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ON BIRATIONAL-INTEGRAL EXTENSION OF RINGS AND PRIME IDEALS OF DEPTH ONE

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Introduction. Let A be a commutative ring with unity and let $\bar{A}$ be the integral closure of A in its total quotient ring $Q(A)$. An intermediate ring R between A and $\bar{A}$ will be called a birational-integral extension of A. In this paper, we are mainly interested in birational-integral extensions of rings. A birational-integral extension of rings is closely related with prime ideals of depth one, and, in section 1, we study various properties associated with prime ideals of depth one. Let A be a noetherian ring satisfying the S_1-condition (i.e., the ideal (0) has no embedded prime divisor),and let p be a prime ideal with depth $A_p = 1$ and of $\operatorname{ht} p > 1$. Then such a prime ideal p is characterized by many equivalent conditions, say, p is a prime divisor of a principal ideal, p is a divisorial ideal and so on. Among others it is proved that if $\bar{A}$ is a finite A-module, then $\{p \in \operatorname{Spec} A \mid \operatorname{ht} p > 1$ and depth $A_p = 1\}$ is a finite set.

Let R be a birational-integral extension of A and assume that R is a finite A-module. For an element $\alpha \in Q(A)$, we set $I_\alpha = \{a \in A \mid a\alpha \in A\}$. I_α is an ideal of A and is called the denominator ideal of α in A. For a non-negative integer i, we set $F_i(A,R) = \{\alpha \in R \mid \operatorname{ht} I_\alpha \geqq i + 1\}$. Then each $F_i(A,R)$ is an intermediate ring between A and R, and $F_0(A,R) = R$ and $F_d(A,R) = A$ for some non-negative integer d. The subrings $F_i(A,R)$'s play important roles in this paper.

In section 2, we mainly treat the seminormality and the glueing of rings which appears in [6]. First, we prove in (2.1) that the proof of seminormality can be reduced to the case of birational-integral extensions. In (2.2), we give a criterion of relative seminormality of A that A is seminormal in R if and only if the conductor ideal $\mathfrak{c}(F_i(A,R)/A)$ is radical in $F_i(A,R)$ for all i, $1 \leqq i \leqq d$. As an application of this criterion, we shall show the stability theorems of seminormality under the étale finite extensions and, in case of an affine domain, the separably generated extensions of the coefficient fields. On the other hand, Traverso proved that if A is seminormal in R, then A is obtained by a sequence of glueings of R with respect to a certain prime ideals of A (c.f. [6]). Here, we prove that A is seminormal in R if and only if A is obtained by a glueing of R with respect to the prime ideals in $\mathrm{Ass}_A(R/A)$.

For an integral extension R/A, let ${}^{+}_{R}A$ be the seminormalization of A (c.f. [6]). Then ${}^{+}_{R}A$ is seminormal in R and the seminormalization of A in ${}^{+}_{R}A$ is equal to ${}^{+}_{R}A$. Hence, for the study of integral extension, we divide it into two cases, the case where A is seminormal in R and the case where $R = {}^{+}_{R}A$. In section 3, we treat birational-integral extension R of A such that $R = {}^{+}_{R}A$. Such an extension is called a cusp type extension.

In section 4, we assume that A is a homomorphic image of a regular ring and satisfies the s_1-condition. For such ring A, it is proved in [2] that $\Delta = \{p \in \mathrm{Spec}\, A \mid \mathrm{ht}\, p > 1$ and $\mathrm{depth}\, A_p = 1\}$ is a finite set. Let x be a non-unit of A and let R be a birational-integral extension of A and assume that R is a finite A-module. If x is a non-zero divisor and $x \notin q$ for all $q \in \mathrm{Ass}_A(R/A) \cup \Delta$, we call such an element x a good section of R/A. With the help of this notion, we prove that, for any $P \in \mathrm{Spec}\, R$, depth $A_{(A \cap P)} \leqq \mathrm{ht}\, P$.

Let A be an integral domain and let K be the quotient field of A and let B be a subring of K containing A. We call such an intermediate ring B an over-ring of A. In the paper [8], we studied the relationship between flat over-rings of a Krull domain and prime ideals of height one. Replacing a Krull domain by a noetherian domain and prime ideals of height one by those of depth one, we develop a similar theory as is done in [8] (c.f. §5).

1. Prime ideals of depth one

Let A be a noetherian ring which satisfies the S_1-condition i.e., the ideal (0) has no embedded prime divisor. We denote the total quotient ring of A by $Q(A)$ and the integral closure of A in $Q(A)$ by $\bar{A}$ and the conductor ideal by $\mathfrak{c}(\bar{A}/A)$.

Proposition - definition 1.1. Let R be a birational-integral extension of A and assume that R is a finite A-module. For an element $\alpha \in Q(A)$, we denote the denominator ideal $\{a \in A \mid a\alpha \in A\}$ in A by I_α. For an integer $i \geqq 0$, define $F_i(A,R) = \{\alpha \in R \mid \text{ht } I_\alpha \geqq i+1\}$. Then $F_i(A,R)$'s are intermediate rings between A and R, and $F_j(A,R) \supsetneqq F_i(A,R)$ for $j < i$ and $F_0(A,R) = R$.

Define $D_i(A,R) = \{p \in Ht_i(A) \mid p \supsetneqq \mathfrak{c}(F_{i-1}(A,R)/A)\}$ and $D(A,R) = \bigcup D_i(A,R)$, where $Ht_i(A) = \{p \in \text{Spec } A \mid \text{ht } p = i\}$. Then $\text{ht } \mathfrak{c}(F_{i-1}(A,R)/A) \geqq i$, so each element of $D_i(A,R)$ is a minimal prime divisor of $\mathfrak{c}(F_{i-1}(A,R)/A)$. There is an integer d $(\geqq 0)$ such that $F_d(A,R) = A$, hence $D(A,R)$ is a finite set.

Proof. Let α and β be elements of $F_i(A,R)$. Since $I_{\alpha+\beta} \supseteqq I_\alpha \cap I_\beta$ and $I_{\alpha\beta} \supseteqq I_\alpha \cdot I_\beta$, we have $\text{ht } I_{\alpha+\beta}$, $\text{ht } I_{\alpha\beta} \geqq i+1$, that is $\alpha+\beta, \alpha\beta \in F_i(A,R)$. Therefore $F_i(A,R)$ is an intermediate ring between A and R.

Next we show that $F_0(A,R) = R$. Let α be an element of R. Then I_α contains an element of A which is not zero-divisor for $\alpha \in Q(A)$. Hence $\text{ht } I_\alpha \geqq 1$ for all $\alpha \in R$. Thus we have $F_0(A,R) = R$.

Since A is noetherian and R is a finite A-module, the submodule $F_{i-1}(A,R)$ is also a finite A-module, say, $F_{i-1}(A,R) = A\alpha_1 + \cdots + A\alpha_n$. Then we have $\mathfrak{c}(F_{i-1}(A,R)/A) = I_{\alpha_1} \cap \cdots \cap I_{\alpha_n}$ and ht $I_{\alpha_j} \geqq i$ for all j. Hence we have ht $\tilde{c}(F_{i-1}(A,R)/A) \geqq i$. Thus each element of $D_i(A,R)$ is a minimal prime divisor of $\tilde{c}(F_{i-1}(A,R)/A)$.

From the sequence $R = F_0(A,R) \supseteqq F_1(A,R) \supseteqq \cdots$, we have a sequence of ideals $\tilde{c}(F_0(A,R)/A) \subseteqq \tilde{c}(F_1(A,R)/A) \subseteqq \cdots$. Since A is noetherian, there is an integer d ($\geqq 0$) such that $\tilde{c}(F_d(A,R)/A) = \tilde{c}(F_{d+1}(A,R)/A) = \cdots$. Suppose that $c(F_d(A,R)/A) \neq A$. Then there exists a prime ideal p containing $\tilde{c}(F_d(A,R)/A)$. Let ht $p = m$. From the choice of d, $p \supseteqq \tilde{c}(F_m(A,R)/A)$, so ht $p > m$ by the above result. This contradicts ht $p = m$. Hence $\tilde{c}(F_d(A,R)/A) = A$, that is $F_d(A,R) = A$.

From the definition of $F_i(A,R)$ and $D_i(A,R)$, the following lemma is easily seen, so we omit the proof.

Lemma 1.2. *Let R_1 and R_2 be birational-integral extensions of A. If $R_1 \supseteqq R_2$, then we have $F_i(A,R_1) \supseteqq F_i(A,R_2)$ for all i, $i \geqq 0$, and $D_i(A,R_1) \supseteqq D_i(A,R_2)$ for $i \geqq 1$. In particular, if $\bar{A}$ is a finite A-module, we have $D(A,R) \subseteqq D(A,\bar{A})$ for any intermediate ring R between A and $\bar{A}$.*

Let $\sqrt{(0)} = p_1 \cap \cdots \cap p_t$. Since A satisfies the S_1-condition, we have $Q(A) = A_{p_1} \times \cdots \times A_{p_t}$. Let p be a prime ideal of A and let $\{e(1), \cdots, e(p)\} = \{ j \mid 1 \leqq j \leqq t \text{ and } p \supseteqq p_j \}$. Then A_p is a subring of $A_{p_{e(1)}} \times \cdots \times A_{p_{e(p)}}$. Denote the projection; $A_{p_1} \times \cdots \times A_{p_t} \longrightarrow A_{p_{e(1)}} \times \cdots \times A_{p_{e(p)}}$ by ϕ_p. Then $\phi_p^{-1}(A_p)$ is a subring of $Q(A)$ containing A.

Proposition 1.3. Let R be a birational-integral extension of A and assume that R is a finite A-module. Then we have $F_i(A,R) = F_{i-1}(A,R) \cap \bigcap_{p \in D_i(A,R)} \phi_p^{-1}(A_p)$ for $i \geqq 1$. Especially, we have $A = R \cap \bigcap_{p \in D(A,R)} \phi_p^{-1}(A_p)$.

Proof. Put $B = F_{i-1}(A,R) \cap \bigcap \phi_p^{-1}(A_p)$, p runs over $D_i(A,R)$. Let $\alpha \in F_i(A,R)$. From the definition, we have $\alpha \in F_{i-1}(A,R)$. Let $p \in D_i(A,R)$. Then $\text{ht } p = i$ and $\text{ht } I_\alpha \geqq i+1$. Thus we have $p \not\supset I_\alpha$. Let $p_1, \cdots, p_t$ be all minimal prime divisor of A. It is well known that if $p_1 \cup \cdots \cup p_t \cup p \supseteqq I_\alpha$ then $p \supseteqq I_\alpha$ or $p_j \supseteqq I_\alpha$ for some j (cf. [3]. Theorem 81, p.55). Hence we have $p_1 \cup \cdots \cup p_t \cup p \not\supset I_\alpha$ i.e., there is a non-zero divisor x of I_α not contained in p. Hence $x\alpha \in A$ and $x \notin p$, that is $\alpha \in \phi_p^{-1}(A_p)$.

