REPLACEMENT POLICIES

FOR

MARKOVIAN DETERIORATING SYSTEMS

(マルコフ型劣化システムの取替方策)

MAMORU OHASHI

CONTENTS

CHAPTER 1

INTRODUCTION

During the last two decades, a great deal of effort has been paid to the development of replacement policies for stochastically deteriorating systems. At the present time, there is a need for the development of replacement policies for modern complex systems such as aircrafts, space vehicles, large scale computer systems and so on. In this thesis we investigate the structure of optimal policies for replacement of such systems.

Derman [14, 1963] considered the basic replacement problem of an equipment whose states deteriorate stochastically during the operating time. The equipmemt is inspected at the beginning of each period and the state of the equipment is classified into one of the (L+1) states, namely, 0,1,...,L. The state 0 means the equipment is new and the state L means the equipment is failed. Either of two actions 0 or 1 is available for the states 1,2,..., L-1, where action 1 is to replace and action 0 is not to replace. If action 1 is taken, then the equipment is instantaneously replaced by a new one. If action 0 is taken, then its state evolves from i to j in one period according to the transition probability P_{ij}.

For a replacement, cost C is incurred, and if the equipment is failed before being replaced, additional cost K is charged. Because of this additional cost K he considered the possibility of replacing the equipment before being failed. Then he has shown under the following Condition A that the optimal replacement policy is a control limit policy such that replace the equipment if its state is in the set $\{i_0, i_0+1, \ldots, L\}$, otherwise, do not replace.

Condition A: The transition probabilities $\{P_{ij}\}$ have a stochastically monotone property, that is , for each $k=0,1,\ldots,L$

$$f_k(i) = \sum_{j=k}^{L} P_{ij}, \qquad i=0,1,\ldots,L-1$$

is nondecreasing function with respect to i.

Here, Condition A asserts that the probability of deterioration increases as the initial state number increases. According to this result, the search for an optimal replacement policy can be restricted within a distinguished subclass of all control limit policies which is very narrow compared with the class of all possible policies. This often enables us to obtain an optimal replacement policy by a simpler procedure. Moreover, control limit policies are generally more tractable and easier for the implementation of maintenance than non-control limit ones.

Barlow and Hunter [2, 1965] considered the replacement problem for stochastically failing equipment whose lifetime has cdf $F(t)$. They have shown under the following Condition B that the optimal replacement policy is an age replacement policy so as to replace the equipment at failure or at age T, whichever comes first.

Condition B: The lifetime distribution $F(t)$ has IFR property, that is, the failure rate

$$\lambda(t) = \frac{dF(t)}{dt} \Big/ (1-F(t))$$

is nondecreasing function of t.

Analogously to Condition A, Condition B states that the probability of failure increases as the age increases. If the age of failed equipment indicates infinity, then it is shown this age replacement policy turns out to have the same structure as the control limit policy.

Jorgenson, McCall and Radner [25, 1967] considered an equipment consisting of two components, labeled by U_1 and U_2. The failure of its components are stochastically independent and the equipment fails when either component fails. Its components can be replaced separately or jointly, and the following costs are considered:

K : the breakdown cost,
C_1 and C_2 : the replacement cost of U_1 and U_2, respectively,
C_{12} : the cost of replacing both components jointly,

where $C_{12}<C_1+C_2$ is assumed. Then they studied the optimal replacement policy for component U_1 in the case when the failure rate of component U_2 is constant, and they have shown under Condition B that the optimal replacement policy for component U_1 is an (n,N) policy such that

(a) if $x<n$, replace component U_1 only if it fails,
(b) if $n\leqq x<N$, replace component U_1 if either component fails,
(c) if $N\leqq x$, replace component U_1 at once,

where x is the age of component U_1. However, the structure of the optimal replacement policy has not be resolved for a general case where the failure rates of both components are increasing and the cost of a breakdown is positive. Vergin [53, 1968] has derived recursive functional equations by the technique of dynamic programming, and has given numerical solutions for certain values of some parameters. Further Berg [8, 1978] has suggested an opportunistic age replacement policy for two components in general case, and has computed some long run operating characteristics of this policy. This policy may be similar to the (n,N) policy for each component.

On the other hand, Tahara and Nishida [50, 1975] considered the so-called minimal repair for replacement problems. In the minimal repair, the equipment stays in the state prior to its failure. They considered the following costs.

K: the breakdown cost,

C: the replacement cost,

M: the minimal repair cost,

where M<C+K is assumed. Then they have shown under Condition B that the optimal replacement policy is a (t,T) policy such that

(a) if $x<t$, carry out minimal repair only if it fails,

(b) if $t\leqq x<T$, replace the equipment only if it fails,

(c) if $T\leqq x$, replace the equipment at once,

where x is the age of the equipment. However, they have not discussed the structure of the optimal replacement policy for equipments consisting of more than two components with minimal repair.

In this thesis we consider replacement problems of an equipment consisting of n components. Most of the previous replacement models have been developed for equipments consisting of single-component. However, most equipments consist of various components. Also, the transition or failure probabilities of these components are usually stochastically dependent and the cost of replacing several components jointly is less than the sum of the costs required for separate replacements. Then the replacement policy for each component may depend upon the states of the other components. Furthermore, we take into consideration of minimal repair at failure of n-component system. Then we investigate the structure of an optimal replacement policy for components in the system possessing stochastic dependence and economic interdependence under the criterion of minimizing the expected total discounted cost, and will provide a simple replacement policy which leads to easier implementation. This policy will be called (ABC)-policy. Moreover, we also study the group replacement policy for the system and obtain the operating characteristics of the simple replacement policies.

In Chapter 2 we consider the structural relationship between a system and its components, and study a replacement problem for components in the system. The system consists of n components subject to Markovian deterioration, though its components are not necessarily independent each other. We define a coherent system consisting of n components. Furthermore, a coherent system with minimal repair will be defined. Then we are interesting in clarifying the structure of an optimal replacement policy for components in these systems under the criterion of minimizing the expected total discounted cost. First we formulate discrete time replacement model for a coherent system. The system is observed at the beginning of each time period and its state is identified. Immediately after each observation an action is taken as to whether or not to replace each component in the coherent system. We assume that the time consumption for replacement is negligible. Then we can find the structural properies of an optimal replacement policy under certain conditions concerning the costs and its transition probabilities. Finally, numerical examples are shown to illustrate the optimal replacement policy.

In Chapter 3 we consider discrete time replacement models for replacing components in a coherent system with minimal repair. The time consumption for replacement and minimal repair is not negligible. After each observation, an action is taken as to whether or not to replace each component in the coherent system with minimal repair, besides an action is taken as to whether or not to carry out minimal repair, also. We first study the properties of the stochastic process representing the behavior of deterioration levels of the coherent system, and clarify the structure of an optimal replacement policy. Furthermore, we suggest a simple replacement policy, called (ABC)-policy, which is easily implementable.

Continuous time replacement models are discussed in Chapter 4. The deterioration process of the coherent system with minimal repair is represented by a jump process in Section 4.2. Then the structure of an optimal replacement policy is clarified similarly for Chapter 3. In Section 4.3 we consider a coherent system consisting of n stochastically failing components with continuous lifetime distributions. Its components are stochastically independent and economically interdependent. Then we investigate the structure of an optimal replacement policy for components in the coherent system. Furthermore, we examine the structure of an optimal group replacement policy for a maintained coherent system consisting of n repairable components in Section 4.4. Finally, we discuss the relation between the replacement problem and optimal stopping problem for the coherent system subject to cumulative damage model.

In Chapter 5 the operating characteristics of several simple replacement policies are obtained for two-component system. Two-component system is one of the important system in reliability or replacement theory. First, for two-component parallel redundant system with repair, we obtain the distribution of the first passage time to the system failure, the stationary availability, the expected number of repair and so on. Besides, for two-component system under (ABC)-policy, we show the stationary availability, the expected number of the opportunistic replacement and so on. Finally, the operating characteristics of two-component system with minimal repair under (ABC)-policy are obtained.

Notes for Chapter 1

The replacement policies for stochastically failing system are surveyed by McCall [34, 1965], and the survey of replacement models for deteriorating system is given by Piershalla and Vaelker [40, 1976]. The fundamental replacement models appear in Derman [14, 1963]. Further results are obtained by Kolesar [30, 1966], Ross [41, 1969], Kalyman [26, 1972], Kao [28, 1973], Feldman [18, 19, 1977], Nummelin [37, 1980] and Siedersleben [48, 1981]. The age replacement policy is discussed by Barlow and Hunter [2, 1965], and extended by Morimura [35, 1970], Wolfe and Subramanian [54, 1974] and Clèroux, Dubuc and Tilquin [13, 1979]. The (n,N) policy is treated by Jorgenson, McCall and Radner [25, 1967], and discussed by Vergin [53, 1968] and Berg [8, 1978]. The (t,T) policy appears in Tahara and Nishida [50, 1975].

CHAPTER 2

REPLACEMENT PROBLEM FOR MARKOVIAN DETERIORATING SYSTEMS

2.1 Introduction

In this chapter we consider the structural relationship between a system and its components, and study the replacement problem for components in the system. So far, replacement theories have been developed for equipments consisting of single component. Most equipment, however, consists of various components. Moreover, the transition probabilities between the states of several components are not stochastically independent and the cost of replacing several components jointly is less than the sum of the costs of these separate replacements. Then the replacement policy for each component may depend upon the states of the other components. Our main interest is the structural properties of an optimal replacement policy with respect to a discrete time replacement model for components in a system cosisting of n components.

In Section 2.2 we consider the dynamic and probabilistic relationship between the deterioration levels of the system and its components, and give a formal definition of the system considered in replacement problems. A discrete time replacement model is defined in Section 2.3. In Section 2.4, we investigate the structural properties of the optimal replacement policy minimizing the expected total discounted cost. Finally in Section 2.5, we show some examples of the optimal replacement policy.

2.2 Coherent System

2.2.1 Definitions

First, we will give a formal definition of a system considered in replacement problems. Let $N=\{1,2,\ldots,n\}$ be a set of components and E_i be a set of deterioration levels of component i for each $i \in N$. And let E_s be a set of deterioration levels of the system composed of n components. A system composed of n components is said to be an *n-component system* if there exists a function $\phi(.)$ with domain $E_c = \prod_{i \in N} E_i$ and range E_s. Using this function $\phi(.)$, the present deterioration level of the n-component system is completely determined by the present deterioration levels of its components.

Let E_i, $i \in N$, and E_s be partially ordered sets with relation $\geq$. They are considered as lattices with maximal and minimal elements. The maximal element e_i of E_i represents the worst state of component i, and the minimal element 0 of E_i represents the best state of component i. Similarly the maximal element e_s and minimal element 0 of E_s represent the worst and best states of the n-component system, respectively.

Definition 2.1. An n-component system is said to be *monotone* if $\phi(x_c^1) \vee \phi(x_c^2) \leq \phi(x_c^1 \vee x_c^2)$ for all x_c^1 and x_c^2 in E_c.

We can easily obtain the following property from Definition 2.1 and assumptions of the state space.

Property 2.1. The structure function $\phi(.)$ of the monotone system is isotone in the sense that $x_c^1 \geq x_c^2$ in E_c implies $\phi(x_c^1) \geq \phi(x_c^2)$ in E_s, and $\phi(e)=e_s$ and $\phi(0)=0$ hold, where $e=(e_1,\ldots,e_n)$ and $0=(0,\ldots,0)$.

Remark 2.1. This property notes that improving the deterioration of a component is not harmful to the monotone system, further notes that a monotone system is in the worst (best) state if every component is the worst (best) state.

Definition 2.2. Component i of an n-component system is said to be *irrelevant* if for each $x_c \in E_c$ there exists $x_s \in E_s$ such that $x_s=\phi(x_i,x_c)$ for all $x_i \in E_i$, where $(._i,x_c)=(x_1,\ldots,x_{i-1},.,x_{i+1},\ldots, x_n)$. Otherwise, component i is said to be *relevant*.

Notice that the relevant component of the n-component system has some role in the system.

In this thesis we restrict our consideration to the n-component system such that the *structure function* $\phi(.)$ is monotone increasing with respect to each argument, and every component is relevant.

Definition 2.3. An n-component system is said to be *coherent* if it is monotone and each component is relevant.

2.2.2 Dynamic Models

In the following we consider dynamic models in which the deterioration levels of the coherent system and its components vary over time. Let $(\Omega,\mathcal{J}, P)$ be a probability space and T be a subset of the extended real numbers. Further let (E_i,β_i) be a measurable state space of component i where β_i contains all singleton events $\{x\}$, $x \in E_i$, and let (E_s,β_s) be a measurable state space of the coherent system. It is assumed that each state space is a non-empty Borel subset of complete separable space. Then for each $t \in T$, let $X_i(t)$, $i \in N$, be a measurable function which maps from $(\Omega,\mathcal{J})$ to (E_i,β_i), and the stochastic process $\{X_i(t);\ t \in T\}$ is a *deterioration process* for component i. Let $X_c(t)=(X_1(t),\ldots,X_n(t))$ denote the vector process of component deterioration.

Definition 2.4. A coherent system is said to be *Markov coherent* if the structure function $\phi(.)$ is a measurable function from (E_c, β_c) to (E_s, β_s) and a stochastic process $\{X_c(t);\ t\in T\}$ is a Markov process with values in a state space $(E_c, \beta_c) = (\prod_{i\in N} E_i, \prod_{i\in N} \beta_i)$.

This definition states that the present deterioration level of a coherent system at time t is completely determined by deterioration levels of components at the present time t irrespective of the past.

Next we shall consider a failure of the coherent system. Let E_0 be a partially ordered set with relation $\geq$ and a lattice with minimal element 0 and maximal element e_0. Further let (E_0, β_0) be a measurable state space, and let $X_0(t)$, $t\in T$, be a measurable function which maps from $(\Omega, \mathcal{J})$ to (E_0, β_0). Now let δ be a failure time of the coherent system. Namely,

$$\delta = \inf\{t\in T \mid X_0(t) \neq 0\}.$$

The stochastic process $\{X_0(t);\ t\in T\}$, called a *damage process*, represents the behavior of the state showing the damage of failure of the coherent system, and may be interdependent upon the stochastic process $\{\phi(X_c(t));\ t\in T\}$ and random damage for the coherent system. Let $X(t) = (X_0(t), X_c(t))$ be the deterioration process of the Markov coherent system with failure and random damage.

Definition 2.5. A Markov coherent system with random failure damage is said to be *M-Markov coherent* if a stochastic process $\{X(t);\ t\in T\}$ is a Markov process with values in a state space (E, β), where $(E, \beta) = (\prod_{i\in N_0} E_i, \prod_{i\in N_0} \beta_i)$ and $N_0 = \{0, 1, \ldots, n\}$.

Now we shall consider a minimal repair for a M-Markov coherent system. When this system fails at state $x = (x_0, x_c)$, $x_0 \neq 0$, the minimal repair brings the system to the state $x = (0, x_c)$.

Example. We assume that the stochastic process $\{X_0(t);\ t\in T\}$ depends upon the stochastic process $\{\phi(X_c(t));\ t\in T\}$ and the random damage for the system. Let $X_0(t)$ denote the magnitude of the shortage of the system performance for the demand at time t. Then the M-Markov coherent system fails if the shortage occurs.

2.2.3 Replacement Policies

Our main interest is replacement problems for components in a coherent system. Let $Z_c=\{X_c(t);\ t\in T\}$ be the deterioration process of a Markov coherent system. The stochastic process Z_c is called a *coherent process* for system deterioration. Similarly, let $Z=\{X(t);\ t\in T\}$ be a *M-coherent process*. At a specified time point $t\in T$, the state $X_c(t)$ of the Markov coherent system is observed, and based on the history of the stochastic process Z_c up to time t, an action is taken to replace each component or to keep it. Let D_c be the action space of a replacement model for components in a Markov coherent system. Similarly when the state $X(t)$ of a M-Markov coherent system is observed, the possible actions are "no action", "replace each component" and "carry out minimal repair for the system". Let $D=D_c\cup\{m\}$ be the action space of a replacement model for a M-Markov coherent system. We assume that given the present state and action, the evolution of the stochastic process Z_c or Z untill the next action is stochastically independent of the past.

We restrict ourselves to nonrandomized Markov policies. A Markov policy $\pi(t,x_c)$ is a $\beta_t\times\beta_c$-measurable function from $T\times E_c$ into D such that $\pi(t,x_c)$ is an action when the state x_c is observed at time t. Let D_{cm} denote the set of all Markov policies. A Markov policy is called *stationary* if it is independent of time, that is $\pi(t,x_c)=\pi(x_c)$ for all $t\in T$ and $x_c\in E_c$. The set of all stationary policies will be denoted by D_{cs}. We restrict policies to a subset D_c in D_{cs}, which may be called a set of *admissible policies* for a

Markov coherent system. Similarly we define D_s as the set of admissible policies for a M-Markov coherent system.

2.3 Statement of the Problem

2.3.1 Explanation of the System

In this section we consider a discrete time replacement model for components in a coherent system. This system consists of n components under Markovian deterioration. The transition probabilities of each component are not independent each other, and the cost of replacing several components concurrently is less than the sum of the costs of replacing them at different time. The state of the coherent system possessing stochastic dependence and economic interdependence is observed at the discrete time periods $t \in T=\{0,1,...\}$. At these time points, an action is taken as to whether or not to replace each component. We are interesting in the structure of the optimal replacement policy of the discrete time replacement model for components in such a system.

2.3.2 Action and Cost

Let $X_c(t)=(X_1(t),...,X_n(t))$ denote the state of n components at time $t \in T$. After observing the state $X_c(t)$ of the coherent system at discrete time t, an action a_c is taken at each time point as to whether or not to replace each component. Now let $a_c=(a_1,...,a_n)$ represent the action taken for components in the coherent system, where $a_i \in D_i=\{0,1\}$ is an action taken for component i, and $D_c = \prod_{i \in N} D_i$. Here $a_i=1$ means replacing component i, and $a_i=0$ means keeping it. The time duration for replacement of each component is negligible.

Let $Z_c^{\pi}=\{X_c^{\pi}(t);\ t \in T\}$ be a stochastic process representing the behavior of the state of the coherent system under a stationary replacement policy $\pi \in D_c$. The transition probability $P^{\pi}(t,x_c,U)$ of the stochastic process Z_c^{π} is given by for each $U \in \beta_c$

$$P^{\pi}(1,x_c,U)=\mathrm{P}[X_c(t+1)\in U \mid X_c(t)=x_c,\ \pi(x_c)=a_c] \tag{2.1}$$
$$=Q(x_c^{a_c},U),$$

where $x_c^{a_c}=(x_1^{a_1},\ldots,x_n^{a_n})$, and $x_i^{a_i}=0$ if $a_i=1$ and $x_i^{a_i}=x_i$ if $a_i=0$. Then we can easily find that this system is a Markov coherent system under each replacement policy $\pi \in D_c$.

For the costs associated with the discrete time replacement model of the Markov coherent system, we consider a replacement cost $C_i(x_i)$ of component i, a set up cost $K(x_c)$ for replacement, and an operating cost $B(x_c)$ per period when the Markov coherent system is in state $x_c=(x_1,\ldots,x_n)$ at the beginning of the period. We assume that all costs and transition probabilities are known, and that all costs are bounded and nonnegative.

2.3.3 Expectation of Discounted Cost

Let $w^{\pi}(t)$, $t\in\mathrm{T}$, be the cost of the discrete time replacement model of the Markov coherent system at time t under a replacement policy $\pi \in D_c$. The expected total discounted cost $V_{\pi}(x_c)$ for an infinite horizon, when we start with the state x_c, is given by

$$V_{\pi}(x_c)=\mathrm{E}\Big[\sum_{t=0}^{\infty} \alpha^t w^{\pi}(t)\Big]. \tag{2.2}$$

We are interested to determine the structure of the optimal replacement policy which minimizes this expected total discounted cost with discount factor $\alpha\in[0,1)$. Let $V_{\alpha}(x_c)$ be the minimum expected total discounted cost when the Markov coherent system is in state x_c at the beginning. Then letting π^* be an optimal replacement policy, we have

$$V_\alpha(x_c)=\inf_{\pi\in D_c} V_\pi(x_c) \tag{2.3}$$
$$=V_{\pi^*}(x_c).$$

Under our assumptions, a stationary optimal replacement policy π^* exists in D_c (see Ross [42, 1970]). Thus $V_\alpha(x_c)$ satisfies the functional equation:

$$V_\alpha(x_c)=\min_{a_c\in D_c}\left[K(x_c)(1-I_0(a_c))+\sum_{i\in A(a_c)} C_i(x_i)+R(x_c^{a_c})\right], \tag{2.4}$$

where

$$R(x_c)=B(x_c)+\alpha\int_{E_c} V_\alpha(u)Q(x_c,du),$$

$$A(a_c)=\{i\in N \mid a_c\in D_c,\ a_i=1\}, \text{ and}$$

$$I_0(a_c)=\begin{cases}1 & \text{if } a_c=0,\\ 0 & \text{otherwise.}\end{cases}$$

2.4 Optimal Replacement Policy

2.4.1 Introduction

In this section we investigate the structural properties of an optimal replacement policy minimizing the expected total discounted cost under some conditions. We can find an optimal replacement policy by solving the functional equation (2.4). We can not, however, obtain a solution explicitly for this system. So some properties on the optimal replacement policy and the corresponding optimal expected total discounted cost are discussed.

2.4.2 Some Lemmas

First we shall examine the structural property of the optimal expected total discounted cost function $V_\alpha(x_c)$ under the following conditions. Let $B(E_c)$ denote a set of all bounded real valued β_c-

measurable function on E_c, and let $F(E_c)$ be a subset of $B(E_c)$ such that for $f \in B(E_c)$, $x_c' \geq x_c$ in E_c implies $f(x_c') \geqq f(x_c)$. An increasing set $U \in E_c$ is a subset for which the indicator function $I_U(x_c)$ is a member of $F(E_c)$. We shall denote the family of all increasing set U by $S(E_c)$.

Condition 2.1. $Q(x_c, U) \in F(E_c)$ for all $U \in S(E_c)$.

This condition asserts that the Markov coherent system has a tendency of monotonically increasing expexted deterioration. The following lemma will be used in the proof of Lemma 2.2 which presents a property of the optimal expected total discounted cost.

Lemma 2.1. If Condition 2.1 holds and $h \in F(E_c)$, then we have $\int_{E_c} h(u) Q(x_c, du) \in F(E_c)$.

Proof: For $h \in F(E_c)$, there exists a nonnegative sequence $\{b_i\}$ and a real number b_c such that for $U_i \in S(E_c)$

$$h_n(x_c) = b_c + \sum_{i=1}^{n} b_i I_{U_i}(x_c)$$

and

$$h(x_c) = \lim_{n \to \infty} h_n(x_c).$$

Then we have that

$$\int_{E_c} h_n(u) Q(x_c, du) = \int_{E_c} [b_c + \sum_{i=1}^{n} b_i I_{U_i}(u)] Q(x_c, du)$$

$$= b_c + \sum_{i=1}^{n} b_i Q(x_c, U_i).$$

Therefore the result follows directly from Condition 2.1 as $n \to \infty$. |

The above lemma is a generalization of an important result obtained by Derman [14, 1963]. The following lemma shows a structure of the optimal expected total discounted cost function under the following condition, and it is used in the proof of theorems which present the structural properties of the optimal replacement policy.

Condition 2.2. $B(x_c) \in F(E_c)$, $K(x_c) \in F(E_c)$ and $\sum_{i \in N} C_i(x_i) \in F(E_c)$.

Lemma 2.2. If Conditions 2.1 and 2.2 hold, then the optimal expected total discounted cost function $V_\alpha(x_c)$ is a member of $F(E_c)$.

Proof: The proof is carried out by the successive approximation technique. Let $V_0(x_c)=0$ and define recurrsively:

$$(2.5) \qquad V_k(x_c) = \min_{a_c \in D_c} [K(x_c)(1-I_0(a_c)) + \sum_{i \in A(a_c)} C_i(x_i) + R_k(x_c^{a_c})]$$

where

$$R_k(x_c) = B(x_c) + \int_{E_c} V_{k-1}(u) Q(x_c, du).$$

We first show $V_k(x_c) \in F(E_c)$ for each k by mathematical induction. For $k=1$ it follows trivially from Condition 2.2. Now suppose that $V_k(x_c)$ is a member of $F(E_c)$ for some k. We show that the same result holds for $k+1$. Under Condition 2.1 and 2.2 we have $R_{k+1}(x_c) \in F(E_c)$ by the induction hypothesis and Lemma 2.1. From equation (2.5) and Condition 2.2 we can easily obtain that $V_{k+1}(x_c)$ is a member of $F(E_c)$. Also it is easy to see that $V_k(x_c) \to V_\alpha(x_c)$ as $k \to \infty$, since all costs are bounded and $\alpha < 1$. Thus $V_\alpha(x_c)$ is a member of $F(E_c)$. ||

Condition 2.2 states that the operating cost, the set up cost for replacement and the replacement cost of components increase as a function of deterioration level of the Markov coherent system.

2.4.3 Structural Properties

The structural properties of the optimal replacement policy for components in the Markov coherent system are investigated. The following theorem shows a simple property of the optimal replacement policy.

Theorem 2.1. If the deterioration level of component i is in the best state 0, then the action to keep component i is optimal.

Proof: For each $i \in \mathrm{N}$, we define

$$[V_\alpha(x_c)]_k^i = \min_{a_c \in \mathrm{D}_0^i} [K(x_c)(1-\mathrm{I}_0(a_c)) + \sum_{j \in \mathrm{A}(a_c)} C_j(x_j) + \mathrm{R}(x_c^{a_c})], \tag{2.6}$$

and

$$[V_\alpha(x_c)]_r^i = \min_{a_c \in \mathrm{D}_1^i} [K(x_c) + \sum_{j \in \mathrm{A}(a_c)} C_j(x_j) + \mathrm{R}(x_c^{a_c})], \tag{2.7}$$

where $\mathrm{D}_j^i = \{a_c \in \mathrm{D}_c \mid a_i = j\}$ for each $j \in \mathrm{D}_i$. Then for $(0_i, x_c) \in \mathrm{E}_c$ we have

$$\begin{aligned} &[V_\alpha(0_i,x_c)]_k^i - [V_\alpha(0_i,x_c)]_r^i \\ &\le \min_{a_c \in \mathrm{D}_0^i} [\sum_{j \in \mathrm{A}(a_c)} C_j(x_j) + \mathrm{R}((0_i,x_c)^{a_c})] \\ &\quad - \min_{a_c \in \mathrm{D}_1^i} [\sum_{j \in \mathrm{A}(a_c)} C_j(x_j) + \mathrm{R}((0_i,x_c)^{a_c})], \end{aligned}$$

and for each $a_c \in \mathrm{D}_0^i$

$$\begin{aligned} &\sum_{j \in \mathrm{A}(a_c)} C_j(x_j) + \mathrm{R}((0_i,x_c)^{a_c}) - [\sum_{j \in \mathrm{A}(a_c)} C_j(x_j) + C_i(0) + \mathrm{R}((0_i,x_c)^{(1_i,a_c)})] \\ &= -C_i(0) + \mathrm{R}((0_i,x_c)^{a_c}) - \mathrm{R}((0_i,x_c)^{(1_i,a_c)}). \end{aligned}$$

Thus from $\mathrm{R}((0_i,x_c)^{a_c}) = \mathrm{R}((0_i,x_c)^{(1_i,a_c)})$ for $a_c \in \mathrm{D}_0^i$ we can easily obtain that $[V_\alpha(0_i,x_c)]_k^i - [V_\alpha(0_i,x_c)]_r^i < 0$. This completes the proof. ||

Remark 2.2. Note the optimal replacement policy is $\pi(0)=0$ from Theorem 2.1. This result is intuitively obvious.

