複素射影空間の全実な平行部分多様体

(Totally real parallel submanifolds in $P^n(\mathbb{C})$)

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Totally real parallel submanifolds in $P^n(\mathbf{C})$

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Introduction. In the study of submanifolds in symmetric spaces, parallel submanifolds often play an important role. For example, in the study of minimal submanifolds in the Euclidean sphere the symmetric R-spaces, which are parallel submanifolds, provide abundant examples for testing various conjectures. Hence it seems to be useful to classify the parallel submanifolds in a specific symmetric space. Actually, these submanifolds have been classified by D.Ferus [5],[6],[7] when the ambient space is the Euclidean space or the Euclidean sphere, and by M.Takeuchi [17] when the ambient space is the real hyperbolic space. Moreover H.Nakagawa and R.Takagi [10] and M.Takeuchi [16] have classified the parallel Kähler submanifolds in the complex projective space with constant holomorphic sectional curvatures.

In this paper we study n-dimensional complete totally real parallel submanifolds in the n-dimensional complex projective space $P^n(\mathbf{C})$ with constant holomorphic sectional curvatures. It is known that a riemannian manifold which admits a parallel isometric immersion into a symmetric space is a locally symmetric space. Fix an n-dimensional simply connected symmetric space M^n. Let $\mathcal{T}_M$ (resp. $\mathcal{S}_M$) be the set of all equivalence classes of totally real parallel isometric immersions of M^n into $P^n(\mathbf{C})$ (resp. of complete totally real parallel submanifolds in $P^n(\mathbf{C})$ with the universal riemannian cover-

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ing M^n). Moreover, in section 3 we define an equivalence relation among symmetric trilinear forms on a tangent space of M satisfying certain conditions, and denote by $\bar{\mathcal{H}}_M$ the set of all equivalence classes of these trilinear forms. In sections 2,3, we shall show that there are the natural correspondences among these sets $\bar{\mathcal{T}}_M$, $\bar{\mathcal{S}}_M$, $\bar{\mathcal{H}}_M$. In section 4,5, we shall determine the set $\bar{\mathcal{H}}_M$ for a symmetric space M without Euclidean factor. Moreover, in section 6, we shall study the set $\bar{\mathcal{H}}_M$ for a symmetric space M with Euclidean factor and an important example in the geometry of totally real surfaces in $P^2(\mathbf{C})$.

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1. Preliminaries

Let $\bar{M}^m$ (resp. M^n) be an m-dimensional (resp. n-dimensional) connected riemannian manifold. Denote by $\bar{\nabla}$ (resp. ∇) the riemannian connection on $\bar{M}^m$ (resp. M^n) and by $\bar{R}$ (resp. R) the riemannian curvature tensor for $\bar{\nabla}$ (resp. ∇). Now let f be an isometric immersion of M^n into $\bar{M}^m$. We denote by the same notation $\langle\ ,\ \rangle$ the riemannian metrics on the both riemannian manifolds. Moreover denote by σ_f the second fundamental form of M^n, by D the normal connection on the normal bundle $N(M)$ of M^n and by $R^{\perp}$ the curvature tensor for D. For a point p in M and a vector ζ in the normal space $N_p(M)$ at p, the shape operator A_ζ is defined by

$$\langle A_\zeta(X), Y\rangle = \langle \sigma_f(X,Y), \zeta\rangle$$

for all vectors $X, Y \in T_p(M)$. The shape operator A_ζ is a symmetric endomorphism on the tangent space $T_p(M)$ at p. It is also characterized by the equation that

$$\overline{\nabla}_X \zeta = - A_\zeta(X) + D_X \zeta$$

for any tangent vector field X of M and any normal vector field ζ of M.

Now we recall the following fundamental equations, called the equations of Gauss, Codazzi-Mainardi, and Ricci respectively.

$$\text{(1.1)} \quad \langle \overline{R}(X,Y)Z, W\rangle = \langle R(X,Y)Z, W\rangle + \langle \sigma_f(X,Z), \sigma_f(Y,W)\rangle - \langle \sigma_f(X,W), \sigma_f(Y,Z)\rangle$$

$$\text{(1.2)} \quad \{\overline{R}(X,Y)Z\}^\perp = (\nabla^*_X \sigma_f)(Y,Z) - (\nabla^*_Y \sigma_f)(X,Z)$$

$$\text{(1.3)} \quad \langle \overline{R}(X,Y)\zeta, \eta\rangle = \langle R^\perp(X,Y)\zeta, \eta\rangle - \langle [A_\zeta, A_\eta](X), Y\rangle$$

for all vectors $X, Y, Z, W \in T_p(M)$ and all vectors $\zeta, \eta \in N_p(M)$. Here we denote by $\{*\}^\perp$ the normal component of $*$ and by ∇^* the covariant derivation associated to the isometric immersion $f : M \to \overline{M}$, defined by

$$(\nabla^*_X \sigma_f)(Y,Z) = D_X(\sigma_f(Y,Z)) - \sigma_f(\nabla_X Y, Z) - \sigma_f(Y, \nabla_X Z)$$

for tangent vector fields X,Y,Z of M. The second fundamental form σ_f as well as the isometric immersion f is said to be *parallel* if $\nabla^*\sigma_f = 0$. Moreover when f is an imbedding, the submanifold $f(M)$ is called a *parallel submanifold* in $\bar{M}$. If the second fundamental form σ_f is parallel, we have

$$(1.4) \quad D_X(\sigma_f(Y,Z)) = \sigma_f(\nabla_X Y,Z) + \sigma_f(Y,\nabla_X Z)$$

for all tangent vector fields X,Y,Z of M.