Conversely, let $\alpha \in B$ and let p be a prime ideal such that $p \supseteqq I_\alpha$. Since $\alpha \in F_{i-1}(A,R)$, $p \supseteqq I_\alpha \supseteqq \mathfrak{C}(F_{i-1}(A,R)/A)$, so we have $\text{ht } p \geqq i$ by (1.1). If $\text{ht } p > i$, then we have nothing to prove. Assume that $\text{ht } p = i$. Then $p \in D_i(A,R)$. Thus $\alpha \in \phi_p^{-1}(A_p)$. Hence we have $p \not\supseteq I_\alpha$. This contradicts $p \supseteqq I_\alpha$.

Remark 1.4. If $p \not\supset \mathfrak{C}(R/A)$, then $R \subseteqq \phi_p^{-1}(A_p)$. Indeed, since

R is a subring of $Q(A)$ and a finite A-module, there exists a non-zero divisor in $\tilde{c}(R/A)$ not contained in p. Thus $R \subseteqq \phi_p^{-1}(A_p)$.

Let A be a ring. We say that A satisfies the S_k-condition if it holds that depth $A_p \geqq \inf(k, \mathrm{ht}\, p)$ for all $p \in \mathrm{Spec}\, A$.

Proposition 1.5. Let A be a noetherian ring satisfying the S_1-condition. The ring A satisfies the S_2-condition if and only if $A = \bigcap_{p \in Ht_1(A)} \phi_p^{-1}(A_p)$. If A satisfies the S_2-condition, then $D(A,R) = D_1(A,R)$ for any intermediate ring R between A and $\bar{A}$ such that R is a finite A-module.

Proof. Let $B = \bigcap_{p \in Ht_1(A)} \phi_p^{-1}(A_p)$. Then $A \subseteqq B$. To show the inverse inclusion, let $\alpha \in B$. Since $\alpha \in \phi_p^{-1}(A_p)$ for all $p \in Ht_1(A)$, the denominator ideal I_α is not contained in a prime ideal of height one. Suppose that A satisfies the S_2-condition and $I_\alpha \neq A$, and let p be a minimal prime divisor of I_α. Since I_α is not contained in a prime ideal of height one, we have $\mathrm{ht}\, p \geqq 2$. By the S_2-condition, depth $A_p \geqq 2$. Since p is a minimal prime divisor of I_α, $I_\alpha A_p$ is a primary ideal belonging to pA_p. Thus there exists an A_p-sequence consisting of two elements of I_α, say, a and b. Since $a, b \in I_\alpha$, we have $c = a\alpha$, $d = b\alpha \in A$, so $ad = bc$. Since a, b is an A_p-sequence, we obtain $c \in aA_p$. Thus we have $\alpha \in A_p$, therefore $I_\alpha \not\subset p$. This contradicts $I_\alpha \subseteqq p$. Hence we have $I_\alpha = A$, that is $\alpha \in A$.

Conversely, let $A = B$ and let p be a prime ideal of A. If $\mathrm{ht}\, p = 0$, then depth $A_p \geqq 0 = \inf(2, \mathrm{ht}\, p)$. If $\mathrm{ht}\, p = 1$, then depth $A_p \geqq 1 = \inf(2, \mathrm{ht}\, p)$. Let $\mathrm{ht}\, p > 1$ and let a be a non-zero divisor contained in p. If it is shown that p is not a prime divisor of aA then depth $A_p \geqq 2 = \inf(2, \operatorname{ht} p)$ and we are done. In the following we shall show that p is not a prime divisor of aA. Before proceeding further we prove a

Lemma 1.6. Let a be a non-zero divisor of A and let $aA = q_1 \cap \cdots \cap q_t$ be an irredundant primary decomposition of aA and $\sqrt{q_i} = p_i$. When $t = 1$, put $\alpha = \frac{1}{a}$. When $t \geqq 2$, take an element b of $q_2 \cap \cdots \cap q_t$ not contained in q_1, and put $\alpha = \frac{b}{a}$. Then the denominator ideal I_α is primary for p_1.

Proof. When $t = 1$, we have $I_\alpha = q_1$, so we may assume $t \geqq 2$. Let x be an element of I_α. Then we have $bx \in aA \subsetneqq q_1$. Since b is not contained in q_1, we have $x \in p_1$, that is $I_\alpha \subsetneqq p_1$. It is obvious that $q_1 \subsetneqq I_\alpha$, so $\sqrt{I_\alpha} = p_1$. Let $xy \in I_\alpha$ and $x \notin p_1$. Then we have $xyb \in aA \subsetneqq q_1$. Since $x \notin p_1$ and $b \in q_2 \cap \cdots \cap q_t$, we obtain $yb \in q_1 \cap \cdots \cap q_t = aA$, that is $y \in I_\alpha$. Hence I_α is a primary ideal belonging to p_1.

Assume that p is a prime divisor of aA. From the lemma we see that there exists an element $\alpha \in Q(A)$ such that I_α is primary for p. Hence I_α is not contained in any prime ideal of height one. Thus $\alpha \in \phi_p^{-1}(A_p)$ for all $p \in Ht_1(A)$. By our hypothesis $A = B$, we have $\alpha \in A$. This contradicts $I_\alpha \subsetneqq p$.

To prove the second half, we assume that A satisfies the S_2-condition. Let $p \in Ht_1(A)$. If $p \notin D_1(A,R)$, then $p \not\supset c(F_0(A,R)/A) = c(R/A)$, hence $F_1(A,R) \subseteqq R \subseteqq \phi_p^{-1}(A_p)$. If $p \in D_1(A,R)$, then $F_1(A,R) \subseteqq \phi_p^{-1}(A_p)$ by (1.3). Therefore we have $F_1(A,R) \subseteqq \bigcap_{p \in Ht_1(A)} \phi_p^{-1}(A_p) = A$. Hence $D(A,R) = D_1(A,R)$.

In (1.11) we shall see that the condition $D(A,\bar{A}) = D_1(A,\bar{A})$ implies the S_2-condition for A when $\bar{A}$ is a finite A-module.

Remark 1.7. $D(A,\bar{A}) = D_1(A,\bar{A})$ implies that the prime divisors of $c(\bar{A}/A)$ are all of height one by the following lemma. But the converse of this assertion is not true in general.

Lemma 1.8. *Let R be a birational-integral extension of A and assume that R is a finite A-module.*

(i) *Let $\alpha \in R$ and let p be a prime divisor of I_α. Then there exists an element $\beta \in R$ such that I_β is primary for p.*

(ii) *Let α be an element of R such that I_α is a primary ideal belonging to p. Then $p \in D(A,R)$.*

(iii) *Let p be a prime divisor of $c(R/A)$. Then there is an element whose denominator ideal is primary for p, so $p \in D(A,R)$.*

Proof. Let $\alpha \in R$ and let p be a prime divisor of I_α. Let $I_\alpha = q \cap q_1 \cap \cdots \cap q_t$ be an irredundant primary decomposition and $\sqrt{q} = p$. If $t = 0$, then $I_\alpha = q$ is primary for p. Assume that $t \geqq 1$. Then there exists an element b of $q_1 \cap \cdots \cap q_t$ not contained in q. Let $\beta = b\alpha \in R$. We shall show that I_β

is primary for p. Let x be any element of I_β. Then $bx \in I_\alpha \subseteqq q$. Since b is not contained in q, $x \in p$. This prove that $I_\beta \subseteqq p$. It is obvious that $q \subseteqq I_\beta$. Hence $\sqrt{I_\beta} = p$. Let $xy \in I_\beta$ and $x \notin p$. Then $bxy \in I_\alpha \subseteqq q$. Since $x \notin p$, $by \in q$. On the other hand $b \in q_1 \cap \cdots \cap q_t$. Hence $by \in I_\alpha$, i.e., $y\beta \in A$. Thus $y \in I_\beta$. Therefore I_β is a primary ideal belonging to p. We have (i).

Let $\alpha \in R$ and I_α primary for p and let ht p = i ($\geqq 1$). Then ht I_α = i for $\sqrt{I_\alpha} = p$. By (1.1), $\alpha \in F_{i-1}(A,R)$, hence we have $p \supseteqq I_\alpha \supseteqq \widehat{c}(F_{i-1}(A,R)/A)$, so $p \in D_i(A,R) \subseteqq D(A,R)$, i.e., (ii).

To prove (iii), let $R = A\alpha_1 + \cdots + A\alpha_n$. Then ©(R/A) = $I_{\alpha_1} \cap \cdots \cap I_{\alpha_n}$. Since p is a prime divisor of ©(R/A), p is a prime divisor of I_{α_j} for some j. The rest follows from (ii).

In the following we shall study prime ideals belonging to $D(A,\bar{A})$ under the assumption that $\bar{A}$ is a finite A-module. Speaking of a prime ideal of height one, the matter is rather simple. In fact, if p is a prime ideal of height one, then $p \in D(A,\bar{A})$ if and only if $p \supseteqq$ ©$(\bar{A}/A)$, i.e., A_p is not integrally closed. We prove in the following that a prime ideal p of height > 1 belongs to $D(A,\bar{A})$ if and only if depth $A_p = 1$.

For the proof of this assertion we need the next two propositions.

Proposition 1.9. Let R be a birational-integral extension of A and assume that R is a finite A-module. Let P be a prime divisor of $\mathfrak{C}(R/A)$ in R. Then $P\cap A$ is an element of $D(A,R)$.

Proof. Let $p = P\cap A$ and let $\mathfrak{C}(R/A) = Q_1\cap\cdots\cap Q_t$ be an irredundant primary decomposition in R and $\sqrt{Q_i} = P_i$, $p_i = P_i\cap A$. We assume that $p = p_1 = \cdots = p_s$ and $p \neq p_j$, $j > s$. Since $\mathfrak{C}(R/A)$ is a ideal of A, we have $c(R/A) = q_1\cap\cdots\cap q_t$, where $q_i = Q_i\cap A$. Let $q = q_1\cap\cdots\cap q_s$. Then q is a primary ideal belonging to p.