Let $E^i = E_1 \times \cdots \times E_{i-1} \times E_{i+1} \times \cdots \times E_n$ for each $i \in N$, and $J_i(x_c^i)$ be a subset of E_i for each $x_c^i \in E^i$. Let $S(E_i)$ be the family of all increasing set U in E_i.

Definition 2.6. Let π_i be a stationary replacement policy for component i in a Markov coherent system. Then π_i is said to be a *control limit policy* with respect to component i if and only if there exists a *replacement set* $J_i(x_c^i) \in S(E_i)$ for each $x_c^i \in E^i$ such that if the state x_i is in the set $J_i(x_c^i)$, replace component i, otherwise, do not replace.

We examine some structural properties of the optimal replacement policy under the following additional condition.

Condition 2.3. $B(x_c) - K(x_c) - \sum_{i \in N} C_i(x_i) \in F(E_c)$.

Theorem 2.2. If Conditions 2.1, 2.2 and 2.3 hold, then there exists a control limit policy π_i with respect to component i minimizing the expected total discounted cost.

Proof: From equations (2.6) and (2.7) we have

$$[V_\alpha(x_c)]_k^i - [V_\alpha(x_c)]_r^i$$

$$= \min_{a_c \in D_0^i} [-K(x_c) I_0(a_c) - C_i(x_i) + \sum_{j \in A(a_c)} C_j(x_j) + R(x_c^{a_c})]$$

$$- \min_{a_c \in D_0^i} [\sum_{j \in A(a_c)} C_j(x_j) + R(x_c^{(1_i, a_c)})].$$

Then the difference $[V_\alpha(x_c)]_k^i - [V_\alpha(x_c)]_r^i$ is a member of $F(E_i)$ from Condition 2.3 and Lemma 2.2. Thus the result follows from Definition 2.6. ||

Remark 2.3. The optimal replacement set $J_i^*(x_c^i)$ of component i is given by

$J_i^*(x_c^i)=\{x_i \in E_i \mid [V_\alpha(x_c)]_k^i \geq [V_\alpha(x_c)]_r^i\}$.

Theorem 2.3. If Conditions 2.1, 2.2 and 2.3 hold, and the action to replace component i in the worst state e_i is optimal, then we have $J_i^*(e^i) \supset J_i^*(0^i)$.

Proof: From equations (2.6) and (2.7) and assumptions of this theorem, we have for each $i \in N$

$$[V_\alpha(x_i,e)]_k^i-[V_\alpha(x_i,e)]_r^i$$
$$=[K(x_i,e)+\sum_{\substack{j\in N\\ j\neq i}} C_j(e_j)+R(x_i,0)]-[K(x_i,e)+\sum_{j\in N} C_j(e_j)+R(0)]$$
$$=-C_i(x_i)+R(x_i,0)-R(0),$$

and from Theorem 2.1 we have

$$[V_\alpha(x_i,0)]_k^i-[V_\alpha(x_i,0)]_r^i=R(x_i,0)-[K(x_i,0)+C_i(x_i)+R(0)].$$

Then the following inequality holds

$$[V_\alpha(x_i,e)]_k^i-[V_\alpha(x_i,e)]_r^i > [V_\alpha(x_i,0)]_k^i-[V_\alpha(x_i,0)]_r^i.$$

Thus we can easily see that $J_i^*(e^i) \supset J_i^*(0^i)$ by the definition of the optimal replacement set $J_i^*(x_c^i)$. ||

Using the following definition we further clarify the structure of the optimal replacement policy for components in a Markov coherent system.

Definition 2.7. Let π^* be an optimal replacement policy. Then the following set $G(a_c)$ is called an *optimal region* of an action $a_c \in D_c$

$$G(a_c)=\{x_c \in E_c \mid \pi^*(x_c)=a_c\}.$$

Property 2.2. If Conditions 2.1, 2.2 and 2.3 hold, then the optimal region G(0) is closed in the sense that $x_c^1 \wedge x_c^2 \in G(0)$ for all x_c^1 and x_c^2 in G(0).

Proof: First if $x_c^1 \wedge x_c^2 = x_c^1$ or $x_c^1 \wedge x_c^2 = x_c^2$, then the result is obvious. Let $x_c^1 \wedge x_c^2 = (x_1^1 \wedge x_1^2, \ldots, x_n^1 \wedge x_n^2)$, then we have $x_c^1 \wedge x_c^2 \leq x_c^1$ or $x_c^1 \wedge x_c^2 \leq x_c^2$ for each $i \in \mathrm{N}$. Thus we have $x_c^1 \wedge x_c^2 \in \mathrm{G}(0)$ from Theorem 2.2. ||

Remark 2.4. Notice that the optimal region G(0) of an action 0 is not always a decreasing set in the sense that for each $x_c \in \mathrm{G}(0)$, $x_c' \leq x_c$ in E_c implies $x_c' \in \mathrm{G}(0)$. This fact is shown in Section 2.5.

Property 2.3. If Conditions 2.1, 2.2 and 2.3 hold, then the optimal region G(11) is closed in the sense that $x_c^1 \vee x_c^2 \in \mathrm{G}(11)$ for all x_c^1 and x_c^2 in G(11).

Proof: The result is easily obtained from Theorem 2.2. ||

Property 2.4. If Conditions 2.1, 2.2 and 2.3 hold, then the optimal region G(11) is an increasing set.

Proof: When $x_c^1 \leq x_c^2$ and $x_c^1 \in \mathrm{G}(11)$, we have for each $a_c \in \mathrm{D}_c$

$$\begin{aligned}
&[K(x_c^2)(1-\mathrm{I}_0(a_c)) + \sum_{j \in \mathrm{A}(a_c)} C_j(x_j) + \mathrm{R}(x_c^{2a_c})] - [K(x_c^2) + \sum_{j \in \mathrm{N}} C_j(x_j) + \mathrm{R}(0)] \\
&\geqq [K(x_c^1)(1-\mathrm{I}_0(a_c)) + \sum_{j \in \mathrm{A}(a_c)} C_j(x_j) + \mathrm{R}(x_c^{1a_c})] \\
&\qquad - [K(x_c^1) + \sum_{j \in \mathrm{N}} C_j(x_j^1) + \mathrm{R}(0)] \\
&> 0.
\end{aligned}$$

The first inequality is true from Condition 2.3, and the second inequality follows from $x_c^1 \in \mathrm{G}(11)$. Thus the result is obtained from equations (2.4). ||

Remark 2.5. We can intuitively expect that the optimal replacement policy $\pi^*(x_c)$ is an isotone function such that $x_c^1 \geq x_c^2$ in E_c implies $\pi^*(x_c^1) \geq \pi^*(x_c^2)$ in D_c. However this conjection is not correct. A counterexample is shown in Section 2.5.

2.5 Example

2.5.1 Model

In this section we consider a two-component system possessing stochastic independence and economic interdependence. Let $E_1=E_2=\{0,1,\ldots,7\}$ be the state space of each component. In this case the transition probability $Q(x_c,y_c)$ of a two-component system is given by

$$Q(x_c,y_c)=P^1_{x_1y_1}\cdot P^2_{x_2y_2},$$

where $P^i_{x_iy_i}$ is the transition probability of component i. We assume that $\sum_{y_i\geq k} P^i_{x_iy_i}$ is nondecreasing in x_i for all $k\in E_i$. Then we can easily obtain that $\sum_{y_c\in U} Q(x_c,y_c)$ for each $U\in S(E_c)$ is a member of $F(E_c)$. Further we assume $B(x_c)=B_1(x_1)+B_2(x_2)$ and $K(x_c)=K\neq 0$.

Table 2.1. Transition matrix P^i

$x_i \backslash y_i$	0	1	2	3	4	5	6	7
0	0.00	0.30	0.20	0.15	0.15	0.10	0.05	0.05
1	0.00	0.25	0.20	0.15	0.15	0.10	0.10	0.05
2	0.00	0.10	0.20	0.20	0.15	0.15	0.10	0.10
3	0.00	0.05	0.10	0.15	0.25	0.20	0.15	0.10
4	0.00	0.05	0.05	0.10	0.25	0.20	0.20	0.15
5	0.00	0.00	0.05	0.10	0.15	0.25	0.25	0.20
6	0.00	0.00	0.00	0.05	0.05	0.10	0.40	0.40
7	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00

Table 2.2. Operating cost $B_i(x_i)$ and replacement cost $C_i(x_i)$.

$\backslash x_i$	0	1	2	3	4	5	6	7
$B_i(x_i)$	0	5	10	15	20	25	30	50
$C_i(x_i)$	30	30	30	30	30	30	30	50

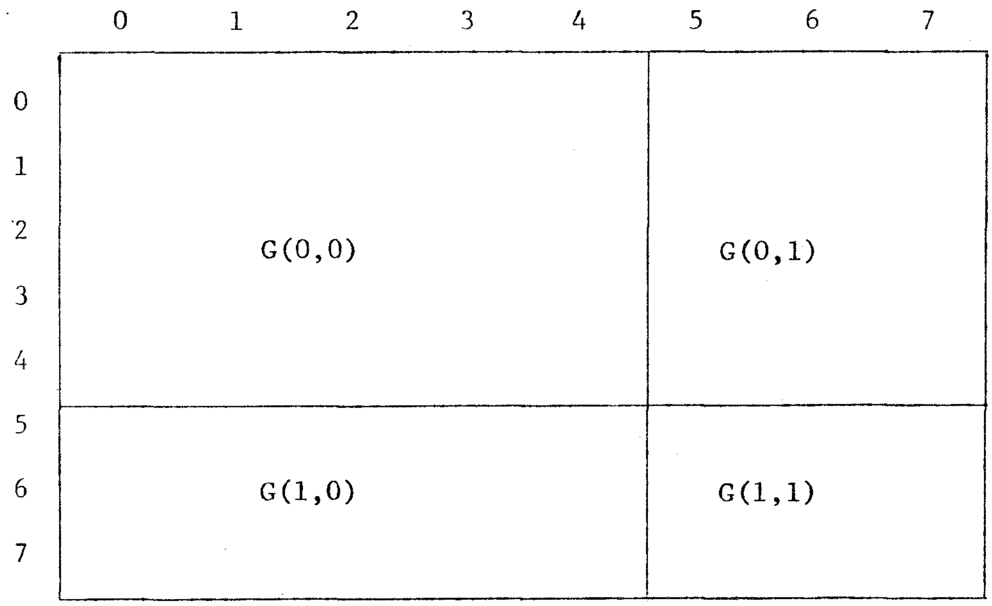

Figure 2.1. Optimal replacement policy for independent components

2.5.2 Numerical Examples

To illustrate an optimal replacement policy for the system of the preceding section, we consider numerical examples. The transition matrix P^i of component i is given in Table 2.1. The operating cost $B_i(x_i)$ and the replacement cost $C_i(x_i)$ are given in Table 2.2. Then Conditions 2.2 and 2.3 are satisfied. First consider the case where the components are stochastically and economically independent, i.e., suppose the set up cost is $2K$ when both components are replaced. Then the optimal replacement policy for components in the two-component system is shown in Figure 2.1 in the case of K=10 and α=0.95. Notice that this optimal replacement policy for one component doesn't depend on the state of the other component (see Vergin [53, 1968]).

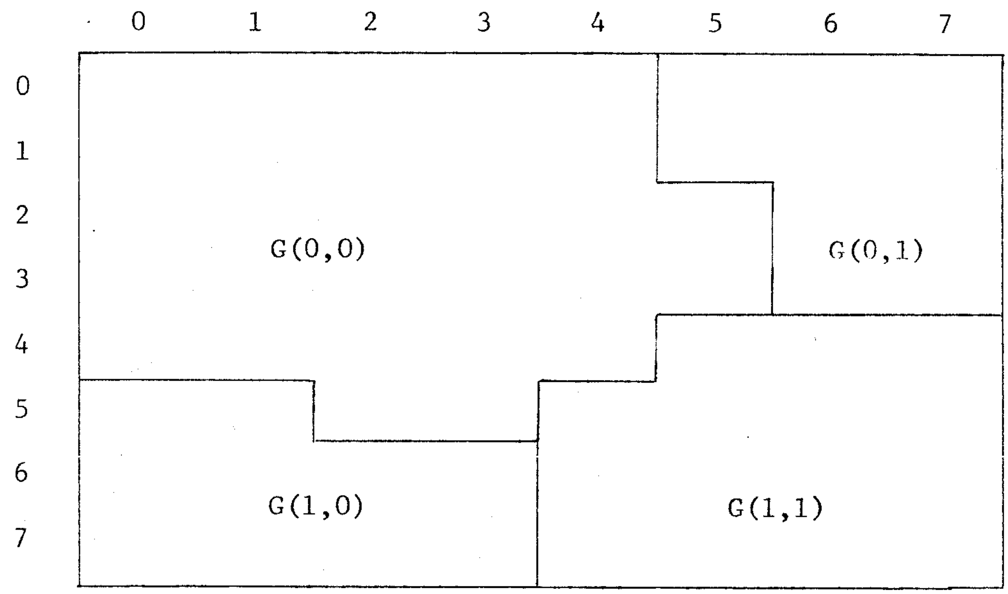

Figure 2.2. Optimal replacement policy for dependent components.

On the other hand the optimal replacement policy for components in the two-component system possessing economic dependence is show in Figure 2.2 in the case of K=10 and α=0.95. This sxample shows that the optimal replacement policy has the form of a control limit policy with four regions. For example $\pi^*(5,1)=(1,0)$ and $\pi^*(5,2)=(0,0)$ are not nondecreasing. Therefore this example also shows that the monotonicity of the optimal replacement policy $\pi^*(x_c)$ does not always hold.

2.5.3 Special Case

Here we consider the above two-component system with $E_1=[0,\infty]$ and $E_2=\{0,1\}$. Then the failure states of components are in state $e_1=\infty$ and $e_2=1$. We have from Remark 2.3

$$J_1^*(0)=\{x_1\in E_1 \mid [V_\alpha(x_1,0)]_k^1 \geq [V_\alpha(x_1,0)]_r^1\},$$

and

$$J_1^*(1)=\{x_1\in E_1 \mid [V_\alpha(x_1,1)]_k^1 \geq [V_\alpha(x_1,1)]_r^1\}.$$

Thus there exist n and N such that

$$J_1^*(0)=[N,\infty],$$

and

$$J_1^*(1)=[n,\infty].$$

Theorem 2.3 states that $J^*(0)\subset J^*(1)$ i.e., $n\leq N$. Therefore the optimal replacement policy π_1 with respect to component 1 is the same structural property of an (n,N) policy introduced by Jorgenson, McCall and Radner [25, 1967] in discrete time case.

In the numerical example, it can be seen that the optimal replacement policy is fairly close to the (n,N) policy with N=5 and n=4. It might be better to employ a simpler (n,N) policy because of easier manipulation than a more complex one.

Notes for Chapter 2

The definition of the coherent system appears in Barlow and Proschan [3, 1975], and is extended by EL-Neweihi, Proschan and Sethuraman [17, 1978], Ross[45, 1979], Griffith [22, 1980] and Buther [11, 1982]. Some standard concepts involving the notion of order and lattices appear in Topkis [52, 1978]. The Markov decision process is studied by Derman [14, 1963], Ross [42,1970] and Howerd [24, 1960]. The control limit policy of two-component system is discussed by Sethi [47, 1977], and is extended by Hatoyama [23, 1977].

CHAPTER 3

DISCRETE TIME REPLACEMENT MODELS

3.1 Introduction

In this chapter we consider discrete time replacement models with certain replacement time. Suppose the coherent system with a random damage consists of n components under Markovian deterioration. Its components are not stochastically independent and economically interindependent, and the replacement time and minimal repair time are not negligible. The objective of this chapter is to clarify the structure of the optimal replacement policy for components in the coherent system with minimal repair. Further we suggest a simple replacement policy, called (ABC)-policy, which is easy for implementation.

In Section 3.2, the discrete time replacement model is considered more elaborately, and the structure of the deterioration process is discussed. Under reasonable conditions the structural properties of the optimal replacement policy for minimizing the expected total discounted cost are investigated in Section 3.3. In Section 3.4, we show an example of an optimal replacement policy. Finally, in Section 3.5 we suggest s simpler (ABC)-policy which is easily manageable.

3.2 Model Formulation

3.2.1 Model of System

In this section, we consider a discrete time replacement model for a coherent system with minimal repair. The coherent system with minimal repair consists of n components under Markovian deterioration, and possesses stochastic dependence and economic interdependence. The system is observed at the beginning of discrete time periods $t \in T=\{0,1,...\}$, and classified into one of the possible number of the states. Further, when the system failure is observed, it is classified into one of the number of states showing the degree of the system failure. Then the possible actions are "no action", "replace each component" and "carry out minimal repair for the coherent system". A replacement of each component means the change of the component to new one, and minimal repair for the coherent system brings the state showing the degree of the system failure back to the best state. The time consumption required for replacement or minimal repair can not be negligible in this chapter. When an action $a \in D$ is taken on the coherent system with state $x=(x_0,...,x_n)$, the time consumption required for replacement or minimal repair, $T(x,a)$, has a probability distribution $G(t;x,a)$ with a finite mean.

3.2.2 Underlying Stochastic Process

Let $X(t)=(X_0(t),...,X_n(t))$ represent the state of the coherent system at time $t \in T$. If an action $a \in D^1=\{a \in D_c \mid a \neq 0\}$ is taken on the coherent system with state $X(t)=x$, then we have $X(t+T(x,a))=(0,x_c^a)$, and if $a=m$ the $X(t+T(x,a))=(0,x_c)$. We are interested in the state of the coherent system. Thus we introduce the following stochastic process $Z^\pi=\{z^\pi(t);\ t \in T\}$ under a replacement $\pi \in D_s$

$$(3.1) \qquad z^\pi(t)=X(T_t)$$

where $0=T_0<T_1<T_2<\ldots,\ T_t=T_{t-1}+S(z^{\pi}(t-1),\pi)$ and

$$S(z^{\pi}(t-1),\pi)=\begin{cases} 1 & \text{if } \pi(z^{\pi}(t-1))=0, \\ T(z^{\pi}(t-1),\pi) & \text{otherwise.} \end{cases}$$

Then the transition probability $P^{\pi}(t,x,U)$ of the stochastic process Z^{π} is given by for each $U\in\beta$

$$\begin{aligned} P^{\pi}(1,x,U)&=\mathrm{P}[z^{\pi}(t+1)\in U \mid z^{\pi}(t)=x,\ \pi(x)=a] \\ &=\begin{cases} Q(x,U) & \text{if } a=0, \\ 1 & \text{if } a=m \text{ and } (0,x_c)\in U, \\ 1 & \text{if } a\in \mathrm{D}^1 \text{ and } (0,x_c^a)\in U, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3.2}$$

Then we can easily find that the coherent system with minimal repair is a M-Markov coherent system under each replacement policy $\pi\in D_s$.

3.2.3 Some Lemmas

We shall study the structural properties of the deterioration process Z of the M-Markov coherent system. Let B(E) be a set of all bounded real valued β-measurable function on E. Let F(E) be a subset of B(E) such that for $f\in$B(E), $x'\geq x$ in E implies $f(x')\geq f(x)$. Furthermore, let S(E) be the family of all increasing set U on E.

Definition 3.1. Let M(E) be a set of probability measure on the state space E. We say that $P_1\in$M(E) is *stochastically smaller* than $P_2\in$M(E), and denote this by $P_1\leq P_2$, if and only if $\int f dP_1 \leq \int f dP_2$ for all $f\in$F(E).

Remark 3.1. A simple approximation argument shows that this is equivalent to the requirement that $P_1(U)\leq P_2(U)$ for all $U\in$S(E).

Definition 3.2. A deterioration process Z of the M-Markov coherent system is said to be *stochastically monotone* if and only if $\mathrm{P}[X(t)\in U \mid X(s)=x]\in$F(E) for all $U\in$S(E) and $t>s$ in T={0,1,...}.

Condition 3.1. $Q(x,U)\in\mathrm{F(E)}$ for all $U\in\mathrm{S(E)}$.

The following lemmas show the structure of the deterioration process Z of the M-Markov coherent system.

Lemma 3.1. The deterioration process Z of the M-Markov coherent system is stochastically monotone if and only if Condition 3.1 holds.

Proof: Assume the stochastic process Z is stochastically monotone. Then Condition 3.1 follows from Definition 3.2 and (3.2). Inversely, from Condition 3.1 we obtain that for each $U\in\mathrm{S(E)}$

$$\mathrm{P}[X(t+1)\in U|\ X(t)=x]=Q(x,U)$$

is a member of F(E). Suppose that for some k, $\mathrm{P}[X(t+k)\in U|\ X(t)=x]$ is a member of F(E). Then we have

$$\mathrm{P}[X(t+k+1)\in U|\ X(t)=x]=\int_{\mathrm{E}}\mathrm{P}[X(t+k+1)\in U|\ X(t+1)=u]Q(x,du)$$

$$=\int_{\mathrm{E}}\mathrm{P}[X(t+k)\in U|\ X(t)=u]Q(x,du).$$

Thus from Lemma 2.1 and the induction hypothesis, we can easily obtain that $\mathrm{P}[X(t+k+1)\in U|\ X(t)=x]$ is a member of F(E). Therefore the result directly follows. ||

Lemma 3.2. The deterioration process Z of the M-Markov coherent system is stochastically monotone if and only if $\mathrm{E}[f(X(t))|\ X(0)=x]$ $\leq\mathrm{E}[f(X(t))|\ X(0)=x']$ for all $f\in\mathrm{F(E)}$ and $x\leq x'$ in E.

Proof: By a simple approximation argument the result follows from Definitions 3.1 and 3.2. ||

3.2.4 Expectation of Discounted Cost

We investigate the structure of an optimal replacement policy which minimizes the expected total discounted cost with discount factor $\alpha\in[0,1)$. For the costs associated with the discrete time replacement model of the coherent system with minimal repair, we consider a replacement cost $C_i(x_i)$ of component i per period, a set up cost $K(x)$ of replacement per period, an operating cost $B(x)$ per period and minimal repair cost $M(x)$ per period when the system is in state x at the beginning of the period. We assume that all costs and transition probabilities are known, and that all costs are bounded and nonnegative. Further we assume for simplicity that each state space E_i is a subset of nonnegative real number R_+.

Now let $V_\alpha(x)$ be the minimum expected total discounted cost when the state of the system is x at the beginning. Then $V_\alpha(x)$ obeys the functional equation:

$$
\begin{aligned}
V_\alpha(x)=\min[&B(x)+\alpha\int V_\alpha(u)Q(x,du),\\
&\int\{M(x)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c)\}dG(t;x,m),\\
&\min_{a\in D^1}\int\{(K(x)+\sum_{i\in A(a)} C_i(x_i))\frac{1-\alpha^t}{1-\alpha}\\
&\qquad+\alpha^t V_\alpha(0,x_c^a)\}dG(t;x,a)]
\end{aligned}
\tag{3.3}
$$

where $D^1=\{a\in D_c \mid a\neq 0\}$, $A(a)=\{i\in N \mid a_i=1,\ a\in D_c\}$, $x_c^a=(x_1^a 1,\dots,x_n^a n)$ for $a\in D^1$, and

$$
x_i^a i=\begin{cases} x_i & if\ a_i=0,\\ 0 & if\ a_i=1.\end{cases}
$$

3.3 Structure of the Optimal Replacement Policy

3.3.1 Property of Optimal Expected Cost

Our aim is to examine the structural properties of the optimal replacement policy for the coherent system with minimal repair, under the criterion of the expected total discounted cost. First we seek the structural property of the optimal expected total discounted cost function. The following theorem shows the structure of $V_\alpha(x)$ under the following condition.

Condition 3.2. (1) $B(x)\in F(E)$, $K(x)\in F(E)$, $M(x)\in F(E)$ and $\sum_{i\in N} C_i(x_i)\in F(E)$.

(2) $1-G(t;x,a)\in F(E)$ for each $t\in T$ and $a\neq 0$.

This condition (1) means that the operating cost, the set up cost of replacement, the minimal repair cost and the replacement cost of components increase as a function of deterioration level of the coherent system with minimal repair. Condition (2) means that the replacement time and minimal repair time have a tendency for monotonically increase as a function of deterioration level of the system.

Theorem 3.1. Assume that the deterioration process Z of the M-Markov coherent system is stochastically monotone. If Condition 3.2 holds, then the optimal expected total discounted cost function $V_\alpha(x)$ is a member of F(E).

Proof: The proof is carried out by the method of Lemma 2.2. Let $V_0(x)=0$ and define recurrsively:

$$V_k(x)=\min[B(x)+\alpha\int V_{k-1}(u)Q(x,du),$$

$$\int\{M(x)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_{k-1}(0,x_c)\}dG(t;x,m), \tag{3.4}$$

$$\min_{a\in D^1}\int\{(K(x)+\sum_{i\in A(a)} C_i(x_i))\frac{1-\alpha^t}{1-\alpha}$$

$$+\alpha^t V_{k-1}(0,x_c^a)\}dG(t;x,a)].$$

We first show $V_k(x)\in F(E)$ for each k. We have $\int\frac{1-\alpha^t}{1-\alpha}\,dG(t;x,a)\in F(E)$ from Condition 3.2 (2) and Lemma 2.1. Therefore for k=1 it follows trivially from Condition 3.2 (1). Suppose that for some k

$V_k(x)\in F(E)$, $V_k(0,x_c)\leqq M(x)/(1-\alpha)$ and $V_k(x)\leqq(K(x)+C_i(x_i))/(1-\alpha)$ for each $i\in N$. Then under Condition 3.2, we obtain that

$$B(x)+\alpha\int V_k(u)Q(x,du)$$

is a member of F(E) by the induction hypothesis and Lemma 2.1. On the other hand, we obtain for each $a\in D^1$

$$(K(x)+\sum_{i\in A(a)} C_i(x_i))\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_k(0,x_c^a)$$

is increasing in t from the induction hypothesis. Thus

$$\int\{(K(x)+\sum_{i\in A(a)} C_i(x_i))\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_k(0,x_c^a)\}dG(t;x,a)$$

is a member of F(E) from Lemma 2.1. Similarly for $a=m$ we obtain that

$$\int\{M(x)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_k(0,x_c)\}dG(t;x,m)$$

is a member of F(E). Thus we have $V_{k+1}(x)\in F(E)$ from equation (3.4). Next we show that $V_{k+1}(0,x_c)\leqq M(x)/(1-\alpha)$ and $V_{k+1}(x)\leqq(K(x)+C_i(x_i))/(1-\alpha)$. From the functional equation (3.4) we have

$$\begin{aligned}V_{k+1}(x)&\leqq\int\{M(x)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_k(0,x_c)\}dG(t;x,m)\\&\leqq M(x)/(1-\alpha).\end{aligned}$$

The last inequality is true since $V_k(0,x_c)\leqq M(x)/(1-\alpha)$. Thus we have $V_{k+1}(0,x_c)\leqq M(x)/(1-\alpha)$ from $V_{k+1}(x)\in F(E)$. Also we have for each $i\in N$

$$\begin{aligned}V_{k+1}(x)&\leqq\int\{(K(x)+C_i(x_i))\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_k(0,(0_i,x_c))\}dG(t;x,(1_i,0))\\&\leqq(K(x)+C_i(x_i))/(1-\alpha).\end{aligned}$$

Thus we obtain that $V_{k+1}(x)$ is a member of F(E). Then we obtain

$V_k(x)\in F(E)$ for each k. Since $\alpha<1$ and all costs are bounded, it is easy to see that $V_k(x)\to V_\alpha(x)$ as $k\to\infty$ for each $x\in E$. Therefore we have $V_\alpha(x)\in F(E)$. ||

3.3.2 Properties of Optimal Replacement Policy

The structural properties of the optimal replacement policy for the discrete time replacement model are investigated. Let $F(D^1)$ be a set of all bounded real valued increasing function on D^1.