Now let $\bar{M}^{2r} = P^r(c)$ be the r-dimensional complex projective space with constant holomorphic sectional curvatures $c\ (>0)$. The complex structure of $P^r(c)$ will be denoted by J. An isometric immersion $f: M^n \to P^r(c)$ is called *totally real* if $JT_p(M) \subset N_p(M)$ for every point p in M. Moreover when f is an imbedding, the submanifold $f(M)$ is called a *totally real submanifold* in $P^r(c)$. Then we have the following

Lemma 1.1 (cf. See [11]). Let f be a totally real isometric immersion of M^n into $P^r(c)$. Then

$$\langle\sigma_f(X,Y),JZ\rangle = \langle\sigma_f(X,Z),JY\rangle$$

for any point $p \in M$ and all vectors $X,Y,Z \in T_p(M)$.

From now on we assume that the complex dimension r equals n. For a totally real isometric immersion $f: M^n \to P^n(c)$ we define the *associated tensor* $\tilde{\sigma}_f$ of M as follows:

$$\tilde{\sigma}_f(X,Y) = J\sigma_f(X,Y)$$

for vectors $X,Y \in T_p(M)$, $p \in M$. If we identify the tangent space $T_p(M)$ with the cotangent space $T_p^*(M)$ through the riemannian metric on M, the associated tensor $\tilde{\sigma}_f$ is a symmetric covariant tensor of degree 3 on M by Lemma 1.1. For a vector X in $T_p(M)$, we define a symmetric endomorphism $\tilde{\sigma}_f(X)$ of $T_p(M)$ by

$$\tilde{\sigma}_f(X)(Y) = \tilde{\sigma}_f(X,Y)$$

for a vector Y in $T_p(M)$. Since the isometric immersion f is totally real in $P^n(c)$, we have $\bar{R}(X,Y)Z \in T_p(M)$ for all vectors $X,Y,Z \in T_p(M)$ and hence the equation of Gauss reduces to

$$(1.5) \quad \bar{R}(X,Y)Z = R(X,Y)Z - [\tilde{\sigma}_f(X),\tilde{\sigma}_f(Y)](Z)$$

for all vectors $X,Y,Z \in T_p(M)$. Moreover we have the following

Lemma 1.2. <u>Let</u> f <u>be a totally real parallel isometric immersion of</u> M^n <u>into</u> $P^n(c)$. <u>Then</u> $\nabla\tilde{\sigma}_f = 0$, <u>that is,</u>

$$\nabla_X(\tilde{\sigma}_f(Y,Z)) = \tilde{\sigma}_f(\nabla_X Y,Z) + \tilde{\sigma}_f(Y,\nabla_X Z)$$

<u>for all tangent vector fields</u> X,Y,Z <u>of</u> M.

Proof. Since $J\zeta$ is a tangent vector field of M for any normal vector field ζ along M,

$$J\bar{\nabla}_X J\zeta = J\nabla_X J\zeta + J\sigma_f(X, J\zeta)$$

for every tangent vector field X of M, while

$$J\bar{\nabla}_X J\zeta = -\bar{\nabla}_X\zeta = A_\zeta(X) - D_X\zeta$$

since $J\circ\bar{\nabla}_X = \bar{\nabla}_X\circ J$. Hence, comparing normal components we get

$$JD_X\zeta = \nabla_X J\zeta .$$

Thus, substituting $\zeta = \sigma_f(Y,Z)$, together with (1.4) we have

$$\nabla_X(\tilde{\sigma}_f(Y,Z)) = \tilde{\sigma}_f(\nabla_X Y, Z) + \tilde{\sigma}_f(Y, \nabla_X Z)$$

for all tangent vector fields X, Y, Z of M.

q.e.d.

Let $\mathfrak{so}(T_p(M))$ be the Lie algebra of all skew symmetric endomorphisms of $T_p(M)$ and $\mathfrak{k}(p)$ the Lie subalgebra in $\mathfrak{so}(T_p(M))$ generated by the set $\{ R_p(X,Y) ; X,Y \in T_p(M) \}$. Since the isometric immersion f is parallel, the manifold M is a locally symmetric space and hence the Lie algebra $\mathfrak{k}(p)$ is spanned by the set $\{ R_p(X,Y) ; X,Y \in T_p(M) \}$ and coincides with the holonomy algebra of M at p. Thus, by Lemma 1.2, we have the following

Corollary 1.3. <u>Let</u> f <u>be a totally real parallel isometric immersion of</u> M^n <u>into</u> $P^n(c)$. <u>Then</u> $\mathfrak{k}(p)\cdot\tilde{\sigma}_f = 0$, <u>that is</u>,

$$T(\tilde{\sigma}_f(X,Y)) = \tilde{\sigma}_f(T(X),Y) + \tilde{\sigma}_f(X,T(Y))$$

<u>for any endomorphism</u> $T \in \mathfrak{k}(p)$ <u>and all vectors</u> $X,Y \in T_p(M)$.

2. Equivariant immersions associated to trilinear forms

Assume that the manifold M^n is a simply connected symmetric space and fix a point o in M^n. Put

$$\mathfrak{p} = T_o(M)\ ,\quad \mathfrak{k} = \mathfrak{k}(o) \quad \text{and} \quad \mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

and define the bracket product [,] on $\mathfrak{g}$ as follows:

$$[T,S] = T\circ S - S\circ T\ ,\quad [T,X] = -[X,T] = T(X)\ ,$$

$$[X,Y] = -R_o(X,Y)$$

for endomorphisms T,S in $\mathfrak{k}$ and vectors X,Y in $\mathfrak{p}$. Then $(\mathfrak{g},[\ ,\ ])$ is a Lie algebra over $\mathbb{R}$ and there exists a simply connected Lie group G acting on the symmetric space M isometrically and transitively, such that the Lie algebra of G is isomorphic to $\mathfrak{g}$ and that the Lie subgroup $K = \{ g \in G\ ;\ g(o) = o \}$ is connected and has the Lie subalgebra $\mathfrak{k}$ (cf. See [8]). Let $\mathcal{U}_M$ be the set of all $\mathfrak{p}$-valued bilinear form $\tilde{\sigma}$ on $\mathfrak{p}$ satisfying the following conditions:

(1) $\tilde{\sigma}$ is a symmetric trilinear form on $\mathfrak{p}$ under the canonical identification of $\mathfrak{p}^* \otimes \mathfrak{p}^* \otimes \mathfrak{p}$ with $\mathfrak{p}^* \otimes \mathfrak{p}^* \otimes \mathfrak{p}^*$ through the riemannian metric $\langle\ ,\ \rangle$ on $\mathfrak{p}$,

(2) $\mathfrak{k} \cdot \tilde{\sigma} = 0$,

(3) $(c/4)(\langle Y,Z\rangle X - \langle X,Z\rangle Y) = R(X,Y)Z - [\tilde{\sigma}(X),\tilde{\sigma}(Y)](Z)$
for all vectors $X,Y,Z \in \mathfrak{p}$.