First, suppose that $q \supseteq q_{s+1}\cap\cdots\cap q_t$. Then $\mathfrak{C}(R/A) = q_{s+1}\cap\cdots\cap q_t$. Let α be an element of $Q_{s+1}\cap\cdots\cap Q_t$ not contained in Q_i, $1 \leqq i \leqq s$. Since $\alpha q \subseteq Q_1\cap\cdots\cap Q_t = \mathfrak{C}(R/A) \subseteq A$, we have $q \subseteq I_\alpha$. We shall prove that $I_\alpha \subseteq p$. Let x be an element of I_α. Then $x\alpha \in Q_{s+1}\cap\cdots\cap Q_t\cap A = q_{s+1}\cap\cdots\cap q_t = \mathfrak{C}(R/A) \subseteq Q_1\cap\cdots\cap Q_s$. If $x \notin p$, then x is not contained in $P_1, \cdots, P_s$, so we have $\alpha\in Q_1\cap\cdots\cap Q_s$, it contradicts the choice of α. Thus $q \subseteq I_\alpha \subseteq p$, that is $\sqrt{I_\alpha} = p$. Hence p is a prime divisor of I_α. By (1.8), we have $p\in D(A,R)$.

Next suppose that $q \not\supseteq q_{s+1}\cap\cdots\cap q_t$. Then p is a prime divisor of $\mathfrak{C}(R/A)$. By (1.8), we have $p\in D(A,R)$.

A prime ideal p of A with ht $p > 1$ and depth $A_p = 1$ is characterized by the following equivalent conditions. Therefore such prime ideals are special one in Spec A and the set of such prime ideals is a finite set.

Proposition 1.10. Assume that A satisfies the S_1-condition. Let $p \in \mathrm{Spec}\, A$ and ht $p \geq 1$. Then the following are equivalent.

(i) Depth $A_p = 1$.

(ii) For any non-zero divisor $a \in p$, p is a prime divisor of aA.

(iii) There exists a non-zero divisor a of A such that p is a prime divisor of aA.

(iv) p is a prime divisor of I_α for some element $\alpha \in Q(A)$.

(v) There exists an element α of $Q(A)$ such that $I_\alpha = p$.

(vi) p is a divisorial ideal over A, i.e., $p = A : (A : p)$.

(vii) p is a prime divisor of a certain divisorial ideal.

Moreover, if $\bar{A}$ is a finite A-module and ht $p = i \geq 2$, then the following are equivalent to each other and to the above conditions.

(viii) There is an element $\alpha \in \bar{A}$ of $I_\alpha = p$, hence $p \in \mathrm{Ass}_A(\bar{A}/A)$.

(ix) There exists an intermediate ring B between A and $\bar{A}$ such that $\mathfrak{c}(B/A) = p$.

(x) There exists an intermediate ring B between A and $\bar{A}$ such that p is a prime divisor of $\mathfrak{c}(B/A)$.

(xi) p is an element of $D_i(A,\bar{A})$.

Proof. First, we prove

(i) → (ii) → (iii) → (iv) → (v) → (vi) → (vii) → (i).

(i) → (ii): Let a be a non-zero divisor of A and let $aA_p = q_1 \cap \cdots \cap q_t$ be an irredundant primary decomposition in A_p. Assume that $\sqrt{q_i} \subsetneqq pA_p$ for all i, that is p is not a prime divisor of aA. Hence there exists an element $b \in pA_p$ such as $b \notin \sqrt{q_i}$ for all i. Then a is a non-zero divisor of A_p and b is a non-zero divisor on A_p/aA_p. Therefore we have depth $A_p > 1$. This is a contradiction. Thus p is a prime divisor of aA.

(ii) → (iii): Since ht $p > 0$ and A satisfies the S_1-condition, there exists a non-zero divisor in p. Hence (ii) implies (iii).

(iii) → (iv): It is proved in (1.6).

(iv) → (v): Let α be an element of $Q(A)$ whose denominator ideal I_α is primary for p. Then there exists a positive integer h such as $p^h \subseteqq I_\alpha$ and $p^{h-1} \not\subset I_\alpha$. Let x be an element of p^{h-1} not contained in I_α and $\beta = x\alpha$. Then $I_\beta \supseteqq p$. Let a be an element of I_β. Then $ax \in I_\alpha$. Since I_α is a primary ideal belonging to p and $x \notin I_\alpha$, we have a p. Therefore $I_\beta = p$

(v) → (vi) : Let α be an element of $Q(A)$ such as $I_\alpha = p$. Then $A : A + A\alpha = I_\alpha = p$. Since $A : (A : A + A\alpha)) \supseteqq A + A\alpha$, we have $A : A + A\alpha \supseteqq A : (A : (A : A + A\alpha)) \supseteqq A : A + A\alpha$. Thus $A : A + A\alpha = p$ is a divisorial ideal over A.

(vi) → (vii) : This is obvious.

(vii) → (i) : Let p be a prime divisor of a divisorial ideal J. Then $A : J$ is a finite A-module, say, $A : J = A\alpha_1 + \cdots + A\alpha_n$.

Hence we have $J = A : (A : J) = I_{\alpha_1} \cap \cdots \cap I_{\alpha_n}$. Since p is a prime divisor of J, p is a prime divisor of I_{α_j} for some j. For a convenience we put $\alpha = \alpha_j$. Let $I_\alpha = q \cap q_1 \cap \cdots \cap q_t$, $\sqrt{q} = p$, be an irredundant primary decomposition and let x be an element of $q_1 \cap \cdots \cap q_t$ not contained in q and let $\beta = x\alpha$. We show that $\sqrt{I_\beta} = p$. Let $a \in I_\beta$. Then $ax \in I_\alpha \subseteqq q$. Since x is not contained in q and q is primary for p, we have $a \in p$. Hence we have $\sqrt{I_\beta} = p$.

Assume that depth $A_p > 1$. Since $p = \sqrt{I_\beta}$, there exists an A_p-sequence $a, b \in I_\beta$. Let $\beta = \frac{c}{a} = \frac{d}{b}$ in $Q(A)$. Hence $ad = bc$. But a, b is an A_p-sequence, so we have $\beta \in A_p$. This contradicts $I_\beta \subseteqq p$. Therefore depth $A_p = 1$, that is (i).

Thus the first half of this theorem is proved. Next we assume that $\bar{A}$ is a finite A-module and ht $p = i \geqq 2$ and we show that

$$(v) \rightarrow (viii) \rightarrow (ix) \rightarrow (x) \rightarrow (vii) \quad \text{and} \quad (viii) \rightarrow (xi) \rightarrow (x).$$

(v) $\rightarrow$ (viii) : We have to show that there exists an element of $\bar{A}$ whose denominator ideal is equal to p.

First, assume that A is an integral domain. Since A is a noetherian domain and $\bar{A}$ is a finite A-module, $\bar{A}$ is a normal domain. Let α be an element of $Q(A)$ with $I_\alpha = p$ and let $\bar{I}_\alpha$ be the denominator ideal of α in $\bar{A}$. If $\bar{I}_\alpha = \bar{A}$, then $\alpha \in \bar{A}$. Assume that $\bar{I}_\alpha \neq \bar{A}$. Since $\bar{A}$ is a normal domain, the denominator ideal has an irredundant primary decomposition and each prime divisor has height one, say, $\bar{I}_\alpha = \bar{Q}_1 \cap \cdots \cap \bar{Q}_t$ and

$\sqrt{\bar{Q}_j} = \bar{P}_j$. Since $I_\alpha \subseteqq \bar{I}_\alpha$, we have $\bar{P}_j \cap A \supseteqq p$. Suppose that $\bar{P}_j \cap A \supsetneqq p$ for all j. Then there exists an element x of $\bar{Q}_1 \cap \cdots \cap \bar{Q}_t \cap A$ not contained in p. Let $\beta = x\alpha$. Then $\beta \in \bar{A}$ and $I_\beta = p$ for $x \notin p$. Suppose that $\bar{P}_j \cap A = p$ for some j. If $p \not\supset c(\bar{A}/A)$, then we have $A_p = \bar{A}_{\bar{P}_j}$. This contradicts $1 = \text{ht}\,\bar{P}_j < \text{ht}\,p$. Hence $\bar{P}_j \supseteqq p \supseteqq \mathfrak{C}(\bar{A}/A)$. Since $\text{ht}\,\bar{P}_j = 1$, $\bar{P}_j$ is a prime divisor of $\mathfrak{C}(\bar{A}/A)$. From (1.9), we have $\bar{P}_j \cap A = p \in D(A,\bar{A})$. Since $\text{ht}\,p = i$, p is a minimal prime divisor of $\mathfrak{C}(F_{i-1}(A,\bar{A})/A)$ by (1.1). By (1.8.(iii)) and the first half of this theorem, our assertion is proved in a domain.

Next, we pass the general case. Let α be an element of $Q(A)$ of $I_\alpha = p$ and let $\bar{I}_\alpha$ be the denominator ideal of α in $\bar{A}$. Let $(0) = q_1 \cap \cdots \cap q_t$ be an irredundant primary decomposition in A and $\sqrt{q_j} = p_j$. Since A satisfies the S_1-condition, the ideal (0) has no embedded prime divisor, so $Q(A) = A_{p_1} \times \cdots \times A_{p_t}$. Then $A_j = A/q_j$ is a subring of A_{p_j} and $A_1 \times \cdots \times A_t$ is an intermediate ring between A and $\bar{A}$. Let $\bar{A}_j$ be the integral closure of A_j in A_{p_j}. Then $\bar{A} = \bar{A}_1 \times \cdots \times \bar{A}_t$. By our assumption that $\bar{A}$ is a finite A-module, each $\bar{A}_j$ is also a finite A_j-module, so $\bar{A}_j$ is noetherian. Let $\alpha = \alpha_1 + \cdots + \alpha_t \in Q(A) = A_{p_1} \times \cdots \times A_{p_t}$ and $\bar{I}_j = \{a \in \bar{A}_j \mid a\alpha_j \in \bar{A}_j\}$. Let $\bar{P}$ be a prime divisor of $\bar{I}_\alpha$. Since $\bar{P}$ is a prime ideal of $\bar{A} = \bar{A}_1 \times \cdots \times \bar{A}_t$, there exists an integer k, $1 \leqq k \leqq t$, and a prime ideal $\bar{Q}$ in $\bar{A}_k$ such as $\bar{P} = \bar{A}_1 \times \cdots \times \bar{Q} \times \cdots \times \bar{A}_t$. Then $\text{ht}\,\bar{P} = \text{ht}\,\bar{Q}$ and $\bar{Q}$ is a prime divisor of $\bar{I}_k$.