Condition 3.3. (1) $M(x)\leqq K(x)+C_i(x_i)$ for each $i\in N$,
(2) $G(t;x,a)\leqq G(t;x,m)$ for each $a\in D^1$,
(3) $1-G(t;x,a)\in F(D^1)$.

Condition 3.3 (1) states that the minimal repair cost is not larger than the replacement cost. Similarly, Condition 3.3 (2) states that the minimal repair time is not larger than the replacement time. Condition 3.3 (3) means that the replacement time has a trend for monotonically increase as a function of the number of the replacement components.

The following theorems show some simple properties of the optimal replacement policy.

Theorem 3.2. Assume that the deterioration process Z of the M-Markov coherent system is stochastically monotone, and Conditions 3.2 and 3.3 hold. If the deterioration level of component i is in the best state 0, then the action to keep component i is optimal.

Proof: Let

$$[V_\alpha(x)]_k^i=\min[B(x)+\alpha\int V_\alpha(u)Q(x,du),$$

$$\int\{M(x)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c)\}dG(t;x,m),$$

$$\min_{a\in D_0^i} \int \{ (K(x)+\sum_{j\in A(a)} C_j(x_j))\frac{1-\alpha^t}{1-\alpha} + \alpha^t V_\alpha(0,x_c^a) \} dG(t;x,a)],$$

$$[V_\alpha(x)]_r^i = \min_{a\in D_1^i} \int \{ (K(x)+\sum_{j\in A(a)} C_j(x_j))\frac{1-\alpha^t}{1-\alpha} + \alpha^t V_\alpha(0,x_c^a) \} dG(t;x,a),$$

where $D_j^i = \{a\in D_c \mid a\neq 0,\ a_i=j\}$ for each $j\in D_i$. Then we have

$$[V_\alpha(x)]_k^i - [V_\alpha(x)]_r^i$$

$$\leqq \min_{a\in D_0^i} \int \{ (K(x)+\sum_{j\in A(a)} C_j(x_j))\frac{1-\alpha^t}{1-\alpha} + \alpha^t V_\alpha(0,x_c^a) \} dG(t;x,a)$$

$$-\min_{a\in D_1^i} \int \{ (K(x)+\sum_{j\in A(a)} C_j(x_j))\frac{1-\alpha^t}{1-\alpha} + \alpha^t V_\alpha(0,x_c^a) \} dG(t;x,a)$$

and for each $a\in D_0^i$

$$\int \{ (K(x)+\sum_{j\in A(a)} C_j(x_j))\frac{1-\alpha^t}{1-\alpha} + \alpha^t V_\alpha(0,x_c^a) \} dG(t;x,a)$$

$$-\int \{ (K(x)+\sum_{j\in A(1_i,a)} C_j(x_j))\frac{1-\alpha^t}{1-\alpha} + \alpha^t V_\alpha(0,x_c^{(1_i,a)}) \} dG(t;x,(1_i,a))$$

$$\leqq \int \{ (K(x)+\sum_{j\in A(a)} C_j(x_j))\frac{1-\alpha^t}{1-\alpha} + \alpha^t V_\alpha(0,x_c^a) \} dG(t;x,a)$$

$$-\int \{ (K(x)+\sum_{j\in A(a)} C_j(x_j)+C_i(x_i))\frac{1-\alpha^t}{1-\alpha}$$

$$+\alpha^t V_\alpha(0,x_c^{(1_i,a)}) \} dG(t;x,a)$$

$$=\int \{ -C_i(x_i)\frac{1-\alpha^t}{1-\alpha} + \alpha^t (V_\alpha(0,x_c^a) - V_c(0,x_\alpha^{(1_i,a)})) \} dG(t;x,a).$$

$$\leqq 0.$$

The first inequality is true from Condition 3.3 (3) and the proof of Theorem 3.1, and the second inequality follows from $V_\alpha(0,(0_i,x_c)^a)$ $=V_\alpha(0,(0_i,x_c)^{(1_i,a)})$. Furthermore from Condition 3.3 (1) (2), we have for $x_i=0$

$$\min[B(x)+\alpha\int V_\alpha(u)Q(x,du),\ \int\{M(x)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c)\}dG(t;x,a)]$$
$$-\int\{(K(x)+C_i(x_i))\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,(0_i,x_c))\}dG(t;x,(1_i,a))$$
$$<0.$$

Thus we can easily obtain that the difference $[V_\alpha(x_0,(0_i,x_c))]_k^i$ $-[V_\alpha(x_0,(0_i,x_c))]_r^i<0$. Therefore the result directly follows. ||

Let $E^i=E_0\text{x}\cdots\text{x}E_{i-1}\text{x}E_{i+1}\text{x}\cdots\text{x}E_n$ and $J_i(x_0,x_c^i)$ be a subset of E_i for each $(x_0,x_c^i)\in E_0\text{x}E^i$. Let $S(E_i)$ be the family of all increasing set in E_i and $D^0=\{a\in D|\ a\neq 0\}$.

Definition 3.3. Let π_i be a stationary replacement policy for component i in a M-Markov coherent system. Then π_i is said to be a *control limit policy* with respect to component i if and only if there exists a *replacement set* $J_i(x_0,x_c^i)\in S(E_i)$ for each $(x_0,x_c^i)\in E_0\text{x}E^i$ such such that if state x_i is in the set $J_i(x_0,x_c^i)$, replace component i, otherwise , do not replace.

The structure of the optimal replacement policy will be clarified under the following additional conditions.

Condition 3.4. (1) $C_i(x_i)\doteq C_i$ for each $i\in N$,

(2) $G(t;x_0)=G(t;x,a)$ for all $a\in D^0$,

(3) $B(x)-K(x)\int\frac{1-\alpha^t}{1-\alpha}\,dG(t;x_0)\in F(E_c)$,

(4) $M(x)-K(x)\in F(E_c)$.

Theorem 3.3. Assume that the deterioration process Z of the M-

Markov coherent system is stochastically monotone. If Conditions 3.2, and 3.4 hold, then there exists a control limit policy π_i with respect to component i minimizing the expected total discounted cost of the discrete time replacement model.

Proof: Under Condition 3.4 (1) and (2), we have

$$[V_\alpha(x)]_k^i-[V_\alpha(x)]_r^i=\min[B(x)-K(x)\int\frac{1-\alpha^t}{1-\alpha}\,dG(t;x_0)+\alpha\int V_\alpha(u)Q(x,du),$$

$$\int\{(M(x)-K(x))\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c)\}dG(t;x_0),$$

$$\min_{a\in D_0^i}\int\{\sum_{j\in A(a)} C_j\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c^a)\}dG(t;x_0)]$$

$$-\min_{a\in D_1^i}\int\{\sum_{j\in A(a)} C_j\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c^a)\}dG(t,x_0).$$

Then from Condition 3.4 (3) and (4) we can easily obtain that the difference $[V_\alpha(x)]_k^i-[V_\alpha(x)]_r^i$ is a member of $F(E_i)$. Thus the result follows from the definition of the control limit policy . ||

Remark 3.2. Let $J_i^*(x_0,x_c^i)$ be the optimal replacement set minimizing the expected total discounted cost. Then since the state space of component i is a subset of nonnegative real number R_+, there exists a control limit $x_i^*(x_0,x_c^i)\in E_i\cup\{\infty\}$ such that $J_i^*(x_0,x_c^i)=[x_i^*(x_0,x_c^i),\infty]\cap E_i$, where $x_i^*(x_0,x_c^i)=\infty$ for $J_i^*(x_0,x_c^i)=\phi$.

Corollary 3.1. Under conditions of Theorem 3.3 if the action to replace component i with the worst state e_i is optimal, and $K(x)-M(x)\geqq 0$, then we have $x_i^*(x_0,e^i)\leqq x_i^*(x_0,0^i)$.

Proof: Since the action to replace component i with the worst state e_i is optimal, we have for each $x_0\in E_0$

$$[V_\alpha(x_0,(x_i,e))]_k^i-[V_\alpha(x_0,(x_i,e))]_r^i$$

$$=\int\{(K(x_0,(x_i,e))+\sum_{j\in N}C_j-C_i)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,(x_i,0))\}dG(t;x_0)$$

$$-\int\{(K(x_0,(x_i,e))+\sum_{j\in N}C_j)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0)\}dG(t;x_0)$$

$$=\int\{\alpha^t(V_\alpha(0,(x_i,0))-V_\alpha(0))-C_i\frac{1-\alpha^t}{1-\alpha}\}dG(t;x_0),$$

and from Theorem 3.2 we have

$$[V_\alpha(x_0,(x_i,0))]_k^i-[V_\alpha(x_0,(x_i,0))]_r^i$$

$$=\min[B(x_0,(x_i,0))+\alpha\int V_\alpha(u)Q((x_0,(x_i,0)),du),$$

$$\int\{M(x_0,(x_i,0))\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,(x_i,0))\}dG(t;x_0)$$

$$-\int\{K(x_0,(x_i,0))+C_i)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0)\}dG(t;x_0)]$$

$$\leqq\int\{(M(x_0,(x_i,0))-K(x_0,(x_i,0))\}\frac{1-\alpha^t}{1-\alpha}dG(t;x_0)$$

$$+\int\{\alpha^t(V_\alpha(0,(x_i,0))-V_\alpha(0))-C_i\frac{1-\alpha^t}{1-\alpha}\}dG(t;x_0).$$

Then from $K(x)-M(x)\geqq 0$, we have

$$[V_\alpha(x_0,(x_i,0))]_k^i-[V_\alpha(x_0,(x_i,0))]_r^i$$

$$\leqq[V_\alpha(x_0,(x_i,e))]_k^i-[V_\alpha(x_0,(x_i,e))]_r^i$$

Thus we can easily obtain that $x_i^*(x_0,e^i)\leqq x_i^*(x_0,0^i)$ by the definition of $x_i^*(x_0,x_c^i)$. $\|$

Remark 3.3. This corollary is concerned with (n,N) policy introduced by Jorgenson, McCall and Radner [25, 1967]. If the coherent system consists of two components and the state space of component 2 is $E_2=\{0,1\}$, then this corollary asserts that the optimal replacement policy for the coherent system with minimal repair is an (n,N) policy with $n=x_1^*(x_0,1)$ and $N=x_1^*(x_0,0)$ for each $x_0 \in E_0$.

Corollary 3.2. Under conditions of Theorem 3.3 if the action to replace component i with the worst state e_i is optimal, and $G(t)=G(t;x,a)$, then we have $x_i^*(x_0,e^i)=x_i^*(e_0,e^i)$ for each $x_0 \in E_0$.

Proof: From the proof of Theorem 3.3, we have

$$[V_\alpha(x_0,(x_i,e))]_k^i-[V_\alpha(x_0,(x_i,e))]_r^i$$

$$=\int\{(K(x_0,(x_i,e))+\sum_{j\neq i} C_j)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,(x_i,0))\}dG(t)$$

$$-\int\{(K(x_0,(x_i,e))+\sum_{j\in N} C_j)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0)\}dG(t)$$

$$=\int\{-C_i\frac{1-\alpha^t}{1-\alpha}+\alpha^t(V_\alpha(0,(x_i,0))-V_\alpha(0))\}dG(t).$$

Thus the result directly follows. ||

Let $F(E_0)$ be a set of all bounded real increasing function on E_0, and $\overline{F}(E_0)$ be a set of all bounded real decreasing function on E_0.

Condition 3.5. (1) $C_i(x_i)=C_i$ for each $i \in N$,

(2) $G(t)=G(t;x,a)$ for all $a \in D^0$,

(3) $B(x)-K(x)\int\frac{1-\alpha^t}{1-\alpha}\,dG(t) \in F(E_0)$,

(4) $M(x)-K(x) \in F(E_0)$.

Theorem 3.4. Assume that the deterioration process Z of the M-Markov coherent system is stochastically monotone. If Conditions 3.2 and 3.5 hold, then a control limit $x_i^*(x_0,x_c^i)$ is a member of $\overline{F}(E_0)$.

Proof: Under Condition 3.5 (1) and (2), we have

$$[V_\alpha(x)]_k^i - [V_\alpha(x)]_r^i$$

$$=\min[B(x)-K(x)\int\frac{1-\alpha^t}{1-\alpha}\,dG(t)+\alpha\int V_\alpha(u)Q(x,du),$$

$$\int\{(M(x)-K(x))\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c)\}dG(t),$$

$$\min_{a\in D_0^i}\int\{\sum_{j\in A(a)} C_j\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c^a)\}dG(t)]$$

$$-\min_{a\in D_1^i}\int\{\sum_{j\in A(a)} C_j\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c^a)\}dG(t).$$

Then from Condition 3.5 (3) and (4) we can easily obtain that the difference $[V_\alpha(x)]_k^i-[V_\alpha(x)]_r^i$ is a member of $F(E_0)$. Thus the result follows from the definition of $x_i^*(x_0,x_c^i)$. ||

Remark 3.4. If the optimal action of the operating state $(0,x_c)$ does not carry out minimal repair, then Theorem 3.4 holds under $M(x)-K(x)\in F(E_0-\{0\})$ in place of Condition 3.5 (4).

Condition 3.6. $B(x)-M(x)\int\frac{1-\alpha^t}{1-\alpha}\,dG(t)\in F(E_0)$.

Theorem 3.5. Assume that the deterioration process Z of the M-Markov coherent system is stochastically monotone. If Conditions 3.2, 3.5 and 3.6 hold, then there exists a control limit $x_0^*(x_c)$ for each $x_c\in E_c$ such that the action to carry out minimal repair for the system is optimal if and only if the failure damage x_0 exceeds $x_0^*(x_c)$.

Proof: From the functional equation (2.4), we have for each $x_c\in E_c$

$$[V_\alpha(x)]_{a=0}-[V_\alpha(x)]_{a=m}$$

$$=B(x)+\alpha\int V_\alpha(u)Q(x,du)-\int\{M(x)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c)\}dG(t)$$

$$=\alpha\int V_\alpha(u)Q(x,du)+\int\{B(x)-M(x)\frac{1-\alpha^t}{1-\alpha}-\alpha^t V_\alpha(0,x_c)\}dG(t)$$

Then we have $[V_\alpha(x)]_{a=0}-[V_\alpha(x)]_{a=m}\in F(E_0)$ by Conditions 3.2 and 3.6. Also under Condition 3.5 we have

$$[V_\alpha(x)]_{a=0}-[V_\alpha(x)]_{a\in D^1}$$

$$= B(x)+\alpha\int V_\alpha(u)Q(x,du)$$

$$-\min_{a\in D^1}\int\{(K(x)+\sum_{j\in A(a)}C_j)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c)\}dG(t)$$

$$=\int\{B(x)-K(x)\frac{1-\alpha^t}{1-\alpha}\}dG(t)+\alpha\int V_\alpha(u)Q(x,du)$$

$$-\min_{a\in D^1}\int\{\sum_{j\in A(a)}C_j\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c)\}dG(t).$$

Then we have $[V_\alpha(x)]_{a=0}-[V_\alpha(x)]_{a\neq 0}\in F(E_0)$. Thus we can easily find the result. ||

Remark 3.5. When the action to replace several components is made in the state $x=(x_0,x_c)$ $(x_0\neq 0)$, this action contains the action to carry out minimal repair. Condition 3.4 (4) can be obtained from Condition 3.5 (3) and Condition 3.6.

The following properties clarify the structure of the *optimal region* $G(a)=\{x\in E \mid \pi(x)=a\}$.

Property 3.1. Assume that the deterioration process Z of the M-Markov coherent system is stocahstically monotone. If Conditions 3.2, 3.4, 3.5 and 3.6 hold, then the optimal region G(0) is closed in the sense that $x^1\wedge x^2\in G(0)$ for all x^1 and x^2 in G(0).

Proof: For x^1 and x^2 in G(0), we have $x_i^1\wedge x_i^2\leq x_i^1$ or $x_i^1\wedge x_i^2\leq x_i^2$ for each $i\in N$. Thus the result follows Theorems 3.3 and 3.5. ||

Propertu 3.2. Assume that the deterioration process Z of the M-Markov coherent system is stochastically monotone. If Conditions 3.2, 3.4, 3.5 and 3.6 hold, then the optimal region $G(\mathbb{1})$ is closed in the sense that $x^1 \vee x^2 \in G(\mathbb{1})$ for all x^1 and x^2 in $G(\mathbb{1})$.

Proof: The proof is similar to that of Property 3.1. ||

Property 3.3. Assume that the deterioration process Z of the M-Markov coherent system is stochastically monotone. If Conditions 3.2, 3.4, 3.5 and 3.6 hold, then the optimal region $G(0_i, \mathbb{1})$ is closed in the sense that $x^1 \vee x^2 \in G(0_i, \mathbb{1})$ for all x^1 and x^2 in $G(0_i, \mathbb{1})$.

Proof: First if $x^1 \vee x^2 = x^1$ or $x^1 \vee x^2 = x^2$, then the result is obvious. Let $x^1 \vee x^2 = (x_0^1 \vee x_0^2, \ldots, x_n^1 \vee x_n^2)$, then we have $x^1 \vee x^2 \in G(0_i, \mathbb{1}) \cup G(\mathbb{1})$ from $x_i^1 \vee x_i^2 \leq x_i^1$ or $x_i^1 \vee x_i^2 \leq x_i^2$ for each $i \in N$ and Theorem 3.3. Then we have

$$[V_\alpha(x^1 \vee x^2)]_{a=\mathbb{1}} - [V_\alpha(x^1 \vee x^2)]_{a=(0_i, \mathbb{1})}$$

$$= \int \{ (K(x^1 \vee x^2) + \sum_{j \in N} C_j) \frac{1-\alpha^t}{1-\alpha} + \alpha^t V_\alpha(0) \} dG(t)$$

$$- \int \{ (K(x^1 \vee x^2) + \sum_{j \neq i} C_j) \frac{1-\alpha^t}{1-\alpha} + \alpha^t V_\alpha(0, (x_i^1 \vee x_i^2, 0)) \} dG(t)$$

$$= \int \{ C_i \frac{1-\alpha^t}{1-\alpha} + \alpha^t (V_\alpha(0) - V_\alpha(0, (x_i^1 \vee x_i^2, 0))) \} dG(t)$$

$$\geqq 0.$$

The last inequality is true since $x_i^1 \vee x_i^2 = x_i^1$ or $x_i^1 \vee x_i^2 = x_i^2$, and x^1 and x^2 in $G(0_i, \mathbb{1})$. Then the result is obvious. ||

Property 3.4. Assume that the deterioration process Z of the M-Markov coherent system is stochastically monotone. If Conditions 3.2, 3.4, 3.5 and 3.6 hold, then the optimal region $G(\mathbb{1})$ is a member of S(E).

Proof: The result is easily obtained by Theorems 3.3 and 3.5. ||

3.3.3 Some Special Cases

Next in the case where the replacement or minimal repair time depends on the action, the structure of the optimel replacement policy is clarified under the following conditions.

Condition 3.7. (1) $C_i(x_i)=C_i$ for each $i\in$N,
(2) $M(x)=M(x_0)$,
(3) $K(x)=K(x_0)$.

Theorem 3.6. Assume that the deterioration process Z of the M-Markov coherent system is stochastically monotone, and Conditions 3.2 and 3.7 hold. If $G(t;x_0,a)=G(t;x,a)$, then there exists a control limit policy π_i with respect to component i minimizing the expected total discounted cost of the discrete time replacement model.

Proof: We have for each $i\in$N

$$[V_\alpha(x)]_k^i-[V_\alpha(x)]_r^i$$

$$=\min[B(x)+\alpha\int V_\alpha(u)Q(x,du),$$

$$\int\{M(x_0)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c)\}dG(t;x_0,a),$$

$$\min_{a\in D_0^i}\int\{(K(x_0)+\sum_{j\in A(a)}C_j)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c^a)\}dG(t;x_0,a)]$$

$$-\min_{a\in D_1^i}\int\{(K(x_0)+\sum_{j\in A(a)}C_j)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c^a)\}dG(t;x_0,a).$$

Then the difference $[V_\alpha(x)]_k^i-[V_\alpha(x)]_r^i$ is a member of $F(E_i)$. Thus we can easily obtain the result. ||

Condition 3.7'. (1) $C_i(x_i)=C_i$ for $i\in$N,
(2) $M(x)=M(x_0)$,
(3) $K(x)=K$.

Theorem 3.7. Assume that the deterioration process Z of the M-Markov coherent system is stochastically monotone, and Conditions 3.2 and 3.7' hold. If $G(t;a)=G(t;x,a)$, then a control limit $x_i^*(x_0,x_c^i)$ is a member of $\overline{\mathrm{F}}(\mathrm{E}_0)$.

Proof: We have

$$[V_\alpha(x)]_k^i-[V_\alpha(x)]_r^i$$

$$=\min[B(x)+\alpha\int V_\alpha(u)Q(x,du),$$

$$\int\{M(x_0)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c)\}dG(t;m),$$

$$\min_{a\in \mathrm{D}_0^i}\int\{(K+\sum_{j\in \mathrm{A}(a)}C_j)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c^a)\}dG(t;a)]$$

$$-\min_{a\in \mathrm{D}_1^i}\int\{(K+\sum_{j\in \mathrm{A}(a)}C_j)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c^a)\}dG(t;a)$$

Then under Condition 3.7' we can easily obtain the result. ||

Condition 3.8. $B(x)-M(x_0)\int\frac{1-\alpha^t}{1-\alpha}\,dG(t;m)\in \mathrm{F}(\mathrm{E}_0)$.

Theorem 3.8. Assume that the deterioration process Z of the M-Markov coherent system is stochastically monotone, and Conditions 3.2, 3.7' and 3.8 hold. If $G(t;a)=G(t;x,a)$, then there exists a control limit $x_0^*(x_c)$ for each $x_c\in \mathrm{E}_c$ such that the action to carry out minimal repair for the system is optimal if and only if the failure damage x_0 exceeds $x_0^*(x_c)$.

Proof: For each $x_c\in \mathrm{E}_c$ we have

$$[V_\alpha(x)]_{a=0}-[V_\alpha(x)]_{a=m}=B(x)+\alpha\int V_\alpha(u)Q(x,du)$$

$$-\int\{M(x_0)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c)\}dG(t;m)$$

Then $[V_\alpha(x)]_{a=0}-[V_\alpha(x)]_{a=m}$ is a member of $F(E_0)$ by Condition 3.8. Further under Condition 3.7' we can show

$$[V_\alpha(x)]_{a\in D^1}-[V_\alpha(x)]_{a=0}$$
$$=\min_{a\in D^1}\int\{(K+\sum_{j\in A(a)}C_j)\frac{1-\alpha^t}{1-\alpha}+\alpha^t V_\alpha(0,x_c^a)\}dG(t;a)$$
$$-\{B(x)+\alpha\int V_\alpha(u)Q(x,du)\}$$

is a member of $F(E_0)$ by Condition 3.2. Thus the result is obvious. ||

The following property, further, show the structure of the optimal region of the replacement policy, but the proof is omitted.

Property 3.5. Assume that the deterioration process Z of the M-Markov coherent system is stochastically monotone, and Conditions 3.2, 3.7' and 3.8 hold. If $G(t;a)=G(t;x,a)$, then

(1) the optimal region G(0) is closed in the sense that $x^1\wedge x^2\in G(0)$ for all x^1 and x^2 in G(0),

(2) the optimal region $G(\mathbb{1})$ is closed in the sense that $x^1\vee x^2\in G(\mathbb{1})$ for all x^1 and x^2 in $G(\mathbb{1})$,

(3) the optimal region $G(\mathbb{1})$ is an increasing set in S(E),

(4) the optimal region $G(0_i,\mathbb{1})$ for each $i\in N$ is closed in the sense that $x^1\vee x^2\in G(0_i,\mathbb{1})$ for all x^1 and x^2 in $G(0_i,\mathbb{1})$,

3.4 Example

3.4.1 Model

In this section we consider a two-component system with minimal repair. Let $E_0=\{0,1\}$ and $E_1=E_2=\{0,1,\ldots,7\}$ be the state space. The transition probability is given by

$$Q(x,y)=P^0_{xy_0}\cdot P^1_{x_1y_1}\cdot P^2_{x_2y_2}$$

where $P^0_{xy_0}$ is the transition probability of the system failure damage, and $P^i_{x_iy_i}$ (i=1,2) is the transition probability of component i. To illustrate the optimal replacement policy, we consider a numerical example. The transition probability matrix P^0 of the system failure damage is given in Table 3.1, and the transition probability matrix P^i of component i is given in Table 3.2. The operating cost $B(x)=B_0(x_0)+B_1(x_1)+B_2(x_2)$, the replacement cost $C_i(x_i)$, the set up cost $K(x)=K(x_0)$, and the minimal repair cost $M(x)=M(x_0)$ are given in Table 3.3. Furthermore the replacement time and minimal repair time are one period. Then Conditions 3.1-3.6 are satisfied except for Condition 3.5 (4).

Table 3.1. Transition probability matrix $P^0=\{P_{(x_0,x_1,x_2)y_0}\}$

(a) $x_0=0$ and $y_0=0$

$x_1\backslash x_2$	0	1	2	3	4	5	6	7
0	1.0	0.9	0.8	0.7	0.6	0.5	0.3	0.1
1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.1
2	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
3	0.7	0.6	0.5	0.4	0.3	0.2	0.2	0.1
4	0.6	0.5	0.4	0.3	0.3	0.2	0.1	0.1
5	0.5	0.4	0.3	0.2	0.2	0.2	0.1	0.0
6	0.3	0.3	0.2	0.2	0.1	0.1	0.1	0.0
7	0.1	0.1	0.1	0.1	0.1	0.0	0.0	0.0

(b) $x_0=0$ and $y_0=1$

$x_1 \backslash x_2$	0	1	2	3	4	5	6	7
0	0.0	0.1	0.2	0.3	0.4	0.5	0.7	0.9
1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.9
2	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
3	0.3	0.4	0.5	0.6	0.7	0.8	0.8	0.9
4	0.4	0.5	0.6	0.7	0.7	0.8	0.9	0.9
5	0.5	0.6	0.7	0.8	0.8	0.8	0.9	1.0
6	0.7	0.7	0.8	0.8	0.9	0.9	0.9	1.0
7	0.9	0.9	0.9	0.9	0.9	1.0	1.0	1.0

(c) $x_0=1$ and $y_0=0$

$P_{(x_0,x_1,x_2)y_0}=0$ for all x_1 and x_2,

(d) $x_0=1$ and $y_0=1$

$P_{(x_0,x_1,x_2)y_0}=1$ for all x_1 and x_2.