Let f be a totally real parallel isometric immersion of M^n into $P^n(c)$. Then

$$\bar{R}(X,Y)Z = (c/4)(\langle Y,Z\rangle X - \langle X,Z\rangle Y)$$

for all vectors $X,Y,Z \in \mathfrak{p}$. Hence we have that $(\tilde{\sigma}_f)_o \in \mathcal{M}_M$ by Lemma 1.1, Corollary 1.3 and (1.5).

Now the riemannian manifold $P^n(c)$ is also a simply connected symmetric space. For $P^n(c)$, we use notations $\bar{o}$, $\bar{\mathfrak{p}}$, $\bar{\mathfrak{k}}$, $\bar{\mathfrak{g}}$, $\bar{G}$, $\bar{K}$ for corresponding objects o, $\mathfrak{p}$, $\mathfrak{k}$, $\mathfrak{g}$, G, K. Note that $\bar{G}$ (resp. $\bar{\mathfrak{g}}$) is isomorphic to the compact Lie group $SU(n+1)$ (resp. the compact Lie algebra $\mathfrak{su}(n+1)$) and that $\bar{\mathfrak{k}}$ is given by

$$\bar{\mathfrak{k}} = \mathfrak{u}(\bar{\mathfrak{p}}) = \{ T \in \mathfrak{so}(\bar{\mathfrak{p}}) ;\ J \circ T = T \circ J \}.$$

A linear subspace $\mathfrak{q}$ in $\bar{\mathfrak{p}}$ is called *totally real* if the subspaces $\mathfrak{q}$ and $J\mathfrak{q}$ are orthogonal. Totally real subspaces in $\bar{\mathfrak{p}}$ of the same dimension are conjugate to each other under the natural action of $\bar{K}$ on $\bar{\mathfrak{p}}$. Fix an n-dimensional totally real subspace $\mathfrak{q}$ in $\bar{\mathfrak{p}}$

and set

$$\bar{\mathfrak{k}}_1 = \{\ T \in \bar{\mathfrak{k}}\ ;\ T(\mathfrak{q}) \subset \mathfrak{q}\ \} \quad \text{and} \quad \bar{\mathfrak{k}}_2 = \{\ T \in \bar{\mathfrak{k}}\ ;\ T(\mathfrak{q}) \subset J\mathfrak{q}\ \}.$$

Then $\bar{\mathfrak{k}}_1$ (resp. $\bar{\mathfrak{k}}_2$) is a Lie subalgebra (resp. linear subspace) in $\bar{\mathfrak{k}}$, and $\bar{\mathfrak{k}}$ is the direct sum of $\bar{\mathfrak{k}}_1$ and $\bar{\mathfrak{k}}_2$. In fact, take an orthonormal basis $\{e_1,\cdots,e_n\}$ of $\mathfrak{q}$ and identify $\bar{\mathfrak{p}}$ with $\mathbb{C}^n$ by the correspondence:

$$\bar{\mathfrak{p}} \ni (\Sigma x_j e_j) + J(\Sigma y_j e_j) \longleftrightarrow (x_j + \sqrt{-1}\,y_j) \in \mathbb{C}^n.$$

Then $\bar{\mathfrak{k}}$, $\bar{\mathfrak{k}}_1$ and $\bar{\mathfrak{k}}_2$ are identified with the Lie algebra $\mathfrak{u}(n)$ of all skew hermitian matrices of degree n, the Lie algebra $\mathfrak{so}(n)$ of all real skew symmetric matrices of degree n, and the linear space $\sqrt{-1}\,S^n(\mathbb{R}) = \{\ \sqrt{-1}A\ ;\ A$ is a real symmetric matrix of degree n $\}$ respectively. This implies the assertion.

Let s be an Euclidean isometry of $\mathfrak{p}$ onto $\mathfrak{q}$. We define an injective Lie homomorphism τ_s of $\mathfrak{so}(\mathfrak{p})$ into $\bar{\mathfrak{k}}_1$ by

$$\tau_s(T)(s(X) + Js(Y)) = s(T(X)) + Js(T(Y))$$

for $T \in \mathfrak{so}(\mathfrak{p})$ and vectors $X, Y \in \mathfrak{p}$. Next, for an element $\tilde{\sigma}$ in $\mathfrak{m}_M$, we define a linear mapping $\mu_{s,\tilde{\sigma}}$ of $\mathfrak{p}$ into $\bar{\mathfrak{k}}_2$ by

$$\mu_{s,\tilde{\sigma}}(X)(s(Y) + Js(Z)) = s(\tilde{\sigma}(X,Z)) - Js(\tilde{\sigma}(X,Y))$$

for vectors $X,Y,Z \in \mathfrak{p}$. Here note that the condition (1) for $\tilde{\sigma}$ implies that $\mu_{s,\tilde{\sigma}}(X) \in \bar{\mathfrak{k}}$. Now we define a linear mapping $\rho_{s,\tilde{\sigma}}$ of $\mathfrak{g}$ into $\bar{\mathfrak{g}}$ by

$$\rho_{s,\tilde{\sigma}}(T+X) = \tau_s(T) + \mu_{s,\tilde{\sigma}}(X) + s(X)$$

for $T \in \mathfrak{k}$ and $X \in \mathfrak{p}$. Then we have the following

Lemma 2.1. _The linear mapping_ $\rho_{s,\tilde{\sigma}}$ _of_ $\mathfrak{g}$ _into_ $\bar{\mathfrak{g}}$ _is an injective Lie homomorphism._

