structured uncertainties: norm-bounded uncertainties and polytopic uncertainties. We formulate robust performance problems using LMI-conditions belonging to the class defined above, and solve those synthesis problems, guaranteeing quadratic stability at the same time. The parametrization of this dissertation reduces those robust synthesis problems to optimization problems on the parameter set. From benefit of the unification of LMI-conditions above, a number of LMIs is applied also to robust synthesis problems. For both uncertainty types, we give algorithms to derive controllers.

Lastly, we give some LMI-based analysis and synthesis results for descriptor systems. In particular, *elimination of differentiating dynamics* is considered as well as stabilization of exponential modes and fulfillment of some specifications. First, we show generalized Lyapunov and Riccati inequalities with equivalent LMIs that give a generalized stability condition and an H_∞-norm condition, respectively. Next, we solve two particular synthesis problems using these inequalities: state-feedback quadratic stabilization and output-feedback H_∞-control.

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Notation

$\mathbf{Z}$	The set of integers.
$\mathbf{R}$	The set of real numbers.
$\mathbf{R}^n$	The set of real vectors with n components.
$\mathbf{R}^{m\times n}$	The set of real $m \times n$ matrices.
$\mathbf{C}$	The set of complex numbers.
$\mathrm{Re}\, c$	The real part of $c \in \mathbf{C}$.
$\mathrm{Im}\, c$	The imaginary part of $c \in \mathbf{C}$.
$\mathrm{trace}\, M$	The trace of a matrix M.
$\mathrm{rank}\, M$	The rank of a matrix M.
M^T	The transpose of a matrix M.
M^{-1}	The inverse of a matrix M.
M^{-T}	The inverse of the transpose of a matrix M. ($M^{-T} = (M^T)^{-1}$.)
$M^\dagger$	The psudoinverse matrix of M.
$M^\perp$	Let $M \in \mathbf{R}^{m\times n}$. If $m < n$, $M^\perp$ represents a matrix that constitutes the basis of the null space of M. If $m > n$, $M^\perp$ represents the transpose of a matrix that constitutes the basis of the null space of M^T.
$\sigma_{\max}(M)$	The largest singular value of a matrix M.
$\lambda_{\max}(M)$	The largest eigenvalue of a symmetric matrix M.

$\lambda_{\min}(M)$	The smallest eigenvalue of a symmetric matrix M.
$M > N$	For symmetric matrices M and N, the inequality $M > N$ represents that $M - N$ is positive definite, i.e., $\lambda_{\min}(M - N) > 0$.
$M < N$	means $N > M$.
$M \geq N$	For symmetric matrices M and N, the inequality $M \geq N$ represents that $M - N$ is positive semidefinite, i.e., $\lambda_{\min}(M - N) \geq 0$.
$M \leq N$	means $N \geq M$.
I_n	The $n \times n$ identity matrix. (Sometimes the subscript n is omitted.)
$0_{m \times n}$	The $m \times n$ zero matrix. (Sometimes the subscript $m \times n$ is omitted.)
$\oplus$	$A_1 \oplus \cdots \oplus A_n = \mathrm{diag}\{A_1, \ldots, A_n\} = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{bmatrix}$.
$PD(n)$	The set of $n \times n$ positive definite matrices.
$\mathbf{RH}_\infty$	The set of proper stable real rational transfer functions.
$\\|G\\|_\infty$	For a transfer function $G \in \mathbf{RH}_\infty$,

$$\|G\|_\infty := \begin{cases} \sup\limits_{0 \leq \omega \leq \infty} \sigma_{\max}(G(j\omega)) & \text{for continuous-time systems,} \\ \sup\limits_{0 \leq \theta < 2\pi} \sigma_{\max}(G(e^{j\theta})) & \text{for discrete-time systems.} \end{cases}$$

$\|G\|_2$ For a transfer function $G \in \mathbf{RH}_\infty$,

$$\|G\|_2 := \begin{cases} \left[\int_0^\infty \mathrm{trace} G^T(-j\omega)G(j\omega)d\omega\right]^{\frac{1}{2}} & \text{for continuous-time systems,} \\ \left[\int_0^{2\pi} \mathrm{trace} G^T(e^{j\theta})G(e^{j\theta})d\theta\right]^{\frac{1}{2}} & \text{for discrete-time systems.} \end{cases}$$

For a function of $t \in \mathbf{R}$ whose value $G(t)$ belongs to $\mathbf{R}^{m \times n}$ and $G(t) \equiv 0, t < 0$, we define

$$\|G\|_2 := \left[\int_0^\infty \mathrm{trace} G^T(t)G(t)dt\right]^{\frac{1}{2}},$$

$G(s;K)$ For a symbol K representing a state-space realization $\{A,B,C,D\}$, we denote the following transfer function of K by $G(s;K)$:

$$G(s;K) = C(sI - A)^{-1}B + D.$$

$K \sim K'$ means that two state-space realizations K and K' have an identical transfer function.
($K \sim K'$ if and only if $G(s;K) = G(s;K')$.)

Contents

Chapter 1

Introduction

1.1 Overview of the dissertation

As progress of technology in all areas, higher performances are demanded for control systems. Thus, in control theory, a more realistic and useful theoretical framework for control system synthesis is necessary to answer more difficult design problems. On the other hand, recent advancement of computers makes it possible to implement complex controllers and to design controllers involving heavy calculations. This brings a new direction to control theory: *Solve more realistic but difficult problems via numerical optimization calculation within admissible time.* We can say that this is a straight extension of controller synthesis through Riccati solutions in LQ and H_∞-control, in the sense of generalizing Riccati equations to various algebraic expressions.

Numerical approaches can relax difficulty of synthesis problems, and we can carry them out with recent fast computers. However, straightforward application of existing numerical algorithms to controller synthesis problems is impossible or returns poor results, because almost all of performance indices in controller synthesis are usually ill functions from the numerical optimization viewpoint. To apply a numerical approach to a controller synthesis problem, the following is necessary:

(1) The control specifications arising in the problem are represented as performance indices or inequality conditions of some *finite-dimensional* variable.

(2) There exists a mapping that gives a controller explicitly from the variable in (1).

(3) There exists an algorithm that derives global optimum from the functions or inequalities in (1) within admissible time (i.e., polynomial-time).

In the item (3), *convex optimization* is guaranteed to converge to a global optimum, and there are many polynomial-time algorithms. Great deal of attention has been paid to approaches to reduce a synthesis problem to a convex optimization problem, recently. In particular, linear matrix inequalities (LMIs in short) provide a lot of algebraic conditions representing control specifications including stability conditions, H_2-norm conditions, H_∞-norm conditions, and so forth [OS94, BGFB94], and perhaps have most applications among controller design methods via convex optimization.

Let us see LMI-based approaches to controller synthesis in detail. Many of LMIs that represent a property of linear systems are inequalities of a positive definite matrix, say P, which gives an Lyapunov function of the system. (Some of LMI-conditions are given with additional variables.) The most standard form of LMI-conditions is:

$$\textit{A system } \Sigma \textit{satisfies a property } \mathcal{S} \Leftrightarrow \exists P > 0, \Phi_{\mathcal{S}}(P, \Sigma) > 0, \tag{1.1}$$

where $\Phi_{\mathcal{S}}$ is a symmetric-matrix-valued function linear (affine) with respect to P and determined by Σ. For a fixed system Σ, to find a solution P to $\Phi_{\mathcal{S}}(P, \Sigma) > 0$ is a convex optimization problem and the solution, if exists, is always found via globally convergent algorithms. Further, several efficient polynomial-time algorithms to solve LMIs have been proposed recently [BG93, NN94, VB94]. On the other hand, the problem to find a controller satisfying the LMI-condition (1.1) is stated as follows: *find a controller K such that the closed-loop system, say $\Sigma(K)$, satisfies $\Phi_{\mathcal{S}}(P, \Sigma(K)) > 0$ for some $P > 0$.* In such problems, which we call **LMI-synthesis problems**, matrix inequalities to solve are not affine with respect to (P, K), and therefore they are never treated via existing convex optimization. Though there are some approaches to solving such nonconvex inequalities directly [SGL94, GTSPL94, GSP94], the computational complexity of them is too large even for recent computers.

Several methods have been proposed for particular LMI-synthesis problems by reducing such a nonconvex inequality to a new LMI of a new variable. Any solution to the new LMI derives a controller satisfying the original inequality condition. There are several results of this type for output-feedback synthesis, solving H_∞-control problems [Gah92, Gah94, IS94, SMN90], H_2-control problems [Rot93], mixed H_2/H_∞-control problems [KR91]. In these results, though LMI-conditions representing control specifications have a lot of common features, solutions are just individually given and relations between any two of solutions are not clear. In other words, these results proposed different parameters and mappings of the above items (1) and (2). On the other hand, results of LMI-synthesis problems for state feedback systems [OK89, GPB91, OMS93b, GPS93a, OMS94, MOS94b] handle much more of LMI-conditions for design, with a single parameter space and mapping. Lastly, there is another previous numerical design method that uses the Youla parametrization [YJB76] and handles functionals of closed-loop transfer functions [BB91]. This approach, called transfer function approach, can employ any convex functionals of transfer functions, but the parameter set ($\mathbf{RH}_\infty$) is infinite-dimensional, and hence finite-dimensional approximations of $\mathbf{RH}_\infty$ are used for numerical optimization. This sometimes implies very high order controllers, difficult to implement even with recent computers.

The purpose of this dissertation is to propose a unified approach to output-feedback linear controller synthesis by using wider variety of LMI-conditions than those ever used. In contrast to previous methods to find a particular solution to some individual LMI-synthesis problem, we will give a unified solution to all the LMI-synthesis problems belonging to a certain class. We will define the class, say $\mathcal{L}$, focusing on a structure shared by many LMIs in control theory. The class $\mathcal{L}$ contains almost all of conventional LMIs for such conditions as several root-clustering conditions, H_2-norm conditions, H_∞-norm conditions and positive-real conditions, and so on. Though there are some previous approaches to controller synthesis for a class of LMIs [Iwa93, Sch95], their classes are included in the class $\mathcal{L}$ as far as full-order controller synthesis is concerned. Further, we propose new multi-objective synthesis methods, by showing that the class $\mathcal{L}$ contains new LMI-conditions representing multiple specifications. In mixed H_2/H_∞-control, the class provides an LMI-condition that does not need those assumptions required in previous approaches to mixed H_2/H_∞-control [BH89, KR91]. Moreover, the class contains more complex LMI-conditions, such as H_2/H_∞/root-clustering conditions, and only our unified solution derives a convex optimization method to solve those complex problems.

As mentioned above, though LMI itself is a convex inequality, an LMI-synthesis problem is not a convex problem. To solve such problems for LMIs in the class $\mathcal{L}$, we give the unified solution as follows. First, we show a new parametrization of stabilizing output-feedback controllers. The parameter set is finite-dimensional and convex, and therefore appropriate to convex optimization on it. Stabilizing controllers and stabilized closed-loop systems are represented explicitly in terms of parameters, say $\boldsymbol{p}$. There have never been parametrizations of output-feedback controllers that have such a parameter set. Next, we show a procedure to derive, from any LMI belonging to the class $\mathcal{L}$, an equivalent LMI-condition, say $\Phi^*(\boldsymbol{p}) > 0$, on the parameter set. All the controllers satisfying the original inequality are parametrized by the parameters $\boldsymbol{p}$ satisfying $\Phi^*(\boldsymbol{p}) > 0$. Thus we give the unified solution to LMI-synthesis problems as a procedure that reduces all the 'nonconvex' inequalities of the class $\mathcal{L}$ to 'convex' LMIs on the parameter set. Many of the results in this dissertation are derived for both continuous- and discrete-time systems.

We show some advantages of the approach as follows:

- From the theoretical point of view, all the LMI-synthesis problems in the class $\mathcal{L}$ turn out to be a single optimization problem of different cost functions but of the same kind; on the parameter set, H_∞-optimization and H_2-optimization are of the same family. Further, we treat continuous- and discrete-time systems simultaneously. These facts can contribute to further progress of theory with LMIs. In this dissertation, we apply the class of LMIs and the unified solution to another formulation of robust controller synthesis problems

- Unlike individual solutions to output feedback LMI-synthesis problems that give only particular solutions, our unified solution completely keeps the original inequality (1.1), i.e., as mentioned above, the set of the solutions satisfying $\Phi^*(\boldsymbol{p}) > 0$ parameterizes all the controllers that meet the specification described by the original LMI-condition.

- The class contains a lot of LMIs. In particular, some of LMIs for multiple specifications have never been used before and less restrictive than what ever used. Therefore the class of LMIs and the unified solution to them provides a theoretical basis for a new CAD framework that includes existing design methods and very flexible in selecting control specifications.

- In contrast to the transfer function approach, our solution always gives a controller at most of full-order if the LMI-synthesis problem is solvable. For the state-feedback systems, there always exists a static state-feedback solution, if solvable.

The key difference between our approach and previous approaches [SMN90, KR91, Gah92, Rot93, Iwa93, IS94, ISG94, Gah94] is that our parametrization completely transforms the variables (P, K) in the inequality $\Phi(P, \Sigma(K)) > 0$ to the new variable $\boldsymbol{p}$ in the inequality $\Phi^*(\boldsymbol{p}) > 0$, while previous approaches eliminate some parts of the freedom of the original variable (P, K) according to particular problems. Our keeping of the freedom brings vast applications of our approach to various LMIs.

Though we have not mentioned robust stability or robust performances so far, LMIs of the class $\mathcal{L}$ can describe robust performance specifications and hence we can handle robust controller synthesis with those LMIs. Furthermore, as we focus on robustness as an important issue in control theory, we will make a unified approach to robust performance problems for plants that have uncertainties in their coefficient matrices of a state space realization. Such uncertainties are called 'structured' uncertainties, and robust performance problems with structured uncertainties have been studied mainly based on so-called quadratic stability [Pet87]. LMIs have been often used to represent specifications in the area of robust controller synthesis based on quadratic stability.

In this dissertation, we will consider two types of structured uncertainties: norm-bounded uncertainties and polytopic uncertainties, and solve LMI-synthesis problems with satisfaction of quadratic stability in presence of those uncertainties. The parametrization of this dissertation reduces synthesis problems to optimization problems on the parameter set. From benefit of the unification of LMI-conditions above, a number of LMIs are applied to robust synthesis problems. Unfortunately, as almost all of previous results for output-feedback robust controller synthesis, matrix inequalities to solve do not completely 'get convex' with respect to all the variables. However, for norm-bounded uncertainties, the complexity does not exceed that of existing methods [YHF95] for norm-bounded uncertain systems, and we show that algorithms for those existing methods are applicable to the result of the dissertation that treats more kinds of LMIs. On the other hand, synthesis problems with polytopic uncertainties have not paid much attention though polytopes can describe uncertainties of plants more exactly than norm-bounded representations. We will show a matrix inequality condition on the parameter set equivalent to the original inequality condition, and propose an algorithm to solve inequalities on the parameter set.

Lastly, we will consider descriptor systems. The descriptor form represents linear systems with differentiating elements and static constraints between state variables, and so on. It is also suitable to describe uncertainties of dynamical systems. In our approach to robust performance problems above, uncertainties are represented in the descriptor form. Another point of this dissertation for descriptor systems is analysis and synthesis of properties of *singular* descriptor systems based on LMIs. In particular, elimination of differentiating dynamics is considered as well as stabilization of exponential modes and some specifications. Though our goal to apply the above unified approach is not attained, we will provide several new useful results for descriptor systems based on LMIs. First, we will show generalized Lyapunov and Riccati inequalities with equivalent LMIs that give a generalized stability condition and an H_∞-norm condition, respectively. Utilizing these LMIs, we solve two particular synthesis problems: state-feedback quadratic stabilization and output-feedback H_∞-control.

1.2 Contribution

In the following list, we summarize the new results of this dissertation.

(i) *A new parametrization of stabilizing output-feedback controllers* is proposed. (Chapter 2, Section 2.3.) The parameter set is finite-dimensional and convex, and therefore suitable for convex optimization on it. We use this parametrization in Chapter 3 and Chapter 4.

(ii) *A unified solution to a class of synthesis problems* is derived through the above parametrization. (Chapter 3, Sections 3.2,3.3.) This solution gives a unified formula to existing synthesis problems to find a controller that satisfies a certain LMI-condition, including H_∞-control, H_2-control, root-clustering, and so on. (Chapter 3, Section 3.4.)

(iii) *New LMI-conditions of multi-objective specifications* are applicable through the above solution to controller design. Though there are several present LMI-conditions used for multi-objective synthesis, we propose a new multi-objective synthesis framework that admits less restrictive problem formulations and wider variety of multiple control specifications. (Chapter 3, Sections 3.2, 3.5.)

(iv) *Robust performance problems are solved* for two types of structured uncertainties. For both types, we apply a larger class of LMI-conditions than those ever used to describe control specifications, and give algorithms to derive robust controllers satisfying such an LMI-condition. (Chapter 4.)

(v) *New results of analysis and synthesis for descriptor systems* are derived based on LMIs. We propose LMI-conditions for a generalized stability and an H_∞-norm condition. Applying these LMIs, we solve state-feedback quadratic stabilization and output-feedback H_∞-control problems. (Chapter 5.)

1.3 Organization

The organization of this dissertation is as follows: In Chapter 2, we give a new parametrization of stabilizing controllers, defining a parameter set and showing an explicit mapping from a parameter to the corresponding controller. In Section 3.2 of Chapter 3, we formulate the LMI-synthesis problem and define the class of LMIs that we handle in this chapter. Next, in Section 3.3, we give a unified solution to the LMI-synthesis problems belonging to the class. Section 3.4 is devoted to list elements of the class, and several subclasses are defined for the following chapters. While Section 3.4 shows only conventional LMI-conditions, in Section 3.5 we indicate new LMI-conditions representing robust multi-objective specifications, and discuss controller design using such LMI-conditions. In the end of this chapter, we show a numerical example of a multi-objective controller design using new LMI-conditions.

In Chapter 4, we formulate problems of robust LMI-synthesis with quadratic stability as a class of robust performance problems. In Section 4.3 and 4.4, we consider synthesis problems for plants that have norm-bounded and polytopic uncertainties, respectively. In both sections, the results are given as optimization problems on the parameter set, and we show an algorithm to solve each optimization problem. Section 4.5 provides a numerical example of quadratic stabilization of a plant with polytopic uncertainties.

Chapter 5 is concerned with descriptor systems. In Section 5.2, we give some definitions used in this chapter, such as a generalized stability. Section 5.3 shows several new algebraic inequalities that represent properties of descriptor systems. In Section 5.4, we solve some selected synthesis problems: robust stabilization of descriptor systems and H_∞-control problems for descriptor systems. We show a numerical example of H_∞-control for a descriptor system with a differentiating element.

All the proofs are stated in Appendix.

Chapter 2

A new parametrization of stabilizing controllers

2.1 Introduction

Controller design through convex optimization has been studied since the late 1980's and now it is paid a great deal of attention. The key to apply convex optimization to controller design is to find a parametrization satisfying items (1)$\sim$(3) in Section 1.1, which we repeat shortly: (1) performance indices described by some finite-dimensional variable, (2) an explicit mapping from the variable in (1) to a controller, and (3) a polynomial-time optimization algorithm for the indices in (1).

In this chapter, we propose a new parametrization of stabilizing output-feedback controllers. The parameter set related to controllers is finite-dimensional and convex, and hence suitable to convex optimization on it. In the following chapter, we show that any of synthesis problems described via LMIs belonging to a certain class is reduced to convex inequality on the parameter set. Unlike all previous parametrizations [YJB76, SIwa93, KKOS93], we parametrize all freedom of the stabilizing full-order controllers, or controllers of any higher order, using the finite-dimensional convex parameter set. Through the parametrization of this chapter, we handle a lot of control system synthesis problems with LMIs in Chapter 3 and Chapter 4.

2.2 Description of control systems

Let us consider the following linear time-invariant plant:

$$sx(t) = Ax(t) + Bu(t) + B_1 w_1(t), \tag{2.1a}$$

$$y(t) = Cx(t) + Du(t) + N_1 w_1(t), \tag{2.1b}$$

$$z_1(t) = C_1 x(t) + H_1 u(t) + D_1 w_1(t), \tag{2.1c}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^m$ is the control input vector, $y(t) \in \mathbf{R}^p$ is the measurement vector, $w_1(t) \in \mathbf{R}^{m_1}$ is the exogenous-input vector and $z_1(t) \in \mathbf{R}^{p_1}$ is the controlled-output vector. We represent by (2.1) both continuous-time and discrete-time systems by defining the variable t and the operator s as

$$\begin{cases} t \in \mathbf{R}, \quad sx(t) = \dot{x}(t) & \text{for continuous-time systems,} \\ t \in \mathbf{Z}, \quad sx(t) = x(t+1) & \text{for discrete-time systems,} \end{cases}$$

respectively. Next, we represent controllers by:

$$sx_c(t) = A_c x_c(t) + B_c y(t), \tag{2.2a}$$

$$u(t) = C_c x_c(t) + D_c y(t), \tag{2.2b}$$

where $x_c(t) \in \mathbf{R}^{n_c}$. We denote the above realization of a controller by $K = \{A_c, B_c, C_c, D_c\}$, and by $G(s; K)$ the transfer function of K:

$$G(s; K) = C_c(sI - A_c)^{-1} B_c + D_c.$$

We define by $\mathcal{K}(n_c)$ the set of state space realization of the form (2.2) with a n_c-th order state variable. If two realizations, say K and K', represent the same transfer function, i.e., if $G(s; K) \equiv G(s; K')$, we write $K \sim K'$. If K or K' is a nonminimal realization, $K \sim K'$ can hold even though the orders of the two realizations are different from each other.

Without loss of generality, we assume that $D = 0$. If this is not satisfied, define the following variable from y and u:

$$y'(t) := y(t) - Du(t) = Cx(t) + N_1 w_1(t),$$

and consider a controller whose input is y':

$$sx_c(t) = A'_c x_c(t) + B'_c y'(t), \tag{2.3a}$$

$$u(t) = C'_c x_c(t) + D'_c y'(t), \tag{2.3b}$$

From any controller (2.3), we get a controller (2.2) from y to u by

$$A_c = A_c' - B_c'(I + DD_c')^{-1}DC_c', \tag{2.4a}$$

$$B_c = B_c'(I + DD_c')^{-1}, \tag{2.4b}$$

$$C_c = (I + D_c'D)^{-1}C_c', \tag{2.4c}$$

$$D_c = (I + D_c'D)^{-1}D_c'. \tag{2.4d}$$

The invertibility of $I + DD_c'$ (or equivalently $I + D_c'D$) is confirmed later when we give a formula of controllers.

We denote a realization of the closed-loop system that consists of the plant (2.1) and a controller $K = \{A_c, B_c, C_c, D_c\}$ as follows:

$$sx_{cl}(t) = A_{cl}(K)x_{cl}(t) + B_{cl(1)}(K)w_1(t), \tag{2.5a}$$

$$z_1(t) = C_{cl(1)}(K)x_{cl}(t) + D_{cl(1)}(K)w_1(t), \tag{2.5b}$$

where $x_{cl}(t) = [x^T(t) \quad x_c^T(t)]^T$ and

$$A_{cl}(K) := \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}, \tag{2.6a}$$

$$B_{cl(1)}(K) := \begin{bmatrix} B_1 + BD_cN_1 \\ B_cN_1 \end{bmatrix}, \tag{2.6b}$$

$$C_{cl(1)}(K) := [\,C_1 + H_1D_cC \quad H_1C_c\,], \tag{2.6c}$$

$$D_{cl(1)}(K) := D_1 + H_1D_cN_1. \tag{2.6d}$$

2.3 Parameter set and mapping

In this section, we show a new parametrization of stabilizing output-feedback controllers. First, we define a parameter set to parametrize stabilizing controllers as follows:

Definition 2.1 Let $n_p \geq n$ and denote by $\mathcal{P}(n_p)$ the set of variables $\boldsymbol{p}$, where $\boldsymbol{p} = \{P_f, P_g, P_h, W_{fa}, W_{ga}, W_h, L_a\}$, $P_f \in PD(n)$, $P_g \in PD(n)$, $P_h \in PD(n_p - n)$, $W_{fa} \in \mathbf{R}^{m\times n_p}$, $W_{ga} \in \mathbf{R}^{n_p\times p}$, $W_h \in \mathbf{R}^{m\times p}$, $L_a \in \mathbf{R}^{n_p\times n_p}$, and P_f and P_g satisfy

$$\begin{bmatrix} P_f & I \\ I & P_g \end{bmatrix} > 0.$$

We also denote

$$L_a = \begin{bmatrix} L & L_{12} \\ L_{21} & L_{22} \end{bmatrix}, \; W_{fa} = [\,W_f \quad W_{f2}\,], \; W_{ga} = \begin{bmatrix} W_g \\ W_{g2} \end{bmatrix},$$

where $L \in \mathbf{R}^{n\times n}$, $W_f \in \mathbf{R}^{m\times n}$ and $W_g \in \mathbf{R}^{n\times p}$. □

The parameter set $\mathcal{P}(n_p)$ is an open convex subset of $\mathbf{R}^{N_p}$, where $N_p := n(n+1) + \frac{1}{2}(n_p - n)(n_p - n + 1) + (n_p + m)(n_p + p)$.

Next, to present parametrizations of controllers for continuous-time and discrete-time systems simultaneously, we define the following function:

$$\Phi_{\rm Lyap}(X, Y) := \begin{cases} \begin{bmatrix} X & 0 \\ 0 & -Y - Y^T \end{bmatrix} & \text{for continuous-time systems,} \\ \begin{bmatrix} X & Y^T \\ Y & X \end{bmatrix} & \text{for discrete-time systems.} \end{cases}$$

Then the LMI $\Phi_{\rm Lyap}(P, PM) > 0$ is equivalent to the Lyapunov inequalities:

$$\begin{cases} P > 0, \ PM + M^T P < 0, & \text{for continuous-time systems,} \\ P > 0, \ M^T PM - P < 0, & \text{for discrete-time systems.} \end{cases}$$

In the following proposition, we give the mapping from the parameter set to the set of stabilizing controllers of order $n_c \geq n$. We show an explicit formula of a closed-loop realization in terms of parameters.

Proposition 2.1 Define the following matrix-valued affine functions on $\mathcal{P}(n_p)$:

$$M_P(\boldsymbol{p}) := \begin{bmatrix} P_f & I_n & 0 \\ I_n & P_g & 0 \\ 0 & 0 & P_h \end{bmatrix}, \tag{2.7a}$$

$$M_A(\boldsymbol{p}) := \begin{bmatrix} AP_f + BW_f & A + BW_hC & BW_{f2} \\ L & P_gA + W_gC & L_{12} \\ L_{21} & W_{g2}C & L_{22} \end{bmatrix}, \tag{2.7b}$$

$$M_{B1}(\boldsymbol{p}) := \begin{bmatrix} B_1 + BW_hN_1 \\ P_gB_1 + W_gN_1 \\ W_{g2}N_1 \end{bmatrix}, \tag{2.7c}$$

$$M_{C1}(\boldsymbol{p}) := \begin{bmatrix} C_1P_f + H_1W_f & C_1 + H_1W_hC & H_1W_{f2} \end{bmatrix}, \tag{2.7d}$$

$$M_{D1}(\boldsymbol{p}) := D_1 + H_1W_hN_1. \tag{2.7e}$$

(1) For any integer $n_p \geq n$, the following two statements are equivalent:

(I) There exists a stabilizing controller of order n_p for the plant (2.1).

(II) There exists a parameter $\boldsymbol{p} \in \mathcal{P}(n_p)$ satisfying the following LMI:

$$\Phi_{\rm Lyap}(M_P(\boldsymbol{p}), M_A(\boldsymbol{p})) > 0. \tag{2.8}$$

(2) Let $n_p \geq n$ and define a mapping that maps a parameter $\boldsymbol{p} \in \mathcal{P}(n_p)$ satisfying (2.8) to a state-space realization as follows:

$$\mathrm{K_{map}}(\boldsymbol{p}) := \{\mathrm{A_c}(\boldsymbol{p}), \mathrm{B_c}(\boldsymbol{p}), \mathrm{C_c}(\boldsymbol{p}), \mathrm{D_c}(\boldsymbol{p})\}, \tag{2.9}$$

$$\left[\begin{array}{c|c} \mathrm{D_c} & \mathrm{C_c} \\ \hline \mathrm{B_c} & \mathrm{A_c} \end{array}\right] := \left[\begin{array}{c|cc} I_n & 0 & 0 \\ \hline B & -P_g^{-1} & 0 \\ 0 & 0 & I_{n_p-n} \end{array}\right] \left[\begin{array}{c|cc} W_h & W_f & W_{f2} \\ \hline W_g & (L - P_g A P_f) & L_{12} \\ W_{g2} & L_{21} & L_{22} \end{array}\right] \left[\begin{array}{c|cc} I_n & -CP_f S^{-1} & 0 \\ \hline 0 & S^{-1} & 0 \\ 0 & 0 & P_h^{-1} \end{array}\right],$$

$$S := P_f - P_g^{-1} (> 0). \tag{2.10}$$

Then, for any parameter $\boldsymbol{p} \in \mathcal{P}(n_p)$ satisfying (2.8), the controller $\mathrm{K_{map}}(\boldsymbol{p})$ stabilizes the plant (2.1). Conversely, for any stabilizing controller K of order n_p there exists a parameter $\boldsymbol{p} \in \mathcal{P}(n_p)$ such that $K \sim \mathrm{K_{map}}(\boldsymbol{p})$.

(3) Let $\boldsymbol{p} \in \mathcal{P}(n_p)$ satisfy (2.8) and $K = \mathrm{K_{map}}(\boldsymbol{p})$ in (2.6). Define the following matrices from $\boldsymbol{p}$:

$$\begin{aligned} P_1 &:= \begin{bmatrix} P_f & S & 0 \\ S & S & 0 \\ 0 & 0 & P_h \end{bmatrix}, \quad U_1 := \begin{bmatrix} I_n & 0 & 0 \\ P_g & -P_g & 0 \\ 0 & 0 & I_{n_p-n} \end{bmatrix}, \\ P_2 &:= P_1^{-1}, \qquad\qquad U_2 := P_1 U_1^T = \begin{bmatrix} P_f & I_n & 0 \\ S & 0 & 0 \\ 0 & 0 & P_h \end{bmatrix}. \end{aligned} \tag{2.11}$$

Then we have

$$M_P(\boldsymbol{p}) = U_1 P_1 U_1^T = U_2^T P_2 U_2, \tag{2.12a}$$

$$\begin{aligned} \begin{bmatrix} M_A(\boldsymbol{p}) & M_{B1}(\boldsymbol{p}) \\ M_{C1}(\boldsymbol{p}) & M_{D1}(\boldsymbol{p}) \end{bmatrix} &= \begin{bmatrix} U_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{cl}(K) P_1 & B_{cl(1)}(K) \\ C_{cl(1)}(K) P_1 & D_{cl(1)}(K) \end{bmatrix} \begin{bmatrix} U_1^T & 0 \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} U_2^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P_2 A_{cl}(K) & P_2 B_{cl(1)}(K) \\ C_{cl(1)}(K) & D_{cl(1)}(K) \end{bmatrix} \begin{bmatrix} U_2 & 0 \\ 0 & I \end{bmatrix}. \end{aligned} \tag{2.12b}$$

These equalities imply the following realization of the closed-loop system:

$$M_P(\boldsymbol{p}) s\tilde{x}(t) = M_A(\boldsymbol{p})\tilde{x}(t) + M_{B1}(\boldsymbol{p}) w_1(t), \tag{2.13a}$$

$$z_1(t) = M_{C1}(\boldsymbol{p})\tilde{x}(t) + M_{D1}(\boldsymbol{p}) w_1(t), \tag{2.13b}$$

where $\tilde{x}(t) = U_2^{-1} x_{cl}(t)$.

