Title	ON THE LIMIT DISTRIBUTIONS -OF DECOMPOSABLE GALTON-WATSON PROCESSES WITH THE PERRON-FROBENIUS ROOT 1
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Citation	大阪大学, 1980, 博士論文
Version Type	VoR
URL	https://hdl.handle.net/11094/24331
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CONTENTS

ON THE LIMIT DISTRIBUTIONS OF DECOMPOSABLE GALTON-WATSON PROCESSES WITH THE PERRON-FROBENIUS ROOT 1

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(Received)

1. Introduction

We consider a decomposable Galton-Watson process (GW process, for short) which contains no supercritical class and at least one critical or final class, that is, such a process for which the mean matrix has the Perron-Frobenius root $\rho = 1$. The principal object of the present paper is to prove several limit theorems for the most general decomposable GW processes with $\rho = 1$ and, among others, give a new characterization of the limit distributions. Some of the main results were announced in [9].

Let us begin with the classification of multitype GW processes. Let $Z(n) = (Z_i(n))_{1 \leqq i \leqq d}$ be a d-type GW process and $M = (m_j^i)_{1 \leqq i, j \leqq d}$, its mean matrix. Type j is said to be accessible from type i if $m_j^{i,(n)}$, the (i,j) component of M^n, is positive for some $n \geqq 0$. This relation is written as $i \rightarrow j$. If $i \rightarrow j$ and $j \rightarrow i$, then i and j are said to communicate with each other. This relation is written as $i \leftrightarrow j$. Since $\leftrightarrow$ is an equivalence relation, we can decompose the set of types $\{1, 2, \cdots, d\}$ into the equivalence classes $C_1, C_2, \cdots, C_N$. Accessibility is a class property, i.e., if $i \rightarrow j$ for some $i \in C_\alpha$ and $j \in C_\beta$, then $i' \rightarrow j'$ for all $i' \in C_\alpha$ and $j' \in C_\beta$. The relation $C_\alpha \longrightarrow C_\beta$ is simply written as $\beta \preceq \alpha$ ($\beta \prec \alpha$ if $\beta \neq \alpha$) and accessibility thus induces a partial order on the classes $C_1, C_2, \cdots, C_N$. The process $Z(n)$ is said to be <u>indecomposable</u> (resp.<u>decomposable</u>) if $N = 1$ (resp.$N \geq 2$).

Set $M_\beta^\alpha = (m_j^i)_{i \in C_\alpha, j \in C_\beta}$. Then,by definition,each M_α^α is irreducible. As in most of the references we assume that each M_α^α is positively regular,i.e.,irreducible and aperiodic. We denote,by ρ_α,the maximal eigenvalue of M_α^α. The class C_α is said to be <u>supercritical</u> if $\rho_\alpha > 1$ and <u>subcritical</u> if $\rho_\alpha < 1$. When $\rho_\alpha = 1$,C_α is said to be <u>final</u> (resp.<u>critical</u>) if the generating function $F^i(s)$,$i \in C_\alpha$,are linear with respect to s^i,$i \in C_\alpha$ (resp.otherwise).

The limit theorems for decomposable GW processes with $\rho = \max \{\rho_1, \rho_2, \cdots, \rho_N\} = 1$ have been studied by several authors.The central problems for such processes are concerned with the limit distributions of random vectors of the form

(1.1) $$\{ \frac{Z_\beta(n)}{a_\beta(n)} ; \beta = 1, \cdots, N | E_n \},$$

where $Z_\beta(n) = (Z_i(n))_{i \in C_\beta}$,$1 \leq \beta \leq N$,are the subvectors of $Z(n)$ on C_β, E_n is a conditioning on $Z(n)$ and $a_\beta(n)$,$1 \leq \beta \leq N$,are certain normalizing sequences. Let $e^i = (0, \cdots, 0, 1, 0, \cdots, 0)$,where the i'th component is 1 and the others are 0,and let P_{e^i} be the measure of the process such that $P_{e^i}[Z(0) = e^i] = 1$. Consider the process starting at $Z(0) = e^i$,$i \in C_\alpha$. Unless $\beta \preceq \alpha$,$Z_\beta(n) = 0$ for every $n \geq 0$. Therefore, without loss of generality,we may assume that

(1.2) $$Z(0) = e^i, i \in C_N, \quad \text{and} \quad \beta \prec N \text{ for any } \beta \neq N.$$

We now list some of those results obtained in the references. Provisionally, $\lim_{n\to\infty} X(n)$ means the limit of random vectors $X(n)$ in distribution. First we state two <u>unconditioned</u> limit theorems.

[A] (Polin [6]). $N = 2$ and $1 < 2$ with C_1 a critical class and C_2 a final class. In this case $\lim_{n\to\infty} (n^{-1}Z_1(n), Z_2(n))$ exists and its components are independent. The limit distribution is given explicitly.

[B] (Foster and Ney [2]). $\{C_1, \cdots, C_N\}$ is linearly ordered; $1 < 2 < \cdots < N$. (Note that the order is converse with that of [2].) C_α is critical for $\alpha \neq N$ and C_N is a one-type final class. Under these assumptions, $\lim_{n\to\infty} \{n^{-N+1}Z_1(n), \cdots, n^{-1}Z_{N-1}(n), Z_N(n)\}$ exists and is non-degenerate. The Laplace transform of the limit distributions is characterized by means of some semi-linear <u>partial</u> differential equation.

Next we state two <u>conditioned</u> limit theorems.

[C] (Foster and Ney [2]). As before, $1 < 2 < \cdots < N$. Every C_α is critical. In this case, the non-degenerate limit of $\{n^{-N}Z_1(n), \cdots, n^{-1}Z_N(n) \mid Z_N(n) \neq 0\}$ exists and the limit distributions are characterized in a way similar to that in [B].

[D] (Ogura [5]). $\{C_1, \cdots, C_N\}$ contains no final class but may not be linearly ordered. Under this assumption, $\lim_{n\to\infty} \{n^{-1}Z_1(n), \cdots, n^{-1}Z_N(n) \mid Z(n) \neq 0\}$ exists and the limit distribution is determined by some recurrence formula with respect to the partial order $<$. In this case, however, the support of the limit distributions are relatively small.

We extend all the above results to the most general GW processes with $\rho = 1$. Theorem 2.3 contains [A] and [B]. Theorem 2.4 contains [C] and, together with Theorem 2.5, solves [C] and [D] (and bridges them). The characterization of the limit distributions in Theorem 2.3 to Theorem 2.5 seems to be new (see [E] below) and the recurrence formula in Theorem 10.1 is simpler than that in [5].

All the main results of this paper heavily depend on Theorem 2.1 which we shall call the <u>fundamental limit theorem</u>. Unlike the limit theorems mentioned above, this theorem and Theorem 2.2 are concerned with the limit of random vectors of the form

$$(1.3)\qquad \{ \frac{Y_\beta(n)}{a_\beta(n)} ; 1 \leqq \beta \leqq N | E_n \},$$

where $Y(n)$ is the sum of n-independent copies of $Z(n)$, $Y_\beta(n)$ is the subvectors of $Y(n)$ on C_β and E_n is a conditioning on $Y(n)$. In the simplest case, Theorem 2.1 and 2.2 are specialized as follows;

[E] As in [B], $1 \langle 2 \langle \cdots \langle N$. Every C_α is critical or final, i.e., $\rho_\alpha = 1$ for $\alpha = 1,2,\cdots,N$. Then, for every $t > 0$, the limit of

$$(1.4)\qquad \{n^{-N}Y_1([nt]),\cdots,n^{-1}Y_N([nt])\}$$

is non-degenerate. The logarithmic Laplace transform $\psi(t,\lambda)$ of the limit distribution of (1.4) is the solution of a first order <u>ordinary</u> differential equation in t having λ as a parameter. Moreover if C_N is critical, a similar result is valid for the conditioned limit of $\{n^{-N}Y_1([nt]),\cdots,n^{-1}Y_N([nt]) | Y_N([nt]) = 0\}$.

The main results are summarized in section 2. Their proofs are given in section 6 to 9. The basic tools in the proof of Theorem 2.1

and 2.3 are the expansion formulas on generating function and an exponential formula on infinite products of matrices which are close to the mean matrix.This exponential formula will be proved in section 4. Standard expansion formulas are given in section 5 and a special expansion formula at a final class,in section 8.The normalized limit M* of products of the mean matrix which is introduced in section 3 is useful for the characterization of limit distributions. Some part in the proof of the limit theorems was much simplified by making use of general results on logarithmic Laplace transform. This was suggested by T.Watanabe. Above all,Lemma 7.2 and its application to conditioned limit theorems are due to him. (The original proof by the author was rather complicated.) In section 10 we shall extend Theorem 2.4 and 2.5 by the method of Ogura [5]. Finally a few examples will be given in section 11.

Acknowledgements. I would like to express my sincere thanks to Professor T.Watanabe for his valuable advices in the course of completing this work. Some of his contributions have been mentioned previously. Most of the results were improved by his advices. Especially he pointed out the fundamental role of Theorem 2.1 whose original version was Lemma 8.1(i) and which I had taken as an auxiliary result for the proof of Theorem 2.3,and suggested the author to reduce all the other limit theorems to Theorem 2.1. The probabilistic interpretation of Theorem 2.1 is also due to him.

2. The main results

The process we consider in this paper is the following;

(A.1) For each α, M_α^α is positively regular,

(A.2) $\rho = \max\{\rho_1,\cdots,\rho_N\} = 1$, that is, there is at least one critical or final class,

(A.3) For each critical class C_α, $\sum_{i,j,k\in C_\alpha} E_{e^i}[Z_j(1)Z_k(1)] < \infty$,

(A.4) $\alpha \prec N$ for every $\alpha \neq N$.

Assumption (A.4) is no essential restriction for our purpose as was mentioned previously. We do not impose any further assumption besides Theorem 2.5 and 10.1.

We define the degree of relationship $\nu(\beta,\alpha)$ between classes C_α and C_β;

$$(2.1)\qquad \nu(\beta,\alpha) = \begin{cases} \max\limits_{\beta=\alpha_1\prec\alpha_2\prec\cdots\prec\alpha_k=\alpha} \#\{i;\rho_{\alpha_i} = 1\} & \text{if } \beta \preceq \alpha, \\ -1 & \text{otherwise,} \end{cases}$$

and

$$(2.2)\qquad \nu(\alpha) = \nu(\alpha,N).$$

Since each M_α^α is positively regular, there exists uniquely a positive row vector $v_\alpha = (v_i)_{i\in C_\alpha}$ and a positive column vector $u^\alpha = (u^i)_{i\in C_\alpha}$ such that

$$(2.3)\qquad v_\alpha M_\alpha^\alpha = \rho_\alpha v_\alpha,\quad M_\alpha^\alpha u^\alpha = \rho_\alpha u^\alpha \quad\text{and}\quad \sum_{i\in C_\alpha} u^i v_i = \sum_{i\in C_\alpha} v_i = 1.$$

If $\rho_\alpha = 1$, we define

$$(2.4)\qquad B_\alpha = \frac{1}{2}\sum_{i,j,k\in C_\alpha} v_i E_{e^i}[Z_j(1)Z_k(1) - \delta_{j,k}Z_k(1)]u^j u^k.$$

In section 3, we shall show the following fact (see Theorem 3.1); for each $i \in C_\alpha$ and $j \in C_\beta$ there exists

(2.5) $$\lim_{n\to\infty} n^{-\nu(\beta,\alpha)+1} m^{i}_{j},(n) = m^{*i}_{j},$$

and $m^{*i}_{j} > 0$ if $\nu(\beta,\alpha) \geqq 1$. We define for $\beta \lneqq \alpha$,

(2.6) $$a_{\alpha,j} = (\nu(\beta,\alpha)-1)\sum_{i\in C_\alpha} v_i m^{*i}_{j} \qquad j \in C_\beta,$$

(2.7) $$b_{\alpha,j} = \sum_{i\in C_\alpha} v_i m^{*i}_{j} \qquad j \in C_\beta,$$

(2.8) $$c^{i}_{\beta} = \sum_{j\in C_\beta} m^{*i}_{j} u^{j} \qquad i \in C_\alpha.$$

Let $\lambda = (\lambda^i)_{1\leqq i\leqq d}$ denote a vector in R^d_+.

Theorem 2.1. (Fundamental limit theorem).

Let C_α be a critical or final class and $i \in C_\alpha$. Then we have

(2.9) $$\lim_{n\to\infty} n(1-E_{e^i}[\exp(-\sum_{\beta\lneqq\alpha}\sum_{j\in C_\beta} n^{-\nu(\beta,\alpha)}\lambda^j Z_j([nt]))]) = \psi_\alpha(t,\lambda)u^i,$$

$$t > 0,$$

where $\psi_\alpha(t,\lambda)$ is the solution of

(2.10) $$\begin{cases} \dfrac{d}{dt}\psi_\alpha(t,\lambda) = -B_\alpha\psi_\alpha(t,\lambda)^2 + \displaystyle\sum_{\nu(\beta,\alpha)\geqq 2}\sum_{j\in C_\beta} a_{\alpha,j}\lambda^j t^{\nu(\beta,\alpha)-2}, \\ \psi_\alpha(0,\lambda) = \displaystyle\sum_{\nu(\beta,\alpha)=1}\sum_{j\in C_\beta} b_{\alpha,j}\lambda^j. \end{cases}$$

$\exp(-\psi_\alpha(t,\lambda))$ is the Laplace transform of an infinitely divisible distribution.

Theorem 2.2. Let C_α be a critical class and $i \in C_\alpha$. Then we have

(2.11) $$\lim_{n\to\infty} n(1-E_{e^i}[\exp(-\sum_{\beta\lneqq\alpha}\sum_{j\in C_\beta} n^{-\nu(\beta,\alpha)}\lambda^j Z_j([nt]));(Z_j([nt]))_{j\in C_\alpha}=0])$$

$$= \psi^\infty_\alpha(t,\lambda)u^i, \qquad t > 0,$$

(2.12) $\lim_{n\to\infty} nE_{e^i}[1-\exp(-\sum_{\beta\leqq\alpha}\sum_{j\in C_\beta} n^{-\nu(\beta,\alpha)}\lambda^j Z_j([nt]);(Z_j([nt]))_{j\in C_\alpha}=0])$

$$= \eta_\alpha(t,\lambda)u^i, \qquad t > 0,$$

where

(2.13) $$\psi_\alpha^\infty(t,\lambda) = \lim_{\substack{\lambda^j\to\infty \\ j\in C_\alpha}} \psi_\alpha(t,\lambda),$$

and $\eta_\alpha(t,\lambda)$ is the solution of

(2.14) $$\begin{cases} \frac{d}{dt}\eta_\alpha(t,\lambda) = -B_\alpha\eta_\alpha(t,\lambda)^2 - \frac{2}{t}\eta_\alpha(t,\lambda) \\ \qquad + \sum_{\nu(\beta,\alpha)\geqq 2}\sum_{j\in C_\beta} a_{\alpha,j}\lambda^j t^{\nu(\beta,\alpha)-2}, \\ \eta_\alpha(0,\lambda) = 0. \end{cases}$$

The relations of ψ_α, ψ_α^∞ and η_α are given by

(2.15) $$\psi_\alpha(t,\lambda) \geqq \eta_\alpha(t,\lambda),$$

(2.16) $$\psi_\alpha^\infty(t,\lambda) = (tB_\alpha)^{-1} + \eta_\alpha(t,\lambda),$$

(2.17) $$tB_\alpha(\psi_\alpha^\infty(t,\lambda) - \psi_\alpha(t,\lambda)) = 1 - tB_\alpha(\psi_\alpha(t,\lambda) - \eta_\alpha(t,\lambda))$$
$$= \exp(-B_\alpha\int_0^t (\psi_\alpha(s,\lambda) + \eta_\alpha(s,\lambda))ds).$$

$\exp(-\eta_\alpha(t,\lambda))$ is the Laplace transform of an infinitely divisible distribution.

With the help of the above theorems we can obtain the limit theorems for the process $Z(n)$ with $Z(0) = e^i, i\in C_N$, under certain normalization which depends on the degree of relationship $\nu(\beta)$ for each class C_β. We first give an unconditioned limit theorem when C_N is a final class.

Theorem 2.3. Let C_N be a final class. Set $D(1) = \{ i \in C_\alpha ; \nu(\alpha) \geqq 2\}$, $D(2) = \{i \in C_\alpha ; \nu(\alpha) = 1\}$ and $d_i = \# D(i), i = 1,2$. Then, for each $i \in C_N$, there exists

$$(2.18) \qquad \lim_{n\to\infty} E_{e^i}[\exp(-\sum_{\beta=1}^{N} \sum_{j \in C_\beta} n^{-\nu(\beta)+1}\lambda^j Z_j(n))] = G(\lambda);$$

the limit is independent of i. $G(\lambda)$ can be decomposed as follows;

$$(2.19) \qquad G(\lambda) = G_1(\lambda) G_2(\lambda),$$

$$(2.20) \qquad G_1(\lambda) = G_1((\lambda^i)_{i\in D(1)}) , \quad G_2(\lambda) = G_2((\lambda^i)_{i\in D(2)}),$$

$$(2.21) \qquad G_1(\lambda) = \prod_{\substack{\nu(\alpha)=2 \\ \rho_\alpha=1}} G_{1,\alpha}(\lambda).$$

$G_2(\lambda)$ is the Laplace transform of a probability measure on $Z_+^{d_2}$. Each $G_{1,\alpha}(\lambda)$ is the Laplace transform of an infinitely divisible distribution on $R_+^{d_1}$ and can be expressed as follows;

$$(2.22) \qquad G_{1,\alpha}(\lambda) = \exp(-c_\alpha \int_0^1 \overline{\Psi}_\alpha(s,\lambda)ds),$$

where

$$(2.23) \qquad c_\alpha = \sum_{i\in C_N} \sum_{j \in C_\alpha} v_i m^{*i}_j u^j > 0,$$

and $\overline{\Psi}_\alpha$ is the solution of

$$(2.24) \qquad \begin{cases} \dfrac{d}{dt} \overline{\Psi}_\alpha(t,\lambda) = -B_\alpha \overline{\Psi}_\alpha(t,\lambda)^2 + \displaystyle\sum_{\substack{\nu(\beta,\alpha) \geqq 2 \\ \nu(\beta,\alpha)+\nu(\alpha)=\nu(\beta)+1}} \sum_{j \in C_\beta} a_{\alpha,j}\lambda^j t^{\nu(\beta,\alpha)-2}, \\ \overline{\Psi}_\alpha(0,\lambda) = \displaystyle\sum_{\substack{\nu(\beta,\alpha) = 1 \\ \nu(\beta,\alpha)+\nu(\alpha)=\nu(\beta)+1}} \sum_{j\in C_\beta} b_{\alpha,j}\lambda^j. \end{cases}$$

We next give two conditioned limit theorems according as C_N is critical or subcritical.

Theorem 2.4. Let C_N be a critical class and $i \in C_N$. Then we have

(2.25) $$\lim_{n\to\infty} E_{e^i}[\exp(-\sum_{\beta=1}^{N} \sum_{j\in C_\beta} n^{-\nu(\beta)}\lambda^j Z_j(n)) \mid (Z_j(n))_{j\in C_N} \neq 0] = H(\lambda);$$

the limit is independent of i. $H(\lambda)$ is the Laplace transform of an infinitely divisible distribution on R_+^d and can be represented as follows;

(2.26) $$H(\lambda) = B_\alpha(\psi_N^\infty(1,\lambda) - \psi_N(1,\lambda)).$$

Theorem 2.5. Let C_N be a subcritical class and $i \in C_N$. Set

(2.27) $$A_1 = \{\alpha;\ \nu(\alpha) = 0 \ \text{ or } \ \nu(\alpha) = 1, \rho_\alpha = 1\}.$$

Assume that the set A_1 has no final class. Then we have

(2.28) $$\lim_{n\to\infty} E_{e^i}[\exp(-\sum_{\beta=1}^{N} \sum_{j\in C_\beta} n^{-\nu(\beta)}\lambda^j Z_j(n)) \mid ((Z_j(n))_{j\in C_\beta})_{\beta\in A_1} \neq 0]$$
$$= H_i(\lambda).$$

$H_i(\lambda)$ is the Laplace transform of a probability measure on R_+^d and can be represented as follows;

(2.29) $$H_i(\lambda) = (\sum_{\substack{\nu(\alpha)=1\\ \rho_\alpha=1}} c_\alpha^i (B_\alpha)^{-1})^{-1} \sum_{\substack{\nu(\alpha)=1\\ \rho_\alpha=1}} c_\alpha^i(\overline{\psi}_\alpha^\infty(1,\lambda) - \overline{\psi}_\alpha(1,\lambda)),$$

where $\overline{\psi}_\alpha(t,\lambda)$ is the solution of (2.24) and

(2.30) $$\overline{\psi}_\alpha^\infty(t,\lambda) = \lim_{\substack{\lambda^j\to\infty\\ j\in C_\alpha}} \overline{\psi}_\alpha(t,\lambda),$$

$\overline{\psi}_{\alpha}^{\infty}(t,\lambda)$ is represented as follows;

(2.31) $$\overline{\psi}_{\alpha}^{\infty}(t,\lambda) = (tB_{\alpha})^{-1} + \overline{\eta}_{\alpha}(t,\lambda),$$

where $\overline{\eta}_{\alpha}(t,\lambda)$ is the solution of

(2.32) $$\begin{cases} \dfrac{d}{dt}\,\overline{\eta}_{\alpha}(t,\lambda) = -B_{\alpha}\overline{\eta}_{\alpha}(t,\lambda)^{2} - \dfrac{2}{t}\,\overline{\eta}_{\alpha}(t,\lambda) \\ \qquad + \displaystyle\sum_{\substack{\nu(\beta,\alpha)\geqq 2 \\ \nu(\beta,\alpha)+\nu(\alpha)=\nu(\beta)+1}} \sum_{j\in C_{\beta}} a_{\alpha,j}\lambda^{j} t^{\nu(\beta,\alpha)-2}, \\ \overline{\eta}_{\alpha}(0,\lambda) = 0. \end{cases}$$

3. The normalized limit M* of products of the mean matrix

Let $M = (m^i_j)_{1 \leqq i,j \leqq d}$ be the mean matrix. Let $M^n = (m^{i,(n)}_j)$ and $(M^n)^\alpha_\beta = (m^{i,(n)}_j)_{i \in C_\alpha, j \in C_\beta}$. Then $(M^n)^\alpha_\alpha = (M^\alpha_\alpha)^n$. In this section we study the asymptotic behavior of M^n. The following lemma is well known (see [7]).