Proof. At first we shall prove the following three formulas:

$$(2.1) \quad [\tau_s(T), \mu_{s,\tilde{\sigma}}(X)] = \mu_{s,\tilde{\sigma}}(T(X))$$

$$(2.2) \quad [\mu_{s,\tilde{\sigma}}(X), \mu_{s,\tilde{\sigma}}(Y)] = \tau_s([\tilde{\sigma}(Y), \tilde{\sigma}(X)])$$

$$(2.3) \quad \bar{R}(s(X), s(Y)) = \tau_s(R(X,Y) - [\tilde{\sigma}(X), \tilde{\sigma}(Y)])$$

for any $T \in \mathfrak{k}$ and all vectors $X,Y \in \mathfrak{p}$. By the condition (2) for $\tilde{\sigma}$ we have

$$\begin{aligned}
&[\tau_s(T), \mu_{s,\tilde{\sigma}}(X)](s(Y) + Js(Z)) \\
&= s(T(\tilde{\sigma}(X,Z))) - Js(T(\tilde{\sigma}(X,Y))) + Js(\tilde{\sigma}(X,T(Y))) - s(\tilde{\sigma}(X,T(Z))) \\
&= s(\tilde{\sigma}(T(X),Z)) - Js(\tilde{\sigma}(T(X),Y)) \\
&= \mu_{s,\tilde{\sigma}}(T(X))(s(Y) + Js(Z))
\end{aligned}$$

for all vectors $Y,Z \in \mathfrak{p}$, and hence (2.1) is proved. Next, by the definitions of τ_s and $\mu_{s,\tilde{\sigma}}$ we have

$$[\mu_{s,\tilde{\sigma}}(X),\mu_{s,\tilde{\sigma}}(Y)](s(Z) + Js(W))$$

$$= - Js(\tilde{\sigma}(X,\tilde{\sigma}(Y,W))) - s(\tilde{\sigma}(X,\tilde{\sigma}(Y,Z))) + Js(\tilde{\sigma}(Y,\tilde{\sigma}(X,W)))$$

$$+ s(\tilde{\sigma}(Y,\tilde{\sigma}(X,Z)))$$

$$= s([\tilde{\sigma}(Y),\tilde{\sigma}(X)](Z)) + Js([\tilde{\sigma}(Y),\tilde{\sigma}(X)](W))$$

$$= \tau_s([\tilde{\sigma}(Y),\tilde{\sigma}(X)])(s(Z) + Js(W))$$

for all vectors Z,W in $\mathfrak{p}$, and hence (2.2) is proved. Since the subspace $\mathfrak{q}$ in $\bar{\mathfrak{p}}$ is totally real, we have

$$\bar{R}(s(X),s(Y))s(Z) = (c/4)(\langle Y,Z\rangle s(X) - \langle X,Z\rangle s(Y))$$

for all vectors $X,Y,Z \in \mathfrak{p}$. By the condition (3) for $\tilde{\sigma}$ we have

$$\bar{R}(s(X),s(Y))(s(Z) + Js(W))$$

$$= \bar{R}(s(X),s(Y))s(Z) + J\bar{R}(s(X),s(Y))s(W)$$

$$= s((c/4)(\langle Y,Z\rangle X - \langle X,Z\rangle Y)) + Js((c/4)(\langle Y,W\rangle X - \langle X,W\rangle Y))$$

$$= s(R(X,Y)Z - [\tilde{\sigma}(X),\tilde{\sigma}(Y)]Z) + Js(R(X,Y)W - [\tilde{\sigma}(X),\tilde{\sigma}(Y)]W)$$

$$= \tau_s(R(X,Y) - [\tilde{\sigma}(X),\tilde{\sigma}(Y)])(s(Z) + Js(W))$$

for all vectors $Z,W \in \mathfrak{p}$. Hence (2.3) is proved.

Now by (2.1), (2.2) and (2.3) we have

$$[\rho_{s,\tilde{\sigma}}(T+X), \rho_{s,\tilde{\sigma}}(S+Y)]$$

$$= [\tau_s(T), \tau_s(S)] + [\tau_s(T), \mu_{s,\tilde{\sigma}}(Y)] + [\tau_s(T), s(Y)]$$

$$+ [\mu_{s,\tilde{\sigma}}(X), \tau_s(S)] + [\mu_{s,\tilde{\sigma}}(X), \mu_{s,\tilde{\sigma}}(Y)] + [\mu_{s,\tilde{\sigma}}(X), s(Y)]$$

$$+ [s(X), \tau_s(S)] + [s(X), \mu_{s,\tilde{\sigma}}(Y)] + [s(X), s(Y)]$$

$$= \tau_s([T,S]) + \mu_{s,\tilde{\sigma}}(T(Y)) + s(T(Y)) - \mu_{s,\tilde{\sigma}}(S(X))$$

$$+ \tau_s([\tilde{\sigma}(Y), \tilde{\sigma}(X)]) - Js(\tilde{\sigma}(X,Y)) - s(S(X)) + Js(\tilde{\sigma}(Y,X))$$

$$- \tau_s(R(X,Y) - [\tilde{\sigma}(X), \tilde{\sigma}(Y)])$$

$$= \tau_s([T,S] - R(X,Y)) + \mu_{s,\tilde{\sigma}}(T(Y) - S(X)) + s(T(Y) - S(X))$$

$$= \rho_{s,\tilde{q}}([T+X, S+Y])$$

for all $T,S \in \mathfrak{k}$ and all $X,Y \in \mathfrak{p}$, and hence $\rho_{s,\tilde{\sigma}}$ is a Lie homomorphism of $\mathfrak{g}$ into $\bar{\mathfrak{g}}$. Moreover, since τ_s and s are injective, $\rho_{s,\tilde{q}}$ is injective.

q.e.d.