Proof. See Appendix. □

Remark 2.1 The above formula of $\mathrm{K}_{\mathrm{map}}$ in (2.10) itself does not guarantee the regularity of $I + DD_c = I + DW_h$, which condition is necessary for the case $D \neq 0$. However, if $\det(I+DW_h) = 0$, we get a perturbed solution of the form cW_h by setting a scalar c such that c^{-1} is not a eigenvalue of $-DW_h$ and that $|c-1|$ is small enough for cW_h to satisfy $\Phi_{\mathrm{Lyap}}(M_P(\boldsymbol{p}), M_A(\boldsymbol{p})) > 0$ instead of W_h. □

The parameter set $\{\boldsymbol{p} \in \mathcal{P}(n_p) | \Phi_{\mathrm{Lyap}}(M_P(\boldsymbol{p}), M_A(\boldsymbol{p})) > 0\}$ is convex and finite-dimensional, and parametrizes all stabilizing controllers of order less than or equal to n_p. None of the previous parametrizations of output-feedback stabilizing controllers [YJB76, SIwa93, Iwa93] has a convex and finite-dimensional parameter set.

The equalities (2.12) suggest that LMIs for closed-loop systems are tractable through the parametrization if they contain the closed-loop realization and P only in the following form:

$$PA_{cl}(K),\ PB_{cl(1)}(K),\ C_{cl(1)}(K), D_{cl(1)}(K)$$

or

$$A_{cl}(K)P,\ B_{cl(1)}(K),\ C_{cl(1)}(K)P,\ D_{cl(1)}(K).$$

In the following chapter, we show that this is in fact true; we define a class of LMIs that have such terms and solve synthesis problems with it. Further, we show that a lot of control specifications are described through such LMI-conditions.

2.4 Issues related to the parametrization

In this section, we discuss some issues related to the above parametrization.

From the LMI $\Phi_{\mathrm{Lyap}}(M_P(\boldsymbol{p}), M_A(\boldsymbol{p})) > 0$, we pick up the following sub-blocks:

$$\Phi_{\mathrm{Lyap}}(P_f, AP_f + BW_f) > 0, \tag{2.14a}$$

$$\Phi_{\mathrm{Lyap}}(P_g, P_gA + W_gC) > 0. \tag{2.14b}$$

These LMIs show that, for both continuous- and discrete-time systems, $W_fP_f^{-1}$ is a stabilizing feedback gain for the pair (A, B) and $P_g^{-1}W_g$ is an observer gain for the pair (A, C). Thus this parametrization is related to the parametrization of stabilizing state-feedback gains [KR91] and observer gains.

We show more about the relation: Consider the following state-feedback system:

$$sx(t) = Ax(t) + Bu(t) + B_1w_1(t), \tag{2.15a}$$

$$y(t) = x(t), \tag{2.15b}$$

$$z_1(t) = C_1x(t) + H_1u(t) + D_1w_1(t). \tag{2.15c}$$

All the stabilizing state-feedback gains F is parametrized as follows [KR91]:

$$F = WP^{-1},\ P = P^T,\ \Phi_{\text{Lyap}}(P, AP + BW) > 0. \tag{2.16}$$

If F is represented as in (2.16), stabilized closed-loop systems have the following realization:

$$Ps\tilde{x}(t) = (AP + BW)\tilde{x}(t) + B_1w_1(t), \tag{2.17a}$$

$$z_1(t) = (C_1P + H_1W)\tilde{x}(t) + D_1w_1(t), \tag{2.17b}$$

where $\tilde{x}(t) = P^{-1}x(t)$. Further, also stable matrices themselves are parametrized through LMI-conditions [OK93]:

$$A = ZP^{-1},\ P = P^T,\ \Phi_{\text{Lyap}}(P, Z) > 0. \tag{2.18}$$

Thus stable matrices themselves, stable matrices attained via state-feedback, and stable matrices attained via output-feedback have the same structure:

$$Z(p)X^{-1}(p),\ X(p) = X^T(p),\ \Phi_{\text{Lyap}}(X(p), Z(p)) > 0, \tag{2.19}$$

where $Z(p)$ and $X(p)$ are appropriate matrix-valued affine functions of p.

Ohara et al. investigated differential geometric structure of stable matrices and stable state-feedback systems [OK93, OA94], based on a one-to-one parametrization of stable matrices or stabilizing feedback gains. In those one-to-one parametrizations, stable matrices attained via stable state-feedback control have the structure of (2.19). However, a one-to-one parametrization of output-feedback controllers is still an open problem. Note that one-to-one parametrization of stabilizing state-feedback gains can be conservative to represent control specifications, however.

Lastly, we discuss lower order controller issues. From recent advancement in implementation technique for control systems such as DSP, using high-order controllers has become much less difficult. But lower-order controllers are needed even today and, if performance does not deteriorate significantly with a constraint on the order, lower-order controller is usually desirable. Though there are several results on lower controller synthesis problems [Gah92, Gah94, GPS93b, Iwa93, IS94, ISG94], none of them give a synthesis procedure that always finds an optimal lower-order controller if exists. A recent result of Sugie et al. [SugT94] shows the condition under which a parameter of our parametrization gives a lower-order controller. (Note that this result needs a little modification of the parameter set [SugT94].) However, they do not give a computational method to derive a parameter that meets the lower-order condition, except for a method reducing their parametrization to those results of [Iwa93], etc.

2.5 Concluding remarks

In this chapter, we showed a new parametrization of output-feedback controllers. The parametrization has a finite-dimensional convex parameter set. In the following chapter, a lot of LMI-synthesis problems, which are not convex problem, are reduced to convex optimization problems on this particular parameter set.

Chapter 3

Unified solution to LMI-synthesis problems

LMIs are useful tools providing algebraic conditions for many control specifications [OS94, BGFB94]. In this chapter, applying the parametrization of the previous chapter, we give a unified solution to LMI-synthesis problems belonging to a certain class, which we define here focusing on structures that many of LMIs in control theory shares. The class contains almost all of conventional LMI-conditions that have been used for controller synthesis, and new LMI-conditions for multiple specifications such as H_2/H_∞-control, H_2/H_∞/root-clustering, and so on. The unified solution provides a procedure to find a controller that satisfies any one of the LMI-conditions belonging to the class.

3.1 Introduction

First, we review previous controller design approaches via convex optimization from the viewpoint of parameter spaces where optimization calculations are carried out.

For state-feedback synthesis problems, the following parametrization of static state-feedback gains [KR91] and equivalent parametrizations [GPB91, OMS93b, OMS94] have been used: for a stabilizable pair (A, B), every stabilizing gain F is given by

$$F = W_f P_f^{-1},\ P_f = P_f^T > 0,\ (AP_f + BW_f) + (AP_f + BW_f)^T < 0. \tag{3.1}$$

In this parametrization, parameter (P_f, W_f) belongs to a finite-dimensional convex set defined by the above LMIs. The mapping $W_f P_f^{-1}$ is surjective to the set of all stabilizing gains. A lot of performance indices are convex on the parameter set, and they are applied to robust controller design via convex optimization [OK89, GPB91, OMS93b, OMS94, MOS94b].

This parametrization is also applied to some output-feedback problems such as mixed H_2/H_∞-control [KR91], H_2-control [Rot93]. In these applications for output-feedback synthesis, only limited freedom of full-order controllers is parametrized via (3.1) and only each of the specifications gets a convex index on the parameter set. Kadoya et al. [KKOS93] proposed a design method of observer controllers. They parametrized estimated-state-feedback gain F as in (3.1) and observer gain G in the following dual form of (3.1):

$$G = P_g^{-1}W_g,\ P_g = P_g^T > 0,\ (P_gA + W_gC) + (P_gA + W_gC)^T < 0, \tag{3.2}$$

where (A, C) is assumed detectable. However, this method does not parameterize all freedom of full-order controllers, and no convex performance indices have been derived through this parametrization of observer controllers.

On the other hand, from context of parametrization of covariance controllers [SIke89], Skelton et al. [SIwa93] derived a parametrization of fixed-order controllers with a nonconvex parameter set. Iwasaki et al. [Iwa93, IS94, ISG94] modified and utilized this parametrization to solve H_∞-control, H_2-control, mixed H_2/H_∞-control problems. These results propose a nonconvex algorithm that can derive lower order controllers, but the algorithm does not enjoy global convergence. To derive full-order controllers, their algorithm is solved via convex optimization, but then these results are reduced to previous results of controller synthesis based on LMIs.

The above parametrizations aim to solve synthesis problems to find a controller that satisfies a certain LMI-condition. On the other hand, Boyd et al. [BB91] proposed a numerical approach using the Youla parametrization [YJB76]. The merit of this method is that it can employ any convex function of closed-loop transfer functions. However, since parameters of the Youla parametrization belong to $\mathbf{RH}_\infty$, a linear but infinite-dimensional space, numerical optimization is only carried out on a finite approximation of $\mathbf{RH}_\infty$. Further, to get a good approximation, the number of the free parameter is often very large. This causes high-order controllers, which is difficult to implement even with recent fast computers [Nob95].

In this chapter, applying the parametrization of the previous chapter, we propose a unified approach to LMI-synthesis. In contrast to many previous methods in output-feedback synthesis to find a particular solution to some individual synthesis problem, we define a class of LMIs, which will turn out to contain both conventional and new LMIs, and give a unified solution to all LMI-synthesis problems belonging to the class. The class contains LMIs for such conditions as several root-clustering conditions, H_2-norm conditions, H_∞-norm conditions and positive-real conditions, and so on. Hence the proposed solution shows that these synthesis problems are equivalent to optimization problems of a performance index of the same family. Further, the class always contains a multi-objective LMI-condition that guarantees any combination of conditions described by LMIs in the class. In particular, some of LMIs for multiple specifications have never been used before and less restrictive than those ever used. With all those LMIs in the class and the unified solution to them, we establish a theoretical basis for a new CAD framework that includes existing design methods and very flexible in selecting control specifications.

We give the results as follows: Section 3.2 formulates LMI-synthesis problems and defines the class of LMIs. In Section 3.3, we give a unified solution to the LMI-synthesis problem, showing a procedure that reduces nonconvex matrix inequalities to equivalent convex LMIs on the parameter set. Section 3.4 is devoted to indicate that the class includes many of conventional LMIs, while in Section 3.5 we present some of new LMI-conditions and discuss robust multi-objective controller design with those LMIs. In Section 3.6, we give an illustrative numerical example using multi-objective LMI-condition for controller design.

3.2 Problem formulation

In this section, we give the exact definition of the LMI-synthesis problem and define a class of LMIs with which we consider synthesis problems.

Let us consider the control system shown in Fig.3.1. The plant has N_s exogenous-input and controlled-output channels and we represent the plant as follows:

$$sx(t) = Ax(t) + Bu(t) + \sum_{i=1}^{N_s} B_i w_i(t), \tag{3.3a}$$

$$y(t) = Cx(t) + Du(t) + \sum_{i=1}^{N_s} N_i w_i(t), \tag{3.3b}$$

$$z_i(t) = C_i x(t) + H_i u(t) + D_i w_i(t),\ i = 1, 2, \ldots, N_s, \tag{3.3c}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^m$ is the control input vector, $y(t) \in \mathbf{R}^p$ is the measurement vector, $w_i(t) \in \mathbf{R}^{m_i}, i = 1, 2, \ldots, N_s$ are the exogenous-input

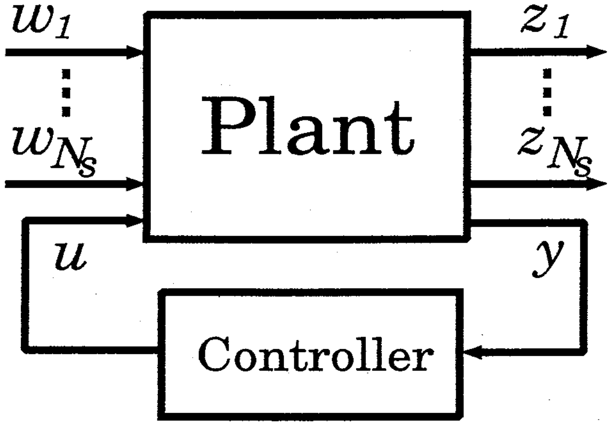

Fig. 3.1 Control system

vectors and $z_i(t) \in \mathbf{R}^{p_i}, i = 1, 2, \ldots, N_s$ are the controlled-output vectors. We assume that $D = 0$ without loss of generality (for a plant with $D \neq 0$, apply (2.4) again). We describe a controller as in (2.2), and in the closed-loop system we consider exogenous-input and controlled-output pairs only from w_i to z_i; we rule out pairs of w_i to z_j for $i \neq j$. We denote a realization of the closed-loop system as follows:

$$sx_{cl}(t) = A_{cl}(K)x_{cl}(t) + B_{cl(i)}(K)w_i(t),\ i = 1, 2, \ldots, N_s, \tag{3.4a}$$

$$z_i(t) = C_{cl(i)}(K)x_{cl}(t) + D_{cl(i)}(K)w_i(t),\ i = 1, 2, \ldots, N_s, \tag{3.4b}$$

where $x_{cl}(t) = [x^T(t)\ \ x_c^T(t)]^T$ and

$$A_{cl}(K) := \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}, \tag{3.5a}$$

$$B_{cl(i)}(K) := \begin{bmatrix} B_i + BD_cN_i \\ B_cN_i \end{bmatrix}, \tag{3.5b}$$

$$C_{cl(i)}(K) := [\,C_i + H_iD_cC \quad H_iC_c\,], \tag{3.5c}$$

$$D_{cl(i)}(K) := D_i + H_iD_cN_i. \tag{3.5d}$$

We denote by $\Sigma_{cl}(K) = \{A_{cl}(K), B_{cl(i)}(K), C_{cl(i)}(K), D_{cl(i)}(K); i = 1, 2, \ldots, N_s\}$ the above realization (3.5), and by $G_{cl(i)}(s;K)$ the closed-loop transfer function from w_i to z_i:

$$G_{cl(i)}(s;K) := C_{cl(i)}(K)(sI - A_{cl}(K))^{-1}B_{cl(i)}(K) + D_{cl(i)}(K). \tag{3.6}$$

Next, we will define the LMI-synthesis problem. Though we showed control-system description above, to define the class of LMIs $\mathcal{L}$ without depending on the structure of systems, we consider the following system that has the same exogenous-input and controlled-output pairs as the plant (3.3):

$$s\bar{x}(t) = \bar{A}\bar{x}(t) + \bar{B}_i w_i(t),\ i = 1, 2, \ldots, N_s, \tag{3.7a}$$

$$z_i(t) = \bar{C}_i\bar{x}(t) + \bar{D}_i w_i(t),\ i = 1, 2, \ldots, N_s, \tag{3.7b}$$

where $\bar{x}(t) \in \mathbf{R}^{n_s}$. Let $\Sigma = \{\bar{A},\ \bar{B}_i,\ \bar{C}_i,\ \bar{D}_i; i = 1, 2, \ldots, N_s\}$ represent the above realization, and denote by $\mathbf{\Sigma}$ all such realizations. We represent the transfer function for the i-th pair by

$$G_{s(i)}(s) := \bar{C}_i(sI - \bar{A})^{-1}\bar{B}_i + \bar{D}_i w_i(t).$$

Consider a symmetric-matrix-valued function $\Phi(P, \Sigma)$ of variables $P \in PD(n_s)$ and $\Sigma \in \mathbf{\Sigma}$. The variable P is to give a Lyapunov function of the form $\bar{x}^T P \bar{x}$ for the system Σ. We consider properties of system Σ represented in the following manner:

System Σ satisfies property $\mathcal{S}$
if (and only if) LMI $\Phi(P, \Sigma) > 0$ holds for some $P \in PD(n_s)$.

Sometimes a property of Σ is described with some additional variables, say V, such as scaling parameters. We denote Φ's for such parametric properties by $\Phi(P, \Sigma; V)$, and assume that V is independent of P and Σ.

Example 3.1 The H_∞-norm condition $\|G_{s(i)}\| < \gamma$ is represented in this form by defining $\Phi = \Phi_{H_\infty(i)}$ as follows:

$$\Phi_{H_\infty(i)}(P, \Sigma; \gamma) := P \oplus \begin{bmatrix} -P\bar{A} - \bar{A}^T P & P\bar{B}_i & \bar{C}_i^T \\ \bar{B}_i^T P & \gamma I_{m_i} & -\bar{D}_i^T \\ \bar{C}_i & -\bar{D}_i & \gamma I_{p_i} \end{bmatrix}. \tag{3.8}$$

□

Let us remind the symbol of the closed-loop system (3.5) above. Substituting $\Sigma_{cl}(K)$ into Σ, we get a matrix inequality condition $\Phi(P, \Sigma_{cl}(K)) > 0$. In the above example, to minimize γ subject to $\Phi_{H_\infty(i)}(P, \Sigma_{cl}(K); \gamma)$ is the H_∞ optimization problem for the plant (3.3). In the following, we give the exact formulation of LMI-synthesis problems:

LMI-synthesis problem:

- *Feasibility problem.* Given $\Phi(\bullet)$, find $P \in PD(n+n_c)$ and a controller K of some order n_c satisfying $\Phi(P, \Sigma_{cl}(K)) > 0$.

- *Minimization problem.* Let $\Phi(\bullet)$ be parametric to a scalar γ, and assume that $\Phi(P, \Sigma; \gamma_1) > \Phi(P, \Sigma; \gamma_2)$ holds for any $\gamma_1 > \gamma_2$, $P \in PD(n_s)$ and $\Sigma \in \mathbf{\Sigma}$. Minimize γ subject to $\Phi(P, \Sigma_{cl}(K); \gamma) > 0$ for some $P \in PD(n+n_c)$ and a controller K of some order n_c.

If K is given and fixed, solving $\Phi(P, \Sigma_{cl}(K)) > 0$ is a convex optimization problem if Φ is affine with respect to P, and solved through efficient globally convergent polynomial-time algorithms [BG93, NN94, VB94]. However, in LMI-synthesis problems the variables to find is all (P, K) and inequalities are never convex, which we see in the following definition of the class of Φ:

Definition 3.1 Denote by $\mathcal{L}$ the set of symmetric-matrix-valued functions $\Phi(P, \Sigma)$, defined for $(P, \Sigma) \in PD(n_s) \times \mathbf{\Sigma}$ with the following property: there exists a positive integer N_b such that, for any $n_s \geq 0$, Φ has the following partition

$$\Phi(\bullet) = \begin{bmatrix} \Phi_{11}(\bullet) & \cdots & \Phi_{1N_b}(\bullet) \\ \vdots & \ddots & \vdots \\ \Phi_{N_b1}(\bullet) & \cdots & \Phi_{N_bN_b}(\bullet) \end{bmatrix}$$

and each block $\Phi_{kl}, k, l = 1, 2, \ldots, N_b$ has the following form:

- *Diagonal blocks*: Two types of blocks are admitted into diagonal entries. One is represented by

$$\Phi_{kk}(P, \Sigma) := c_{kk}^{(1)} P + c_{kk}^{(2)}(P\bar{A} + \bar{A}^T P), \tag{3.9}$$

  and we call such Φ_{kk} a *state* block. The other, which we call *anti-state* block, is constant (and we write it as $\Phi_{kk}(P, \Sigma) \equiv \Phi_{kk}^{(1)}$ for later convenience.)

- *Off-diagonal blocks* ($k > l$):

$$\Phi_{kl}(P,\Sigma) = \begin{cases} c_{kl}^{(1)} P + c_{kl}^{(2)} P\bar{A} + c_{kl}^{(3)} \bar{A}^T P \\ \qquad \text{if both } \Phi_{kk} \text{ and } \Phi_{ll} \text{ are } \textit{state} \text{ blocks,} \\ P\bar{B}_i R_{Bi} + \bar{C}_i^T R_{Ci}^T \\ \qquad \text{if } \Phi_{kk} \textit{ state} \text{ and } \Phi_{ll} \textit{ anti-state,} \\ R_{Bi}^T \bar{B}_i^T P + R_{Ci} \bar{C}_i \\ \qquad \text{if } \Phi_{kk} \textit{ anti-state} \text{ and } \Phi_{ll} \textit{ state,} \\ \Phi_{kl}^{(1)} + \Phi_{kl}^{(2)} \bar{D}_i \Phi_{kl}^{(3)}, \\ \qquad \text{if both } \Phi_{kk} \text{ and } \Phi_{ll} \textit{ anti-state,} \end{cases}$$

where $c_{kl}^{(i)}$'s are scalar and R_{Bi}'s, R_{Ci}'s and $\Phi_{kl}^{(i)}$'s are matrices with appropriate sizes. Lastly, we assume that the inequality $\Phi(P, \Sigma) > 0$ implies $P > 0$. □

In this definition, the terms that depend on P and/or Σ appear only in the form of P, $P\bar{A}$, $P\bar{B}_i$, $\bar{C}_i$ and $\bar{D}_i$ or some linear combinations of them. If Φ depends on some additional variable V, some of the constants in Φ in the above definition is replaced with a function of V, such as $c_{kl}^{(i)}(V)$, $R_{Bi}(V)$, $R_{Ci}(V)$ and $\Phi_{kl}^{(i)}(V)$.

From the above definition, finding a solution (P, K) to an inequality $\Phi(P, \Sigma_{cl}(K)) > 0$, $\Phi \in \mathcal{L}$ is so-called bilinear (biaffine) matrix inequality (BMI), which is never solved via convex optimization. Throughout this dissertation, however, we call such an inequality condition 'LMI-condition' because it is not just any BMI-condition but originally an LMI-condition that decides whether a fixed system satisfies a certain property.

The function $\Phi_{H_\infty(i)}$ defined above in (3.8) is obviously an element of $\mathcal{L}$. We will give more examples of $\mathcal{L}$ in Section 3.4 and 3.5. One can see that from Section 3.4 a lot of previous LMI-conditions are included in the formulation of this dissertation, and Section 3.5 shows new LMI-conditions for multi-objective controller synthesis in the class $\mathcal{L}$.

In the rest of this section, we show some properties that LMIs of $\mathcal{L}$ commonly have. First, for any $\Phi_1, \Phi_2 \in \mathcal{L}$, the direct sum $\Phi_1 \oplus \Phi_2$ always belongs to $\mathcal{L}$, which fact is immediately checked from the definition of the class. This property derives new LMI-conditions for multiple specifications. In Section 3.5, we will discuss multiobjective controller design that uses such LMIs.

Next, any condition described by an inequality $\Phi(P,\Sigma) > 0$ is independent of arbitrariness of state-space realization:

Lemma 3.1 Let T be a matrix with n_s rows and define $\Xi(T)$ according to $\Phi \in \mathcal{L}$ as follows:

$$\Xi(T) = \Xi_1(T) \oplus \Xi_2(T) \oplus \cdots \oplus \Xi_{Nb}(T), \tag{3.10a}$$

$$\Xi_k(T) = \begin{cases} T & \text{if } \Phi_{kk} \text{ is a state block,} \\ I & \text{if } \Phi_{kk} \text{ is an anti-state block.} \end{cases} \tag{3.10b}$$

Then, if T is square and nonsingular, we have

$$\begin{aligned} &\Xi(T^T)\Phi(P, \{\bar{A}, \bar{B}_i, \bar{C}_i, \bar{D}_i, i = 1, 2, \ldots, N_s\})\Xi(T) \\ &\quad = \Phi(\hat{P}, \{\hat{A}, \hat{B}_i, \hat{C}_i, \hat{D}_i, i = 1, 2, \ldots, N_s\}), \end{aligned} \tag{3.11}$$

where $\hat{P} = T^T P T$ and

$$\begin{bmatrix} \hat{A} & \hat{B}_i \\ \hat{C}_i & \hat{D}_i \end{bmatrix} = \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A} & \bar{B}_i \\ \bar{C}_i & \bar{D}_i \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}.$$

□

This lemma implies a dual class of LMIs: Define $\mathcal{L}'$ in the same way to that of $\mathcal{L}$ in Definition 3.1 with the following replacements:

$$\begin{aligned} P &\to P', \\ P\bar{A} &\to \bar{A}P', \\ P\bar{B}_i &\to \bar{B}_i, \\ \bar{C}_i &\to \bar{C}_i P'. \end{aligned}$$

For any $\Phi(P,\Sigma) \in \mathcal{L}$ (resp. $\Phi'(P',\Sigma) \in \mathcal{L}'$), there exists $\Phi'(P',\Sigma) \in \mathcal{L}'$ (resp. $\Phi(P,\Sigma) \in \mathcal{L}$) such that

$$\Xi(P^{-1})\Phi(P,\Sigma)\Xi(P^{-1}) = \Phi'(P',\Sigma) \tag{3.12}$$

holds for $P' = P^{-1}$. Therefore $\Phi(P,\Sigma)$ and $\Phi'(P',\Sigma)$ are equivalent. We mainly use the class $\mathcal{L}$ in this dissertation.

3.3 Unified solution to LMI-synthesis problems

In this section, we show a solution to the LMI-synthesis problem. The solution is a procedure to derive from any $\Phi \in \mathcal{L}$ a new LMI on the parameter set, denote it by $\Phi^*(\boldsymbol{p}) > 0$, equivalent to the original inequality $\Phi(P, \Sigma) > 0$. First, we show the definition of Φ^*.

Definition 3.2 For $\Phi \in \mathcal{L}$, define a symmetric-matrix-valued function $\Phi^*(\boldsymbol{p})$ with the following replacements of the terms in Φ:

$$\begin{aligned} P &\to M_P(\boldsymbol{p}), \\ P\bar{A} &\to M_A(\boldsymbol{p}), \\ P\bar{B}_i &\to M_{Bi}(\boldsymbol{p}), \\ \bar{C}_i &\to M_{Ci}(\boldsymbol{p}), \\ \bar{D}_i &\to M_{Di}(\boldsymbol{p}), \end{aligned}$$

where functions $M_P(\boldsymbol{p})$, $M_A(\boldsymbol{p})$, etc. are defined by:

$$M_P(\boldsymbol{p}) := \begin{bmatrix} P_f & I_n & 0 \\ I_n & P_g & 0 \\ 0 & 0 & P_h \end{bmatrix}, \tag{3.13a}$$

$$M_A(\boldsymbol{p}) := \begin{bmatrix} AP_f + BW_f & A + BW_hC & BW_{f2} \\ L & P_gA + W_gC & L_{12} \\ L_{21} & W_{g2}C & L_{22} \end{bmatrix}, \tag{3.13b}$$

$$M_{Bi}(\boldsymbol{p}) := \begin{bmatrix} B_i + BW_hN_i \\ P_gB_i + W_gN_i \\ W_{g2}N_i \end{bmatrix}, \tag{3.13c}$$

$$M_{Ci}(\boldsymbol{p}) := \begin{bmatrix} C_iP_f + H_iW_f & C_i + H_iW_hC & H_iW_{f2} \end{bmatrix}, \tag{3.13d}$$

$$M_{Di}(\boldsymbol{p}) := D_i + H_iW_hN_i. \tag{3.13e}$$

□

Example 3.2 The corresponding function to $\Phi_{H_\infty(i)}$ in Example 3.1 by Definition 3.2 is:

$$\Phi^*_{H_\infty(i)}(\boldsymbol{p};\gamma) = M_P(\boldsymbol{p}) \oplus \begin{bmatrix} -M_A(\boldsymbol{p}) - M_A^T(\boldsymbol{p}) & M_{Bi}(\boldsymbol{p}) & M_{Ci}^T(\boldsymbol{p}) \\ M_{Bi}^T(\boldsymbol{p}) & \gamma I_{m_i} & -M_{Di}^T(\boldsymbol{p}) \\ M_{Ci}(\boldsymbol{p}) & -M_{Di}(\boldsymbol{p}) & \gamma I_{p_i} \end{bmatrix}.$$

□

Using Φ^*, now we show the main result of this chapter:

Theorem 3.1 [MOS95a, MOS96b, MOS95b] Given $\Phi \in \mathcal{L}$, define Φ^* by Definition 3.2.

(1) For any $n_p \geq n$, the following two statements are equivalent:

(I) For some $n_c \geq 0$, there exist $P \in PD(n + n_c)$ and a controller K of order n_c satisfying

$$\Phi(P, \Sigma_{cl}(K)) > 0. \tag{3.14}$$

(II) There exists $\boldsymbol{p} \in \mathcal{P}(n_p)$ such that

$$\Phi^*(\boldsymbol{p}) > 0. \tag{3.15}$$

(2) For any $n_p \geq n$, the set of all controllers of order n_p satisfying (3.14) for some P is represented by:

$$\begin{aligned} &\{G_c(s; K) | K \in \mathcal{K}(n_p), \Phi(P, \Sigma_{cl}(K)) > 0, \exists P \in PD(n + n_p)\} \\ &= \{G_c(s; \mathrm{K}_{\mathrm{map}}(\boldsymbol{p})) | \boldsymbol{p} \in \mathcal{P}(n_p), \Phi^*(\boldsymbol{p}) > 0, \} \end{aligned} \tag{3.16}$$

Proof. See Appendix. □

This theorem asserts that, for any LMI of the class $\mathcal{L}$, solving the nonconvex inequality $\Phi(P, \Sigma(K)) > 0$ is equivalent to solving the convex inequality $\Phi^*(\boldsymbol{p}) > 0$. This fact reduces the complexity of the original problem, which is a BMI problem, in general NP-complete and impossible to solve, to polynomial-time globally convergent problem. As shown in the following section, the class $\mathcal{L}$ contains a lot of LMI-conditions that have been utilized to represent properties of linear systems. Though conventional results solved synthesis problems described such LMI-conditions or equivalent algebraic equations or inequalities, our solution gives a formula unifying those results. Further, since $\mathcal{L}$ contains LMI-conditions that have not been solved as an LMI-synthesis problem, the solution enables new formulations for controller design. We will discuss such new design in Section 3.5.

Setting $n_p = n$ in Theorem 3.1, we always have a full-order solution to any of the LMI-synthesis problems formulated above whenever it is solvable. (If there exists a controller of order $n_c \geq n$ satisfying (3.14), the statement (1) in the theorem derives parameter $\boldsymbol{p} \in \mathcal{P}(n)$ satisfying (3.15), which then gives a controller of order n that meets (3.14).) Hence we will often treat only the case $n_p = n$ below. If Φ is a parametric LMI to some variables V, the corresponding Φ^* is also parametric to V, and Φ^* can be nonconvex with respect to $(\boldsymbol{p}, V)$. Then we have to use some special algorithm even for (3.15), but it is much easier to solve than (3.14).