Lemma 3.1. Let M^α_α be a positively regular matrix. Then there exists uniquely a positive row vector $v_\alpha = (v_i)_{i \in C_\alpha}$ and a positive column vector $u^\alpha = (u^i)_{i \in C_\alpha}$ such that

(3.1) $$v_\alpha M^\alpha_\alpha = \rho_\alpha v_\alpha, \quad M^\alpha_\alpha u^\alpha = \rho_\alpha u^\alpha \quad \text{and} \quad \sum_{i \in C_\alpha} u^i v_i = \sum_{i \in C_\alpha} v_i = 1.$$

Set

(3.2) $$P^\alpha_\alpha = u^\alpha \otimes v_\alpha = (u^i v_j)_{i,j \in C_\alpha}.$$

Then we have

(3.3) $$(M^\alpha_\alpha)^n = \rho^n_\alpha P^\alpha_\alpha + O(\rho^n) \quad \text{for some } 0 < \rho < \rho_\alpha.$$

The mean matrix $M = (M^\alpha_\beta)_{1 \leqq \alpha, \beta \leqq N}$ satisfies the condition that $M^\alpha_\beta = 0$ unless $\beta \leqq \alpha$. If $L = (L^\alpha_\beta)$ also satisfies the same condition, then

(3.4) $$(LM)^\alpha_\beta = \sum_{\gamma=1}^{N} L^\alpha_\gamma M^\gamma_\beta = \sum_{\beta \leqq \gamma \leqq \alpha} L^\alpha_\gamma M^\gamma_\beta.$$

From this we have

(3.5) $$(M^n)^\alpha_\beta = 0 \quad \text{unless } \beta \leqq \alpha,$$

(3.6) $$(M^n)^\alpha_\beta = \sum_{\beta \leqq \gamma \leqq \alpha} (M^{n-1})^\alpha_\gamma M^\gamma_\beta = \sum_{\beta \leqq \gamma \leqq \alpha} M^\alpha_\gamma (M^{n-1})^\gamma_\beta \quad \text{for } \beta \leqq \alpha.$$

A more useful form is this;

$$(3.7)\qquad (M^n)^{\alpha}_{\beta} = \sum_{k=0}^{n-1} \sum_{\beta \leqq \gamma < \alpha} (M^{\alpha}_{\alpha})^{k} M^{\alpha}_{\gamma} (M^{n-k-1})^{\gamma}_{\beta}$$

$$= \sum_{k=0}^{n-1} \sum_{\beta \leqq \gamma < \alpha} (M^{\alpha}_{\alpha})^{n-k-1} M^{\alpha}_{\gamma} (M^{k})^{\gamma}_{\beta}$$

if $\beta < \alpha$ and $n \geqq 1$.

Theorem 3.1. Let $\beta \leqq \alpha$. Then the following statements are valid;

(i) If $\nu(\beta,\alpha) = 0$, then there exists $0 < \rho < 1$ such that

$$(3.8)\qquad (M^n)^{\alpha}_{\beta} = O(\rho^n),$$

and hence

$$(3.9)\qquad 0 < \sum_{n=0}^{\infty} (M^n)^{\alpha}_{\beta} < \infty .$$

(ii) If $\nu(\beta,\alpha) \geqq 1$, then there exists a finite $M^{*\alpha}_{\beta} > 0$ such that

$$(3.10)\qquad \lim_{n\to\infty} n^{-\nu(\beta,\alpha)+1} (M^n)^{\alpha}_{\beta} = M^{*\alpha}_{\beta}.$$

(iii) Let $\rho_{\alpha} = 1$; if $\nu(\beta,\alpha) = 1$ and $\beta < \alpha$,

$$(3.11)\qquad M^{*\alpha}_{\beta} = \sum_{k=0}^{\infty} \sum_{\beta \leqq \gamma < \alpha} P^{\alpha}_{\alpha} M^{\alpha}_{\gamma} (M^k)^{\gamma}_{\beta}.$$

and if $\nu(\beta,\alpha) \geqq 2$,

$$(3.12)\qquad M^{*\alpha}_{\beta} = (\nu(\beta,\alpha)-1)^{-1} \sum_{\substack{\beta \leqq \gamma < \alpha \\ \nu(\beta,\gamma) = \nu(\beta,\alpha)-1}} P^{\alpha}_{\alpha} M^{\alpha}_{\gamma} M^{*\gamma}_{\beta}.$$

(iv) If $\rho_{\alpha} < 1$ and $\nu(\beta,\alpha) \geqq 1$,

$$(3.13)\qquad M^{*\alpha}_{\beta} = \sum_{\substack{\beta \leqq \gamma < \alpha \\ \nu(\beta,\gamma) = \nu(\beta,\alpha)}} (I-M^{\alpha}_{\alpha})^{-1} M^{\alpha}_{\gamma} M^{*\gamma}_{\beta}.$$

Remark. $M^{*\alpha}_{\alpha} = P^{\alpha}_{\alpha}$ if $\rho_{\alpha} = 1$.

Proof. We shall prove this theorem by induction with respect to the partial order $\langle$. First we shall show (i). If α is a minimal element, then $\rho_\alpha < 1$ and $\alpha = \beta$. In this case, (3.8) is clear by Lemma 3.1. We have to show (3.8) assuming that $(M^n)^\gamma_\beta = O(\rho^n)$ for any $\beta \leqq \gamma \langle \alpha$ with $\nu(\beta,\gamma) = 0$. But by (3.7) we have

$$(M^n)^\alpha_\beta = O(\sum_{k=0}^{n-1} \rho_\alpha^k \rho^{n-k-1}) = O(\bar{\rho}^n),$$

where $\max(\rho_\alpha,\rho) < \bar{\rho} < 1$. Positivity of (3.9) follows from the definition of the partial order $\langle$.

Next we shall show (ii),(iii) and (iv) simultaneously. If α is a minimal element, then $\rho_\alpha = 1$ and $\beta = \alpha$. (3.10) is clear by Lemma 3.1. For general α, we have to show (3.10) assuming that $\lim_{n\to\infty} n^{-\nu(\beta,\gamma)+1}(M^n)^\gamma_\beta = M^{*\gamma}_\beta > 0$ for any $\beta \leqq \gamma \langle \alpha$ with $\nu(\beta,\gamma) \geqq 1$. If $\rho_\alpha = 1$ and $\nu(\beta,\alpha) = 1$, then $\nu(\beta,\gamma) = 0$ for any $\beta \leqq \gamma \langle \alpha$. By (3.7),(3.8) and Lemma 3.1, we have

$$\lim_{n\to\infty} (M^n)^\alpha_\beta = \sum_{k=0}^{\infty} \sum_{\beta \leqq \gamma \langle \alpha} P^\alpha_\alpha M^\alpha_\gamma (M^k)^\gamma_\beta .$$

This proves (3.11). Positivity follows from (3.9), $P^\alpha_\alpha > 0$ and $M^\alpha_\gamma \neq 0$ for some $\beta \leqq \gamma \langle \alpha$. If $\rho_\alpha = 1$ and $\nu(\beta,\alpha) \geqq 2$, then $\nu(\beta,\alpha) = \max_{\beta \leqq \gamma \langle \alpha} \nu(\beta,\gamma) + 1$. Then by (3.7) we have

$$\lim_{n\to\infty} n^{-\nu(\beta,\alpha)+1}(M^n)^\alpha_\beta$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n-1} \sum_{\beta \leqq \gamma \langle \alpha} n^{-\nu(\beta,\alpha)+1}(M^\alpha_\alpha)^k M^\alpha_\gamma (M^{n-k-1})^\gamma_\beta$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{n}\left(\frac{n-k-1}{n}\right)^{\nu(\beta,\alpha)-2} \sum_{\substack{\beta\leq\gamma<\alpha \\ \nu(\beta,\gamma)=\nu(\beta,\alpha)-1}} P^{\alpha}_{\alpha} M^{\alpha}_{\gamma} M^{*\gamma}_{\beta}$$

$$= (\nu(\beta,\alpha)-1)^{-1} \sum_{\substack{\beta\leq\gamma<\alpha \\ \nu(\beta,\gamma)=\nu(\beta,\alpha)-1}} P^{\alpha}_{\alpha} M^{\alpha}_{\gamma} M^{*\gamma}_{\beta},$$

which proves (3.12). Positivity follows from $P^{\alpha}_{\alpha} > 0, M^{*\gamma}_{\beta} > 0$ and $M^{\alpha}_{\gamma} \neq 0$ for some $\beta \leq \gamma < \alpha$ with $\nu(\beta,\gamma) = \nu(\beta,\alpha) - 1$. If $\rho_{\alpha} < 1$ and $\nu(\beta,\alpha) \geq 1$, then $\nu(\beta,\alpha) = \max_{\beta\leq\gamma<\alpha} \nu(\beta,\gamma)$. By (3.7) we have

$$\lim_{n\to\infty} n^{-\nu(\beta,\alpha)+1} (M^{n})^{\alpha}_{\beta}$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n-1} \sum_{\beta\leq\gamma<\alpha} (M^{\alpha}_{\alpha})^{k} M^{\alpha}_{\gamma} n^{-\nu(\beta,\alpha)+1} (M^{n-k-1})^{\gamma}_{\beta}$$

$$= \lim_{n\to\infty} \sum_{k=0}^{n-1} \sum_{\substack{\beta\leq\gamma<\alpha \\ \nu(\beta,\gamma)=\nu(\beta,\alpha)}} (M^{\alpha}_{\alpha})^{k} M^{\alpha}_{\gamma} \left(\frac{n-k-1}{n}\right)^{\nu(\beta,\alpha)-1} M^{*\gamma}_{\beta}$$

$$= \sum_{\substack{\beta\leq\gamma<\alpha \\ \nu(\beta,\gamma)=\nu(\beta,\alpha)}} (I-M^{\alpha}_{\alpha})^{-1} M^{\alpha}_{\gamma} M^{*\gamma}_{\beta}.$$

This proves (3.13). Positivity follows from $(I-M^{\alpha}_{\alpha})^{-1} > 0, M^{*\gamma}_{\beta} > 0$ and $M^{\alpha}_{\gamma} \neq 0$ for some $\beta \leq \gamma < \alpha$ with $\nu(\beta,\gamma) = \nu(\beta,\alpha)$.

4. Infinite products of matrices close to the mean matrix

In this section we shall establish some general results on infinite products of matrices close to the mean matrix. The exponential formula (4.12) or, more generally, (4.26) is a basic tool for the proof of Theorem 2.3 and has its own interest. For any sequence $\{M(n)\}_{n \geqq 0}$ of matrices and $n \geqq m$, we make the following convention;

$$(4.1) \qquad \prod_{k=m}^{n} M(k) = \prod_{k=0}^{n-m} M(n-k) = M(n)M(n-1)\cdots M(m).$$

We shall also define $\prod_{k=m}^{m-1} M(k) = I$ (the identity matrix).

Lemma 4.1. Let M_α^α be a positively regular matrix with $\rho_\alpha = 1$ and $\{a(n)\}_{n \geqq 1}$, a sequence of positive integers going to infinity. Assume that the matrices $\{M_\alpha^\alpha(k,n)\}_{0 \leqq k \leqq a(n), n \geqq 1}$ and $\{M_\alpha^\alpha(k)\}_{k \geqq 0}$ satisfy the following conditions;

$$(4.2) \qquad 0 \leqq M_\alpha^\alpha(k,n) \leqq M_\alpha^\alpha(k), \quad 0 \leqq k \leqq a(n), n \geqq 1,$$

$$(4.3) \qquad \lim_{n\to\infty} M_\alpha^\alpha(k,n) = M_\alpha^\alpha(k) \leqq M_\alpha^\alpha, \quad k \geqq 0,$$

$$(4.4) \qquad \sum_{k=0}^{\infty} M\,(k) < \infty.$$

Then there exists a row vector w_α such that

$$(4.5) \qquad \lim_{n\to\infty} \prod_{k=0}^{a(n)} (M_\alpha^\alpha - M_\alpha^\alpha(k,n)) = \lim_{n\to\infty} \prod_{k=0}^{a(n)} (M_\alpha^\alpha - M_\alpha^\alpha(k))$$

$$= \lim_{n\to\infty} \prod_{k=0}^{n} (M_\alpha^\alpha - M_\alpha^\alpha(k)) = u^\alpha \otimes w_\alpha.$$

Proof. To be short we omit the suffix α. First we shall show the last equality of (4.5). Set $L_n = \prod_{k=0}^{n-1} (M-M(k))$ and $c(k) =$

$\max \{m_j^i(k)(m_j^i)^{-1};\ i,j \in C_\alpha, m_j^i > 0\}$. Then by (4.4) we get

$$(4.6) \qquad \sum_{k=0}^{\infty} c(k) < \infty.$$

Since $(1-c(k))M \leqq M - M(k) \leqq M$, we have $\prod_{k=n}^{n+m-1} (1-c(k)) \cdot M^m L_n \leqq L_{m+n} \leqq M^m L_n$.

Therefore by Lemma 3.1,

$$(4.7) \qquad \prod_{k=n}^{\infty} (1-c(k)) \cdot (PL_n)_j^i \leqq \varliminf_{n\to\infty} (L_n)_j^i \leqq \varlimsup_{n\to\infty} (L_n)_j^i \leqq (PL_n)_j^i.$$

Since $\lim_{n\to\infty} \prod_{k=n}^{\infty} (1-c(k)) = 1$ by (4.6), there exists $Q = \lim_{n\to\infty} L_n$. By (4.7) we get $Q = PQ = (u \otimes v)Q = u \otimes vQ$. Setting $w = vQ$, we have the last equality of (4.5).

The second equality in (4.5) is obvious.

The first equality of (4.5) follows from the following inequality;

$$(4.8) \qquad 0 \leqq \prod_{k=0}^{a(n)} (M-M(k,n)) - \prod_{k=0}^{a(n)} (M-M(k))$$

$$= \sum_{k=0}^{a(n)} \prod_{\ell=k+1}^{a(n)} (M-M(\ell)) \cdot (M(k)-M(k,n)) \cdot \prod_{\ell=0}^{k-1} (M-M(\ell,n))$$

$$\leqq \sum_{k=0}^{a(n)} M^{a(n)-k} (M(k)-M(k,n)) M^k,$$

because the last display goes to zero with n by assumption (4.4).

Lemma 4.2. Let M_α^α be a positively regular matrix with $\rho_\alpha = 1$ and $\{a(n)\}_{n\geqq 1}$, a sequence of positive integers going to infinity. Assume that the matrices $\{M_\alpha^\alpha(k,n)\}_{0\leqq k\leqq a(n), n\geqq 1}$ satisfy the following conditions;

(4.9) $$0 \leqq M_\alpha^\alpha(k,n) \leqq M_\alpha^\alpha,$$

(4.10) $$\lim_{n\to\infty} \max_{0\leqq k\leqq a(n)} v_\alpha M_\alpha^\alpha(k,n)u^\alpha = 0,$$

and there exists a finite limit

(4.11) $$\lim_{n\to\infty} \sum_{k=0}^{a(n)} v_\alpha M_\alpha^\alpha(k,n)u^\alpha = c(\alpha).$$

Then we have

(4.12) $$\lim_{n\to\infty} \prod_{k=0}^{a(n)} (M_\alpha^\alpha - M_\alpha^\alpha(k,n)) = e^{-c(\alpha)} P_\alpha^\alpha.$$

Proof. As before we omit the suffix α. Set

(4.13) $$Q(m,n) = \prod_{k=m}^{a(n)} (M-M(k,n)), \quad q(m,n) = vQ(m,n)u.$$

To show (4.12), it suffices to prove the following two relations;

(4.14) $$\lim_{n\to\infty} \frac{Q(0,n)}{q(0,n)} = P = u \otimes v,$$

(4.15) $$\lim_{n\to\infty} q(0,n) = e^{-c(\alpha)}.$$

Since $\lim_{n\to\infty} M^n = u \otimes v > 0$, there exists a sequence of positive numbers $\{r_n\}$ such that

(4.16) $$\lim_{n\to\infty} r_n = 0 \quad \text{and} \quad (1-r_n)u \otimes v \leqq M^n \leqq (1+r_n)u \otimes v.$$

Set $\delta_n = \max_{0\leqq k\leqq a(n)} \max_{i,j\in C_\alpha} \{m_j^i(k,n)(m_j^i)^{-1} \ ; \ m_j^i > 0\}$. Then we have

(4.17) $$(1-\delta_n)M \leqq M - M(k,n) \leqq M, \quad 0 \leqq k \leqq a(n), \ n \geqq 1,$$

and by (4.10),

(4.18) $$\lim_{n\to\infty} \delta_n = 0.$$

First we shall prove (4.14). Let m be fixed arbitrarily and set $R(m,n) = \prod_{k=m}^{a(n)-m} (M-M(k,n))$. By (4.16) and (4.17) we have

$$((1-\delta_n)^m(1-r_m)u \otimes v)R(m,n)((1-\delta_n)^m(1-r_m)u \otimes v)$$

$$\leqq Q(0,n) \leqq ((1+r_m)u \otimes v)R(m,n)((1+r_m)u \otimes v).$$

It then follows that

$$(4.19) \quad (1-\delta_n)^{2m}(1-r_m)^2(vR(m,n)u)\ u \otimes v$$
$$\leqq Q(0,n) \leqq (1+r_m)^2(vR(m,n)u)u \otimes v,$$

$$(4.20) \quad (1-\delta_n)^{2m}(1-r_m)^2 vR(m,n)u \leqq q(0,n) \leqq (1+r_m)^2 vR(m,n)u.$$

Hence we get

$$\frac{(1-\delta_n)^{2m}(1-r_m)^2}{(1+r_m)^2}\, u \otimes v \leqq \frac{Q(0,n)}{q(0,n)} \leqq \frac{(1+r_m)^2}{(1-\delta_n)^{2m}(1-r_m)^2}\, u \otimes v.$$

Then letting $n \longrightarrow \infty$ and $m \longrightarrow \infty$,we obtain (4.14).

Next we shall prove (4.15). Set

$$(4.21) \qquad u(k,n) = \frac{Q(k,n)u}{q(k,n)} - u.$$

If we can show that

$$(4.22) \qquad \lim_{n\to\infty} \max_{0\leqq k\leqq a(n)} \max_{i\in C_\alpha} |u^i(k,n)| = 0,$$

then (4.15) is proved as follows. By (4.13) we have

$$q(0,n) = q(1,n) - vM(0,n)Q(1,n)u$$
$$= q(1,n)\left(1 - \frac{vM(0,n)Q(1,n)u}{q(1,n)}\right)$$
$$= \dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$= q(a(n),n) \prod_{k=0}^{a(n)-1} \left(1 - \frac{vM(k,n)Q(k+1,n)u}{q(k+1,n)}\right)$$

$$= (1-vM(a(n),n)u) \prod_{k=0}^{a(n)-1} (1-vM(k,n)(u+u(k+1,n))).$$

Then by (4.22), $\log(1-r) = -r + O(r^2)$ and the assumptions of this lemma we obtain (4.15);

$$\lim_{n\to\infty} \log q(0,n)$$

$$= \lim_{n\to\infty} \{\log(1-vM(a(n),n)u) + \sum_{k=0}^{a(n)-1} \log(1-vM(k,n)(u+u(k+1,n)))\}$$

$$= \lim_{n\to\infty} \{vM(a(n),n)u + \sum_{k=0}^{a(n)-1} vM(k,n)u\}$$

$$= -c(\alpha).$$

It remains to prove (4.22). Let m be fixed and consider two cases; $k \leq a(n) - m$ or $k > a(n) - m$. If $k \leq a(n) - m$, then by (4.16) and (4.17),

$$(1-\delta_n)^m(1-r_n)(u \otimes v)Q(k+m,n)u \leq Q(k,n)u \leq (1+r_m)(u \otimes v)Q(k+m,n)u.$$

Hence

$$(1-\delta_n)^m(1-r_m)q(k+m,n)u \leq Q(k,n)u \leq (1+r_m)q(k+m,n)u,$$

$$(1-\delta_n)^m(1-r_m)q(k+m,n) \leq q(k,n) \leq (1+r_m)q(k+m,n).$$

Then

$$\frac{(1-\delta_n)^m(1-r_m)}{1+r_m} u \leq \frac{Q(k,n)u}{q(k,n)} \leq \frac{1+r_m}{(1-\delta_n)^m(1-r_m)} u,$$

and therefore

$$(4.23) \quad |u^i(k,n)| \leq \left(\frac{1+r_m}{(1-\delta_n)^m(1-r_m)} - \frac{(1-\delta_n)^m(1-r_m)}{1+r_m}\right)u^i, \quad k\leq a(n)-m, i\in C_\alpha.$$

If $k > a(n) - m$, then by (4.17) we get

$$(1-\delta_n)^m u \leqq (1-\delta_n)^{a(n)-k+1} M^{a(n)-k+1} u$$

$$\leqq Q(k,n)u \leqq M^{a(n)-k+1} u = u,$$

$$(1-\delta_n)^m \leqq q(k,n) \leqq 1.$$

Then

$$(1-\delta_n)^m u \leqq \frac{Q(k,n)u}{q(k,n)} \leqq \frac{1}{(1-\delta_n)^m} u,$$

and therefore

$$(4.24) \qquad |u^i(k,n)| \leqq \left(\frac{1}{(1-\delta_n)^m} - (1-\delta_n)^m\right) u^i, \quad k > a(n)-m, i \in C_\alpha.$$

By (4.23) and (4.24) we get

$$\overline{\lim_{n\to\infty}} \max_{0\leqq k\leqq a(n)} \max_{i\in C_\alpha} |u^i(k,n)| \leqq \left(\frac{1+r_m}{1-r_m} - \frac{1-r_m}{1+r_m}\right) \max_{i\in C_\alpha} u^i.$$

Letting $m \to \infty$ we obtain (4.22).