Now, we shall show that each prime divisor of $\bar{I}_\alpha$ has height one in $\bar{A}$. By the above discussion, it's enough to show the proof under the assumption that the nilpotent ideal of A is prime in A.

Denote the residue class of α in $Q(A)/\mathrm{nil}(Q(A))$ by $\bar{\alpha}$. Let $\bar{P}$ be a prime divisor of $\bar{I}_\alpha$ and assume that $\mathrm{ht}\,\bar{P} > 1$. From the first half of this theorem, there exists an element $\beta \in Q(A)$ of $\bar{I}_\beta = \bar{P}$. Then we have $\bar{I}_{\bar{\beta}} = \{\bar{a} \in \bar{A}/\mathrm{nil}(\bar{A}) \mid \bar{a}\bar{\beta} \in \bar{A}/\mathrm{nil}(\bar{A})\} \supseteqq \bar{P}/\mathrm{nil}(\bar{A})$ and $\mathrm{ht}\,\bar{P}/\mathrm{nil}(\bar{A}) > 1$. Since $\bar{A}/\mathrm{nil}(\bar{A})$ is the integral closure of a noetherian domain $A/\mathrm{nil}(A)$ and a finite $A/\mathrm{nil}(A)$-module, we have $\bar{\beta} \in \bar{A}/\mathrm{nil}(\bar{A})$. Hence there is an element $\xi \in \dot{\bar{A}}$ such that $\beta - \xi \in \mathrm{nil}(\bar{A})$, so β is integral over $\bar{A}$, that is $\beta \in \bar{A}$. This contadicts $\bar{I}_\beta = \bar{P} \neq \bar{A}$. Thus we have $\mathrm{ht}\,\bar{P} = 1$.

Let $\bar{I}_\alpha = \bar{I}_1 \cap \cdots \cap \bar{I}_n$ be an irredundant primary decomposition in $\bar{A}$ and $\sqrt{\bar{I}_j} = \bar{P}_j$. Let $p_j = \bar{P}_j \cap A$. Then $I_\alpha \subseteqq \bar{I}_\alpha \subseteqq \bar{P}_j$, hence $p \subseteqq p_j$ for all j. If $p \subsetneqq p_j$ for all j, then there exists an element x in $\bar{I}_\alpha \cap A$ not contained in p and let $\beta = x\alpha$. Then $\beta \in \bar{A}$ and $I_\beta = p$ for $x \notin p$. Suppose that $p = p_j$ for some j. Then $\bar{P}_j$ is lying over p and $1 = \mathrm{ht}\,\bar{P}_j < \mathrm{ht}\,p$. If $p \not\supset \mathfrak{C}(\bar{A}/A)$, then $A_p = \bar{A}_{\bar{P}_j}$, so $\mathrm{ht}\,p = \mathrm{ht}\,\bar{P}_j = 1$. This contradicts $\mathrm{ht}\,p = i > 1$. Hence $\bar{P}_j \supseteqq p \supseteqq \mathfrak{C}(\bar{A}/A)$. Since $\mathrm{ht}\,\bar{P}_j = 1$ and $\mathrm{ht}\,\mathfrak{C}(\bar{A}/A) \geqq 1$, $\bar{P}_j$ is a prime divisor of $\mathfrak{C}(\bar{A}/A)$. By (1.9) we have $p \in D_i(A,\bar{A})$, so p is a prime divisor of $\mathfrak{C}(F_{i-1}(A,\bar{A})/A)$ by (1.1). From (1.8.(iii)) and the first half of this theorem, we see that there exists an element $\beta \in \bar{A}$ of $I_\beta = p$. Hence we have (viii).

(viii) → (ix) : Put $B = \{\beta \in \bar{A} \mid \beta p \subseteq A\}$. We show that B is a ring. Let β be an element of B. Since $\beta \in \bar{A}$, there exists some elements $a_1, \cdots, a_n \in A$ such that $\beta^n + a_1\beta^{n-1} + \cdots + a_n = 0$. Let x be any element of p. Then we have $(x\beta)^n + a_1 x(x\beta)^{n-1} + \cdots + x^n a_n = 0$. Since $x\beta \in A$ and $x \in p$, we have $(x\beta)^n \in p$, so $x\beta \in p$. Hence if $\alpha, \beta \in B$ then $\alpha + \beta, \alpha\beta \in B$, i.e., B is a subring of $\bar{A}$. By the assumption (viii), there exists an element of $\bar{A}$ whose denominator ideal is equal to p. Hence $B \supsetneq A$ and $Ⓒ(B/A) = p$. Thus we have (ix).

(ix) → (x) : This is obvious.

(x) → (vii) : Let B be an intermediate ring between A and $\bar{A}$. Then $Ⓒ(B/A) = A : B$. Since $A : (A : B) \supseteq B$ follows in general, $A : B \subseteq A : (A : (A : B)) \subseteq A : B$, so $A : B = Ⓒ(B/A)$ is divisorial.

(viii) → (xi) : Let α be an element of $\bar{A}$ with $I_\alpha = p$. Then $\alpha \in F_{i-1}(A,\bar{A})$ by the definition. Hence we have $Ⓒ(F_{i-1}(A,\bar{A})/A) \subseteq I_\alpha = p$. Thus $p \in D_i(A,\bar{A})$.

(xi) → (x) : If p is an element of $D_i(A,\bar{A})$, then p is a prime divisor of $Ⓒ(F_{i-1}(A,\bar{A})/A)$ by (1.1). Hence we have (x).

Corollary 1.11. Assume that $\bar{A}$ is a finite A-module. If $D(A,\bar{A}) = D_1(A,\bar{A})$, then A satisfies the S_2-condition.

Proof. For the proof, it suffices to show that if $\operatorname{ht} p > 1$, then $\operatorname{depth} A_p > 1$. Let $p \in \operatorname{Spec} A$ of $\operatorname{ht} p = i > 1$ and assume that $\operatorname{depth} A_p \leqq 1$. If $\operatorname{depth} A_p = 0$, then $\operatorname{ht} p = 0$ for A satisfies the S_1-condition. Hence $\operatorname{depth} A_p = 1$ and $i > 1$.

By (1.10), $p \in D_i(A,\bar{A})$. This contradicts $D(A,\bar{A}) = D_1(A,\bar{A})$.

Corollary 1.12. *If $\bar{A}$ is a finite A-module, then the set of embedded prime divisors of a principal ideal of A is a finite subset of Spec A.*

Proof. Let p be an embedded prime divisor of aA. Since A satisfies the S_1-condition, a is a non-zero divisor. Hence ht p > 1, and $p \in \mathrm{Ass}_A(\bar{A}/A)$ by (1.10). If $\bar{A}$ is a finite A-module, then $\mathrm{Ass}_A(\bar{A}/A)$ is a finite subset of Spec A, hence we are done.

Proposition 1.13. *Let R be a birational-integral extension of A and assume that R is a finite A-module. Then we have $D(A,R) = \mathrm{Ass}_A(R/A)$.*

Proof. Let p be an element of $D_i(A,R)$, $i \geqq 1$. Then p is a prime divisor of $\mathfrak{C}(F_{i-1}(A,R)/A)$, hence there is an element α in $F_{i-1}(A,R) \subseteqq R$ such that $I_\alpha = p$ by (1.8) and (1.10). Therefore we have $p \in \mathrm{Ass}_A(R/A)$.

Conversely, let $p \in \mathrm{Ass}_A(R/A)$ and let ht p = i. Then there is an element α of R whose denominator ideal is equal to p. Since R is a subring of $Q(A)$ and a finite A-module, there exists a non-zero divisor x in $\mathfrak{C}(R/A)$. Hence $I_\alpha = p \supseteqq \mathfrak{C}(R/A) \ni x$. Thus ht p = i $\geqq$ 1. By (1.1) we have $\alpha \in F_{i-1}(A,R)$.

Hence $p = I_\alpha \supseteqq \hat{c}(F_{i-1}(A,R)/A)$. Thus $p \in D_i(A,R) \subseteqq D(A,R)$.

2. A criterion of the seminormality

We recall the definition of the seminormalization and the seminormality in [6] : Let R be a ring containing a ring A and assume that R is integral over A. The seminormalization of A in R is defined by

" $^+_R A = \{\alpha \in R \mid \alpha_p \in A_p + J(R_p)$, for all prime ideal p of $A\}$, where $J(R)$ is the Jacobson radical of R."

If $A = {^+_R A}$, then we say that A is seminormal in R.

Proposition 2.1. Let R/A be an integral extension of domains and $C = Q(A) \cap R$, where $Q(A)$ is the quotient field of A. Then we have

(i) $^+_R A = {^+_C A}$, and

(ii) A is seminormal in R if and only if A is also seminormal in C.

Proof. Let p be a prime ideal of A. We show that $J(C_p) = J(R_p) \cap C_p$. Let M be a maximal ideal of R_p and let $m = M \cap C_p$. Since R_p is integral over C_p, m is a maximal ideal of C_p. Hence we have $J(C_p) \subseteqq J(R_p) \cap C_p$.

Conversely, let m be a maximal ideal of C_p. Since R_p is integral over C_p, there exists a maximal ideal of R_p lying over m, that is $m = M \cap C_p$. Hence $J(C_p) \supseteqq J(R_p) \cap C_p$. Thus we have $J(C_p) = J(R_p) \cap C_p$. Let β be an element of ${}_C^+A$. Then $\beta \in A_p + J(C_p) \subseteqq A_p + J(R_p)$ for all $p \in \mathrm{Spec}\ A$. Therefore we have ${}_C^+A \subseteqq {}_R^+A$.

Since (0) is a prime ideal of A, we have ${}_R^+A \subseteqq A_{(0)} + J(R_{(0)}) = Q(A)$. Hence ${}_R^+A \subseteqq Q(A) \cap R = C$. Let β be an element of ${}_R^+A$ and let p be a prime ideal of A. Then there exists an element $\alpha \in A_p$ such as $\beta - \alpha \in J(R_p)$. Since $\alpha, \beta \in C_p$, $\beta - \alpha \in J(R_p) \cap C_p = J(C_p)$. Therefore we have $\beta \in A_p + J(C_p)$ for all $p \in \mathrm{Spec}\ A$. Thus we have (i).

It is obvious that (ii) follows from (i).