In the following, we consider state-feedback solutions. Assume that the state of the plant (3.3) is available, i.e., $C = I$, $D = 0$, $N_i = 0$, $i = 1, \ldots, N_s$. The following corollary claims that, if the state is available, there always exists a static state-feedback solution to any LMI-synthesis problem that is solvable.

Definition 3.3 For $\Phi \in \mathcal{L}$, define a function $\Phi^{*sf}(\boldsymbol{p})$ by the following replacements of the elements in Φ:

$$
\begin{aligned}
P &\to P_f, \\
P\bar{A} &\to AP_f + BW_f, \\
P\bar{B}_i &\to B_i, \\
\bar{C}_i &\to C_iP_f + H_iW_f, \\
\bar{D}_i &\to D_i.
\end{aligned}
$$

Corollary 3.1 [MOS95b] The following statements (i) and (ii) are equivalent if the state is available in the plant (3.3):

(i) $\Phi(P, \Sigma_{cl}(K)) > 0$.

(ii) $\Phi^{*sf}(\boldsymbol{p}) > 0$.

If (ii) holds, $u(t) = W_f P_f^{-1} x(t)$ is a controller satisfying (i). □

3.4 Elements and subclasses of $\mathcal{L}$

In this section, we first show several properties of linear systems represented by an LMI belonging to $\mathcal{L}$. Next, we define some subclasses of $\mathcal{L}$, which we employ in the following chapter. LMIs shown here are widely known and used for controller design. We will show new LMI conditions later in Section 3.5.

3.4.1 Root-clustering conditions

Let us consider the following regions in the complex plane (See Fig.3.2):

$$\mathcal{C}_E(\beta_E) := \{\lambda \in \mathbf{C} \,|\, \mathrm{Re}\lambda < -\beta_E\}, \tag{3.17a}$$

$$\mathcal{C}_F(\beta_F) := \{\lambda \in \mathbf{C} \,|\, \mathrm{Re}\lambda > -\beta_F\}, \tag{3.17b}$$

$$\mathcal{C}_I(\beta_I) := \{\lambda \in \mathbf{C} \,|\, |\mathrm{Im}\lambda| < \beta_I\}, \tag{3.17c}$$

$$\mathcal{C}_D(c, \mu) := \{\lambda \in \mathbf{C} \,|\, |\lambda + c| < \mu\}, \tag{3.17d}$$

where β_E, β_F, c are real constants, and β_I, μ are positive constants. Set $Q := -\bar{A}^T P - P\bar{A}$, $J := \bar{A}^T P - P\bar{A}$ and define the following functions:

$$\Phi_E(P, \Sigma; \beta_E) := P \oplus (Q - 2\beta_E P), \tag{3.18a}$$

$$\Phi_F(P, \Sigma; \beta_F) := P \oplus (2\beta_F P - Q), \tag{3.18b}$$

$$\Phi_I(P, \Sigma; \beta_I) := \begin{bmatrix} 2\beta_I P & J \\ J^T & 2\beta_I P \end{bmatrix}, \tag{3.18c}$$

$$\Phi_D(P, \Sigma; \{c, \mu\}) := \begin{bmatrix} \mu P & \bar{A}^T P + cP \\ P\bar{A} + cP & \mu P \end{bmatrix}. \tag{3.18d}$$

Denote by $\Lambda(\bar{A})$ all the eigenvalues of $\bar{A}$. Then we have

$$\Lambda(\bar{A}) \subset \mathcal{C}_E \leftrightarrow \Phi_E(P, \Sigma; \beta_E) > 0, \tag{3.19}$$

$$\Lambda(\bar{A}) \subset \mathcal{C}_F \leftrightarrow \Phi_F(P, \Sigma; \beta_F) > 0, \tag{3.20}$$

$$\Lambda(\bar{A}) \subset \mathcal{C}_I \leftrightarrow \Phi_I(P, \Sigma; \beta_I) > 0, \tag{3.21}$$

$$\Lambda(\bar{A}) \subset \mathcal{C}_D \leftrightarrow \Phi_D(P, \Sigma; \{c, \mu\}) > 0. \tag{3.22}$$

These results are proved in [ONS91] for (3.19)~(3.21) and [Yed93] for (3.19),(3.20) and (3.22). We note that

$$\Phi'_E(P, \Sigma; \beta_E) := \begin{bmatrix} Q & P \\ P & \frac{1}{2\beta} P \end{bmatrix} > 0 \tag{3.23}$$

gives an equivalent condition to (3.19). We will mention this LMI later in Subsection 3.4.4, and use it in Section 4.5.

3.4.2 Performances of continuous-time systems

- H_∞-norm conditions [BGFB94]

  The H_∞-norm from w_i to z_i is given by

$$\|G_{s(i)}\|_\infty = \inf_P \{\gamma | \Phi_{H_\infty(i)}(P, \Sigma; \gamma) > 0\},$$

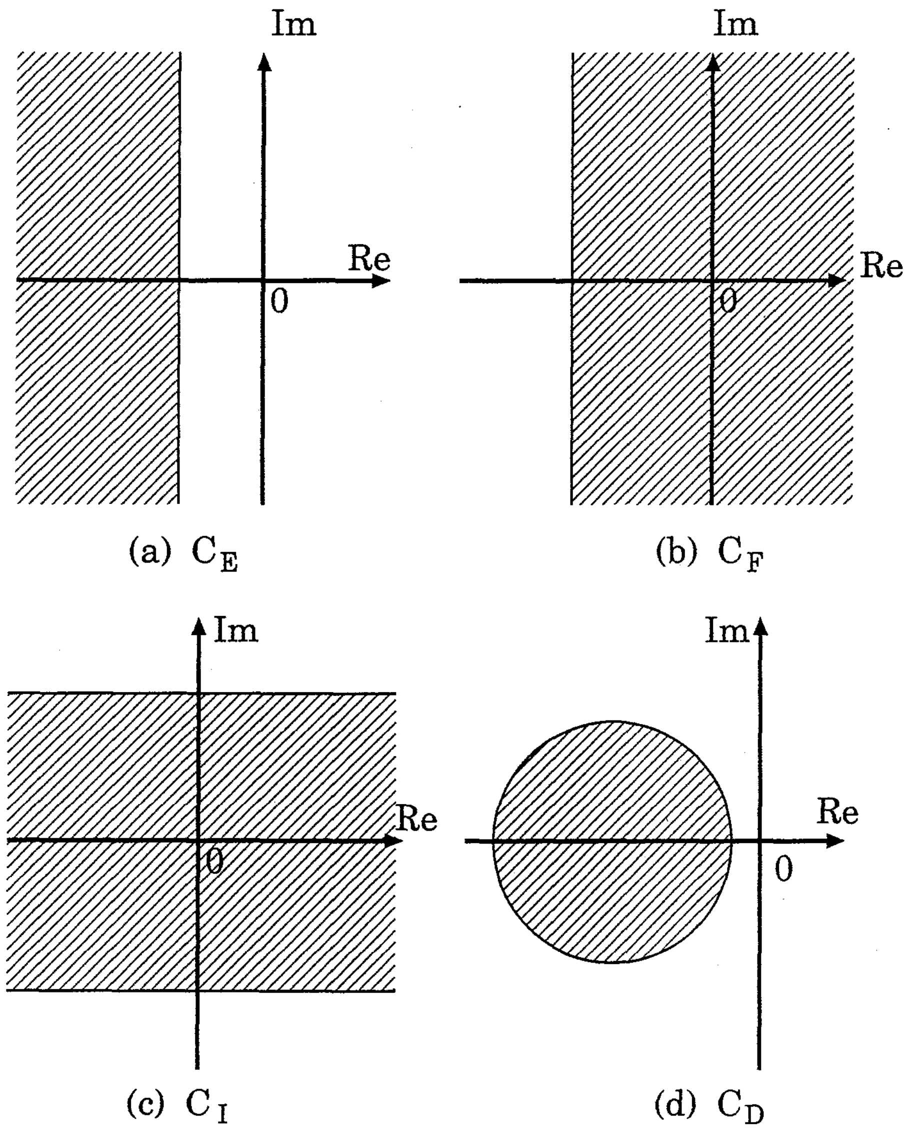

Fig. 3.2 Subregions of the complex plane

where $\Phi_{H_\infty(i)}(\bullet)$ is defined in (3.8).

- H_2-norm conditions [Rot93, BGFB94]
  Assume that $\bar{D}_i = 0$. The following function $\Phi_{H_2(i)}$ satisfies
  $$\|\bar{G}_{s(i)}\|_2 = \inf_{P,R}\{\gamma | \Phi_{H_2(i)}(P,\Sigma;\{\gamma,R\}) > 0\},$$
  where $R \in PD(m_i)$ and
  $$\Phi_{H_2(i)}(P,\Sigma;\{\gamma,R\}) := \begin{bmatrix} -P\bar{A} - \bar{A}^T P & \bar{C}_i^T \\ \bar{C}_i & \gamma I_{p_i} \end{bmatrix} \oplus \begin{bmatrix} P & P\bar{B}_i \\ \bar{B}_i^T P & R \end{bmatrix} \oplus (\gamma - \mathrm{trace} R).$$
  There is an alternative LMI-condition for the H_2-norm, given by
  $$\Phi_{H_2C(i)}(P,\Sigma;\{\gamma,R\}) := \begin{bmatrix} -P\bar{A} - \bar{A}^T P & P\bar{B}_i \\ \bar{B}_i^T P & \gamma I_{m_i} \end{bmatrix} \oplus \begin{bmatrix} P & \bar{C}_i^T \\ \bar{C}_i & R \end{bmatrix} \oplus (\gamma - \mathrm{trace} R),$$
  where $R \in PD(p_i)$. Rotea [Rot93] proposed several generalized H_2-conditions, represented by similar LMIs to the above ones.

- Positive-real conditions [BGFB94]
  Assume that $m_i = p_i$. The transfer function $G_{s(i)}$ is strictly positive-real if and only if
  $$\Phi_{PR(i)}(P,\Sigma) := P \oplus \begin{bmatrix} -P\bar{A} - \bar{A}^T P & P\bar{B}_i + \bar{C}_i^T \\ \bar{B}_i^T P + \bar{C}_i & \bar{D}_i + \bar{D}_i^T \end{bmatrix} > 0.$$

3.4.3 Performances for discrete-time systems

- H_∞-norm condition [SX92]
  The H_∞-norm from w_i to z_i is given by $\|G_{s(i)}\|_\infty = \inf_P \{\gamma | \Phi^d_{H_\infty(i)}(P,\Sigma;\gamma) > 0\}$ with the following function:
  $$\Phi^d_{H_\infty(i)}(P,\Sigma;\gamma) := \begin{bmatrix} P & 0 & P\bar{A} & P\bar{B}_i \\ 0 & \gamma I_{p_i} & \bar{C}_i & \bar{D}_i \\ \bar{A}^T P & \bar{C}_i^T & P & 0 \\ \bar{B}_i^T P & \bar{D}_i^T & 0 & \gamma I_{m_i} \end{bmatrix}.$$

- H_2-norm [GPS93a]
  The following function $\Phi_{H_2(i)}$ satisfies $\|\bar{G}_{s(i)}\|_2 = \inf_{P,R} \{\gamma | \Phi_{H_2(i)}(P,\Sigma;\{\gamma,R\}) > 0\}$, where $R \in PD(m_i)$ and
  $$\Phi^d_{H_2(i)}(P,\Sigma;\{\gamma,R\}) := \begin{bmatrix} P & P\bar{A} & \bar{C}_i^T \\ \bar{A}^T P & P & 0 \\ \bar{C}_i & 0 & \gamma I_{p_i} \end{bmatrix} \oplus \begin{bmatrix} R & \bar{B}_i^T P & \bar{D}_i^T \\ P\bar{B}_i & P & 0 \\ \bar{D}_i & 0 & I_{m_i} \end{bmatrix}$$
  $$\oplus(\gamma - \mathrm{trace} R).$$

Instead of $\Phi^d_{H_2(i)}$, the same condition holds for

$$\Phi^d_{H_2C(i)}(P,\Sigma;\{\gamma,R\}) := \begin{bmatrix} P & P\bar{A} & P\bar{B}_i \\ \bar{A}^TP & P & 0 \\ \bar{B}_i^TP & 0 & \gamma I_{m_i} \end{bmatrix} \oplus \begin{bmatrix} R & \bar{C}_i & \bar{D}_i \\ \bar{C}_i^T & P & 0 \\ \bar{D}_i^T & 0 & I_{p_i} \end{bmatrix} \oplus(\gamma - \mathrm{trace}R).$$

where $R \in PD(p_i)$.

3.4.4 Subclasses of $\mathcal{L}$

In the rest of this section, we give three subclasses of $\mathcal{L}$: $\mathcal{L}_S$, $\mathcal{L}_M$ and $\mathcal{L}_C$. The subclasses $\mathcal{L}_M$ and $\mathcal{L}_C$ are used in the following chapter, while we discuss relations between our result and previous results for LMIs belonging to $\mathcal{L}_S$.

Definition 3.4

(i) Let $\mathcal{L}_S$ be a subset of $\mathcal{L}$ whose elements have the following representation:

$$\begin{aligned}\Phi(P,\Sigma) =& \Phi_0(P,\{\bar{B}_i,\bar{C}_i,\bar{D}_i, i=1,2,\ldots,N_s\}) \\ &+\Phi_1 P\bar{A}\Phi_2 + \Phi_2^T\bar{A}^TP\Phi_1^T,\end{aligned} \tag{3.24}$$

where Φ_1 and Φ_2 are constant matrices. In (3.24), each $P\bar{A}$ and $\bar{A}^TP$ appears once in $\Phi \in \mathcal{L}_S$.

(ii) Let $\mathcal{L}_M$ be a subset of $\mathcal{L}_S$ whose elements are shown by:

$$\begin{aligned}\Phi(P,\Sigma) =& \Phi_0(P,\{\bar{C}_i,\bar{D}_i, i=1,2,\ldots,N_s\}) \\ &+\Phi_1 \begin{bmatrix} P\bar{A} & P\bar{B}_1 & \cdots & P\bar{B}_{N_s} \end{bmatrix}\Phi_2 + \Phi_2^T \begin{bmatrix} \bar{A}^TP \\ \bar{B}_1^TP \\ \vdots \\ \bar{B}_{N_s}^TP \end{bmatrix}\Phi_1^T,\end{aligned} \tag{3.25}$$

where Φ_1 and Φ_2 are constant matrices.

(iii) Let $\mathcal{L}_C$ be a subset of $\mathcal{L}_M$ whose elements satisfy the following constraints in addition to those of the definition of $\mathcal{L}$:

- *Diagonal blocks*: If a diagonal block is a *state* block, either $c^{(1)}_{kk} \equiv 0$ or $c^{(2)}_{kk} \equiv 0$, i.e., a diagonal block is represented by

$$\Phi_{kk}(P,\Sigma) = c^{(1)}_{kk}P, \text{ or } c^{(2)}_{kk}(P\bar{A} + \bar{A}^TP).$$

- *Off-diagonal blocks* ($k > l$): If both Φ_{kk} and Φ_{ll} are *state* blocks,

$$\Phi_{kl}(P, \Sigma) = c_{kl}^{(1)} P.$$

(iv) Denote by $\mathcal{L}_{S\oplus}$ (resp. $\mathcal{L}_{M\oplus}, \mathcal{L}_{C\oplus}$) a set that contains all the elements of $\mathcal{L}_S$ (resp. $\mathcal{L}_M, \mathcal{L}_C$) and any direct sum of them, i.e., for example,

$$\Phi \in \mathcal{L}_{S\oplus} \to \exists N_0, \exists \Phi_1, \Phi_2, \ldots, \Phi_{N_0} \in \mathcal{L}_S, \Phi = \Phi_1 \oplus \Phi_2 \oplus \cdots \oplus \Phi_{N_0}.$$

□

Remark 3.1 From the definition of $\mathcal{L}_C$, $\Phi \in \mathcal{L}_C$ has only one term of $P\bar{A} + \bar{A}^T P$, and, if one picks up all the state blocks from Φ, they are arranged as:

$$\begin{bmatrix} & \vdots & & \vdots & & \vdots & \\ \cdots & P & \cdots & P & \cdots & P & \cdots \\ & \vdots & & \vdots & & \vdots & \\ \cdots & P & \cdots & P\bar{A} + \bar{A}^T P & \cdots & P & \cdots \\ & \vdots & & \vdots & & \vdots & \\ \cdots & P & \cdots & P & \cdots & P & \cdots \\ & \vdots & & \vdots & & \vdots & \end{bmatrix},$$

where we dropped coefficients $c_{kl}^{(i)}(\bullet)$. □

In Fig.3.3, we show which subset of $\mathcal{L}$ each LMI in Subsection 3.4.1~3.4.3 belongs to. The subset $\mathcal{L}_C$ contains no root-clustering LMI-conditions except for Φ'_E, which we will use in Section 4.5.

All the LMI-conditions that have been solved in synthesis problems in the literature of [KR91, Gah92, Rot93, Iwa93, IS94, Gah94, ISG94] belong to $\mathcal{L}_S$. However, there have been no previous solutions to the LMIs of $\mathcal{L}_{S\oplus} \setminus \mathcal{L}_S$, which contains many new LMI-conditions, especially for multi-objective control. In Section 3.5, we show some of LMI-conditions belonging to $\mathcal{L}_{S\oplus} \setminus \mathcal{L}_S$ and discuss multi-objective design using those LMIs. Obviously $\mathcal{L}_{S\oplus}$ is the largest subclass. Note that we have no examples of LMIs in $\mathcal{L} \setminus \mathcal{L}_{S\oplus}$ now. However, if a performance index is represented in terms of LMIs belonging to any part of $\mathcal{L}$, Theorem 3.1 solves the LMI-synthesis problem with it.

On the other hand, for LMIs in the subclass $\mathcal{L}_S$, there exists a simpler condition equivalent to (3.15). In the rest of this section we show some of such conditions for

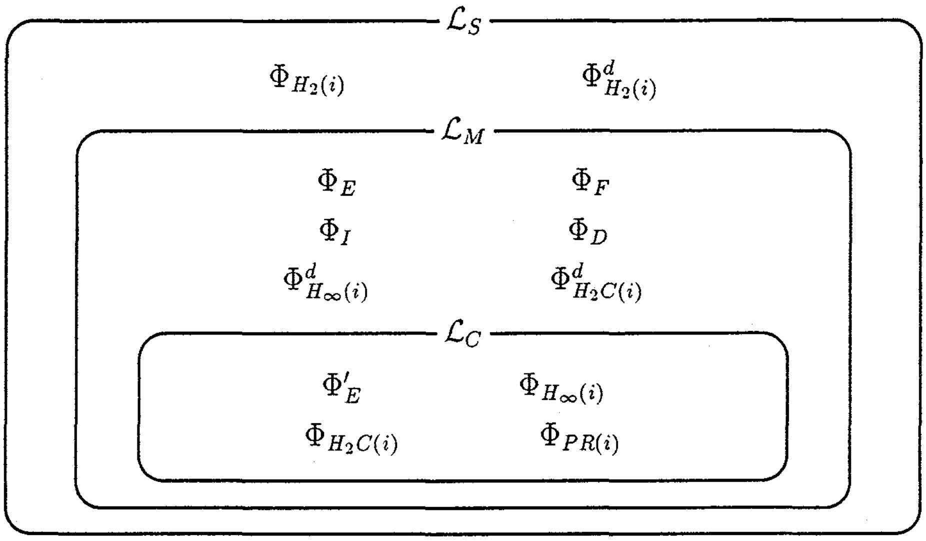

Fig. 3.3 Subsets and elements

$\mathcal{L}_S$. This simpler solution is related to previous results based on eliminating variables [SugT94] in LMIs.

Suppose that Φ belongs to the subclass $\mathcal{L}_S$. From the definition of $\mathcal{L}_S$, each L and L^T appears once in the corresponding Φ^*. Through some permutation of lows and columns, it is represented in the following form:

$$\Phi^*(\boldsymbol{p}) = \begin{bmatrix} \phi_{11} & \phi_{12} + L^T & \phi_{13} \\ \phi_{12}^T + L & \phi_{22} & \phi_{23} \\ \phi_{13}^T & \phi_{23}^T & \phi_{33} \end{bmatrix},$$

where we set $n_p = n$, and ϕ_{ij}'s depend on $\tilde{\boldsymbol{p}} = \{P_f,\ P_g,\ W_f,\ W_g,\ W_h\}$, not on L. (Note that P_h is eliminated if $n_p = n$.) Then (3.15) holds if and only if

$$\Phi_f^* := \begin{bmatrix} \phi_{11} & \phi_{13} \\ \phi_{13}^T & \phi_{33} \end{bmatrix} > 0 \text{ and } \Phi_g^* := \begin{bmatrix} \phi_{22} & \phi_{23} \\ \phi_{23}^T & \phi_{33} \end{bmatrix} > 0,$$

and one of the solutions to (3.15) is given by $\boldsymbol{p} = \{\tilde{\boldsymbol{p}},\ L\}$, where

$$L = -\phi_{12} + \phi_{13}\phi_{33}^{-1}\phi_{23}^T. \tag{3.26}$$

Thus LMIs in $\mathcal{L}_S$ are reduced to LMIs of only P_f, P_g and W_h.

Example 3.3 Consider $\Phi_{H_\infty(i)}(\bullet)$ defined in (3.8). Obviously this belongs to $\mathcal{L}_S$, and $\Phi_f^*(\bullet)$ and $\Phi_g^*(\bullet)$ are as follows:

$$\Phi_f^*(\boldsymbol{p};\gamma) = \Phi_{f1}^*(\boldsymbol{p};\gamma) \oplus M_P(\boldsymbol{p}),$$

$$
\begin{aligned}
&\Phi^*_{f1}(\boldsymbol{p};\gamma)\\
&=\begin{bmatrix} -(AP_f+BW_f)-(AP_f+BW_f)^T & B_i+BW_hN_i & (C_iP_f+H_iW_f)^T \\ (B_i+BW_hN_i)^T & \gamma I & -(D_i+H_iW_hN_i)^T \\ C_iP_f+H_iW_f & -(D_i+H_iW_hN_i) & \gamma I \end{bmatrix}\\
&=\begin{bmatrix} -AP_f-P_fA^T & B_i+BW_hN_i & (C_iP_f)^T \\ (B_i+BW_hN_i)^T & \gamma I & -(D_i+H_iW_hN_i)^T \\ C_iP_f & -(D_i+H_iW_hN_i) & \gamma I \end{bmatrix}\\
&\qquad +\begin{bmatrix} B \\ 0 \\ H_i \end{bmatrix} W_f\,[\,I_n \quad 0 \quad 0\,] + \begin{bmatrix} I_n \\ 0 \\ 0 \end{bmatrix} W_f^T\,[\,B^T \quad 0 \quad H_i^T\,],\\
&\Phi^*_g(\boldsymbol{p};\gamma)=\Phi^*_{g1}(\boldsymbol{p};\gamma)\oplus M_P(\boldsymbol{p}),\\
&\Phi^*_{g1}(\boldsymbol{p};\gamma)\\
&=\begin{bmatrix} -(P_gA+W_gC)-(P_gA+W_gC)^T & P_gB_i+W_gN_i & (C_i+H_iW_hC)^T \\ (P_gB_i+W_gN_i)^T & \gamma I & -(D_i+H_iW_hN_i)^T \\ C_i+H_iW_hC & -(D_i+H_iW_hN_i) & \gamma I \end{bmatrix}\\
&=\begin{bmatrix} -P_gA-A^TP_g & P_gB_i & (C_i+H_iW_hC)^T \\ (P_gB_i)^T & \gamma I & -(D_i+H_iW_hN_i)^T \\ C_i+H_iW_hC & -(D_i+H_iW_hN_i) & \gamma I \end{bmatrix}\\
&\qquad +\begin{bmatrix} I_n \\ 0 \\ 0 \end{bmatrix} W_g\,[\,C \quad N_i \quad 0\,] + \begin{bmatrix} C^T \\ N_i^T \\ 0 \end{bmatrix} W_g^T\,[\,I_n \quad 0 \quad 0\,].
\end{aligned}
$$

Cutting redundant blocks, permuting some rows and columns, and applying 'variable elimination' [SugT94], we get the following equivalent LMI-condition:

$$
M_P(\boldsymbol{p})=\begin{bmatrix} P_f & I_n \\ I_n & P_g \end{bmatrix} > 0, \tag{3.27a}
$$

$$
\begin{bmatrix} \begin{bmatrix} B \\ H_i \end{bmatrix}^{\perp} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} -AP_f-P_fA^T & P_fC_i^T & B_i \\ C_iP_f & \gamma I & -D_i \\ B_i^T & -D_i^T & \gamma I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} B \\ H_i \end{bmatrix}^{\perp} & 0 \\ 0 & I \end{bmatrix} > 0, \tag{3.27b}
$$

$$
\begin{bmatrix} [\,C \quad N_i\,]^{\perp} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -P_gA-A^TP_g & P_gB_i & C_i^T \\ B_i^TP_g & \gamma I & -D_i^T \\ C_i & -D_i & \gamma I \end{bmatrix} \begin{bmatrix} [\,C \quad N_i\,]^{\perp} & 0 \\ 0 & I \end{bmatrix}^T > 0. \tag{3.27c}
$$

□

The last LMI-condition (3.27) is used in [Gah92, Gah94] and an equivalent condition is found in [IS94]. From any solution P_f, P_g to (3.27), we get W_f, W_g satisfying

$\Phi_f^*(p;\gamma) \oplus \Phi_g^*(p;\gamma) > 0$ with $W_h = 0$ through, for example, Theorem 1 in [IS94], and L satisfying $\Phi_{H_\infty(i)}^*(p;\gamma) > 0$ by (3.26). Thus, as far as only an H_∞-norm condition is concerned, the inequality (3.27) gives a necessary and sufficient condition and we obtain one H_∞-suboptimal controller from any solution to (3.27). If we use the freedom of H_∞-suboptimal controllers, only the inequality $\Phi_{H_\infty(i)}^*(p;\gamma) > 0$ provides all the freedom of full-order H_∞-suboptimal controllers via a convex subset of the parameter set.

If each L and L^T appears more than once in Φ^*, which situation comes with a combined LMI for multiple specifications, we do not have the separation above, and then the freedom of L can act to make such a combined LMI hold.

3.5 Robust multiobjective controller synthesis based on the class $\mathcal{L}$

We showed in Section 3.3 a solution to LMI-synthesis problems for any LMIs of the class $\mathcal{L}$, and in Section 3.4 we listed several LMI-conditions belonging to $\mathcal{L}$. Those LMIs have been used in LMI-based synthesis problems such as H_2, H_∞-control problems, and the solution in Section 3.3 gives a unified formula to those previous solutions. On the other hand, the class $\mathcal{L}$ contains LMIs that give robust multi-objective performance conditions. Further, some of those LMIs have not been used in output-feedback controller synthesis before. In this section, we show those new robust multi-objective conditions in the class $\mathcal{L}$ and compare them with conditions used in previous synthesis methods.

3.5.1 H_2/H_∞-control, H_2/H_∞/root-clustering

The original setting of H_2/H_∞ problems that we consider is stated as follows: *Let $N_s = 2$ in (3.3). Find a controller K that stabilizes the closed-loop system (3.5) and minimizes $\|G_{cl(2)}\|_2$ subject to $\|G_{cl(1)}\|_\infty < 1$.* This problem has been recognized as a difficult problem to get the exact optimal solution [Nob95].

Boyd et al. proposed an approach to multi-objective synthesis that is now so-called transfer function approach [BB91]. Using the Youla's parametrization [YJB76], the closed-loop transfer functions $G_{cl(i)}, i = 1, 2$ attained via a stabilizing controller is parametrized as follows:

$$G_{cl(i)} = \bar{T}_{1(i)} + T_{2(i)} Q T_{3(i)}, \; i = 1, 2,$$

where $Q \in \mathbf{RH}_\infty$ is a Youla parameter, a stable transfer function matrix, and $T_{j(i)}$,'s are immediately computed from the plant. Since the free parameter Q belongs to the infinite-dimensional parameter set $\mathbf{RH}_\infty$, they approximated Q by the following form:

$$Q = c_0 + \sum_{i=1}^{n_B} c_i \frac{1}{(s+\alpha)^i}, \; \alpha > 0$$

where we assume for simplicity Q is scalar. Since both H_2-norm and H_∞-norm are convex functions of transfer functions, $\|G_{cl(2)}\|_2$ and $\|G_{cl(1)}\|_\infty$ are convex functions of Q, and hence of c_i. Thus we get convex performance indices of the parameters $(c_0, c_1, \ldots, c_{n_B})$. The optimization is carried out with algorithms such as the ellipsoid algorithm [BB91]. However, to get a good approximation, the number of the parameters n_B is often large and this causes high-order controllers.

On the other hand, several other approaches have been also proposed based on algebraic equations or inequalities. Bernstein et al. [BH89] proposed an approach to minimize an upper-bound of H_2-norm and reduced the auxiliary problem to solving a triple of Riccati equations. Khargonekar et al. [KR91] solved the auxiliary problem via convex optimization. In these results, in addition to the assumptions for the H_2-norm to be finite, they assume that $B_1 = B_2$ and $N_1 = N_2$, which is necessary to use the upper-bound. This upper-bound, say f_{BH}, is represented by an LMI of the subclass $\mathcal{L}_S$ as follows:

$$f_{BH} = \inf_{P,R} \left\{ \gamma \,\middle|\, \Phi_{H_\infty(1)}(P, \Sigma; 1) \oplus \begin{bmatrix} P & \bar{C}_2^T \\ \bar{C}_2 & R \end{bmatrix} \oplus (\gamma - \mathrm{trace} R) > 0 \right\}^{\frac{1}{2}}. \tag{3.28}$$

Instead of the above upper-bound, the class $\mathcal{L}$ has the following LMI-condition that gives an upper-bound of the H_2-norm without the assumptions on B_i and N_i:

$$f_L = \inf_{P,R} \left\{ \gamma \,\middle|\, \Phi_{H_\infty(1)}(P, \Sigma; 1) \oplus \Phi_{H_2(2)}(P, \Sigma; \{R, \gamma\}) > 0 \right\} \tag{3.29}$$

If $B_1 = B_2$ and $N_1 = N_2$ hold, (3.29) is reduced to (3.28).

Thus an LMI-condition in the class $\mathcal{L}$ enables a less restrictive mixed H_2/H_∞ formulation. Another merit of our approach is the fact that it is easy to add other specifications: For example, suppose that a controller derived by optimizing f_{BH} or f_L is too oscillatory. To improve the behavior of closed-loop system within the present framework of H_2-control or H_2/H_∞-control, what can be adjusted are only weighting functions, and finding appropriate weighting functions is not easy. In contrast, the class $\mathcal{L}$ can provide an LMI-condition that directly assigns a performance condition for the closed-loop system not to be oscillatory. For example,

$$f_{LP} = \inf_{P,R} \left\{ \gamma \,\middle|\, \Phi_{H_\infty(1)}(P,\Sigma;1) \oplus \Phi_{H_2(2)}(P,\Sigma;\{R,\gamma\}) \oplus \Phi_D(P,\Sigma;\{c,\mu\}) > 0 \right\} \quad (3.30)$$

can be useful in such a situation, where Φ_D gives a root-clustering condition into a disk (see Subsection 3.4.1). We will give a numerical example of this triple-objective LMI-condition in the following section.