Combining the above two lemmas, we have

Lemma 4.3. Let M_α^α be a positively regular matrix with $\rho_\alpha = 1$ and $\{a(n)\}$, a sequence of positive integers going to infinity. Let $\{M_\alpha^{\alpha,(i)}(k,n)\}_{0\leqq k\leqq a(n), n\geqq 1}$, $i = 1,2$, and $\{M_\alpha^{\alpha,(2)}(k)\}_{k\geqq 0}$ be those matrices satisfying the following conditions;

$\{M_\alpha^{\alpha,(1)}(k,n)\}$ satisfy the conditions in Lemma 4.2, $\{M_\alpha^{\alpha,(2)}(k,n)\}$ and $\{M_\alpha^{\alpha,(2)}(k)\}$ satisfy the conditions in Lemma 4.1 and

$$(4.25) \qquad 0 \leqq M_\alpha^{\alpha,(1)}(k,n) + M_\alpha^{\alpha,(2)}(k,n) \leqq M_\alpha^\alpha, \quad 0 \leqq k \leqq a(n), n \geqq 1.$$

Then we have

$$(4.26)\qquad \lim_{n\to\infty} \prod_{k=0}^{a(n)} (M_\alpha^\alpha - M_\alpha^{\alpha,(1)}(k,n) - M_\alpha^{\alpha,(2)}(k,n)) = e^{-c(\alpha)} u^\alpha \otimes w_\alpha,$$

where

$$(4.27)\qquad c(\alpha) = \lim_{n\to\infty} \sum_{k=0}^{a(n)} v_\alpha M_\alpha^{\alpha,(1)}(k,n) u^\alpha,$$

$$(4.28)\qquad w_\alpha = v_\alpha \cdot \lim_{n\to\infty} \prod_{k=0}^{n} (M_\alpha^\alpha - M_\alpha^{\alpha,(2)}(k)).$$

Proof. As before we omit the suffix α. Let m be fixed arbitrarily and set

$$(4.29)\quad \begin{cases} M(k,n) = M^{(1)}(k,n) + M^{(2)}(k,n), \quad L(m,n) = \prod_{k=0}^{m-1} (M - M(k,n)), \\ Q(m,n) = \prod_{k=m}^{a(n)} (M - M(k,n)), \quad Q^{(1)}(m,n) = \prod_{k=m}^{a(n)} (M - M^{(1)}(k,n)). \end{cases}$$

Then we have

$$(4.30)\qquad \prod_{k=0}^{a(n)} (M - M(k,n)) = Q(m,n) L(m,n),$$

$$(4.31)\qquad \lim_{n\to\infty} L(m,n) = \prod_{k=0}^{m-1} (M - M^{(2)}(k)).$$

By Lemma 4.2,

$$(4.32)\qquad \lim_{n\to\infty} Q^{(1)}(m,n) = e^{-c(\alpha)} P,$$

and by a calculation similar to (4.8),

$$(4.33)\qquad 0 \leqq Q^{(1)}(m,n) - Q(m,n)$$
$$= \sum_{k=m}^{a(n)} \prod_{\ell=k+1}^{a(n)} (M - M(\ell,n)) \cdot M^{(2)}(k,n) \cdot \prod_{\ell=m}^{k-1} (M - M^{(1)}(\ell,n))$$
$$\leqq \sum_{k=m}^{a(n)} M^{a(n)-k} M^{(2)}(k,n)\, M^{k-m}.$$

Hence

$$(4.34)\qquad e^{-c(\alpha)}P^i_j \geqq \overline{\lim_{n\to\infty}}\, Q^i_j(m,n) \geqq \underline{\lim_{n\to\infty}}\, Q^i_j(m,n)$$

$$\geqq e^{-c(\alpha)}P^i_j - \sum_{k=m}^{\infty} (PM^{(2)}(k)M^{k-m})^i_j.$$

Since $\lim\limits_{n\to\infty} \sum\limits_{k=n}^{\infty} M^{(2)}(k) = 0$, it follows from (4.31) and (4.34) that

$$\lim_{n\to\infty} \prod_{k=0}^{a(n)} (M-M(k,n)) = \lim_{m\to\infty} \lim_{n\to\infty} Q(m,n)L(m,n)$$

$$= e^{-c(\alpha)}P\cdot\lim_{n\to\infty} \prod_{k=0}^{n} (M-M^{(2)}(k)) = e^{-c(\alpha)}u \otimes w.$$

Finally we shall generalize Theorem 3.1. This lemma will be often used in later sections.

Lemma 4.4. Let α be fixed and $\{a(n)\}$, a sequence of positive integers going to infinity. Assume that $\{M(k,n)\}_{0\leqq k\leqq a(n),\, n\geqq 1}$ satisfy

$$(4.35)\qquad 0 \leqq M^\beta_\gamma(k,n) \leqq M^\beta_\gamma \quad \text{for any } \beta,\gamma \lesseqq \alpha \text{ and } 0 \leqq k \leqq a(n), n \geqq 1,$$

$$(4.36)\qquad \lim_{n\to\infty} \max_{0\leqq k\leqq a(n)} m^i_j(k,n) = 0 \quad \text{for any } i,j \in \bigcup_{\gamma\lesseqq\alpha} C_\gamma,$$

$$(4.37)\qquad \lim_{n\to\infty} \sum_{k=0}^{a(n)} M^\gamma_\gamma(k,n) = 0 \quad \text{for any } \gamma \lesseqq \alpha \text{ with } \rho_\gamma = 1.$$

Then, for any $\gamma \lesseqq \beta \lesseqq \alpha$, the following statements are valid.

(i) If $\nu(\gamma,\beta) = 0$, then

$$(4.38)\qquad \lim_{n\to\infty} \prod_{k=0}^{m} (M-M(k,n))^\beta_\gamma = (M^{m+1})^\beta_\gamma, \quad m \geqq 0,$$

$$(4.39)\qquad \lim_{n\to\infty} \sum_{k=0}^{a(n)} \prod_{\ell=0}^{k-1} (M-M(\ell,n))^\beta_\gamma = \sum_{k=0}^{\infty} (M^k)^\beta_\gamma.$$

(ii) If $\nu(\gamma,\beta) \geqq 1$, then

(4.40) $$\lim_{n\to\infty} a(n)^{-\nu(\gamma,\beta)+1}\Big(\prod_{k=0}^{a(n)} (M-M(k,n))\Big)_{\gamma}^{\beta} = M*_{\gamma}^{\beta}.$$

Proof. (4.38) is obvious and (4.39) follows from (3.8),(4.38) and the dominated convergence theorem. We next prove (4.40). In fact we shall show the following relation by induction;if $\nu(\gamma,\beta) \geqq 1$,then for any $b(n) \leqq a(n)$ such that $\lim_{n\to\infty} b(n) = \infty$,

(4.41) $$\lim_{n\to\infty} b(n)^{-\nu(\gamma,\beta)+1}\Big(\prod_{k=0}^{b(n)} (M-M(k,n))\Big)_{\gamma}^{\beta} = M*_{\gamma}^{\beta}.$$

If β is a minimal element,then $\rho_\beta = 1$ and $\beta = \gamma$. In this case,(4.41) is nothing but (4.12) in Lemma 4.2 ($c(\beta)=0$). For general β ($\precneqq \alpha$),we have to show (4.41),assuming that $\lim_{n\to\infty} c(n)^{-\nu(\gamma,\delta)+1}\Big(\prod_{k=0}^{c(n)} (M-M(k,n))\Big)_{\gamma}^{\delta}$ $= M*_{\gamma}^{\delta}$ for any $\gamma \preceq \delta \prec \beta$ with $\nu(\gamma,\delta) \geqq 1$ and any $c(n) \leqq a(n)$ with $\lim_{n\to\infty} c(n) = \infty$. This is proved in a way similar to the proof of Theorem 3.1 by using,instead of (3.7),the following formula;

(4.42) $$\Big(\prod_{k=0}^{m} (M-M(k,n))\Big)_{\gamma}^{\beta}$$

$$= \sum_{k=0}^{m} \sum_{\gamma\preceq\delta\prec\beta} \prod_{\ell=m-k}^{m} (M_\beta^\beta - M_\beta^\beta(\ell,n))\cdot(M_\delta^\beta - M_\delta^\beta(m-k,n))\cdot\prod_{\ell=0}^{m-k-1} (M-M(\ell,n)))_{\gamma}^{\delta}$$

$$\gamma \prec \beta.$$

For example,if $\rho_\beta < 1$ and $\nu(\gamma,\beta) \geqq 1$,then we have

$$\lim_{n\to\infty} b(n)^{-\nu(\gamma,\beta)+1}\Big(\prod_{k=0}^{b(n)} (M-M(k,n))\Big)_{\gamma}^{\beta} = \sum_{\substack{\gamma\preceq\delta\prec\beta \\ \nu(\gamma,\delta)=\nu(\gamma,\beta)}} (I-M_\beta^\beta)^{-1} M_\delta^\beta M*_{\gamma}^{\delta}$$

$$= M*_{\gamma}^{\beta},$$

where the last equality is due to (3.13) in Theorem 3.1. The cases when $\rho_\beta = 1, \nu(\gamma,\beta) = 1$ and $\rho_\beta = 1, \nu(\gamma,\beta) \geqq 2$ are proved similarly.

5. An auxiliary limit theorem

Let ν be a nonnegative integer and $i \in C_\beta, \beta \leqq \alpha$. An important step for the fundamental limit theorem is to estimate

$$(5.1) \qquad 1 - E_{e^i}[\exp(-\sum_{\gamma \leqq \beta} \sum_{j \in C_\gamma} n^{-\nu(\gamma,\alpha)-\nu+1} \lambda^j Z_j(k))].$$

Let X be a random vector taking values in R_+^d. The Laplace transform and the logarithmic Laplace transform of X (or the distribution of X) are defined by

$$f(\lambda) = E[\exp(-\sum_{i=1}^d \lambda^i X_i)]$$

$$\psi(\lambda) = -\log E[\exp(-\sum_{i=1}^d \lambda^i X_i)] \qquad \text{for } \lambda \in R_+^d .$$

We need the following facts from the general theory of Laplace transform.

Lemma 5.1. Let $\{X(n)\}$ be a sequence of random vectors taking values in R_+^d and $\{a(n)\}$, a sequence of positive numbers increasing to infinity.

(i) If $\{a(n)\sum_{i=1}^d E[X_i(n)]\}$ is bounded, then

$$(5.2) \qquad \psi_n(\lambda) = a(n)(1 - E[\exp(-\sum_{i=1}^d \lambda^i X_i(n))])$$

is uniformly bounded and equicontinuous in λ on compact subsets of R_+^d.

(ii) The sequence (5.2) is convergent if and only if $E[\exp(-\sum_{i=1}^d \lambda^i X_i(n))]^{a(n)}$ is convergent. In this case, the limit

$$(5.3) \qquad \psi(\lambda) = \lim_{n\to\infty} a(n)(1 - E[\exp(-\sum_{i=1}^d \lambda^i X_i(n))])$$

$$= -\lim_{n\to\infty} a(n)\cdot\log E[\exp(-\sum_{i=1}^d \lambda^i X_i(n))]$$

is the logarithmic Laplace transform of an infinitely divisible distribution on R_+^d, if $\psi(\lambda)$ is continuous. Then the convergence of (5.3) is uniform on compact sets of R_+^d, so that for $\lambda_n \to \lambda$ we have

$$(5.4) \qquad \psi(\lambda) = \lim_{n\to\infty} a(n)(1-E[\exp(-\sum_{i=1}^{d} \lambda_n^i X_i(n))]).$$

We shall give several estimates of (5.1).

Lemma 5.2. Let $i \in C_\alpha$, $\nu \geqq 0$ and $t > 0$. Then there exists $c > 0$ and $0 < \rho < 1$ such that

$$(5.5) \qquad 1 - E_{e^i}[\exp(-\sum_{\gamma \leqq \alpha} \sum_{j\in C_\gamma} n^{-\nu(\gamma,\alpha)-\nu+1}\lambda^j Z_j(k))]$$

$$\leqq cn^{-\nu} \sum_{\nu(\gamma,\alpha)\geqq 1} \sum_{j \in C_\gamma} \lambda^j + cn^{-\nu+1}\rho^k \sum_{\nu(\gamma,\alpha)=0} \sum_{j\in C_\gamma} \lambda^j,$$

$$0 \leqq k \leqq [nt].$$

Proof. Since $1 - \exp(-r) \leqq r$ on R_+, we have

$$(5.6) \qquad 1 - E_{e^i}[\exp(-\sum_{\gamma \leqq \alpha} \sum_{j\in C_\gamma} n^{-\nu(\gamma,\alpha)-\nu+1}\lambda^j Z_j(k))]$$

$$\leqq \sum_{\gamma \leqq \alpha} \sum_{j\in C_\gamma} n^{-\nu(\gamma,\alpha)-\nu+1}\lambda^j E_{e^i}[Z_j(k)]$$

$$= \sum_{\gamma \leqq \alpha} \sum_{j\in C_\gamma} n^{-\nu(\gamma,\alpha)-\nu+1}\lambda^j m_j^{i,(k)}.$$

By Theorem 3.1, there exists $c > 0$ and $0 < \rho < 1$ such that for any $k \geqq 0$ and $j \in C_\gamma$,

$$(5.7) \qquad \begin{cases} m_j^{i,(k)} \leqq c\rho^k & \text{if } \nu(\gamma,\alpha) = 0, \\ m_j^{i,(k)} \leqq c(k+1)^{\nu(\gamma,\alpha)-1} & \text{if } \nu(\gamma,\alpha) \geqq 1. \end{cases}$$

(5.5) is an easy consequence of (5.6) and (5.7).

Lemma 5.3. Let $\beta \leqq \alpha, i \in C_\beta, \nu \geqq 0$ and $t > 0$.

(i) If $\rho_\beta = 1$, then there exists $c > 0$ such that

$$(5.8)\qquad 1 - E_{e^i}[\exp(-\sum_{\gamma \leqq \beta} \sum_{j \in C_\gamma} n^{-\nu(\gamma,\alpha)-\nu+1} \lambda^j Z_j(k))]$$

$$\leqq cn^{-\nu(\beta,\alpha)-\nu+1} \sum_{\gamma \leqq \beta} \sum_{j \in C_\gamma} \lambda^j, \quad 0 \leqq k \leqq [nt].$$

(ii) If $\rho_\beta < 1$, then there exists $c > 0$ and $0 < \rho < 1$ such that

$$(5.9)\qquad 1 - E_{e^i}[\exp(-\sum_{\gamma \leqq \beta} \sum_{j \in C_\gamma} n^{-\nu(\gamma,\alpha)-\nu+1} \lambda^j Z_j(k))]$$

$$\leqq cn^{-\nu(\beta,\alpha)-\nu} \sum_{\nu(\gamma,\beta) \geqq 1} \sum_{j \in C_\gamma} \lambda^j + cn^{-\nu(\beta,\alpha)-\nu+1} \rho^k \sum_{\nu(\gamma,\beta)=0} \sum_{j \in C_\gamma} \lambda^j,$$

$$0 \leqq k \leqq [nt].$$

Proof. For any $\gamma \leqq \beta \leqq \alpha$, we have

$$(5.10)\qquad \nu(\gamma,\beta) + \nu(\beta,\alpha) \leqq \nu(\gamma,\alpha) + 1 \quad \text{if } \rho_\beta = 1,$$

$$(5.11)\qquad \nu(\gamma,\beta) + \nu(\beta,\alpha) \leqq \nu(\gamma,\alpha) \quad \text{if } \rho_\beta < 1.$$

Therefore

$$-\nu(\gamma,\alpha) - \nu + 1 \leqq -\nu(\gamma,\beta) - (\nu(\beta,\alpha) + \nu - 1) + 1 \quad \text{if } \rho_\beta = 1,$$

$$-\nu(\gamma,\alpha) - \nu + 1 \leqq -\nu(\gamma,\beta) - (\nu(\beta,\alpha) + \nu) + 1 \quad \text{if } \rho_\beta < 1.$$

Hence we obtain (5.8) (resp. (5.9)) by taking $\nu(\beta,\alpha) + \nu - 1$ (resp. $\nu(\beta,\alpha) + \nu$) for ν in (5.5).

Theorem 5.1. Let $i \in C_\alpha, \nu \geqq 2$ and $\lim_{n\to\infty} \lambda_n = \lambda$. Then

(5.12) $$\lim_{n\to\infty} n^{\nu}(1-E_{e^i}[\exp(-\sum_{\gamma\leq\alpha}\sum_{j\in C_\gamma} n^{-\nu(\gamma,\alpha)-\nu+1}\lambda_n^j Z_j([nt]))])$$

$$= \sum_{\nu(\gamma,\alpha)\geq 1}\sum_{j\in C_\gamma} m^{*i}_j\lambda^j t^{\nu(\gamma,\alpha)-1}, \quad t > 0.$$

Before the proof of this theorem,we derive some expansion formulas on generating function. Let I be a finite index set and $0 \leq s^i \leq 1, x_i \in Z_+$ for $i \in I$. For $s = (s^i)_{i\in I}$ and $x = (x_i)_{i\in I}$, s^x means $\prod_{i\in I}(s^i)^{x_i}$. Let $p^i(x)$ be the transition function of $Z(n)$ and $F(s)$ be the generating function;

$$F(s) = (F^i(s))_{1\leq i\leq d}, \quad F^i(s) = E_{e^i}[s^{Z(1)}] = \sum_{x\in Z_+^d} p^i(x)s^x.$$

Let $F(n:s) = (F^i(n:s))_{1\leq i\leq d}$ be the n-th iteration of $F(s)$,i.e.,

(5.13) $$F^i(n:s) = F^i(F(n-1:s)) = F^i(n-1:F(s)) = E_{e^i}[s^{Z(n)}], \quad 1\leq i\leq d.$$

We denote

(5.14) $$s^\alpha = (s^i)_{i\in C_\alpha}, \quad s^{(0,\alpha)} = (s^\beta)_{\beta<\alpha}, \quad s^{(0,\alpha]} = (s^\beta)_{\beta\leq\alpha},$$

(5.15) $$x_\alpha = (x_i)_{i\in C_\alpha}, \quad x_{(0,\alpha)} = (x_\beta)_{\beta<\alpha}, \quad x_{(0,\alpha]} = (x_\beta)_{\beta\leq\alpha}.$$

Then we may write

(5.16) $$s^{(0,\alpha]} = (s^{(0,\alpha)}, s^\alpha), \quad x_{(0,\alpha]} = (x_{(0,\alpha)}, x_\alpha).$$

Unless $\beta \leq \alpha$, $P_{e^i}[Z_j(n)=0] = 1$ for every $i \in C_\alpha, j\in C_\beta$ and $n \geq 0$. Therefore we may write $F^\alpha(n:s) = F^\alpha(n:s^{(0,\alpha]})$.

Expanding $F^\alpha(s)$ at $s = 1$, we have

$$(5.17)\qquad 1^\alpha - F^\alpha(s) = 1^\alpha - F^\alpha(s^{(0,\alpha]})$$

$$= (1^\alpha - F^\alpha(1^{(0,\alpha)}, s^\alpha)) + (F^\alpha(1^{(0,\alpha)}, s^\alpha) - F^\alpha(s^{(0,\alpha)}, s^\alpha))$$

$$= (M^\alpha_\alpha - M^\alpha_\alpha(s))(1^\alpha - s^\alpha) + \sum_{\beta < \alpha} (M^\alpha_\beta - M^\alpha_\beta(s))(1^\beta - s^\beta),$$

where

$$(5.18)\qquad M^\alpha_\beta(s) = (m^i_j(s))_{i\in C_\alpha, j\in C_\beta},$$

and

$$(5.19)\qquad \begin{cases} m^i_j(s) = m^i_j(s^\alpha) = \displaystyle\sum_{x\in Z^d_+} p^i(x)x_j\{1 - \int_0^1 (1^\alpha - (1^\alpha - s^\alpha)\xi)^{x_\alpha - (e^j)_\alpha} d\xi\}, \\ \qquad\qquad i,j \in C_\alpha, \\ m^i_j(s) = m^i_j(s^{(0,\alpha]}) \end{cases}$$

$$= \sum_{x\in Z^d_+} p^i(x)x_j\{1 - \int_0^1 (1^{(0,\alpha)} - (1^{(0,\alpha)} - s^{(0,\alpha)})\xi)^{x_{(0,\alpha)} - (e^j)_{(0,\alpha)}} d\xi \cdot (s^\alpha)^{x_\alpha}\},$$

$$i \in C_\alpha,\ j \in C_\beta,\ \alpha \neq \beta.$$

We set

$$(5.20)\qquad M(s) = (M^\alpha_\beta(s))_{1\leq\alpha,\beta\leq N}.$$

Then by (5.17),

$$(5.21)\qquad 1 - F(s) = (M - M(s))(1 - s).$$

Therefore by induction we get

$$(5.22)\qquad 1^\alpha - F^\alpha(n:s) = \prod_{k=1}^{\ell} (M^\alpha_\alpha - M^\alpha_\alpha(F(n-k:s)))\cdot(1^\alpha - F^\alpha(n-\ell:s))$$

$$+ \sum_{\beta<\alpha}\sum_{k=1}^{\ell}\prod_{m=1}^{k-1} (M^\alpha_\alpha - M^\alpha_\alpha(F(n-m:s)))\cdot(M^\alpha_\beta - M^\alpha_\beta(F(n-k:s)))(1^\beta - F^\beta(n-k:s)),$$

$$1 \leq \ell \leq n,$$

or

$$(5.23)\qquad 1^{\alpha}-F^{\alpha}(n:s) = \prod_{k=\ell}^{n-1} (M_{\alpha}^{\alpha}-M_{\alpha}^{\alpha}(F(k:s)))\cdot(1^{\alpha}-F^{\alpha}(\ell:s))$$

$$+\sum_{\beta\langle\alpha}\sum_{k=\ell}^{n-1}\prod_{m=k+1}^{n-1}(M_{\alpha}^{\alpha}-M_{\alpha}^{\alpha}(F(m:s)))\cdot(M_{\beta}^{\alpha}-M_{\beta}^{\alpha}(F(k:s)))(1^{\beta}-F^{\beta}(k:s)),$$

$$0 \leqq \ell \leqq n-1,$$

and

$$(5.24)\qquad 1 - F(n:s) = \prod_{k=0}^{n-1} (M - M(F(k:s)))\cdot(1-s).$$

Moreover we prepare an expansion formula which will be used in later sections. If C_α is a final class, then $F^{\alpha}(1^{(0,\alpha)},s^{\alpha}) = M_{\alpha}^{\alpha}s^{\alpha}$. Therefore

$$(5.25)\qquad M_{\alpha}^{\alpha}(s) = 0 \quad\text{if}\quad C_\alpha \quad\text{is final.}$$

If C_α is critical, the expansion of $m_j^i(s^{\alpha})$ at $s^{\alpha} = 1^{\alpha}$ leads us to

$$(5.26)\qquad m_j^i(s) = \sum_{k\in C_\alpha}(q_{j,k}^{\,i} - q_{j,k}^{\,i}(s))(1 - s^k),\quad i,j\in C_\alpha,$$

where

$$(5.27)\qquad q_{j,k}^{\,i} = \frac{1}{2}E_{e^i}[Z_j(1)Z_k(1)-\delta_{j,k}Z_k(1)] = \frac{1}{2}\sum_{x\in Z_+^d} p^i(x)(x_jx_k-\delta_{j,k}x_k),$$

$$i,j,k\in C_\alpha,$$

$$(5.28)\qquad q_{j,k}^{\,i}(s) = q_{j,k}^{\,i}(s^{\alpha})$$

$$= \sum_{x\in Z_+^d} p^i(x)(x_jx_k-\delta_{j,k}x_k)\{\frac{1}{2} - \int_0^1 (1^{\alpha}-(1^{\alpha}-s^{\alpha})\xi)^{x_\alpha-(e^j)_\alpha-(e^k)_\alpha}(1-\xi)d\xi\}.$$

By (5.25) and (5.26) we obtain

$$(5.29)\qquad M_{\alpha}^{\alpha}(s) = O(\sum_{i\in C_\alpha}(1-s^i)) \quad\text{if } \rho_\alpha = 1.$$

Remark. Expansion formulas (5.17) and (5.26) are slightly different from (and more convenient than) the standard ones (see, e.g., [5]).