In [1] it is proved that the seminormality is equivalent to the (2,3)-closedness, i.e., if $\alpha \in R$ and $\alpha^2, \alpha^3 \in A$, then $\alpha \in A$. By using our notaion we have another criterion of the seminormality when R is birational integral over A. and a finite A-module.

Theorem 2.2. *Let A be a noetherian domain and let R be a birational-integral extension of A and assume that R is a finite A-module. Then the following are equivalent;*

(i) *A is seminormal in R.*

(ii) *A_p is seminormal in R_p for all $p \in D(A,R)$.*

(iii) *The conductor ideal $\mathfrak{C}(F_i(A,R)/A)$ is a radical ideal in $F_i(A,R)$ for all $i \geqq 0$.*

Proof. (i) → (ii) and (i) → (iii) are proved in [6]. Since $A = R \cap \bigcap_{p \in D(A,R)} A_p$, (ii) implies (i) by the (2,3)-closedness.

We shall show that (iii) implies (i). For this purpose it suffices to show that ${}_R^+A \subseteq F_{i-1}(A,R)$ implies ${}_R^+A \subseteq F_i(A,R)$ for $F_d(A,R) = A$ by (1.1). Assume that ${}_R^+A \subseteq F_{i-1}(A,R)$ and let $p \in D_i(A,R)$ and let α be an element of ${}_R^+A$. Then $\alpha \in A_p + J(R_p)$. Therefore $\alpha = x + \beta$ for some $x \in A_p$ and $\beta \in J(R_p)$. Since R_p is integral over A_p, we have $J(R_p) \cap F_{i-1}(A,R)_p = J(F_{i-1}(A,R)_p)$. Since p is an element of $D_i(A,R)$ and $\mathfrak{c}(F_{i-1}(A,R)/A)$ is a radical ideal in $F_{i-1}(A,R)$, we obtain $J(F_{i-1}(A,R)_p) = pA_p \subseteq A_p$. Therefore β is contained in A_p. Hence we see that ${}_R^+A \subseteq F_i(A,R)$.

Corollary 2.3. Suppose that A satisfies the S_2-condition. Then A is seminormal in R if and only if $\mathfrak{c}(R/A)$ is a radical ideal in R.

In [7] it is proved that

$$F_i(A[[X]],R[[X]]) = F_i(A,R)[[X]],$$
$$F_i(A[X],R[X]) = F_i(A,R)[X],$$
$$\mathfrak{c}(F_i(A[[X]],R[[X]])/A[[X]]) = \mathfrak{c}(F_i(A,R)/A)A[[X]]$$

and $\mathfrak{c}(F_i(A[X],R[X])/A[X]) = \mathfrak{c}(F_i(A,R)/A)A[X]$.

Using this result and (2.2), we have the following;

Proposition 2.4. Let A be a noetherian domain and let R be a birational-integral extension of A and assume that R is a finite A-module. If A is seminormal in R, then $A[X]$ is seminormal in $R[X]$ and $A[[X]]$ is seminormal in $R[[X]]$.

In the following proposition (2.5), we give a useful result to prove that A is seminormal in R.

Proposition 2.5. A is seminormal in R if and only if $F_i(A,R)$ is seminormal in $F_{i-1}(A,R)$ for all i.

Proof. If A is seminormal in R, then $F_i(A,R)$ is seminormal in R for $F_i(A,R) = R \cap \bigcap_{p \in D_j(A,R)} A_p$, $j \leq i$. Hence $F_i(A,R)$ is seminormal in $F_{i-1}(A,R)$. Next, suppose that $F_i(A,R)$ is seminormal in $F_{i-1}(A,R)$ for all i. By (1.1) there exists an integer d such that $F_d(A,R) = A$. From ([6], Lemma 1.2), we see that A is seminormal in $F_0(A,R) = R$.

Theorem 2.6. Let A be a noetherian domain and let R be an intermediate ring between A and $\bar{A}$ and assume that R is a finite A-module. Let B be a ring and étale (unramified and flat) and finite over A. Suppose that $R \otimes_A B$ is a domain and the canonical homomorphisms of rings; $B \longrightarrow R \otimes_A B$ and $R \longrightarrow R \otimes_A B$ are injective, hence we may assume that B and R are subrings of $R \otimes_A B$. Then A is seminormal in R if and only if B is seminormal in $R \otimes_A B$.

Proof. To see the seminormality, we have only to show the (2,3)-closedness. Let $\alpha \in R$ and $\alpha^2, \alpha^3 \in A$. Then $\alpha \in R \otimes_A B$ and $\alpha^2, \alpha^3 \in B$. If B is seminormal in $R \otimes_A B$, then $\alpha \in B$. Since R and

B are linearly disjoint over A, $R \cap B = A$. Hence we see that if B is seminormal in $R \otimes_A B$ then A is seminormal in R.

Suppose that A is seminormal in R. We prove that B is seminormal in $R \otimes_A B$. By (2.2) it is enough to prove this assertion when A is a local ring. Let A be a local ring. Then B is a free A-module. Let $e_1, \cdots, e_n$ be a basis of B over A. For a convenience we denote $R \otimes_A B$ by R_B.

First, we show that $F_i(B, R_B) = F_i(A, R) \otimes_A B$. Let α be an element of $F_i(B, R_B)$ and $I_{\alpha,B} = \{a \in B \mid a\alpha \in B\}$. Then $\operatorname{ht} I_{\alpha,B} \geqq i + 1$. Let $\alpha = \beta_1 e_1 + \cdots + \beta_n e_n$ for $\beta_j \in R$. Since B is flat and integral over A, we have $\operatorname{ht} I_{\alpha,B} \cap A = \operatorname{ht} I_{\alpha,B} \geqq i + 1$ ([4]. Theorem 20. p.81). Let x be an element of $I_{\alpha,B} \cap A$. Then $x\alpha = x\beta_1 e_1 + \cdots + x\beta_n e_n \in Ae_1 \oplus \cdots \oplus Ae_n$. Hence we have $x\beta_j \in A$ for all j, $1 \leqq j \leqq n$. Thus $I_{\alpha,B} \cap A \subseteqq I_{\beta_j}$. Therefore $\operatorname{ht} I_{\beta_j} \geqq i + 1$, so we have $\beta_j \in F_i(A, R)$. Hence $\alpha \in F_i(A, R) \otimes_A B$.

Conversely, let $\alpha = \beta_1 e_1 + \cdots + \beta_n e_n$, for $\beta_1, \cdots, \beta_n \in F_i(A, R)$. Then $\operatorname{ht} I_{\beta_j} \geqq i + 1$, so we have $\operatorname{ht} I_{\beta_1} \cap \cdots \cap I_{\beta_n} \geqq i + 1$. Since B is flat and integral over A, we obtain $\operatorname{ht} (I_{\beta_1} \cap \cdots \cap I_{\beta_n})B \geqq i + 1$ ([4]. Theorem 19. p.79). Since $I_{\alpha,B} \supseteqq (I_{\beta_1} \cap \cdots \cap I_{\beta_n})B$, we have $\operatorname{ht} I_{\alpha,B} \geqq i + 1$. Hence $\alpha \in F_i(B, R_B)$. Thus $F_i(B, R_B) = F_i(A, R) \otimes_A B$.

Since $F_i(B, R_B)$ is a free $F_i(A, R)$-module, we see that $\mathfrak{C}(F_i(B, R_B)/B) = \mathfrak{C}(F_i(A, R)/A) F_i(B, R_B)$. Since A is seminormal in R, $\mathfrak{C}(F_i(A, R)/A)$ is a radical ideal in $F_i(A, R)$. Since B is unramified over A, $F_i(A, R) \otimes_A B$ is unramified over $F_i(A, R)$. Therefore $\mathfrak{C}(F_i(A, R)/A) F_i(B, R_B)$ is a radical ideal in $F_i(B, R_B)$. Hence B is seminormal in R_B by (2.2).

Let A be an affine domain over a field k and let K be a field separably generated over k. Suppose that A and K are linearly disjoint over k. Since K is separably generated over k, there exist some transcendental elements $t_1, \cdots, t_n \in K$ over k such that K is separable algebraic over $k(t_1, \cdots, t_n)$.

Proposotion 2.7. Let A be an affine domain over k and let K be a field separably generated over k and let R be a birational-integral extension of A and a finite A-module. If A is seminormal in R, then $A \otimes_k K$ is seminormal in $R \otimes_k K$.

Proof. Let A be seminormal in R. By (2.4) $A \otimes_k k[t_1, \cdots, t_n] = A[t_1, \cdots, t_n]$ is seminormal in $R \otimes_k k[t_1, \cdots, t_n] = R[t_1, \cdots, t_n]$. Let $S = k[t_1, \cdots, t_n] \setminus \{0\}$. Then S is a multiplicatively closed set and $S^{-1}A[t_1, \cdots, t_n] = A \otimes_k k(t_1, \cdots, t_n)$ and $S^{-1}R[t_1, \cdots, t_n] = R \otimes_k k(t_1, \cdots, t_n)$. Hence $A \otimes_k k(t_1, \cdots, t_n)$ is seminormal in $R \otimes_k k(t_1, \cdots, t_n)$. Since $A \otimes_k K$ is étale finite over $A \otimes_k k(t_1, \cdots, t_n)$, we have that $A \otimes_k K$ is seminormal in $R \otimes_k K$.

In [6], Traverso defined a glueing of R with respect to one prime ideal of A and proved that if A is seminormal in R then A is obtained by a sequence of glueings of R. We shall define a "glueing" of R with respect to a finite subset of Spec A as follows. If the subset consists of only one prime ideal of A, then the definition of Traverso and ours are the same.

Definition 2.8. Let R be an intermediate ring between A and $\bar{A}$ and let $p_1, \cdots, p_t$ be prime ideals of A. Let $P_{i1}, \cdots, P_{ie(i)}$ be all prime ideals of R lying over p_i. Denote the residue field of p_i by $k(p_i)$ and the residue class of an element f with respect to a prime ideal p by $f(p)$. Let

$$G(p_1,\cdots,p_t;A;R) = \{f \in R \mid f(P_{i1}) = \cdots = f(P_{ie(i)}) \in k(p_i) \text{ for all } i\}.$$

Then $G(p_1,\cdots,p_t;A;R)$ is a subring of R which will be called a glueing of R over A with respect to $p_1, \cdots, p_t$.

Lemma 2.9. Let $G = G(p_1,\cdots,p_t;A;R)$. Then $P_{i1} \cap G = \cdots = P_{ie(i)} \cap G = P_i'$, hence P_i' is the only prime ideal of G lying over p_i, and we have $k(p_i) = k(P_i')$.