Table 3.2. Transition probability matrix $P^i=\{P_{x_iy_i}\}$

$x_1 \backslash x_2$	0	1	2	3	4	5	6	7
0	0.00	0.30	0.20	0.15	0.15	0.10	0.05	0.05
1	0.00	0.25	0.20	0.15	0.15	0.10	0.10	0.05
2	0.00	0.10	0.20	0.20	0.15	0.15	0.10	0.10
3	0.00	0.05	0.10	0.15	0.25	0.20	0.15	0.10
4	0.00	0.05	0.05	0.10	0.25	0.20	0.20	0.15
5	0.00	0.00	0.05	0.10	0.15	0.25	0.25	0.20
6	0.00	0.00	0.00	0.05	0.05	0.10	0.40	0.40
7	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00

Table 3.3. Costs $B_i(x_i)$, $C_i(x_i)$, $B_0(x_0)$, $K(x_0)$ and $M(x_0)$.

x_i	0	1	2	3	4	5	6	7
$B_i(x_i)$	0	5	10	15	20	25	30	35
$C_i(x_i)$	70	70	70	70	70	70	70	70

x_0	0	1
$B_0(x_0)$	0	300
$K(x_0)$	100	160
$M(x_0)$	160	160

3.4.2 Numerical Example

The optimal replacement policy for components in a two-component system with minimal repair is shown in Figure 3.1 in the case of α=0.95. This example shows that the optimal replacement policy is similar to the (n,N) policy with N=6 and n=5 in the case of $x_0=0$, and is fairly close to the (n,N) policy with N=5 and n=5 in the case of $x_0=1$.

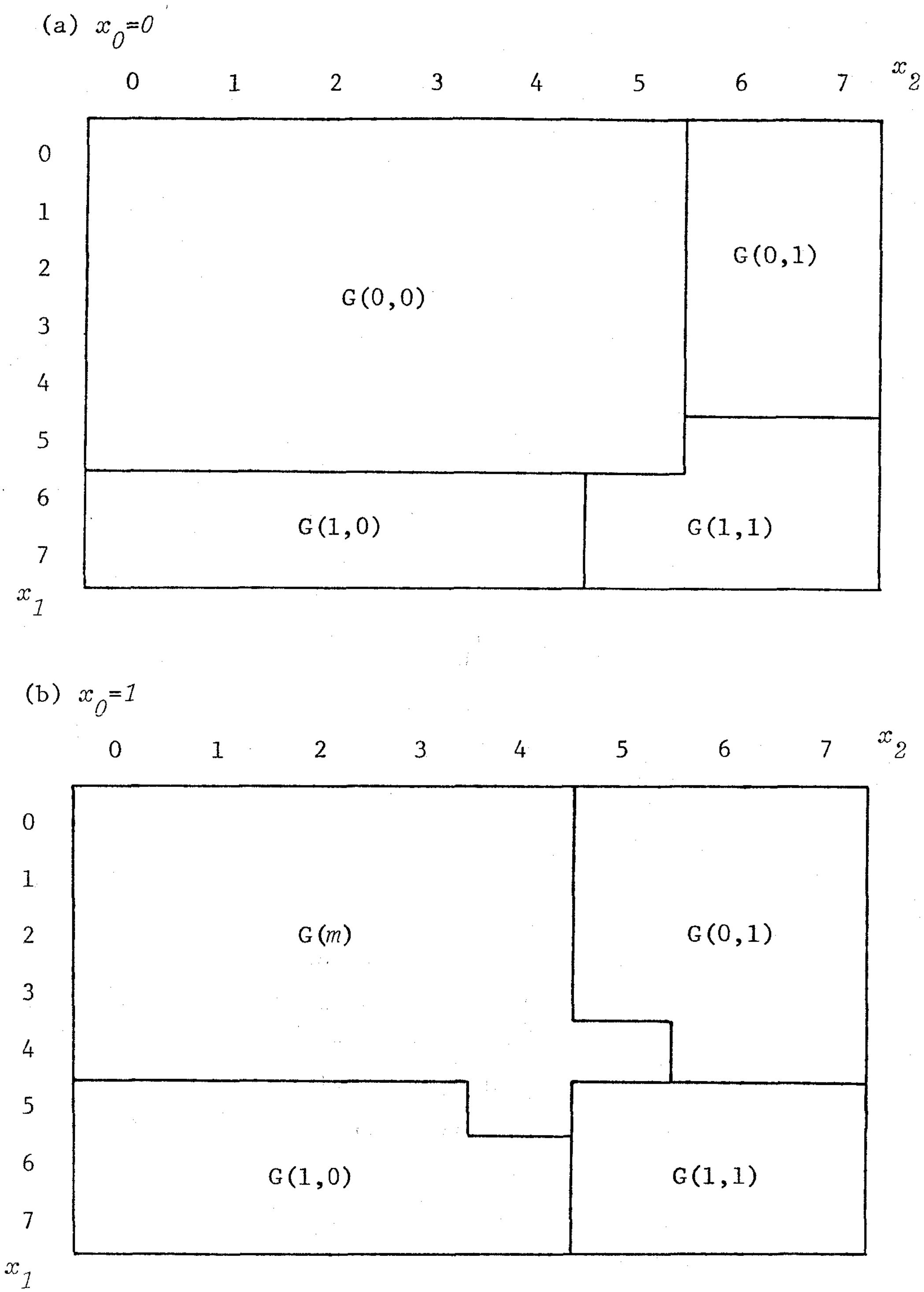

Figure 3.1. Optimal replacement policy.

3.5 (ABC)-Policy

In this section we are interesting in an elegant and simple replacement policy which lets to easily implementable policy. The optimal replacement policy is not so simple, and requires large scale computations for implementation. Thus we consider a simple replacement policy for component i, called (ABC)-policy, such that;

(1) if $0 \leq x_i < A$, keep component i,

(2) if $A \leq x_i < C$, replace component i concurrently if other components are replaced,

(3) if $B \leq x_i < C$, replace component i if the reliability system fails,

(4) if $C \leq x_i$, replace component i at once,

where x_i is the state of component i.

If the failure of the coherent system is not considered, then this (ABC)-policy is similar to (n,N) policy with A=n and C=N. Furthermore if the opportunistic replacement is not considered, then this (ABC)-policy is similar to (t,T) policy with B=t and C=T in the case of single-component system.

In the previous example, it can be seen that the optimal replacement policy is fairly close to the (ABC)-policy with A=5, B=5 and C=6. Thus in some cases it might be better to use a simple (ABC)-policy than a more complex one.

Notes for Chapter 3

The random minimal repair cost is discussed by Cléroux, Dubuc and Tilquin [13, 1980]. We extended this to a concept of the random failure damage, and we considered minimal repair for the coherent system with random failure damage. The (ABC)-policy suggested in this chapter is a generalization of an (n,N) policy and a (t,T) policy. The other typical replacement policies are an OARP(opportunistic age replacement policy) introduced by Berg [8, 1978], a trigger-off replacement policy suggested in Bansard, Descamps, Maarek and Morihain [1, 1970], replacement policy III, IV, V suggested in Morimura [35, 1970] and so on.

CHAPTER 4

CONTINUOUS TIME REPLACEMENT MODELS

4.1 Introduction

In this chapter we consider continuous time replacement models for components in a coherent system. The coherent system consisting of n components is monitored continuously. The time consumption required for replacement is not negligible except for the cases in Section 4.5. Our aim is to clarify the structure of the optimal replacement policy for the following systems. (1) The deterioration process of the coherent system is a jump process. (2) The coherent system consists of n stochastically failing components. (3) The maintained coherent system consists of n repairable components. Furthermore, we will study the optimal stopping time for a replacement problem of the coherent system consisting of n components under additive damage.

In Section 4.2 we consider a continuous time replacement model for the coherent system consisting of n components under jump deterioration. The binary coherent system, which consists of n stochastically failing components with some lifetime distributions, is dealt with in Section 4.3. In Section 4.4 we discuss the group replacement problem for the coherent system whose repairable components are separately maintained. Finally, we study the optimal stopping problem for replacement of the coherent system under additive damage.

4.2 Jump Deterioration System

4.2.1 Explanation of Model

In this section we consider a continuous time replacement model for the coherent system with minimal repair. The coherent system consisting of n components is stochastically dependent and economically interdependent. The coherent system is monitored continuously in infinite time interval $T=[0,\infty)$, and classified into one of the possible number of states. The possible actions are "no action", "replace each component" and "carry out minimal repair for the coherent system". Let $X(t)=(X_0(t),\ldots,X_n(t))$ donote the state of the coherent system with minimal repair. We assume that a stochastic process $Z=\{X(t);\ t\in T\}$ is a jump process, and an action is taken after a jump has occurred. Further we assume that each state space E_i, $i\in N$, is a subset of R_+.

4.2.2 Jump Deterioration Process

Let $Z^\pi=\{X^\pi(t);\ t\in T\}$ be a stochastic process representing the state of the coherent system with minimal repair under a replacement policy $\pi\in D_s$. Let $\lambda(x)$ be a β-measurable function from E into R_+. It is interpreted as the jump rate when the current state is x and the current action is "do nothing". Let $Q(x,U)$ be a β-measurable function from E into $[0,1]$ such that $Q(x,\cdot)$ is a probability measure on (E,β) for each $x\in E$. It means that if a jump occurs when the current state is x and the current action is no doing, then the state after the jump is determined by the probability measure $Q(x,\cdot)$. Similarly let $\mu(x,a)$ be a β-measurable function from E into R_+ for each $a\in D^0$ and it is interpreted as the jump rate of replacement or minimal repair when the current state is x and the current action is $a\epsilon D^0$. If a jump of replacement occurs then the state after the jump is $(0,x_c^a)$, and if a jump of minimal repair occurs then the state after the jump is $(0,x_c)$. Thus we can easily find that the stochastic process Z^π is a M-coherent process.

4.2.3 Monotonicity

We shall examine some properties of the deterioration process Z of the system with jump deterioration.

Definition 4.1. The deterioration process Z of the M-Markov coherent system is said to be *stochastically monotone* if and only if $P[X(t)\epsilon U|\ X(s)=x]$ is a member of F(E) for all $U\epsilon S(E)$ and all $t>s$ in $T=[0,\infty)$.

Lemma 4.1. Assume that the jump rate $\lambda(x)$ is constant. If $Q(x,U)\epsilon F(E)$ for each $U\epsilon S(E)$, then the deterioration process Z of the M-Markov coherent system is stochastically monotone.

Proof: Since the jump rate is constant $\lambda(x)=\lambda$, the transition probability $P(t,x,U)$ is given by for each $U\epsilon S(E)$

$$P(t,x,U)=\sum_{k=0}^{\infty}\frac{(\lambda t)^k}{k!}e^{-\lambda t}Q^k(x,U),$$

where $Q^0(x,U)=I_U(x)$ and

$$Q^k(x,U)=\int_E Q^{k-1}(u,U)Q(x,du),\quad k\geq 1.$$

Now we prove $Q^k(x,U)\epsilon F(E)$ for $U\epsilon S(E)$ by induction. For $k=1$ it follows trivially from $Q(x,U)\epsilon F(E)$. Suppose $Q^k(x,U)\epsilon F(E)$ for some k. Then we can easily obtain $Q^{k+1}(x,U)\epsilon F(E)$ from Lemma 2.1. Thus for each k $Q^k(x,U)$ is a member of F(E) for $U\epsilon S(E)$ and $P(t,x,U)$ is a member of F(E) for $U\epsilon S(E)$. Then the result is obvious from Definition 4.1. ||

Definition 4.2. Assume that the deterioration process Z of the M-Markov coherent system is a jump process, then the stochastic process Z is said to be *stochastically quasi-monotone* if and only if $\lambda(x)\in F(E)$ and $Q(x,U)\in F(E)$ for each $U\in S(E)$.

Remark 4.1. The jump process Z is a Markov process with state space (E,β) (see Blumenthel and DeGroot [10, 1968]). It is intuitively expected that the stochastic process Z being stochastically quasi-monotone is stochastically monotone, but this prediction is not proved.

4.2.4 Expected Total Discounted Cost

Let $w^{\pi}(t)$, $t\in T$, be the cost rate of the continuous time replacement model with jump deterioration in time t under a replacement policy π. For the cost rates associated with the continuous time replacement model, we consider a replacement cost rate $C_i(x_i)$ of component i, a fixed cost rate $K(x)$ of replacement, an operating cost $B(x)$, and a minimal repair cost rate $M(x)$. We assume that $M(x)\leqq K(x)+C_i(x_i)$ for each $i\in N$, and all cost rates are bounded and nonnegative.

The expected total discounted cost $V_{\pi}(x)$ of the continuous time replacement model for an infinite horizon when we start with the M-Markov coherent system in state x is given by

$$(4.1)\qquad V_{\pi}(x)=E[\int e^{-\alpha t}w^{\pi}(t)dt].$$

The objective is to investigate the structure of an optimal replacement policy which minimizes the expected total discounted cost with discount factor $\alpha\in(0,\infty)$. Let $V_{\alpha}(x)$ be the minimum expected total discounted cost when the initial state of the M-Markov coherent system is x. Then letting π^* be an optimal replacement policy we have

$$(4.2)\qquad V_{\alpha}(x)=\inf_{\pi\in D_s} V_{\pi}(x)$$

$$=V_{\pi^*}(x).$$

First we begin by introducing the weak infinitesimal operator A_π of the stochastic process Z^π for each $\pi\in D_s$. For a function f in the domain of A_π we have

$$A_\pi f(x)=\lim_{t\downarrow 0} t^{-1}\mathrm{E}[f(X^\pi(t))-f(x)] \tag{4.3}$$

$$=\begin{cases} \lambda(x)\int f(u)Q(x,du)-\lambda(x)f(x) & \text{if } \pi(x)=0 \\ \mu(x,m)f(0,x_c)-\mu(x,m)f(x) & \text{if } \pi(x)=m \\ \mu(x,a)f(0,x_c^a)-\mu(x,a)f(x) & \text{if } \pi(x)=a\in \mathrm{D}^1 \end{cases}$$

Of great importance is Doshi's formula (see Doshi [15, 1976])

$$\alpha V_\alpha(x)=\min_{a\in \mathrm{D}}[r(x,a)+A_\pi V_\alpha(x)], \tag{4.4}$$

where $r(x,a)$ is a cost rate when the state is x and action is a. Then the minimum expected total discounted cost $V_\alpha(x)$ satisfies the following functional equation:

$$\begin{aligned} \alpha V_\alpha(x)=\min[&B(x)+\lambda(x)\int V_\alpha(u)Q(x,du)-\lambda(x)V_\alpha(x), \\ &M(x)+\mu(x,m)V_\alpha(0,x_c)-\mu(x,m)V_\alpha(x), \\ &\min_{a\in \mathrm{D}^1}\{K(x)+\sum_{j\in \mathrm{A}(a)} C_j(x_j)+\mu(x,a)V_\alpha(0,x_c^a)-\mu(x,a)V_\alpha(x)\}]. \end{aligned} \tag{4.5}$$

4.2.5 Structural Properties of Optimal Policy

We investigate the structural properties of the optimal replacement policy for the M-Markov coherent system under jump deterioration. First we shall examine the property of the optimal expected total discounted cost function under the following condition.

Condition 4.1. (1) $B(x)\in \mathrm{F(E)}$, $M(x)\in \mathrm{F(E)}$, $K(x)\in \mathrm{F(E)}$ and $\sum_{i\in \mathrm{N}} C_i(x_i)\in \mathrm{F(E)}$.

(2) $1/\mu(x,a)\in \mathrm{F(E)}$.

Theorem 4.1. Assume that the deterioration process Z of the M-Markov coherent system is stochastically quasi-monotone. If Condition 4.1 holds, then the optimal expected total discounted cost function $V_\alpha(x)$ is a member of F(E).

Proof: The functional equation (4.5) can be written as

$$
\begin{aligned}
V_\alpha(x) = \min \frac{1}{\alpha+\Lambda} \Big[& B(x)+\lambda(x)\int V_\alpha(u)Q(x,du)+(\Lambda-\lambda(x))V_\alpha(x), \\
& M(x)+\mu(x,m)V_\alpha(0,x_c)+(\Lambda-\lambda(x,m))V_\alpha(x), \\
& \min_{a\in D^1}\{K(x)+\sum_{i\in A(a)} C_i(x_i)+\mu(x,a)V_\alpha(0,x_c^a)+(\Lambda-\mu(x,a))V_\alpha(x)\}\Big]
\end{aligned}
\tag{4.6}
$$

where Λ is any value larger than $\max\{\sup\lambda(x),\ \sup\mu(x,a)\}$. We can calculate by using the successive approximation technique:

$$
\begin{aligned}
V_{k+1}(x) = \min \frac{1}{\alpha+\Lambda} \Big[& B(x)+\lambda(x)\int V_k(u)Q(x,du)+(\Lambda-\lambda(x))V_k(x), \\
& M(x)+\mu(x,m)V_k(0,x_c)+(\Lambda-\mu(x,m))V_k(x), \\
& \min_{a\in D^1}\{K(x)+\sum_{i\in A(a)} C_i(x_i)+\mu(x,a)V_\alpha(0,x_c^a)+(\Lambda-\mu(x,a))V_\alpha(x)\}\Big]
\end{aligned}
\tag{4.7}
$$

where $V_0(x)=0$ for all $x\in E$. Then the proof for the monotonicity of $V_\alpha(x)\in F(E)$ is carried out by mathematical induction. For $k=1$ it follows trivially from Condition 4.1 (1). Suppose $V_k(x)\in F(E)$ for some k. Then from Lemma 2.1 we have

$$\int V_k(u)Q(x,du)\in F(E).$$

Further from Condition 4.1 and the definition of Λ, we can easily obtain $V_{k+1}(x)\in F(E)$. Thus we have $V_k(x)\in F(E)$ for all k. Since all cost rates and jump rates are bounded, it is easy to see that $V_k(x)\to V_\alpha(x)$ as $k\to\infty$ for each $x\in E$. Therefore the result is obtained. ||

The following theorem shows a simple property of the optimal replacement policy.

Theorem 4.2. Assume that the deterioration process Z of the M-Markov coherent system is stochastically quasi-monotone and Condition 4.1 holds. If the deterioration level of component i is in the best state 0, $\mu(x,m)\geqq\mu(x,a)$ for $a\in D^1$ and $1/\mu(x,a)\in F(D^1)$, then the action to keep component i is optimal.

Proof: Let for each $i\in N$

$$[V_\alpha(x)]_k^i=\min \frac{1}{\alpha+\Lambda}\ [B(x)+\lambda(x)\int V_\alpha(u)Q(x,du)+(\Lambda-\lambda(x))V_\alpha(x),$$

$$M(x)+\mu(x,m)V_\alpha(0,x_c)+(\Lambda-\mu(x,m))V_\alpha(x),$$

$$\min_{a\in D_0^i}\{K(x)+\sum_{j\in A(a)} C_j(x_j)+\mu(x,a)V_\alpha(0,x_c^a)+(\Lambda-\mu(x,a))V_\alpha(x)\}], \tag{4.8}$$

$$[V_\alpha(x)]_r^i=\min_{a\ D_1^i}\frac{1}{\alpha+\Lambda}\ \{K(x)+\sum_{j\in A(a)} C_j(x_j)+\mu(x,a)V_\alpha(0,x_c^a)$$
$$+(\Lambda-\mu(x,a))V_\alpha(x)\}].$$

Then we have

$$[V_\alpha(x)]_k^i-[V_\alpha(x)]_r^i$$

$$\leqq\min_{a\in D_0^i}\frac{1}{\alpha+\Lambda}\ [K(x)+\sum_{j\in A(a)} C_j(x_j)+\mu(x,a)V_\alpha(0,x_c^a)+(\Lambda-\mu(x,a))V_\alpha(x)]$$

$$-\min_{a\in D_1^i}\frac{1}{\alpha+\Lambda}\ [K(x)+\sum_{j\in A(a)} C_j(x_j)+\mu(x,a)V_\alpha(0,x_c^a)+(\Lambda-\mu(x,a))V_\alpha(x)]$$

and for each $a\in D_0^i$

$$[K(x)+\sum_{j\in A(a)} C_j(x_j)+\mu(x,a)V_\alpha(0,x_c^a)+(\Lambda-\mu(x,a))V_\alpha(x)$$

$$-[K(x)+\sum_{j\in A(a)} C_j(x_j)+C_i(x_i)+\mu(x,(1_i,a))V_\alpha(0,x_c^{(1_i,a)})$$
$$+(\Lambda-\mu(x,(1_i,a))V_\alpha(x)]$$

$$\leqq -C_i(x_i)+\mu(x,a)\{V_\alpha(0,x_c^a)-V_\alpha(0,x_c^{(1_i,a)})\}.$$

$$<0.$$

The first inequality follows from the assumption $1/\mu(x,a)\in F(D^1)$ and Theorem 4.1, and the second inequality is true since $(0_i,x)^a=(0_i,x)^{(1_i,a)}$. Furthermore since $\mu(x,m)\geqq\mu(x,a)$ for each $a\in D^1$ and $M(x)\leqq K(x)+C_i(x_i)$, we have for $x_i=0$

$$\min \frac{1}{\alpha+\Lambda}\,[B(x)+\lambda(x)\int V_\alpha(u)Q(x,du)+(\Lambda-\lambda(x))V_\alpha(x),$$

$$M(x)+\mu(x,m)V_\alpha(0,x_c)+(\Lambda-\mu(x,m))V_\alpha(x)]$$

$$-\frac{1}{\alpha+\Lambda}\,[K(x)+C_i(x_i)+\mu(x,(1_i,0))V_\alpha(0,(0_i,x_c))$$

$$+(\Lambda-\mu(x,(1_i,a))V_\alpha(x)]$$

$$<0.$$

Thus we can easily obtain that $[V_\alpha(x_0,(0_i,x_c))]_k^i-[V_\alpha(x_0,(0_i,x_c))]_r^i<0$. Then the result is obvious. $\|$

Here we add the following condition.

Condition 4.2. (1) $C_i(x_i)=C_i$ for each $i\in N$.

(2) $B(x)/(\alpha+\Lambda)-K(x)/(\alpha+\mu(x_0))\in F(E_c)$.

(3) $M(x)-K(x)\in F(E_c)$.

The following theorems show the structure of the optimal replacement policy under reasonable conditions.

Theorem 4.3. Assume that the deterioration process Z of the M-Markov coherent system is stochastically quasi-monotone. If Conditions 4.1 and 4.2 hold and $\mu(x,a)=\mu(x_0)$, then there exists a control limit policy π_i with respect to component i minimizing the expected total discounted cost of the continuous time replacement model for the system with jump deterioration.

Proof: The functional equation (4.5) can be written as

$$(4.9)\quad V_\alpha(x)=\min\Big[\frac{1}{\alpha+\Lambda}\{B(x)+\lambda(x)\int V_\alpha(u)Q(x,du)+(\Lambda-\lambda(x))V_\alpha(x)\},$$
$$\frac{1}{\alpha+\mu(x,m)}\{M(x)+\mu(x,m)V_\alpha(0,x_c)\},$$
$$\min_{a\in D^1}\frac{1}{\alpha+\mu(x,a)}\{K(x)+\sum_{j\in A(a)}C_j(x_j)+\mu(x,a)V_\alpha(0,x_c^a)\}\Big].$$

Let for each $i\in N$

$$[V_\alpha(x)]^i_{\underline{k}}=\min\Big[\frac{1}{\alpha+\Lambda}\{B(x)+\lambda(x)\int V_\alpha(u)Q(x,du)+(\Lambda-\lambda(x))V_\alpha(x)\},$$
$$(4.10)\quad \frac{1}{\alpha+\mu(x,m)}\{M(x)+\mu(x,m)V_\alpha(0,x_c)\},$$
$$\min_{a\in D_0^i}\frac{1}{\alpha+\mu(x,a)}\{K(x)+\sum_{j\in A(a)}C_j(x_j)+\mu(x,a)V_\alpha(0,x_c^a)\}\Big],$$
$$[V_\alpha(x)]^i_{\underline{r}}=\min_{a\in D_1^i}\frac{1}{\alpha+\mu(x,a)}\{K(x)+\sum_{j\in A(a)}C_j(x_j)+\mu(x,a)V_\alpha(0,x_c^a)\}.$$

Then we have under Condition 4.2 and $\mu(x,a)=\mu(x_0)$

$$[V_\alpha(x)]^i_{\underline{k}}-[V_\alpha(x)]^i_{\underline{r}}$$
$$=\min\Big[\frac{B(x)}{\alpha+\Lambda}-\frac{K(x)}{\alpha+\mu(x_0)}+\frac{\lambda(x)}{\alpha+\Lambda}\int V_\alpha(u)Q(x,du)+\frac{\Lambda-\lambda(x)}{\alpha+\Lambda}V_\alpha(x),$$
$$\frac{1}{\alpha+\mu(x_0)}\{M(x)-K(x)+\mu(x_0)V_\alpha(0,x_c)\},$$
$$\min_{a\in D_0^i}\frac{1}{\alpha+\mu(x_0)}\{\sum_{j\in A(a)}C_j+\mu(x_0)V_\alpha(0,x_c^a)\}\Big]$$
$$-\min_{a\in D_1^i}\frac{1}{\alpha+\mu(x_0)}\{\sum_{j\in A(a)}C_j+\mu(x_0)V_\alpha(0,x_c^a)\}$$

So we can easily obtain that the difference $[V_\alpha(x)]^i_{\underline{k}}-[V_\alpha(x)]^i_{\underline{r}}$ is a member of $F(E_i)$. Thus the result follows from Definition 2.6. ||

Theorem 4.4. Assume that the deterioration process Z of the M-Markov coherent system is stochastically quasi-monotone and Conditions 4.1 and 4.2 hold. If the action to replace component i with the worst state e_i for each $i\in N$ is optimal, $M(x)-K(x)\leqq 0$ and $\mu(x,a)=\mu(x_0)$, then we have $x_i^*(x_0,e^i)\leqq x_i^*(x_0,0^i)$.

Proof: Since the action to replace component i with the worst state e_i is optimal, we have for each $x_0\in E_0$

$$[V_\alpha(x_0,(x_i,e))]_k^i-[V_\alpha(x_0,(x_i,e))]_r^i$$
$$=\frac{1}{\alpha+\Lambda}[-C_i+\mu(x_0)\{V_\alpha(0,(x_i,0))-V_\alpha(0)\}]$$

and from Theorem 4.2 we have

$$[V_\alpha(x_0,(x_i,0))]_k^i-[V_\alpha(x_0,(x_i,0))]_r^i$$
$$\leqq\frac{1}{\alpha+\Lambda}[M(x_0,(x_i,0))-K(x_0,(x_i,0))$$
$$-C_i+\mu(x_0)\{V_\alpha(0,(x_i,0))-V_\alpha(0)\}].$$

Then from $M(x)-K(x)\leqq 0$ we have

$$[V_\alpha(x_0,(x_i,e))]_k^i-[V_\alpha(x_0,(x_i,e))]_r^i$$
$$\geqq[V_\alpha(x_0,(x_i,0))]_k^i-[V_\alpha(x_0,(x_i,0))]_r^i$$

Thus we can get $x_i^*(x_0,0^i)\geqq x_i^*(x_0,e^i)$ by the definition of $x_i^*(x_0,x_c^i)$. ||

Theorem 4.5. Assume that the deterioration process Z of the M-Markov coherent system is stochastically quasi-monotone and Conditions 4.1 and 4.2 hold. If the action to replace component i with the worst state e_i is optimal, and $\mu(x,a)=\mu$, then $x_i^*(x_0,e^i)=x_i^*(e_0,e^i)$ for each $x_0\in E_0$.