Proof of Theorem 5.1. By Lemma 5.1 (ii), it is enough to consider the case $\lambda_n \equiv \lambda$. We denote $e_n(\lambda) = ((\exp(-n^{-\nu(\gamma,\alpha)-\nu+1}\lambda^j))_{j\in C_\gamma})_{\gamma\leqq\alpha}$. Then it follows that

$$\text{(5.30)} \qquad E_{e^i}[\exp(-\sum_{\gamma\leqq\alpha}\sum_{j\in C_\gamma} n^{-\nu(\gamma,\alpha)-\nu+1}\lambda^j Z_j(k))] = F^i(k:e_n(\lambda))$$

$$\text{for } i\in C_\beta, \beta \leqq \alpha.$$

By (5.24) we have

$$\text{(5.31)} \qquad 1^\alpha - F^\alpha([nt]:e_n(\lambda))$$

$$= \sum_{\beta\leqq\alpha} (\prod_{k=0}^{[nt]-1} (M-M(F(k:e_n(\lambda)))))^\alpha_\beta \cdot (1^\beta - e_n^\beta(\lambda))$$

$$= \sum_{\nu(\beta,\alpha)=0} + \sum_{\nu(\beta,\alpha)\geqq 1} .$$

The first part $\sum_{\nu(\beta,\alpha)=0}$ vanishes as $n\to\infty$ for

$$(\prod_{k=0}^{[nt]-1} (M-M(F(k:e_n(\lambda)))))^\alpha_\beta \leqq (M^{[nt]})^\alpha_\beta = O(\rho^{[nt]}), 0<\rho<1, \text{by (5.7)}.$$

To evaluate the limit of the second part $\sum_{\nu(\beta,\alpha)\geqq 1}$, we shall apply Lemma 4.4 to $M(F(k:e_n(\lambda)))$. Let $\beta \leqq \alpha$ with $\rho_\beta = 1$ be fixed. If $j\in C_\beta$, then by Lemma 5.3 (i) we get

(5.32) $$1 - F^j(k:e_n(\lambda))$$

$$= 1 - E_{e^j}[\exp(-\sum_{\gamma\leqq\beta}\sum_{h\in C_\gamma} n^{-\nu(\gamma,\alpha)-\nu+1}\lambda^h Z_h(k))]$$

$$\leqq cn^{-\nu(\beta,\alpha)-\nu+1}\sum_{\gamma\leqq\beta}\sum_{h\in C_\gamma}\lambda^h, \qquad 0 \leqq k \leqq [nt].$$

But since $\nu \geqq 2$ and $\nu(\beta,\alpha) \geqq 1$, we have

(5.33) $$\lim_{n\to\infty}\sum_{k=0}^{[nt]} M_\gamma^\gamma(F(k:e_n(\lambda))) = 0 \quad \text{for } \gamma \leqq \alpha \text{ with } \rho_\gamma = 1 \text{ and } t > 0.$$

For each $j \in C_\gamma, \gamma \leqq \alpha$, it follows that

(5.34) $$\lim_{n\to\infty}\max_{0\leqq k\leqq[nt]}(1-F^j(k:e_n(\lambda))) = 0.$$

In fact this follows from (5.33) if $\rho_\gamma = 1$. If $\rho_\gamma < 1$, (5.34) follows from Lemma 5.3 (ii) by the same method. Therefore

$$\lim_{n\to\infty}\max_{0\leqq k\leqq[nt]} m_j^i(F(k:e_n(\lambda))) = 0 \text{ for any } i,j \in \bigcup_{\gamma\leqq\alpha} C_\gamma.$$

We can apply Lemma 4.4 (ii) to each term of $\sum_{\nu(\gamma,\alpha)\geqq 1}$ in (5.31);

(5.35) $$\lim_{n\to\infty} n^\nu(1^\alpha - F^\alpha([nt]:e_n(\lambda)))$$

$$= \lim_{n\to\infty} n^\nu \sum_{\nu(\beta,\alpha)\geqq 1}[nt]^{\nu(\beta,\alpha)-1}(M^{*\alpha}_\beta + o(1))n^{-\nu(\beta,\alpha)-\nu+1}(\lambda^\beta + o(1))$$

$$= \sum_{\nu(\beta,\alpha)\geqq 1} M^{*\alpha}_\beta \lambda^\beta t^{\nu(\beta,\alpha)-1}.$$

We have proved (5.12).

6. Proof of Theorem 2.1

Let us start with

Lemma 6.1. Let $\rho_\alpha = 1$. If $\lambda^j > 0$ for any $j \in C_\alpha$, then

$$(6.1)\quad \lim_{n\to\infty} \frac{1-E_{e^i}[\exp(-\sum_{\beta\leq\alpha}\sum_{j\in C_\beta} n^{-\nu(\beta,\alpha)}\lambda^j Z_j([nt]))]}{\sum_{i\in C_\alpha} v_i(1-E_{e^i}[\exp(-\sum_{\beta\leq\alpha}\sum_{j\in C_\beta} n^{-\nu(\beta,\alpha)}\lambda^j Z_j([nt]))])} = u^i,$$

$$i \in C_\alpha,\quad t > 0.$$

Proof. Set

$$(6.2)\qquad e_n(\lambda) = ((\exp(-n^{-\nu(\beta,\alpha)}\lambda^j))_{j\in C_\beta})_{\beta\leq\alpha}.$$

Then (6.1) is equivalent to

$$(6.3)\qquad \lim_{n\to\infty} \frac{1^\alpha - F^\alpha([nt]:e_n(\lambda))}{v_\alpha(1^\alpha - F^\alpha([nt]:e_n(\lambda)))} = u^\alpha, \quad t > 0.$$

Let $n = [nt]$, $\ell = 0$ and $s = e_n(\lambda)$ in (5.23). Then

$$(6.4)\qquad 1^\alpha - F^\alpha([nt]:e_n(\lambda)) = \prod_{k=0}^{[nt]-1} (M_\alpha^\alpha - M_\alpha^\alpha(F(k:e_n(\lambda))))\cdot(1^\alpha - e_n^\alpha(\lambda))$$

$$+ \sum_{\beta<\alpha}\sum_{k=0}^{[nt]-1}\prod_{m=k+1}^{[nt]-1} (M_\alpha^\alpha - M_\alpha^\alpha(F(m:e_n(\lambda))))\cdot(M_\beta^\alpha - M_\beta^\alpha(F(k:e_n(\lambda))))(1^\beta - F^\beta(k:e_n(\lambda))).$$

Since $\rho_\alpha = 1$ it follows that

$$1 - F^i(k:e_n(\lambda)) \leq cn^{-1}, \quad i\in C_\alpha, 0 \leq k \leq [nt],$$

by Lemma 5.2 . (Note that $\{\gamma;\nu(\gamma,\alpha) = 0\}$ is empty.) Hence by (5.29) we get $M_\alpha^\alpha(F(k:e_n(\lambda))) = O(n^{-1})$. Therefore there exists $a > 0$ such that

$$(6.5)\qquad M_\alpha^\alpha \geq M_\alpha^\alpha - M_\alpha^\alpha(F(k:e_n(\lambda))) \geq (1 - \frac{a}{n})M_\alpha^\alpha, \quad 0 \leq k \leq [nt].$$

By (6.4) and (6.5) we have

$$1^\alpha - F^\alpha([nt]:e_n(\lambda)) \geqq (1 - \frac{a}{n})^{[nt]} (M_\alpha^\alpha)^{[nt]} (1^\alpha - e_n^\alpha(\lambda)).$$

Since $\lambda^j > 0$ for $j \in C_\alpha$, we get

(6.6) $$\lim_{n\to\infty} n^2(1 - F^i([nt]:e_n(\lambda))) = \infty, \quad i \in C_\alpha.$$

On the other hand we claim that

(6.7) $$\sup_{n\geqq 1} n^2(1 - F^j([nt]:e_n(\lambda))) < \infty, \quad j \in C_\beta, \beta \prec \alpha.$$

If $\rho_\beta = 1$, then $\nu(\beta,\alpha) \geqq 2$ and if $\rho_\beta < 1$, then $\nu(\beta,\alpha) \geqq 1$. To see (6.7) it is enough to take $\nu = 1$ in Lemma 5.3.

By (5.22) we have

(6.8) $$\begin{aligned} &1^\alpha - F^\alpha([nt]:e_n(\lambda)) \\ &= \prod_{k=1}^{\ell} (M_\alpha^\alpha - M_\alpha^\alpha(F([nt]-k:e_n(\lambda)))) \cdot (1^\alpha - F^\alpha([nt]-\ell:e_n(\lambda))) \\ &+ \sum_{\beta \prec \alpha} \sum_{k=1}^{\ell} \prod_{m=1}^{k-1} (M_\alpha^\alpha - M_\alpha^\alpha(F([nt]-m:e_n(\lambda)))) \cdot (M_\beta^\alpha - M_\beta^\alpha(F([nt]-k:e_n(\lambda)))) \\ &\qquad (1^\beta - F^\beta([nt]-k:e_n(\lambda))) \\ &= I_1(\ell,n) + I_2(\ell,n), \qquad \ell \geqq 1. \end{aligned}$$

By (6.5) and (6.6),

(6.9) $$\lim_{n\to\infty} \frac{I_2(\ell,n)}{v_\alpha I_1(\ell,n)} = 0, \quad \ell \geqq 1,$$

which implies (6.3) by Lemma 5.2 of [5]. For the completeness, let us reproduce the proof of Ogura [5]. By (6.5) we have

$$(1-\frac{a}{n})^{\ell}(M_{\alpha}^{\alpha})^{\ell}\frac{1^{\alpha}-F^{\alpha}([nt]-\ell:e_n(\lambda))}{v_{\alpha}(1^{\alpha}-F^{\alpha}([nt]-\ell:e_n(\lambda)))}$$

$$\leqq \frac{I_1(\ell,n)}{v_{\alpha}I_1(\ell,n)} \leqq (1-\frac{a}{n})^{-\ell}(M_{\alpha}^{\alpha})^{\ell}\frac{1^{\alpha}-F^{\alpha}([nt]-\ell:e_n(\lambda))}{v_{\alpha}(1^{\alpha}-F^{\alpha}([nt]-\ell:e_n(\lambda)))}.$$

Therefore by Lemma 3.1,

(6.10) $$u^i-c\rho^{\ell} \leqq \varliminf_{n\to\infty}\frac{I_1^i(\ell,n)}{v_{\alpha}I_1(\ell,n)} \leqq \varlimsup_{n\to\infty}\frac{I_1^i(\ell,n)}{v_{\alpha}I_1(\ell,n)} \leqq u^i+c\rho^{\ell},$$

$$i\in C_{\alpha}, \ell > 0, 0 < \rho < 1.$$

Since

$$\frac{1^{\alpha}-F^{\alpha}([nt]:e_n(\lambda))}{v_{\alpha}(1^{\alpha}-F^{\alpha}([nt]:e_n(\lambda)))} = \left(\frac{I_1(\ell,n)}{v_{\alpha}I_1(\ell,n)}+\frac{I_2(\ell,n)}{v_{\alpha}I_1(\ell,n)}\right)\left(1+\frac{v_{\alpha}I_2(\ell,n)}{v_{\alpha}I_1(\ell,n)}\right)^{-1},$$

we have

$$u^i - c\rho^{\ell} \leqq \varliminf_{n\to\infty}\frac{1-F^i([nt]:e_n(\lambda))}{v_{\alpha}(1^{\alpha}-F^{\alpha}([nt]:e_n(\lambda)))}$$

$$\leqq \varlimsup_{n\to\infty}\frac{1-F^i([nt]:e_n(\lambda))}{v_{\alpha}(1^{\alpha}-F^{\alpha}([nt]:e_n(\lambda)))} \leqq u^i + c\rho^{\ell}.$$

Letting $\ell \longrightarrow \infty$,we obtain (6.3).

Proof of Theorem 2.1. Let $e_n(\lambda)$ be the one in (6.2). We have to show

(6.11) $$\lim_{n\to\infty} n(1^{\alpha}-F^{\alpha}([nt]:e_n(\lambda))) = \psi_{\alpha}(t,\lambda)u^{\alpha}, \quad t > 0.$$

By Lemma 5.2 and 5.1 (i), the sequence

$$\{n(1-E_{e^i}[\exp(-\sum_{\gamma\leqq\alpha}\sum_{j\in C_{\gamma}} n^{-\nu(\gamma,\alpha)}\lambda^j z_j([nt]))])\}_{n\geqq 1}, \quad i\in C_{\alpha},$$

is uniformly bounded and equicontinuous on compact subsets of R_+^d.

Therefore it is enough to show (6.11) only for λ such that $\lambda^j > 0$ for all j. Set

(6.12) $$\psi_\alpha(\ell,n) = nv_\alpha(1^\alpha - F^\alpha(\ell:e_n(\lambda))).$$

Then since $v_\alpha M_\alpha^\alpha = v_\alpha$, we have by (5.17) and (5.26),

(6.13) $$\psi_\alpha(\ell+1,n) = nv_\alpha(1^\alpha - F^\alpha(\ell+1:e_n(\lambda)))$$

$$= nv_\alpha M_\alpha^\alpha(1^\alpha - F^\alpha(\ell:e_n(\lambda))) - nv_\alpha M_\alpha^\alpha(F(\ell:e_n(\lambda)))(1^\alpha - F^\alpha(\ell:e_n(\lambda)))$$
$$+ \sum_{\beta \prec \alpha} nv_\alpha(M_\beta^\alpha - M_\beta^\alpha(F(\ell:e_n(\lambda))))(1^\beta - F^\beta(\ell:e_n(\lambda)))$$
$$= \psi_\alpha(\ell,n)$$
$$- \sum_{i,j,k \in C_\alpha} nv_i(q_{j,k}^i - q_{j,k}^i(F(\ell:e_n(\lambda))))(1-F^j(\ell:e_n(\lambda)))\cdot(1-F^k(\ell:e_n(\lambda)$$
$$+ \sum_{\beta \prec \alpha} nv_\alpha(M_\beta^\alpha - M_\beta^\alpha(F(\ell:e_n(\lambda)))(1^\beta - F^\beta(\ell:e_n(\lambda))).$$

We shall show that there exists $c > 0$ and $0 < \rho < 1$ such that

(6.14) $$|\psi_\alpha(\ell+1,n) - \psi_\alpha(\ell,n)| \leqq c(n^{-1} + \rho^\ell), \quad 0 \leqq \ell \leqq [nt].$$

By Lemma 5.2, the second part on the last side of (6.13) is $O(n^{-1})$. Next consider the third part. If $\beta \prec \alpha$ and $\rho_\beta = 1$, then $\nu(\beta,\alpha) \geqq 2$. By Lemma 5.3 (i) we have $1^\beta - F^\beta(\ell:e_n(\lambda)) \leqq cn^{-2}\cdot 1^\beta$. If $\beta \prec \alpha$ and $\rho_\beta < 1$, then $\nu(\beta,\alpha) \geqq 1$. By Lemma 5.3 (ii) we have $1^\beta - F^\beta(\ell:e_n(\lambda))$ $\leqq c(n^{-2} + n^{-1}\rho^\ell)\cdot 1^\beta$. From these estimates, (6.14) is valid.

Set

(6.15) $$\psi_\alpha^{(n)}(t) = \begin{cases} \psi_\alpha(\ell,n) & \text{for } t = \ell n^{-1}, \\ \psi_\alpha(\ell,n) + (nt-\ell)(\psi_\alpha(\ell+1,n) - \psi_\alpha(\ell,n)) & \\ & \text{for } \ell n^{-1} \leqq t \leqq (\ell+1)n^{-1}. \end{cases}$$

Then, on compact subsets of $(0,\infty)$, $\{\psi_\alpha^{(n)}(t)\}_{n\geq 1}$ is uniformly bounded by Lemma 5.2 and equicontinuous by (6.14). Therefore we can use the compactness argument. Let $\{\psi_\alpha^{(n_j)}\}$ be any convergent subsequence and set

$$\psi_\alpha(t) = \lim_{j\to\infty} \psi_\alpha^{(n_j)}(t), \quad t > 0. \tag{6.16}$$

We shall show that $\psi_\alpha(t)$ is the solution of (2.10). This combined with Lemma 6.1 proves (2.9).

Set

$$\overline{M}_\gamma^\beta(\ell,n) = \begin{cases} M_\gamma^\beta(F(\ell:e_n(\lambda))) & \text{if } \beta \lneqq \alpha \text{ or } \gamma \lneqq \alpha, \\ 0 & \text{if } \beta = \gamma = \alpha, \end{cases} \tag{6.17}$$

$$q^\beta(\ell,n) \tag{6.18}$$

$$= \begin{cases} 0 & \text{if } \beta \lneqq \alpha, \\ \left(\sum_{j,k\in C_\alpha} (q_{j,k}^i - q_{j,k}^i(F(\ell:e_n(\lambda))))(1-F^j(\ell:e_n(\lambda)))(1-F^k(\ell:e_n(\lambda)))\right)_{i\in C_\alpha} & \text{if } \beta = \alpha, \end{cases}$$

and

$$\begin{cases} \overline{M}(\ell,n) = (\overline{M}_\gamma^\beta(\ell,n))_{\beta\leqq\alpha,\gamma\leqq\alpha}, \\ q(\ell,n) = (q^\beta(\ell,n))_{\beta\leqq\alpha}. \end{cases} \tag{6.19}$$

Let $M^{(0,\alpha]} = (M_\gamma^\beta)_{\beta\leqq\alpha,\gamma\leqq\alpha}$. Then, by (5.17) and (5.26),

$$1^{(0,\alpha]} - F^{(0,\alpha]}(\ell+1:e_n(\lambda))$$

$$= (M^{(0,\alpha]} - \overline{M}(\ell,n))(1^{(0,\alpha]} - F^{(0,\alpha]}(\ell:e_n(\lambda))) - q(\ell,n).$$

Therefore

$$(6.20)\qquad 1^{(0,\alpha]} - F^{(0,\alpha]}([nt]:e_n(\lambda))$$

$$= \prod_{\ell=0}^{[nt]-1} (M^{(0,\alpha]} - \overline{M}(\ell,n))\cdot(1^{(0,\alpha]} - e_n(\lambda))$$

$$- \sum_{\ell=0}^{[nt]-1} \prod_{m=\ell+1}^{[nt]-1} (M^{(0,\alpha]} - \overline{M}(m,n))\cdot q(\ell,n).$$

By (6.17) and (6.18) we obtain

$$(6.21)\qquad nv_\alpha(1^\alpha - F^\alpha([nt]:e_n(\lambda)))$$

$$= \sum_{\gamma\leqq\alpha} nv_\alpha(\prod_{\ell=0}^{[nt]-1} (M^{(0,\alpha]}-\overline{M}(\ell,n)))^\alpha_\gamma(1^\gamma-e_n^\gamma(\lambda)) - \sum_{\ell=0}^{[nt]-1} nv_\alpha q^\alpha(\ell,n).$$

In the same way as in the proof of Theorem 5.1,we can see that $\lim_{n\to\infty} \sum_{\ell=0}^{[nt]} \overline{M}^\gamma_\gamma(\ell,n) = 0$ for any $\gamma \leqq \alpha$ with $\rho_\gamma = 1$ and $\lim_{n\to\infty} \max_{0\leqq\ell\leqq[nt]} m^i_j(F(\ell:e_n(\lambda))) = 0$ for any $i,j \in \bigcup_{\gamma\leqq\alpha} C_\gamma$. Therefore we can apply Lemma 4.4 to each term of the first part on the right side of (6.21) to obtain;

$$(6.22)\qquad \lim_{n\to\infty} \sum_{\gamma\leqq\alpha} nv_\alpha(\prod_{\ell=0}^{[nt]-1} (M^{(0,\alpha]} - \overline{M}(\ell,n))^\alpha_\gamma(1^\gamma-e_n^\gamma(\lambda))$$

$$= \lim_{n\to\infty} \sum_{\gamma\leqq\alpha} nv_\alpha[nt]^{\nu(\gamma,\alpha)-1}(M^{*\alpha}_\gamma+o(1))n^{-\nu(\gamma,\alpha)}(\lambda^\gamma+o(1))$$

$$= \sum_{\gamma\leqq\alpha} v_\alpha M^{*\alpha}_\gamma \lambda^\gamma t^{\nu(\gamma,\alpha)-1}.$$

Since

$$\sum_{\ell=0}^{[nt]-1} nv_\alpha q^\alpha(\ell,n) = \int_0^{\frac{[nt]}{n}} \sum_{i,j,k\in C_\alpha} v_i(q^i_{j,k}-q^i_{j,k}(F([ns]:e_n(\lambda))))$$

$$\cdot n(1-F^j([ns]:e_n(\lambda)))\cdot n(1-F^k([ns]:e_n(\lambda)))ds,$$

we have by Lemma 6.1 and (6.16),

$$(6.23)\qquad \lim_{j\to\infty}\sum_{\ell=0}^{[n_j t]} n_j v_\alpha q^\alpha(\ell,n_j) = \int_0^t \sum_{i,j,k\in C_\alpha} v_i q_{j,k}^i u^j u^k \psi_\alpha(s)^2 ds$$

$$= B_\alpha \int_0^t \psi_\alpha(s)^2 ds.$$

Therefore $\psi_\alpha(t)$ is the solution of

$$(6.24)\qquad \psi_\alpha(t) = -B_\alpha \int_0^t \psi_\alpha(s)^2 ds + \sum_{\gamma\leq\alpha} v_\alpha M^{*\alpha}_\gamma \lambda^\gamma t^{\nu(\gamma,\alpha)-1}.$$

This is equivalent to (2.10) and we have completed the proof of the first part of Theorem 2.1. The second half of Theorem 2.1 is obvious from Lemma 5.1 (ii).