There are some previous synthesis methods of H_∞ or H_2/H_∞ control with some root-clustering control. Saeki [Sae92] proposed an H_∞ / root-clustering controller synthesis method using Riccati equations. We can find alternative LMI-conditions in the class $\mathcal{L}$ to the specifications considered in the method. On the other hand, Bambang et al. [BSU93] showed an optimization approach to H_2/H_∞/root-clustering synthesis with various root regions. However, their formulation is nonconvex and the computational complexity is not clear.

The flexibility of our approach in assigning multiple specifications comes from the fact that, as mentioned in Section 3.2, for any $\Phi_1, \Phi_2 \in \mathcal{L}$, the direct sum $\Phi_1 \oplus \Phi_2$ always belongs to $\mathcal{L}$. We remark that such a direct sum requires each Φ_i to have a common solution P, which causes conservatism. However, as shown above, the class $\mathcal{L}$ contains less restrictive conditions than what has been solved by using convex optimization algorithms, and enables to consider more direct specifications for controller design.

3.5.2 Robust controller design with multiple specifications

We have not discussed *robustness* of any specifications explicitly so far. However, some of the specifications that we showed above, such as represented by H_∞-norm conditions, can describe some robustness. Here we show a simple example.

Example 3.4

- Suppose $\bar{D}_2 = 0$ and that the plant has an uncertainty denoted by:

$$w_2 = \Delta z_2, \ \sigma_{\max}(\Delta) \leq 1,$$

An upper bound for the L_2 gain from w_1 to z_1 subject to the quadratic stability with this uncertainty is given by $\inf\{\gamma | \Phi_Q(P, \Sigma; \{\gamma, \alpha\}) > 0\}$, where:

$$\Phi_{Q-H_\infty}(P, \Sigma; \{\gamma, \alpha\}) := \begin{bmatrix} -P\bar{A} - \bar{A}^T P & P\bar{B}_1 & \bar{C}_1^T & P\bar{B}_2 & \bar{C}_2^T \\ \bar{B}_1^T P & \gamma I_{m_1} & -\bar{D}_1^T & 0 & 0 \\ \bar{C}_1 & -\bar{D}_1 & \gamma I_{p_1} & 0 & 0 \\ \bar{B}_2^T P & 0 & 0 & \alpha I_{m_2} & 0 \\ \bar{C}_2 & 0 & 0 & 0 & \alpha^{-1} I_{p_2} \end{bmatrix}.$$

In this case Φ_{Q-H_∞} is not affine with respect to α. This condition is proposed by Xie et al. [XFS92] for robust H_∞-control.

- Assume the presence of the above uncertainty. If the following inequality:

$$\Phi_{Q-D}(P, \Sigma; \{c, \mu, \alpha\}) := \begin{bmatrix} \mu P & \bar{A}^T P + cP & 0 & \bar{C}_2^T \\ P\bar{A} + cP & \mu P & P\bar{B}_2 & 0 \\ 0 & \bar{B}_2^T P & \alpha I_{m_2} & 0 \\ \bar{C}_2^T & 0 & 0 & \alpha^{-1} I_{p_2} \end{bmatrix} > 0$$

  holds for some P and α, all the closed-loop poles sit inside the disk $\mathcal{C}_D$ (see (3.17d)) for any Δ, $\sigma_{\max}(\Delta) \le 1$ (See Section 4.3 for proof). □

We can find similar LMIs in $\mathcal{L}$. As in the previous subsection, direct sums of LMIs provide robust multiobjective LMI-conditions. For example, if there exist P, α_1 and α_2 such that

$$\Phi_{Q-H_\infty}(P; \Sigma, \{\gamma, \alpha_1\}) \oplus \Phi_{Q-D}(P, \Sigma; \{c, \mu, \alpha_2\}) > 0$$

holds, the system Σ satisfies a robust H_∞-norm condition and a robust root-clustering condition simultaneously. The equivalent inequality on the parameter set is:

$$\begin{bmatrix} -M_A(\boldsymbol{p}) - M_A(\boldsymbol{p})^T & M_{B1}(\boldsymbol{p}) & M_{C1}^T(\boldsymbol{p}) & M_{B2}(\boldsymbol{p}) & M_{C2}^T(\boldsymbol{p}) \\ M_{B1}^T(\boldsymbol{p}) & \gamma I_{m_1} & -M_{D1}(\boldsymbol{p})^T & 0 & 0 \\ M_{C1}(\boldsymbol{p}) & -M_{D1}(\boldsymbol{p}) & \gamma I_{p_1} & 0 & 0 \\ M_{B2}^T(\boldsymbol{p}) & 0 & 0 & \alpha_1 I_{m_2} & 0 \\ \bar{C}_2 & 0 & 0 & 0 & \alpha_1^{-1} I_{p_2} \end{bmatrix}$$
$$\oplus \begin{bmatrix} \mu M_P(\boldsymbol{p}) & M_A^T(\boldsymbol{p}) + cM_P(\boldsymbol{p}) & 0 & M_{C2}(\boldsymbol{p})^T \\ M_A(\boldsymbol{p}) + cM_P(\boldsymbol{p}) & \mu M_P(\boldsymbol{p}) & M_{B2}(\boldsymbol{p}) & 0 \\ 0 & M_{B2}^T(\boldsymbol{p}) & \alpha_2 I_{m_2} & 0 \\ M_{C2}(\boldsymbol{p}) & 0 & 0 & \alpha_2^{-1} I_{p_2} \end{bmatrix} > 0.$$

This inequality has nonconvexity in α_1 and α_2. In the following chapter, We will make more systematic approach to robust performance problems and discuss detail of robust performance problems with $\mathcal{L}$.

3.6 Numerical example (Multi-objective Controller Design)

In this section, we show a numerical example to demonstrate controller design using multi-objective LMI-conditions in the class $\mathcal{L}$. Let us show a two-mass system in Fig.3.4, where θ_i, $i = 1, 2$ are the angles of the motor and the load, respectively. The physical parameters are: $J_1 = J_2 = 0.01$, $d_1 = d_2 = 0.001$ and $K = 50$. Assume that the motor velocity $\dot{\theta}_1$ is available with additive sensor noise w_2, while w_1 is the impulsive exogenous input on the load.

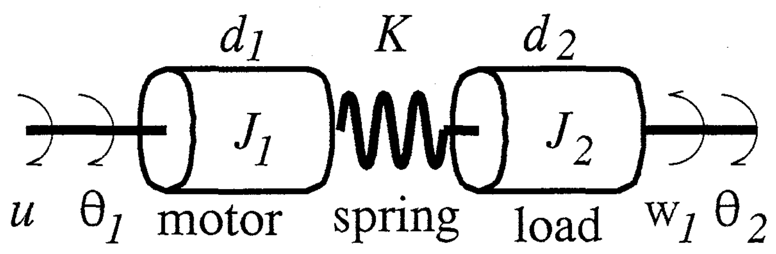

Fig. 3.4 Two-mass system

We give a state-space model of the two-mass system as follows, with $x := [\dot{\theta}_1,\ \dot{\theta}_2,\ \theta_1 - \theta_2]^T$, $z_1 := \dot{\theta}_1$ and $z_2 := \dot{\theta}_2$:

$$A = \begin{bmatrix} -d_1/J_1 & 0 & -K/J_1 \\ 0 & -d_2/J_2 & K/J_2 \\ 1 & -1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1/J_1 \\ 0 \\ 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ -1/J_2 \\ 0 \end{bmatrix},\ B_2 = 0,$$

$$C = [1 \quad 0 \quad 0],\ D = 0,\ N_1 = 0,\ N_2 = 1,$$

$$C_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ H_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ D_1 = 0,$$

$$C_2 = [1 \quad 0 \quad 0],\ H_2 = 0,\ D_2 = 0.$$

First, we consider minimization of the H_2-norm $\|G_{cl(1)}\|_2$ with the H_∞ norm constraint $\|G_{cl(2)}\|_\infty < 2$. This problem is formulated using $\mathcal{L}$ as follows:

$$\underset{P,R,K}{\text{Minimize}}\, \gamma \text{ subject to}$$

$$\Phi_{H_2C(1)}(P, \Sigma_{cl}(K); \{\gamma, R\}) \oplus \Phi_{H_\infty(2)}(P, \Sigma_{cl}(K); 2) > 0.$$

Applying Theorem 3.1 and eliminating a redundant block, we get the following LMI to solve on the parameter set:

$$\begin{aligned}&\underset{\boldsymbol{p},R}{\text{Minimize}}\ \gamma \text{ subject to}\\ &\begin{bmatrix} -M_A(\boldsymbol{p})-M_A^T(\boldsymbol{p}) & M_{B1}(\boldsymbol{p}) \\ M_{B1}^T(\boldsymbol{p}) & \gamma I \end{bmatrix} \oplus \begin{bmatrix} M_P(\boldsymbol{p}) & M_{C1}^T(\boldsymbol{p}) \\ M_{C1}(\boldsymbol{p}) & R \end{bmatrix} \oplus (\gamma - \text{trace}R)\\ &\oplus \begin{bmatrix} -M_A(\boldsymbol{p})-M_A^T(\boldsymbol{p}) & M_{B2}(\boldsymbol{p}) & M_{C2}^T(\boldsymbol{p}) \\ M_{B2}^T(\boldsymbol{p}) & 2I & -M_{D2}^T(\boldsymbol{p}) \\ M_{C2}(\boldsymbol{p}) & -M_{D2}(\boldsymbol{p}) & 2I \end{bmatrix} > 0.\end{aligned}$$

Using a tool that we had developed to handle LMIs (named 'MAKELMI' [MOS96a]), we reduced this LMI of $\boldsymbol{p}$ to the standard form of LMI:

$$\Phi_0 + \sum_{i=1}^{N_P} p_i \Phi_i > 0, \tag{3.31}$$

where p_i's are elements of $\boldsymbol{p}$. The coefficients Φ_i's are automatically calculated from the plant data by the tool. We used a standard LMI solver by Vandenberghe et al. [VLB94] on MATLAB version 4.0a to solve (3.31). The result of the convex optimization is:

$$\begin{aligned} P_f &= \begin{bmatrix} 0.1827 & -0.0262 & -0.0006 \\ -0.0262 & 0.1879 & 0.0418 \\ -0.0006 & 0.0418 & 0.1908 \end{bmatrix}, \\ P_g &= \begin{bmatrix} 5.8097 & 0.8092 & -0.1734 \\ 0.8092 & 5.7753 & -1.2588 \\ -0.1734 & -1.2588 & 5.5404 \end{bmatrix}, \\ W_f &= [0.4464 \quad -0.0798 \quad -0.1058], \\ W_g &= \begin{bmatrix} 19.39 \\ 0.0314 \\ -4.6312 \end{bmatrix}, \\ W_h &= -0.2283, \\ L &= \begin{bmatrix} 31.4928 & -1.1104 & -1.2338 \\ -1.7068 & -0.0102 & 0.4517 \\ -1.7053 & -4.9476 & 27.2586 \end{bmatrix}. \end{aligned}$$

The derived controller is:

$$A_c = \begin{bmatrix} -88.16 & -136.02 & 36.44 \\ 89.58 & -0.6561 & 52.53 \\ 20.92 & -39.71 & -11.30 \end{bmatrix},$$

$$B_c = \begin{bmatrix} -1.1815 \\ 0.6617 \\ 0.3555 \end{bmatrix},$$
$$C_c = [6.535 \quad 5.124 \quad -3.296],$$
$$D_c = -0.2283.$$

We show the impulse responses $G_{cl(1)}$ and the singular value plot of $G_{cl(2)}$ in Fig.3.5 and Fig.3.6, respectively. The H_∞ norm condition is satisfied, but the impulse responses are oscillatory. From the plot of the closed-loop poles in Fig.3.7, the closed-loop system is found to have poles close to the imaginably axis.

To avoid such oscillatory responses, we add a specification to cluster the closed-loop poles into the disk whose center is $-500 + 0j$ and radius is 485. Hence real parts of the closed-loop poles are less than -15, and poles near the imaginary axis with a large imaginary part are avoided because the region is a disk. We consider the following optimization problem described by an LMI in $\mathcal{L}$:

$$\begin{aligned}
&\underset{P,R,K}{\text{Minimize}}\ \gamma \text{ subject to} \\
&\Phi_{H_2C(1)}(P, \Sigma_{cl}(K); \{\gamma, R\}) \oplus \Phi_{H_\infty(2)}(P, \Sigma_{cl}(K); 2) \\
&\oplus \Phi_D(P, \Sigma_{cl}(K); \{500, 485\}) > 0,
\end{aligned}$$

and the corresponding LMI on the parameter set is:

$$\begin{aligned}
&\underset{\boldsymbol{p},R}{\text{Minimize}}\ \gamma \text{ subject to} \\
&\begin{bmatrix} -M_A(\boldsymbol{p}) - M_A^T(\boldsymbol{p}) & M_{B1}(\boldsymbol{p}) \\ M_{B1}^T(\boldsymbol{p}) & \gamma I \end{bmatrix} \oplus \begin{bmatrix} M_P(\boldsymbol{p}) & M_{C1}^T(\boldsymbol{p}) \\ M_{C1}(\boldsymbol{p}) & R \end{bmatrix} \oplus (\gamma - \text{trace} R) \\
&\oplus \begin{bmatrix} -M_A(\boldsymbol{p}) - M_A^T(\boldsymbol{p}) & M_{B2}(\boldsymbol{p}) & M_{C2}^T(\boldsymbol{p}) \\ M_{B2}^T(\boldsymbol{p}) & 2I & -M_{D2}^T(\boldsymbol{p}) \\ M_{C2}(\boldsymbol{p}) & -M_{D2}(\boldsymbol{p}) & 2I \end{bmatrix} \\
&\oplus \begin{bmatrix} 485 M_P(\boldsymbol{p}) & M_A^T(\boldsymbol{p}) + 500 M_P(\boldsymbol{p}) \\ M_A(\boldsymbol{p}) + 500 M_P(\boldsymbol{p}) & 485 M_P(\boldsymbol{p}) \end{bmatrix} > 0.
\end{aligned}$$

The result of the convex optimization is:

$$P_f = \begin{bmatrix} 1.2101 & -0.5410 & 0.0435 \\ -0.5410 & 1.5866 & 0.3642 \\ 0.0435 & 0.3642 & 1.5336 \end{bmatrix},$$
$$P_g = \begin{bmatrix} 1.4535 & 0.0781 & -0.6693 \\ 0.0781 & 1.0389 & -0.0754 \\ -0.6693 & -0.0754 & 2.1189 \end{bmatrix},$$

$$
\begin{aligned}
W_f &= \begin{bmatrix} 13.81 & -9.213 & -0.4916 \end{bmatrix}, \\
W_g &= \begin{bmatrix} 10.04 \\ 1.003 \\ -2.652 \end{bmatrix}, \\
W_h &= -0.2283, \\
L &= \begin{bmatrix} 128.9778 & 5.2147 & -27.7253 \\ -52.8790 & 3.0122 & 12.7320 \\ -32.8268 & -0.6459 & 14.9785 \end{bmatrix}.
\end{aligned}
$$

The derived controller is:

$$
\begin{aligned}
A_c &= \begin{bmatrix} -270.76 & -281.3 & 77.801 \\ 129.39 & 41.20 & 12.05 \\ 43.83 & 14.32 & -30.26 \end{bmatrix}, \\
B_c &= \begin{bmatrix} -7.3602 \\ 4.2725 \\ -1.0328 \end{bmatrix}, \\
C_c &= \begin{bmatrix} 17.36 & 17.55 & -2.749 \end{bmatrix}, \\
D_c &= -0.5632.
\end{aligned}
$$

We show the impulse responses of $G_{cl(1)}$ and the singular value plot of $G_{cl(2)}$ in Fig.3.8 and Fig.3.9, respectively. Fig.3.8 shows much less oscillatory impulse responses, and the H_∞ norm condition is confirmed in Fig.3.9. The closed-loop poles are placed at -107.4, $-52.94 \pm 114.3i$, -46.42, $-23.32 \pm 75.22i$ (see Fig.3.10), which belong to the prescribed disk.

3.7 Concluding remarks

In this chapter, we formulated the LMI-synthesis problem with the class of LMIs, and provided a unified solution to it. This result includes a large part of existing LMI-based approaches and unifies them. Further, the class contains new LMI-conditions for multiple specifications, and along with the unified solution to all LMIs in the class enable a new multi-objective controller design framework. A numerical example is presented showing feature of multi-objective design using our approach.

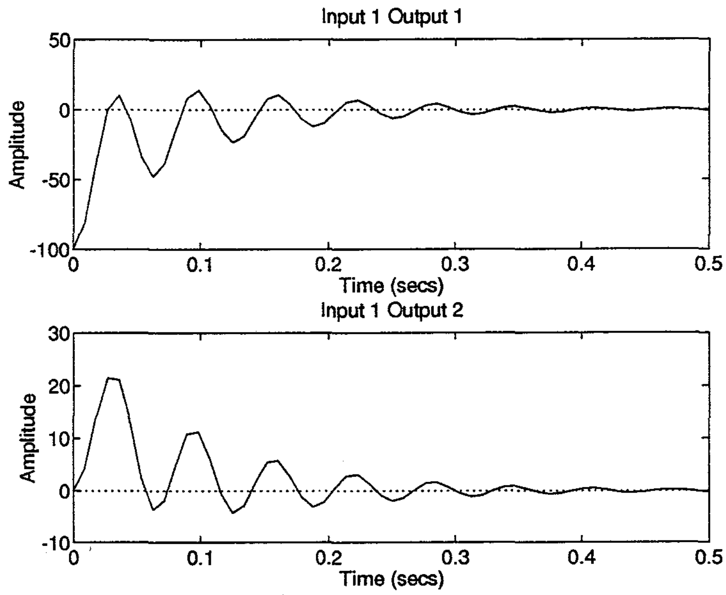

Fig. 3.5 Impulse responses (H_2/H_∞-control)

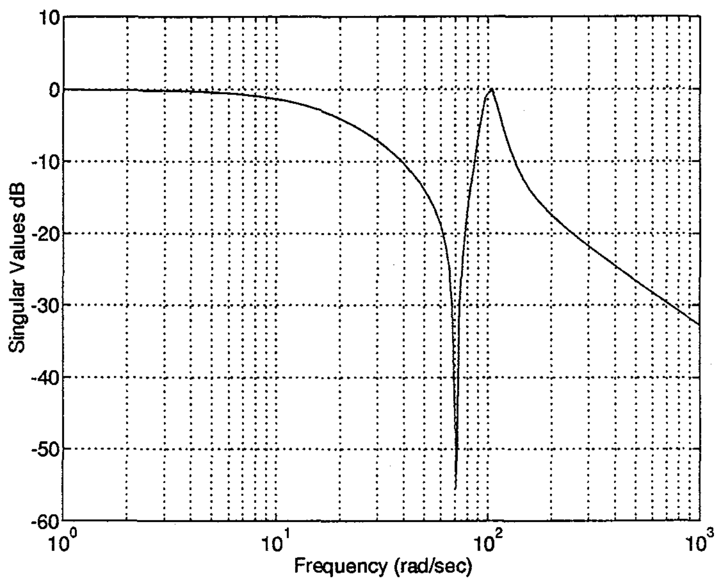

Fig. 3.6 Singular value plot (H_2/H_∞-control)

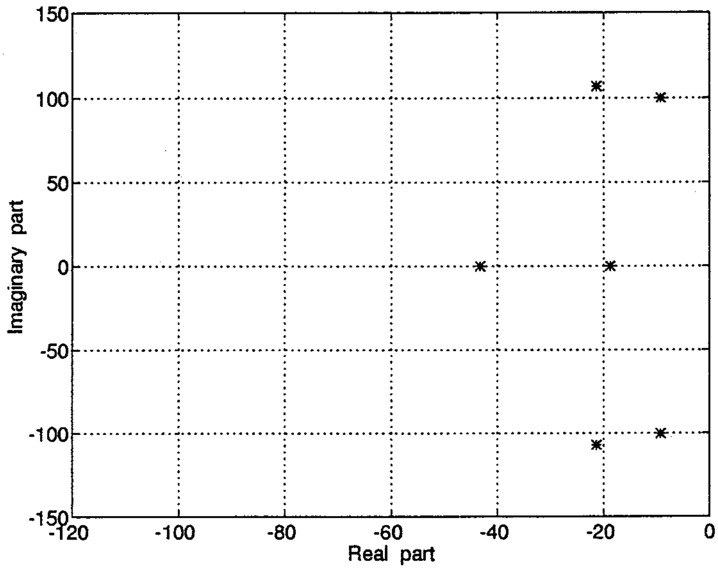

Fig. 3.7 Closed-loop poles (H_2/H_∞-control)

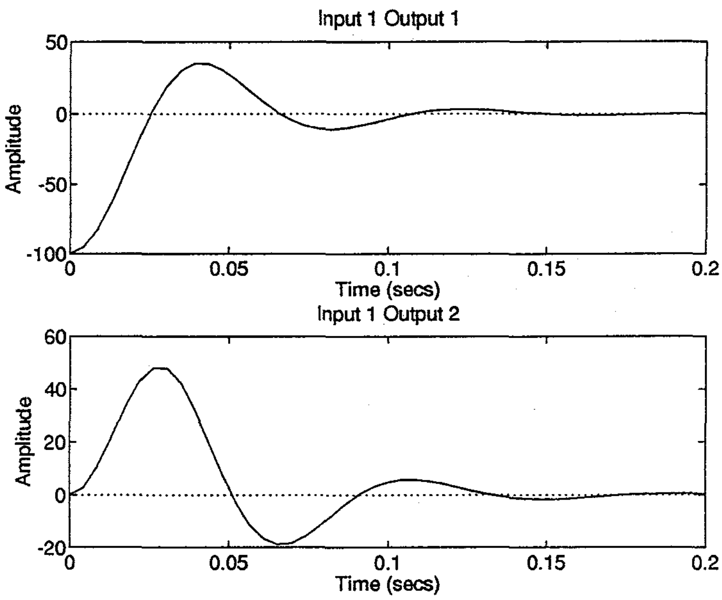

Fig. 3.8 Impulse responses (H_2/H_∞/root-clustering)

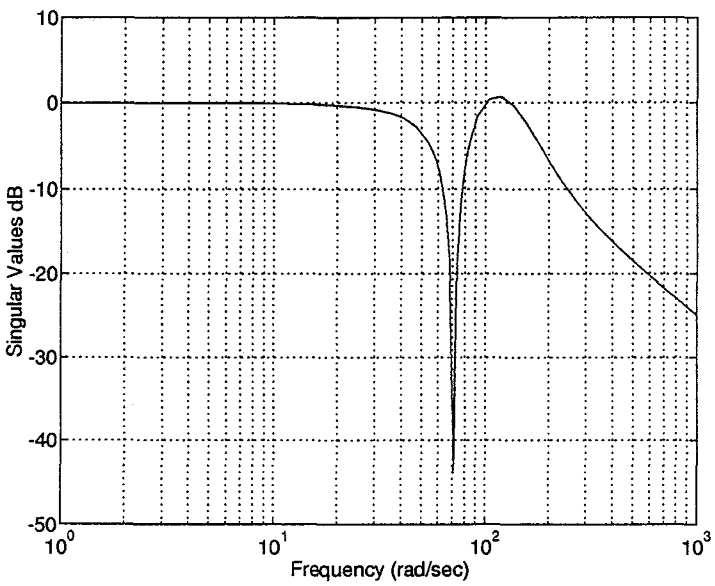

Fig. 3.9 Singular value plot (H_2/H_∞/root-clustering)

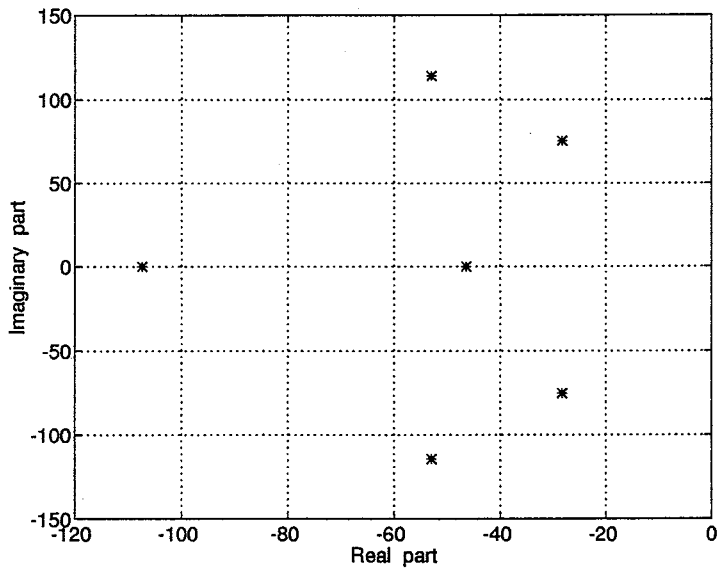

Fig. 3.10 Closed-loop poles (H_2/H_∞/root-clustering)

Chapter 4

Robust performance problems with structured uncertainties

4.1 Introduction

In Section 3.5, we mentioned just a little about robust performance specifications described via LMIs in $\mathcal{L}$. However, we did not consider expression of uncertainties so far. In this chapter, we will formulate robust performance problems for plants that have uncertainties in coefficient matrices of a state-space realization, that is, so-called structured uncertainties. Robust controller synthesis problems for plants with such uncertainties are investigated through quadratic stabilization approaches. Though robust controller synthesis based on quadratic stability can cause conservatism in designing controllers, this approach has been derived fruitful results for robust performance problems.

We will consider two types of structured uncertainties. One is norm-bounded uncertainty, represented the maximum singular value of matrices that describe uncertainties of plants. Petersen [Pet87] solved the quadratic stabilization problem with norm-bounded uncertainties reducing the quadratic stability condition to a Riccati equation condition. Quadratic stabilization with an H_∞-norm condition was solved by Xie et al. [XFS92]. Asai et al. [AH92, AH95] and Yamamoto et al. [YK93, YK94, YK96] proposed more general representations of norm-bounded uncertainties. Those problems are reduced to constant scaling H_∞ problems, which are represented in terms of LMIs but not convex problem with respect to the scaling parameters for output-feedback synthesis [YFH94].

The other type of uncertainty that we will treat here is polytopic uncertainty, which is represented by matrix polytopes in a state space realization. There are a number of results in robust state-feedback controller design via convex optimization with this type of uncertainty [OK89, GPB91, OMS93b, GPS93a, OMS94, MOS94b], while there have been no convex formulations of output-feedback problems: Kadoya et al. [KKOS93] and Geromel et al. [GPS93b, PGS93] proposed output-feedback robust controller design algorithms for polytopic uncertain systems, but global convergence of their algorithms is not guaranteed. Though it is difficult to handle polytopic uncertainty, it has advantage in representing parametric uncertainties exactly.

The purpose of this chapter is to solve robust performance problems described via LMIs of the class $\mathcal{L}$ by applying the parametrization of Chapter 2. The results here extend previous results in the area of robust performance synthesis with quadratic stability via output-feedback to wider variety of LMIs representing various specifications. In norm-bounded uncertainty case, we reduce synthesis problems to solving an inequality on the parameter set with scaling parameters. We show that the algorithm proposed by Yamada et al. [YHF95] for the constant scaling H_∞-control problem is applicable to synthesis with a larger class of LMI-conditions. On the other hand, a new alternation algorithm is proposed for robust synthesis with polytopic uncertain plants.

Section 4.2 gives the problem formulation of robust performance problems with structured uncertainties. Norm-bounded uncertainties and polytopic uncertainties are treated in Section 4.3 and 4.4, respectively.

4.2 LMI-synthesis with quadratic stability

First, we give the definition of quadratic stability.

Definition 4.1 [Pet87] Let $\mathcal{M}$ be a compact subset of $\mathbf{R}^{n\times n}$ and $M(t) \in \mathcal{M}$ be a function of time t. Consider the following linear time-variant system:

$$sx(t) = M(t)x(t), \quad M(t) \in \mathcal{M}. \tag{4.1}$$

For continuous-time systems, we say that the system (4.1) is *quadratically stable* if there exists a matrix $P \in PD(n)$ and a scalar $\alpha > 0$ that satisfy

$$\begin{cases} x^T(M^TP + PM)x \le -\alpha\|x\|^2, & \text{for continuous-time systems,} \\ x^T(M^TPM - P)x \le -\alpha\|x\|^2, & \text{for discrete-time systems,} \end{cases} \tag{4.2}$$

for $\forall M \in \mathcal{M}$. □

If (4.2) holds, $x^T Px$ is a Lyapunov function of the system (4.1), and then (4.1) is asymptotically stable for any function $M(t)$ whose value belongs in $\mathcal{M}$ for any t. Note that (4.2) is equivalent to

$$\Phi_{\text{Lyap}}(P, MP) > 0, \ \forall M \in \mathcal{M}. \tag{4.3}$$

Let us denote by $\Sigma_{cl}(K, \Delta)$ a control system that consists of a controller K and an uncertain plant whose uncertainties are represented Δ. Suppose that Δ belongs to a certain compact set $\mathcal{U}$. We formulate a robust controller synthesis problem, say QS-LMI-synthesis problem, based on the quadratic stability and LMIs $\Phi \in \mathcal{L}$ as follows:

QS-LMI-synthesis problem:

- *Feasibility problem.* Given $\Phi(\bullet)$, find $P \in PD(n + n_c)$ and a controller K of some order n_c satisfying $\Phi(P, \Sigma_{cl}(K, \Delta)) > 0$ for all $\Delta \in \mathcal{U}$.

- *Minimization problem.* Let $\Phi(\bullet)$ be parametric to a scalar γ and assume that $\Phi(P, \Sigma; \gamma_1) > \Phi(P, \Sigma; \gamma_2)$ holds for any $\gamma_1 > \gamma_2$, $P \in PD(n_s)$ and $\Sigma \in \mathbf{\Sigma}$. Minimize γ subject to $\Phi(P, \Sigma_{cl}(K, \Delta); \gamma) > 0$ for some $P \in PD(n + n_c)$ and a controller K of some order n_c and all $\Delta \in \mathcal{U}$.

LMI-conditions used in control theory usually imply internal stability, i.e., $\Phi(\bullet) > 0$ induces $\Phi_{\text{Lyap}}(\bullet) > 0$. Hence, the condition $\Phi(P, \Sigma_{cl}(K, \Delta)) > 0$, $\forall \Delta \in \mathcal{U}$ guarantees the quadratic stability of $\Sigma_{cl}(K, \Delta)$.

For any LMI-condition $\Phi(P, \Sigma) > 0$ shown in the previous chapter, the inequality condition $\Phi(P, \Sigma_{cl}(K, \Delta)) > 0$, $\exists P > 0$, $\forall \Delta \in \mathcal{U}$ at least guarantees that the closed-loop system $\Sigma_{cl}(K, \Delta)$ satisfies the specification represented by Φ for all *fixed* $\Delta \in \mathcal{U}$ as well as the quadratic stability. On the other hand, $\Phi_{H_\infty(1)}(P, \Sigma(K, \Delta); \gamma) > 0$ guarantees that, for any $\Delta(t)$ that varies in $\mathcal{U}$, $\|z_1\|_2$ (L_2-norm of $z_1(t)$) is less than γ for any $w_1(t)$ with $\|w_1\|_2 < 1$. This fact is proved in [XFS92], and the discrete-time version of this L_2 gain condition is found in [SFX93].