Proof: From the proof of Theorem 4.4 we have

$$[V_\alpha(x_0,(x_i,e))]^i_k-[V_\alpha(x_0,(x_i,e))]^i_r$$

$$=\frac{1}{\alpha+\Lambda}\{K(x_0,(x_i,e))+\sum_{j\in N}C_j-C_i+\mu V_\alpha(0,(x_i,0))\}$$

$$-\frac{1}{\alpha+\Lambda}\{K(x_0,(x_i,e))+\sum_{j\in N}C_j+\mu V_\alpha(0)\}$$

$$=\frac{1}{\alpha+\Lambda}\{\mu(V_\alpha(0,(x_i,0))-V_\alpha(0))-C_i\}.$$

Then the result directly follows. ||

Remark 4.2. We can now obtain the results similar to 3.3.3 when the jump rate of replacement or minimal repair depends upon the action a. Furthermore, it might be better to employ a simpler (ABC)-policy because of easier implementation than a more complex optimal replacement policy.

We shall consider the following condition instead of Condition 4.2.

Condition 4.3. (1) $C_i(x_i)=C_i$ for each $i\in N$,

(2) $\mu=\mu(x,a)$ for each $x\in E$ and $a\in D$,

(3) $M(x)-K(x)\in F(E_0)$,

(4) $B(x)/(\alpha+\Lambda)-K(x)/(\alpha+\mu)\in F(E_0)$.

Theorem 4.6. Assume that the deterioration process Z of the M-Markov coherent system is stochastically quasi-monotone. If Conditions 4.1 and 4.3 hold, then $x^*_i(x_0,x^i_c)$ is a member of $\overline{F}(E_0)$.

Proof: From the proof of Theorem 4.3 we have

$$[V_\alpha(x)]^i_{\underline{k}}-[V_\alpha(x)]^i_{\underline{r}}$$

$$=\min[\frac{1}{\alpha+\Lambda}\{B(x)+\lambda(x)\int V_\alpha(u)Q(x,du)+(\Lambda-\lambda(x))V_\alpha(x)\},$$

$$\frac{1}{\alpha+\mu}\{M(x)+\mu V_\alpha(0,x_c)\},$$

$$\frac{1}{\alpha+\mu}\min_{a\in D_0^i}\{K(x)+\sum_{j\in A(a)} C_j+\mu V_\alpha(0,x_c^a)\}]$$

$$-\frac{1}{\alpha+\mu}\min_{a\in D_1^i}\{K(x)+\sum_{j\in A(a)} C_j+\mu V_\alpha(0,x_c^a)].$$

Then from Condition 4.3 we can easily obtain that $[V_\alpha(x)]^i_{\underline{k}}-[V_\alpha(x)]^i_{\underline{r}}$ is a member of $F(E_0)$. Thus the result is obvious. ||

Further we consider the following condition.

Condition 4.4. $B(x)/(\alpha+\Lambda)-M(x)/(\alpha+\mu)\in F(E_0)$.

Theorem 4.7. Assume that the deterioration process Z of the M-Markov coherent system is stochastically quasi-monotone. If Conditions 4.1, 4.3 and 4.4 hold, then there exists a control limit $x_0^*(x_c)$ for each $x_c\in E_c$ such that the action to carry out minimal repair for the system is optimal if and only if the failure damage x_0 exceeds $x_0^*(x_c)$.

Proof: By using the functional equation (4.9) under Conditions 4.1, 4.3 and 4.4, we have

$$[V_\alpha(x)]_{a=0}-[V_\alpha(x)]_{a=m}$$
$$=\frac{1}{\alpha+\Lambda}\{B(x)+\lambda(x)\int V_\alpha(u)Q(x,du)+(\Lambda-\lambda(x))V_\alpha(x)$$
$$-\frac{1}{\alpha+\mu}\{M(x)+\mu V_\alpha(0,x_c)\}$$

and

$$[V_\alpha(x)]_{a=0}-[V_\alpha(x)]_{a\in D^1}$$
$$=\frac{1}{\alpha+\Lambda}\{B(x)+\lambda(x)\int V_\alpha(u)Q(x,du)+(\Lambda-\lambda(x))V_\alpha(x)\}$$
$$-\min_{a\in D^1}\frac{1}{\alpha+\mu}\{K(x)+\sum_{j\in A(a)} C_j+\mu V_\alpha(0,x_c^a)\}.$$

Then we have $[V_\alpha(x)]_{a=0}-[V_\alpha(x)]_{a\in D^1}\in F(E_0)$ from Conditions 4.3 and 4.4. Thus we can get the result. ||

The following property can be proved similarly to those in Chapter 3.

Property 4.1. Assume that the deterioration process Z of the M-Markov coherent system is stochastically quasi-monotone. If Conditions 4.1, 4.2, 4.3 and 4.4 hold, then

(1) the optimal region G(0) is closed in the sense that $x^1\wedge x^2\in G(0)$ for all x^1 and x^2 in G(0).

(2) the optimal region G(11) is closed in the sense that $x^1\vee x^2\in G(11)$ for all x^1 and x^2 in G(11).

(3) the optimal region G(11) is an increasing set in S(E).

(4) the optimal region $G(0_i,11)$ for each $i\in N$ is closed in the sense that $x^1\vee x^2\in G(0_i,11)$ for all x^1 and x^2 in $G(0_i,11)$.

4.3 Binary Coherent System

4.3.1 Model Formulation

We consider the following continuous time replacement model for a binary coherent system. The binary coherent system consisting of n components is stochastically independent, and it is economically interdependent. The binary coherent system is monitored continuously in time interval $T=[0,\infty)$, and the failure of components is detected immediately. The time to failure of component i has a continuous pdf $f_i(t)$ with finite mean $1/\lambda_i$. The failure time cdf of component i is denoted by $F_i(t)$ and the failure rate function is $\lambda_i(t)=f_i(t)/\overline{F}_i(t)$, where $\overline{F}_i(t)=1-F_i(t)$.

An action is taken as to whether or not to replace each component, based upon the result of monitoring. When an action a_c is taken, the replacement time has an exponential distribution function $G(t;x_c,a_c)=1-\exp\{-\mu(x_c,a_c)t\}$ for each $a_c \in D^1$.

4.3.2 Age Deterioration Process

Let $Z_c^\pi=\{X_c^\pi(t);\ t\in T\}$ be a stochastic process representing the ages of components in the binary coherent system under a replacement policy $\pi\in D_c$. The state space of this process Z_c^π is $E_c=\prod_{i\in N} E_i$ and $E_i=[0,\infty]$, where the state $x_i=\infty$ represents the failure state of component i.

Now we consider component i in the binary coherent system. The following lemma shows the property of the deterioration process $\{X_i(t);\ t\in T\}$ of component i.

Lemma 4.2. If the life distribution of component i is IFR, then the deterioration process $\{X_i(t);\ t\in T\}$ is stochastically monotone.

Proof: The transition function $P(t,x_i,U_i)$ of the stochastic process $\{X_i(t);\ t\in T\}$ is given by for $U_i\in S(E_i)$

$$
(4.11)\qquad P(t,x_i,U_i)=P[X_i(t+s)\in U_i \mid X_i(s)=x_i]
$$

$$
=\begin{cases} 1 & \textit{if } x_i+t\in U_i, \\ (\overline{F}_i(x_i)-\overline{F}_i(x_i+t))/\overline{F}_i(x_i) & \textit{otherwise}. \end{cases}
$$

Thus the result follows from IFR property. ||

The following lemma shows the property of the deterioration process $Z_c=\{X_c(t);\ t\in T\}$ of the binary coherent system.

Lemma 4.3. If the life distribution of each component is IFR, then the deterioration process Z_c of the binary coherent system is stochastically monotone.

Proof: The result directly follows from Proposition 2 of Kamae, Krengel and O'brien [27, 1977]. ||

4.3.3 Cost Structure

The objective of this section is to investigate the structure of the optimal replacement policy which minimizes the expected total discounted cost with discount factor $\alpha>0$. For the cost rates associated with the continuous time replacement model of the binary coherent system, we consider a replacement cost rate $C_i(x_i)$ of component i with age x_i, a fixed cost rate $K(x_c)$ of replacement, and an operating cost rate $B(x_c)$ when the binary coherent system is in state x_c. We assume that all cost rates are bounded and non-negative.

Let $V_\alpha(x_c)$ be the minimum expected total discounted cost when the initial state of the binary coherent system is x_c. Then from the Doshi's formula (4.4) we can find the structural properties of the optimal replacement policy. However, it is hard to deal with this functional equation. Consequently, in this section we approximate $Z_c=\{X_c(t);\ t\in T\}$ by a discrete time Markov chain with unit time interval h and investigate the structural properties of the optimal replacement policy without this functional equation.

4.3.4 Structure of Optimal Policy

We examine the structure of the optimal expected total discounted cost and the structural properties of the optimal replacement policy for the binary coherent system.

Condition 4.5. (1) $B(x_c)\in F(E_c)$, $K(x_c)\in F(E_c)$, and $\sum_{i\in N} C_i(x_i)\in F(E_c)$. (2) $1/\mu(x_c,a_c)\in F(E_c)$.

Theorem 4.8. Assume that the life distribution of each component is IFR. If Condition 4.5 holds, then the optimal expected total discounted cost function $V_\alpha(x_c)$ is a member of $F(E_c)$.

Proof: We approximate $Z_c^\pi=\{X_c^\pi(t);\ t\in T\}$ by a discrete time Markov process with unit time interval h. Now define the discrete time Markov process $Z_N^\pi=\{X_c(k);\ k\in N\}$ as follows;

$$P[X_c^\pi(k+1)=((l_1+1),\ldots(l_n+1))\mid X_c^\pi(k)=(l_1,\ldots,l_n),\ \pi(l)=0]$$
$$=(1-h\sum_{i\in N}\lambda_i(l_i)),$$

$$(4.12)\qquad P[X_c^\pi(k+1)=((l_1+1),\ldots,(l_{i-1}+1),\infty,(l_{i+1}+1),\ldots,(l_n+1))$$
$$|X_c^\pi(k)=(l_1,\ldots,l_n),\ \pi(l)=0]=\lambda_i(l_i)h,$$
$$P[X_c^\pi(k+1)=l\mid X_c^\pi(k)=l,\ \pi(l)=a_c\in D^1]=(1-h\mu(l,a_c)),$$
$$P[X_c^\pi(k+1)=l^a\mid X_c^\pi(k)=l,\ \pi(l)=a_c\in D^1]=h\mu(l,a_c),$$

where $\lambda_i(\infty)=0$, $\infty+l_i=\infty$ and $0<h<[\max\{n\sup\lambda(x_c),\sup\mu(x_c,a)\}]^{-1}$. Then it can be found that the transition matrix $Q(h)$ of this Markov chain satisfies Condition 3.1 in Chapter 3. Thus the expected total discounted cost function $V_\alpha^N(x_c)$ for the discrete time Markov chain is a member of $F(E_c)$ similarly to Theorem 3.1. Letting $h\downarrow 0$, we see that the result holds for Condition 4.5 similarly to that of Barlow and Proschan [4, 1976]. ||

The structural properties of the optimal replacement policy for components in the binary coherent system are shown in the following theorems. The proofs are similar to those of Chapter 3.

Theorem 4.9. Assume that the life distribution of each component is IFR, and Condition 4.5 holds. If component i is new with state 0 and $1/\mu(x_c,a_c)\in F(E_c)$, then the action to keep component i is optimal.

Condition 4.6. (1) $C_i(x_i)=C_i$ for each $i\in N$.
(2) $B(x_c)-K(x_c)\in F(E_c)$.

Theorem 4.10. Assume that the life distribution of each component is IFR. If Conditions 4.5 and 4.6 hold, then there exists a control limit policy π_i with respect to component i minimizing the expected total discounted cost.

Theorem 4.11. Assume that the life distribution of each component is IFR, and Conditions 4.5 and 4.6 hold. If the action to replace failed component i is optimal, then $x_i^*(\infty^i) \leqq x_i^*(0^i)$.

Property 4.2. Assume that the life distribution of each component is IFR. If Conditions 4.5 and 4.6 hold, then
(1) the optimal region G(0) is closed in the sense that $x_c^1 \wedge x_c^2 \in G(0)$ for all x_c^1 and x_c^2 in G(0).
(2) the optimal region G(11) is closed in the sense that $x_c^1 \vee x_c^2 \in G(11)$ for all x_c^1 and x_c^2 in G(11).
(3) the optimal region G(11) is an increasing set in $S(E_c)$.
(4) the optimal region $G(0_i, 11)$ for $i \in N$ is closed in the sense that $x_c^1 \vee x_c^2 \in G(0_i, 11)$ for all x_c^1 and x_c^2 in $G(0_i, 11)$.

Remark 4.3. In this case of binary coherent system, the simple (ABC)-policy is reduced to the (n,N) policy with A=n and C=N.

4.4 Group Replacement Problem

4.4.1 Group Replacement Policy

In this section we consider a continuous time replacement model for a maintained coherent system which is not stochastically monotone. This coherent system consists of n repairable components. Each component is subject to random failure. Upon failure the component is repaired and recovers its function perfectly. Ross[44, 1976] has proved that the distribution of the time to first system failure has NBU (new better than used) property when all components are initially good, and have exponential life distributions with parameter λ_i, $i \in N$, and repair time distributions with parameter μ_i, $i \in N$. Thus efforts to replace the maintained coherent system before system failure may be advantageous. On the other hand, as compared with individual repair upon failed components, the group replacement may cause a loss for some good components. However, we can expect to obtain advantage of scale merit.

The coherent systen is monitored continuously, and based upon the hysteresis of monitoring, an action is taken as to whether or not to replace the maintained coherent system. The objective of this section is to study the structure of the optimal group replacement policy minimizing the expected total discounted cost.

4.4.2 Maintained Coherent System

Consider a maintained coherent system. The coherent system consists of n components and has n repair facilities. Each of its components is either up or down, and acts independently each other. When component i goes up (down), it remains up (down) for exponentially distributed time interval with parameter λ_i (μ_i) and then goes down (up). Let uptimes and downtimes be independent. The state of the coherent system at any time depends upon the states of components through a coherent structure function $\phi(\cdot)$.

Let for $i \in \mathrm{N}$

$$X_i(t) = \begin{cases} 0 & \textit{if component } i \textit{ is up at time } t, \\ 1 & \textit{otherwise.} \end{cases}$$

Then the evolution of the state of component i is described by the stochastic process $\{X_i(t);\ t \in \mathrm{T}\}$. Let

$$X_c(t) = (X_1(t), \ldots, X_n(t))$$

and

$$\phi(X_c(t)) = \begin{cases} 0 & \textit{if the coherent system is up at time } t, \\ 1 & \textit{otherwise,} \end{cases}$$

then the state of the coherent system is summarized by binary n-vector $X_c(t)$ with a state space $\mathrm{E}_c = \{0,1\}^n$ and the actual state of the maintained coherent system is described by the stochastic process $\{\phi(X_c(t));\ t\in T\}$.

At each time epoch $t\in T$, observing the state $X_c(t)=x_c$, an action can be done whether to replace the maintained coherent system, or to keep it. We assume that the time needed to replace the coherent system is exponential with parameter μ_0. Let $\pi(x_c)\in D_c=\{0,1\}$ represent the action taken for the coherent system at any time t, where $\pi(x_c)=1$ means to replace the coherent system and $\pi(x_c)=0$ means to keep it. For the cost rate $r(x_c,\pi(x_c))$ associated with the maintained coherent system, we consider the following cost rate. At time t when the state is x_c and an action $\pi(x_c)=0$ is taken on the maintained coherent system, then the cost is incurred at the rate $r(x_c,0)=P\phi(x_c)+\sum_{i\in C_0(x_c)} r_i$, where P is the system down cost rate, r_i, $i\in N$, is the repair cost rate of component i and $C_0(x_c)$ denotes the set of currectly failed components. On the other hand, When an action $\pi(x_c)=1$ is taken, then the cost is incurred at the rate $r(x_c,1)=R$, where R is the replacement cost rate (i.e., R/μ_0 is the expected replacement cost).

The objective is to investigate the structure of the optimal group replacement policy minimizing the expected total discounted cost with discount factor $\alpha>0$. Now let $V_\alpha(x_c)$ be the minimum expected total discounted cost when the initial state of the coherent system is $X_c(0)=x_c$. We derive the weak infinitesimal operator A_π of the stochastic process $Z_c^\pi=\{X_c^\pi(t);\ t\in T\}$ for each $\pi\in D_c$. For a function f in the domain of A_π we have

$$A_\pi f(x_c)=\lim_{t\downarrow 0} t^{-1}E_{x_c}[f(X_c^\pi(t))-f(x_c)]$$

$$=\begin{cases} \sum_{i\in C_1(x_c)} \lambda_i(f(1_i,x_c)-f(x_c))+\sum_{i\in C_0(x_c)} \mu_i(f(0_i,x_c)-f(x_c)), & \\ & if\ \ \pi(x_c)=0, \\ \mu_0(f(0)-f(x_c)), & if\ \ \pi(x_c)=1 \end{cases}$$

where $C_j(x_c)=\{i\in N|\ x_i=j\}$, $(j=0,1)$. From Doshi's formula (4.4) the function $V_\alpha(x_c)$ satisfies the following functional equation:

$$\alpha V_\alpha(x_c)=\min\ [P\phi(x_c)+\sum_{i\in C_1(x_c)}(r_i+\mu_i V_\alpha(0_i,x_c)-V_\alpha(x_c))$$

$$+\sum_{i\in C_0(x_c)}\lambda_i(V_\alpha(1_i,x_c)-V_\alpha(x_c)), \tag{4.13}$$

$$R+\mu_0(V_\alpha(0)-V_\alpha(x_c))].$$

4.4.3 Properties of Optimal Policy

Some properties on the optimal group replacement policy and the corresponding optimal expected total discounted cost function are discussed. The following two theorems show the structure of the optimal total discounted cost function, and they are used in the proof of theorems which present the structural properties of the optimal group replacement policy.

Theorem 4.12. The minimum expected total discounted cost function $V_\alpha(x_c)$ is a member of $F(E_c)$.

Proof: The functional equation (4.13) can be written as

$$V_\alpha(x_c)=\min\ [P\phi(x_c)+\sum_{i\in C_1(x_c)}(r_i+\mu_i V_\alpha(0_i,x_c))$$

$$+\sum_{i\in C_0(x_c)}\lambda_i V_\alpha(1_i,x_c)$$

$$+(\Lambda-\sum_{i\in C_1(x_c)}\mu_i-\sum_{i\in C_0(x_c)}\lambda_i)V_\alpha(x_c)\}/(\alpha+\Lambda), \tag{4.14}$$

$$\{R+\mu_0 V_\alpha(0)+(\Lambda-\mu_0)V_\alpha(x_c)\}/(\alpha+\Lambda)],$$

where Λ is any value larger than $\max\{\mu_0,\ \max_{x_c\in E_c}\{\sum_{i\in C_1(x_c)}\mu_i+\sum_{i\in C_0(x_c)}\lambda_i + \max_{i\in C_1(x_c)}\lambda_i\}\}$. We can calculate by using the successive approximation technique:

$$V_{k+1}(x_c)=\min\Big[\{P\phi(x_c)+\sum_{i\in C_1(x_c)}(r_i+\mu_i V_k(0_i,x_c))$$
$$+\sum_{i\in C_0(x_c)}\lambda_i V_k(1_i,x_c)$$
$$+(\Lambda-\sum_{i\in C_1(x_c)}\mu_i-\sum_{i\in C_0(x_c)}\lambda_i)V_k(x_c)\}/(\alpha+\Lambda),$$
$$\{R+\mu_0 V_k(0)+(\Lambda-\mu_0)V_k(x_c)\}/(\alpha+\Lambda)\Big],$$

where $V_0(x_c)=0$ for all $x_c\in E_c$. Then to prove the monotonicity of $V_\alpha(x_c)$ we use mathematical induction. For $k=1$ the result follows easily from the properties of the structure function ϕ and the definition of $C_0(x_c)$ and $C_1(x_c)$. Suppose the result is true for some k. At the $k+1$ stage, if an optimal action is to keep the maintained coherent system for $(1_i,x_c)\in E_c$, $i\in N$, then

$$V_{k+1}(1_i,x_c)-V_{k+1}(0_i,x_c)$$
$$\geq[P\phi(1_i,x_c)+\sum_{j\in C_1(1_i,x_c)}(r_j+\mu_j V_k(1_i,0_j,x_c))$$
$$+\sum_{j\in C_0(1_i,x_c)}\lambda_j V_k(1_i,1_j,x_c)$$
$$+(\Lambda-\sum_{j\in C_1(1_i,x_c)}\mu_j-\sum_{j\in C_0(1_i,x_c)}\lambda_j)V_k(1_i,x_c)]/(\alpha+\Lambda)$$
$$-[P\phi(0_i,x_c)+\sum_{j\in C_1(0_i,x_c)}(r_j+\mu_j V_k(0_i,0_j,x_c))$$
$$+\sum_{j\in C_0(0_i,x_c)}\lambda_j V_k(0_i,1_j,x_c)$$
$$+(\Lambda-\sum_{j\in C_1(0_i,x_c)}\mu_j-\sum_{j\in C_0(0_i,x_c)}\lambda_j)V_k(0_i,x_c)]/(\alpha+\Lambda)$$
$$\geq[P(\phi(1_i,x_c)-\phi(0_i,x_c))+r_i$$
$$+\sum_{j\in C_1(0_i,x_c)}\mu_j(V_k(1_i,0_j,x_c)-V_k(0_i,0_j,x_c))$$

$$+\sum_{j\in C_0(0_i,x_c)} \lambda_j(V_k(1_i,1_j,x_c)-V_k(0_i,1_j,x_c))$$
$$+(\Lambda-\lambda_i-\sum_{j\in C_1(1_i,x_c)}\mu_j-\sum_{j\in C_0(0_i,x_c)}\lambda_j)(V_k(1_i,x_c)-V_k(0_i,x_c))]$$
$$/(\alpha+\Lambda)$$
$$\geqq 0.$$

On the other hand, even if an optimal action is to replace the main-maintained coherent system, $V_{k+1}(1_i,x_c)-V_{k+1}(0_i,x_c)\geqq 0$ is proved similarly to the above. Thus for each k, $V_k(x_c)$ is a member of $F(E_c)$. Then from the successive approximation technique, as

$$\lim_{k\to\infty} V_k(x_c)=V_\alpha(x_c),$$

$V_\alpha(x_c)$ is a member of $F(E_c)$. ||

Theorem 4.13. The minimum expected total discounted cost function $V_\alpha(x_c)$ is not larger than R/α.

The above theorem is easily proved by the functional equation (4.14). Nex the structural properties of the optimal group replacement policy for a maintained coherent system are characterized.

Theorem 4.14. If all components are operating, then an optimal action is to keep the maintained coherent system.

Proof: This follows directly from the functional equation (4.14) and Theorem 4.13. ||

Theorem 4.15. An optimal group replacement policy $\pi^*(x_c)$ is a member of $F(E_c)$.

Proof: The functional equation (4.13) can be written as

$$V_\alpha(x_c)=\min\ [\{P\phi(x_c)+\sum_{i\in C_1(x_c)}(r_i+\mu_i V_\alpha(0_i,x_c)$$

(4.15)
$$+\sum_{i\in C_0(x_c)}\lambda_i V_\alpha(1_i,x_c)$$

$$+(\Lambda-\sum_{i\ C_1(x_c)}\mu_i-\sum_{i\ C_0(x_c)}\lambda_i)V_\alpha(x_c)\}/(\alpha+\Lambda)$$

$$\{R+\mu_0 V_\alpha(0)\}/(\alpha+\mu_0)].$$

Notice that the latter quantity does not contain variable x_c. From Theorem 4.12 and the above fact, the result is easily obtained. ||

Property 4.3. If $\mu_0 \geqq \sum_{i\in C_1(x_c)}\mu_i + \sum_{i\in C_0(x_c)}\lambda_i$ and $R \leqq P\phi(x_c) + \sum_{i\in C_1(x_c)} r_i$ for $x_c \in E_c$, then an optimal action is to replace the maintained coherent system.

Proof: The functional equation (4.13) can be written as

$$V_\alpha(x_c)=\min\ [\{P\phi(x_c)+\sum_{i\in C_1(x_c)}(r_i+\mu_i V_\alpha(0_i,x_c))$$

$$+\sum_{i\in C_0(x_c)}\lambda_i V_\alpha(1_i,x_c)\}/\{\alpha+\sum_{i\in C_1(x_c)}\mu_i+\sum_{i\in C_0(x_c)}\lambda_i\}, \tag{4.16}$$

$$\{R+\mu_0 V_\alpha(0)\}/(\alpha+\mu_0)].$$

The result is shown by comparing each term with one in functional equation (4.16). Thus we see

$$[V_\alpha(x_c)]_{\pi(x_c)=0}-[V_\alpha(x_c)]_{\pi(x_c)=1}$$

$$=\{P\phi(x_c)+\sum_{i\in C_1(x_c)}(r_i+\mu_i V_\alpha(0_i,x_c))+\sum_{i\in C_0(x_c)}\lambda_i V_\alpha(1_i,x_c)\}$$

$$/(\alpha+\sum_{i\in C_1(x_c)}\mu_i+\sum_{i\in C_0(x_c)}\lambda_i)-\{R+\mu_0 V_\alpha(0)\}/(\alpha+\mu_0)$$

$$\geqq[1/(\alpha+\sum_{i\in C_1(x_c)}\mu_i+\sum_{i\in C_0(x_c)}\lambda_i)-1/(\alpha+\mu_0)]R$$

$$+[(\sum_{i\in C_1(x_c)}\mu_i+\sum_{i\in C_0(x_c)}\lambda_i)/(\alpha+\sum_{i\in C_1(x_c)}\mu_i+\sum_{i\in C_0(x_c)}\lambda_i)$$

$$-\mu_0/(\alpha+\mu_0)]V_\alpha(0)$$

$$\geqq [1/(+ \sum_{i\in C_1(x_c)} \mu_i + \sum_{i\in C_0(x_c)} \lambda_i) - 1/(\alpha+\mu_0)]\alpha V_\alpha(0)$$

$$+[(\sum_{i\in C_1(x_c)} \mu_i + \sum_{i\in C_0(x_c)} \lambda_i)/(\alpha + \sum_{i\in C_1(x_c)} \mu_i + \sum_{i\in C_0(x_c)} \lambda_i)$$

$$-\mu_0/(\alpha+\mu_0)]V_\alpha(0)$$

$$\geqq 0$$

The first inequality follows from the assumptions and Theorem 4.12. The second inequality is true from Theorem 4.13. ||

Property 4.4. If $\mu_0 < \sum_{i\in C_1(x_c)} \mu_i + \sum_{i\in C_0(x_c)} \lambda_i$ and $R/\mu_0 \leqq [P\phi(x_c) + \sum_{i\in C_1(x_c)} r_i]/[\sum_{i\in C_1(x_c)} \mu_i + \sum_{i\in C_0(x_c)} \lambda_i]$ for $x_c \in E_c$, then an optimal action is to replace the maintained coherent system.

Proof: The result is proved similarly to Property 4.3. ||

Remark 4.4. The results of this section remain valid even when we extend the cost rate $r(x_c,0)$ is a member of $F(E_c)$ and $r(0,0)=0$.

Remark 4.5. We notice that the monotone property of the optimal group replacement policy holds irrespective of failure and repair rates, but of course the actual policy $\pi(x_c)$ depends an the values of failure and repair rates.