7. Proof of Theorem 2.2

The following lemma is well known (see [3]).

Lemma 7.1. Let C_α be a critical class. Then

$$\lim_{n\to\infty} n(1-P_{e^i}[(Z_j(n))_{j\in C_\alpha} = 0]) = (B_\alpha)^{-1}u^i, \qquad i\in C_\alpha. \tag{7.1}$$

We next give a lemma from the general theory of Laplace transform.

Lemma 7.2. (T.Watanabe) Let $\{X(n)\}$ be a sequence of random vectors taking values in R_+^d and $\{a(n)\}$,a sequence of positive numbers increasing to infinity. Assume that the following conditions are satisfied:

(a) There exists

$$\psi(\lambda) = \lim_{n\to\infty} a(n)(1 - E[\exp(-\sum_{i=1}^{d} \lambda^i X_i(n))]), \tag{7.2}$$

where $\psi(\lambda)$ is the logarithmic Laplace transform of a random vector X on R_+^d.

(b) For an integer c such that $1 \leqq c < d$ it holds that

$$\lim_{n\to\infty} a(n)(1 - P[(X_i(n))_{c+1\leqq i\leqq d} = 0]) = \psi(0,\cdots,0,\infty,\cdots,\infty), \tag{7.3}$$

where $\psi(\lambda^1,\cdots,\lambda^c,\infty,\cdots,\infty) = \lim_{\substack{\lambda^j\to\infty \\ c+1\leqq j\leqq d}} \psi(\lambda^1,\cdots,\lambda^d)$.

Then we can conclude that

$$\lim_{n\to\infty} a(n)(1 - E[\exp(-\sum_{i=1}^{c} \lambda^i X_i(n)); (X_i(n))_{c+1\leqq i\leqq d} = 0]) \tag{7.4}$$

$$= \psi(\lambda^1,\cdots,\lambda^c,\infty,\cdots,\infty),$$

(7.5) $$\lim_{n\to\infty} a(n)E[1-\exp(-\sum_{i=1}^{c}\lambda^i X_i(n);(X_i(n))_{c+1\leqq i\leqq d}=0]$$

$$= \psi(\lambda^1,\cdots,\lambda^c,\infty,\cdots,\infty) - \psi(0,\cdots,0,\infty,\cdots,\infty).$$

Proof. First we remark the following fact;if $\lim_{n\to\infty} X(n) = X$ in distribution and $\lim_{n\to\infty} P[X(n)\in A] = P[X\in A]$ for a closed set A,then for any bounded continuous function $f(x)$ on R_+^d we have $\lim_{n\to\infty} E[f(X(n));X(n)\in A] = E[f(X);X\in A]$.

Let $\{X^{(k)}(n)\}_{1\leqq k\leqq [a(n)]}$ be the independent copies of $X(n)$ and set $Y(n) = \sum_{k=1}^{[a(n)]} X^{(k)}(n)$. Then by assumption (b),

$$\psi(0,\cdots,0,\infty,\cdots,\infty) = -\lim_{n\to\infty} \log P[(X_i(n))_{c+1\leqq i\leqq d}=0]^{[a(n)]}$$

$$= -\lim_{n\to\infty} \log P[(Y_i(n))_{c+1\leqq i\leqq d}=0].$$

By assumption (a),we have $\lim_{n\to\infty} Y(n) = X$ in distribution. Therefore by the first remark we have

$$\psi(\lambda^1,\cdots,\lambda^c,\infty,\cdots,\infty)$$

$$= -\lim_{n\to\infty} \log E[\exp(-\sum_{i=1}^{c}\lambda^i Y_i(n));(Y_i(n))_{c+1\leqq i\leqq d}=0]$$

$$= -\lim_{n\to\infty} \log E[\exp(-\sum_{i=1}^{c}\lambda^i X_i(n));(X_i(n))_{c+1\leqq i\leqq d}=0]^{[a(n)]}$$

$$= \lim_{n\to\infty} a(n)(1-E[\exp(-\sum_{i=1}^{c}\lambda^i X_i(n));(X_i(n))_{c+1\leqq i\leqq d}=0]),$$

which proves (7.4). (7.5) follows from (7.3) and (7.4).

Proof of Theorem 2.2. Let $i \in C_\alpha$ and $t > 0$. Then by Theorem 2.1,

$$(7.6)\qquad \lim_{n\to\infty} n(1-E_{e^i}[\exp(-\sum_{\beta \leqq \alpha}\sum_{j\in C_\beta} n^{-\nu(\beta,\alpha)}\lambda^j Z_j([nt]))]) = \psi_\alpha(t,\lambda)u^i.$$

It follows from Lemma 7.1 that

$$(7.7)\qquad \lim_{n\to\infty} n(1-P_{e^i}[(Z_j([nt]))_{j\in C_\alpha}=0]) = (tB_\alpha)^{-1}u^i.$$

But $\psi_\alpha(t,0^{(0,\alpha)},\lambda^\alpha)$ is the solution of

$$\frac{d}{dt}\psi_\alpha(t,0^{(0,\alpha)},\lambda^\alpha) = -B_\alpha\psi_\alpha(t,0^{(0,\alpha)},\lambda^\alpha)^2, \psi_\alpha(0,0^{(0,\alpha)},\lambda^\alpha) = v_\alpha\lambda^\alpha.$$

Hence

$$(7.8)\qquad \psi_\alpha(t,0^{(0,\alpha)},\infty^\alpha) = \lim_{\substack{\lambda^j\to\infty \\ j\in C_\alpha}} \psi_\alpha(t,0^{(0,\alpha)},\lambda^\alpha)$$

$$= \lim_{\substack{\lambda^j\to\infty \\ j\in C_\alpha}} \frac{v_\alpha\lambda^\alpha}{1+tB_\alpha v_\alpha\lambda^\alpha} = (tB_\alpha)^{-1}.$$

Therefore by Lemma 7.2,

$$(7.9)\qquad \lim_{n\to\infty} n(1-E_{e^i}[\exp(-\sum_{\beta<\alpha}\sum_{j\in C_\beta} n^{-\nu(\beta,\alpha)}\lambda^j Z_j([nt]));(Z_j([nt])_{j\in C_\alpha}=0])$$

$$= \psi_\alpha(t,\lambda^{(0,\alpha)},\infty^\alpha).$$

This proves (2.11) and (2.13). (2.12) and (2.16) follows from Lemma 7.2, (7.8) and (7.9). (2.15) is obvious from (2.9) and (2.12).

Next we shall prove (2.14). By (2.16) we have

$$(7.10)\qquad t\psi_\alpha^\infty(t,\lambda) = (B_\alpha)^{-1} + t\eta_\alpha(t,\lambda).$$

By (2.10) we get

$$\frac{d}{dt}(t^2\psi_\alpha(t,\lambda)) = -B_\alpha(t\psi_\alpha(t,\lambda))^2 + 2t\psi_\alpha(t,\lambda) + \sum_{\nu(\beta,\alpha)\geqq 2}\ \sum_{j\in C_\beta} a_{\alpha,j}\lambda^j t^{\nu(\beta,\alpha)},$$

i.e.,

$$t\cdot t\psi_\alpha(t,\lambda) = -B_\alpha\int_0^t (s\psi_\alpha(s,\lambda))^2 ds + 2\int_0^t s\psi_\alpha(s,\lambda)ds + \int_0^t \sum_{\nu(\beta,\alpha)\geqq 2}\ \sum_{j\in C_\beta} a_{\alpha,j}\lambda^j s^{\nu(\beta,\alpha)} ds.$$

Therefore

$$t\cdot t\psi^\infty_\alpha(t,\lambda) = -B_\alpha\int_0^t (s\psi^\infty_\alpha(s,\lambda))^2 ds + 2\int_0^t s\psi^\infty_\alpha(s,\lambda)ds + \int_0^t \sum_{\nu(\beta,\alpha)\geqq 2}\ \sum_{j\in C_\beta} a_{\alpha,j}\lambda^j s^{\nu(\beta,\alpha)} ds. \tag{7.11}$$

Substituting (7.10) in (7.11) and differentiating with respect to t, we obtain (2.14).

The proof of (2.17) is also easy. First equality is obvious from (2.16). By (2.10) and (2.14) we have

$$\frac{d}{dt}\{t(\psi_\alpha(t,\lambda)-\eta_\alpha(t,\lambda))\} = (\psi_\alpha(t,\lambda)+\eta_\alpha(t,\lambda))\{1-tB_\alpha(\psi_\alpha(t,\lambda)-\eta_\alpha(t,\lambda))\}.$$

Second equality follows from this.

Finally the last statement of Theorem 2.2 is a consequence of Lemma 5.1, for $\lim_{n\to\infty} P_{e^i}[(Z_j([nt]))_{j\in C_\alpha} = 0] = 1$ by (7.1) and the left hand side of (2.12) equals

$$\lim_{n\to\infty} nE_{e^i}[1-\exp(-\sum_{\beta<\alpha}\ \sum_{j\in C_\beta} n^{-\nu(\beta,\alpha)}\lambda^j Z_j([nt])\,|\,(Z_j([nt]))_{j\in C_\alpha}=0].$$

8. Proof of Theorem 2.3

We first prepare an expansion formula on generating function at the final class.

Let C_N be a final class. Then the generating functions $F^i(s), i \in C_N$, have the form

$$F^i(s) = \sum_{j \in C_N} k_j^i(s^{(0,N)}) s^j, \quad i \in C_N. \tag{8.1}$$

The function $k_j^i(s^{(0,N)})$ can be written as

$$k_j^i(s^{(0,N)}) = \sum p_j^i(x_{(0,N)})(s^{(0,N)})^{x_{(0,N)}}, \quad i,j \in C_N, \tag{8.2}$$

where the summation is taken over all $x_{(0,N)} = ((x_k)_{k \in C_\alpha})_{\alpha < N}$, such that $x_k \in Z_+$. It is easy to see that

$$\begin{aligned} p_j^i(x_{(0,N)}) &= p^i(x_{(0,N)}, (e^j)_N) \\ &= P_{e^i}[Z_{(0,N)}(1) = x_{(0,N)}, Z_N(1) = (e^j)_N]. \end{aligned} \tag{8.3}$$

If we define

$$K_N^N(s^{(0,N)}) = (k_j^i(s^{(0,N)}))_{i,j \in C_N}, \tag{8.4}$$

then we have

$$F^N(s) = K_N^N(s^{(0,N)}) s^N. \tag{8.5}$$

Noting that

$$K_N^N(1^{(0,N)}) = M_N^N, \tag{8.6}$$

we expand $k_j^i(s^{(0,N)})$ at $1^{(0,N)}$ as follows, (8.7) $$m_j^i - k_j^i(s^{(0,N)}) = \sum_{\alpha \prec N} \sum_{k \in C_\alpha} \ell_{j,k}^i(s^{(0,N)})(1-s^k), \quad i,j \in C_N,$$

where

(8.8) $$\ell_{j,k}^i(s^{(0,N)})$$

$$= \sum_{x_{(0,N)}} p_j^i(x_{(0,N)})x_k \int_0^1 (1^{(0,N)}-(1^{(0,N)}-s^{(0,N)})\xi)^{x_{(0,N)}-(e^k)_{(0,N)}} d\xi.$$

By (8.3),

(8.9) $$\sum_{j \in C_N} \ell_{j,k}^i(1^{(0,N)}) = m_k^i.$$

Setting

(8.10) $$L_{N,\alpha}^N(s^{(0,N)};w^\alpha) = (\sum_{k \in C_\alpha} \ell_{j,k}^i(s^{(0,N)})w^k)_{i,j \in C_N}, \quad \alpha \prec N,$$

we have

(8.11) $$F^N(s) = (M_N^N - \sum_{\alpha \prec N} L_{N,\alpha}^N(s^{(0,N)};1^\alpha-s^\alpha))s^N.$$

By induction,

(8.12) $$F^N(n:s) = \prod_{k=0}^{n-1} (M_N^N - \sum_{\alpha \prec N} L_{N,\alpha}^N(F^{(0,N)}(k:s);1^\alpha-F^\alpha(k:s)))\cdot s^N.$$

By (8.9),

(8.13) $$L_{N,\alpha}^N(1^{(0,N)};w^\alpha)1^N = (\sum_{j \in C_N} \sum_{k \in C_\alpha} \ell_{j,k}^i(1^{(0,N)})w^k)_{i \in C_N}$$

$$= (\sum_{k \in C_\alpha} m_k^i w^k)_{i \in C_N} = M_\alpha^N w^\alpha.$$

Set

(8.14) $$e_n(\lambda) = ((\exp(-n^{-\nu(\beta)+1}\lambda_j))_{j \in C_\beta})_{1 \leq \beta \leq N}.$$

Then by (8.12) we have

$$(8.15)\qquad (E_{e^i}[\exp(-\sum_{\alpha=1}^{N}\sum_{j\in C_\alpha} n^{-\nu(\alpha)+1}\lambda^j Z_j([nt]))])_{i\in C_N}$$

$$= F^N([nt]:e_n(\lambda))$$

$$= \prod_{k=0}^{[nt]-1} (M_N^N - \sum_{\alpha\langle N} L_{N,\alpha}^N (F^{(0,N)}(k:e_n(\lambda));1^\alpha - F^\alpha(k:e_n(\lambda))))\cdot e_n^N(\lambda).$$

Therefore to prove Theorem 2.3,it will be necessary to estimate

$$(8.16)\qquad 1^\alpha - F^\alpha(k:e_n(\lambda)) = (1-E_{e^i}[\exp(-\sum_{\gamma\leqq\alpha}\sum_{j\in C_\gamma} n^{-\nu(\gamma)+1}\lambda^j Z_j(k))])_{i\in C_\alpha},$$

for every $\alpha \langle N$.

In the following lemma,<u>we do not assume</u> that C_N is a final class.

Lemma 8.1. Suppose that $i\in C_\alpha$.

(i) If $\rho_\alpha = 1$,then

$$(8.17)\qquad \lim_{n\to\infty} n(1-E_{e^i}[\exp(-\sum_{\gamma\leqq\alpha}\sum_{j\in C_\gamma} n^{-\nu(\gamma)+\nu(\alpha)-1}\lambda^j Z_j([nt]))])$$

$$= \overline{\psi}_\alpha(t,\lambda)u^i, \quad t > 0,$$

where $\overline{\psi}_\alpha$ is the solution of (2.24).

(ii) If $\rho_\alpha < 1$,then

$$(8.18)\qquad \lim_{n\to\infty} n(1-E_{e^i}[\exp(-\sum_{\gamma\leqq\alpha}\sum_{j\in C_\gamma} n^{-\nu(\gamma)+\nu(\alpha)}\lambda^j Z_j([nt]))])$$

$$= \sum_{\substack{\nu(\beta)=\nu(\alpha)+1\\ \rho_\beta=1}} \sum_{j\in C_\beta} m^{*i}_{j} u^j \overline{\psi}_\beta(t,\lambda), \quad t > 0.$$

Proof. First we shall show (i). Choose any convergent sequence $\{\lambda_n\}$ in R_+^d. By Theorem 2.1 and Lemma 5.1,we have

$$(8.19)\qquad \lim_{n\to\infty} n(1-E_{e^i}[\exp(-\sum_{\gamma\leq\alpha}\sum_{j\in C_\gamma} n^{-\nu(\gamma,\alpha)}\lambda_n^j Z_j([nt]))])$$

$$= \psi_\alpha(t,\lim_{n\to\infty}\lambda_n)u^i.$$

Take $\lambda_n^j = n^{\nu(\gamma,\alpha)-\nu(\gamma)+\nu(\alpha)-1}\lambda^j, j\in C_\gamma, \gamma\leq\alpha$. Then by (5.10) we have

$$\lim_{n\to\infty}\lambda_n^j = \begin{cases}\lambda^j & \text{if } \nu(\gamma,\alpha)+\nu(\alpha) = \nu(\gamma)+1,\\ 0 & \text{if } \nu(\gamma,\alpha)+\nu(\alpha) < \nu(\gamma)+1.\end{cases}$$

From (2.10), $\psi_\alpha(t,\lim_{n\to\infty}\lambda_n) = \overline{\psi}_\alpha(t,\lambda)$.

Next we shall show (ii) by induction. The induction hypothesis is this; for any $\beta < \alpha$ with $\rho_\beta < 1$,

$$(8.20)\qquad \lim_{n\to\infty} n(1-E_{e^i}[\exp(-\sum_{\delta\leq\beta}\sum_{j\in C_\delta} n^{-\nu(\delta)+\nu(\beta)}\lambda^j Z_j([nt]))])$$

$$= \sum_{\substack{\nu(\gamma)=\nu(\beta)+1\\ \rho_\gamma=1}}\sum_{j\in C_\gamma} m^*{}_j^i u^j\overline{\psi}_\gamma(t,\lambda), \qquad t>0,\ i\in C_\beta.$$

We first prove the following relation; if $\beta < \alpha$ and $i\in C_\beta$,

$$(8.21)\qquad \lim_{n\to\infty} n(1-E_{e^i}[\exp(-\sum_{\delta\leq\beta}\sum_{j\in C_\delta} n^{-\nu(\delta)+\nu(\alpha)}\lambda^j Z_j([nt]))])$$

$$= \begin{cases}\overline{\psi}_\beta(t,\lambda)u^i & \text{if } \rho_\beta = 1 \text{ and } \nu(\beta) = \nu(\alpha)+1,\\ \sum_{\substack{\nu(\gamma)=\nu(\beta)+1\\ \rho_\gamma=1}}\sum_{j\in C_\gamma} m^*{}_j^i u^j\overline{\psi}_\gamma(t,\lambda) & \text{if } \nu(\beta)=\nu(\alpha),\\ 0 & \text{otherwise.}\end{cases}$$

The first two cases on the right hand side of (8.21) are immediate from (8.17) and the induction hypothesis (8.20). In the other cases

$(\rho_\beta = 1, \nu(\beta) \geqq \nu(\alpha) + 2$ or $\rho_\beta < 1, \nu(\beta) \geqq \nu(\alpha) + 1)$ it follows from (5.10) and (5.11) that

(8.22) $$\nu(\gamma,\beta) + 1 - \nu(\gamma) + \nu(\alpha) \leqq 0.$$

By substituting $\beta, \nu = 2$ and $n^{\nu(\gamma,\beta)+1-\nu(\gamma)+\nu(\alpha)}\lambda$ for α, ν and λ in Lemma 5.2, we have

$$1 - E_{e^i}[\exp(-\sum_{\delta \leqq \beta} \sum_{j \in C_\delta} n^{-\nu(\delta)+\nu(\alpha)}\lambda^j Z_j([nt]))] = O(n^{-2}),$$

which proves the last part on the right side of (8.21).