Proof. Let α be an element of G. Then $\alpha(P_{i1}) = \cdots = \alpha(P_{ie(i)}) \in k(p_i)$. Hence we have $P_{i1} \cap G = \cdots = P_{ie(i)} \cap G$ and $k(P_i') = k(p_i)$.

Using these notations, the result given by Traverso [6] is stated in the following form;

Theorem 2.10. Let A be a noetherian domain and let R be a birational-integral extension of A and assume that R is a finite A-module. Let $\mathrm{Ass}_A(R/A) = \{p_1,\cdots,p_t\}$. Then A is seminormal in R if and only if $A = G(p_1,\cdots,p_t;A;R)$.

Proof. We shall show that if $A = G(p_1, \cdots, p_t; A; R)$ then A is seminormal in R. By [6], the seminormalization is characterized by the following ;

$^{+}_{R}A$ is the largest subring B of R such that :

(i) for any prime ideal p of A, there is exactly one prime ideal p' of B above p.

(ii) the canonical homomorphism $k(p) \to k(p')$ is an isomorphism.

Let α be any element of $^{+}_{R}A$. There is exactly one prime ideal p_i' of $^{+}_{R}A$ above p_i. Therefore $P_{i1}, \cdots, P_{ie(i)}$ are exactly all prime ideal of R above p_i' and $k(p_i) = k(p_i')$. Hence we have $\alpha(P_{i1}) = \cdots = \alpha(P_{ie(i)}) \in k(p_i') = k(p_i)$. Thus $\alpha \in G(p_1, \cdots, p_t; A; R) = A$ by the definition (2.8). Therefore $^{+}_{R}A = A$, this means that A is seminormal in R.

Next we show the converse of this theorem as the following lemma.

Lemma 2.11. Assume that A is seminormal in R and let $\mathrm{Ass}_A(R/A) = \{p_{ij} \mid \mathrm{ht}\, p_{ij} = i,\ 1 \leqq j \leqq n(i)\}$. Let $G_u = G(p_{ij}, i \leqq u; A; R)$ for an integer $u \geqq 0$. Then $G_u = F_u(A,R)$.

Proof. We show this lemma by an induction on u. If $u = 0$, then $G_0 = R = F_0(A,R)$. Suppose that $G_{u-1} = F_{u-1}(A,R)$. Let α $F_u(A,R)$. Since $F_u(A,R) \subseteqq F_{u-1}(A,R)$, $\alpha \in F_{u-1}(A,R) = G_{u-1}$. Let $p = p_{un}$ for an integer n and let $P_1, \cdots, P_v$ be all prime ideals of R lying over p. Since $\alpha \in F_u(A,R)$ and $p = p_{un}$ $D_u(A,R)$, we have $\alpha \in A_p$ by (1.3). Thus there are elements a and b of A such that $\alpha = \frac{b}{a}$ and $a \notin p$. Then $a \notin P_1, \cdots, P_v$ and $\alpha(P_k) = \frac{b(P_k)}{a(P_k)}$. Thus we have $\alpha \in G_u$. Hence $F_u(A,R) \subseteqq G_u$.

Let α be an element of G_u. Since $\alpha \in G_u \subseteqq G_{u-1} = F_{u-1}(A,R)$, it suffices to show that $\alpha \in A_q$ for all $q \in D_u(A,R)$. Let q be an element of $D_u(A,R)$ and let $P_1, \cdots, P_v$ be all prime ideals of R above q. Since $\alpha \in G_u = G(p_{ij}, i \leqq u; A; R)$ and $q = p_{un}$ for some integer n, there is an element $\frac{b}{a}$ of A_q such that $\alpha(P_1) = \cdots = \alpha(P_v) = \frac{b}{a}(qA_q)$. Hence $\alpha - \frac{b}{a} \in (P_1 \cap \cdots \cap P_v)R_q = J(R_q)$. Therefore we have $\alpha \in A_q + J(R_q)$. Let p be a prime ideal of A properly contained in q. Then $\operatorname{ht} p < u$. Since $\alpha \in F_{u-1}(A, R)$, we have $\alpha \in A_p$, thus $\alpha \in {}^{+}_{R_q}A_q$. By the seminormality of A, A_q is also seminormal in R_q. Therefore we have $\alpha \in {}^{+}_{R_q}A_q = A_q$. Hence $G_u = F_u(A,R)$. By (1.1), we have $A = G$.

Remark 2.12. Let $p_1, \cdots, p_t$ be prime ideals of A and $G = G(p_1, \cdots, p_t; A; R)$ and let P_{ij}, $1 \leqq j \leqq e(i)$, be all prime ideals of R above p_i. By (2.9) we see that $P_i = P_{i1} \cap G = \cdots = P_{ie(i)} \cap G$ is the only prime ideal of G above p_i, and $k(p_i) = k(P_i)$. Hence we have $G = G(P_1, \cdots, P_t; G; R)$ and G is seminormal in R.

Proposition 2.13. Let $p_1, \cdots, p_t$ be prime ideals of A and $G = G(p_1, \cdots, p_t; A; R)$. For this ring G, we have

$$D(G,R) = \mathrm{Ass}_G(R/G) \subseteq \{P_1, \cdots, P_t\}, \text{ where } P_i = P_{i1} \cap G.$$

Proof. In the above remark we have $G = G(P_1, \cdots, P_t; G; R)$. We may assume that $G(P_1, \cdots, P_{i-1}, P_{i+1}, \cdots, G; R) \supsetneq G$ for each $i = 1, \cdots, t$. Changing the notation, let $\{P_1, \cdots, P_t\} = \{P_{ij} \mid \text{ht } P_{ij} = i,\ 1 \leqq i \leqq d,\ 1 \leqq j \leqq n(i)\}$ and $G_u = G(P_{ij},\ i \leqq u; G; R)$.

We show that $\{P \in \mathrm{Ass}_G(R/G) \mid \text{ht } P = u\} = \{P_{u1}, \cdots, P_{un(u)}\}$ inductively. Suppose that $\{P \in \mathrm{Ass}_G(R/G) \mid \text{ht } P \leqq u-1\} = \{P_{ij},\ i \leqq u-1\}$. Then we have $G_{u-1} = F_{u-1}(G,R)$ by (2.11). We show that $©(F_{u-1}(G,R)/G) = \bigcap_{i \geqq u} P_{ij}$. Let $x \in \bigcap_{i \geqq u} P_{ij}$ and $f \in F_{u-1}(G,R)$. For a proof of $xf \in G$, let $P \in \{P_{ij}\}$, say, $P = P_{nj}$, $1 \leqq n \leqq d$ and let $Q_1, \cdots, Q_m$ be all prime ideals of R lying over P. First, let $n \leqq u-1$. By our assumption, $P \in \mathrm{Ass}_G(R/G)$ and $\text{ht } P = n \leqq u-1$, hence we have $F_{u-1}(G,R) \subseteq G_P$ by (1.3). Since $xf \in F_{u-1}(G,R) \subseteq G_P$, we have $xf(Q_1) = \cdots = xf(Q_m) \in k(P)$. Next, let $n \geqq u$. Then $x \in \bigcap_{i \geqq u} P_{ij} \subseteq P \subseteq Q_1 \cap \cdots \cap Q_m$. Hence $xf(Q_1) = \cdots = xf(Q_m) = 0$. Therefore we have $xf \in G$, that is $x \in ©(F_{u-1}(G,R)/G)$.

Conversely, let $x \in ©(F_{u-1}(G,R)/G)$ and suppose that $x \notin P_{nj}$ for some integers $n (\geqq u)$ and j. Therefore $x(P_{nj}) \neq 0$.

Case (i) : there is exactly one prime ideal Q of G_{n-1} above P_{nj}. If $k(Q) = k(P_{nj})$, then we can omit P_{nj} from the set $\{P_{ij}\}$, but $G(P_1, \cdots, P_{i-1}, P_{i+1}, \cdots, P_t; G; R) \supsetneq G$. This is a contradiction. Hence there is an element f of G_{u-1} such that $f(Q) \notin k(P_{nj})$. So we have $xf(Q) \notin k(P_{nj})$ for $x(Q) \in k(P_{nj})$ and $x(P_{nj}) \neq 0$. Thus $xf \notin G$, it is contrary to $x \in ©(F_{u-1}(G,R)/G)$.

Case (ii): Let $Q_1, \cdots, Q_m$, $m \geqq 2$, be all prime ideals of G_{n-1} lying over P_{nj}. Then there exists an element f of G_{n-1} such that $f \notin Q_1$ and $f \in Q_2$. Hence we have $xf(Q_1) \neq 0$ and $xf(Q_2) = 0$. This contradicts $xf \in G$.

Therefore we have $\hat{c}(F_{u-1}(G,R)/G) = \bigcap_{i \geqq u} P_{ij}$. By (1.1) and (1.13), we have $\{P \in \mathrm{Ass}_G(R/G) \mid \mathrm{ht}\, P = u\} = \{P_{u1}, \cdots, P_{un(u)}\}$.

3. A cusp type extensions

If ${}^{+}_{R}A = R$, then we call that the extension R/A is a cusp type. From the characterization of the seminormalization, R/A is a cusp type if and only if the canonical map $\mathrm{Spec}\, R \longrightarrow \mathrm{Spec}\, A$ is bijective and the corresponding residue fields are coincident.

The following elementary results are easily seen by the above fact, so we omit the proof.

Proposition 3.1. (i) Let S be a multiplicatively closed set in A. If R/A is a cusp type, then R_S/A_S is also.
(ii) Let R/A is a cusp type and let p be a prime ideal of A. Thus there exists a unique prime ideal P of R lying over p, then the extension $(R/P)/(A/p)$ is a cusp type.
(iii) If B/A and R/B are cusp types, then R/A is also.
(iv) Let B be an intermediate ring between A and R. If R/A is a cusp type, then B/A and R/B are also cusp types.