4.3 Robust performance problems with norm-bounded uncertainties

Consider the following linear time-invariant system that has uncertainties, denoted by Δ, in its coefficient matrices:

$$E(\Delta)sx(t) = A(\Delta)x(t) + B(\Delta)u(t) + \sum_{i=1}^{N_s} B_i(\Delta)w_i(t), \tag{4.4a}$$

$$y_i(t) = Cx(t) + Du(t) + \sum_{i=1}^{N_s} N_i w_i(t), \tag{4.4b}$$

$$z_i(t) = C_i x(t) + H_i u(t) + D_i w_i(t),\ i = 1, 2, \ldots, N_s, \tag{4.4c}$$

where $x(t) \in \mathbf{R}^n$ is the state vector, $u(t) \in \mathbf{R}^m$ is the control input vector, $y(t) \in \mathbf{R}^p$ is the measurement vector, $w_i(t) \in \mathbf{R}^{m_i}, i = 1, 2, \ldots, N_s$, are the exogenous-input vectors and $z_i(t) \in \mathbf{R}^{p_i}, i = 1, 2, \ldots, N_s$, are the controlled-output vectors. The size of $E(\Delta)$ is $n \times n$, and the other matrices have appropriate sizes. We consider the following uncertainties of the coefficient matrices:

$$\begin{aligned} &[\, E(\Delta) \quad A(\Delta) \quad B(\Delta) \quad B_1(\Delta) \quad \cdots \quad B_{N_s}(\Delta) \,] \\ &= [\, E \quad A \quad B \quad B_1 \quad \cdots \quad B_{N_s} \,] \\ &\quad + \tilde{B}_x \Delta (I - \tilde{H}\Delta)^{-1} [\, \tilde{C}_d \quad \tilde{C}_x \quad \tilde{C}_u \quad \tilde{C}_{(1)} \quad \cdots \quad \tilde{C}_{(N_s)} \,], \end{aligned} \tag{4.5}$$

and let Δ to the following set:

$$\mathcal{U}_{nb} = \{\Delta \in \mathbf{R}^{d_r \times d_c} | \sigma_{\max}(\Delta) \le 1\}. \tag{4.6}$$

We assume that $E(\Delta)$ is regular for all $\Delta \in \mathcal{U}_{nb}$.

This form of norm-bounded uncertainties is more useful than that of the first paper of quadratic stabilization [Pet87] in two points. First, uncertainties are described in descriptor form [AH92, AH95], which is suitable to represent uncertainties of physical systems, such as:

$$\hat{M}\ddot{\xi} + \hat{D}\dot{\xi} + \hat{K}\xi = \hat{B}u$$

with uncertainties in the coefficients $\hat{M}$, $\hat{D}$, $\hat{K}$ and $\hat{B}$. This differential equation is rewritten by

$$\begin{bmatrix} \hat{M} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \ddot{\xi} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} -\hat{D} & -\hat{K} \\ I & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi} \\ \xi \end{bmatrix} + \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix} u.$$

The other point is the form $\Delta(I - \tilde{H}\Delta)$, which Yamamoto et al. [YK93, YK94, YK96] first proposed. This form can offer more exact approximation of complex parametric uncertainties in coefficient matrices than the simple form of Δ. There can be other ways to describe norm-bounded uncertain systems, some of which represent uncertainties of more parts of the plant. However, we consider uncertainties (4.5) to keep simplicity; our main purpose of this section is to apply existing methods of quadratic stabilization (with H_∞-control) to LMIs that have never used for robust controller synthesis with norm-bounded uncertainties. Lastly, as in the previous chapter, we assume $D = 0$ without loss of generality.

The following lemma gives a simpler representation of the same uncertainties defined in (4.5) and (4.6).

Lemma 4.1 Assume that $\sigma_{\max}(\tilde{H}) < 1$ and $\sigma_{\max}(\Xi_H) < 1$, where

$$\Xi_H := \tilde{H} - \tilde{C}_d E^{-1} \tilde{B}_x.$$

Define the following set:

$$\mathcal{E} := \{(E^{-1}(\Delta)A(\Delta), E^{-1}(\Delta)B(\Delta), E^{-1}(\Delta)B_i(\Delta), i = 1, 2, \ldots, N_s) | \Delta \in \mathcal{U}_{nb}\}.$$

Then

$$\mathcal{E} = \{(A^* + \tilde{B}_x^* \Delta^* \tilde{C}_x^*, B^* + \tilde{B}_x^* \Delta^* \tilde{C}_u^*, B_i^* + \tilde{B}_x^* \Delta^* \tilde{C}_{(i)}^*, i = 1, 2, \ldots, N_s) | \Delta^* \in \mathcal{U}_{nb}\} \quad (4.7)$$

holds, where

$$\begin{aligned}
J_r &:= (I - \Xi_H^T \Xi_H)^{-\frac{1}{2}}, \\
J_c &:= (I - \Xi_H \Xi_H^T)^{-\frac{1}{2}}, \\
\Xi_x &:= \tilde{C}_x - \tilde{C}_d E^{-1} A, \\
\Xi_u &:= \tilde{C}_u - \tilde{C}_d E^{-1} B, \\
\Xi_{(i)} &:= \tilde{C}_{(i)} - \tilde{C}_d E^{-1} B_i, \\
A^* &:= E^{-1}(A + \tilde{B}_x J_r \Xi_H^T J_c \Xi_x), \\
B^* &:= E^{-1}(B + \tilde{B}_x J_r \Xi_H^T J_c \Xi_u), \\
B_i^* &:= E^{-1}(B + \tilde{B}_x J_r \Xi_H^T J_c \Xi_{(i)}), \\
\tilde{B}_x^* &:= E^{-1} \tilde{B}_x J_r, \\
\tilde{C}_x^* &:= J_c \Xi_x, \\
\tilde{C}_u^* &:= J_c \Xi_u, \\
\tilde{C}_{(i)}^* &:= J_c \Xi_{(i)}.
\end{aligned} \quad (4.8)$$

Proof. See Appendix.

This lemma reduces the uncertain plant (4.4) to a special case of itself with $E(\Delta) \equiv I$ and $\tilde{H} = 0$:

$$sx(t) = A(\Delta)x(t) + B(\Delta)u(t) + \sum_{i=1}^{N_s} B_i(\Delta)w_i(t), \tag{4.9a}$$

$$y_i(t) = Cx(t) + Du(t) + \sum_{i=1}^{N_s} N_i w_i(t), \tag{4.9b}$$

$$z_i(t) = C_i x(t) + H_i u(t) + D_i w_i(t),\ i = 1, 2, \ldots, N_s. \tag{4.9c}$$

where we have replaced A^*, B^*, B_i^* with A, B, B_i, respectively. Also in the following we drop stars. The uncertainties of the new coefficient matrices are represented by:

$$\begin{aligned} &[\,A(\Delta) \quad B(\Delta) \quad B_1(\Delta) \quad \cdots \quad B_{N_s}(\Delta)\,] \\ &= [\,A \quad B \quad B_1 \quad \cdots \quad B_{N_s}\,] + \tilde{B}_x \Delta\, [\,\tilde{C}_x \quad \tilde{C}_u \quad \tilde{C}_1 \quad \cdots \quad \tilde{C}_{N_s}\,], \\ &\Delta \in \mathcal{U}_{nb}. \end{aligned} \tag{4.10}$$

We consider exogenous-input and controlled-output pairs only from w_i to z_j in closed-loop systems with a controller (2.2), and give a realization of closed-loop systems as follows:

$$sx_{cl}(t) = (A_{cl}(K) + \tilde{B}_{cl}\Delta\Gamma_0(K))x_{cl}(t) + (B_{cl(i)}(K) + \tilde{B}_{cl}\Delta\Gamma_i(K))w_i(t), \tag{4.11a}$$

$$z_i(t) = C_{cl(i)}(K)x_{cl}(t) + D_{cl(i)}(K)w_i(t), \tag{4.11b}$$

$i = 1, 2, \ldots, N_s$, where $x_{cl}(t) = [x^T(t) \;\; x_c^T(t)]^T$ and

$$\tilde{B}_{cl} := \begin{bmatrix} \tilde{B}_x \\ 0 \end{bmatrix}, \tag{4.12a}$$

$$\Gamma_0(K) := [\,\tilde{C}_x + \tilde{C}_u D_c C \quad \tilde{C}_u C_c\,], \tag{4.12b}$$

$$\Gamma_i(K) := [\,\tilde{C}_{(i)} + \tilde{C}_u D_c C \quad \tilde{C}_u N_i\,]. \tag{4.12c}$$

We denote (4.11) by $\Sigma_{nb}(K, \Delta)$.

First, we consider $\Phi \in \mathcal{L}_M$. From the structure of LMIs in $\mathcal{L}_M$, any $\Phi \in \mathcal{L}_M$ has the following representation of $\Phi(P, \Sigma_{nb}(K, \Delta))$:

$$\begin{aligned} \Phi(P, \Sigma_{nb}(K,\Delta)) = \Phi(P, \Sigma_{cl}(K)) &\\ + \Phi_1 P \tilde{B}_{cl} \Delta\, [\,\Gamma_0(K) \quad \Gamma_1(K) \quad \cdots \Gamma_{N_s}(K)\,]\, \Phi_2 &\\ + \Phi_2^T \begin{bmatrix} \Gamma_0^T(K) \\ \Gamma_1^T(K) \\ \vdots \\ \Gamma_{N_s}^T(K) \end{bmatrix} \Delta^T \tilde{B}_{cl}^T \Phi_1^T, & \end{aligned} \tag{4.13}$$

where $\Sigma_{cl}(K)$ refers to the realization (3.5). Thus the Petersen's lemma [Pet87], shown below, is applicable to the inequality $\Phi(P, \Sigma_{nb}(K, \Delta)) > 0$:

Lemma 4.2 [Pet87] Let $\hat{Q} \in PD(N)$, $\hat{B} \in \mathbf{R}^{N \times d_r}$ and $\hat{C} \in \mathbf{R}^{d_c \times N}$. Then $\hat{Q} + \hat{B}\Delta\hat{C} + \hat{C}^T\Delta^T\hat{B}^T > 0$ holds for all $\Delta \in \mathcal{U}_{nb}$ if and only if $\hat{Q} + \varepsilon\hat{B}\hat{B}^T + \frac{1}{\varepsilon}\hat{C}^T\hat{C} > 0$ for some $\varepsilon > 0$.

Proof. See [Pet87]. □

In the following, we reduce the LMI-conditions for plants with norm-bounded uncertainties to an LMI of $\boldsymbol{p}$ that does not have Δ with it. First, we show the result for the class $\mathcal{L}_S$

Lemma 4.3 Let $n_p \geq n$ and define the following functions:

$$M_{\tilde{B}x}(\boldsymbol{p}) := \begin{bmatrix} \tilde{B}_x^T \\ P_g\tilde{B}_x \\ 0_{(n_p-n)\times d_r} \end{bmatrix}, \tag{4.14a}$$

$$M_\Gamma(\boldsymbol{p}) := [\, M_{\Gamma 0}(\boldsymbol{p}) \quad M_{\Gamma 1}(\boldsymbol{p}) \quad \cdots \quad M_{\Gamma N_s}(\boldsymbol{p}) \,], \tag{4.14b}$$

$$M_{\Gamma 0}(\boldsymbol{p}) := [\, \tilde{C}_x P_f + \tilde{C}_u W_f \quad \tilde{C}_x \quad \tilde{C}_u W_{f2} \,], \tag{4.14c}$$

$$M_{\Gamma i}(\boldsymbol{p}) := [\, \tilde{C}_{(i)} P_f + \tilde{C}_u W_f \quad \tilde{C}_{(i)} \quad \tilde{C}_u W_{f2} \,],\ i = 1, 2, \ldots, N_s. \tag{4.14d}$$

Suppose that $\Phi \in \mathcal{L}_M$.

(1) There exist $P \in PD(n + n_c)$ and a controller K of some order n_c satisfying

$$\Phi(P, \Sigma_{nb}(K, \Delta)) > 0. \tag{4.15}$$

for all $\Delta \in \mathcal{U}_{nb}$ if and only if there exist $\boldsymbol{p} \in \mathcal{P}(n_p)$ and $q \in \mathrm{R}$ such that

$$\Phi^{*nb}(\boldsymbol{p}; q) := \begin{bmatrix} \Phi^*(\boldsymbol{p}) & \Phi_1 M_{\tilde{B}x}(\boldsymbol{p}) & \Phi_2^T M_\Gamma^T(\boldsymbol{p}) \\ M_{\tilde{B}x}^T(\boldsymbol{p})\Phi_1^T & qI_{d_r} & 0 \\ M_\Gamma(\boldsymbol{p})\Phi_2 & 0 & q^{-1}I_{d_c} \end{bmatrix} > 0. \tag{4.16}$$

(2) If $\boldsymbol{p}$ is a solution to (4.16), a controller satisfying (4.15) is given by $\mathrm{K_{map}}(\boldsymbol{p})$.

Proof. See Appendix. □

The above lemma is easily applied to the class $\mathcal{L}_{M\oplus}$:

Theorem 4.1 [MOS96c] Let $\Phi \in \mathcal{L}_{M\oplus}$. Then, from the definition, $\Phi = \Phi_1 \oplus \Phi_2 \oplus \cdots \oplus \Phi_{N_0}$ for $\exists\Phi_i \in \mathcal{L}_M$. Suppose that $N_0 > 1$.

(1) There exist $P \in PD(n+n_c)$ and a controller K of some order n_c satisfying

$$\Phi(P, \Sigma_{nb}(K, \Delta)) > 0 \tag{4.17}$$

for all $\Delta \in \mathcal{U}_{nb}$ if there exist $\boldsymbol{p} \in \mathcal{P}(n_p)$ and $q_j, j = 1, 2, \ldots, N_0$ such that

$$\Phi_j^{*nb}(\boldsymbol{p}; q_j) > 0,\ j = 1, 2, , \ldots, N_0, \tag{4.18}$$

where

$$\Phi_j^{*nb}(\boldsymbol{p}; q_j) := \begin{bmatrix} \Phi_j^*(\boldsymbol{p}) & \Phi_1 M_{\tilde{B}x}(\boldsymbol{p}) & \Phi_2^T M_\Gamma^T(\boldsymbol{p}) \\ M_{\tilde{B}x}^T(\boldsymbol{p})\Phi_1^T & q_j I_{d_r} & 0 \\ M_\Gamma(\boldsymbol{p})\Phi_2 & 0 & q_j^{-1} I_{d_c} \end{bmatrix} > 0. \tag{4.19}$$

(2) If $\boldsymbol{p}$ is a solution to (4.16), a controller satisfying (4.15) is given by $\mathrm{K}_{\mathrm{map}}(\boldsymbol{p})$.

Example 4.1 Let us consider the following QS-LMI-synthesis problem:

$$\begin{aligned} &\underset{P,R,K}{\text{Minimize}}\ \gamma \text{ subject to} \\ &\Phi_{H_2C(1)}(P, \Sigma_{nb}(K, \Delta); \{\gamma, R\}) \oplus \Phi_D(P, \Sigma_{nb}(K, \Delta); \{c, \mu\}) > 0, \\ &\text{for all } \Delta \in \mathcal{U}_{nb}. \end{aligned}$$

Theorem 4.1 provides an optimization problem whose optimum is an upper-bound of the optimum of the above problem:

$$\begin{aligned} &\underset{\boldsymbol{p}, R, q_1, q_2}{\text{Minimize}}\ \gamma \text{ subject to} \\ &\begin{bmatrix} -M_A(\boldsymbol{p}) - M_A^T(\boldsymbol{p}) & M_{B1}(\boldsymbol{p}) & M_{\tilde{B}x}(\boldsymbol{p}) & M_\Gamma^T(\boldsymbol{p}) \\ M_{B1}^T(\boldsymbol{p}) & \gamma I_{m_i} & 0 & 0 \\ M_{\tilde{B}x}^T(\boldsymbol{p}) & 0 & q_1 I_{d_r} & 0 \\ M_\Gamma & 0 & 0 & q_1^{-1} I_{d_c} \end{bmatrix} \\ &\oplus \begin{bmatrix} M_P(\boldsymbol{p}) & M_{C1}^T(\boldsymbol{p}) \\ M_{C1}(\boldsymbol{p}) & R \end{bmatrix} \oplus (\gamma - \mathrm{trace} R) \\ &\oplus \begin{bmatrix} \mu M_P(\boldsymbol{p}) & M_A^T(\boldsymbol{p}) + c M_P(\boldsymbol{p}) & 0 & M_\Gamma^T(\boldsymbol{p}) \\ M_A(\boldsymbol{p}) + c M_P(\boldsymbol{p}) & \mu M_P(\boldsymbol{p}) & M_{\tilde{B}x}(\boldsymbol{p}) & 0 \\ 0 & M_{\tilde{B}x}^T(\boldsymbol{p}) & q_2 I_{d_r} & 0 \\ M_\Gamma(\boldsymbol{p}) & 0 & 0 & q_2^{-1} I_{d_c} \end{bmatrix} > 0. \end{aligned}$$

□

The derived inequality (4.16) (or (4.19)) is not convex with respect to $(\boldsymbol{p}, q)$ (or $(\boldsymbol{p}, (q_1, q_2, \ldots, q_{N_0}))$). Yamada et al. [YHF95] proposed an algorithm for an inequality of this type arising in a constantly scaled H_∞-control problem, which belongs to $\mathcal{L}_S$. Below we show a basic idea to solve (4.19) for any LMI in $\mathcal{L}_{M\oplus}$ by modifying and applying Yamada's algorithm.

First, it is easily checked that (4.19) is equivalent to

$$\hat{\Phi}_j^{*nb}(\boldsymbol{p}; \{q_j, r_j\}) := \begin{bmatrix} \Phi_j^*(\boldsymbol{p}) & M_{\tilde{B}x}(\boldsymbol{p})\Phi_1 & \Phi_2^T M_\Gamma^T(\boldsymbol{p}) \\ M_{\tilde{B}x}^T(\boldsymbol{p})\Phi_1^T & q_j I_{d_r} & 0 \\ M_\Gamma(\boldsymbol{p})\Phi_2 & 0 & r_j I_{d_c} \end{bmatrix} > 0, \tag{4.20a}$$

$$q_j > 0, r_j > 0, q_j r_j < 1. \tag{4.20b}$$

Thus the inequality condition for $\boldsymbol{p}$ in (4.18) is separated into the convex part: $\hat{\Phi}_j^{*nb}(\boldsymbol{p}; \{q_j, r_j\}) > 0, j = 1, 2, \ldots, N_0$ and nonconvex part: $q_j r_j < 1, j = 1, 2, \ldots, N_0$. Next, the set

$$\mathcal{B} = \{(q_1, r_1, \ldots, q_{N_0}, r_{N_0}) | q_j > 0, r_j > 0, q_j r_j < 1\}$$

is represented by

$$\mathcal{B} = \bigcup_{0<\lambda_j<\infty} \{(q_1, r_1, \ldots, q_{N_0}, r_{N_0}) | q_j > 0, r_j > 0, \lambda_j r_j + \frac{1}{\lambda_j} q_j < 1, j = 1, \ldots, N_0\},$$

which is an infinite union of convex sets. Consider the following finite approximation of the set $\mathcal{B}$:

$$\hat{\mathcal{B}} = \bigcup_{1\le k\le N_\lambda} \hat{\mathcal{B}}_k,$$

$$\hat{\mathcal{B}}_k := \{(q_1, r_1, \ldots, q_{N_0}, r_{N_0}) | q_j > 0, r_j > 0, \lambda_{j(k)} r_j + \frac{1}{\lambda_{j(k)}} q_j < 1, j = 1, \ldots, N_0\},$$

for appropriate $\lambda_{j(k)} > 0$. The set $\hat{\mathcal{B}}$ provides an approximated problem:

> Find $\boldsymbol{p}$ such that
> $\hat{\Phi}_j^{*nb}(\boldsymbol{p}; \{q_j, r_j\}) > 0, \lambda_{j(k)} r_j + \frac{1}{\lambda_{j(k)}} q_j < 1, j = 1, \ldots, N_0,$
> for some k,

which is solved at most through N_λ times of convex optimizations.

In [YHF95], a method to generate $\lambda_{j(k)}$ and worst case computational complexity are shown for the constantly scaled H_∞-control problem. Though a generation technique of $\lambda_{j(k)}$ for LMIs in $\mathcal{L}_{M\oplus}$ and computational complexity are still open, a similar result is expected.

4.4 Robust performance problems with polytopic uncertainties

In this section, we consider a different type of structured uncertainties. Consider the following plant:

$$E(\eta)sx(t) = A(\theta)x(t) + B(\theta)u(t) + \sum_{i=1}^{N_s} B_i(\theta)w_i(t), \tag{4.21a}$$

$$y(t) = C(\theta)x(t) + Du(t) + \sum_{i=1}^{N_s} N_i(\theta)w_i(t), \tag{4.21b}$$

$$z_i(t) = C_i(\theta)x(t) + H_i(\theta)u(t) + D_i(\theta)w_i(t),\ i = 1, 2, \ldots, N_s, \tag{4.21c}$$

where variables x, u, y, w_i, z_i are the same as in (4.4). The coefficient matrices are defined as follows:

$$\begin{aligned}
E(\eta) &= \sum_{j=1}^{n_E} \eta_l E_{(l)},\ E(\eta) \in \mathbf{R}^{n\times n},\\
A(\theta) &= \sum_{k=1}^{n_A} \theta_k A_{(k)},\\
B(\theta) &= \sum_{k=1}^{n_A} \theta_k B_{(k)},\\
C(\theta) &= \sum_{k=1}^{n_A} \theta_k C_{(k)},\\
B_i(\theta) &= \sum_{k=1}^{n_A} \theta_k B_{i(k)},\ i = 1, 2, \ldots, N_s,\\
N_i(\theta) &= \sum_{k=1}^{n_A} \theta_k N_{i(k)},\ i = 1, 2, \ldots, N_s,\\
C_i(\theta) &= \sum_{k=1}^{n_A} \theta_k C_{i(k)},\ i = 1, 2, \ldots, N_s,\\
H_i(\theta) &= \sum_{k=1}^{n_A} \theta_k H_{i(k)},\ i = 1, 2, \ldots, N_s,\\
D_i(\theta) &= \sum_{k=1}^{n_A} \theta_k D_{i(k)},\ i = 1, 2, \ldots, N_s,
\end{aligned}$$

where

$$\eta = (\eta_1, \eta_2, \ldots, \eta_{n_E}) \in \Omega^{n_E}, \tag{4.22a}$$

$$\theta = (\theta_1, \theta_2, \ldots, \theta_{n_A}) \in \Omega^{n_A}, \tag{4.22b}$$

$$\Omega^{n_0} := \{(v_1, v_2, \ldots, v_{n_0}) | \sum_{j=1}^{n_0} v_j = 1,\ v_j \geq 0, j = 1, 2, \ldots, n_0\}. \tag{4.22c}$$

Let $\eta_0 \in \Omega^{n_E}$ and $\theta_0 \in \Omega^{n_A}$ correspond to the nominal plant of (4.21) and denote the nominal coefficients by $E = E(\eta_0)$, $A = A(\theta_0)$, $B = B(\theta_0)$ and so forth. We assume that $D = 0$ and that $E(\eta)$ is nonsingular for all $\eta \in \Omega^{n_E}$. We call $E_{(l)}, A_{(k)}, B_{(k)}$, etc. vertex matrices.

Next, we show a realization of the closed-loop system with the controller (2.2) with assuming $D_c = 0$ and $n_c = n$ for simplicity:

$$sx_{cl}(t) = A_{cl}(K,\eta,\theta)x_{cl}(t) + B_{cl(i)}(K,\eta,\theta)w_i(t),\ i = 1,2,\ldots,N_s, \qquad (4.23a)$$

$$z_i(t) = C_{cl(i)}(K,\theta)x_{cl}(t) + D_{cl(i)}(K,\theta)w_i(t),\ i = 1,2,\ldots,N_s, \qquad (4.23b)$$

where $x_{cl}(t) = [x^T(t)\ \ x_c^T(t)]^T$ and

$$A_{cl}(K,\eta,\theta) := \begin{bmatrix} E^{-1}(\eta)A(\theta) & E^{-1}(\eta)B(\theta)C_c \\ B_cC(\theta) & A_c \end{bmatrix}, \qquad (4.24a)$$

$$B_{cl(i)}(K,\eta,\theta) := \begin{bmatrix} E^{-1}(\eta)B_i(\theta) \\ B_cN_i(\theta) \end{bmatrix}, \qquad (4.24b)$$

$$C_{cl(i)}(K,\theta) := [\,C_i(\theta)\quad H_i(\theta)C_c\,], \qquad (4.24c)$$

$$D_{cl(i)}(K,\theta) := D_i(\theta). \qquad (4.24d)$$

Denote the closed-loop system (4.23) by $\Sigma_p(K,\theta,\eta)$. We also prepare notation for vertexes of the uncertainties:

$$sx_{cl}(t) = A_{cl}(K;[j,k])x_{cl}(t) + B_{cl(i)}(K;[j,k])w_i(t),\ i = 1,2,\ldots,N_s, \qquad (4.25a)$$

$$z_i(t) = C_{cl(i)}(K;[k])x_{cl}(t) + D_{cl(i)}(K;[k])w_i(t),\ i = 1,2,\ldots,N_s, \qquad (4.25b)$$

where

$$A_{cl}(K;[j,k]) := \begin{bmatrix} E_{(j)}^{-1}A_{(k)} & E_{(j)}^{-1}B_{(k)}C_c \\ B_cC_{(k)} & A_c \end{bmatrix}, \qquad (4.26a)$$

$$B_{cl(i)}(K;[j,k]) := \begin{bmatrix} E_{(j)}^{-1}B_{i(k)} \\ B_cN_{i(k)} \end{bmatrix}, \qquad (4.26b)$$

$$C_{cl(i)}(K;[k]) := [\,C_{i(k)}\quad H_{i(k)}C_c\,], \qquad (4.26c)$$

$$D_{cl(i)}(K;[k]) := D_{i(k)}. \qquad (4.26d)$$

Denote (4.25) by $\Sigma_p(K;[j,k])$.

Definition 4.2 For $i = 1,2,\ldots,N_s$, $j = 1,2,\ldots,n_E$ and $k = 1,2,\ldots,n_A$, define the following matrix-valued functions:

$$M_A(\boldsymbol{p};[j,k]) := \left[\begin{array}{c|c} E_{(j)}^{-1}(A_{(k)}P_f + B_{(k)}W_f) & E_{(j)}^{-1}A_{(k)} \\ \hline L + [\,P_g\quad W_g\,]\,Z_{jk}\begin{bmatrix} P_f \\ W_f \end{bmatrix} & P_gE_{(j)}^{-1}A_{(k)} + W_gC_{(k)} \end{array}\right], \qquad (4.27a)$$

$$M_{Bi}(\boldsymbol{p};[j,k]) := \begin{bmatrix} E_{(j)}^{-1}B_{i(k)} \\ P_g B_{i(k)} + W_g N_{i(k)} \end{bmatrix}, \tag{4.27b}$$

$$M_{Ci}(\boldsymbol{p};[k]) := \begin{bmatrix} C_{i(k)}P_f + H_{i(k)}W_f & C_{i(k)} \end{bmatrix}, \tag{4.27c}$$

$$M_{Di}(\boldsymbol{p};[k]) := D_{i(k)}, \tag{4.27d}$$

where

$$Z_{jk} := \begin{bmatrix} E_{(j)}^{-1}A_{(k)} - E^{-1}A & E_{(j)}^{-1}B_{(k)} - E^{-1}B \\ C_{(k)} - C & 0 \end{bmatrix}.$$

If $E(\eta)$ is constant, we can redefine $E(\eta) \equiv I$, and then replace '$[j,k]$' by '$[k]$' in the above notation.

The following theorem gives a necessary and sufficient condition described on the parameter set to QS-LMI-synthesis problems for uncertain control system $\Sigma_p(K,\eta,\theta)$.

Definition 4.3 For $\Phi \in \mathcal{L}$, define a symmetric-matrix-valued function $\Phi^{*p}(\boldsymbol{p};[j,k])$ with the following replacements in the definition of Φ:

$$\begin{aligned} P &\to M_P(\boldsymbol{p}), \\ P\bar{A} &\to M_A(\boldsymbol{p};[j,k]), \\ P\bar{B}_i &\to M_{Bi}(\boldsymbol{p};[j,k]), \\ \bar{C}_i &\to M_{Ci}(\boldsymbol{p};[k]), \\ \bar{D}_i &\to M_{Di}(\boldsymbol{p};[k]). \end{aligned}$$

□

Theorem 4.2 [MOS94c]

(1) Suppose that $E(\eta) \equiv I$ and $\Phi \in \mathcal{L}$. Then there exist $P \in PD(2n)$ and a controller K of order n that have a realization of order n satisfying

$$\Phi(P, \Sigma_p(K,\theta)) > 0 \tag{4.28}$$

for all $\theta \in \Omega^{n_A}$ if and only if there exists $\boldsymbol{p} \in \mathcal{P}(n)$ such that

$$\Phi^{*p}(\boldsymbol{p};[k]) > 0,\ k = 1,2,\ldots,n_A. \tag{4.29}$$

(2) Suppose that $E(\eta)$ is not constant and $\Phi \in \mathcal{L}_{C\oplus}$. Then there exist $P \in PD(2n)$ and a controller K that have a realization of order n satisfying

$$\Phi(P, \Sigma_p(K,\eta,\theta)) > 0 \tag{4.30}$$

for all $(\eta,\theta) \in \Omega^{n_E} \times \Omega^{n_A}$ if and only if there exists $\boldsymbol{p} \in \mathcal{P}(n)$ such that

$$\Phi^{*p}(\boldsymbol{p};[j,k]) > 0,\ j = 1,2,\ldots,n_E, k = 1,2,\ldots,n_A. \tag{4.31}$$

A controller that satisfies (4.28) (resp. (4.30)) is given by $\mathrm{K_{map}}(\boldsymbol{p})$ from a parameter that solves (4.29) (resp. (4.31)). □

In contrast to norm-bounded uncertainties, we have a necessary and sufficient condition on the parameter set. The derived inequality on the parameter set still keeps nonconvexity. Later we show an algorithm to solve such nonconvex BMI and discuss some related issues.

In the following, we consider state-feedback solutions. Assume that the state of the plant (3.3) is available, i.e., $C(\theta) \equiv I$, $D = 0$, $N_i(\theta) \equiv 0$, $i = 1, \ldots, N_s$. The following corollary asserts that, if the state is available, there always exists a static state-feedback solution to any QS-LMI-synthesis problem solvable at all.