4.4.4 Example

To illustrate the optimal group replacement policy, we give a numerical example. We consider a so called bridge structure system shown in Figure 4.1. The failure and repair rates of components are given in Table 4.1. The repair cost rate of components are also given in Table 4.1. The system down cost rate, replacement cost rate, and replacement rate are P=5.0, R=10.0, and μ_0=2.0, respectiely. Then we obtain the optimal group replacement policy for the maintained coherent system by the value iteration method. Also to illustrate

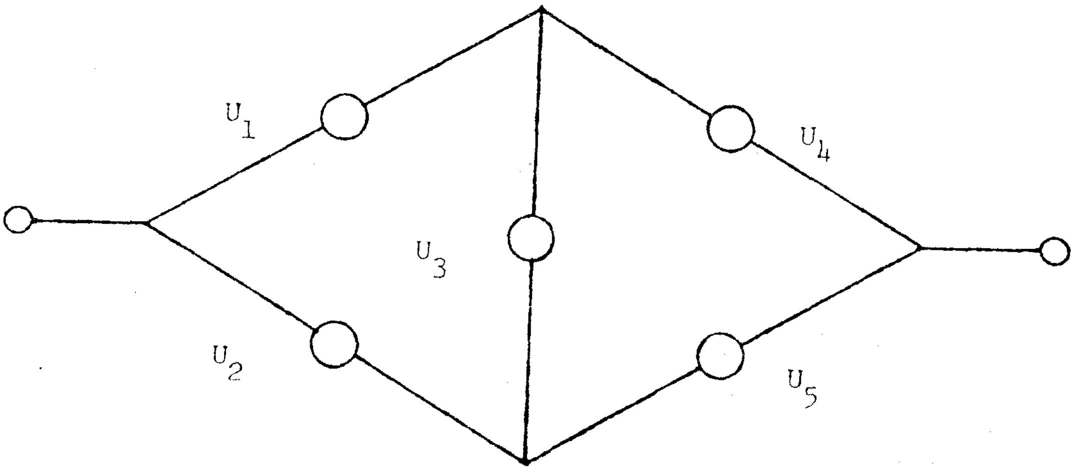

Figure 4.1. Bridge structure system

Table 4.1. Failure rates, repair rates, and cost rates.

	λ	μ	r
U_1	0.1	1.0	2.0
U_2	0.1	2.0	2.0
U_3	0.2	1.0	2.0
U_4	0.1	1.0	2.0
U_5	0.1	2.0	2.0

the results of Properties 4.3 and 4.4, the values $m_1(x_c)=\sum_{i\in C_1(x_c)}\mu_i + \sum_{i\in C_0(x_c)}\lambda_i$, $m_2(x_c)=P\phi(x_c)+\sum_{i\in C_1(x_c)} r_i$, and $m_2(x_c)/m_1(x_c)$ are computed. The results of these computations are given in Table 4.2 in the case of the discount factor α=0.05. No.4,6,11,13,18, and 25 satisfy the condition of Property 4.3, and No.1,2,3,5,7,9,17, and 21 satisty the condition of Property 4.4. But No.8,10,12,14,15,19,

Table 4.2. Optimal replacement policy

No.	x_1	x_2	x_3	x_4	x_5	$\phi(x_c)$	$\pi(x_c)$	$m_1(x_c)$	$m_2(x_c)$	$\frac{m_2(x_c)}{m_1(x_c)}$	$V_\alpha(x_c)$
1	1	1	1	1	1	1	1	2.80	15.0	5.35	18.60
2	1	1	1	1	0	1	1	2.04	13.0	6.37	18.60
3	1	1	1	0	1	1	1	2.44	13.0	5.33	18.60
4	1	1	1	0	0	1	1	1.68	11.0	6.55	18.60
5	1	1	0	1	1	1	1	2.48	13.0	5.24	18.60
6	1	1	0	1	0	1	1	1.72	11.0	6.40	18.60
7	1	1	0	0	1	1	1	2.12	11.0	5.19	18.60
8	1	1	0	0	0	1	1	1.36	9.0	6.62	18.60
9	1	0	1	1	1	1	1	2.04	13.0	6.37	18.60
10	1	0	1	1	0	0	1	1.28	6.0	4.69	18.60
11	1	0	1	0	1	1	1	1.68	11.0	6.55	18.60
12	1	0	1	0	0	0	1	0.92	4.0	4.35	18.60
13	1	0	0	1	1	1	1	1.72	11.0	6.40	18.60
14	1	0	0	1	0	0	1	0.96	4.0	4.17	18.60
15	1	0	0	0	1	0	1	1.36	4.0	2.94	18.60
16	1	0	0	0	0	0	0	0.60	2.0	3.33	17.64
17	0	1	1	1	1	1	1	2.44	13.0	5.33	18.60
18	0	1	1	1	0	1	1	1.68	11.0	6.55	18.60
19	0	1	1	0	1	0	1	2.08	6.0	2.88	18.60
20	0	1	1	0	0	0	1	1.32	4.0	3.03	18.60
21	0	1	0	1	1	1	1	2.12	11.0	5.19	18.60
22	0	1	0	1	0	0	1	1.36	4.0	2.94	18.60
23	0	1	0	0	1	0	0	1.76	4.0	2.27	18.13
24	0	1	0	0	0	0	0	1.00	2.0	2.00	16.15
25	0	0	1	1	1	1	1	1.68	11.0	6.55	18.60
26	0	0	1	1	0	0	1	0.96	4.0	4.17	18.60
27	0	0	1	0	1	0	1	1.32	4.0	3.03	18.60
28	0	0	1	0	0	0	0	0.56	2.0	3.57	17.38
29	0	0	0	1	1	1	1	1.36	9.0	6.62	18.60
30	0	0	0	1	0	0	0	0.60	2.0	3.33	17.46
31	0	0	0	0	1	0	0	1.00	2.0	2.00	16.15
32	0	0	0	0	0	0	0	0.24	0.0	0.00	14.07

20,22,26,27, and 29 don't satisfy these conditions when the optimal action is to replace the maintained coherent system. This shows that the conditions of Properties 4.3 and 4.4 are not necessary for replacing the system. No.10,12,14,15,19,20,22,26, and 27 show that a preventive replacement is optimal.

4.4.5 Concluding Remark

We have been examined the structure of optimal group replacement policy for a coherent system consisting of n repairable compoments. We showed that the optimal group replacement policy minimizing the expected total discounted cost is a monotone policy without regard to the failure and repair rates. Further we discussed sufficient conditions for the group replacement of the maintained coherent system. It is a furture problem to find the structure of an optimal group replacement policy when we have to consider fixed costs for turning on or turning off repair facilities.

4.5 Optimal stopping problem

4.5.1 Cumulative Damage Model

In this section we consider a coherent system in a random enviroment. The coherent system consists of n components which are subject to a sequence of random shock occurring in a Poisson stream at rate λ. Each shock causes a random amount of damage and these damages accumulate additively. The successive shocks of magnitudes Y(1), Y(2),..., for components are positive, independent, identically distributed random variables having a known distribution function $F(y)$. A system failure can only happen at the time of a shock arrival and occurs with probability depending on the amount of accumulated damages in each component. The failure probability is a nondecreasing function of the accumulated damages caused by all previous shocks. More precisely, if the cummulative damage is $X_c(t)=x_c$ for components at time t and a shock of magnitude y_c occurs, Then the coherent system fails with known probability $1-s(x_c+y_c)$. The function $s(.)$ is referred to as a survival function. Upon failure, the coherent system must be replaced by a new one having the same properties, and at this time a failure cost is incurred.

If the coherent system is replaced before failure, a smaller cost is incurred. Thus, there is an incentive to attempt to replace the coherent system before failure. We allow a controller to replace the coherent system at any stopping time $T \leqq \delta$, where δ is the failure time of the coherent system.

The purpose of this section is to derive an optimal replacement policy which minimizes the total long-run average cost per unit time for a coherent system.

4.5.2 Formulation

We will describe the damage process in detail. For $t \leqq \delta$, a stochastic process $Z_c = \{X_c(t) \ ; \ t \in T\}$ represents the cumulative damages attributed to shocks during $[0,t]$. The state space is $E_c = [0,\infty)^n$. Then we allow a controller to intend a planned replacement at any stopping time $T < \delta$. Upon failure, the coherent system must be replaced by a new similar one, and the replacement cycles are repeated indefinitely. Every planned replacement cost equals C and replacement by failure incurs an additional cost of K. We study the group replacement policy which is optimal in the sense that it minimizes the total long-run average cost per unit time.

We consider a renewal process formed by successive replacements of similar systems. Using familiar results in renewal theorem, we see that the long-run average cost is the expected cost over a replacement cycle divided by the expected duration between replacements. That is, the average cost associated with a stopping time $T \leqq \delta$ will be

$$(4.17) \qquad \psi_T = \frac{C + K\mathrm{P}[\delta = T]}{\mathrm{E}[T]} .$$

In our application the coherent system always start with $X_c(0)=0$, Let E_{x_c} denote expectation under the condition $X_c(0)=x_c$, and E without suffix represents expectation under the condition $X_c(0)=0$.

Let A be the infinitesimal operator of the stochastic process

Z_c. For a function f in the domain of A, the infinitesimal operator is defined as follows:

$$Af(x_c)=\lim_{t\downarrow 0} t^{-1}E_{x_c}[f(X_c(t))-f(x_c)].$$

We use Dynkin's formula

$$E_{x_c}[f(X_c(T))]-f(x_c)=E_{x_c}[\int_0^T Af(X_c(s))ds]$$

which is valid for any f in the domain of A and any stopping time T having finite expectation (see Dynkin [16, 1965]).

4.5.3 Optimal Stopping Time

We derive the optimal group replacement policy which minimizes the total long-run average cost per unit time. The damage process Z_c is clearly a strong Markov process (see Blumenthal and Getoor [10, 1968]). Let $\psi^*=\inf_T \psi_T$ be the optimal average cost. A stopping time T^* is said to be optimal if $\psi^*=\psi_{T^*}$. Then in order to prove the main theorem we need the following lemmas.

Lemma 4.4. Let $d(x_c)$ be defined by $d(x_c)=E_{x_c}[\delta]$, then

$$d(x_c)=\frac{1}{\lambda}+\int_{E_c} d(x_c+y_c)s(x_c+y_c)dF(y_c).$$

Proof: Let τ be the intershock time. Then by applying a remewal theorem we have

$$\begin{aligned} d(x_c)&=E_{x_c}[\tau+E_{X_c(\tau)}[\delta]] \\ &=E_{x_c}[\tau]+E_{x_c}[E_{X_c(\tau)}[\delta]] \\ &=\frac{1}{\lambda}+\int_{E_c} d(x_c+y_c)s(x_c+y_c)dF(y_c). \quad || \end{aligned}$$

Lemma 4.5. If the function $1-s(x_c)$ is a member of $F(E_c)$ and there exists x_c such that $s(x_c)\neq 1$ and $x_c\neq 0$, then the function $d(x_c)$

is bounded.

Proof: By the assumption there exists x_c' such that $\beta=s(x_c')<1$ and $x_c'\neq 0$. For $x_c \geq x_c'$, since the function $1-s(x_c)$ is a member of $F(E_c)$, we have from Lemma 4.4

$$d(x_c) \leqq \frac{1}{\lambda} + \int_{E_c} \beta w(x_c+y_c)dF(y_c).$$

Thus we have

$$d(x_c) \leqq \frac{1}{\lambda} \sum_{n=0}^{\infty} \beta^n = \frac{1}{\lambda(1-\beta)},$$

while by using the renewal function $M_i(x_c') = \sum_{n=0}^{\infty} P[x_i + Y_i(1) + \ldots + Y_i(n) \leqq x_i']$, the mean time needed to achieve $X_i(t) \geqq x_c'$ is no more than $M_i(x_c')/\lambda$. That is, for each $x_c \in E_c$ we have

$$d(x_c) \leqq \frac{1}{\lambda}\left[\sum_{n=1}^{\infty} M_i(x_c') + \frac{1}{1-\beta}\right],$$

and thus the proof of Lemma 4.5 is complete. ||

Lemma 4.6. For every stopping time $T \leqq \delta$, we have the following equation:

$$E_{x_c}[d(X_c(t))I_{(T<\delta)}] = E_{x_c}[\delta - T]$$

where I(.) is an indicator function.

Proof: By the strong Markov property we have

$$E_{x_c}[d(X_c(T))I_{(T<\delta)}] = E_{x_c}[E_{X_c(T)}[\delta I_{(T<\delta)}]]$$
$$= E_{x_c}[\delta - T]. \quad ||$$

Lemma 4.7. If T maximizes the following function θ_T

$$\theta_T = \psi^* E[\delta] - C - K - E[\{\psi^* d(X_c(T)) - k\} I_{(T<\delta)}]$$

then T minimizes ψ_T.

Proof: For every stopping time T, the following inequality holds

$$\psi^* \leqq \frac{C+KP[T=\delta]}{E[T]},$$

and a stopping time $T \leqq \delta$ minimizes the average cost if it maximizes

$$\theta_T = \psi^* E[T] - C - KP[T=\delta]$$

Using Lemma 4.6, we obtain

$$\theta_T = \psi^*\{E[\tau]-E[\delta-T]\} - C - K + KE[I_{(T<\delta)}]$$

$$= \psi^* E[\tau] - C - K + E[\{\psi^* w(X_c(T)) + K\} I_{(T<\delta)}]. \quad ||$$

Let $S(x_c) = \int_{E_c} s(x_c+y_c)dF(y_c)$. Now we have the main result of this section.

Theorem 4.16. An optimal stopping time T^* is

$$T^* = \min[\inf\{t \geqq 0 \mid X_c(t) \in G\}, \delta]$$

where

$$G = \{x_c \in E_c \mid \psi^* - \lambda K(1-S(x_c)) \leqq 0\}.$$

Proof: Let $\Psi(x_c) = \psi^* d(x_c) - K$. By Lemma 4.5 the function $\Psi(x_c)$ is bounded. Then we have

$$A\Psi(x_c) = \lim_{t \downarrow 0} t^{-1} E_{x_c}[\Psi(X_c(t)) I_{(t<\delta)} - \Psi(x_c)]$$

$$= -\lambda[\Psi(x_c) - \int_{E_c} \Psi(x_c+y_c)s(x_c+y_c)dF(y_c)].$$

From $\Psi(x_c) = \psi^* d(x_c) - K$, we obtain

$$A\Psi(x_c) = -\lambda[\psi^* d(x_c) - K - \int_{E_c} \{\psi^* d(x_c+y_c) - K\} s(x_c+y_c)dF(y_c)].$$

Using Lemma 4.4, we have

$$A\Psi(x_c) = -\lambda[\psi^*\{\frac{1}{\lambda} + \int_{E_c} d(x_c+y_c)s(x_c+y_c)dF(y_c)\} - K$$

$$-\psi^*\int_{E_c} d(x_c+y_c)s(x_c+y_c)dF(y_c)+K\int_{E_c} s(x_c+y_c)dF(y_c)]$$

$$=-\psi^*+\lambda K[1-\int_{E_c} s(x_c+y_c)dF(y_c)]$$

$$=-\psi^*+\lambda K(1-S(x_c)).$$

From Lemma 4.5 and $T\leqq\delta$, we have

$$E[T]\leqq E[\delta]<\infty.$$

Then since $E[T]<\infty$, we may apply Dynkin's formula to yield

$$E[\psi^* w(X_c(T))-K]=E[\int_0^T\{-\psi^*+\lambda K(1-S(X_c(s)))\}ds]+\psi^* E[\delta]-K.$$

Using Lemma 4.7, for every stopping time $T\leqq\delta$, we have

$$\theta_T=-C+E[\int_0^T\{\psi^*-\lambda K(1-S(X_c(s)))\}ds].$$

By the definition of the optimal stopping time T^* and $1-S(x_c)\in F(E_c)$, we have

$$\psi^*\lambda K(1-S(X_c(t)))>0 \quad \textit{if and only if } t<T^*.$$

For every stopping time $T\leqq\delta$, we have

$$\theta_{T^*}-\theta_T=E[\int_0^{T^*}\{\psi^*-\lambda K(1-S(X_c(s)))\}ds]-E[\int_0^T\{\psi^*-\lambda K(1-S(X_c(s)))\}ds]$$

$$=E[\int_T^{T^*}\{\psi^*-\lambda K(1-S(X_c(s)))\}ds]-E[\int_0^T\{\psi^*-\lambda K(1-S(X_c(s)))\}ds]$$

$$=E[\int_T^{T^*}\{\psi^*-\lambda K(1-S(X_c(s)))\}I_{(T<T^*)}ds]$$

$$-E[\int_{T^*}^T\{\psi^*-\lambda K(1-S(X_c(s)))\}I_{(T\geqq T^*)}ds]$$

$$\geqq 0.$$

Thus T maximizes θ_T, and this completes the proof of the optimality of T^*. ||

4.5.4 Example

We consider a two-component system. A component i fails as soon as the cumulative damage exceeds a fixed threshold L_i. Then the survival function $s(x_c)$ is

$$s(x_1,x_2)=\begin{cases}0 & \textit{if } x_1 \geqq L_1 \textit{ and } x_2 \geqq L_2, \\ 1 & \textit{otherwise.}\end{cases}$$

Also we assume that each shock causes a random amount of damage which is exponentially distributed, $F_i(y_i)=1-\exp\{-\mu_i y_i\}$. Then by the definition of $S(x_1,x_2)$ we have

$$S(x_1,x_2)=\begin{cases}1-exp\{-\mu_1(L_1-x_1)-\mu_2(L_2-x_2)\} & \textit{if } 0 \leqq x_1 < L_1 \textit{ and } 0 \leqq x_2 < L_2, \\ 1-exp\{-\mu_1(L_1-x_1)\} & \textit{if } 0 \leqq x_1 < L_1 \textit{ and } x_2 \geqq L_2, \\ 1-exp\{-\mu_2(L_2-x_2)\} & \textit{if } x_1 \geqq L_1 \textit{ and } 0 \leqq x_2 < L_2, \\ 0 & \textit{otherwise.}\end{cases}$$

By Theorem 4.16 we have the optimal stopping region G shown in Figure 4.2.

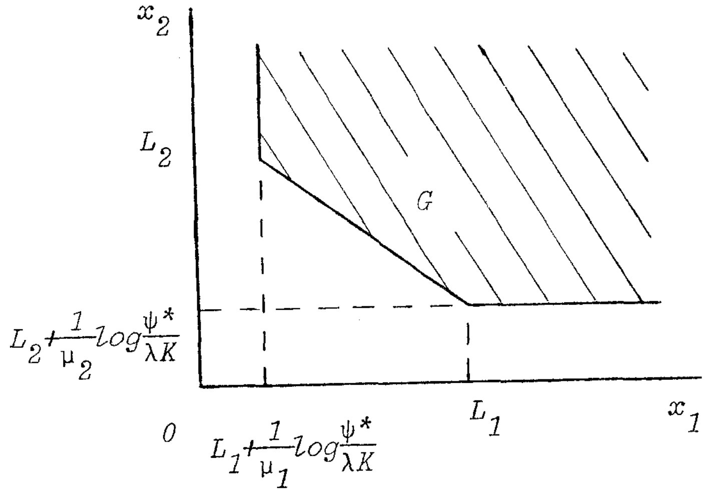

Figure 4.2. Optimal stopping region G

4.5.5 Concluding Remark

Theorem 4.16 states that it is optimal to replace the coherent system when the process $Z_c=\{X_c(t);\ t\in T\}$ enters into the optimal stopping region G, or the system fails, whichever occurs first. The stopping problem considered here has a structure similar to stopping problems treated by Ross[43, 1971] and Bergman[9, 1978]. Ross investigated the optimality of so-called infinitesimal-look-ahead (ILA) stopping rules, and Bergman studied the expected infinitesimal-look-ahead (EILA) stopping rules. It is a future proplem to find effective algorithms to obtain an optimal stopping region G.

Notes for Chapter 4

Optimal replacement policies for two-component system with increasing running cost are discussed by Berg [7,8, 1976] when the lifetimes of components are exponentially distributed. The structure of jump process is discussed by Blumenthal and Getoor [10, 1968]. The functional equation (4.5) can also be obtained by using the semi-Markov decision process in Ross [42, 1970]. Furthermore the functional equation (4.6) can be obtained by using the equivalence between continuous and discrete time Markov decision process in Serfozo [46, 1979]. The method of approximation of the coherent system deterioration process Z is discussed by Barlow and Proschan [4, 1976]. Moreover, the distribution of time to first failure in multi-component system is discussed by Ross [44, 1976] and Chiang and Nie [12, 1980]. The group replacement of a multi-component system with deterioration only appears in Sivazlian and Mahoney [49, 1978]. The optimal stopping time for replacement problem is treated by Feldman [18, 19, 1977] and Taylor [51, 1975].

CHAPTER 5

OPERATING CHARACTERISTICS OF REPLACEMENT POLICIES

5.1 Introduction

In this chapter we investigate the operating characteristic of replacement policies for a coherent system. The operating characteristic of a replacement policy is a measure defined on the induced stochastic process when a replacement policy is implemented. Some of the operating characteristics examined in this chapter are the reliability; the availability; the expected rate of system failure; the expected rate of joint replacement of components; the expected rate of replacement. These operating characteristics are the information needed to establish a suitable replacement policy.

In Section 5.2 and 5.3, the operating characteristics of the simple failure replacement policy for a two-component system are obtained. The operating characteristics of the (ABC)-policy for a two-component system are shown in Section 5.4, furthermore the operating characteristics of the (ABC)-policy for a two-component system with minimal repair are discussed in Section 5.5.

5.2 Two-Component Parallel System with Repair: I

5.2.1 Explanation of Model

In this section we consider a two-component redundant system with repair. The system consists of two identical components in parallel. A failure of a component is detected immediately, and repair (or replacement) begins. Then the other component continues the job. However, if an operating component fails when the other component is under repair, then the failed component must wait for repair until a repairman becomes free. Of course, this situation means the system failure. It is assumed that a repaired component goes into operation immediately. The time to failure and the repair time have the general continuous pdf's $f(t)$ and $g(t)$, respectively. The cdf of the time to failure is denoted by $F(t)$ and the hazard rate function is $\lambda(t)=f(t)/\overline{F}(t)$, where $\overline{F}(t)=1-F(t)$. Similarly, the cdf of of the repair time is $G(t)$ and $\mu(t)=g(t)/\overline{G}(t)$. We assume that the failure and repair processes for two components are entirely independent, and the repaired component is as good as new.

Let us now define $X_i(t)$ as the age of component i, i=1,2, at time $t \in T$, and set $X_i(0)=0$. Let $Y(t)$ represent the time that has elapsed up to time t since the beginning of the current repair job. Further let $N(t)$ denote a random variable that assumes values 0, 1 or 2. In this model, we shall set $N(t)=0$ when both components are operating at time t, $N(t)=1$ when one is operating and the other is under repair or replacement at time t, and $N(t)=2$ when two components are inoperated. These variables $Y(t)$, $N(t)$ and $X_c(t)=(X_1(t),X_2(t))$ define Markov processes in continuous time.

5.2.2 Equations of the System

We first consider the operating characteristic of a two-component system with repair when the system failure doesn't ocuur. We define the following state probabilities;

$$P_0(t,x)dx = \mathrm{P}[N(t)=0,\ 0 \leq N(s) \leq 1 \text{ for all } s \leq t,\ x<X_1(t)=X_2(t)<x+dx \mid X_c(0)=0],$$

$$P_1(t,x,y)dxdy = \mathrm{P}[N(t)=0,\ 0 \leq N(s) \leq 1 \text{ for all } s \leq t,\ x<X_i(t)<x+dx,\ y<X_{3-i}(t)<y+dy \text{ for some } i \text{ and } x>y \mid X_c(0)=0],$$

$$P_2(t,x,y)dxdy = \mathrm{P}[N(t)=1,\ 0 \leq N(s) \leq 1 \text{ for all } s \leq t,\ x<X_i(t)<x+dx \text{ for some } i,\ y<Y(t)<y+dy \mid X_c(0)=0].$$

By connecting the above state probabilities at time $t+h$ with those at time t and taking limits as $h \to 0$, we get the following differential equations governing the behaviour of the system

$$\begin{aligned} &\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + 2\lambda(x)\right]P_0(t,x)=0, \\ (5.1)\qquad &\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \lambda(x) + \lambda(y)\right]P_1(t,x,y)=0, \\ &\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \lambda(x) + \mu(y)\right]P_2(t,x,y)=0. \end{aligned}$$

These equations are to be solved under the following boundary and initial conditions.

Boundary conditions:

$$\begin{aligned} &P_1(t,x,0)=\int_0^t P_2(t,x,u)\mu(u)du, \\ (5.2)\qquad &P_2(t,x,0)=2\lambda(x)P_0(t,x)+\int_0^x P_1(t,x,u)\lambda(u)du+\int_x^t P_1(t,u,x)\lambda(u)du, \\ &P_0(t,0)=P_1(t,0,y)=P_2(t,0,y)=0. \end{aligned}$$

Initial conditions:

$$\begin{aligned} &P_0(0,0)=1, \\ (5.3)\qquad &P_1(0,x,y)=P_2(0,x,y)=0. \end{aligned}$$

Taking Laplace transforms of equations (5.1) and (5.2) with respect to t and using initial conditions, we have

$$(5.4.a)\qquad [s+\frac{\partial}{\partial x}+2\lambda(x)]P_0^*(s,x)=0,$$

$$(5.4.b)\qquad [s+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\lambda(x)+\lambda(y)]P_1^*(s,x,y)=0,$$

$$(5.4.c)\qquad [s+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\lambda(x)+\mu(y)]P_2^*(s,x,y)=0,$$

$$(5.5.a)\qquad P_0^*(s,0)=1,$$

$$(5.5.b)\qquad P_1^*(s,x,0)=\int_0^x P_2^*(s,x,u)\mu(u)du,$$

$$(5.5.c)\qquad P_2^*(s,x,0)=2\lambda(x)P_0^*(s,x)+\int_0^x P_1^*(s,x,u)\lambda(u)du+\int_x^\infty P_1^*(s,u,x)\lambda(u)du,$$

where "*" denotes the Laplace transform. The solution of equation (5.4.a), using (5.5.a), is given by

$$(5.6)\qquad P_0^*(s,x)=\overline{F}(x)^2 e^{-sx}.$$

The partial differential equation (5.4.b) is Lagrange's linear equation. Thus Lagrange's auxiliary equations are given by

$$\frac{dx}{1}=\frac{dy}{1}=\frac{-dP_1^*(s,x,y)}{[s+\lambda(x)+\lambda(y)]P_1^*(s,x,y)}.$$

Solving these equations, we have

$$x=y+c_1,$$

$$P_1^*(s,x,y)=c_2\overline{F}(x)\overline{F}(y)e^{-sy}/\overline{F}(c_1),$$

where c_1 and c_2 are arbitrary constants. Therefore the general solution of equation (5.4.b) is given by

$$(5.7)\qquad P_1^*(s,x,y)=H_1(s,x-y)\overline{F}(x)\overline{F}(y)e^{-sy},$$

where $H_1(s,x-y)$ is an arbitrary function. Similarly the general solution of equation (5.4.c) is given by

(5.8) $\qquad P_2^*(s,x,y)=H_2(s,x-y)\overline{F}(x)\overline{G}(y)e^{-sy}$,

where $H_2(s,x-y)$ is also an arbitrary function.

In order to find the form of functions $H_1(s,x-y)$ and $H_2(s,x-y)$ in (5.7) and (5.8), we proceed as follows: The probability densities along x axis in (5.7) and (5.8), setting $y=0$, are given by

(5.7') $\qquad P_1^*(s,x,0)=H_1(s,x)\overline{F}(x)$,

(5.8') $\qquad P_2^*(s,x,0)=H_2(s,x)\overline{F}(x)$.