Set $e_n(\lambda) = ((\exp(-n^{-\nu(\gamma)+\nu(\alpha)}\lambda^j))_{j \in C_\gamma})_{\gamma \leqq \alpha}$. For every $i \in C_\beta$, $\beta \lneqq \alpha$, we have

(8.23) $$E_{e^i}[\exp(-\sum_{\gamma \leqq \beta} \sum_{j \in C_\gamma} n^{-\nu(\gamma)+\nu(\alpha)}\lambda^j Z_j(k)] = F^i(k:e_n(\lambda)).$$

Substituting $s = e_n(\lambda), n = [nt]$ and $\ell = 0$ in (5.23), we have

(8.24) $$1^\alpha - F^\alpha([nt]:e_n(\lambda)) = \prod_{k=0}^{[nt]-1} (M_\alpha^\alpha - M_\alpha^\alpha(F(k:e_n(\lambda)))) \cdot (1^\alpha - e_n^\alpha(\lambda))$$
$$+ \sum_{\beta \lneqq \alpha} \sum_{k=0}^{[nt]-1} \prod_{m=k+1}^{[nt]-1} (M_\alpha^\alpha - M_\alpha^\alpha(F(m:e_n(\lambda)))) \cdot (M_\beta^\alpha - M_\beta^\alpha(F(k:e_n(\lambda)))) (1^\beta - F^\beta(k:e_n(\lambda))).$$

By using (8.21) we can complete the proof of (8.18) as follows;

(8.25) $$\lim_{n\to\infty} n(1^\alpha - F^\alpha([nt]:e_n(\lambda)))$$
$$= \sum_{\substack{\beta \lneqq \alpha \\ \nu(\beta)=\nu(\alpha)}} (I-M_\alpha^\alpha)^{-1} M_\beta^\alpha \sum_{\substack{\nu(\gamma)=\nu(\beta)+1 \\ \rho_\gamma=1}} \bar{\Psi}_\gamma(t,\lambda) M^{*}{}_\gamma^\beta u^\gamma$$
$$+ \sum_{\substack{\nu(\beta)=\nu(\alpha)+1 \\ \rho_\beta=1}} \bar{\Psi}_\beta(t,\lambda)(I-M_\alpha^\alpha)^{-1} M_\beta^\alpha u^\beta$$

$$= \sum_{\substack{\nu(\beta)=\nu(\alpha)+1 \\ \rho_\beta=1}} \overline{\Psi}_\beta(t,\lambda)\Big(\sum_{\substack{\beta \langle \gamma \langle \alpha \\ \nu(\beta,\gamma)=\nu(\beta,\alpha)=1}} (I-M^\alpha_\alpha)^{-1} M^\alpha_\gamma M{*}^\gamma_\beta$$

$$+ (I-M^\alpha_\alpha)^{-1} M^\alpha_\beta P^\beta_\beta \Big) u^\beta$$

$$= \sum_{\substack{\nu(\beta)=\nu(\alpha)+1 \\ \rho_\beta=1}} \overline{\Psi}_\beta(t,\lambda) M{*}^\alpha_\beta u^\beta,$$

where the last equality is due to (3.13).

To complete the induction argument we have to show that (8.18) is valid for each minimal element α in $\{\alpha;\rho_\alpha < 1\}$. It is easy to see that, if α is minimal in the whole set $\{1,2,\cdots,N\}$, both sides of (8.18) are zero. Unless α is minimal in the whole set, the argument in the preceding paragraph is still valid. (In this case, there is no β such that $\beta \langle \alpha$ with $\rho_\beta < 1$, so that there does not occur the second case in the right hand side of (8.21).)

Proof of Theorem 2.3. First note that $u^N = 1^N$, since M^N_N is a probability matrix. Let $e_n(\lambda)$ be the vector defined in (8.14). Set

$$(8.26) \qquad \overline{\lambda}^j = \begin{cases} \lambda^j & \text{if } j \in C_\alpha, \nu(\alpha) \geqq 2, \\ 0 & \text{if } j \in C_\alpha, \nu(\alpha) = 1. \end{cases}$$

We shall apply Lemma 4.3 to the right side of (8.15). To this end let us define

$$(8.27) \quad m^N_\alpha(k,n) = \begin{cases} v_N L^N_{N,\alpha}(F^{(0,N)}(k:e_n(\lambda)); 1^\alpha - F^\alpha(k:e_n(\lambda)))1^N & \text{if } \nu(\alpha) \geqq 2, \\ v_N L^N_{N,\alpha}(F^{(0,N)}(k:e_n(\lambda)); 1^\alpha - F^\alpha(k:e_n(\overline{\lambda})))1^N & \text{if } \nu(\alpha)=1. \end{cases}$$

We investigate the asymptotic behavior of these values. Recalling (8.16) and substituting $\nu = 0, \alpha = N$ and $\beta = \alpha$ in Lemma 5.3, we get

$$(8.28)\qquad 1^{\alpha}-F^{\alpha}(k:e_n(\lambda)) \leqq c(n^{-2}+n^{-1}\rho^{k}), \quad 0 \leqq k \leqq [nt],$$
$$\text{for } \nu(\alpha) \geqq 3 \text{ or } \nu(\alpha) = 2, \rho_{\alpha} < 1.$$

Then by the definition (8.10) of $L^{N}_{N.\alpha}$, it follows that

$$(8.29)\qquad \lim_{n\to\infty} \sum_{k=0}^{[nt]-1} m^{N}_{\alpha}(k,n) = 0 \quad \text{if } \nu(\alpha) \geqq 3 \text{ or } \nu(\alpha) = 2, \rho_{\alpha} < 1.$$

By Lemma 5.3 (i) we have

$$(8.30)\qquad 1^{\alpha}-F^{\alpha}(k:e_n(\lambda)) \leqq cn^{-1}, \quad 0 \leqq k \leqq [nt], \quad \text{for } \nu(\alpha) = 2, \rho_{\alpha} = 1.$$

Therefore by Lemma 8.1 (i),

$$(8.31)\qquad \lim_{n\to\infty} \sum_{k=0}^{[nt]-1} m^{N}_{\alpha}(k,n) = \int_0^t v_N L^{N}_{N,\alpha}(1^{(0,N)};\overline{\psi}_{\alpha}(s,\lambda)u^{\alpha})1^{N}ds$$
$$= v_N M^{N}_{\alpha} u^{\alpha}\int_0^t \overline{\psi}_{\alpha}(s,\lambda)ds, \quad \text{for } \nu(\alpha)=2, \rho_{\alpha}=1,$$

where the last equality follows from (8.13). By (8.28) and (8.30) it follows that

$$(8.32)\qquad \lim_{n\to\infty} \max_{0\leqq k\leqq [nt]} m^{N}_{\alpha}(k,n) = 0 \quad \text{for } \nu(\alpha) \geqq 2.$$

A slightly careful examination of the proof of Lemma 5.3 tells us that

$$(8.33)\qquad 1^{\alpha}-F^{\alpha}(k:e_n(\overline{\lambda})) \leqq cn^{-1}, \quad 0\leqq k\leqq [nt], \quad \text{for } \nu(\alpha)=1, \rho_{\alpha} < 1.$$

(Note that (8.33) is not valid for $1^{\alpha}-F^{\alpha}(k:e_n(\lambda))$.) By Lemma 8.1 (ii) we have

(8.34) $$\lim_{n\to\infty} n(1^{\alpha}-F^{\alpha}([nt]:e_n(\bar{\lambda}))) = \lim_{n\to\infty} n(1^{\alpha}-F^{\alpha}([nt]:e_n(\lambda)))$$

$$= \sum_{\substack{\nu(\beta)=\nu(\alpha)+1 \\ \rho_\beta=1}} \bar{\psi}_\beta(t,\lambda)M^*{}^{\alpha}_{\beta}u^{\beta} \qquad \text{for } \nu(\alpha)=1, \rho_\alpha < 1.$$

Therefore

(8.35) $$\lim_{n\to\infty} \sum_{k=0}^{[nt]-1} m^N_\alpha(k,n)$$

$$= \int_0^t v_N L^N_{N,\alpha}(1^{(0,N)}; \sum_{\substack{\nu(\beta)=\nu(\alpha)+1 \\ \rho_\beta=1}} M^*{}^{\alpha}_{\beta}u^{\beta}\bar{\psi}_\beta(s,\lambda))1^N ds$$

$$= \sum_{\substack{\nu(\beta)=\nu(\alpha)+1 \\ \rho_\beta=1}} v_N M^N_\alpha M^*{}^{\alpha}_{\beta}u^{\beta}\int_0^t \bar{\psi}_\beta(s,\lambda)ds$$

$$\text{for } \nu(\alpha)=1, \rho_\alpha < 1.$$

Set

(8.36) $$\begin{cases} M^{N,(1)}_N(k,n) = \sum_{\nu(\alpha)\geqq 2} L^N_{N,\alpha}(F^{(0,N)}(k:e_n(\lambda)); 1^{\alpha}-F^{\alpha}(k:e_n(\lambda))) \\ \qquad + \sum_{\substack{\alpha \prec N \\ \nu(\alpha)=1}} L^N_{N,\alpha}(F^{(0,N)}(k:e_n(\lambda)); 1^{\alpha}-F^{\alpha}(k:e_n(\bar{\lambda}))), \\ M^{N,(2)}_N(k,n) = \sum_{\substack{\alpha \prec N \\ \nu(\alpha)=1}} L^N_{N,\alpha}(F^{(0,N)}(k:e_n(\lambda)); F^{\alpha}(k:e_n(\bar{\lambda}))-F^{\alpha}(k:e_n(\lambda))). \end{cases}$$

By those results obtained previously we have

(8.37) $$\lim_{n\to\infty} \max_{0\leqq k\leqq [nt]-1} v_N M^{N,(1)}_N(k,n)1^N = 0,$$

and

(8.38) $$\lim_{n\to\infty} \sum_{k=0}^{[nt]-1} v_N M^{N,(1)}_N(k,n)1^N$$

$$= \sum_{\substack{\nu(\alpha)=2\\ \rho_\alpha=1}} v_N M_\alpha^N u^\alpha \int_0^t \overline{\psi}_\alpha(s,\lambda)ds$$

$$+ \sum_{\substack{\alpha \lneqq N\\ \nu(\alpha)=1}} \sum_{\substack{\nu(\beta)=\nu(\alpha)+1\\ \rho_\beta=1}} v_N M_\alpha^N M^{*\alpha}_{\beta} u^\beta \int_0^t \overline{\psi}_\beta(s,\lambda)ds$$

$$= \sum_{\substack{\nu(\alpha)=2\\ \rho_\alpha=1}} v_N M^{*N}_{\alpha} u^\alpha \int_0^t \overline{\psi}_\alpha(s,\lambda)ds,$$

where the last equality follows from (3.12).

We next discuss $M_N^{N,(2)}(k,n)$. Let $\alpha \lneqq N, \nu(\alpha)=1$ and $i \in C_\alpha$. It then follows that

(8.39) $$F^i(k:e_n(\overline{\lambda})) - F^i(k:e_n(\lambda))$$

$$= E_{e^i}[\exp(-\sum_{\substack{\beta \leqq \alpha\\ \nu(\beta)\geqq 2}} \sum_{j\in C_\beta} n^{-\nu(\beta)+1}\lambda^j Z_j(k)) - \exp(-\sum_{\beta \leqq \alpha} \sum_{j\in C_\beta} n^{-\nu(\beta)+1}\lambda^j Z_j(k))]$$

$$\leqq E_{e^i}[1 - \exp(-\sum_{\substack{\beta \leqq \alpha\\ \nu(\beta)=1}} \sum_{j\in C_\beta} \lambda^j Z_j(k))]$$

$$\leqq c\rho^k,$$

(8.40) $$\lim_{n\to\infty} (F^i(k:e_n(\overline{\lambda})) - F^i(k:e_n(\lambda)))$$

$$= 1 - E_{e^i}[\exp(-\sum_{\substack{\beta \leqq \alpha\\ \nu(\beta)=1}} \sum_{j\in C_\beta} \lambda^j Z_j(k))] = 1 - F^i(k:e(\lambda)),$$

where $e(\lambda) = \lim_{n\to\infty} e_n(\lambda)$, i.e.,

$$(8.41) \qquad e^{i}(\lambda) = \begin{cases} 1 & \text{for } i \in C_\beta, \nu(\beta) \geqq 2, \\ \exp(-\lambda^{i}) & \text{for } i \in C_\beta, \nu(\beta) = 1. \end{cases}$$

Set $M_N^{N,(2)}(k) = \sum_{\substack{\alpha \prec N \\ \nu(\alpha)=1}} L^{N}_{N,\alpha}(F^{(0,N)}(k:e(\lambda)); 1^\alpha - F^\alpha(k:e(\lambda)))$. It is obvious from (8.39) and (8.40) that

$$(8.42) \qquad M_N^{N,(2)}(k,n) \leqq M_N^{N,(2)}(k) \quad \text{and} \quad \lim_{n\to\infty} M_N^{N,(2)}(k,n) = M_N^{N,(2)}(k).$$

Moreover, each component of $M_N^{N,(2)}(k)$ is bounded by $c\rho^{k}$ by (8.39).

Hence

$$(8.43) \qquad \sum_{k=0}^{\infty} M_N^{N,(2)}(k) < \infty.$$

On the other hand

$$(8.44) \qquad M_N^{N,(2)}(k) = \sum_{\alpha \prec N} L^{N}_{N,\alpha}(F^{(0,N)}(k:e(\lambda)); 1^\alpha - F^\alpha(k:e(\lambda)))$$

$$= M_N^N - K_N^N(F^{(0,N)}(k:e(\lambda))),$$

where the first equality follows from $F^\alpha(k:e(\lambda)) = 1^\alpha$ whenever $\nu(\alpha) \geqq 2$ and the second equality follows from (8.7).

By (8.38), (8.39), (8.43) and (8.44) we can apply Lemma 4.3 to (8.15) to obtain

$$(8.45) \qquad \lim_{n\to\infty} F^N([nt]:e_n(\lambda))$$

$$= \exp\Big(-\sum_{\substack{\nu(\alpha)=2 \\ \rho_\alpha=1}} v_N M^N_\alpha u^\alpha \int_0^t \overline{\psi}_\alpha(s,\lambda)\,ds\Big) \cdot \lim_{n\to\infty} \prod_{k=0}^{n} K_N^N(F^{(0,N)}(k:e(\lambda))) \cdot e^N(\lambda)$$

$$= \prod_{\substack{\nu(\alpha)=1 \\ \rho_\alpha=1}} G_{1,\alpha}(t,\lambda) \cdot G_2^N(\lambda),$$

where

(8.46) $$G_2^N(\lambda) = \lim_{n\to\infty} \prod_{k=0}^{n} K_N^N(F^{(0,N)}(k:e(\lambda)))\cdot e^N(\lambda).$$

Since M_N^N is a probability matrix, the component of $G_2^N(\lambda)$ is identical. Let $G_2(\lambda)$ be the common component function of $G_2^N(\lambda)$. The formula (8.45) proves (2.17) to (2.23). By (2.17) and (2.19) we have

(8.47) $$\lim_{n\to\infty} E_{e^i}[\exp(-\sum_{\nu(\beta)=1} \sum_{j\in C_\beta} \lambda^j z_j(n))] = G_2(\lambda), \quad i\in C_N,$$

so that $G_2(\lambda) = G_2((\lambda^j)_{j\in D(2)})$ is the Laplace transform of a probability measure on $Z_+^{d_2}$. Infinite divisibility of $G_{1,\alpha}(t,\lambda)$ is a consequence of Theorem 2.1. Thus we have completed the proof of Theorem 2.3.

9. Proof of Theorem 2.4 and 2.5

We first prove Theorem 2.4. For each $i \in C_N$ we have

$$(9.1)\qquad E_{e^i}[\exp(-\sum_{\alpha=1}^{N}\sum_{j\in C_\alpha} n^{-\nu(\alpha)}\lambda^j Z_j(n)) \mid (Z_j(n))_{j\in C_N} \neq 0]$$

$$= (P_{e^i}[(Z_j(n))_{j\in C_N} \neq 0])^{-1}\{(1-E_{e^i}[\exp(-\sum_{\alpha=1}^{N}\sum_{j\in C_\alpha} n^{-\nu(\alpha)}\lambda^j Z_j(n)); (Z_j(n))_{j\in C_N}=0]) - (1-E_{e^i}[\exp(-\sum_{\alpha=1}^{N}\sum_{j\in C_\alpha} n^{-\nu(\alpha)}\lambda^j Z_j(n))])\}.$$

Since C_N is critical, $n\times$ the denominator in (9.1) converges to $(B_N)^{-1}u^i$ by Lemma 7.1 and $n\times$ the numerator converges to $(\psi_N^\infty(1,\lambda)-\psi_N(1,\lambda))u^i$ by Theorem 2.1 and 2.2. Infinite divisibility of $H(\lambda)$ is obvious from (2.16).

We next prove Theorem 2.5. As in (9.1) we have

$$(9.2)\qquad E_{e^i}[\exp(-\sum_{\alpha=1}^{N}\sum_{j\in C_\alpha} n^{-\nu(\alpha)}\lambda^j Z_j(n)) \mid ((Z_j(n))_{j\in C_\alpha})_{\alpha\in A_1} \neq 0]$$

$$= (P_{e^i}[((Z_j(n))_{j\in C_\alpha})_{\alpha\in A_1} \neq 0])^{-1} \{(1-E_{e^i}[\exp(-\sum_{\alpha\in A_1}\sum_{j\in C_\alpha} n^{-\nu(\alpha)}\lambda^j Z_j(n)); ((Z_j(n))_{j\in C_\alpha})_{\alpha\in A_1}=0]) - (1-E_{e^i}[\exp(-\sum_{\alpha=1}^{N}\sum_{j\in C_\alpha} n^{-\nu(\alpha)}\lambda^j Z_j(n))])\}.$$

We have already proved that

$$(9.3)\qquad \lim_{n\to\infty} n(1-E_{e^i}[\exp(-\sum_{\alpha=1}^{N}\sum_{j\in C_\alpha} n^{-\nu(\alpha)}\lambda^j Z_j(n))])$$

$$= \sum_{\substack{\nu(\beta)=1\\ \rho_\beta=1}}\sum_{j\in C_\beta} m^{*i}_{j} u^j \overline{\psi}_\beta(1,\lambda),$$

in Lemma 8.1 (ii) and that

(9.4) $$\lim_{\substack{\lambda^j\to\infty \\ j\in C_\beta}} \overline{\psi}_\beta(1,0^{(0,\beta)},\lambda^\beta) = \lim_{\substack{\lambda^j\to\infty \\ j\in C_\beta}} \psi_\beta(1,0^{(0,\beta)},\lambda^\beta) = (B_\beta)^{-1}$$

for $\rho_\beta = 1, \nu(\beta) = 1,$

in (7.8). Therefore, if we prove

(9.5) $$\lim_{n\to\infty} n(1-P_{e^i}[((Z_j(n))_{j\in C_\alpha})_{\alpha\in A_1}=0]) = \sum_{\substack{\nu(\beta)=1 \\ \rho_\beta=1}} \sum_{j\in C_\beta} m^{*i}_{j} u^j (B_\beta)^{-1},$$

we can use Lemma 7.2 to obtain

(9.6) $$\lim_{n\to\infty} n(1-E_{e^i}[\exp(-\sum_{\alpha\in A_1} \sum_{j\in C_\alpha} n^{-\nu(\alpha)}\lambda^j Z_j(n)); ((Z_j(n))_{j\in C_\alpha})_{\alpha\in A_1}=0])$$

$$= \sum_{\substack{\nu(\beta)=1 \\ \rho_\beta=1}} \sum_{j\in C_\beta} m^{*i}_{j} u^j \overline{\psi}^{\infty}_\beta(1,\lambda),$$

which, together with (9.3) and (9.5), completes the proof of Theorem 2.5.

Lemma 9.1. The relation (9.5) is valid.

The proof is quite similar to the proof of Lemma 8.1 (ii) and even simpler. We make the induction hypothesis as follows; for any $\beta \prec N$ such that $\rho_\beta < 1, \beta \in A_1$,

(9.7) $$\lim_{n\to\infty} n(1 - P_{e^i}[((Z_j(n))_{j\in C_\beta})_{\delta \preceq \beta, \delta\in A_1}=0])$$

$$= \sum_{\substack{\nu(\gamma)=\nu(\beta)+1 \\ \rho_\gamma=1}} \sum_{j\in C_\gamma} m^{*i}_{j} u^j (B_\gamma)^{-1}, \qquad i\in C_\beta.$$

Then the following relation is valid; if $\beta \prec N$ and $i\in C_\beta$, (9.8) $$\lim_{n\to\infty} n(1 - P_{e^i}[((Z_j(n))_{j\in C_\delta})_{\delta \leqq \beta, \delta\in A_1}=0])$$

$$= \begin{cases} (B_\beta)^{-1} & \text{if } \rho_\beta = 1, \beta\in A_1, \\ \sum_{\substack{\nu(\gamma)=\nu(\beta)+1 \\ \rho_\gamma=1}} \sum_{j\in C_\gamma} m^{*i}_j u^j (B_\gamma)^{-1} & \text{if } \rho_\beta < 1, \beta\in A_1, \\ 0 & \text{if } \beta \notin A_1. \end{cases}$$

If $\rho_\beta = 1$ and $\beta\in A_1$, there is no δ such that $\delta \lneqq \beta, \delta\in A_1$. Hence the first case on the right hand side of (9.8) is valid by Lemma 7.1. The second equality is (9.7) itself. The third case is also obvious, since there is no $\delta \leqq \beta, \delta\in A_1$ if $\beta\notin A_1$. Let $e(\lambda) = (\exp(-\lambda^i))_{1\leqq i\leqq d}$ and define λ_0 by

(9.9) $$\lambda_0^j = \begin{cases} \infty & \text{if } j\in C_\beta, \beta\in A_1, \\ 0 & \text{if } j\in C_\beta, \beta\notin A_1. \end{cases}$$

Then for any $\beta \leqq N$ we have

(9.10) $$P_{e^i}[((Z_j(n))_{j\in C_\delta})_{\delta\leqq\beta, \delta\in A_1}=0] = F^i(n:e(\lambda_0)) \qquad \text{for } i\in C_\beta.$$

The rest of the proof is the same as in Lemma 8.1 (ii), so it is omitted.

Remark. To complete the induction in the above proof we need the fact that

(9.11) $$\lim_{n\to\infty} n(1-P_{e^i}[(Z_j(n))_{j\in C_\alpha}=0]) = 0, \ i\in C_\alpha, \text{ if } \rho_\alpha < 1,$$

which is immediate from Lemma 3.1.

10. More on conditioning

In this section we shall extend Theorem 2.4 and 2.5 in the present paper and Theorem 5.1 in [5] by the method of Ogura [5].

For each $k \geqq 1$, let

(10.1) $$\nu_k(\alpha) = \max\{\nu(\alpha)-k+1, 1\},$$

(10.2) $$A_k = \{\alpha; \nu(\alpha) \leqq k-1 \text{ or } \nu(\alpha) = k, \rho_\alpha = 1\}.$$

Theorem 10.1. Let $1 \leqq k \leqq \max_{1\leqq\alpha\leqq N} \nu(\alpha)$. Assume that the set A_k has no final class. $c^i_\beta, i \in C_\alpha$, are the constants defined by (2.8).