Since $\bar{\mathfrak{g}}$ is a compact Lie algebra, we have the following

Corollary 2.2. <u>If the set $\mathcal{M}_M$ is not empty, the Lie algebra $\mathfrak{g}$ is a compact Lie algebra.</u>

We call $\rho_{s,\tilde{\sigma}}$ the *Lie homomorphism associated to* s *and* $\tilde{\sigma}$.

Since G is a simply connected Lie group, there exists the unique Lie homomorphism $\hat{\rho}_{s,\tilde{\sigma}}$ of G into $\bar{G}$ such that the differential $d\hat{\rho}_{s,\tilde{\sigma}}$ is $\rho_{s,\tilde{\sigma}}$. The associated homomorphism $\rho_{s,\tilde{\sigma}}$ maps the Lie subalgebra $\mathfrak{k}$ into the Lie subalgebra $\bar{\mathfrak{k}}$ and the isotropy subgroup K is connected. Hence we can define a G-equivariant C^{∞}-mapping $f_{s,\tilde{\sigma}}$ of M^n into $P^n(c)$ by

$$f_{s,\tilde{\sigma}}(g(o)) = \hat{\rho}_{s,\tilde{\sigma}}(g)(\bar{o})$$

for $g \in G$. Then we have the following

Theorem 2.3. Let M^n be a simply connected symmetric space. Then, for any Euclidean isometry s and any $\tilde{\sigma} \in \mathcal{H}_M$, the associated G-equivariant mapping $f_{s,\tilde{\sigma}}$ of M^n into $P^n(c)$ is a totally real parallel isometric immersion such that

$$(f_{s,\tilde{\sigma}})_{*o} = s \quad \text{and} \quad (\tilde{\sigma}_{f_{s,\tilde{\sigma}}})_o = \tilde{\sigma}.$$

Proof. Note that $\bar{G}$ divided by the center is the group of all holomorphic isometries of $P^n(c)$. The claim $(f_{s,\tilde{\sigma}})_{*o} = s$ is obvious by the definition of $f_{s,\tilde{\sigma}}$. Now we show that $f_{s,\tilde{\sigma}}$ is a totally real parallel isometric immersion. Since $f_{s,\tilde{\sigma}}$ is G-equivariant, it is sufficient to see our claim at o. The linear mapping s is a isometry and the image $\mathfrak{q}$ of s is a totally real subspace in $\bar{\mathfrak{p}}$. Hence $f_{s,\tilde{\sigma}}$ is a totally real and isometric immersion at o. Moreover, to show that $f_{s,\tilde{\sigma}}$ is parallel, it is sufficient to see that

$$(2.4)\qquad [\rho_{s,\tilde{\sigma}}(X)_{\bar{\mathfrak{k}}}, [\rho_{s,\tilde{\sigma}}(X)_{\bar{\mathfrak{k}}}, \rho_{s,\tilde{\sigma}}(X)_{\bar{\mathfrak{p}}}]] \in \mathfrak{q}$$

for any vector X in $\mathfrak{p}$ (See Proposition 5.2 in [11]). Here the suffix $\bar{\mathfrak{k}}$ (resp. $\bar{\mathfrak{p}}$) means the $\bar{\mathfrak{k}}$-component (resp. $\bar{\mathfrak{p}}$-component) with respect to the decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{k}} + \bar{\mathfrak{p}}$. In fact, since

$$\rho_{s,\tilde{\sigma}}(X)_{\bar{\mathfrak{k}}} = \mu_{s,\tilde{\sigma}}(X) \quad \text{and} \quad \rho_{s,\tilde{\sigma}}(X)_{\bar{\mathfrak{p}}} = s(X) ,$$

the left hand of (2.4) equals $-s(\tilde{\sigma}(X,\tilde{\sigma}(X,X))) \in \mathfrak{q}$. Now the second fundamental form at o of the G-equivariant immersion $f_{s,\tilde{\sigma}}$ is given by

$$(2.5)\qquad (\tilde{\sigma}_{f_{s,\tilde{\sigma}}})_o(X,Y) = [(\rho_{s,\tilde{\sigma}})(X)_{\bar{\mathfrak{k}}}, (\rho_{s,\tilde{\sigma}})(Y)_{\bar{\mathfrak{p}}}]_{J\mathfrak{q}}$$

for all vectors X,Y in $\mathfrak{p}$ (See Proposition 5.1 in [11]). Here the suffix $J\mathfrak{q}$ means the $J\mathfrak{q}$-component with respect to the decomposition $\bar{\mathfrak{p}} = \mathfrak{q} + J\mathfrak{q}$. Hence we have $(\tilde{\sigma}_{f_{s,\tilde{\sigma}}})_o = -Js(\tilde{\sigma}(X,Y))$. This implies $(\tilde{\sigma}_{f_{s,\tilde{\sigma}}})_o = \tilde{\sigma}$.

q.e.d.

3. Frenet curves and rigidity problems

Let $\bar{M}$ be a riemannian manifold and $c(t)$ be a C^∞-curve in $\bar{M}$ defined on an open interval I containing 0 which is parametrized by arc-length. The curve $c(t)$ is called a *Frenet curve* in $\bar{M}$ of osculating rank r (≥ 1) if for all $t \in I$ its higher order derivatives

$$\dot{c}(t) = (\overline{\nabla}^0_{\frac{\partial}{\partial t}}\dot{c})(t),\ (\overline{\nabla}_{\frac{\partial}{\partial t}}\dot{c})(t),\cdots,(\overline{\nabla}^{r-1}_{\frac{\partial}{\partial t}}\dot{c})(t)$$

are linearly independent but

$$\dot{c}(t) = (\overline{\nabla}^0_{\frac{\partial}{\partial t}}\dot{c})(t),\ (\overline{\nabla}_{\frac{\partial}{\partial t}}\dot{c})(t),\cdots,(\overline{\nabla}^{r}_{\frac{\partial}{\partial t}}\dot{c})(t)$$