Substituting (5.8) and (5.7') into (5.5.b), the following integral equation holds

(5.9) $\qquad H_1(s,x)=\int_0^x H_2(s,x-u)g(u)e^{-su}du.$

Similarly, substituting (5.6), (5.7) and (5.8') into (5.5.c), we have the following integral equation

(5.10)
$$H_2(s,x)=2f(x)e^{-sx}+\int_0^x H_1(s,x-u)f(u)e^{-su}du$$
$$+\int_x^\infty H_1(s,u-x)f(u)e^{-su}du.$$

Further, substituting (5.9) into the right side of (5.19), we get the following equation

(5.11) $\qquad H_2(s,x)=2f(x)e^{-sx}+\int_0^\infty K(s,x,u)H_2(s,u)du,$

where

$$K(s,x,u)=\int_0^\infty [f(x-u-y)e^{-sy}+f(x+u+y)e^{-s(x+u)}]g(y)dy,$$

and $f(t)=0$ for $t<0$. Equation (5.11) is the well-known Fredholm integral equation. Once we have obtained $H_2(s,x)$, $P_1^*(s,x,y)$ and $P_2^*(s,x,y)$ by (5.7), (5.8) and (5.9), we can compute various operating characteristics.

5.2.3 Various Characteristics of the System

Consider the reliability of the system starting from $X_i(0)=0$, $i=0,\ 1$. Now let $R(t)$ denote the reliability of the system and δ denote the time to the first system failure. Then we have

$$\begin{aligned}R(t)&=\mathrm{P}[\delta>t]\\&=\mathrm{P}[0\leqq N(s)\leqq 1 \text{ for all } s\leqq t \mid X_c(0)=0].\end{aligned}$$

From the definition of $P_0(t,x)$, $P_1(t,x,y)$ and $P_2(t,x,y)$, integrating these functions with respect to x and y, and adding, the reliability of the system can be written as

$$R(t)=\int_0^t[P_0(t,x)+\int_0^x\{P_1(t,x,y)+P_2(t,x,y)\}dy]dx.$$

Then using (5.6)-(5.8) we obtain the Laplace transform of the reliability of the system

$$(5.12)\qquad R^*(s)=\int_0^\infty[\overline{F}(x)^2e^{-sx}+H_1(s,x)a_1(s,x)+H_2(s,x)a_2(s,x)]dx,$$

where

$$a_1(s,x)=\int_0^\infty\overline{F}(x+u)\overline{F}(u)e^{-su}du,$$

$$a_2(s,x)=\int_0^\infty\overline{F}(x+u)\overline{G}(u)e^{-su}du,$$

and the mean time to the first system failure is given by

$$\begin{aligned}(5.13)\qquad \mathrm{E}[\delta]&=\lim_{s\to 0}R^*(s)\\&=\int_0^\infty[\overline{F}(x)^2+H_1(x)a_1(x)+H_2(x)a_2(x)]dx,\end{aligned}$$

where $a_i(x)=a_i(0,x)$ and $H_i(x)=H_i(0,x)$ for $i=1$ or 2. Then integral equation (5.11) is given by

$$(5.13')\qquad H_2(x)=2f(x)+\int_0^\infty K(x,u)H_2(u)du,$$

where

$$K(x,u)=\int_0^\infty[f(x-u-y)+f(x+u+y)]g(y)dy.$$

The above integral equation can also be written as

$$H_2(x)=2f(x)+2\sum_{k=1}^{\infty}\int_0^{\infty}K_k(x,u)f(u)du,$$

where

$$K_1(x,u)=K(x,u),$$

$$K_k(x,u)=\int_0^{\infty}K(x,y)K_{k-1}(y,u)dy, \qquad k\geq 2.$$

Thus using the equation (5.9), we have

$$H_1(x)=2\int_0^{x}f(x-u)g(u)du+ 2\sum_{k=1}^{\infty}\int_0^{\infty}\int_0^{\infty}K_k(x-u,y)f(y)g(u)dydu.$$

We consider the probability that a repairman is idle at time t when the system failure doesn't occur in $[0,t]$. Denote by $I(t)$ the above probability starting from $X_c(0)=0$. We have

$$I(t)=\mathrm{P}[N(t)=0|\ X_c(0)=0].$$

Thus from the definitions of $P_0(t,x)$ and $P_2(t,x,y)$, the Laplace tranceform of $I(t)$ is obtained by integrating (5.6) and (5.7) over x and y.

$$(5.14) \qquad I^*(s)=\int_0^{\infty}[F(x)^2e^{-sx}+H_1(s,x)a_1(s,x)]dx.$$

Next consider the expected total idle time of a repairman during the interval $[0,t]$ $(t<\delta)$. Denote by $I_e(t)$ the expected total idle time of the repairman during $[0,t]$. Then the Laplace transform of $I_e(t)$ is given by

$$I_e^*(s)=I^*(s)/s,$$

and the expected total idle time of the repairman prior to the system failure is

(5.15) $\quad I_e=\lim_{s\to 0} sI_e^*(s)$

$$=\int_0^\infty[\overline{F}(x)^2+H_1(x)a_1(x)]dx.$$

We consider the expected number of replacement or repair during the interval $[0,t]$ $(t<\delta)$. Using the definition of $P_2(t,x,y)$, the expected number of replacement is given by

$$R_e(t)=\int_0^t\int_0^x P_2(x,y,0)dydx.$$

The Laplace transform of $R_e(t)$, using (5.8), is obtained

(5.16) $\quad R_e^*(s)=\dfrac{1}{s}\int_0^\infty H_2(s,x)\overline{F}(x)dx,$

and the expected number of replacements prior to the system failure is

(5.17) $\quad R_e=\lim_{s\to 0} sR_e^*(s)$

$$=\int_0^\infty H_2(x)\overline{F}(x)dx.$$

5.2.4 Examples

Example 5.2.1

Consider the case where time to failure obeys a k-Erlang distribution and the repair time is arbitrarily distributed. Then the pdf of the time to failure is given by

(5.18) $\quad f(t)=\dfrac{\lambda^k t^{k-1}}{(k-1)!}\,e^{-\lambda t}.$

We shall not solve the Fredholm integral equation (5.11), but will obtain some operating characteristics using the relation of integral equations (5.9) and (5.10). Inserting (5.18) into (5.9) and (5.10), we have

(5.19) $\quad H_1(s,x)=\int_0^\infty H_2(s,x-u)g(u)e^{-su}du,$

$$H_2(s,x)=\frac{\lambda^k x^{k-1}}{(k-1)!}\,e^{-(s+\lambda)x}$$

$$(5.20)\qquad +\frac{\lambda^k}{(k-1)!}\,e^{-(s+\lambda)x}\int_0^x H_1(s,u)(x-u)^{k-1}e^{(s+\lambda)u}du$$

$$+\frac{\lambda^k}{(k-1)!}\,e^{-(s+\lambda)x}\int_0^\infty H_1(s,u)(x+u)^{k-1}e^{-\lambda u}du.$$

In order to obtain the operating characteristics, let us define $H_{ij}(s)$ as,

$$H_{ij}(s)=\int_0^\infty H_i(s,x)x^j e^{-\lambda x}dx, \qquad i=1,2,\ j=0,1,\dots,k-1.$$

Then, multiplying equations (5.19) and (5.20) by $x^j e^{-\lambda x}$ and integrating with respect to x, we obtain

$$(5.21)\qquad H_{1j}(s)=\sum_{n=0}^{j}\binom{j}{n}(-1)^{j-n}g^{(j-n)}(s+\lambda)H_{2j}(s),$$

$$H_{2j}(s)=\frac{(k+j-1)!}{(k-1)!}\,\frac{2\lambda^k}{(2\lambda+s)^{k+j}}$$

$$(5.22)\qquad +\sum_{n=0}^{j}\binom{j}{n}\frac{(k+n-1)!}{(k-1)!}\,\frac{\lambda^k}{(2\lambda+s)^{k+n}}H_{1j-n}(s)$$

$$+\sum_{n=0}^{k-1}\binom{k-1}{n}\frac{(n+j)!}{(k-1)!}\,\frac{\lambda^k}{(2\lambda+s)^{n+j+1}}H_{1k-n-1}(s),$$

where

$$g^{(j)}(s)=\int_0^\infty(-x)^j g(x)e^{-sx}\,dx, \qquad j=0,1,\dots,k-1.$$

The above equations are the set of $2k$ linear equations in $2k$ variables $H_{ij}(s)$ ($i=1,2.$, $j=0,1,\dots k-1$) with known coefficients.

Then we have some operating characteristics from the solution of $2k$ linear equations. In order to obtain the reliability $R(t)$, inserting (5.18) into (5.12) we have

$$R^*(s)=\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\binom{i+j}{i}\frac{\lambda^{i+j}}{(2\lambda+s)^{i+j+1}}$$

$$(5.23)\qquad +\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\sum_{n=0}^{j}\binom{i+j-n}{i}\frac{\lambda^{i+j}H_{1n}(s)}{n!(2\lambda+s)^{i+j-n+1}}$$

$$+\sum_{i=0}^{k-1}\sum_{n=0}^{i}\frac{\lambda^{i}}{n!}\left[\frac{1}{(s+\lambda)^{i-n+1}}-\sum_{m=0}^{i-n}\frac{(-1)^m g^{(m)}(s+\lambda)}{m!(s+\lambda)^{i-n-m+1}}\right]H_{2n}(s).$$

Then, inserting the solution of the foregoing set of linear equations into (5.23), we find the Laplace transform of the reliability, and the mean time to the first system failure by setting s=0. Similarly, we obtain the other operating characteristics as follows.

$$(5.24)\qquad I_e^*(s)=\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\binom{i+j}{i}\frac{\lambda^{i+j}}{s(2\lambda+s)^{i+j+1}} +\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\sum_{n=0}^{j}\binom{i+j-n}{i}\frac{\lambda^{i+j}H_{1n}(s)}{n!s(2\lambda+s)^{i+j-n+1}},$$

$$(5.25)\qquad R_e^*(s)=\sum_{i=0}^{k-1}\frac{\lambda^i}{i!s}H_{2i}(s).$$

The above reliability $R^*(s)$ can be obtained by using the regenerative properties (see Kodama and Deguchi [29, 1974]), however, the other operating characteristics are new.

Next, consider a two-stage Erlang failure distribution as a special case. Then, the pdf of the time to failure is given by

$$f(x)=\lambda^2 xe^{-\lambda x}.$$

Then, the solution of linear equations (5.21) and (5.22) is given as

$$H_{10}(s)=2\lambda^2(2\lambda+s)g(s+\lambda)c,$$

$$H_{11}(s)=2\lambda^2[2g(s+\lambda)-(2\lambda+s)g^{(1)}(s+\lambda)]c,$$

$$H_{20}(s)=2\lambda^2(2+s)c,$$
$$H_{21}(s)=4\lambda^2c,$$

where

$$c^{-1}=(2\lambda+s)[(2\lambda+s)(2\lambda+s+\lambda^2g^{(1)}(s+\lambda)-4\lambda^2g(s+\lambda)],$$
$$g(s+\lambda)=g^{(0)}(s+\lambda).$$

And, inserting $H_{ij}(s)$ (i=1,2., j=0,1.) into equation (5.23), we find the reliability $R^*(s)$;

$$R^*(s)=\frac{10\lambda^2+6\lambda s+s^2}{(2\lambda+s)^3}$$
$$+\frac{2\lambda^2c}{(2\lambda+s)^2}[(16\lambda^2+8\lambda s+s^2)g(s+\lambda)-\lambda(6\lambda^2+5\lambda s+s^2)g^{(1)}(s+\lambda)]$$
$$+\frac{2\lambda^2c}{(2\lambda+s)^2}[(6\lambda^2+6\lambda s+s^2)(1-g(s+\lambda))+\lambda(2\lambda^2+3\lambda s+s^2)g^{(1)}(s+\lambda)],$$

and the mean time to the first system failure is

$$\mathrm{E}[\delta]=\frac{5}{4\lambda}+\frac{12-4g(\lambda)+g^{(1)}(\lambda)}{4\lambda\{2(1-g(\lambda))+\lambda g^{(1)}(\lambda)\}}.$$

The above result agrees with that of Kodama and Deguchi [29, 1974]. The other operating characteristcs are

$$I_e^*(s)=\frac{10\lambda^2+6\lambda s+s^2}{(2\lambda+s)^3}$$
$$+\frac{2\lambda^2c}{(2\lambda+s)^2}[(16\lambda^2+8\lambda s+s^2)g(s+\lambda)-\lambda(6\lambda^2+5\lambda s+s^2)g^{(1)}(s+\lambda)],$$

$$I_e=\frac{5}{4\lambda}+\frac{8g(\lambda)-3\lambda g^{(1)}(\lambda)}{4\lambda\{2(1-g(\lambda))+\lambda g^{(1)}(\lambda)\}},$$
$$R_e^*(s)=2\lambda^2(4\lambda+s)c/s,$$
$$R_e=2/\{2(1-g(\lambda))+\lambda g^{(1)}(\lambda)\}.$$

Example 5.2.2

Consider the case where time to failure is distributed uniformly in the interval [0,1] and the repair time is constant

$$F(t)=\begin{cases} t & 0\leq t<1, \\ 1 & 1\leq t, \end{cases} \tag{5.26}$$
$$G(t)=\begin{cases} 0 & 0\leq t<1/2, \\ 1 & 1/2\leq t. \end{cases}$$

In example 5.2.1 we obtained the operating characteristic by solving the $2k$ linear equations instead of the Fredholm integral equation. In this example we solve the integral equation explicitly and obtain the mean time to the first system failure, the expected total idle time of a repairman prior to the system failure, and the expected number of repairs prior to the system failure.

First let us solve the integral equation (5.13'). For $0\leq x<1$, inserting (5.26) into (5.13'), we have

$$H_2(x)=\begin{cases} 2+\int_0^x H_2(u)du & 0\leq x<1/4, \\ 2+\int_0^{-x+1/2} H_2(u)du & 1/4\leq x<1/2, \\ 2+\int_0^{x-1/2} H_2(u)du & 1/2\leq x<1. \end{cases} \tag{5.27}$$

For $0\leq x<1/4$, differentiating both sides of integral equation (5.27) with respect to x, we obtain

$$\frac{d}{dx} H_2(x)=H_2(x).$$

Thus, using $H_2(0)=2$, the solution of the above differential equation is given by

$$H_2(x)=2e^x. \tag{5.28}$$

For $1/4\leq x<1/2$, using (5.28), the solution of the integral equation (5.27) is given by

$$(5.29) \qquad H_2(x)=2e^{-x+1/2}.$$

Similarly for $1/2 \leqq x<1$, using (5.28) and (5.29), we have

$$(5.30) \qquad H_2(x)= \begin{cases} 2e^{x-1/2} & 1/2 \leqq x<3/4, \\ 2(2e^{1/4}-e^{1-x}) & 3/4 \leqq x<1. \end{cases}$$

Furthermore, using the relation of equation (5.9), we have

$$(5.31) \qquad H_1(x)= \begin{cases} 0 & 0 \leqq x<1/2, \\ 2e^{-x-1/2} & 1/2 \leqq x<3/4, \\ 2e^{1-x} & 3/4 \leqq x<1. \end{cases}$$

Next let us calculate $a_1(x)$ and $a_2(x)$. From equation (5.26) we have

$$(5.32) \qquad \begin{aligned} a_1(x)&= \begin{cases} (1-x)^2(2+x)/6 & 0 \leqq x<1, \\ 0 & 1 \leqq x, \end{cases} \\ a_2(x)&= \begin{cases} (3-4x)/8 & 0 \leqq x<1/2, \\ (1-x)^2/2 & 1/2 \leqq x<1, \\ 0 & 1 \leqq x. \end{cases} \end{aligned}$$

Then inserting (5.28)-(5.32) into (5.13), the mean time to the first system failure $E[\delta]$ is given by

$$E[\delta]= (123e^{1/4} -141)/24.$$

Similarly, the other operating characteristics are given by

$$I_e=(299e^{1/4} -348)/96,$$

$$R_e=(49e^{1/4} -64)/8.$$

5.2.5 Remarks

The reliability of two-component parallel redundant system with general distribution is obtained by solving the integral equation, based upon the pdf's of the time to failure and the repair time.

However, the integral equation is not easily solved. The procedure developed in this section, in principle, can be applicable to the other models such as the process which describes the behavior of the system does not have regenerative states. Note that other techniques, based upon Markov renewal processes, are not applicable to such systems.

5.3 Two-Component Parallel System with Repair: II

5.3.1 Fundamental Equations

In this section we consider the stationary operating characteristics of the two-component system with repair. We define the following state probabilities,

$$P_0(t,x,y)dxdy = \mathrm{P}[N(t)=0,\ x<X_i(t)<x+dx,\ y<X_{3-i}(t)<y+dy \text{ for } x>y \text{ and some } i \mid X_1(0)=Y(0)=0],$$

$$P_1(t,x,y)dxdy = \mathrm{P}[N(t)=1,\ x<X_i(t)<x+dx,\ y<Y(t)<y+dy \text{ for some } i \mid X_1(0)=Y(0)=0],$$

$$P_2(t,x)dx = \mathrm{P}[N(t)=1,\ x<X_i(t)<x+dx,\ X_i(t)=Y(t) \text{ for some } i \mid X_1(0)=Y(0)=0],$$

$$P_3(t,x)dx = \mathrm{P}[N(t)=2,\ x<Y(t)<x+dx \mid X_1(0)=Y(0)=0].$$

Using these probabilities, we have the following differential equations governing the behavior of the system

$$\begin{aligned}
&\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\lambda(x)+\lambda(y)\right]P_0(t,x,y)=0,\\
&\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\lambda(x)+\mu(y)\right]P_1(t,x,y)=0,\\
&\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\lambda(x)+\mu(x)\right]P_2(t,x)=0,\\
&\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu(x)\right]P_3(t,x)=\lambda(x)P_2(t,x)+\int_x^t \lambda(u)P_1(t,u,x)du.
\end{aligned} \tag{5.33}$$

These equations are solved under the following boundary and initial conditions.

Boundary conditions:

$$
(5.34)\quad
\begin{aligned}
&P_0(t,x,0)=\int_0^x \mu(u)P_1(t,x,u)du+\mu(x)P_2(t,x),\\
&P_1(t,x,0)=\int_0^x \lambda(u)P_0(t,x,u)du+\int_x^t \lambda(u)P_0(t,u,x)du,\\
&P_2(t,0)=\int_0^t \mu(u)P_3(t,u)du,\\
&P_0(t,0,y)=P_1(t,0,y)=P_3(t,0)=0.
\end{aligned}
$$

Initial conditions:

$$
(5.35)\quad
\begin{aligned}
&P_2(0,0)=1,\\
&P_0(0,x,y)=P_1(0,x,y,)=P_3(0,x)=0.
\end{aligned}
$$

5.3.2 Various Characteristics

We are interested in the stationary operating characteristics of a two-component system with repair. Thus, we consider the following steady state probabilities

$$
\begin{aligned}
&P_0(x,y)=\lim_{t\to\infty} P_0(t,x,y),\\
&P_1(x,y)=\lim_{t\to\infty} P_1(t,x,y),\\
&P_2(x)=\lim_{t\to\infty} P_2(t,x),\\
&P_3(x)=\lim_{t\to\infty} P_3(t,x).
\end{aligned}
$$

General properties of Markov processes assure the existence of such probabilities when distributions of time to failure and the repair time have finite moments. The above probabilities are free from t, and thus, the stationary probabilities are obtained by solving equations (5.33) since the derivatives with respect to t are all zero. The general solution of such equations is given by

$$
(5.36)\quad
\begin{aligned}
&P_0(x,y)=cH_0(x-y)\overline{F}(x)\overline{F}(y),\\
&P_1(x,y)=cH_1(x-y)\overline{F}(x)\overline{G}(y),
\end{aligned}
$$

$$P_2(x)=c\,\overline{F}(x)\overline{G}(x),$$

$$P_3(x)=cF(x)\overline{G}(x)+c\overline{G}(x)\int_0^x\int_u^\infty H_1(y-u)f(y)dydu.$$

In order to find the functions $H_0(.)$ and $H_1(.)$, and the constant c in equation (5.36), we proceed as follows. Setting $y=0$, the probability densities along the x-axis in (5.36) are given as,

$$P_0(x,0)=cH_0(x)\overline{F}(x), \tag{5.37}$$

$$P_1(x,0)=cH_1(x)\overline{F}(x).$$

Substituting equations (5.36) and (5.37) into both sides of equation (5.34), we have the following integral equation for $H_0(.)$ and $H_1(.)$

$$H_0(x)=\int_0^x H_1(x-u)g(u)du+g(x), \tag{5.38}$$

$$H_1(x)=\int_0^x H_0(x-u)f(u)du+\int_x^\infty H_0(u-x)f(u)du. \tag{5.39}$$

Further, substituting equation (5.38) into the right side of equation (5.39), we get the following integral equation

$$H_1(x)=a(x)+\int_0^\infty K(x,u)H_1(u)du, \tag{5.40}$$

where

$$a(x)=\int_0^\infty\{g(x-u)+g(u-x)\}f(u)du,$$

$$K(x,u)=\int_0^\infty\{g(x-u-y)+g(y-x-u)\}f(y)dy,$$

and $g(t)=0$ for $t<0$. Equation (5.40) is the well-known Fledholm integral equation. A constant c is obtainable by the normalizing equation

$$\int_0^\infty\int_0^x P_0(x,y)dydx+\int_0^\infty\int_0^x P_1(x,y)dydx+\int_0^\infty P_2(x)dx+\int_0^\infty P_3(x)dx=1. \tag{5.41}$$

Once we have obtained $H_0(x)$, $H_1(x)$ and c, then we can compute various operating characteristics.

We obtain the operating characteristics using various state probabilities and densities in case of the steady state two-component system. Let A_v be the stationary system availability. Then we have

$$A_v = \lim_{t\to\infty} P[N(t)=0 \text{ or } 1]$$

From the definitions of $P_0(t,x,y)$, $P_1(t,x,y)$ and $P_2(t,x)$, integrating these functions with respect to x and y, and adding, the stationary availability is obtained as following.

$$A_v = \int_0^\infty \int_0^x P_0(x,y)\,dy\,dx + \int_0^\infty \int_0^x P_1(x,y)\,dy\,dx + \int_0^\infty P_2(x)\,dx.$$

Then using equations (5.36) and (5.41) we obtain the stationary availability

$$(5.42)\qquad A_v = c\int_0^\infty \int_0^x [\{H_0(x-y)\overline{F}(y)+H_1(x-y)\overline{G}(y)\}dy+\overline{G}(x)]dx,$$

where

$$c^{-1} = \frac{1}{\mu}[1+\int_0^\infty H_1(x)\overline{F}(x)dx]+\int_0^\infty\int_0^\infty H_0(x)\overline{F}(x+u)\overline{F}(u)\,du\,dx,$$

and

$$\mu^{-1} = \int_0^\infty t\,dG(t).$$

Now we consider the probability that a repairman is free. Denote by I_s such probability. Then we have

$$I_s = \lim_{t\to\infty} P[N(t)=0].$$

Thus from the definition of $P_0(t,x,y)$, I_s can be written as

$$(5.43)\qquad I_s = \int_0^\infty \int_0^x H_0(x-y)\overline{F}(x)\overline{F}(y)\,dx\,dy.$$

Next, we consider the expected number of system failures per unit time. Using the definition of $P_2(t,x)$, we have the expected number of system failures

$$(5.44) \qquad F_e = P_2(0) = c.$$

5.3.3 Some Special Cases

We shall consider the case in which times to failure obeys a k-Erlang distribution and the repair time is arbitrarily distributed. In this case, the pdf of the life time is given by

$$(5.45) \qquad f(t) = \frac{\lambda^k t^{k-1}}{(k-1)!} e^{-\lambda t}.$$

We shall not solve the integral equation (5.40), but will obtain the operating characteristics using the relation of the integral equations (5.38) and (5.39) instead of (5.40). Inserting equation (5.45) into equations (5.38) and (5.39), we have

$$H_0(x) = \int_0^x H_1(u) g(x-u) du + g(x),$$

$$(5.46) \qquad H_1(x) = \frac{\lambda^k}{(k-1)!} e^{-\lambda u} [\int_0^x H_0(u)(x-u)^{k-1} e^{\lambda u} du + \int_0^\infty H_0(u)(x+u)^{k-1} e^{-\lambda u} du].$$

In order to obtain some operating characteristics, let us define

$$H_{ij} = \int_0^\infty H_i(u) u^j e^{-\lambda u} du, \qquad i=0,1,\ j=0,1,\dots,k-1,$$

and multiplying equation (5.46) by $x^j e^{-\lambda x}$ and intergrating, we obtain

$$H_{0j} = \sum_{n=0}^{j} \{\binom{j}{n} (-1)^{j-n} g^{(j-n)}(\lambda) H_{1j} + (-1)^j g^{(j)}(\lambda)\},$$

$$(5.47) \qquad H_{1j} = \sum_{n=0}^{j} \binom{j}{n} \frac{(k+n-1)!}{(k-1)!} \lambda^{-n} 2^{-(k+n)} H_{0j-n} + \sum_{n=0}^{k-1} \binom{k-1}{n} \frac{(n+j)!}{(k+1)!} \lambda^{-(n+j+1-k)} 2^{-(n+j+1)} H_{0k-n-1},$$

where

$$g^{(j)}(\lambda)=\int_0^\infty(-u)^j e^{-\lambda u} g(u)du,$$

The above equations are the set of $2k$ linear equations in $2k$ variables H_{ij} with known coefficients.

Then, we have the operating characteristics from the solution of $2k$ linear equations. In order to obtain the system availability, inserting (5.45) into (5.42) we have

$$\begin{aligned}A_v=c\sum_{i=0}^{k-1}[i!\lambda^{-i-1}&-\sum_{m=0}^{i} i!(m!)^{-1}\lambda^{-i+m-1}g^{(m)}(\lambda)\\ &+\sum_{j=0}^{k-1}\sum_{n=0}^{j}\binom{i+j-n}{i}(n!)^{-1}\lambda^{n-1}2^{-i-j+n-1}H_{0n}\\ &+\sum_{n=0}^{i}(n!)^{-1}H_{1n}\{\lambda^{n-1}-\sum_{m=0}^{i-n}(m!)^{-1}(-1)^m\, {}^{n+m-1}g^{(m)}(\lambda)\}],\end{aligned}\tag{5.48}$$

where

$$\begin{aligned}c^{-1}=\mu^{-1}\{1+\sum_{i=0}^{k-1}(i!)^{-1}\lambda^{i}H_{1i}\}\\ +\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\sum_{n=0}^{j}\binom{i+j-n}{i}(n!)^{-1}\lambda^{n-1}2^{-i-j+n-1}H_{0n}.\end{aligned}$$

Then, inserting the solution of the foregoing set of linear equations into (5.48), we obtain the system availability A_v. Similarly, we obtain the idle time probability for the repairman I_s and the expected number of system failures per unit time F_e

$$I_s=c\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\sum_{n=0}^{j}\binom{i+j-n}{i}(n!)^{-1}\lambda^{n-1}2^{-i-j+n-1}H_{0n},$$

$$F_e=c.$$

Example 5.3.1

Consider an exponential failure time distribution as a special case.

$$f(t)=\lambda e^{-\lambda t}.$$

Then equation (5.46) becomes

$$H_{00}=g(\lambda)H_{10}+g(\lambda),$$
$$H_{10}=H_{00}.$$

The solution of the above equation is given by

$$H_{00}=H_{10}=g(\lambda)/(1-g(\lambda)).$$

Thus inserting H_{00} and H_{10} into equation (5.48) we find the stationary availability

$$A_v=(2-g(\lambda))/(g(\lambda)+2\lambda/\mu),$$

and similarly we obtain the idle time probability for the repairman and the expected number of system failures

$$I_s=g(\lambda)/(g(\lambda)+2\lambda/\mu),$$
$$F_e=2\lambda(1-g(\lambda))/(g(\lambda)+2\lambda/\mu).$$

The above resluts coincide with those obtained by Gaver [20, 1963].