A criterion of a cusp type is given by the following;

Proposition 3.2. *Let A be a noetherian domain and let R be a birational-integral extension of A and a finite A-module. Let $D(A,R) = \{p_1, \cdots, p_t\}$. Then R/A is a cusp type if and only if there exists a unique prime ideal P_i of R above p_i for all $i = 1, \cdots, t$, and $k(p_i) = k(P_i)$, and $(R/P_i)/(A/p_i)$ is a cusp type.*

Proof. We prove the "if part" by an induction on t. If $t = 0$, then $A = R$ by (1.3). Hence R/A is a cusp type. Let $t > 0$ and $p = p_t$ a minimal element of $D(A,R)$. Put $B = R \cap A_p$. Then $D(A,B) \subseteqq \{p_1, \cdots, p_t\}$ by $B \subseteqq R$ and (1.2). If $p = p_t \in D(A,B)$, then there is an element of B whose denominator ideal is primary for p, but $B \subseteqq A_p$. Thus $D(A,B) \subseteqq \{p_1, \cdots, p_{t-1}\}$ and B satisfies the necessary condition. By the inductive hypothesis, B/A is a cusp type. We show that R/B is a cusp type.

Since $A \subseteqq B = R \cap A_p \subset A_p$, we have $A_p = B_P$, where $P = pA_p \cap B$. Hence $B = R \cap B_P$. We show that $D(B,R) = \{P\}$. Let Q be a prime ideal in $D(B,R)$ and $Q \neq P$. Suppose that $Q \subsetneqq P$ and let $q = Q \cap A$. Then $q \subsetneqq p$ for $A_p = B_P$. Since p is a minimal element of $D(A,R)$, $q \not\supset \mathfrak{C}(R/A)$ by (1.8). Hence we have $B_Q \supseteqq A_q \supseteqq R$. Thus $Q \notin D(B,R)$. Therefore $Q \not\subset P$. Since $Q \in D(B,R)$, there exists an element α of R whose denominator ideal I_α is equal to Q. Then $I_\alpha = Q \not\subset P$, hence we have $\alpha \in B_P \cap R = B$, so $I_\alpha = B$, this is a contradiction. If $B = R$, then $D(A,R) = D(A,B) \subseteqq \{p_1,\cdots,p_{t-1}\}$, this contradicts $p_t \in D(A,R)$. Hence $B \subsetneqq R$, so we have $D(B,R) = \{P\}$.

At last, we show that ${}^+_R B = R$ by using the characterization of the seminormalization. Let Q be a prime ideal of B. Assume that $Q \not\supset P$. Since $D(B,R) = \{P\}$, $\mathfrak{C}(R/B)$ is a primary ideal belonging to P, hence we have $Q \not\supset \mathfrak{C}(R/B)$. Thus $B_Q \supseteqq R$. Therefore there exists a unique prime ideal $\bar{Q}$ of R above Q, and $k(Q) = k(\bar{Q})$. Next assume that $Q \supset P$ and let $\bar{Q}$ be a prime ideal of R above Q. By our hypothesis there exists a unique prime ideal $\bar{P}$ of R above p. Then $\bar{P}$ is a unique prime ideal above P. From (1.9) and $D(B,R) = \{P\}$ we see that $c(R/B)$ is a primary ideal of R belonging to $\bar{P}$. Since $\bar{Q} \supset Q \supset P \supset \mathfrak{C}(R/B)$, we have $\bar{Q} \supset \bar{P}$. By our hypothesis; $(R/\bar{P})/(A/p)$ is a cusp type, $\bar{Q}$ is a unique prime ideal of R above Q and $k(Q) = k(\bar{Q})$. Hence R/B and B/A are cusp types, so R/A is a cusp type by (3.1).

The "only if part" is easily seen by (3.1).

<u>Corollary 3.3.</u> <u>If $D(A,R) = \{p_1,\cdots,p_t\}$ and each residue ring A/p_i is integrally closed in it's quotient field, then we have</u>

$$\underline{{}^+_R A = G(p_1,\cdots,p_t,A,R).}$$

<u>Proof.</u> Let $B = G(p_1,\cdots,p_t,A,R)$. Since B is seminormal in R and contains A, we have ${}^+_R A \subseteqq B$. By (1.2) we have $D(A,B) \subseteqq D(A,R)$. Since B is given by a glueing at p_1, $\cdots$, p_t

over A, there exists a unique prime ideal P_i of B above p_i, for all i, and $k(p_i) = k(P_i)$. Since the residue ring A/p_i is integrally closed in $k(p_i)$, we have $A/p_i = B/P_i$, hence A/p_i is a cusp type in B/P_i. By (3.2) we see that A is a cusp type in B. Therefore we obtain that ${}^{+}_{R}A \supseteqq B$, so we have ${}^{+}_{R}A = G(p_1,\cdots,p_t,A,R)$.

Let R be an integral domain containing a field k. Let P be a prime ideal and Q a primary ideal belonging to P. Then the total quotient ring $Q(R/Q)$ is an Artin local ring, so a complete local ring of equicharacteristic. Hence $Q(R/Q)$ has a coefficient field K satisfying the following commutative diagram;

$$\begin{array}{ccc} R_P/QR_P & \underset{i}{\overset{p_2}{\rightleftarrows}} & K \\ \uparrow i_1 & & \uparrow i_2 \\ R/Q & \xrightarrow{p_1} & R/P \end{array} \quad , \; p_2 \circ i = I_K .$$

Let F be an intermediate ring between K and $Q(R/Q) = R_P/QR_P$, and let $F \neq Q(R/Q)$. Let $A_F = \{f \in R \mid f(Q) \in F\}$, where $f(Q)$ is the residue class of an element f in R/Q. Then we have the following;

Proposition 3.4. *Let R be a noetherian domain containing a field and let P be a prime ideal of R and let Q be a primary ideal belonging to P. Let K be a coefficient field of $Q(R/Q)$ and F a proper subring of $Q(R/Q)$ such that $F \supsetneqq K$. Let $A_F = \{f \in R \mid f(Q) \in F\}$. If R/Q is integral over $(R/Q) \cap K$, then the following holds;*

(i) <u>R is integral over A_K, hence over A_F</u>,

(ii) <u>$D(A_F, R) = \{P \cap A_F\}$, and</u>

(iii) <u>A_F is a cusp type in R.</u>

Proof. First, we prove (i): By the definition of A_K, we have $A_K \supseteq Q$ and so we easily see that $(R/Q) \cap K = A_K/Q$. Since R/Q is integral over $(R/Q) \cap K$, R is integral over A_K. Hence R is integral over A_F.

Next, we prove (ii). Denote A_F by A and $P \cap A_F$ by p. By the definition of A_F, we have $©(R/A) \supseteq Q$ and Q is a primary ideal belonging to p. Let p' be a prime ideal in $D(A,R)$. By (1.1) we have $p' \supseteq ©(R/A)$, so $p' \supseteq p$. Suppose that $p' \supsetneqq p$. Since $p' \in D(A,R)$, there exists an element α of R whose denominator ideal I_α is equal to p'. Since $p' \supsetneqq p$, there exists an element x of p' not contained in p. Then $x(Q)$ is an invertible element of F. On the other hand $x\alpha \in A$ implies $x(Q)\alpha(Q) \in F$. Thus we have $\alpha(Q) \in F$. By the definition, $\alpha \in A$, this contradicts $I_\alpha \neq A$. From the hypothesis $F \neq Q(R/Q)$, $R \neq A$. Hence we obtain that $D(A_F, R) = \{P \cap A_F\}$.

We prove (iii). Applying (3.1) it suffices to show that A_K is a cusp type in R. Since K is a field and $(R/Q) \cap K = A_K/Q$, Q is a prime ideal of A_K. Therefore $P \cap A_K = Q$, hence we have $D(A_K, R) = \{Q\}$, and $Q(A_K/Q) = Q(R/P)$. Let P' be a prime ideal of R above Q. Then $P' \supseteq Q$ and Q is a primary ideal belonging to P, thus $P' \supseteq P$. Since R is integral over

A_K and $P \cap A_K = P' \cap A_K$, we have $P = P'$. Hence P is a unique prime ideal of R lying over Q. Let f be an element of R. Then we have $i \cdot i_2(f(P)) = i \cdot i_2 \cdot p_1(f(Q)) = i_1(f(Q)) \in i_1(R/Q) \cap i(K)$. Since $(R/Q) \cap K = A_K/Q$, we obtain $f(P) \in A_K/Q$. Hence $A_K/Q = R/P$. Applying (3.2), we conclude that R/A_K is a cusp type, so R/A_F is also.

4. Good sections

In this section assume that A is a homomorphic image of a regular ring and satisfies the S_1-condition. In [2], for such ring A, it is proved that $\Delta = \{p \in \operatorname{Spec} A \mid \operatorname{ht} p > 1$ and $\operatorname{depth} A_p = 1\}$ is a finite set. Let x be a non-unit of A and let R be an intermediate ring between A and $\bar{A}$ and assume that R is a finite A-module. If $x \notin q$ for all $q \in \operatorname{Ass}_A(R/A) \cup \Delta$, then we call the element x a good section of R/A. If $\operatorname{depth} A_p > 1$, then there is a good section of R_p/A_p.

Proposition 4.1. Let x be a good section of R/A. If p is a prime divisor of xA, then we have $\operatorname{ht} p = 1$. Hence the principal ideal xA has no embedded prime divisors.

Proof. Suppose that p is a prime divisor of xA and $\operatorname{ht} p > 1$. By (1.10) we have $\operatorname{depth} A_p = 1$, i.e., $p \in \Delta$. But, since x is a good section, $x \notin p$. This is a contradiction.

Proposition 4.2. Let x be a good section of R/A. Then $xR \cap A = xA$.

Proof. Let x be a good section of R/A and let $xA = q_1 \cap \cdots \cap q_t$ be an irredundant primary decomposition and $\sqrt{q_i} = p_i$. Let $y = x\alpha \in xR \cap A$ for some $\alpha \in R$. By (4.1), ht $p_i = 1$ for all i. Hence, if $p_i \supseteqq ©(R/A)$, then p_i is a prime divisor of $©(R/A)$, so $p_i \in D(A,R) = \mathrm{Ass}_A(R/A)$ by (1.8) and (1.13). But x is a good section of R/A, so we have $x \notin q$ for all $q \in \mathrm{Ass}_A(R/A)$. Thus we have $p_i \not\supseteq ©(R/A)$, that is $A_{p_i} \supseteqq R$ for all i. Put $P_i = p_i A_{p_i} \cap R$. Then $A_{p_i} = R_{P_i}$. Hence we have $y \in xR_{P_i} = xA_{p_i}$. Therefore there exists an element a_i of A not contained in p_i and $a_i y \in xA$ for each i. Then $a_i y \in q_i$ and $a_i \notin p_i$. Since q_i is primary for p_i, we have $y \in q_i$. Thus $y \in xA$, i.e., $xR \cap A = xA$.