(i) Let C_N be a critical class. Then for each $i \in C_N$ there exists

(10.3) $$\lim_{n\to\infty} E_{e^i}[\exp(-\sum_{\alpha=1}^{N} \sum_{j\in C_\alpha} n^{-\nu_k(\alpha)} \lambda^j Z_j(n)) | ((Z_j(n))_{j\in C_\alpha})_{\alpha\in A_k} \neq 0]$$

$$= I(\lambda);$$

the limit is independent of i. $I(\lambda)$ is the Laplace transform of a probability measure on R^d_+ and is represented as follows;

(10.4) $$I(\lambda) = (b^k_N)^{-1}(\psi^{k,\infty}_N(\lambda) - \psi^k_N(\lambda)).$$

Here $b^k_\alpha, \psi^k_\alpha$ and $\psi^{k,\infty}_\alpha$ for $\nu(\alpha) \leqq k$ with $\rho_\alpha = 1$ are determined by the same recurrence formula with respect to the partial order $\langle$ as follows; if $\nu(\alpha) = k$ and $\rho_\alpha = 1$, then

(10.5) $$b^k_\alpha = (B_\alpha)^{-1}, \quad \psi^k_\alpha(\lambda) = \overline{\psi}_\alpha(1,\lambda) \quad \text{and} \quad \psi^{k,\infty}_\alpha(\lambda) = \overline{\psi}^\infty_\alpha(1,\lambda),$$

and if $\nu(\alpha) < k$ and $\rho_\alpha = 1$, then

(10.6) $$\begin{cases} b^k_\alpha = R_\alpha(b^k_\beta \,; \beta \langle \alpha), \quad \psi^k_\alpha(\lambda) = R_\alpha(\psi^k_\beta(\lambda); \beta \langle \alpha) \\ \qquad \text{and} \\ \psi^{k,\infty}_\alpha(\lambda) = R_\alpha(\psi^{k,\infty}_\beta(\lambda); \beta \langle \alpha), \end{cases}$$

where $R_\alpha(x_\beta;\beta\langle\,\alpha)$ is defined by

$$(10.7)\qquad R_\alpha(x_\beta;\beta\langle\,\alpha) = ((B_\alpha)^{-1} \sum_{\substack{\nu(\beta)=\nu(\alpha)+1\\ \rho_\beta=1}} \sum_{i\in C_\alpha} v_i c_\beta^i x_\beta)^{\frac{1}{2}}.$$

(ii) Let C_N be a subcritical class. Then, for each $i\in C_N$, there exists

$$(10.8)\qquad \lim_{n\to\infty} E_{e^i}[\exp(-\sum_{\alpha=1}^{N} \sum_{j\in C_\alpha} n^{-\nu_k(\alpha)} \lambda^j Z_j(n)) \,|\, ((Z_j(n))_{j\in C_\alpha})_{\alpha\in A_k} \neq 0]$$

$$= I_i(\lambda).$$

$I_i(\lambda)$ is the Laplace transform of a probability measure on R_+^d and is represented as follows;

$$(10.9)\qquad I_i(\lambda) = (\sum_{\substack{\nu(\alpha)=1\\ \rho_\alpha=1}} c_\alpha^i b_\alpha^k)^{-1} \sum_{\substack{\nu(\alpha)=1\\ \rho_\alpha=1}} c_\alpha^i (\psi_\alpha^{k,\infty}(\lambda)-\psi_\alpha^k(\lambda)).$$

Note that $I(\lambda)$ and $I_i(\lambda)$ depend only on $(\lambda^i; i\in C_\alpha, \nu(\alpha)\geq k)$. The case of $k = 1$ is nothing but Theorem 2.4 and 2.5. The case of $k = \max_{1\leq\alpha\leq N} \nu(\alpha)$ was shown in Theorem 5.1 of [5], although Ogura [5] uses more complicated recurrence formulas than ours. The simplification of recurrence formulas is due to the recurrence formula (3.13) for m^{*i}_j (see the proof of Lemma 10.1).

Actually we can prove a little more than Theorem 10.1. Let $\alpha\in A_k$ and $i\in C_\alpha$. As in section 9 we have

$$(10.10)\qquad E_{e^i}[\exp(-\sum_{\gamma\leqq\alpha}\sum_{j\in C_\gamma} n^{-\nu_k(\gamma)}\lambda^j Z_j(n))\,|\,((Z_j(n))_{j\in C_\gamma})_{\substack{\gamma\leqq\alpha\\ \gamma\in A_k}}\neq 0]$$

$$= (P_{e^i}[((Z_j(n))_{j\in C_\gamma})_{\substack{\gamma\leqq\alpha\\ \gamma\in A_k}}\neq 0])^{-1}$$

$$\{(1-E_{e^i}[\exp(-\sum_{\substack{\gamma\leqq\alpha\\ \gamma\notin A_k}}\sum_{j\in C_\gamma} n^{-\nu_k(\gamma)}\lambda^j Z_j(n));((Z_j(n))_{j\in C_\gamma})_{\substack{\gamma\leqq\alpha\\ \gamma\in A_k}}=0])$$

$$-(1-E_{e^i}[\exp(-\sum_{\gamma\leqq\alpha}\sum_{j\in C_\gamma} n^{-\nu_k(\gamma)}\lambda^j Z_j(n))])\}.$$

The following three lemmas enable us to find the limit of (10.10). Taking $\alpha = N$ we obtain (10.3) and (10.8). The first two lemmas are essentially due to Ogura [5]. Set

$$(10.11)\qquad \mu_k(\alpha) = \begin{cases} 2^{\nu(\alpha)-k} & \text{if } \rho_\alpha = 1, \alpha\in A_k, \\ 2^{\nu(\alpha)-k+1} & \text{if } \rho_\alpha < 1, \alpha\in A_k. \end{cases}$$

Note that $\mu_k(\alpha) \leqq 1$.

Lemma 10.1. (i) If C_α is a critical class and $\alpha\in A_k$, then

$$(10.12)\qquad \lim_{n\to\infty} n^{\mu_k(\alpha)}(1-E_{e^i}[\exp(-\sum_{\gamma\leqq\alpha}\sum_{j\in C_\gamma} n^{-\nu_k(\gamma)}\lambda^j Z_j(n))])$$

$$= \psi_\alpha^k(\lambda)u^i, \qquad i\in C_\alpha.$$

(ii) If C_α is a subcritical class and $\alpha\in A_k$, then

$$(10.13)\qquad \lim_{n\to\infty} n^{\mu_k(\alpha)}(1-E_{e^i}[\exp(-\sum_{\gamma\leqq\alpha}\sum_{j\in C_\gamma} n^{-\nu_k(\gamma)}\lambda^j Z_j(n))])$$

$$= \sum_{\substack{\nu(\beta)=\nu(\alpha)+1\\ \rho_\beta=1}} \sum_{j\in C_\beta} m^{*i}_{j} u^j \psi^k_\beta(\lambda), \qquad i\in C_\alpha.$$

Lemma 10.2. (i) If C_α is a critical class and $\alpha\in A_k$, then

$$(10.14)\qquad \lim_{n\to\infty} n^{\mu_k(\alpha)}(1-P_{e^i}[((Z_j(n))_{j\in C_\gamma})_{\substack{\gamma\leqq\alpha\\ \gamma\in A_k}}=0]) = b^k_\alpha u^i, \qquad i\in C_\alpha.$$

(ii) If C_α is a subcritical class and $\alpha\in A_k$, then

$$(10.15)\qquad \lim_{n\to\infty} n^{\mu_k(\alpha)}(1-P_{e^i}[((Z_j(n))_{j\in C_\gamma})_{\substack{\gamma\leqq\alpha\\ \gamma\in A_k}}=0])$$

$$= \sum_{\substack{\nu(\beta)=\nu(\alpha)+1\\ \rho_\beta=1}} \sum_{j\in C_\beta} m^{*i}_{j} u^j b^k_\beta, \qquad i\in C_\alpha.$$

Lemma 10.3. (i) If C_α is a critical class and $\alpha\in A_k$, then

$$(10.16)\qquad \lim_{n\to\infty} n^{\mu_k(\alpha)}(1-E_{e^i}[\exp(-\sum_{\substack{\gamma\leqq\alpha\\ \gamma\notin A_k}}\sum_{j\in C_\gamma} n^{-\nu_k(\gamma)}\lambda^j Z_j(n));$$

$$((Z_j(n))_{j\in C_\gamma})_{\substack{\gamma\leqq\alpha\\ \gamma\in A_k}}=0]) = \psi^{k,\infty}_\alpha(\lambda)u^i, \qquad i\in C_\alpha.$$

(ii) If C_α is a subcritical class and $\alpha\in A_k$, then

(10.17) $$\lim_{n\to\infty} n^{\mu_k(\alpha)}\Bigl(1-E_{e^i}[\exp(-\sum_{\substack{\gamma\leq\alpha\\ \gamma\notin A_k}}\sum_{j\in C_\gamma} n^{-\nu_k(\gamma)}\lambda^j Z_j(n));$$

$$((Z_j(n))_{j\in C_\gamma})_{\substack{\gamma\leq\alpha\\ \gamma\in A_k}}=0]\Bigr)$$

$$=\sum_{\substack{\nu(\beta)=\nu(\alpha)+1\\ \rho_\beta=1}}\sum_{j\in C_\beta} m^{*i}_{\ j}u^j\psi_\beta^{k,\infty}(\lambda),\qquad i\in C_\alpha.$$

Lemma 10.3 is a consequence of Lemma 10.1 and 10.2,if Lemma 7.2 is applicable to this case. Since $\psi_\alpha^k(\lambda)$ depends only on $(\lambda^\beta)_{\nu(\beta)\geq k}$, write $\psi_\alpha^k(\lambda) = \psi_\alpha^k(\lambda^\gamma$ for $\nu(\gamma) > k, \lambda^\beta$ for $\nu(\beta)=k)$. It is enough to show that

(10.18) $$\lim_{\substack{\lambda^j\to\infty\\ j\in C_\beta,\nu(\beta)=k}} \psi_\alpha^k(0^\gamma \text{ for } \nu(\gamma) > k, \lambda^\beta \text{ for } \nu(\beta)=k) = b_\alpha^k.$$

This is true by (7.8) if $\nu(\alpha) = k$ and by the recurrence formula (10.6) if $\nu(\alpha) < k$.

The proof of Lemma 10.1 is the same as that of Ogura [5] except the proof of the following relation;

(10.19) $$\lim_{n\to\infty}(1-E_{e^i}[\exp(-\sum_{\gamma\leq\alpha}\sum_{j\in C_\gamma} n^{-\nu_k(\gamma)}\lambda^j Z_j(n))]) = 0,$$

$$i\in C_\alpha, \alpha\in A_k, \rho_\alpha = 1,$$

which we need in the course of the proof. Ogura [5] used his Lemma 4.7 to show (10.19). But the lemma does not apply to our case,for we allow the existence of final classes in $\{1,2,\cdots,N\} - A_k$. Although our substantial task is to prove (10.19) and our recurrence formula (10.6), we will outline the whole proof.

Proof of Lemma 10.1. Since the right side in (10.12) and (10.13) are continuous in $\lambda \in R_+^d$ we have only to consider the case when $\lambda^j > 0, 1 \leqq j \leqq d$. The relation (10.12) for $\nu(\alpha) = k$ and (10.13) for $\nu(\alpha) = k-1$ have been already shown in Lemma 8.1. For general $\alpha \in A_k$, we use induction with respect to the partial order $\prec$.

First we shall show (10.13). Let α with $\rho_\alpha < 1$ and $\nu(\alpha) = \ell (\leqq k-2)$ be fixed. Assume that (10.12) is valid for any $\alpha' \prec \alpha$ such that $\rho_{\alpha'} = 1, \alpha' \in A_k$, and (10.13) is valid for any $\alpha' \prec \alpha$ such that $\rho_{\alpha'} < 1$, $\alpha' \in A_k$. It is easy to see that, for any $\beta \prec \alpha$ and $i \in C_\beta$,

$$(10.20) \qquad \lim_{n\to\infty} n^{\mu_k(\alpha)} (1-E_{e^i}[\exp(-\sum_{\gamma \preceq \beta} \sum_{j \in C_\gamma} n^{-\nu_k(\gamma)} \lambda^j z_j(n))])$$

$$= \begin{cases} \psi_\beta^k(\lambda) u^i & \text{if } \rho_\beta = 1 \text{ and } \nu(\beta) = \nu(\alpha) + 1, \\ \sum\limits_{\substack{\nu(\gamma)=\nu(\beta)+1 \\ \rho_\gamma = 1}} \sum\limits_{j \in C_\gamma} m^{*i}_{\ j} u^j \psi_\gamma^k(\lambda) & \text{if } \nu(\beta) = \nu(\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$(10.21) \qquad e_n(\lambda) = ((\exp(-n^{-\nu_k(\gamma)} \lambda^j))_{j \in C_\gamma})_{\gamma \preceq \alpha}.$$

Then, for any $i \in C_\beta, \beta \preceq \alpha$, we have

$$(10.22)\qquad E_{e^i}[\exp(-\sum_{\gamma\underline{\leq}\beta}\sum_{j\in C_\gamma} n^{-\nu_k(\gamma)}\lambda^j Z_j(k))] = F^i(k:e_n(\lambda)).$$

Then we can show (10.13) by the same method as in the proof of Lemma 8.1 (ii), so the proof is omitted.

It remains to show (10.12). Let α with $\rho_\alpha = 1$ and $\nu(\alpha) = \ell(\underline{\leq}k-1)$ be fixed. Assume that (10.12) is valid for any $\alpha' \langle \alpha$ such that $\rho_{\alpha'} = 1$, $\alpha' \in A_k$ and (10.13) is valid for any $\alpha' \langle \alpha$ such that $\rho_{\alpha'} < 1, \alpha' \in A_k$. It then follows that, for any $\beta \langle \alpha$ and $i \in C_\beta$,

$$(10.23)\qquad \lim_{n\to\infty} n^{2\mu_k(\alpha)}(1-E_{e^i}[\exp(-\sum_{\delta\underline{\leq}\beta}\sum_{j\in C_\delta} n^{-\nu_k(\delta)}\lambda^j Z_j(n))])$$

$$= \begin{cases} \psi^k_\beta(\lambda)u^i & \text{if } \rho_\beta = 1 \text{ and } \nu(\beta) = \nu(\alpha)+1, \\ \displaystyle\sum_{\substack{\nu(\gamma)=\nu(\beta)+1 \\ \rho_\gamma = 1}} \sum_{j\in C_\gamma} m^{*i}_{j} u^j \psi^k_\gamma(\lambda) & \text{if } \nu(\beta) = \nu(\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

Set $s = F(n:e_n(\lambda))$ in (5.17). Since $v_\alpha M^\alpha_\alpha = v_\alpha$ we have by (5.26),

$$(10.24)\qquad v_\alpha(1^\alpha - F^\alpha(n+1:e_n(\lambda))) - v_\alpha(1^\alpha - F^\alpha(n:e_n(\lambda)))$$

$$= -\sum_{i,j,k\in C_\alpha} v_i(q^{\ i}_{j,k} - q^{\ i}_{j,k}(F(n:e_n(\lambda))))(1-F^j(n:e_n(\lambda)))(1-F^k(n:e_n(\lambda)))$$

$$+ \sum_{\beta\langle\alpha} v_\alpha(M^\alpha_\beta - M^\alpha_\beta(F(n:e_n(\lambda))))(1^\beta - F^\beta(n:e_n(\lambda))).$$

If we set

$$(10.25)\quad \begin{cases} a_n = v_\alpha(1^\alpha - F^\alpha(n:e_n(\lambda))), \\ b_n = (a_n)^{-2} \sum_{i,j,k\in C_\alpha} v_i(q^i_{j,k} - q^i_{j,k}(F(n:e_n(\lambda)))) \\ \qquad (1-F^j(n:e_n(\lambda)))(1-F^k(n:e_n(\lambda))), \\ c_n = \sum_{\beta<\alpha} v_\alpha(M^\alpha_\beta - M^\alpha_\beta(F(n:e_n(\lambda))))(1^\beta - F^\beta(n:e_n(\lambda))) \\ \qquad + v_\alpha(F^\alpha(n+1:e_n(\lambda)) - F^\alpha(n+1:e_{n+1}(\lambda))), \end{cases}$$

then by (10.24) we get

$$(10.26)\qquad a_{n+1} - a_n = -b_n(a_n)^2 + c_n.$$

Let $\mu = 2^{\nu(\alpha)-k+1} \leqq 1$. Then $\frac{\mu}{2} = \mu_k(\alpha)$ by (10.11). We shall prove the following relations;

$$(10.27)\qquad \lim_{n\to\infty} a_n = 0,$$

$$(10.28)\qquad \lim_{n\to\infty} n^\mu c_n = B_\alpha \psi^k_\alpha(\lambda)^2 > 0,$$

$$(10.29)\qquad \lim_{n\to\infty} b_n = B_\alpha > 0,$$

$$(10.30)\qquad \lim_{n\to\infty} (a_n)^{-1}(1^\alpha - F^\alpha(n:e_n(\lambda))) = u^\alpha.$$

By (10.27),(10.28) and (10.29) we can apply Lemma 4.6 of [5] to obtain

$$(10.31)\qquad \lim_{n\to\infty} n^{\frac{\mu}{2}} a_n = \{B_\alpha^{-1}\cdot B_\alpha \psi^k_\alpha(\lambda)^2\}^{\frac{\mu}{2}} = \psi^k_\alpha(\lambda).$$

This,combined with (10.30),proves (10.12).

It remains to prove relations (10.27) to (10.30). First of all, by (5.31) of [5], there exists some $c > 0$ such that

(10.32) $$F^i(n+1:e_{n+1}(\lambda)) - F^i(n+1:e_n(\lambda)) \leqq cn^{-1}(1-F^i(n+1:e_{n+1}(\lambda)),$$

$$i \in C_\alpha.$$

We shall prove (10.27) (which is equivalent to (10.19)). Set $a = \overline{\lim_{n\to\infty}} a_n$. Then we can choose a subsequence $\{n_j\}$ so that

(10.33) $$\lim_{j\to\infty} a_{n_j} = a,$$

and there exists

(10.34) $$\xi^\alpha_{(k)} = \lim_{j\to\infty} (1^\alpha - F^\alpha(n_j-k:e_{n_j-k}(\lambda))),$$

for every $k \geqq 0$. Substituting $s = F(n_j-1:e_{n_j-1}(\lambda))$ in (5.17), we have

(10.35) $$1^\alpha - F^\alpha(n_j:e_{n_j-1}(\lambda))$$

$$= (M^\alpha_\alpha - M^\alpha_\alpha(F(n_j-1:e_{n_j-1}(\lambda))))(1^\alpha - F^\alpha(n_j-1:e_{n_j-1}(\lambda)))$$

$$+ \sum_{\beta<\alpha} (M^\alpha_\beta - M^\alpha_\beta(F(n_j-1:e_{n_j-1}(\lambda))))(1^\beta - F^\beta(n_j-1:e_{n_j-1}(\lambda))).$$

By (10.23) and (10.33) we get

(10.36) $$\xi^\alpha_{(0)} = (M^\alpha_\alpha - M^\alpha_\alpha(1^\alpha - \xi^\alpha_{(1)}))\xi^\alpha_{(1)},$$

(10.37) $$a = v_\alpha \xi^\alpha_{(0)} = v_\alpha \xi^\alpha_{(1)} - v_\alpha M^\alpha_\alpha(1^\alpha - \xi^\alpha_{(1)})\xi^\alpha_{(1)}$$

$$\leqq a - v_\alpha M^\alpha_\alpha(1^\alpha - \xi^\alpha_{(1)})\xi^\alpha_{(1)}.$$

Therefore $v_\alpha M_\alpha^\alpha(1^\alpha-\xi^\alpha_{(1)})\xi^\alpha_{(1)} = 0$ and hence

(10.38) $$M_\alpha^\alpha(1^\alpha-\xi^\alpha_{(1)})\xi^\alpha_{(1)} = 0.$$

Hence

(10.39) $$\xi^\alpha_{(0)} = M_\alpha^\alpha\xi^\alpha_{(1)}, \quad a = v_\alpha\xi^\alpha_{(1)}.$$

Similarly we have

(10.40) $$\xi^\alpha_{(1)} = M_\alpha^\alpha\xi^\alpha_{(2)}, \quad a = v_\alpha\xi^\alpha_{(2)}.$$

Repeating this argument we obtain

(10.41) $$\xi^\alpha_{(1)} = (M_\alpha^\alpha)^k\xi^\alpha_{(k+1)}, \quad a = v_\alpha\xi^\alpha_{(k+1)}, \qquad k \geqq 0.$$

Then, by (3.3), it follows that

(10.42) $$\xi^\alpha_{(1)} = au^\alpha.$$

If $a > 0$, then $\xi^\alpha_{(1)} > 0$. But since C_α is a critical class we have $m^i_j(1^\alpha-\xi^\alpha_{(1)}) > 0$ for some $i,j \in C_\alpha$. This contradicts to (10.38).

Next we shall show (10.28). By (10.27) and (10.32), $\lim_{n\to\infty} nv_\alpha(F^\alpha(n+1:e_n(\lambda))-F^\alpha(n+1:e_{n+1}(\lambda))) = 0$. Therefore by (10.23),

(10.43) $$\lim_{n\to\infty} n^\mu c_n$$

$$= \sum_{\substack{\beta \prec \alpha \\ \nu(\beta)=\nu(\alpha)}} v_\alpha M^\alpha_\beta \sum_{\substack{\nu(\gamma)=\nu(\beta)+1 \\ \rho_\gamma=1}} \psi^k_\gamma(\lambda)M^{*\beta}_\gamma u^\gamma$$

$$+ \sum_{\substack{\nu(\beta)=\nu(\alpha)+1 \\ \rho_\beta=1}} \psi^k_\beta(\lambda)v_\alpha M^\alpha_\beta u^\beta$$

$$= \sum_{\substack{\nu(\beta)=\nu(\alpha)+1 \\ \rho_\beta=1}} \psi_\beta^k(\lambda) v_\alpha \Big(\sum_{\substack{\beta \lessdot \gamma \lessdot \alpha \\ \nu(\beta,\gamma)=\nu(\beta,\alpha)=1}} \rho_\alpha^\alpha M_\gamma^\alpha M^{*\gamma}_\beta + \rho_\alpha^\alpha M_\beta^\alpha P^\beta \Big) u^\beta$$

$$= \sum_{\substack{\nu(\beta)=\nu(\alpha)+1 \\ \rho_\beta=1}} \psi_\beta^k(\lambda) v_\alpha M^{*\alpha}_\beta u^\beta = B_\alpha \psi_\alpha^k(\lambda)^2,$$

where the third equality follows from (3.12) and the last equality is due to the definition (10.6) and (10.7) for $\psi_\alpha^k(\lambda)$. Hence we have proved (10.28).