Example 5.3.2

Consider a two stage Erlang distribution. By setting $k=2$ in equation (5.45) we get a two-stage Erlang distribution

$$f(t)=\lambda^2 te^{-\lambda t}.$$

Hence, by equation (5.47), we have

$$H_{00}=g(\lambda)+g(\lambda)H_{10},$$
$$H_{01}=-g^{(1)}(\lambda)-g^{(1)}(\lambda)H_{10}+g(\lambda)H_{11},$$
$$H_{10}=2^{-1}(H_{00}+\lambda H_{01}),$$

$$H_{11}=2^{-1}(H_{01}+H_{00}/\lambda).$$

The solution of the above equations is given by

$$H_{00}=g(\lambda)(2-g(\lambda))/c_1,$$

$$H_{01}=(g(\lambda)^2-2\lambda g^{(1)}(\lambda))/\lambda c_1,$$

$$H_{10}=(g(\lambda)-\lambda g^{(1)}(\lambda))/c_1,$$

$$H_{11}=(g(\lambda)-\lambda g^{(1)}(\lambda))/\lambda c_1,$$

where

$$c_1=2(1-g(\lambda))+\lambda g^{(1)}(\lambda).$$

Then, inserting H_{00}, H_{01}, H_{10} and H_{11} into equation (5.48) we obtain the stationary availability

$$\begin{aligned}A_v=\{&8(1+\lambda)/\lambda+2(3\lambda-8)g(\lambda)/\lambda+2(4-3\lambda)g(\lambda)^2/\lambda+2(6-7\lambda)g^{(1)}(\lambda)\\&+12(\lambda-1)g(\lambda)g^{(1)}(\lambda)+4\lambda(1-\lambda)(g^{(1)}(\lambda))^2\}\\&/\{4\lambda(2-\lambda g^{(1)}(\lambda))/\lambda+10g(\lambda)-2g(\lambda)^2-16\lambda g^{(1)}(\lambda)\},\end{aligned}$$

and the idle time probability for the repairman and the expected number of system failures as,

$$\begin{aligned}I_s=&\{10g(\lambda)-2g(\lambda)^2-6\lambda g^{(1)}(\lambda)\}/\{4\lambda(2-\lambda g^{(1)}(\lambda))/\mu\\&+10g(\lambda)-2g(\lambda)^2-6\lambda g^{(1)}(\lambda)\},\end{aligned}$$

$$\begin{aligned}F_e=&\ \lambda\{2(1-g(\lambda))+\lambda g^{(1)}(\lambda)\}/\{4\lambda(2-\lambda g^{(1)}(\lambda))/\mu\\&+10g(\lambda)-2g(\lambda)^2-6\lambda g^{(1)}(\lambda)\}.\end{aligned}$$

Example 5.3.3

We shall solve the integral equation (5.39) where we assume that the time to failure and the repair time are exponentially distributed. The pdf's of the time to failure and the repair time are

$$f(t)=\lambda e^{-\lambda t},$$

$$g(t)=\mu e^{-\mu t},$$

and thus, integral equations (5.38) and (5.39) are written to be as

$$H_0(x)=\int_0^x H_1(x-u)\mu e^{-\mu u}du+\mu e^{-\mu x},$$

$$H_1(x)=\int_0^x H_0(x-u)\lambda e^{-\lambda u}du+\int_x^\infty H_0(u-x)\lambda e^{-\lambda u}du.$$

Differenting both sides of the above equations with respect to x, we obtain the following equation

$$\frac{d}{dx}H_0(x)=\mu\{H_1(x)-H_0(x)\},$$

$$\frac{d}{dx}H_1(x)=\lambda\{H_0(x)-H_1(x)\}.$$

The general solution of the differential equation is given by

$$H_0(x)= \mu + e^{-(\lambda+\mu)x},$$

$$H_1(x)= \mu + e^{-(\lambda+\mu)x}.$$

Then we obtain the state probabilitis as,

$$P_0(x,y)=c_2 e^{-\lambda(x+y)},$$

$$P_1(x,y)=c_2 e^{-\lambda x-\mu y},$$

$$P_2(x)=c_2 e^{-(\lambda+\mu)x},$$

$$P_3(x)=c_2\frac{\lambda+\mu}{\lambda}(e^{-\lambda x}-e^{-(\lambda+\mu)x}),$$

where

$$c_2^{-1}=(\mu^2+2\lambda^2+2\lambda\mu)/2\lambda^2\mu.$$

Thus, we have the following operating characteristics

$$A_v=\mu(\mu+2\lambda)/(\mu^2+2\lambda^2+2\lambda\mu),$$

$$I_s=\mu^2/(\mu^2+2\lambda^2+2\lambda\mu),$$

$$F_e=2\lambda^2\mu/(\mu^2+2\lambda^2+2\lambda\mu).$$

5.4 Simple Replacement Policy for Two-Component System

5.4.1 (ABC)-Policy without minimal Repair

In this section we consider the operating characteristics of a simple replacement policy for two-component system. The system consists of two identical components working in series. We don't consider minimal repair. In this case, a simple replacement policy implemented is an (ABC)-policy auch as,

(a) if component i reaches at age C and the other component $3-i$ is operating in the interval $0 \leqq x_{3-i} < A$, then replace component i only,

(b) if component i reaches at age C and the other component $3-i$ is operating in the interval $A \leqq x_{3-i} < C$, then replace both components together,

where $x_i = \infty$ means that component i is failed. This policy is similar to an oppotunistic age replacement policy (OARP).

The cdf's of the time to failure and the repair time of single component are similar to Section 5.2. Furthermore, the cdf of the time consumption required for preventive replacement of single component is denoted by $G_1(t)$, and the failure replacement and preventive replacement of bothe components have the general distributions $G_2(t)$ and $G_2(t)$, respectively.

5.4.2 Fundamental Equations

We consider the operating characteristic of the two-component system under (ABC)-policy. Let $Y_1(t)$ represent the time that has elapsed up to time t since the beginning of the current preventive replacement job of of single component. Further, let $Y_2(t)$ denote the time that has elapsed up to time t since the beginning of the current failure replacement job of both components, and let $Y_3(t)$ denote that of the current preventive replacement job. The other notations are similar to Section 5.2. Then we define the following state probabilities:

$$P_0(t,x)dx = \mathrm{P}[x<X_1(t)=X_2(t)<x+dx,\ N(t)=0 \mid X_c(0)=0],$$

$$P_1(t,x,y)dxdy = \mathrm{P}[x<X_1(t)<x+dx,\ y<X_2(t)<y+dy,\ N(t)=0 \mid X_c(0)=0],$$

$$P_2(t,x,y)dy = \mathrm{P}[X_1(t)=x,\ y<Y(t)<y+dy,\ N(t)=2 \mid X_c(0)=)],$$

$$P_3(t,x,y)dy = \mathrm{P}[X_2(t)=x,\ y<Y(t)<y+dy,\ N(t)=2 \mid X_c(0)=2],$$

$$P_4(t,x,y)dy = \mathrm{P}[X_1(t)=x,\ y<Y_1(t)<y+dy,\ N(t)=2 \mid X_c(0)=2],$$

$$P_5(t,x,y)dy = \mathrm{P}[X_2(t)=x,\ y<Y_2(t)<y+dy,\ N(t)=2 \mid X_c(0)=2],$$

$$P_6(t,y)dy = \mathrm{P}[y<Y_2(t)<y+dy \mid X_c(0)=0],$$

$$P_7(t,y)dy = \mathrm{P}[y<Y_3(t)<y+dy \mid X_c(0)=0].$$

Using these probabilities, we have the following differential equations governing the behavior of the system under (ABC)-policy:

$$
(5.49)\qquad
\begin{aligned}
&[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + 2\lambda(x)]P_0(t,x)=0,\\
&[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \lambda(x)+\lambda(y)]P_1(t,x,y)=0,\\
&[\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \mu(y)]P_2(t,x,y)=0,\\
&[\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \mu(y)]P_3(t,x,y)=0,\\
&[\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \mu_1(y)]P_4(t,x,y)=0,\\
&[\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \mu_1(y)]P_5(t,x,y)=0,\\
&[\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \mu_2(y)]P_6(t,y)=0,\\
&[\frac{\partial}{\partial t} + \frac{\partial}{\partial y} + \mu_3(y)]P_7(t,y)=0.
\end{aligned}
$$

These equations are to be solved under the following boundary and initial conditions.

Boundary conditions:

$$P_0(t,0)=\int_0^t \mu_2(y)P_6(t,y)dy+\int_0^t \mu_3(y)P_7(t,y)dy,$$

$$P_1(t,x,0)=\int_0^{t-x} \mu(y)P_2(t,x,y)dy+\int_0^{t-x} \mu_1(y)P_4(t,x,y)dy,$$

$$P_1(t,0,x)=\int_0^{t-x} \mu(y)P_3(t,x,y)dy+\int_0^{t-x} \mu_1(y)P_5(t,x,y)dy,$$

$$P_2(t,x,0)=\lambda(x)P_0(t,x)+\int_0^{\min(t,x+A,C)} \lambda(y)P_1(t,x,y)dy,$$

$$P_3(t,x,0)=\lambda(x)P_0(t,x)+\int_0^{\min(t,x+A,C)} \lambda(y)P_1(t,y,x)dy,$$

$$P_4(t,x,0)=P_1(t,x,C),$$

$$P_5(t,x,0)=P_1(t,C,x), \tag{5.50}$$

$$P_6(t,0)=\int_A^{\min(t,C)} \lambda(x)P_0(t,x)dx$$

$$+\int_A^C\int_{\max(A,x-A)}^{\min(C,x+A)} \{\lambda(x)+\lambda(y)\}P_1(t,x,y)dydx$$

$$+\int_0^A\int_A^{\min(t,x+A,C)} \lambda(x)P_1(t,x,y)dydx$$

$$+\int_0^A\int_A^{\min(t,y+A,C)} \lambda(y)P_1(t,x,y)dxdy,$$

$$P_7(t,0)=P_0(t,C)+\int_{\max(A,C-A)}^C P_1(t,x,C)dx$$

$$+\int_{\max(A,C-A)}^C P_1(t,C,y)dy.$$

Initial conditions:

$$P_0(0,0)=1,$$

$$P_1(0,x,y)=P_2(0,x,y)=P_3(0,x,y)=0, \tag{5.51}$$

$$P_4(0,x,y)=P_5(0,x,y)=P_6(0,y)=P_7(0,y)=0.$$

Similarly to Section 5.3, we will obtain the following steady state probabilities:

$$P_0(x)=\lim_{t\to\infty} P_0(t,x),$$

$$P_1(x,y)=\lim_{t\to\infty} P_1(t,x,y),$$

$$P_2(x,y)=\lim_{t\to\infty} P_2(t,x,y),$$

$$P_3(x,y)=\lim_{t\to\infty} P_3(t,x,y),$$

$$P_4(x,y)=\lim_{t\to\infty} P_4(t,x,y),$$

$$P_5(x,y)=\lim_{t\to\infty} P_5(t,x,y),$$

$$P_6(y)=\lim_{t\to\infty} P_6(t,y),$$

$$P_7(y)=\lim_{t\to\infty} P_7(t,y).$$

The above probabilities are obtained by solving equation (5,49) since the derivatives with respect to t are all zero. The general solution of such equations is given by

$$P_0(x)=c\overline{F}(x)^2, \qquad \text{for } 0\leqq x<\mathrm{C},$$

$$P_1(x,y)=cH(x-y)\overline{F}(x)\overline{F}(y), \qquad \text{for } 0\leqq x,\ y<\mathrm{C},$$

$$P_2(x,y)=c\{\overline{F}(x)f(x)+\int_0^{\min(x+\mathrm{A},\mathrm{C})} H(x-u)\overline{F}(x)f(u)du\}\overline{G}(y), \qquad \text{for } 0\leqq x<\mathrm{A},$$

$$P_3(x,y)=c\{\overline{F}(x)f(x)+\int_0^{\min(x+\mathrm{A},\mathrm{C})} H(x-u)\overline{F}(x)f(u)du\}\overline{G}(y), \qquad \text{for } 0\leqq x<\mathrm{A},$$

$$P_4(x,y)=cH(x-\mathrm{C})\overline{F}(x)\overline{F}(\mathrm{C})\overline{G}_1(y), \qquad \text{for } 0\leqq x<\mathrm{A},$$

$$P_5(x,y)=cH(\mathrm{C}-x)\overline{F}(\mathrm{C})\overline{F}(x)\overline{G}_1(y), \qquad \text{for } 0\leqq x<\mathrm{A},$$

$$\begin{aligned}P_6(y)=c\{&\int_{\mathrm{A}}^{\mathrm{C}}\overline{F}(x)f(x)dx\\&+\int_{\mathrm{A}}^{\mathrm{C}}\int_{\max(\mathrm{A},x-\mathrm{A})}^{\min(\mathrm{C},x+\mathrm{A})} H(x-u)\{\overline{F}(x)f(u)+\overline{F}(u)f(x)\}du dx\\&+\int_0^{\mathrm{A}}\int_{\mathrm{A}}^{\min(\mathrm{C},x+\mathrm{A})} H(x-u)\overline{F}(u)f(x)du dx\\&+\int_0^{\mathrm{A}}\int_{\mathrm{A}}^{\min(\mathrm{C},u+\mathrm{A})} H(x-u)\overline{F}(x)f(u)du dx\}\overline{G}_2(y),\end{aligned}$$

$$P_7(y)=c\{\overline{F}(\mathrm{C})^2+\int_{\max(\mathrm{A},\mathrm{C}-\mathrm{A})}^{\mathrm{C}} H(x-C)\overline{F}(x)\overline{F}(\mathrm{C})dx$$

$$+\int_{\max(\mathrm{A},\mathrm{C}-\mathrm{A})}^{\mathrm{C}} H(\mathrm{C}-x)\overline{F}(\mathrm{C})\overline{F}(x)dx\}\overline{G}_3(y).$$

where

$$H(x)=f(x)+\int_0^{\min(x+\mathrm{A},\mathrm{C})} H(x-u)f(u)du+H(\mathrm{C}-x)\overline{F}(\mathrm{C}),$$

$$H(-x)=H(x),$$

$$\begin{aligned}
\frac{1}{c} = {} & \int_0^{\mathrm{C}}\overline{F}(x)^2dx+\int_0^{\mathrm{C}}\int_0^{\mathrm{C}} H(x-y)\overline{F}(x)\overline{F}(y)dydx \\
& + \frac{2}{\mu}\int_0^{\mathrm{A}}\{\overline{F}(x)f(x)+\int_0^{\min(x+\mathrm{A},\mathrm{C})} H(x-y)\overline{F}(x)f(y)dy\}dx \\
& + \frac{2}{\mu_1}\int_0^{\mathrm{A}} H(x-\mathrm{C})\overline{F}(x)\overline{F}(\mathrm{C})dx \\
& + \frac{1}{\mu_2}\{\int_{\mathrm{A}}^{\mathrm{C}}\overline{F}(x)f(x)dx \\
& \qquad +\int_{\mathrm{A}}^{\mathrm{C}}\int_{\max(\mathrm{C},x-\mathrm{A})}^{\min(\mathrm{C},x+\mathrm{A})} H(x-y)\{\overline{F}(x)f(y)+\overline{F}(y)f(x)\}dydx \\
& \qquad + 2\int_0^{\mathrm{A}}\int_{\mathrm{A}}^{\min(\mathrm{C},x+\mathrm{A})} H(x-y)\overline{F}(y)f(x)dydx \\
& + \frac{1}{\mu_3}\{\overline{F}(\mathrm{C})^2+2\int_{\max(\mathrm{A},\mathrm{C}-\mathrm{A})}^{\mathrm{C}} H(x-\mathrm{C})\ \overline{F}(x)\overline{F}(\mathrm{C})dx\}.
\end{aligned}$$

5.4.3 The Operating Characteristics

We shall obtain the following operating characteristics; the system availability, the expected number of replacements of single component per unit time and the expected number of replacements of both components per unit time. They are of great importance in replacement theory.

Consider the stationary abailability. Now denote by A_v the stationary availability. We have

$$A_v=\lim_{t\to\infty} \mathrm{P}[N(t)=0].$$

From the definitions of $P_0(x)$ and $P_1(x,y)$, integrating these functions, the stationary availability A_v is given as

$$A_v=\lim_{t\to\infty} \{\int_0^C P_0(x)dx+\int_0^C\int_0^C P_1(x,y)dxdy\}.$$

Then using equation (5.52) we obtain the stationary availability

$$A_v=c\{\int_0^C \overline{F}(x)^2dx+\int_0^C\int_0^C H(x-y)\overline{F}(x)\overline{F}(y)dxdy\}.$$

Let R_{sf} denote the expected number of failure replacements of single component per unit time in the long run. Then we have

$$R_{sf}=\lim_{t\to\infty} \{\int_0^A P_2(t,x,0)dx+\int_0^A P_3(t,x,0)dx\}.$$

Thus from equation (5.52) we have

$$R_{sf}=2c\{\int_0^C \overline{F}(x)f(x)dx+\int_0^A\int_0^{\min(x+A,C)} H(x-y)\overline{F}(x)f(y)dydx\}.$$

Similarly, let R_{sp} denote the expected number of preventive replacements of single component per unit time. Then we have

$$R_{sp}=\lim_{t\to\infty} \{\int_0^A P_4(t,x,0)dx+\int_0^A P_5(t,x,0)dx\}$$
$$=2c\int_0^A H(x-C)\overline{F}(x)\overline{F}(C)dx.$$

Next, let R_{ef} denote the expected number of common failure replacements of both components per unit time. Then we have

$$R_{ef}=\lim_{t\to\infty} P_6(t,0)$$
$$=c\{2\int_A^C \overline{F}(x)f(x)dx$$
$$+\int_A^C\int_{\max(C,x-A)}^{\min(C,x+A)} H(x-y)\{\overline{F}(x)f(y)+\overline{F}(y)f(x)\}dudx$$

$$+2\int_0^A\int_A^{\min(C,x+A)} H(x-y)\overline{F}(y)f(x)\,dy\,dx\}.$$

Further, let R_{ep} denote the expected number of common preventive replacements of both components per unit time. Then we have

$$R_{ep}=\lim_{t\to\infty} P_7(t,0)$$

$$=c\{\overline{F}(C)^2+2\int_{\max(A,C-A)}^{C} H(x-C)\overline{F}(x)\overline{F}(C)\,dx\}$$

5.4.4 Special Case

Consider the case where the time to failure obeys an exponential distribution. Then, the pdf of the time to failure is given by

$$f(t)=\lambda e^{-\lambda t}.$$

In this case, the solution of integral equation (5.52) is given by

$$H(x)=\begin{cases} \dfrac{\lambda(1-e^{-\lambda C})}{e^{-\lambda A}(1-e^{2\lambda(A-C)})} & \text{for } 0\leqq x<C-A,\ A\leqq C<2A, \\ \dfrac{\lambda(1-e^{-\lambda C})}{e^{-\lambda A}(1-e^{2\lambda(A-C)})(1-e^{\lambda C})} & \text{for } C-A\leqq x\leqq A,\ A\leqq C<2A, \\ \lambda e^{\lambda A} & \text{for } 2A\leqq C. \end{cases}$$

Thus we can obtain the operating characteristics.

5.5 Simple Replacement Policy for Two-Component System with Minimal Repair

5.5.1 (ABC)-Policy

In this section we consider the operating characteristics of a two-component system with minimal repair. The system consists of two identical components working in series. The simple replacement policy implemented is an (ABC)-policy such as,

(1) when the two-component system is operating,

(a) if component i reaches at age C and the other component $3-i$ is operating in the interval $0 \leqq x_{3-i} < A$, then replace component i only,

(b) if component i reaches at age C and the other component $3-i$ is operating in the interval $A \leqq x_{3-i} < C$, then replace components 1 and 2 together.

(2) when the two-component system fails,

(a) if the age x_i of component i is in the interval $0 \leqq x_i < B$, for $i=1$ and 2, then carry out minimal repair,

(b) if the age x_i of component i is in the interval $0 \leqq x_i < A$ and the other component is in the interval $B \leqq x_{3-i} < C$, then replace component $3-i$ only and carry out minimal repair,

(c) in the other case replace components 1 and 2 together.

The time consumption required minimal repair has the general distribution $G_0(t)$. The other distributions are similar to Section 5.4.

5.5.2 Fundamental Equations

We consider the operating characteristics of the two-component system under (ABC)-policy. Let $Y_0(t)$ represent the time that has elapsed up to time t since the beginning of the current minimal repair job. We define the following state probabilities

$$P_0(t,x)dx=\mathrm{P}[x<X_1(t)=X_2(t)<x+dx,\ N(t)=0 \mid X_c(0)=0],$$
$$P_1(t,x,y)dy=\mathrm{P}[X_1(t)=X_2(t)=x,\ y<Y_0(t)<y+dy,\ N(t)=1 \mid X_c(0)=0],$$
$$P_2(t,y)dy=\mathrm{P}[y<Y_2(t)<y+dy \mid X_c(0)=0],$$
$$P_3(t,y)dy=\mathrm{P}[y<Y_3(t)<y+dy \mid X_c(0)=0].$$

Using these probabilities, we have the following differential equations governing the behaviour of the two-component system under (ABC)-policy,

$$(5.56)\quad \begin{aligned} &\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+2\lambda(x)\right]P_0(t,x)=\mathrm{I}_{[0,\mathrm{B})}(x)\int_0^t \mu_0(y)P_1(t,x,y)dy, && \text{for } 0\leqq x<\mathrm{C},\\ &\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\mu_0(y)\right]P_1(t,x,y)=0, && \text{for } 0\leqq x<\mathrm{B},\\ &\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\mu_2(y)\right]P_2(t,y)=0,\\ &\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\mu_3(y)\right]P_3(t,y)=0. \end{aligned}$$

These equation are to be solved under the following boundary and initial conditions.

Boundary conditions:

$$(5.57)\quad \begin{aligned} &P_0(t,0)=\int_0^t\{\mu_2(y)P_2(t,y)\ +\mu_3(y)P_3(t,y)\}dy,\\ &P_1(t,x,0)=2\lambda(x)P_0(t,x), && \text{for } 0\leqq x<\mathrm{B},\\ &P_2(t,0)=\int_{\mathrm{B}}^{\mathrm{C}}2\lambda(x)P_0(t,x)dx,\\ &P_3(t,0,0)=0. \end{aligned}$$

Initial conditions:

$$(5.58)\quad \begin{aligned} &P_0(0,0)=1,\\ &P_1(0,x,y)=P_2(0,y)=P_3(0,y)=0. \end{aligned}$$

Similarly to Section 5.3, we obtain the following steady state probabilities

$$P_0(x)=\lim_{t\to\infty} P_0(t,x),$$
$$P_1(x,y)=\lim_{t\to\infty} P_1(t,x,y),$$
$$P_2(y)=\lim_{t\to\infty} P_2(t,y),$$
$$P_3(y)=\lim_{t\to\infty} P_3(t,y).$$

Then the general solution of these equations is given by

$$P_0(x)=\begin{cases} c & 0\leqq x<\mathrm{B}, \\ c\overline{F}(x)^2\overline{F}(\mathrm{B})^{-2} & \mathrm{B}\leqq x<\mathrm{C}, \end{cases}$$

$$\begin{aligned} &P_1(x,y)=2c\lambda(x)\overline{G}_1(y), && 0\leqq x<\mathrm{B}, \\ &P_2(y)=c\overline{F}(\mathrm{C})^2\overline{F}(\mathrm{B})^{-2}\overline{G}_2(y), \\ &P_3(y)=2c\overline{G}_3(y)\int_{\mathrm{B}}^{\mathrm{C}}\overline{F}(x)\overline{F}(\mathrm{B})^{-2}f(x)dx, \end{aligned} \tag{5.59}$$

where

$$c^{-1}=\mathrm{B}+\int_{\mathrm{B}}^{\mathrm{C}}(\overline{F}(x)\overline{F}(\mathrm{B})^{-1})^2dx+\frac{2}{\mu_0}\int_0^{\mathrm{B}}\lambda(x)dx+\frac{1}{\mu_2}(\overline{F}(\mathrm{C})\overline{F}(\mathrm{B})^{-1})^2$$
$$+\frac{2}{\mu_3}\int_{\mathrm{B}}^{\mathrm{C}}\overline{F}(x)\overline{F}(\mathrm{B})^{-2}f(x)dx,$$

$$\mu_i^{-1}=\int_0^\infty tg_i(t)dt, \qquad i=0,2,3.$$

5.5.3 The Operating Characteristics

We shall obtain the following operating characteristics; the stationary availability, the expected number of preventive replacements, the expected number of failure replacements and the expected number of minimal repair. They are of great importance in replacement theory.

Consider the stationary availability. Now denote by A_v the stationary availability. We have

$$A_v=\lim_{t\to\infty} \mathrm{P}[N(t)=0].$$

From the definition of $P_0(t,x)$, integrating this function, the stationay availability can be written as

$$A_v=\lim_{t\to\infty} \int_0^C P_0(t,x)dx.$$

Then, using equation (5.59), we obtain the stationary availability

$$A_v=c\{\mathrm{B}+\int_{\mathrm{B}}^{\mathrm{C}}\overline{F}(x)^2\overline{F}(\mathrm{B})^{-2}dx\}. \tag{5.60}$$

Let M_e denote the expected number of minimal repair per unit time in the long run. Then we have

$$M_e=\lim_{t\to\infty} \mathrm{P}[N(t)=1,Y_1(t)=0].$$

Thus, from the definition of $P_1(t,x,y)$, M_e is given as,

$$\begin{aligned} M_e&=\lim_{t\to\infty} \int_0^{\mathrm{B}} P_1(t,x,0)dx \\ &=2c\int_0^{\mathrm{B}}\lambda(x)dx. \end{aligned} \tag{5.61}$$

Similarly, let R_{ep} denote the expected number of preventive replacements per unit time. Then, using the definition of $P_2(t,y)$, we have

$$R_{ep}=c\overline{F}(\mathrm{C})^2\overline{F}(\mathrm{B})^{-2} \tag{5.62}$$

Let R_{ef} denote the expected number of failure replacements per unit time. Then we have

$$R_{ef}=2c\int_{\mathrm{B}}^{\mathrm{C}}\overline{F}(x)\overline{F}(\mathrm{B})^{-2}f(x)dx. \tag{5.63}$$

Notes for Chapter 5

The two-component redundant system has been treated previously by using the supplementary variable method (see Gaver [20, 1963], Liebowity [31, 1966], Linton and Saw [33, 1974], and Linton [32, 1976]), semi-Markov processes (see Osaki [38, 1970]) and regenerative properties (see Gnedenko, Belyaev and Solovyev [21, 1969], Kodama and Deguchi [29, 1974], Osaki [39, 1970]). The operating characteristics of the (n,N) policy are computed byJorgenson, McCall and Radner [25, 1967]. Bisides those of (t,T) policy appear in Tahara and Nishida [50, 1975], and those of OAPR appear in Berg [8, 1978].

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LIST OF PUBLICATIONS

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10 Replacement policies for components in a coherent system with minimal repair. (submitted).

11 Optimal replacement for binary coherent systems. (submitted).

12 Optimal replacement policy for components in a coherent system under jump deterioration. (in preparation).

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