Definition 4.4 For $\Phi \in \mathcal{L}$, define a function $\Phi^{*p-sf}(\boldsymbol{p}; [j,k])$ by the following replacements of the elements in Φ:

$$\begin{aligned} P &\rightarrow P_f, \\ P\bar{A} &\rightarrow E_{(j)}^{-1}(A_{(k)}P_f + B_{(k)}W_f), \\ P\bar{B}_i &\rightarrow E_{(j)}^{-1}B_{k(i)}, \\ \bar{C}_i &\rightarrow C_{i(k)}P_f + H_{i(k)}W_f, \\ \bar{D}_i &\rightarrow D_{i(k)}. \end{aligned}$$

Corollary 4.1 If $E(\eta) \equiv I$, suppose $\Phi \in \mathcal{L}$, and otherwise suppose $\Phi \in \mathcal{L}_{C\oplus}$. Then the following statements (i) and (ii) are equivalent if the state is available in the plant (3.3):

(i) $\Phi(P, \Sigma_{cl}(K, \theta, \eta)) > 0$ for all $\eta \in \Omega^{r_E}$ and all $\theta \in \Omega_{r_A}$.

(ii) $\Phi^{*p-sf}(\boldsymbol{p}; [j,k]) > 0$, $j = 1, 2, \ldots, n_E, k = 1, 2, \ldots, n_A$.

If (ii) holds, $u(t) = W_f P_f^{-1} x(t)$ is a controller satisfying (i).

Proof. Similar to the proof of Corollary 3.1. □

In contrast to output-feedback case, the results of state-feedback synthesis get a convex formulation. As the state-feedback solution in Section 3.3, the existence of a static state-feedback solution is shown for any QS-LMI-synthesis problem solvable at all. The derived LMI-conditions coincide with those in [OK89, GPB91, OMS93b, GPS93a, OMS94, MOS94b].

In the rest of this section, we show an alternation algorithm to solve (4.29) or (4.31). First, we consider to get an initial parameter. The derived inequality (4.29) or (4.31) is not convex, containing the bilinear term:

$$L + [\,P_g \quad W_g\,]\, Z_{jk} \begin{bmatrix} P_f \\ W_f \end{bmatrix}.$$

On the other hand, the *state-feedback block* and *observer block* defined below is affine with respect to $\boldsymbol{p}$:

Definition 4.5 For Φ^{*p} in Definition 4.3, define Φ^{*p-f} and Φ^{*p-g} by

$$\Phi^{*pf-init}(P_f, W_f; [j,k]) := \Xi\left([\,I_n \quad 0\,]\right) \Phi^{*p}(\boldsymbol{p}; [j,k]) \Xi\left(\begin{bmatrix} I_n \\ 0 \end{bmatrix}\right), \tag{4.32a}$$

$$\Phi^{*pg-init}(P_g, W_g; [j,k]) := \Xi\left([\,0 \quad I_n\,]\right) \Phi^{*p}(\boldsymbol{p}; [j,k]) \Xi\left(\begin{bmatrix} 0 \\ I_n \end{bmatrix}\right). \tag{4.32b}$$

□

Note that the definition of $\Phi^{*pf-init}$ is the same as that of Φ^{*p-sf}. Both of the inequalities $\Phi^{*pf-init}(P_f, W_f) > 0$ and $\Phi^{*pg-init}(P_g, W_g) > 0$ are necessary for $\Phi^{*p}(\boldsymbol{p}) > 0$, and these two are convex inequalities with respect to (P_f, W_f) and (P_g, W_g), respectively.

Next, we define functions of $\boldsymbol{p}$ for alternation of convex optimizations.

Definition 4.6 For Φ^{*p} in Definition 4.3, define Φ^{*pF} and Φ^{*pG} by

$$\Phi^{*pF}(\boldsymbol{p}; [j,k]) := \Xi\left(\begin{bmatrix} I_n & 0 \\ 0 & P_g^{-1} \end{bmatrix}\right) \Phi^{*p}(\boldsymbol{p}; [j,k]) \Xi\left(\begin{bmatrix} I_n & 0 \\ 0 & P_g^{-1} \end{bmatrix}\right) \tag{4.33a}$$

$$\Phi^{*pG}(\boldsymbol{p}; [j,k]) := \Xi\left(\begin{bmatrix} P_f^{-1} & 0 \\ 0 & I_n \end{bmatrix}\right) \Phi^{*p}(\boldsymbol{p}; [j,k]) \Xi\left(\begin{bmatrix} P_f^{-1} & 0 \\ 0 & I_n \end{bmatrix}\right). \tag{4.33b}$$

□

Setting $\tilde{P}_f := P_f^{-1}$, $\tilde{P}_g := P_g^{-1}$, $\tilde{L}_f := LP_f^{-1}$, $\tilde{L}_g := P_g^{-1}L$, $F := W_f P_f^{-1}$ and $G := P_g^{-1} W_g$, we get the following representation of Φ^{*p-F} and Φ^{*p-G}:

$$\text{For } \Phi^{*pF}: \begin{cases} P \rightarrow \begin{bmatrix} P_f & \tilde{P}_g \\ \tilde{P}_g & \tilde{P}_g \end{bmatrix}, \\ P\bar{A} \rightarrow \left[\begin{array}{c|c} E_{(j)}^{-1}[\,A_{(k)} \quad B_{(k)}\,] \begin{bmatrix} P_f \\ W_f \end{bmatrix} & E_{(j)}^{-1} A_{(k)} \tilde{P}_g \\ \hline [\,I \quad G\,] Z_{jk} \begin{bmatrix} P_f \\ W_f \end{bmatrix} + \tilde{L}_g & [\,I \quad G\,] \begin{bmatrix} E_{(j)}^{-1} A_{(k)} \\ C_{(k)} \end{bmatrix} \tilde{P}_g \end{array}\right], \\ P\bar{B}_i \rightarrow \begin{bmatrix} E_{(j)}^{-1} B_{i(k)} \\ B_{i(k)} + G N_{i(k)} \end{bmatrix}, \\ \bar{C}_i \rightarrow [\,C_{i(k)} P_f + H_{i(k)} W_f \quad C_{i(k)\tilde{P}_g}\,], \\ \bar{D}_i \rightarrow D_{i(k)}. \end{cases}$$

For Φ^{*pG} :
$$\left\{\begin{array}{lcl} P & \to & \begin{bmatrix} \tilde{P}_f & \tilde{P}_f \\ \tilde{P}_f & P_g \end{bmatrix}, \\ P\bar{A} & \to & \left[\begin{array}{c|c} \tilde{P}_f E_{(j)}^{-1} [\, A_{(k)} \quad B_{(k)} \,] \begin{bmatrix} I \\ F \end{bmatrix} & \tilde{P}_f E_{(j)}^{-1} A_{(k)} \\ \hline [\, P_g \quad M_g \,] Z_{jk} \begin{bmatrix} I \\ F \end{bmatrix} + \tilde{L}_f & [\, P_g \quad M_g \,] \begin{bmatrix} E_{(j)}^{-1} A_{(k)} \\ C_{(k)} \end{bmatrix} \end{array}\right]. \\ P\bar{B}_i & \to & \begin{bmatrix} \tilde{P}_f E_{(j)}^{-1} B_{i(k)} \\ P_g B_{i(k)} + W_g N_{i(k)} \end{bmatrix}, \\ \bar{C}_i & \to & [\, C_{i(k)} + H_{i(k)} F \quad C_{i(k)} \,], \\ \bar{D}_i & \to & D_{i(k)}. \end{array}\right.$$

From the above relations, we see that Φ^{*pF} and Φ^{*pG} are affine with respect to $(P_f, \tilde{P}_g, W_f, \tilde{L}_g)$ and $(\tilde{P}_f, P_g, W_g, \tilde{L}_f)$, respectively.

Lastly, we give the following functions:

Definition 4.7

$$\Phi^{*pf-cnst}(P_f, F; [j,k]) := \Xi\left([\, I_n \quad 0 \,]\right) \Phi^{*pG}(\bullet) \Xi\left(\begin{bmatrix} I_n \\ 0 \end{bmatrix}\right), \tag{4.34a}$$

$$\Phi^{*pg-cnst}(P_g, G; [j,k]) := \Xi\left([\, 0 \quad I_n \,]\right) \Phi^{*pF}(\bullet) \Xi\left(\begin{bmatrix} 0 \\ I_n \end{bmatrix}\right). \tag{4.34b}$$

□

Using these functions and new variables, we give an algorithm to solve (4.31).

Algorithm 1 [MOS94c]

Step 1 Find P_f, P_g, W_f, W_g satisfying

$$\Phi^{*pf-init}(P_f, W_f; [j,k]) > 0, \ \Phi^{*pg-init}(P_g, W_g; [j,k]) > 0.$$

If there are no solutions, the inequality (4.31) is not solvable. Next, fix P_f, P_g, W_f, W_g and minimize the following function

$$\phi_0(L) := \max_{j,k}\{-\lambda_{\min}(\Phi^{*p}(L; [j,k]))\}.$$

Whenever $\phi_0 < 0$, $\mathrm{K}_{\mathrm{map}}(\boldsymbol{p})$ gives a satisfactory controller. If the minimum is not negative, go to *Step 2* or *Step 3*.

Step 2 Set $\tilde{P}_g := P_g^{-1}$, $\tilde{L}_g := P_g^{-1} L$, $G := P_g^{-1} W_g$ and fix G. Minimize:

$$\phi_g := \max_{j,k}\{-\lambda_{\min}(\Phi^{*pF}(P_f, \tilde{P}_g, W_f, \tilde{L}_g; [j,k]))\}$$

subject to $\Phi^{*pf-init}(P_f, W_f) > 0$ and $\Phi^{*pg-cnst}(\tilde{P}_g, G) > 0$. From the minimizing arguments, set $P_f := \tilde{P}_f^{-1}$, $L := \tilde{L}_f P_f$, $M_f := F P_f$. Whenever $\phi_g < 0$, $\mathrm{K}_{\mathrm{map}}(\boldsymbol{p})$ gives a satisfactory controller. If the value of ϕ_g did not decrease enough, stop the algorithm. Otherwise go to *Step 3*.

Step 3 Set $\tilde{P}_f := P_f^{-1}$, $\tilde{L}_f := L P_f^{-1}$, $F := W_f P_f^{-1}$, and fix F. Minimize:

$$\phi_f := \max_{j,k}\{-\lambda_{\min}(\Phi^{*pG}(\tilde{P}_f, P_g, W_g, \tilde{L}_f; [j,k]))\}$$

subject to $\Phi^{*pf}(\tilde{P}_f, F) > 0$ and $\Phi^{*pg-cnst}(P_g, W_g) > 0$. From the minimizing arguments, set $P_g := \tilde{P}_g^{-1}$, $L := P_g \tilde{L}_g$, and $W_g := P_g G$. Whenever $\phi_g < 0$, $\mathrm{K}_{\mathrm{map}}(\boldsymbol{p})$ gives a satisfactory controller. If the value of ϕ_g did not decrease enough, stop the algorithm. Otherwise go to *Step 2*.

This algorithm unfortunately has no proof of convergence to a global optimum. In the following section, we show a numerical example, where this algorithm solves a quadratic stabilization problem for a polytopic uncertain system. There is another algorithm that can solve quadratic stabilization problems with polytopic uncertainties, proposed by Asai et al. [AH94]. Their algorithm alternates approximation of polytopic uncertainties via norm-bounded uncertainties and H_∞-controller synthesis. However, their method has no proof of global convergence, and gets conservatism from approximating uncertainties. On the other hand, our direct method uses necessary and sufficient conditions to original inequalities.

In Algorithm 1, the sequence from Step 1 to Step 3 is similar to linear transfer recovery (LTR) design methods. Ohara et al. [OK89] showed the existence of an observer that recovers the quadratic stability and a robust root-clustering condition attained by a state-feedback gain for plants with polytopic uncertainties under certain assumptions including a minimum-phase condition. We expect a guarantee of convergence for plants with certain assumptions, but it is still open.

There are direct BMI approaches [SGL94, GTSPL94, GSP94] to controller synthesis, which aim to solve the original inequality $\Phi(P, \Sigma(K)) > 0$ or, if polytopic uncertainty is concerned, $\Phi(P, \Sigma_p(K; [j,k])) > 0$. These approaches use optimization methods such as the branch and bound method to find a global optimum of a nonconvex problem. In those algorithms, a certain LMI is solved in each iteration to get an upper or lower bound needed in branch and bound algorithms. Global convergence is confirmed in those algorithms, and they are applicable to our problems. However, the computational complexity and amount of memory needed are still large ever for recent computers.

4.5 Numerical Example (Quadratic stabilization of a polytopic uncertain system)

In this section, we show a numerical using the algorithm proposed in Section 4.4. Consider the following plant with uncertain parameters.

$$\begin{bmatrix} 0.01 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} -0.1 & 0 & -b \\ 0 & -0.001 & b \\ 1 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} x,$$

$$\begin{aligned} a &= 0.008\eta_1 + 0.01(1-\eta_1), \\ b &= 50\theta_1 + 60(1-\theta_1), \\ &\quad 0 \le \eta_1 \le 1,\ 0 \le \theta_1 \le 1. \end{aligned}$$

The eigenvalues of the plant sit around $\{-0.11,\ -0.05 \pm 100i\}$ over the uncertainty. The open-loop poles at the vertexes are shown in Fig.4.1.

Let us find a controller with which all the closed-loop poles have real parts less than -2 and the closed-loop system is quadratically stable. For this specification, we consider the following LMI-condition:

$$\Phi'_E(P, \Sigma_p(K, \eta_1, \theta_1), 2) > 0,\ \forall(\eta, \theta) \in \Omega^{n_E} \times \Omega^{n_A},$$

where Φ'_E belongs to $\mathcal{L}_C(\subset \mathcal{L}_{C\oplus})$ and hence the statement (2) of Theorem 4.2 is applicable.

To carry out Algorithm 1 that alternates solving LMIs, we used MATLAB version 4.0a, SP [VLB94] and MAKELMI [MOS96a]. The derived parameters are:

$$P_f = \begin{bmatrix} 382.2 & 166.8 & 22.19 \\ 166.8 & 165.0 & 15.07 \\ 22.19 & 15.07 & 9.983 \end{bmatrix},$$

$$W_f = [0.1164 \quad 0.2999 \quad -2.468] \times 10^3,$$

$$P_g = \begin{bmatrix} 0.0416 & 0.0175 & -0.0030 \\ 0.0175 & 0.0308 & -0.0231 \\ -0.0030 & -0.0231 & 0.2628 \end{bmatrix},$$
$$W_g = \begin{bmatrix} 0.0409 & -173.75 \\ 0.3286 & 39.74 \\ -2.9568 & -10.94 \end{bmatrix},$$
$$L = \begin{bmatrix} -0.5744 & -0.3022 & -1.4052 \\ 0.4104 & 0.3000 & -0.3702 \\ -1.1000 & -0.4993 & -0.1008 \end{bmatrix} \times 10^4,$$

and we get the controller as follows:

$$A_c = \begin{bmatrix} -354.4 & 253.8 & 3154 \\ -360.5 & 99.0 & 4352 \\ 192.9 & -250.7 & 196.2 \end{bmatrix},$$
$$B_c = \begin{bmatrix} 1.1 & 6317 \\ -3.0 & -5134 \\ 11.0 & -338.9 \end{bmatrix},$$
$$C_c = \begin{bmatrix} 53.40 & -25.69 & -586.9 \end{bmatrix}.$$

The closed-loop poles at vertexes, shown in Fig.4.2, have real parts less than −2, which meets the specification. In the following table, we show the feature of convex optimization for this example:

	Values of ϕ_f, ϕ_g	FLOPs †
Step 1	−	120635
Step 2	150 → −22.7	365762

† the number of the floating point operations.

The algorithm found a solution with 2 steps, and it worked out well.

4.6 Concluding remarks

In this chapter, we considered robust performance problems, named QS-LMI-synthesis problems, for two types of uncertainties. In both types, we applied the result of the previous chapter and proposed equivalent or guaranteeing matrix inequality conditions on the parameter set. We showed algorithms to solve those inequalities.

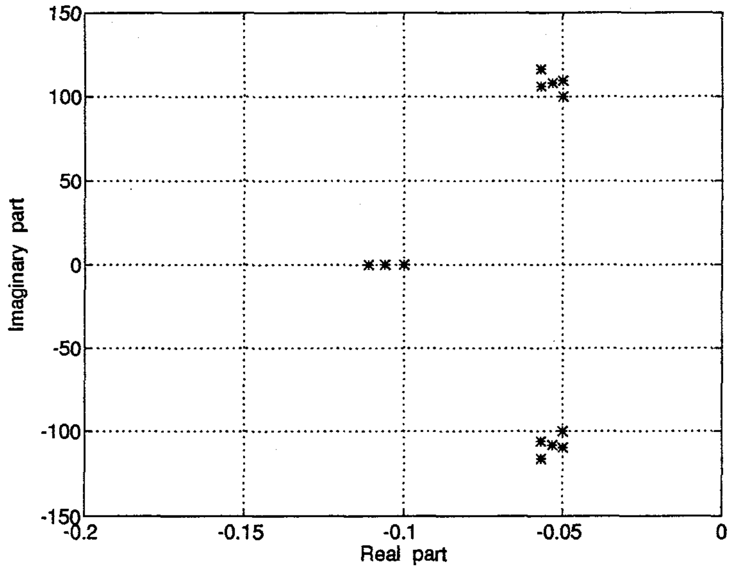

Fig. 4.1 Open-loop poles

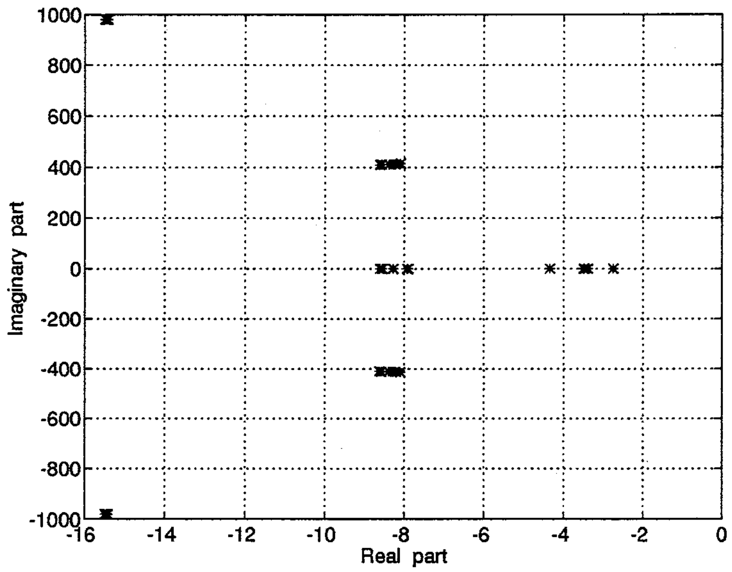

Fig. 4.2 Closed-loop poles

Chapter 5

Descriptor Systems

5.1 Introduction

The descriptor form ($E\dot{x} = Ax + \cdots$) is a natural representation of linear dynamical systems and makes it possible to analyze a larger class of systems than state space equations ($\dot{x} = Ax + \cdots$) do [Lew86, Shi95]. In particular, the descriptor form explicitly represents static constraints on physical variables and impulsive elements. A considerable number of notions and results in the control theory based on the state space representation have been generalized for descriptor systems, such as controllability and observability [Cob84b], pole-assignment, [Cob84a], LQ-problems [Cob83, Lew86, BL87, KM92] and Lyapunov equations [Lew86, TMK95].

In this chapter, we give several new results on analysis and synthesis for descriptor systems based on LMIs. Section 5.2 shows some examples of descriptor systems and gives definitions concerned with behaviors of descriptor systems. In Section 5.3, we propose algebraic inequality conditions for a stability generalized to descriptor systems and H_∞-norm conditions. Though there are some algebraic conditions for the generalized stability [Lew86, TMK94], our LMI-conditions need no assumptions, such as regularity, that those previous conditions required. Using those algebraic conditions, we give two results of controller synthesis for descriptor systems: state-feedback quadratic stabilization and output-feedback H_∞-control problem. In both results, we reduce the synthesis problems to LMIs of certain parameters.

5.2 Properties of descriptor systems

Let us consider a linear time-invariant descriptor system represented as follows:

$$Esx(t) = Ax(t) + Bu(t), \tag{5.1a}$$

$$y(t) = Cx(t) + Du(t), \tag{5.1b}$$

where $x(t) \in \mathbf{R}^n$ is the descriptor variable vector, $u(t) \in \mathbf{R}^m$ is the input vector, and $y(t) \in \mathbf{R}^p$ is the output vector. We assume that $E \in \mathbf{R}^{n\times n}$, i.e., E is square. If E is nonsingular, the equation (5.1a) is rewritten to the form: $sx(t) = E^{-1}Ax(t) + E^{-1}Bu(t)$. In contrast, the nonsingularity is not assumed here, and this makes it possible for descriptor systems to describe a larger class of dynamical systems than state-space systems.

Example 5.1 [Shi95]

- Let $x(t) \in \mathbf{R}^2$ and consider the following descriptor system:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} sx(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{5.2a}$$

$$y(t) = [0 \quad 1]\, x(t). \tag{5.2b}$$

  These equations are reduced to $y(t) = su(t)$, which represents a differentiator for continuous-time case ($su(t) = du(t)/dt$) and a forward time-shift operator for discrete-time case ($su(t) = u(t+1)$), respectively.

- Consider a dynamical system $sx(t) = Ax(t) + Bu(t)$ under a static constraint $0 = Cx(t) + Du(t)$. These two equations are assembled as:

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} sx(t) \\ su(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}. \tag{5.3}$$

  If $D = -I$ and C is a feedback gain, this equation represents a static state-feedback control system with control input $u(t) = Cx(t)$.

- The inverse of a state-space system is not always represented via state-space equation. In contrast, the inverse of every descriptor system is represented by a descriptor form; the inverse of (5.1) is:

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} sx(t) \\ su(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} + \begin{bmatrix} 0 \\ -I \end{bmatrix} y(t), \tag{5.4a}$$

$$u(t) = [0 \quad I] \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}. \tag{5.4b}$$

As shown in the above examples, descriptor systems can represent dynamics that state-space systems can not describe, such as differentiating elements and static constraints between the descriptor variable. However, we have to take care of the solvability of the differential equation (5.1a); in the following, we show some notions concerned with properties arising in singular (rank$E < n$) descriptor systems.

Definition 5.1

(I) [Lew86] A pencil $sE - A$ (or a pair (E, A)) is *regular* if $\det(sE - A)$ is not identically zero for $s \in \mathbf{C}$.

(II) [TMK94] For regular pencil $sE-A$, a complex number s that satisfies $\mathrm{rank}(sE-A) < n$ is said to be the *finite modes* of (E, A). Suppose that $Ev_1 = 0$. Then the infinite eigenvalues associated with the generalized principal vectors v_k satisfying $Ev_k = Av_{k-1}$, $k = 2, 3, \ldots$ are *impulsive modes* of (E, A).

(III) A pair (E, A) is *admissible* if it is regular and has neither impulsive modes nor unstable finite modes. □

If (E, A) is regular, the differential equation (5.1a) has a unique solution. In the third definition, we defined the property 'admissibility' as a generalization of stability for state-space systems. In control system synthesis with descriptor plants, the admissibility plays a role of the stability for state-space systems, as the minimum demand for control systems.

Lastly, if $sE - A$ is regular, the transfer function:

$$G(s) = C(sE - A)^{-1}B + D \tag{5.5}$$

is defined and we can consider input-output properties through $G(s)$.

5.3 LMI-conditions for descriptor systems

In this section, we present algebraic inequalities that characterize some properties of descriptor systems. From now on, we consider only continuous-time systems (Let $s = \frac{d}{dt}$).

Since the equations (5.1) is always rewritten as:

$$\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{\zeta}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \zeta(t) \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} u(t), \tag{5.6a}$$

$$y(t) = \begin{bmatrix} C & I \end{bmatrix} \begin{bmatrix} x(t) \\ \zeta(t) \end{bmatrix}, \tag{5.6b}$$

we can assume that $D = 0$ without loss of generality: the augmentation(5.6) does not change the finite modes and impulsive modes of the original descriptor system [VLK81].

5.3.1 Admissibility

Several algebraic conditions have been proposed for admissibility of descriptor systems.

Lemma 5.1 [Lew86] Let (E, A) be regular and (E, A, C) be observable [Lew86]. Then (E, A) is admissible if and only if there exists a positive definite solution P to

$$A^T PE + E^T PA + E^T C^T CE = 0 \tag{5.7}$$

and, if P_1 and P_2 are two such solutions, then $E^T P_1 E = E^T P_2 E$. □

Lemma 5.2 [TMK94] Suppose that the pencil $sE - A$ is regular and that (E, A, C) is impulse observable and finite dynamics detectable [TMK94]. Then (E, A) is admissible if and only if there exists a solution X to GLE:

$$\begin{aligned} E^T X = X^T E \geq 0, \\ A^T X + X^T A + C^T C = 0. \end{aligned}$$

□

In these lemmas, the pair (E, A) is assumed to be *regular*. However, since the regularity of the plant can be destroyed by feedback input, an algebraic condition that guarantees the regularity as well is important especially to develop controller synthesis theory using the descriptor form. Modifying the GLE condition in Lemma 5.2, we propose a matrix inequality condition equivalent to the admissibility of (E, A) without assuming that (E, A) is regular.

Theorem 5.1 [MOS94d, MKOS96] A pair (E, A) is admissible if and only if there exists $X \in \mathbf{R}^{n \times n}$ such that

$$E^T X = X^T E \geq 0, \tag{5.8a}$$

$$A^T X + X^T A < 0. \tag{5.8b}$$

Proof. See Appendix.

If (5.8) holds, A and X are nonsingular.

5.3.2 H_∞-norm condition with admissibility

The following theorem gives a generalized algebraic Riccati inequality (GARI) condition necessary and sufficient that $sE - A$ is admissible and that $||G||_\infty < \gamma$ for given $\gamma > 0$.

Theorem 5.2 [MKOS96] The pair (E, A) is admissible and $||G||_\infty < \gamma$ if and only if (I) there exists $X \in \mathbf{R}^{n\times n}$ such that

$$E^T X = X^T E \geq 0, \tag{5.9a}$$
$$A^T X + X^T A + C^T C + \frac{1}{\gamma^2} X^T B B^T X < 0, \tag{5.9b}$$

or (II) there exists $Y \in \mathbf{R}^{n\times n}$ such that

$$Y E^T = E Y^T \geq 0, \tag{5.10a}$$
$$Y A^T + A Y^T + B B^T + \frac{1}{\gamma^2} Y C^T C Y^T < 0. \tag{5.10b}$$

Proof. See Appendix. □

We easily get equivalent LMIs to (5.9) and (5.10):

Corollary 5.1 The pair (E, A) is admissible and $||G||_\infty < \gamma$ if and only if (I) there exists $\hat{Y} \in \mathbf{R}^{n\times n}$ such that

$$E^T \hat{Y} = \hat{Y}^T E \geq 0, \tag{5.11a}$$
$$\begin{bmatrix} -A^T\hat{Y} - \hat{Y}^T A & \hat{Y}^T B & C^T \\ B^T \hat{Y} & \gamma I_m & 0 \\ C & 0 & \gamma I_p \end{bmatrix} > 0, \tag{5.11b}$$

or (II) there exists $\hat{X} \in \mathbf{R}^{n\times n}$ such that

$$\hat{X} E^T = E \hat{X}^T \geq 0, \tag{5.12a}$$
$$\begin{bmatrix} -\hat{X}A^T - A\hat{X}^T & B & \hat{X}C^T \\ B^T & \gamma I_m & 0 \\ C\hat{X}^T & 0 & \gamma I_p \end{bmatrix} > 0. \tag{5.12b}$$

□

5.4 Some selected LMI-synthesis problems for descriptor systems

In this section, we propose LMI-based synthesis methods for descriptor systems using algebraic inequalities shown in the previous section. In Subsection 5.4.1, we show a state-feedback quadratic stabilization, while Subsection 5.4.2 proposes H_∞-controller design for descriptor systems. In Section 5.5, we will give a numerical example of descriptor H_∞-control for plants that have a differentiating element and integrating element.

5.4.1 Robust stabilization of descriptor systems

This section investigates stabilization and robust stabilization problems of descriptor systems. We assume here that the descriptor variable is available; we set $C = I$ and $D = 0$ in (5.1). Consider static state-feedback $u(t) = Fx(t)$. The closed-loop system is:

$$E\dot{x}(t) = (A + BF)x(t). \tag{5.13}$$

The following proposition gives an LMI-condition to get a controller that makes (5.13) admissible.

Proposition 5.1 [MOS94d] Let K be any regular matrix satisfying $EK^T = KE^T \geq 0$ and fix it. Then there exists F that makes $(E, A + BF)$ admissible if and only if there exist Y and M satisfying

$$(AY^T + BM) + (AY^T + BM)^T < 0, \tag{5.14a}$$

$$EY^T = YE^T \geq 0. \tag{5.14b}$$

Then there exists $\delta \geq 0$ such that $\tilde{Y} = Y + \delta K$ is a regular matrix and $F = M\tilde{Y}^{-T}$ makes $(E, A + BF)$ admissible.

Proof. See Appendix. □

Remark 5.1 (1) If Y is regular, $\delta = 0$ (i.e., $F = MY^{-T}$) gives a gain with which $(M, A+BF)$ is admissible. (2) One of such K used in Proposition 5.1 is derived from a singular value decomposition $E = U_E \Sigma_E V_E^T$ by setting $K = U_E V_E^T$.

Next, let us consider quadratic stabilization of uncertain descriptor systems given as follows:

$$E(\eta)\dot{x}(t) = A(\theta)x(t) + B(\theta)u(t), \tag{5.15}$$

where we assume that $E(\eta)$ is square and, in contrast to in Chapter 4, $E(\eta)$ is singular. Let the coefficients $E(\eta)$, $A(\theta)$ and $B(\theta)$ represent parametric uncertainties of plants, and we assume that the plant keeps its structure over the possible uncertainties, such as the number of exponential modes. We give the notation of $E(\eta)$, $A(\theta)$ and $B(\theta)$ for such uncertainties as follows:

$$E(\eta) = G(\eta)H, \qquad G(\eta) = \sum_{j=1}^{n_E} \eta_j G_{(j)}, \tag{5.16a}$$

$$A(\theta) = \sum_{k=1}^{n_A} \theta_k A_{(k)}, \qquad B(\theta) = \sum_{k=1}^{n_A} \theta_k B_{(k)}, \tag{5.16b}$$

$$\eta \in \Omega^{n_E}, \quad \theta \in \Omega^{n_A}. \tag{5.16c}$$

We assume that $G(\eta)$ is regular for any $\eta \in \Omega^{n_E}$, which implies $\mathrm{Ker}E(\eta) \equiv \mathrm{Ker}H$.

Theorem 5.3 [MOS94d] Let K be any regular matrix satisfying $HK^T = KH^T \geq 0$ and fix it. Suppose that there exist Y and M satisfying the following LMI for $j = 1, 2, \ldots, n_E$, $k = 1, 2, \ldots, n_A$:

$$(A_{(k)}Y^T + B_{(k)}M)G_{(j)}^T + G_{(j)}(A_{(k)}Y^T + B_{(k)}M)^T < 0, \tag{5.17a}$$

$$HY^T = YH^T \geq 0. \tag{5.17b}$$

Then there exists $\delta \geq 0$ such that $\tilde{Y} = Y + \delta K$ is a regular matrix and that $F = M\tilde{Y}^{-T}$ makes (5.15) admissible for all $(\eta, \theta) \in \Omega^{n_E} \times \Omega^{n_A}$. Further, the dynamic part of the closed-loop system is quadratically stable.