are linearly dependent in $T_{c(t)}(\overline{M})$. Then there exist the unique C^∞-positive functions $\kappa_1(t),\cdots,\kappa_{r-1}(t)$ on I and the unique C^∞-orthonormal vector fields $V_1(t),\cdots,V_r(t)$ along the curve $c(t)$ such that

$$(3.1)\quad \begin{cases} \dot{c}(t) = V_1(t) \\ (\overline{\nabla}_{\frac{\partial}{\partial t}}V_1)(t) = \kappa_1(t)V_2(t) \\ (\overline{\nabla}_{\frac{\partial}{\partial t}}V_2)(t) = -\kappa_1(t)V_1(t) + \kappa_2(t)V_3(t) \\ \quad\vdots \\ (\overline{\nabla}_{\frac{\partial}{\partial t}}V_j)(t) = -\kappa_{j-1}(t)V_{j-1}(t) + \kappa_j(t)V_{j+1}(t) \\ \quad\vdots \\ (\overline{\nabla}_{\frac{\partial}{\partial t}}V_{r-1})(t) = -\kappa_{r-2}(t)V_{r-2}(t) + \kappa_{r-1}(t)V_r(t) \\ (\overline{\nabla}_{\frac{\partial}{\partial t}}V_r)(t) = -\kappa_{r-1}(t)V_{r-1}(t). \end{cases}$$

Here we call $\kappa_j(t)$ $(1 \le j \le r-1)$ the *Frenet curvature functions* on I, the vector fields $\{V_j(t)\ ;\ 1 \le j \le r\}$ the *Frenet r-frame* along $c(t)$, and the equations (3.1) the *Frenet formulas.* For a given integer r (≥ 1)

and given C^∞-positive functions $\kappa_1(t),\cdots,\kappa_{r-1}(t)$ on I, the Frenet formulas (3.1) may be regarded as a system of differential equations with variables $c, V_1,\cdots,V_r$. It is known that this system of differential equations has the unique local solution for given initial conditions; a point $c(0) = p \in \bar{M}$ and an orthonormal r-frame $\{ V_1(0) = V_1,\cdots,V_r(0) = V_r \}$ of $T_p(\bar{M})$. If the riemannian manifold $\bar{M}$ is complete, the Frenet curve $c(t)$ is defined for $-\infty < t < +\infty$ (cf. See [4] and [15]). Now we have the following

Lemma 3.1 (W.Strübing [15]). Let M and $\bar{M}$ be riemannian manifolds and f a parallel isometric immersion of M into $\bar{M}$. Suppose that a curve $c(t)$ defined on I containing 0 is a geodesic in M parametrized by arc-length. Then

a) the curve $(f\circ c)(t)$ on I is a Frenet curve in $\bar{M}$,

b) the Frenet curvature functions $\kappa_1(t),\cdots,\kappa_{r-1}(t)$ are constant (and positive), where r denotes the osculating rank of $(f\circ c)(t)$,

c) the integer r (≥ 1), the constant positive numbers $\kappa_1,\cdots,\kappa_{r-1}$ and the orthonormal vectors $V_1 = V_1(0),\cdots,V_r = V_r(0)$ are determined only by the initial point $p = c(0)$ of $c(t)$, the initial tangent vector $X = \dot{c}(0)$ of $c(t)$, the differential $(f_*)_p$ at p, and the second fundamental form $(\sigma_f)_p$ at p.

Now, by Lemma 3.1, we have the following fundamental lemma.

Lemma 3.2. Let g and f be parallel isometric immersions of a complete riemannian manifold M into another riemannian manifold $\bar{M}$.

<u>If there exists a point</u> o <u>in</u> M <u>such that</u>

$$g(o) = f(o) = \bar{o}, \ (g_*)_o = (f_*)_o : T_o(M) \to T_{\bar{o}}(\bar{M}), \ (\sigma_g)_o = (\sigma_f)_o,$$

<u>then the mapping</u> g <u>and</u> f <u>coincide on</u> M.

Proof. For any point p in M, there exists a geodesic $c(t)$ in M parametrized by arc-length, such that $c(0) = o$ and $c(\ell) = p$. Then $(g \circ c)(t)$ and $(f \circ c)(t)$ are Frenet curves in $\bar{M}$ by Lemma 3.1,a). By Lemma 3.1,c), the above assumption implies that the Frenet curves $(f \circ c)(t)$ and $(g \circ c)(t)$ are solutions of same Frenet formulas for the same initial conditions. Hence, by the uniqueness for solutions of the system of differential equations, we have $(f \circ c)(t) = (g \circ c)(t)$ and paticularly $f(p) = g(p)$.

q.e.d.

Now let $\mathcal{J}_M$ be the set of all totally real parallel isometric immersions of a simply connected symmetric space M^n into the riemannian manifold $P^n(c)$, $I(M)$ the group of all isometries of M, and $\bar{G}$ the group of all holomorphic isometries of $P^n(c)$. Then we can define an action of $\bar{G} \times I(M)$ on $\mathcal{J}_M$ by

$$(\bar{g}, g) \cdot f = \bar{g} \circ f \circ g^{-1}$$

for $\bar{g} \in \bar{G}$, $g \in I(M)$ and $f \in \mathcal{J}_M$. Let $\bar{\mathcal{J}}_M$ be the set of all orbits of the $\bar{G} \times I(M)$-action on $\mathcal{J}_M$. The orbit $[f]_{\mathcal{J}}$ of f in $\mathcal{J}_M$ is called the *equivalence class* of f.

Secondly, let $\mathcal{S}_M$ be the set of all complete totally real parallel submanifolds with the universal riemannian covering M^n. Then we can define an action of $\overline{G}$ on $\mathcal{S}_M$ by

$$\overline{g}\cdot N = \overline{g}(N)$$

for $\overline{g}\in\overline{G}$ and $N\in\mathcal{S}_M$. Let $\overline{\mathcal{S}}_M$ be the set of all orbits of the $\overline{G}$-action on $\mathcal{S}_M$. The orbit $[N]_{\mathcal{S}}$ of N in $\mathcal{S}_M$ is called the *equivalence class* of N.