By (10.26),(10.27) and (10.28) we have $\varliminf_{n\to\infty} n^\mu a_n = \varliminf_{n\to\infty} n^\mu a_{n+1}$

$= \varliminf_{n\to\infty} \{n^\mu a_n(1-a_n b_n) + n^\mu c_n\} = \varliminf_{n\to\infty} n^\mu a_n + B_\alpha \psi_\alpha^k(\lambda)^2$. This means that

$\varliminf_{n\to\infty} n^\mu a_n = \infty$. Therefore by (10.23) we see that

$$\lim_{n\to\infty} \frac{1^\beta - F^\beta(n:e_n(\lambda))}{v_\alpha(1^\alpha - F^\alpha(n:e_n(\lambda)))} = 0 \text{ for any } \beta \lessdot \alpha.$$ Then, as in the latter half of the proof of Lemma 6.1, we obtain (10.30). Therefore $\lim_{n\to\infty} b_n = B_\alpha > 0$

by definition of b_n.

For the proof of Lemma 10.2 we remark that as in Lemma 9.1, for any $\beta \lneq \alpha$,

$$P_{e^i}[((Z_j(n))_{j\in C_\delta})_{\delta\leq\beta,\delta\in A_k}=0] = F^i(n:e(\lambda_0)), \quad i\in C_\delta, \tag{10.44}$$

where $e(\lambda) = (\exp(-\lambda^i))_{1\leq i\leq d}$ and λ_0 is defined by

$$\lambda_0^i = \begin{cases} \infty & \text{if } i\in C_\gamma, \gamma\in A_k, \\ 0 & \text{if } i\in C_\gamma, \gamma\notin A_k. \end{cases} \tag{10.45}$$

Then the proof of Lemma 10.2 is the same as that of Lemma 10.1.

11. Examples

In the first three examples, the mean matrix is assumed to have the form

$$(11.1)\qquad M = \begin{pmatrix} 1 & 0 & 0 & & & \\ 1 & 1 & 0 & & \bigcirc & \\ 0 & 1 & 1 & & & \\ & & \ddots & \ddots & \ddots & \\ & \bigcirc & & 1 & 1 & 0 \\ & & & 0 & 1 & 1 \end{pmatrix}.$$

Then it is easy to see that

$$(11.2)\qquad C_i = \{i\}, \quad \rho_i = 1,$$

$$(11.3)\qquad 1 \langle 2 \langle \cdots \langle d,$$

$$(11.4)\qquad \nu(j,i) = \begin{cases} i-j+1 & \text{if } i \geqq j, \\ -1 & \text{if } i < j, \end{cases}$$

and

$$(11.5)\qquad m^{*i}_j = \lim_{n\to\infty} n^{-\nu(j,i)+1} m^i_j,(n) = \begin{cases} \frac{1}{(i-j)!} & \text{if } i \geqq j, \\ 0 & \text{if } i < j. \end{cases}$$

Let $f(s)$ be a one-dimensional generating function such that

$$(11.6)\qquad f'(1) = 1, f''(1) = 2B > 0.$$

Recall that s,λ denotes the d-dimensional vectors: $s = (s^i)_{1 \leqq i \leqq d}$, $0 \leqq s^i \leqq 1$ and $\lambda = (\lambda^i)_{1\leqq i\leqq d}$, $\lambda^i \geqq 0$.

EXAMPLE 1. Let $F^1(s) = f(s^1), F^i(s) = s^{i-1}f(s^i), 2 \leqq i \leqq d-1$ and $F^d(s) = s^{d-1}s^d$. Then C_i $(i \leqq d-1)$ is a critical class and C_d is a final class. We have $a_{i,j} = \dfrac{1}{(i-j-1)!}$ $,b_{i,j} = \dfrac{1}{(i-j)!}$. By Theorem 2.1, we get

$$(11.7) \qquad \psi_i(t,\lambda) = \lim_{n\to\infty} n(1-E_{e^i}[\exp(-\sum_{j=1}^{i} n^{-i+j-1}\lambda^j Z_j([nt]))]),$$

and $\psi_i(t,\lambda)$ is the solution of

$$(11.8) \qquad \begin{cases} \dfrac{d}{dt}\psi_i(t,\lambda) = -B\psi_i(t,\lambda)^2 + \displaystyle\sum_{j=1}^{i-1} \frac{\lambda^j}{(i-j-1)!} t^{i-j-1}, \\ \psi_i(0,\lambda) = \lambda^i, \end{cases}$$

for $i \leqq d-1$.

It is easy to see that $\overline{\psi}_i = \psi_i$. Then by Theorem 2.3 we obtain

$$(11.9) \qquad \lim_{n\to\infty} E_{e^d}[\exp(-\sum_{j=1}^{d} n^{j-d}\lambda^j Z_j(n))] = \exp(-\int_0^1 \psi_{d-1}(t,\lambda)dt)e^{-\lambda^d}.$$

This is already given in [2] with the different formulation.

In case it is necessary to distinguish ψ_i for various d's, we shall write $\psi_{d,i}$, $i = 1,2,\cdots$.

If $d = 2$ or 3, we can solve (11.8) explicitly for $i = d-1$; if $d = 2$,

$$(11.10) \qquad \psi_{2,1}(t,\lambda) = \frac{\lambda^1}{1+tB\lambda^1},$$

$$(11.11) \qquad \lim_{n\to\infty} E_{e^2}[\exp(-n^{-1}\lambda^1 Z_1(n)-\lambda^2 Z_2(n))] = (1+B\lambda^1)^{-\frac{1}{B}}e^{-\lambda^2},$$

and if $d = 3$,

$$(11.12) \qquad \psi_{3,2}(t,\lambda) = \sqrt{\frac{\lambda^1}{B}}\,\frac{\sqrt{B}\lambda^2\cosh(\sqrt{B\lambda^1}t)+\sqrt{\lambda^1}\sinh(\sqrt{B\lambda^1}t)}{\sqrt{B}\lambda^2\sinh(\sqrt{B\lambda^1}t)+\sqrt{\lambda^1}\cosh(\sqrt{B\lambda^1}t)},$$

$$\text{(11.13)} \qquad \lim_{n\to\infty} E_{e^3}[\exp(-\sum_{j=1}^{3} n^{j-3}\lambda^j Z_j(n))]$$

$$= (\cosh(\sqrt{B\lambda^1}) + \lambda^2\sqrt{\frac{B}{\lambda^1}}\sinh(\sqrt{B\lambda^1}))^{-\frac{1}{B}} e^{-\lambda^3}.$$

Let $a_i \geqq 0$, $1 \leqq i \leqq d-1$ and $\lambda \geqq 0$. By substituting $\lambda^j = (d-j-2)!a_j\lambda$, $(j \leqq d-2), \lambda^{d-1} = a_{d-1}\lambda, \lambda^d = 0$ in (11.8) for $i = d-1$ and (11.9), we have

$$\text{(11.14)} \qquad \lim_{n\to\infty} E_{e^d}[\exp(-\lambda\sum_{j=1}^{d-2} n^{j-d}(d-j-2)!a_j Z_j(n) - \lambda n^{-1}a_{d-1}Z_{d-1}(n))]$$

$$= \exp(-\int_0^1 \tilde{\psi}_{d-1}(t,\lambda)dt),$$

where $\tilde{\psi}_{d-1}(t,\lambda)$ is the solution of

$$\text{(11.15)} \qquad \begin{cases} \dfrac{d}{dt}\tilde{\psi}_{d-1}(t,\lambda) = -B\tilde{\psi}_{d-1}(t,\lambda)^2 + \displaystyle\sum_{j=1}^{d-2} a_j t^{d-j-2}\lambda, \\ \tilde{\psi}_{d-1}(0,\lambda) = a_{d-1}\lambda. \end{cases}$$

If $a_i = 0, 1 \leqq i \leqq d-2$, then (11.14) is the Laplace transform of a gamma distribution. If $d \geqq 3$ and $a_i > 0$ for some $1 \leqq i \leqq d-2$, then (11.14) is the Laplace transform of an infinitely divisible distribution with a smooth density (see [10]). A class of ordinary differential equations closely related to the infinitely divisible distributions is given in [11].

EXAMPLE 2. Let $F^1(s) = f(s^1), F^i(s) = s^{i-1}f(s^i), 2 \leqq i \leqq d-2$, and $F^i(s) = s^{i-1}s^i$, $i = d-1, d$. In this case, C_i $(i \leqq d-2)$ is critical and C_i $(i = d-1, d)$ is final. By Theorem 2.1 we have

(11.16) $$\psi_{d-1}(t,\lambda) = \lim_{n\to\infty} n(1-E_{e^{d-1}}[\exp(-\sum_{j=1}^{d-1} n^{j-d}\lambda^j Z_j([nt]))])$$

where ψ_{d-1} is the solution of

$$\frac{d}{dt}\psi_{d-1}(t,\lambda) = \sum_{j=1}^{d-2} \frac{\lambda^j}{(d-j-2)!} t^{d-j-2}, \quad \psi_{d-1}(0,\lambda) = \lambda^{d-1},$$

i.e.,

$$\psi_{d-1}(t,\lambda) = \sum_{j=1}^{d-1} \frac{\lambda^j}{(d-j-1)!} t^{d-j-1}.$$

Hence by Theorem 2.3,

(11.17) $$\lim_{n\to\infty} E_{e^d}[\exp(-\sum_{j=1}^{d} n^{j-d}\lambda^j Z_j(n))] = \exp(-\sum_{j=1}^{d} \frac{\lambda^j}{(d-j)!}).$$

This is the Laplace transform of a delta measure.

EXAMPLE 3. Let $F^1(s) = f(s^1), F^i(s) = s^{i-1}f(s^i), 2 \leq i \leq d$. Then all the classes are critical. By Theorem 2.4 we have

(11.18) $$\lim_{n\to\infty} E_{e^d}[\exp(-\sum_{j=1}^{d} n^{-d+j-1}\lambda^j Z_j(n)) \mid Z_d(n) \neq 0]$$

$$= B(\psi_d^{\infty}(1,\lambda) - \psi_d(1,\lambda)),$$

where ψ_d is the solution of

(11.19) $$\begin{cases} \dfrac{d}{dt}\psi_d(t,\lambda) = -B\psi_d(t,\lambda)^2 + \displaystyle\sum_{j=1}^{d-1} \frac{\lambda^j}{(d-j-1)!} t^{d-j-1}, \\ \psi_d(0,\lambda) = \lambda^d, \end{cases}$$

and

(11.20) $$\psi_d^\infty(t,\lambda) = \lim_{\lambda^d \to \infty} \psi_d(t,\lambda).$$

This is already shown in [2] by a different formulation. If d = 1 or 2, then we can solve (11.19) explicitly; if d = 1,

(11.21) $$\psi_{1,1}(t,\lambda) = \frac{\lambda^1}{1+tB\lambda^1}, \quad \psi_{1,1}^\infty(t,\lambda) = \frac{1}{tB},$$

(11.22) $$\lim_{n\to\infty} E_{e^1}[\exp(-\lambda^1 Z_1(n)) \mid Z_1(n) \neq 0] = (1+B\lambda^1)^{-1},$$

and if d = 2,

(11.23) $$\begin{cases} \psi_{2,2}(t,\lambda) = \sqrt{\dfrac{\lambda^1}{B}}\, \dfrac{\sqrt{B}\lambda^2\cosh(\sqrt{B\lambda^1}t) + \sqrt{\lambda^1}\sinh(\sqrt{B\lambda^1}t)}{\sqrt{B}\lambda^2\sinh(\sqrt{B\lambda^1}t) + \sqrt{\lambda^1}\cosh(\sqrt{B\lambda^1}t)}, \\[2ex] \psi_{2,2}^\infty(t,\lambda) = \sqrt{\dfrac{\lambda^1}{B}}\, \dfrac{\cosh(\sqrt{B\lambda^1}t)}{\sinh(\sqrt{B\lambda^1}t)}, \end{cases}$$

(11.24) $$\lim_{n\to\infty} E_{e^2}[\exp(-n^{-2}\lambda^1 Z_1(n) - n^{-1}\lambda^2 Z_2(n)) \mid Z_2(n) \neq 0]$$

$$= \sqrt{B\lambda^1}\left(\frac{\cosh(\sqrt{B\lambda^1})}{\sinh(\sqrt{B\lambda^1})} - \frac{\sqrt{B}\lambda^2\cosh(\sqrt{B\lambda^1}) + \sqrt{\lambda^1}\sinh(\sqrt{B\lambda^1})}{\sqrt{B}\lambda^2\sinh(\sqrt{B\lambda^1}) + \sqrt{\lambda^1}\cosh(\sqrt{B\lambda^1})} \right).$$

Letting $k \geqq 2$, we apply Theorem 10.1. In this case, $\nu(\alpha) = k$ is valid only for $\alpha = d-k+1$. Hence

(11.25) $$\begin{cases} b_{d-k+1}^k = B^{-1}, \\ \psi_{d-k+1}^k(\lambda) = \overline{\psi}_{d-k+1}(1,\lambda) = \psi_{d-k+1}(1,\lambda), \\ \psi_{d-k+1}^{k,\infty}(\lambda) = \overline{\psi}_{d-k+1}^\infty(1,\lambda) = \psi_{d-k+1}^\infty(1,\lambda), \end{cases}$$

where $\psi_{d-k+1}(t,\lambda)$ and $\psi^{\infty}_{d-k+1}(t,\lambda)$ are given by

$$(11.26)\quad \begin{cases} \frac{d}{dt}\psi_{d-k+1}(t,\lambda) = -B\psi_{d-k+1}(t,\lambda)^2 + \sum_{j=1}^{d-k} \frac{\lambda^j}{(d-k-j)!} t^{d-k-j}, \\ \psi_{d-k+1}(0,\lambda) = \lambda^{d-k+1}, \\ \psi^{\infty}_{d-k+1}(t,\lambda) = \lim_{\lambda^{d-k+1}\to\infty} \psi_{d-k+1}(t,\lambda). \end{cases}$$

Recurrence formula (10.7) is given by

$$(11.27)\qquad R_i(x_j;j< i) = (B^{-1}x_{i-1})^{\frac{1}{2}}.$$

It then follows that

$$(11.28)\qquad \lim_{n\to\infty} E_{e^d}[\exp(-\sum_{j=1}^{d-k+1} n^{-d+k+j-2}\lambda^j Z_j(n) - \sum_{j=d-k}^{d} n^{-1}\lambda^j Z_j(n)) \mid (Z_j(n))_{d-k+1\leq j\leq d} \neq 0]$$

$$= (B\psi^{\infty}_{d-k+1}(1,\lambda))^{2^{-k+1}} - (B\psi_{d-k+1}(1,\lambda))^{2^{-k+1}}.$$

If $k = d$, then $\psi_1(t,\lambda) = \dfrac{\lambda^1}{1+tB\lambda^1}$ and $\psi^{\infty}_1(t,\lambda) = \dfrac{1}{tB}$.

Therefore we have

$$\lim_{n\to\infty} E_{e^d}[\exp(-\sum_{j=1}^{d} n^{-1}\lambda^j Z_j(n)) \mid (Z_j(n))_{1\leq j\leq d} \neq 0]$$

$$= 1 - \left(1 - \frac{1}{1+B\lambda^1}\right)^{2^{1-d}}.$$

The case of $d = 2$ is already given in [5].

We shall give an example such that C_N is a final class with $\# C_N > 1$.

EXAMPLE 4. Let $F^1(s) = f(s^1), F^2(s) = s^1 f(s^2), F^3(s) = \frac{1}{2} f(s^2)s^3 + \frac{1}{2} f(s^1)s^4$ and $F^4(s) = \frac{1}{4} f(s^1)s^3 + \frac{3}{4} f(s^2)s^4$. Then the mean matrix is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} & \frac{3}{4} \end{pmatrix}. \tag{11.29}$$

It is easy to see that

$$C_1 = \{1\}, C_2 = \{2\}, C_3 = \{3,4\}, \; \rho_1 = \rho_2 = \rho_3 = 1, \tag{11.30}$$

$$1 \prec 2 \prec 3, \tag{11.31}$$

$$\nu(\beta,\alpha) = \begin{cases} \alpha-\beta+1 & \text{if } \alpha \succeq \beta, \\ -1 & \text{if } \alpha \prec \beta, \end{cases} \tag{11.32}$$

and by Theorem 3.1,

$$M^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}. \tag{11.33}$$

Obviously, C_1, C_2 are critical and C_3 is final. It follows from (11.33) that $a_{2,1} = b_{2,1} = c_1^2 = 1$. By Theorem 2.1 we get

$$\psi_2(t,\lambda) = \lim_{n\to\infty} n(1-E_{e^2}[\exp(-\sum_{j=1}^{2} n^{j-3}\lambda^j Z_j([nt]))]) \tag{11.34}$$

and ψ_2 is the solution of

$$\frac{d}{dt}\,\psi_2(t,\lambda) = -B\psi_2(t,\lambda)^2 + \lambda^1, \qquad \psi_2(0,\lambda) = \lambda^2,$$

i.e.,

(11.35) $$\psi_2(t,\lambda) = \sqrt{\frac{\lambda^1}{B}}\;\frac{\sqrt{B}\lambda^2\cosh(\sqrt{B\lambda^1}t)+\sqrt{\lambda^1}\sinh(\sqrt{B\lambda^1}t)}{\sqrt{B}\lambda^2\sinh(\sqrt{B\lambda^1}t)+\sqrt{\lambda^1}\cosh(\sqrt{B\lambda^1}t)}\,.$$

Therefore, by Theorem 2.3, we have

(11.36) $$\lim_{n\to\infty} E_{e^i}[\exp(-n^{-2}\lambda^1 Z_1(n)-n^{-1}\lambda^2 Z_2(n)-\lambda^3 Z_3(n)-\lambda^4 Z_4(n))]$$

$$= \exp\left(-\frac{2}{3}\int_0^1 \psi_2(t,\lambda)dt\right)\left(\frac{1}{3}e^{-\lambda^3} + \frac{2}{3}e^{-\lambda^4}\right)$$

$$= \frac{1}{3}\left(\cosh(\sqrt{B\lambda^1}) + \lambda^2\sqrt{\frac{B}{\lambda^1}}\sinh(\sqrt{B\lambda^1})\right)^{-\frac{2}{3B}}(e^{-\lambda^3}+2e^{-\lambda^4}),$$

for $i = 3,4$.

Finally we shall give an example such that $\{C_1,\cdots,C_N\}$ is not linearly ordered.

EXAMPLE 5. Let $F^1(s) = f(s^1)$, $F^2(s) = s^1 s^2$, $F^3(s) = s^1 f(s^3)$ and $F^4(s) = s^2 s^3 s^4$. Then

(11.37) $$C_i = \{i\}, \quad \rho_i = 1,$$

(11.38) $$1 \prec 2 \prec 4, \quad 1 \prec 3 \prec 4 \quad \text{and neither } 2 \prec 3 \text{ nor } 3 \prec 2,$$

(11.39) $$\nu(4) = 1, \ \nu(3) = \nu(2) = 2, \ \nu(1) = 3.$$

In this case, C_1 and C_3 are critical and C_2 and C_4 are final. By Theorem 2.1 we have

(11.40) $$\lim_{n\to\infty} n(1-E_{e^i}[\exp(-n^{-2}\lambda^1 Z_1([nt])-n^{-1}\lambda^i Z_i([nt]))])$$

$$= \psi_i(t,\lambda), \quad i = 2,3,$$

where ψ_2 and ψ_3 are the solutions of

(11.41) $$\frac{d}{dt}\psi_2(t,\lambda) = \lambda^1, \quad \psi_2(0,\lambda) = \lambda^2,$$

(11.42) $$\frac{d}{dt}\psi_3(t,\lambda) = -B\psi_3(t,\lambda)^2 + \lambda^1, \quad \psi_3(0,\lambda) = \lambda^3,$$

i.e.,

$$\psi_2(t,\lambda) = t\lambda^1 + \lambda^2,$$

$$\psi_3(t,\lambda) = \sqrt{\frac{\lambda^1}{B}}\ \frac{\sqrt{B}\lambda^3\cosh(\sqrt{B\lambda^1}t)+\sqrt{\lambda^1}\sinh(\sqrt{B\lambda^1}t)}{\sqrt{B}\lambda^3\sinh(\sqrt{B\lambda^1}t)+\sqrt{\lambda^1}\cosh(\sqrt{B\lambda^1}t)}.$$

Therefore, by Theorem 2.3, we obtain

$$\lim_{n\to\infty} E_{e^4}[\exp(-n^{-2}\lambda^1 Z_1(n)-n^{-1}\lambda^2 Z_2(n)-n^{-1}\lambda^3 Z_3(n)-\lambda^4 Z_4(n))]$$

$$= \exp(-\int_0^1 \psi_2(t,\lambda)dt)\exp(-\int_0^1 \psi_3(t,\lambda)dt)\exp(-\lambda^4)$$

$$= \exp(-\tfrac{1}{2}\lambda^1-\lambda^2-\lambda^4)(\cosh(\sqrt{B\lambda^1})+\lambda^3\sqrt{\frac{B}{\lambda^1}}\sinh(\sqrt{B\lambda^1}))^{-\frac{1}{B}}.$$

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