Proof. See Appendix. □

To make the closed-loop exponential modes less than $-\beta$ ($\beta > 0$), apply Proposition 5.1 and Theorem 5.3 with replacing A with $A + \beta E$ and $A_{(k)}$ with $A_{(k)} + \beta H$, respectively.

5.4.2 Descriptor H_∞-control problem via LMIs

There are several results on H_∞-control for descriptor systems. Morihira et al. [MTK93] and Takaba et al. [TMK94] generalized the J-spectral factorization [GGLD90] to solve the descriptor H_∞-control problem, while Wen et al. [WY93] applied the generalized eigenvalue problem [GLDKS91]. These results give H_∞-norm conditions for descriptor systems in terms of generalized algebraic Riccati equations and, as in state space H_∞ problems based on Riccati equations, require the assumption that descriptor plants have no $j\omega$-axis zeros and satisfy some rank conditions.

In this section, we solve the H_∞-control problem for descriptor systems using algebraic inequalities shown in Theorem 5.2 and Corollary 5.1. The H_∞-norm condition of closed-loop systems is reduced to an equivalent LMI-condition of two positive definite matrices, and we give a procedure to get an H_∞-suboptimal controller from a solution to the LMI. Though previous descriptor H_∞-synthesis methods [TMK94, WY93] assume several conditions on descriptor plants, our solution to be shown here does not need any prescribed restrictions on plants. Further, we show that, if the descriptor H_∞ problem is solvable, there always exists an H_∞-suboptimal controller whose order is at most the number of the finite modes of the plant.

Let us consider the following descriptor system:

$$E\dot{x}(t) = Ax + Bu + B_1 w_1(t), \tag{5.18a}$$

$$y(t) = Cx(t) + Du(t) + N_1 w_1(t), \tag{5.18b}$$

$$z_1(t) = C_1 x(t) + H_1 u(t) + D_1 w_1(t), \tag{5.18c}$$

where $x \in \mathbf{R}^n$ is the descriptor variable, $u \in \mathbf{R}^m$ is the control input, $y \in \mathbf{R}^p$ is the measured output $w_1 \in \mathbf{R}^{m_1}$ is the exogenous input, and $z_1 \in \mathbf{R}^{p_1}$ is the controlled output. The rank of the matrix $E \in \mathbf{R}^{n\times n}$ is $r(\leq n)$. Without loss of generality, we assume $D = 0, N_1 = 0, H_1 = 0$ and $D_1 = 0$. If this is not satisfied, we can rewrite the descriptor system as in the way of (5.6). Though such an augmentation of a descriptor system brings additional components in the descriptor variable, it does not increase the computational complexity of the LMI-based synthesis method shown later.

We represent a dynamic output-feedback controller as follows:

$$\hat{E}\dot{\xi}(t) = \hat{A}\xi(t) + \hat{B}y(t), \quad u(t) = \hat{C}\xi(t), \tag{5.19}$$

where $\xi(t) \in \mathbf{R}^{n_\xi}$ and $\hat{E} \in \mathbf{R}^{n_\xi \times n_\xi}$. Note that any proper controller, which can have a feed-through term, are represented in the form of (5.19) with an appropriate integer n_ξ. Later we will discuss the regularity and infinite modes of the controller (5.19). The closed-loop system is:

$$E_{cl}\dot{x}_{cl}(t) = A_{cl}x_{cl}(t) + B_{cl}w_1(t), \quad z_1(t) = C_{cl}x_{cl}(t), \tag{5.20}$$

where $x_{cl}(t) = [\,x^T(t) \quad \xi^T(t)\,]^T$ and

$$\begin{aligned} E_{cl} &:= \begin{bmatrix} E & 0 \\ 0 & \hat{E} \end{bmatrix}, \quad A_{cl} := \begin{bmatrix} A & B\hat{C} \\ \hat{B}C & \hat{A} \end{bmatrix}, \\ B_{cl} &:= \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \qquad C_{cl} := [\,C_1 \quad 0\,]. \end{aligned} \tag{5.21}$$

To give solutions, first we follow the result of [SMN90] and generalize it to descriptor systems, representing H_∞-norm conditions via Riccati inequalities (5.9),(5.10). From Theorem 5.2, the admissibility of the closed-loop system and the H_∞-norm condition $\|C_{cl}(sE_{cl}-A_{cl})^{-1}B_{cl}\|_\infty < \gamma$ is equivalent to the existence of $X_{cl} \in \mathbf{R}^{(n+n_\xi)\times(n+n_\xi)}$ such that

$$E_{cl}^T X_{cl} = X_{cl}^T E_{cl} \geq 0, \tag{5.22a}$$

$$\mathcal{G}_\gamma(X_{cl}; A_{cl}, B_{cl}, C_{cl}) < 0, \tag{5.22b}$$

where

$$\mathcal{G}_\gamma(X; A, B, C) := A^T X + X^T A + C^T C + \frac{1}{\gamma^2} X^T B B^T X.$$

Below we show an algebraic condition of the "state-feedback" and "observer" form, which is a generalization of the result of [SMN90]:

Lemma 5.3 [MKOS96]

(1) The following statements (I) and (II) are equivalent:

(I) Given $\gamma > 0$, there exists a controller of the form (5.19) such that the closed-loop system (5.20) is admissible and that $\|C_{cl}(sE_{cl} - A_{cl})^{-1}B_{cl}\|_\infty < \gamma$.

(II) (a) There exist $F \in \mathbf{R}^{m\times n}$ and $X \in \mathbf{R}^{n\times n}$ such that

$$E^T X = X^T E \geq 0, \tag{5.23a}$$

$$\mathcal{G}_\gamma(X; A + BF, B_1, C_1) < 0. \tag{5.23b}$$

(b) There exist $G \in \mathbf{R}^{n\times p}$ and $Y \in \mathbf{R}^{n\times n}$ such that

$$EY^T = YE^T \geq 0, \tag{5.24a}$$

$$\mathcal{G}_\gamma(Y^T; (A + GC)^T, C_1^T, B_1^T) < 0 \tag{5.24b}$$

(c) The matrices X and Y satisfy

$$E^T(\gamma^2 Y^{-T} - X) \geq 0, \tag{5.25}$$

where $Y^{-T} := (Y^T)^{-1}$. (Note that X and Y are nonsingular if (a) and (b) hold, respectively.)

(2) If (II) holds, we can assume that $\Delta := \gamma^2 Y^{-T} - X$ is nonsingular. (If this is not true, for appropriate $\kappa \in (0,1)$, κX instead of X satisfies (b) and the nonsingularity of Δ.) A controller that satisfies (I) is given by

$$\hat{E} = E, \tag{5.26a}$$

$$\hat{C} = F, \tag{5.26b}$$

$$\hat{B} = -\gamma^2 \Delta^{-T} Y^{-1} G, \tag{5.26c}$$

$$\begin{aligned}\hat{A} = {} & A + B\hat{C} - \hat{B}C + \tfrac{1}{\gamma^2} B_1 B_1^T X - \Delta^{-T} \hat{C}^T B^T X \\ & + \Delta^{-T} \mathcal{G}_\gamma(X; A + BF, B_1, C_1).\end{aligned} \tag{5.26d}$$

Proof. See Appendix. □

Remark 5.2 Though Lemma 5.3 gives an H_∞-controller of the descriptor form, we can always derive a *proper* controller in the following way. Suppose that the statement (II) in Lemma 5.3 holds and set $\{\hat{E}, \hat{A}, \hat{B}, \hat{C}\}$ as in (5.26). From a singular value decomposition of $\hat{E}$, we represent $\hat{E}$ and $\hat{A}$ by

$$\hat{E} = U_E \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V_E^T, \quad \hat{A} = U_E \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} V_E^T,$$

where $\Sigma > 0$. If $\hat{A}_{22}$ is nonsingular, the controller is regular and has no impulsive modes. If $\hat{A}_{22}$ is singular, redefine $\hat{A}$ by

$$\hat{A} := U_E \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} + \kappa I \end{bmatrix} V_E^T,$$

where κ is a scalar such that $\hat{A}_{22} + \kappa I$ is nonsingular and that the new $\hat{A}$ satisfies (5.22b). Such κ always exists. Thus we can always obtain a proper controller whenever (II) holds. Note that the order of the controller is no more than $r (= \mathrm{rank} E)$.

Now we summarize the results above in this subsection in the following theorem.

Theorem 5.4 [MKOS96]

(1) The following statements (I) and (II) are equivalent:

(I) Given $\gamma > 0$, there exists a proper controller of the form (5.19) such that the closed-loop system is admissible and that $||C_{cl}(sE_{cl} - A_{cl})^{-1} B_{cl}||_\infty < \gamma$.

(II) There exist matrices X, Y and positive scalars σ, δ that satisfy the following inequalities:

$$\begin{aligned}
&E^TX = X^TE \geq 0,\\
&A^TX + X^TA + C_1^TC_1 + X^T(\tfrac{1}{\gamma^2}B_1B_1^T - \sigma BB^T)X < 0,\\
&EY^T = YE^T \geq 0, \qquad\qquad (5.27)\\
&AY^T + YA^T + B_1B_1^T + Y(\tfrac{1}{\gamma^2}C_1^TC_1 - \delta C^TC)Y^T < 0,\\
&E^T(\gamma^2Y^{-T} - X) \geq 0.
\end{aligned}$$

(2) If (5.27) holds, we can assume that X, Y and $\Delta := \gamma^2Y^{-T} - X$ are nonsingular. A proper controller that satisfies (I) is calculated from (5.25) and the procedure in Remark 5.2, where $F = -\frac{\sigma}{2}B^TX$ and $G = -\frac{\delta}{2}YC^T$. The McMillan degree of the controller is at most r.

Proof. Obvious from Lemma 5.3, Remark 5.2 and the Finsler's theorem [Jac77]. □

Theorem 5.4 reduces the output-feedback descriptor H_∞-control problem to solving the GARIs with the coupling inequality (5.27). If $E = I$, the above results coincide with the results of the state space H_∞ synthesis based on Riccati inequalities [SMN90].

We have to present a computational method to solve these coupled GARIs, but this type of quadratic inequalities has never been studied. In order to get a solution to (5.23)~(5.25), we give an equivalent LMI-condition below. From now on, we assume that

$$E = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \ \Sigma \in \mathbf{R}^{r\times r}, \Sigma = \Sigma^T > 0 \qquad (5.28)$$

for simplicity as well as $D = 0, N_1 = 0, H_1 = 0$ and $D_1 = 0$. Every descriptor system is represented in a form satisfying these assumptions. We will use the following notation:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \ B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \ C_1 = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix},$$

$$M_XU_X^T := \left[\begin{array}{c|c} \begin{matrix} A_{12} \\ A_{22} \end{matrix} & B \\ \hline C_{12} & 0 \\ \hline \multicolumn{2}{c}{0_{p_1\times(n+m-r)}} \end{array}\right], \quad M_YU_Y^T := \left[\begin{array}{c|c} \begin{matrix} A_{21}^T \\ A_{22}^T \end{matrix} & C^T \\ \hline B_{12}^T & 0 \\ \hline \multicolumn{2}{c}{0_{m_1\times(n+m-r)}} \end{array}\right],$$

where $A_{11} \in \mathbf{R}^{r\times r}$, $B_{11} \in \mathbf{R}^{r\times m_1}$, $C_{11} \in \mathbf{R}^{p_1\times r}$, and the four matrices M_X, U_X, M_Y, U_Y are column full rank. Let N_X and N_Y column full rank matrices with maximum sizes to satisfy $N_X^T M_X = 0$, $N_Y^T M_Y = 0$, respectively.

Proposition 5.2 [MKOS96]
(1) The following statements (I) and (II) are equivalent:

(I) The inequalities (5.23)∼(5.25) have a solution $\{X, Y, F, G\}$.

(II) There exist symmetric matrices $P_X, P_Y \in \mathbf{R}^{r\times r}$ such that

$$\begin{bmatrix} P_X & \Sigma^{-1} \\ \Sigma^{-1} & P_Y \end{bmatrix} \geq 0, \quad N_X^T Q_X N_X > 0, \quad N_Y^T Q_Y N_Y > 0, \tag{5.29}$$

where

$$Q_X := \begin{bmatrix} -X_0A^T - AX_0^T & -X_0C_1^T & -B_1 \\ -C_1X_0^T & \gamma I & 0 \\ -B_1^T & 0 & \gamma I \end{bmatrix}, \tag{5.30a}$$

$$Q_Y := \begin{bmatrix} -Y_0^TA - A^TY_0 & -Y_0^TB_1 & -C_1^T \\ -B_1^TY_0 & \gamma I & 0 \\ -C_1 & 0 & \gamma I \end{bmatrix}, \tag{5.30b}$$

$$X_0 := \begin{bmatrix} \Sigma P_X & 0 \\ 0 & 0 \end{bmatrix}, \ Y_0 := \begin{bmatrix} P_Y\Sigma & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.30c}$$

(2) If (5.29) is solvable, a solution to (5.23)∼(5.25) is

$$X := \gamma\hat{X}^{-T},\ Y := \gamma\hat{Y}^{-T},\ F := (\hat{X}^{-1}W)^T, G := (Z\hat{Y}^{-1})^T, \tag{5.31}$$

where

$$\hat{X} := \left[\begin{array}{c|c} \Sigma P_X & X_r \\ 0 & \end{array}\right],\ \hat{Y} := \left[\begin{array}{cc} P_Y\Sigma & 0 \\ \hline \multicolumn{2}{c}{Y_l} \end{array}\right], \tag{5.32}$$

$$[\,X_r \quad W\,] := -\rho_X(J^T\Phi_X J)^{-1}J^T\Phi_X M_X U_X^{\dagger}, \tag{5.33}$$

$$[\,Y_l^T \quad Z^T\,] := -\rho_Y(J^T\Phi_Y J)^{-1}J^T\Phi_Y M_Y U_Y^{\dagger}, \tag{5.34}$$

$$\Phi_X := (Q_X + \rho_X M_X M_X^T)^{-1},$$

$$\Phi_Y := (Q_Y + \rho_Y M_Y M_Y^T)^{-1},$$

$$\rho_X > \lambda_{\max}\{M_X^{\dagger}(Q_X N_X(N_X^T Q_X N_X)^{-1}M_X^T Q_X - Q_X)(M_X^{\dagger})^T\},$$

$$\rho_Y > \lambda_{\max}\{M_Y^{\dagger}(Q_Y N_Y(N_Y^T Q_Y N_Y)^{-1}M_Y^T Q_Y - Q_Y)(M_Y^{\dagger})^T\},$$

and $J := [\, I_n \quad 0_{n\times(m_1+p_1)}\,]^T$. If $\hat{X}$ or $\hat{Y}$ is singular, modify it by adding $\mathrm{diag}\{0_{r\times r}, \alpha I_{n-r}\}$ with $|\alpha|$ small enough not to destroy (5.29).

Proof. See Appendix. □

Most of the computational complexity to solve (5.29) depends on $r(r+1)$, the number of the elements of the two symmetric matrices P_X and P_Y. This ensures that such an augmentation of descriptor forms as (5.6) does not increase the complexity, since it keeps the number of the finite modes.

5.5 Numerical Example (H_∞-control for a descriptor system)

This section shows a numerical example using the results of the previous section. Let us consider a plant in Fig.5.1, which has a differentiating element and an integrating element.

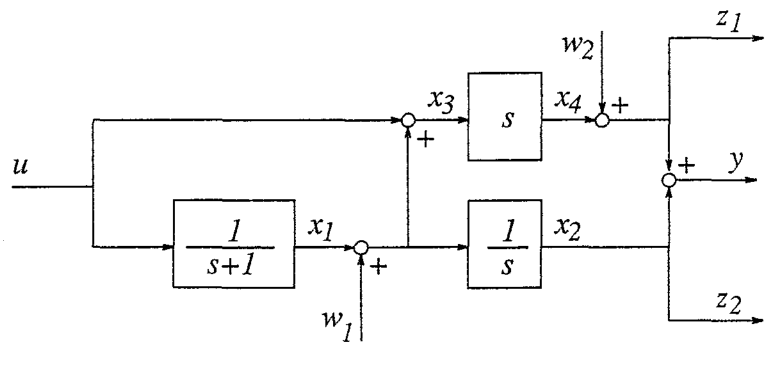

Fig. 5.1 Plant with a differentiating element and an integrating element.

The following equations describe the plant:

$$\dot{x}_1 = -x_1 + u,\ \dot{x}_2 = x_1 + w_1,$$
$$\dot{x}_3 = x_4,\ 0\dot{x}_4 = -x_3 + u + x_1 + w_1,$$
$$z_1 = x_4 + w_2,\ z_2 = x_2,\ y = x_2 + x_4 + w_2,$$

where the second line of the above equations indicates the differentiating element: $x_4 = \frac{d}{dt}(u + x_1 + w_1)$. To avoid feed-through terms, we set $x_5 := w_2$ and define

$x = [x_1, x_2, x_3, x_4, x_5]^T$ as a descriptor variable of the plant. Then each matrix in (5.18) is given by

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix},$$

$$B = [1 \quad 0 \quad 0 \quad 1 \quad 0]^T, B_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T,$$

$$C = [0 \quad 1 \quad 0 \quad 1 \quad 1], \ C_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$D = 0, \ N_1 = 0, \ H_1 = 0, \ D_1 = 0.$$

This plant has one $j\omega$-axis zero caused by the integrating element. The result of convex optimization to minimize γ subject to (5.29) is:

$$P_X = \begin{bmatrix} 7.2607 & -2.9437 & 0.3036 \\ -2.9437 & 2.8013 & -1.9834 \\ 0.3036 & -1.9834 & 3.8013 \end{bmatrix},$$

$$P_Y = \begin{bmatrix} 13.0675 & 0.1796 & 0.4672 \\ 0.1796 & 2.0757 & 0.5671 \\ 0.4672 & 0.5671 & 2.1246 \end{bmatrix},$$

$$\gamma = 3.1241,$$

where we used MATLAB version 4.0a, SP [VLB94] and MAKELMI [MOS96a]. Since $\hat{X}$ and $\hat{Y}$ derived from these matrices were singular, we added $\mathrm{diag}\{0_{3\times3}, 0.01I_2\}$ to them. The modified $\hat{X}$ and $\hat{Y}$ are nonsingular and X, Y, F, G computed from them by (5.31) satisfies (5.27). The realization $\{\hat{A}_s, \hat{B}_s, \hat{C}_s, \hat{D}_s\}$ of the derived controller is given by:

$$\hat{A}_s = 10^3 \begin{bmatrix} -0.8831 & -1.2198 & -2.0557 \\ 2.8968 & 4.0076 & 6.7599 \\ -2.2226 & -3.0780 & -5.1892 \end{bmatrix}, \ \hat{B}_s = 10^3 \begin{bmatrix} -1.1624 \\ 3.8213 \\ -2.9321 \end{bmatrix},$$

$$\hat{C}_s = 10^3 [-1.3058 \quad -1.8064 \quad -3.0451], \ \hat{D}_s = -1.7217 \times 10^4.$$

The singular value plot of the closed-loop transfer function is shown in Fig.5.2.

5.6 Concluding remarks

This chapter provided new algebraic inequality conditions for descriptor systems: the generalized Lyapunov inequality for the admissibility, and the generalized Riccati inequality for H_∞-norm conditions. Using these inequalities, we proposed quadratic stabilization via state-feedback and H_∞-control via output-feedback for descriptor systems. In these synthesis methods, a derived controller satisfies not only quadratic stability or H_∞-norm condition but also elimination of possible impulsive modes of descriptor plants.

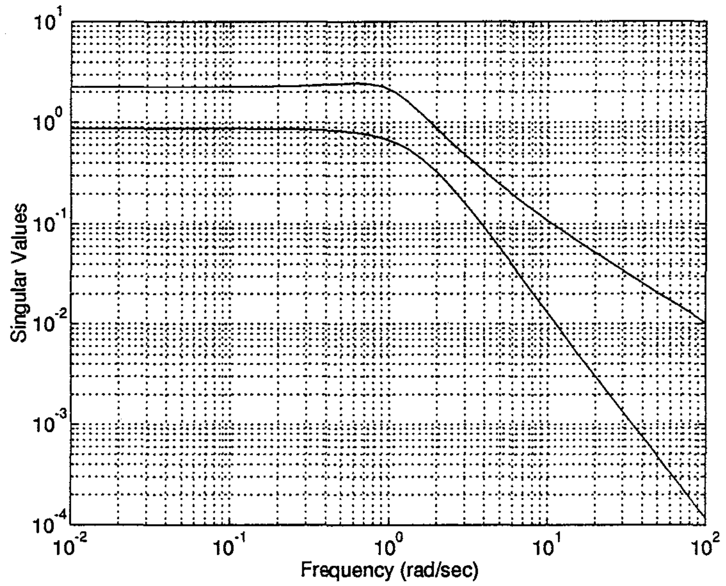

Fig. 5.2 Singular plot

Chapter 6

Conclusion

LMIs have been recognized as powerful and comprehensive mathematical tool to represent properties (specifications in design) of linear systems. In this dissertation, we have proposed a unified approach to solve control system synthesis problems described in terms of LMIs.

In Chapter 2, we have presented a new parametrization of stabilizing output-feedback controllers. The set of all stabilizing controllers of some order is connected to a finite-dimensional convex parameter set, and this parametrization opened the way to optimization calculation on the convex parameter set. In Chapter 3, we have formulated the LMI-synthesis problem and defined a class of LMIs representing performances of closed-loop systems. We have provided a solution to nonconvex problems to synthesize a controller that satisfies any of the LMIs in the class. The solution is a procedure to reduce them to convex optimization problems on the parameter set defined in Chapter 2. The class contains almost all of conventional LMI-conditions, such as H_∞-norm conditions, H_2-norm conditions, root-clustering conditions and so on, for both continuous- and discrete-time systems. Therefore the solution is a unified formula to synthesis problems for one of those LMIs. Furthermore, the class contains new LMI-conditions for multi-objective design such as H_2/H_∞/root-clustering, and thus the solution enables controller design with more complex specifications.

In Chapter 4, we have considered robust performance problems formulated by LMIs in the class defined in Chapter 3. We have treated two types of structured uncertainties: norm-bounded uncertainties and polytopic uncertainties. For both types, we have proposed algorithms to solve those synthesis problems via convex optimizations on the parameter set. As in Chapter 3, a larger class of LMI-conditions than those ever considered are applicable with the results of Chapter 4 to representing design specifications.

In Chapter 5, we have provided results of analysis and synthesis for descriptor systems based on LMIs. We have proposed LMI-conditions for the admissibility, a generalized stability, and H_∞-norm condition with admissibility. Applying these LMIs, we have solved two synthesis problems: state-feedback quadratic stabilization and output-feedback H_∞-control. In these problems, impulse elimination is attained as well as each quadratic stability and H_∞-norm condition.

The results of this dissertation derive a new and more flexible CAD including many existing methods such as H_∞ design. It enables design with complex multiple specifications, which enlarges what designers can do in designing control systems. The existing fast convex optimization algorithms [BG93, NN94, VB94] solve LMIs on the parameter set. We have developed a tool [MOS96a] that derives LMIs of the standard form (3.31) from such description of LMIs in a text file as $P = P^T > 0$, $(AP_f + BW_f) + (AP_f + BW_f)^T < 0$. We have used them in the three numerical examples of this dissertation. Lastly, if one find a new LMI-condition and it belongs to the class $\mathcal{L}$, Theorem 3.1 immediately derives a procedure to find a controller that satisfies the LMI-condition.

Appendix

Proof of the results

Proof of Proposition 2.1.

This proposition is a special case of Theorem 3.1 in Section 2.3, aimed to explain the parametrization. See the proof of Theorem 3.1, setting

$$\Phi(P,\Sigma) = \Phi_{\text{Lyap}}(P, P\bar{A}),\ \Xi(T) = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}.$$

□

Proof of Theorem 3.1.

First, we show some technical lemmas.

Lemma A.1 Assume that for some n_c there exist $P \in PD(n+n_c)$ and $K \in \mathcal{K}(n_c)$ satisfying $\Phi(P, \Sigma(K)) > 0$. Then for *any* $n_c' \geq n_c$, there exist $P' \in PD(n + n_c')$ and $K' \in \mathcal{K}(n_c')$ that satisfy $\Phi(P', \Sigma(K')) > 0$ and $K \sim K'$.

Proof. From $K = \{A_c, B_c, C_c, D_c\}$ and a matrix $A_0 \in \mathbf{R}^{(n_c'-n_c)\times(n_c'-n_c)}$, define a controller $K' := \{A_c', B_c', C_c', D_c\}$ as follows:

$$A_c' := \begin{bmatrix} A_c & 0 \\ 0 & A_0 \end{bmatrix},\ B_c' := \begin{bmatrix} B_c \\ 0_{(n-n_0)\times p} \end{bmatrix},\ C_c' := \begin{bmatrix} C_c & 0_{m\times(n-n_0)} \end{bmatrix}.$$

Then obviously $K \sim K'$, and we have

$$\bar{A}(K') = \begin{bmatrix} \bar{A}(K) & 0 \\ 0 & A_0 \end{bmatrix},\ \bar{B}_i(K') = \begin{bmatrix} \bar{B}_i(K) \\ 0 \end{bmatrix},\ \bar{C}_i(K') = \begin{bmatrix} C_i(K) & 0 \end{bmatrix}.$$

Next, define

$$P' := \begin{bmatrix} P & 0 \\ 0 & P_0 \end{bmatrix},\quad P_0 \in PD(n_c' - n_c),$$

which implies

$$P'\bar{A}(K') = \begin{bmatrix} P\bar{A}(K) & 0 \\ 0 & P_0 A_0 \end{bmatrix}, \; P'\bar{B}_i(K') = \begin{bmatrix} P\bar{B}_i(K) \\ 0 \end{bmatrix}.$$

Hence, for some T_0, a permutation matrix, and Φ_0, a function of (P_0, A_0), we have

$$\Phi(P', \Sigma(K')) = T_0^T \begin{bmatrix} \Phi(P, \Sigma(K)) & 0 \\ 0 & \Phi_0(P_0, A_0) \end{bmatrix} T_0.$$

Next, define a function $\Phi_{\text{state}}(P, \Sigma)$ as follows:

- If $\Phi(P, \Sigma) = \Phi_{\text{Lyap}}(P, P\bar{A})$, define $\Phi_{\text{state}}(P, \Sigma) := \Phi_{\text{Lyap}}(P, \bar{A})$.
- Otherwise, define $\Phi_{\text{state}}(P, \Sigma)$ by picking up all the *state* blocks (3.9) and the corresponding off-diagonal blocks.

Then obviously $\Phi_{\text{state}}(P, \Sigma) > 0$ holds and, if we set

$$\begin{aligned} E_0 &:= \left[I_{n_c'-n_c} \quad 0_{(n_c'-n_c)\times(n+n_c)} \right]^T, \\ E &:= E_0 \oplus \cdots \oplus E_0, \\ P_0 &:= E_0^T P E_0 (> 0), \\ A_0 &:= P_0^{-1} E_0^T P \bar{A}(K) E_0, \end{aligned}$$

we get $P_0 A_0 = E_0^T P \bar{A}(K) E_0$, which implies $\Phi_0(P_0, A_0) = E^T \Phi_{\text{state}}(P, \Sigma) E > 0$. □

Lemma A.2 Assume that for some $n_c \geq n$ there exist $P \in PD(n + n_c)$ and $K \in \mathcal{K}(n_c)$ that satisfy $\Phi(P, \Sigma(K)) > 0$. Then we have $P' \in PD(n + n_c)$ satisfying $\Phi(P', \Sigma(K)) > 0$ and, when $(P')^{-1}$ is partitioned as

$$(P')^{-1} = \begin{bmatrix} P_{(1)} & P_{(12)} \\ P_{(12)}^T & P_{(2)} \end{bmatrix}, \; P_{(1)} \in PD(n), \, P_{(2)} \in PD(n_c), \tag{A.1}$$

$P_{(12)}$ is row full rank.

Proof. Obvious. □

Lemma A.3 Suppose that $\Phi(P, \Sigma(K)) > 0$ holds for some $P \in PD(n + n_c)$ and $K \in \mathcal{K}(n_c)$. Then, for any $n_c' \geq \max\{n, n_c\}$, there exist $K' \in \mathcal{K}(n_c')$ and $P' \in \mathcal{P}(n + n_c')$ satisfying $\Phi(P', \Sigma(K)) > 0$, $K \sim K'$ and

$$P' = \begin{bmatrix} P_f & S & 0 \\ S & S & 0 \\ 0 & 0 & P_h \end{bmatrix}^{-1}, \; P_f, S \in PD(n), P_f > S, \; P_h \in PD(n_c - n). \tag{A.2}$$

Proof. From Lemma A.1 and Lemma A.2, for any $n_c' \geq \max\{n, n_c\}$, we have $\hat{P} \in PD(n+n_c')$ and $\hat{K} = \{A_c, B_c, C_c, D_c\} \in \mathcal{K}(n_c')$ satisfying $\Phi(\hat{P}, \Sigma(\hat{K})) > 0$ and, when $\hat{P}^{-1}$ is partitioned as

$$\hat{P}^{-1} = \begin{bmatrix} P_{(1)} & P_{(12)} \\ P_{(12)}^T & P_{(2)} \end{bmatrix},\ P_{(1)} \in PD(n),\ P_{(2)} \in PD(n_c),$$

$P_{(12)}$ is row full rank. Define the following matrix:

$$T := \begin{bmatrix} P_{(12)} P_{(2)}^{-1} \\ (P_{(12)}^{\perp})^T \end{bmatrix}^{-1},$$

whose nonsingularity is easily checked. Setting $P_f = P_{(1)}$, $S = P_{(12)} P_{(2)}^{-1} P_{(12)}^T$ and $P_h = (P_{(12)}^{\perp})^T P_{(2)} P_{(12)}^{\perp}$, we have

$$\begin{aligned} P' &:= \begin{bmatrix} I_n & 0 \\ 0 & T^T \end{bmatrix} \hat{P} \begin{bmatrix} I_n & 0 \\ 0 & T \end{bmatrix} \\ &= \left\{ \begin{bmatrix} I_n & 0 \\ 0 & \begin{bmatrix} P_{(12)} P_{(2)}^{-1} \\ (P_{(12)}^{\perp})^T \end{bmatrix} \end{bmatrix}^{-1} \begin{bmatrix} P_{(1)} & P_{(12)} \\ P_{(12)}^T & P_{(2)} \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & \begin{bmatrix} P_{(2)}^{-1} P_{(12)}^T & P_{(12)}^{\perp} \end{bmatrix} \end{bmatrix} \right\}^{-1} \\ &= \begin{bmatrix} P_f & S & 0 \\ S & S & 0 \\ 0 & 0 & P_h \end{bmatrix} \end{aligned}$$

and $P_f - S = P_{(1)} - P_{(12)} P_{(2)}^{-1} P_{(12)}^T > 0$. Lastly, $K' := \{T^{-1} A_c T, T^{-1} B_c, C_c T, D_c\}$ implies

$$\Phi(P', \Sigma(K')) = \Xi\left(\begin{bmatrix} I_n & 0 \\ 0 & T^T \end{bmatrix} \right) \Phi(\hat{P}, \Sigma(\hat{K})) \Xi\left(\begin{bmatrix} I_n & 0 \\ 0 & T \end{bmatrix} \right) > 0$$

and $K \sim K'$. □

Now we are ready for the proof of Theorem 3.1. We prove the statement (2) first. Let us denote

$$\begin{aligned} \mathcal{T}_L &:= \{G_c(s; K) | K \in \mathcal{K}(n_p), \Phi(P, \Sigma(K)) > 0, \exists P \in PD(n+n_p)\}, \\ \mathcal{T}_R &:= \{G_c(s; \mathrm{K}_{\mathrm{map}}(\boldsymbol{p})) | \boldsymbol{p} \in \mathcal{P}(n_p), \Phi^*(\boldsymbol{p}) > 0\}, \end{aligned}$$

which are the left hand side and the right hand side of (3.16), respectively.