Lastly, set

$$F_o(M) = \{\, g\in I(M)\ ;\ g(o) = o\,\}.$$

Then we can define an action of $F_o(M)$ on $\mathcal{H}_M$ by

$$(k\cdot\tilde{\sigma})(X,Y) = (k_*)_o(\tilde{\sigma}((k_*)_o^{-1}X, (k_*)_o^{-1}Y))$$

for $k\in F_o(M)$, $\tilde{\sigma}\in\mathcal{H}_M$ and $X,Y\in\mathfrak{p}$. Let $\overline{\mathcal{H}}_M$ be the set of all orbits of the $F_o(M)$-action on $\mathcal{H}_M$. The orbit $[\tilde{\sigma}]_{\mathcal{H}}$ of $\tilde{\sigma}$ in $\mathcal{H}_M$ is called the *equivalence class* of $\tilde{\sigma}$.

Now we study the relations among three kinds of equivalences. At first we have the following

Lemma 3.3. For any $\overline{g}\in\overline{G}$, $g\in I(M)$ and $f\in\mathcal{J}_M$, there exists some $k\in F_o(M)$ such that

$$(\tilde{\sigma}_{\bar{g}\circ f\circ g^{-1}})_o = k\cdot(\tilde{\sigma}_f)_o .$$

Moreover, if $g \in F_o(M)$, the very same element g can be taken as the above element k.

Proof. Since $\bar{g}_*$ and J are comutative, we have

$$(3.2)\quad (\tilde{\sigma}_{\bar{g}\circ f\circ g^{-1}})_o(X,Y) = (\tilde{\sigma}_{f\circ g^{-1}})_o(X,Y)$$
$$= g_*((\tilde{\sigma}_f)_{g^{-1}(o)}((g_*)^{-1}X,(g_*)^{-1}Y))$$

for all vectors $X,Y \in \mathfrak{p}$. Let $\gamma(t)$ be a geodesic joining o to $g^{-1}(o)$. Since M is a symmetric space, there exists some $h \in I(M)$ such that $h(o) = g^{-1}(o)$ and that $h^{-1}\cdot(\tilde{\sigma}_f)_{h(o)}$ is the parallel translate of $(\tilde{\sigma}_f)_{h(o)}$ along the geodesic $\gamma(t)$, where

$$h^{-1}\cdot(\tilde{\sigma}_f)_{h(o)}(X,Y) = h_*^{-1}((\tilde{\sigma}_f)_{h(o)}(h_*X,h_*Y))$$

for all vectors $X,Y \in \mathfrak{p}$ (cf. See [8]). Putting $k = g\circ h$, we have $k \in F_o(M)$. Since $\tilde{\sigma}_f$ is parallel by Lemma 1.2, we have

the last term of (3.2)

$$= k_*(h_*^{-1}((\tilde{\sigma}_f)_{h(o)}(h_*(k_*^{-1}X),h_*(k_*^{-1}Y)))$$

$$= k_*((\tilde{\sigma}_f)_o(k_*^{-1}X,k_*^{-1}Y)) = (k\cdot(\tilde{\sigma}_f)_o)(X,Y).$$

The second assertion is clear from the above proof.

q.e.d.

Now we define a mapping i_M of $\tilde{\mathcal{J}}_M$ into $\bar{\mathcal{K}}_M$ by

$$i_M([f]_{\mathcal{J}}) = [(\tilde{\sigma}_f)_o]_{\mathcal{K}}$$

for f in $\mathcal{J}_M$. By Lemma 3.3 the mapping i_M is well-defined. Then we have the following

Theorem 3.4. <u>The mapping</u> i_M <u>of</u> $\tilde{\mathcal{J}}_M$ <u>into</u> $\bar{\mathcal{K}}_M$ <u>is bijective</u>.

Proof. By Theorem 2.3 it is obvious that i_M is onto. We show that the mapping i_M is injective. Take two mappings f_1, f_2 in $\mathcal{J}_M$ and suppose that $(\tilde{\sigma}_{f_1})_o = k\cdot(\tilde{\sigma}_{f_2})_o$ for some $k \in F_o(M)$. Then, putting $f_3 = f_2 \circ k^{-1}$, we have $(\tilde{\sigma}_{f_1})_o = (\tilde{\sigma}_{f_3})_o$ by Lemma 3.3. Since f_1 and f_3 are totally real, there exists some $\bar{g} \in \bar{G}$ such that

$$(\bar{g}\circ f_3)(o) = f_1(o) = \bar{o} \quad \text{and} \quad (\bar{g}\circ f_3)_*(T_o(M)) = (f_1)_*(T_o(M)) = ⓠ.$$

Moreover, since any Euclidean isometry of the totally real subspace ⓠ is the differential at $\bar{o}$ of some holomorphic isometry of $P^n(c)$, we may assume that $(\bar{g}\circ f_3)_{*o} = (f_1)_{*o}$. Here note that $(\tilde{\sigma}_{\bar{g}\circ f_3})_o = (\tilde{\sigma}_{f_3})_o$ by Lemma 3.3. Hence, by Lemma 3.2, we have $\bar{g}\circ f_3 = f_1$ on M and thus $[f_1]_{\mathcal{J}} = [f_3]_{\mathcal{J}} = [f_2]_{\mathcal{J}}$.

q.e.d.

Theorem 3.5. <u>Any totally real parallel isometric immersion of</u> M^n <u>into</u> $P^n(c)$ <u>is G-equivariant.</u>

Proof. Let f be a totally real parallel isometric immersion and put $f(o) = \bar{o}$. Then we have $f = f_{(f_*)_o, (\tilde{\sigma}_f)_o}$ by Theorem 2.3 and Lemma 3.2. This implies the theorem.

q.e.d.