Put $Q_i = xR_{P_i} \cap R$ and $J = Q_1 \cap \cdots \cap Q_t$. Then Q_i is a primary ideal belonging to P_i and $Q_i \cap A = q_i$. Hence $J \cap A = xA$, i.e., A/xA is a subring of R/J. Since ht $P_i = 1$ for all i, we have $Q(R/J) = R_{P_1}/Q_1R_{P_1} \times \cdots \times R_{P_t}/Q_tR_{P_t} = R_{P_1}/xR_{P_1} \times \cdots \times R_{P_t}/xR_{P_t} = A_{p_1}/xA_{p_1} \times \cdots \times A_{p_t}/xA_{p_t} = Q(A/xA)$. Therefore R/J is birational integral over A/xA.

Corollary 4.3. <u>Assume that $\bar{A}$ is a finite A-module. Let x be a non-unit and non-zero divisor of A. Then x is a good section of $\bar{A}/A$ if and only if $x\bar{A} \cap A = xA$.</u>

Proof. It is already proved that if x is a good section of $\bar{A}/A$ then we have $x\bar{A} \cap A = xA$.

Conversely, let $x\bar{A} \cap A = xA$. By (1.10), we have $\Delta \subseteqq \mathrm{Ass}_A(\bar{A}/A)$. Hence it suffices to prove that $x \notin q$ for all $q \in \mathrm{Ass}_A(\bar{A}/A)$. Let q be an element of $\mathrm{Ass}_A(\bar{A}/A)$ and suppose that $x \in q$. Since $q \in \mathrm{Ass}_A(\bar{A}/A)$, there is an element $\alpha \in \bar{A}$ with $I_\alpha = q$. Let $y = x\alpha$. Then y is an element of $x\bar{A} \cap A = xA$. Since x is a non-zero divisor of A, we have $\alpha \in A$. This contradicts $I_\alpha = q$.

Theorem 4.4. Let A be a homomorphic image of a regular ring and satisfy the S_1-condition and let R be a birational-integral extension of A and assume that R is a finite A-module. Then, for any $P \in \mathrm{Spec}\, R$, it holds that

$$\mathrm{depth}\, A_{A\cap P} \leqq \mathrm{ht}\, P.$$

Proof. Let $p = P \cap A$. For this proof, we may assume that A is a local ring and p is the only maximal ideal of A. If $\mathrm{ht}\, P = 0$, then we have $\mathrm{ht}\, p = 0$ for R is a subring of $Q(A)$, so $\mathrm{depth}\, A = 0 \leqq \mathrm{ht}\, P$. Assume that $\mathrm{ht}\, P > 0$. If $\mathrm{depth}\, A = 1$, then $1 = \mathrm{depth}\, A \leqq \mathrm{ht}\, P$. Suppose that $\mathrm{depth}\, A > 1$. Hence there exists a good section x of R/A. Then A/xA is a homomorphic image of a regular ring and satisfies the S_1-condition. Let $xA = q_1 \cap \cdots \cap q_t$ be an irredundant primary decomposition and $\sqrt{q_i} = p_i$. Then $A_{p_i} \supseteqq R$. Put $P_i = p_i A_{p_i} \cap R$ and $Q_i = xR_{p_i} \cap R$ and $J = Q_1 \cap \cdots \cap Q_t$. Then R/J is birational integral over A/xA and a finite A/xA-module, and we have $(P/J) \cap (A/xA) = p/xA$.

Applying the induction on depth A, we have $\mathrm{depth}\, A/xA = \mathrm{depth}\, A - 1 \leqq \mathrm{ht}\, P/J \leqq \mathrm{ht}\, P - 1$. Therefore we have $\mathrm{depth}\, A \leqq \mathrm{ht}\, P$.

§5. Flat over-rings of an integral domain

Let A be an integral domain and let K be the quotient field of A. Hereafter, we are mainly concerned with a subring B of K containing A. We call such an intermediate ring an over-ring of A. The purpose of this section is to study the relationship between an over-ring B and subsets $F_A(B)$ (resp. $F_A^*(B)$) of Spec A defined by

$$F_A(B) = \{p \in \text{Spec } A \mid A_p \not\supset B\} \quad \text{and}$$

$$F_A^*(B) = \{p \in F_A(B) \mid \text{depth } A_p = 1\}$$

respectively.

In the paper [8], we studied the relationship between flat over-rings over a Krull domain and prime ideals of height one. Replacing a Krull domain by a noetherian domain and prime ideals of height one by those of depth one, we develop a similar study as is done in [8].

Lemma. 5.1. *Any minimal element of $F_A(B)$ has depth 1. Hence the subset of $F_A(B)$ consisting all minimal elements of $F_A(B)$ is contained in $F_A^*(B)$.*

Proof. Let p be a minimal element of $F_A(B)$ and suppose that depth $A_p > 1$. Then $A_p = \cap A_q$, q runs all prime ideals properly contained in p and depth $A_q = 1$ (cf. [3]. Theorem 123, p90). Since p is a minimal element of $F_A(B)$, we have $q \notin F_A(B)$ when q is properly contained in p. Therefore we have $A_p = \cap A_q \supseteq B$, it contradicts $p \in F_A(B)$.

For a convenience we write down the following well-known fact without proof (cf. [5]).

Lemma 5.2. *Let A be a ring and B an A-algebra contained in the total quotient ring of A. Then the following four conditions are equivalent to each other;*

(i) *B is flat over A.*

(ii) *$B_p = B \otimes_A A_p$ is flat over A_p for any $p \in \text{Spec } A$.*

(iii) *$A_{A \cap P} = B_P$ for any $P \in \text{Spec } B$.*

(iv) *For every $p \in \text{Spec } A$, either $pB = B$ or $A_p = B_p$.*

Using this result we have the following;

Theorem 5.3. *Let B be an over-ring of A. Then B is flat over A if and only if either $B_p = A_p$ or $pB = B$ holds for any prime ideal p of A with depth $A_p = 1$.*

Proof. From (5.2), it suffices to prove the "if part" of this theorem. If q is a prime ideal of A not contained in $F_A(B)$, then we have $A_q = B_q$ by the definition of $F_A(B)$. Hence to prove the theorem it is sufficient to show that for any $q \in F_A(B)$ we see $qB = B$. Let q be an element of $F_A(B)$. By (5.1) there exists an element p of $F_A{}^*(B)$ with $q \supseteqq p$. Since $A_p \neq B_p$, we have $pB = B$ by the assumption. Hence we conclude that $qB = B$.

Remark 5.4. If $p \notin F_A(B)$, then we have $A_p = B_p$. Hence we see that B is flat over A if and only if $pB = B$ for all $p \in F_A{}^*(B)$.

Prposition 5.5. Let A be a noetherian domain and let B be an over-ring of A. If B is flat over A, then $\bar{B}$ is also flat over $\bar{A}$.

Proof. Let P be a prime ideal of $\bar{A}$ and let $p = P \cap A$. If $p \notin F_A(B)$, then we have $A_p \supseteqq B$, hence $\overline{(A_p)} \supseteqq \bar{B}$, so we have $\bar{A}_p = \bar{B}_p$. If $p \in F_A(B)$, then we have $pB = B$, hence $p\bar{B} = P\bar{B} = B$. Therefore $\bar{B}$ is flat over $\bar{A}$.

In the following we see that if B is flat over A then B is uniquely determined by the set $F_A{}^*(B)$.

Proposition 5.6. Let B be an over-ring of A. If B is flat over A, then $B = \bigcap_{p \in \Delta} A_p$, where $\Delta = \{p \in \operatorname{Spec} A \mid \operatorname{depth} A_p = 1$ and $p \notin F_A{}^*(B)\}$.

Proof. Put $R = \bigcap_{p \in \Delta} A_p$. If $p \in \Delta$, then $p \notin F_A(B)$. Hence we have $A_p \supseteqq B$. Thus we have $R \supseteqq B$ always. It is well known that $B = \bigcap B_M$, M runs all maximal ideals of B. Let M be a maximal ideal of B and let $m = M \cap A$. Then we have $B_M = A_m$ by (5.2). Hence $m \notin F_A(B)$. By ([8], 1.2) the set $F_A(B)$ is closed under the specializations, therefore all prime ideals of depth one contained in m are elements of Δ. Thus we see that $B_M = A_m \supseteqq R$ by ([3]. Theorem 123. p90). Hence we have $B \supseteqq R$. Thus $B = R = \bigcap_{p \in \Delta} A_p$.

Let a be a non-zero element of A. Then it is easily seen that $F_A^*(A[\frac{1}{a}]) = \{p \in \text{Spec } A \mid p$ is a prime divisor of $aA\}$. Let B be a finitely generated over-ring of A. Then there exists a non-zero element a of A such that B is contained in $A[\frac{1}{a}]$. Applying ([8], Lemma 1.4), we have $F_A^*(B) \subseteqq F_A^*(A[\frac{1}{a}])$. Hence the set $F_A^*(B)$ is a finite set. But the converse assertion does not hold in general. Here is a counterexample ; let k be a field and let x, y be indeterminants over k. Let $A = k[x, y]$ and $B = A[\frac{y}{x}, \frac{y}{x^2}, \cdots]$. Then $F_A^*(B) = \{xA\}$, but B is not finitely generated over A. Under an additional assumption, the following holds;

Theorem. 5.7. Let A be a noetherian domain and let B be an over-ring of A. If B is finitely generated over A, then $F_A^*(B)$ is a finite set. If we impose an additional assumption that B is flat over A, then the converse also holds.

Proof. The first half is already proved. Let B be flat over A and $F_A^*(B) = \{p_1, \cdots, p_t\}$. By the definition of $F_A^*(B)$, we have $A_{p_i} \neq B_{p_i}$. Apllying (5.4), we obtain $(p_1 \cap \cdots \cap p_t)B = B$. Therefore there exist some elements $a_1, \cdots, a_u \in p_1 \cap \cdots \cap p_t$ and $\alpha_1, \cdots, \alpha_u \in B$ such that $\Sigma a_i \alpha_i = 1$. Let $C = A[\alpha_1, \cdots, \alpha_u]$. Then we have $p_i C = C$, $1 \leqq i \leqq t$. Hence $F_A^*(B) \subseteqq F_A^*(C)$. On the other hand we have $F_A^*(B) \supseteqq F_A^*(C)$ for $C \subseteqq B$. Therefore $F_A^*(B) = F_A^*(C)$. Hence we see that B and C are flat over A. Since $F_A^*(B) = F_A^*(C)$, we have $B = C$ by (5.6). Hence B is finitely generated over A.

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