Suppose that $G_{c0}(s) \in \mathcal{T}_L$. Then, from Lemmas A.1~A.3, there exist $K' = \{A_c, B_c, C_c, D_c\} \in \mathcal{K}(n_p)$ and $P' \in \mathcal{P}(n+n_p)$ of the form (A.2) such that $K \sim K'$,

$G_{c0}(s) = C_c(sI - A_c)^{-1}B_c + D_c$ and $\Phi(P', \Sigma(K')) > 0$. Using P' and K', set a parameter $\boldsymbol{p} := \{P_f,\, P_g,\, P_h,\, W_{fa},\, W_{ga},\, W_h,\, L_a\}$, where $P_g := (P_f - S)^{-1}$ and

$$\left[\begin{array}{c|c} W_h & W_{fa} \\ \hline W_{ga} & L_a \end{array}\right] := \left[\begin{array}{c|cc} I_n & 0 & 0 \\ \hline P_gB & -P_g & 0 \\ 0 & 0 & I_{n_p-n} \end{array}\right] \left[\begin{array}{c|c} D_c & C_c \\ \hline B_c & A_c - \begin{bmatrix} AP_fS^{-1} & 0 \\ 0 & 0 \end{bmatrix} \end{array}\right] \left[\begin{array}{c|cc} I_n & CP_f & 0 \\ \hline 0 & S & 0 \\ 0 & 0 & P_h \end{array}\right]. \tag{A.3}$$

Then we get (2.13) for matrices defined in (2.11) through some manipulations. From the equalities in (2.13) and Lemma 3.1,

$$\Phi^*(\boldsymbol{p}) = \Xi(U_2^T)\Phi(P', \Sigma(K'))\Xi(U_2) > 0$$

holds. Since (A.3) is the inverse of (2.10), the above parameter $\boldsymbol{p}$ implies $K' = \{A_c,\, B_c,\, C_c,\, D_c\} = \mathrm{K_{map}}(\boldsymbol{p})$. Thus, from a controller $G_{c0}(s) \in \mathcal{T}_L$, we have found a parameter $\boldsymbol{p}$ such that $\Phi^*(\boldsymbol{p}) > 0$ and $G_{c0}(s) = G_c(s; \mathrm{K_{map}}(\boldsymbol{p}))$, which shows that $G_{c0}(s)$ belongs to $\mathcal{T}_R$.

Next, let $G_{c0}(s) = G_c(s; \mathrm{K_{map}}(\boldsymbol{p}))$ belong to $\mathcal{T}_R$. Then, setting $K := \mathrm{K_{map}}(\boldsymbol{p})$ and

$$P := \begin{bmatrix} P_f & S & 0 \\ S & S & 0 \\ 0 & 0 & P_h \end{bmatrix}^{-1},$$

we can easily see that

$$\Phi(P, \Sigma(K)) = \Xi(U_2^{-T})\Phi^*(\boldsymbol{p})\Xi(U_2^{-1}) > 0,$$

which implies that $G_{c0}(s)$ belongs to $\mathcal{T}_L$, and proves the statement (2).

Lastly, we prove the statement (1). The part (II)→(I) is clear. Conversely, if (I) holds, Lemma A.1 guarantees that, for any $n_0 \geq \max\{n, n_c\}$, $\Phi(P', \Sigma(K')) > 0$ holds for some $P' \in PD(n_0)$ and $K' \in (n_0)$. From this pair of P' and K', we derive a parameter $\boldsymbol{p} = \{P_f,\, P_g,\, P_h,\, W_{fa},\, W_{ga},\, W_h,\, L_a\} \in \mathcal{P}(n_0)$ satisfying (3.15). Further, we easily get a parameter $\boldsymbol{p}' \in \mathcal{P}(n_0')$, $n \leq \forall n_0' < n_0$ from the above $\boldsymbol{p}$ by:

$$\begin{aligned} &\boldsymbol{p}' := \{P_f, P_g, P_h', W_{fa}', W_{ga}', L_a'\}, \\ &L_a' := [\, I_{n_0'} \quad 0\,]\, L_a \begin{bmatrix} I_{n_0'} \\ 0 \end{bmatrix},\ P_h := [\, I_{(n_0'-n)} \quad 0\,]\, P_h \begin{bmatrix} I_{(n_0'-n)} \\ 0 \end{bmatrix}, \\ &W_{fa}' := W_{fa}\, [\, I_{n_0'} \quad 0\,],\ W_{ga}' := \begin{bmatrix} I_{n_0'} \\ 0 \end{bmatrix} W_{ga}. \end{aligned}$$

Thus, for any $n_p \geq n$, we get a parameter satisfying (3.15) in the above procedure by setting $n_0 := n_p$ or $n_0' := n_p$. □

Proof of Corollary 3.1.

Suppose that (i) holds. From theorem 3.1, there exists $p \in \mathcal{P}(n)$ that satisfies $\Phi^*(p) > 0$. Setting E_0 by

$$E_0 := \begin{bmatrix} I_n \\ 0_{n_p \times n} \end{bmatrix},$$

we get

$$\Phi^{*sf}(p, X) = \Xi^T(E_0)\Phi^*(p, X)\Xi(E_0) > 0,$$

which proves (ii). Conversely, assume that (ii) holds and set $D_c := W_f P_f^{-1}$. Then we have

$$\Phi(P, \Sigma_{cl}(K)) = \Xi(P_f)\Phi^{*sf}(p, X)\Xi(P_f) > 0,$$

and this completes the proof. □

Proof of Lemma 4.1.

Define $\tilde{w}(t)$ and $\tilde{z}(t)$ as follows:

$$\tilde{w}(t) = \Delta\tilde{z}(t), \tag{A.4a}$$

$$\tilde{z}(t) = (I - \tilde{H}\Delta)^{-1}(\tilde{C}_x x(t) + \tilde{C}_u u(t) + \sum_{i=1}^{N_s} \tilde{C}_{(i)} w_i(t) - \tilde{C}_d s x(t)). \tag{A.4b}$$

Then (4.5) is represented by

$$Esx(t) = Ax(t) + Bu(t) + \sum_{i=1}^{N_s} B_i w_i(t) + \tilde{B}_x \tilde{w}(t). \tag{A.5}$$

From (A.4) and (A.5), we get

$$\tilde{z}(t) = \Xi_x x(t) + \Xi_u u(t) + \sum_{i=1}^{N_s} \Xi_{(i)} w_i(t) + \Xi_H \tilde{w}(t),$$

where (4.8) defines Ξ_x, Ξ_u, etc. This implies

$$\tilde{w}(t) = \Delta(I - \Xi_H\Delta)^{-1}(\Xi_x x(t) + \Xi_u u(t) + \sum_{i=1}^{N_s} \Xi_{(i)} w_i(t)). \tag{A.6}$$

Now consider a mapping defined on $\mathcal{U}_{nb}$:

$$\Delta^* = -\Xi_H^T + (I - \Xi_H^T \Xi_H)^{\frac{1}{2}} \Delta (I - \Xi_H \Delta)^{-1} (I - \Xi_H \Xi_H^T)^{\frac{1}{2}}.$$

It is shown in [YK96] that this mapping is bijective from $\mathcal{U}_{nb}$ to $\mathcal{U}_{nb}$, and we get

$$\Delta (I - \Xi_H \Delta)^{-1} = (I - \Xi_H^T \Xi_H)^{-\frac{1}{2}} (\Delta^* + \Xi_H^T)(I - \Xi_H \Delta)^{-\frac{1}{2}}. \tag{A.7}$$

Substituting (A.6) and (A.7) into (A.5) completes the proof. □

Proof of Lemma 4.3.

From (4.13) and Lemma 4.2, the inequality (4.15) holds for all $\Delta \in \mathcal{U}_{nb}$ if and only if

$$\begin{bmatrix} \Phi(P,\Sigma) & \Phi_1 P \tilde{B}_{cl} & \Phi_2^T \begin{bmatrix} \Gamma_0^T(K) \\ \Gamma_1^T(K) \\ \vdots \\ \Gamma_{N_s}^T(K) \end{bmatrix} \\ \tilde{B}_{cl}^T P \Phi_1^T & q I_{d_r} & 0 \\ [\,\Gamma_0(K) \quad \Gamma_1(K) \quad \cdots \Gamma_{N_s}(K)\,]\,\Phi_2 & 0 & q^{-1} I_{d_c} \end{bmatrix} > 0. \tag{A.8}$$

This matrix inequality belongs to $\mathcal{L}$ if one defines an appropriate augmented plant as Σ. Therefore Theorem 3.1 shows that this inequality is equivalent to (4.16). □

Proof of Theorem 4.2.

We prove only (2) here; the proof of (1) easily follows that of (2). Since $\Phi \in \mathcal{L}_{C\oplus}$, there exists some $\Phi_j \in \mathcal{L}_C$ such that $\Phi = \Phi_1 \oplus \Phi_2 \oplus \cdots \oplus \Phi_{N_0}$ for some N_0. From the definition of $\mathcal{L}_C$, each Φ_j has only one diagonal block of the form $c_{kk}^{(i)}(P\bar{A} + \bar{A}^T P)$ and $\bar{A}$ does not appear in any other parts. According to Φ, define $\Pi(T)$ as follows:

$$\Pi(T) = \Pi_1(T) \oplus \Pi_2(T) \oplus \cdots \oplus \Pi_{Nb}(T), \tag{A.9a}$$

$$\Pi_k(T) = \begin{cases} T & \text{if } \Phi_{kk} \text{ has the form of } c_{kk}^{(i)}(P\bar{A} + \bar{A}^T P), \\ I & \text{otherwise.} \end{cases} \tag{A.9b}$$

Then, since $\mathcal{L}_C \subset \mathcal{L}_M$ and $\Phi_j \in \mathcal{L}_M$ is defined to have $P\bar{B}_i$ terms in the form of (3.25), we see that (4.30) is equivalent to the following matrix inequality and that its left hand side is biaffine with respect to (η, θ):

$$\Pi\left(\begin{bmatrix} E(\eta) & 0 \\ & I_{n_c} \end{bmatrix} P^{-1}\right) \Phi(P, \Sigma_p(K,\eta,\theta)) \Pi\left(P^{-1} \begin{bmatrix} E^T(\eta) & 0 \\ & I_{n_c} \end{bmatrix}\right) > 0.$$

Therefore the above inequality is equivalent to

$$\Pi\left(\begin{bmatrix} E_{(j)} & 0 \\ & I_{n_c} \end{bmatrix} P^{-1}\right) \Phi(P, \Sigma_p(K; [j,k])) > 0, \Pi\left(P^{-1}\begin{bmatrix} E_{(j)}^T & 0 \\ & I_{n_c} \end{bmatrix}\right) > 0,$$
$$j = 1, 2, \ldots, n_E,\ k = 1, 2, \ldots, n_A.$$

Lastly, this is equivalent to (4.31), which is proved in a similar way to that of Theorem 3.1. □

Proof of Theorem 5.1.

(*Sufficiency*) Suppose that (5.8) has a solution X. Since A and X are nonsingular from (5.8b), for any non-zero complex number s,

$$\det(sE - A) = (-s)^n \det A \det(\frac{1}{s}I - A^{-1}E)$$

holds. The right hand side of this equality, which is not identically zero, proves the regularity of (E, A). Next, the matrix $L := -A^TX - X^TA$ is nonsingular and satisfies that $(E, A, L^{\frac{1}{2}})$ is observable [Cob84b]. From Lemma 5.2, (E, A) has neither impulsive nor unstable finite modes.

(*Necessity*) Obvious from Lemma 5.2. □

Proof of Theorem 5.2.

We prove only (I) here.

(*Sufficiency*) Theorem 5.1 guarantees the admissibility of (E, A). The H_∞ norm condition is derived by simple manipulations.

(*Necessity*) Since (E, A) is admissible, there exist nonsingular matrices L, R such that

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = LER, \quad \begin{bmatrix} A_d & 0 \\ 0 & I \end{bmatrix} = LAR \tag{A.10}$$

hold for a stable matrix A_1. We also denote that

$$\begin{bmatrix} B_d \\ B_s \end{bmatrix} = LB, \quad [C_d \quad C_s] = CR.$$

The transfer function in (5.5) is represented by $G(s) = C_d(sI - A_d)^{-1}B_d - C_sB_s$. From $||G||_\infty < \gamma$, there exists $P > 0$ such that

$$\begin{bmatrix} -A_d^TP - PA_d - C_d^TC_d & PB_d - C_d^TC_sB_s \\ B_d^TP - B_s^TC_s^TC_d & \gamma^2 I - B_s^TC_s^TC_sB_s \end{bmatrix} > 0$$

holds [ZK88]. Let $\alpha > 0$ and define $W := C_s^T C_s + \alpha I$, $Z := -C_s^T C_d$. Then $W > 0$, and for sufficiently small α,

$$\begin{bmatrix} -A_d^T P - PA_d - C_d^T C_d & PB_d + Z^T B_s \\ B_d^T P + B_s^T Z & \gamma^2 I - B_s^T W B_s \end{bmatrix} > 0$$

is also valid. Using this inequality, we can easily verify that

$$X = L^T \begin{bmatrix} P & 0 \\ Z & -W \end{bmatrix} R^{-1}$$

satisfies (5.9b). The condition (5.9a) obviously holds for this X. □

Proof of Proposition 5.1.

The necessity is obvious. We prove only sufficiency. Suppose that (5.14) holds. If Y is regular, $F = MY^{-T}$ makes $(E, A+BF)$ admissible. Otherwise, consider $\tilde{Y} = Y + \delta K$ with $\delta > 0$. We have $E\tilde{Y}^T = EY^T + \delta EK^T \geq 0$ for any $\delta > 0$ and

$$\begin{aligned} &(A\tilde{Y}^T + BM) + (A\tilde{Y}^T + BM)^T \\ &= (AY^T + BM) + (AY^T + BM)^T + \delta(AK^T + KA^T) \\ &< 0 \end{aligned}$$

for enough small $\delta > 0$ with $\det(Y + \delta K)$ is nonsingular. □

Proof of Theorem 5.3.

Suppose that (5.17) holds. We get $\delta \geq 0$ as in the proof of Proposition 5.1 and

$$G^{-1}(\theta)(A(\theta) + B(\theta)F)\tilde{Y}^T + \tilde{Y}(A(\theta) + B(\theta)F)^T G^{-T}(\eta) < 0. \tag{A.11}$$

This implies the admissibility of $(H, G^{-1}(\eta)(A(\theta)+B(\theta)F))$ for all $(\eta, \theta) \in \Omega^{n_E} \times \Omega^{n_A}$, and hence $(E(\eta), (A(\theta) + B(\theta)F))$ is admissible, satisfying (A.11) and (5.17b).

Next, we prove the quadratic stability of the dynamic part of the closed-loop system, showing that $q(x) := x^T \tilde{Y}^{-1} Hx$ is a Lyapunov function for any function $(\theta(t), \delta(t))$ that varies in $\Omega^{n_E} \times \Omega^{n_A}$. We get $q(x) \geq 0$ easily. Since $\tilde{Y}$ is nonsingular and $\tilde{Y}^{-1}H = H^T \tilde{Y}^{-T} \geq 0$, we have $Hx = 0$ if $q(x) = 0$. Further, we see that $Hx = 0$ induces $x = 0$ as follows: Consider the following singular value decomposition of H

and partition of $G^{-1}(\eta)(A(\theta)+B(\theta)F)$:

$$\begin{bmatrix} \Sigma_H & 0 \\ 0 & 0 \end{bmatrix} = U_H^T H V_H,$$
$$\hat{A}(\eta,\theta) = \begin{bmatrix} \hat{A}_{11}(\eta,\theta) & \hat{A}_{12}(\eta,\theta) \\ \hat{A}_{21}(\eta,\theta) & \hat{A}_{22}(\eta,\theta) \end{bmatrix} = U_H^T G(\eta)^{-1}(A(\theta)+B(\theta)F)V_H.$$

Then the closed-loop system is represented as follows:

$$\Sigma_H \dot{\hat{x}}_1(t) = \hat{A}_{11}(\eta,\theta)\hat{x}_1(t) + \hat{A}_{12}(\eta,\theta)\hat{x}_2(t) \tag{A.12a}$$
$$0 = \hat{A}_{21}(\eta,\theta)\hat{x}_1(t) + \hat{A}_{22}(\eta,\theta)\hat{x}_2(t), \tag{A.12b}$$

where $[\,\hat{x}_1^T(t) \quad \hat{x}_2^T(t)\,]^T = V_H^T x(t)$. Since $\hat{A}(\eta,\theta)$ has no impulsive modes for all $(\eta,\theta) \in \Omega^{n_E} \times \Omega^{n_A}$, the block $\hat{A}_{22}(\eta,\theta)$ is nonsingular [Shi95]. The condition $Hx = 0$ is represented in (A.12) as $\hat{x}_1 = 0$, and we have

$$\hat{x}_2 = -\hat{A}_{22}^{-1}(\eta,\theta)\hat{A}_{21}(\eta,\theta)\hat{x}_1 = 0.$$

Thus $q(x) = 0$ derives $x = 0$. Lastly,

$$\frac{d}{dt}q(x(t)) = 2x^T(t)\tilde{Y}^{-1}H\dot{x}(t) = 2x^T(t)\tilde{Y}^{-1}G^{-1}(\theta)(A(\theta)+B(\theta)F)x(t),$$

which shows that $\dot{q}(x)$ is negative-definite from (A.11). □

Proof of Lemma 5.3.

(*Necessity*) Suppose that (I) holds. Then, from Theorem 5.2, there exists X_{cl} that satisfies (5.22). Moreover, when X_{cl} is partitioned as

$$X_{cl} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix},\ X_{11} \in \mathbf{R}^{n\times n},\ X_{22} \in \mathbf{R}^{n_\xi \times n_\xi}, \tag{A.13}$$

we can assume that X_{cl} and the diagonal blocks X_{11} and X_{22} are nonsingular. If not, we can derive another solution X_c' that meets this assumption as follows: denote by K and $\hat{K}$ fixed nonsingular matrices such that $E^TK = K^TE \geq 0$ and $\hat{E}^T\hat{K} = \hat{K}^T\hat{E} \geq 0$, and define

$$K_{cl} := \begin{bmatrix} K & 0 \\ 0 & \hat{K} \end{bmatrix}.$$

Next, let α be a positive scalar that is small enough to satisfy $\mathcal{G}_\gamma(X_{cl}+\alpha K_{cl}; A_{cl}, B_{cl}, C_{cl}) < 0$ and not an eigenvalue of $-K^{-1}X_{11}$, $-\hat{K}^{-1}X_{22}$ nor $-K_{cl}^{-1}X_{cl}$. Then we can easily see that (5.22) and the above assumption of nonsingularity holds for $X'_{cl} := X_{cl} + \alpha K_{cl}$ instead of X_{cl}.

First, define the following matrices:

$$\begin{aligned}
\tilde{S} &:= \begin{bmatrix} I & 0 \\ -X_{22}^{-T} X_{12}^T & I \end{bmatrix}, \quad \tilde{T} := \begin{bmatrix} I & 0 \\ -X_{22}^{-1} X_{21} & I \end{bmatrix}, \\
\tilde{X}_{cl} &:= \tilde{S}^T X_{cl} \tilde{T} = \begin{bmatrix} X_{11} - X_{12} X_{22}^{-1} X_{21} & 0 \\ 0 & X_{22} \end{bmatrix}, \\
\tilde{E}_{cl} &:= \tilde{S}^{-1} E_{cl} \tilde{T}, \quad \tilde{A}_{cl} := \tilde{S}^{-1} A_{cl} \tilde{T}, \\
\tilde{B}_{cl} &:= \tilde{S}^{-1} B_{cl}, \quad \tilde{C}_{cl} := C_{cl} \tilde{T}.
\end{aligned}$$

Then, from (5.22),

$$\tilde{T}^T (E_{cl}^T X_{cl}) \tilde{T} = \tilde{E}_{cl}^T \tilde{X}_{cl} = \tilde{X}_{cl}^T \tilde{E}_{cl} \geq 0, \tag{A.14a}$$

$$\tilde{T}^T \mathcal{G}_\gamma (X_{cl}; A_{cl}, B_{cl}, C_{cl}) \tilde{T} = \mathcal{G}_\gamma (\tilde{X}_{cl}; \tilde{A}_{cl}, \tilde{B}_{cl}, \tilde{C}_{cl}) < 0 \tag{A.14b}$$

hold. The (1,1)-blocks of (A.14) imply (5.23) for $X := X_{11} - X_{12} X_{22}^{-1} X_{21}$ and $F := -\hat{C} X_{22}^{-1} X_{21}$.

Next, define the following matrices:

$$\begin{aligned}
\bar{S} &:= \begin{bmatrix} I & -X_{11}^{-T} X_{21}^T \\ 0 & I \end{bmatrix}, \quad \bar{T} := \begin{bmatrix} I & -X_{11}^{-1} X_{12} \\ 0 & I \end{bmatrix}, \\
\bar{X}_{cl} &:= \bar{S}^T X_{cl} \bar{T} = \begin{bmatrix} X_{11} & 0 \\ 0 & * \end{bmatrix}, \\
\bar{E}_{cl} &:= \bar{S}^{-1} E_{cl} \bar{T}, \quad \bar{A}_{cl} := \bar{S}^{-1} A_{cl} \bar{T}, \\
\bar{B}_{cl} &:= \bar{S}^{-1} B_{cl}, \quad \bar{C}_{cl} := C_{cl} \bar{T},
\end{aligned}$$

where the '$*$' part is not used in the following discussion. From (5.22),

$$\bar{T}^T (E_{cl}^T X) \bar{T} = \bar{E}_{cl}^T \bar{X}_{cl} = \bar{X}_{cl}^T \bar{E}_{cl} \geq 0, \tag{A.15a}$$

$$\bar{T}^T \mathcal{G}_\gamma (X_{cl}; A_{cl}, B_{cl}, C_{cl}) \bar{T} = \mathcal{G}_\gamma (\bar{X}_{cl}; \bar{A}_{cl}, \bar{B}_{cl}, \bar{C}_{cl}) < 0 \tag{A.15b}$$

hold. Define

$$\bar{Y}_{cl} := \gamma^2 \bar{X}_{cl}^{-T} = \begin{bmatrix} \gamma^2 X_{11}^{-T} & 0 \\ 0 & * \end{bmatrix}.$$

Then we can easily verify that (A.15) implies

$$\bar{E}_{cl} \bar{Y}_{cl}^T = \bar{Y}_{cl} \bar{E}_{cl}^T \geq 0, \tag{A.16a}$$

$$\mathcal{G}_\gamma (\bar{Y}_{cl}^T; \bar{A}_{cl}^T, \bar{C}_{cl}^T, \bar{B}_{cl}^T) < 0. \tag{A.16b}$$

The (1,1)-blocks of (A.16) guarantee that (5.24) holds for $Y := \gamma^2 X_{11}^{-T}$ and $G := X_{11}^{-T} X_{21}^T \hat{B}$.

Lastly, from (5.22a),

$$E^T(\gamma^2 Y^{-T} - X) = E^T(X_{12}X_{22}^{-1}X_{21}) = X_{21}^T X_{22}^{-T}(X_{22}^T E)X_{22}^{-1}X_{21} \geq 0.$$

(*Sufficiency*) Suppose that (II) holds. Then we can assume that $\Delta := \gamma^2 Y^{-T} - X$ is nonsingular. We will see that the matrix

$$X_{cl} := \begin{bmatrix} \gamma^2 Y^{-T} & -\Delta \\ -\Delta & \Delta \end{bmatrix}$$

satisfies (5.22) with a controller (5.26). First, if we set $\hat{E} = E$,

$$E_{cl}^T X_{cl} = \begin{bmatrix} \gamma^2 E^T Y^{-T} & -E^T\Delta \\ -E^T\Delta & E^T\Delta \end{bmatrix}.$$

The equalities in (5.23a) and (5.24a) guarantee that $E_{cl}^T X_{cl} = X_{cl}^T E_{cl}$. Moreover, for any $x, y \in \mathbf{R}^n$,

$$\begin{aligned} &[x^T \quad y^T] \begin{bmatrix} \gamma^2 E^T Y^{-T} & -E^T\Delta \\ -E^T\Delta & E^T\Delta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \gamma^2 x^T E^T X x + (x-y)^T E^T \Delta (x-y) \geq 0 \end{aligned}$$

holds from (5.23a) and (5.25), and this implies (5.22a). Next, define the following matrices:

$$\begin{aligned} &\tilde{S} := \tilde{T} := \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}, \\ &\bar{S} := \begin{bmatrix} I & \frac{1}{\gamma^2} Y\Delta^T \\ 0 & I \end{bmatrix}, \quad \bar{T} := \begin{bmatrix} I & \frac{1}{\gamma^2} Y^T\Delta \\ 0 & I \end{bmatrix}, \\ &\tilde{X}_{cl} := \tilde{S}^T X_{cl} \tilde{T} = \begin{bmatrix} X & 0 \\ 0 & \Delta \end{bmatrix}, \quad \bar{X}_{cl} := \bar{S}^T X_{cl} \bar{T} = \begin{bmatrix} \gamma^2 Y^{-T} & 0 \\ 0 & * \end{bmatrix}, \\ &\tilde{A}_{cl} := \tilde{S}^{-1} A_{cl} \tilde{T}, \quad \tilde{B}_{cl} := \tilde{S}^{-1} B_{cl}, \quad \tilde{C}_{cl} := C_{cl} \tilde{T}, \\ &\bar{A}_{cl} := \bar{S}^{-1} A_{cl} \bar{T}, \quad \bar{B}_{cl} := \bar{S}^{-1} B_{cl}, \quad \bar{C}_{cl} := C_{cl} \bar{T}, \end{aligned}$$

and consider the following partitions:

$$\begin{bmatrix} \tilde{V}_1 & \tilde{V}_{12} \\ \tilde{V}_{12}^T & \tilde{V}_2 \end{bmatrix} = \mathcal{G}_\gamma(\tilde{X}_{cl}; \tilde{A}_{cl}, \tilde{B}_{cl}, \tilde{C}_{cl}), \quad \begin{bmatrix} \bar{V}_1 & \bar{V}_{12} \\ \bar{V}_{12}^T & \bar{V}_2 \end{bmatrix} = \mathcal{G}_\gamma(\bar{X}_{cl}; \bar{A}_{cl}, \bar{B}_{cl}, \bar{C}_{cl}),$$

where $\tilde{V}_1, \bar{V}_1 \in \mathbf{R}^{n \times n}$. Then $\tilde{V}_2 = \bar{V}_1 - \tilde{V}_1 + \tilde{V}_{12} + \tilde{V}_{12}^T$ holds (see Lemma 4 in [SMN90]). Further, setting $\hat{A}$, $\hat{B}$ and $\hat{C}$ by (5.26), we derive $\tilde{V}_1 = \tilde{V}_{12} = \tilde{V}_{12}^T = \mathcal{G}_\gamma(X; A +$

$B_2F, B_1, C_1) < 0$ from (5.23b) and $\bar{V}_1 < 0$ from (5.24b). Finally, we obtain (5.22) from

$$\mathcal{G}_\gamma(\tilde{X}_{cl}; \tilde{A}_{cl}, \tilde{B}_{cl}, \tilde{C}_{cl}) = \begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \begin{bmatrix} \tilde{V}_1 & 0 \\ 0 & \bar{V}_1 \end{bmatrix} \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} < 0.$$

□

Proof of Proposition 5.2.

(*Necessity*) Suppose that (5.23)~(5.25) hold. Then X and Y are nonsingular. Setting $\hat{X} := \gamma X^{-T}$, $\hat{Y} := \gamma Y^{-T}$, $W := \gamma(FX^{-1})^T$ and $Z := \gamma(Y^{-1}G)^T$, we have the following LMIs:

$$\hat{X}E^T = E\hat{X}^T \geq 0,\ E^T\hat{Y} = \hat{Y}^T E \geq 0,\ E^T(\hat{Y} - \hat{X}^{-T}) \geq 0, \tag{A.17a}$$

$$\begin{bmatrix} -A\hat{X}^T - \hat{X}A^T - BW^T - WB^T & -\hat{X}C_1^T & -B_1 \\ -C_1\hat{X}^T & \gamma I & 0 \\ -B_1^T & 0 & \gamma I \end{bmatrix} > 0, \tag{A.17b}$$

$$\begin{bmatrix} -A^T\hat{Y} - \hat{Y}^T A - C^T Z - Z^T C & -\hat{Y}^T B_1 & -C_1^T \\ -B_1^T\hat{Y} & \gamma I & 0 \\ -C_1 & 0 & \gamma I \end{bmatrix} > 0. \tag{A.17c}$$

Since E is assumed to be the diagonal form of (5.28), the equality conditions in (A.17a) imply that $\hat{X}$ and $\hat{Y}$ have the following form:

$$\hat{X} = \left[\begin{array}{c|c} \hat{X}_1 & \multirow{2}{*}{X_r} \\ 0 & \end{array}\right],\quad \hat{Y} = \left[\begin{array}{cc} \hat{Y}_1 & 0 \\ \hline \multicolumn{2}{c}{Y_l} \end{array}\right],$$

where $\hat{X}_1, \hat{Y}_1 \in \mathbf{R}^{r\times r}$ are nonsingular. Defining $P_X := \Sigma^{-1}\hat{X}_1$ and $P_Y := \hat{Y}_1\Sigma^{-1}$, we get the first inequality in (5.29) from the inequality conditions in (A.17a). Next, if we set X_0 as in (5.30c), it is easy to check that (A.17b) is identical to

$$Q_X - M_X U_X^T \begin{bmatrix} X_r^T \\ W^T \end{bmatrix} J^T - J\,[\,X_r \quad W\,]\,U_X M_X^T > 0, \tag{A.18}$$

which gives $N_X^T Q_X N_X > 0$. We get $N_Y^T Q_Y N_Y > 0$ from (A.17c) in the same way.

(*Sufficiency*) Suppose that (5.29) holds and define X, Y by (5.31). Then (5.23a), (5.24a) and (5.25) are easily checked. Next, applying Theorem 1 in [IS94], we see that $[\,X_r \quad W\,]$ defined in (5.33) satisfies (A.17b). The proof of (A.17c) is similar. Setting F, G as in (5.31), we get (5.23b) and (5.24b) from (A.17b) and (A.17c), respectively.

□

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