Now let j_M be a mapping of $\mathcal{J}_M$ into $\mathcal{S}_M$ defined by

$$j_M([f]_{\mathcal{J}}) = [f(M)]_{\mathcal{S}}$$

for $f \in \mathcal{J}_M$. Here note that the image $f(M)$ is a submanifold in $P^n(c)$ by Theorem 3.5. Then we have the following

Theorem 3.6. <u>The mapping</u> j_M <u>of</u> $\mathcal{J}_M$ <u>into</u> $\mathcal{S}_M$ <u>is bijective.</u>

Proof. It is obvious that j_M is onto. We show that the mapping j_M is injective. Take two mappings $f_1, f_2 \in \mathcal{J}_M$ and suppose that $f_1(M) = \bar{g}(f_2(M))$ for some $\bar{g} \in \bar{G}$. Put $\bar{o} = f_1(o)$ and $N = f_1(M)$. Taking some $g \in I(M)$ and putting $f_3 = \bar{g} \circ f_2 \circ g$, we have

$$f_1(o) = f_3(o) = \bar{o} \quad \text{and} \quad f_1(M) = f_3(M) = N.$$

Let $(\sigma_N)_{\bar{o}}$ be the second fundamental form at $\bar{o}$ of the submanifold N. Then we have

$$(\sigma_N)_{\bar{o}}(\bar{X},\bar{Y}) = (\sigma_{f_1})_o((f_1)_*^{-1}\bar{X},(f_1)_*^{-1}\bar{Y})$$
$$= (\sigma_{f_3})_o((f_3)_*^{-1}\bar{X},(f_3)_*^{-1}\bar{Y})$$

for all vectors $X, Y \in T_{\bar{o}}(N)$. Hence we have

$$(\tilde{\sigma}_{f_3})_o(X,Y) = ((f_3)_*^{-1}\circ(f_1)_*)((\tilde{\sigma}_{f_1})_o((f_1)_*^{-1}\circ(f_3)_*X,(f_1)_*^{-1}\circ(f_3)_*Y))$$

for all vectors $X, Y \in T_o(M)$. Note that $f_3{}^{-1}\circ f_1$ defines a local isometry of M around o. Since M is a simply connected symmetric space, there exists a unique element $k \in F_o(M)$ that coincides with $f_3{}^{-1}\circ f_1$ around o. Hence we have $(\tilde{\sigma}_{f_3})_o = k\cdot(\tilde{\sigma}_{f_1})_o$. By Theorem 3.4 we have $[f_3]_{\mathcal{J}} = [f_1]_{\mathcal{J}}$ and thus $[f_2]_{\mathcal{J}} = [f_1]_{\mathcal{J}}$.

q.e.d.

4. The set $\bar{\mathcal{H}}_M$ for a simply connected symmetric space M without Euclidean factor

In this section we assume that M^n is a simply connected symmetric space without Euclidean factor, thus, M is decomposed as a riemannian manifold as follows:

$$M^n = M_1{}^{n_1}\times\cdots\times M_r{}^{n_r} \quad (n = \Sigma_{j=1}^{r} n_j)$$

where $M_j{}^{n_j}$ is an n_j-dimensional irreducible simply connected symmetric space for each j. Then the tangent space $T_o(M) = \mathfrak{P}$ (resp. the holonomy algebra $\mathfrak{K}$) is decomposed as follows:

$$\mathfrak{p} = \Sigma_{j=1}^{r} \mathfrak{p}_j \quad (\text{resp. } \mathfrak{k} = \Sigma_{j=1}^{r} \mathfrak{k}_j)$$

where the subspace $\mathfrak{p}_j \subset \mathfrak{p}$ (resp. the subalgebra $\mathfrak{k}_j \subset \mathfrak{k}$) denotes the tangent space $T_o(M_j)$ (resp. the holonomy algebra of M_j). For a $\mathfrak{p}$-valued symmetric bilinear form $\tilde{\sigma}$ on $\mathfrak{p}$ and any ordered triple $\{i,j,k\}$ ($1 \leq i,j,k \leq r$), a mapping $\tilde{\sigma}_{ij}^{k} : \mathfrak{p}_i \times \mathfrak{p}_j \to \mathfrak{p}_k$ is defined by

$$\tilde{\sigma}_{ij}^{k}(X_i, Y_j) = \text{the } \mathfrak{p}_k\text{-component of } \tilde{\sigma}(X_i, Y_j)$$

for $X_i \in \mathfrak{p}_i$ and $Y_j \in \mathfrak{p}_j$. Then we may write symbolically as

$$\tilde{\sigma} = \Sigma_{i,j,k=1}^{r} \tilde{\sigma}_{ij}^{k} .$$

Assume that $\tilde{\sigma} \in \mathcal{H}_M$. Since each holonomy algebra $\mathfrak{k}_j$ ($1 \leq j \leq r$) acts on the subspace $\mathfrak{p}_j$ irreducibly and on the other subspaces $\mathfrak{p}_k$ ($j \neq k$) trivially, the condition (2) for $\tilde{\sigma}$ implies that

$$(4.1) \quad \tilde{\sigma} = \Sigma_{j=1}^{r} \tilde{\sigma}_{jj}^{j} .$$

Now we have the following

Lemma 4.1. <u>Assume that the set</u> $\mathcal{H}_M$ <u>is not empty. Then the simply connected symmetric space</u> M <u>without Euclidean factor is irreducible and of compact type.</u>

Proof. Suppose that $r \geq 2$ and $\tilde{\sigma} \in \mathcal{H}_M$. In the condition (3) for $\tilde{\sigma}$, let X be a nonzero vector in $\mathfrak{p}_j$ and $Y = Z$ a nonzero vector in $\mathfrak{p}_k$ with $j \neq k$. Then, by (4.1), we have

$$(c/4)\langle Y,Y\rangle X = R(X,Y)Y - [\tilde{\sigma}(X),\tilde{\sigma}(Y)]Y = - [\tilde{\sigma}(X),\tilde{\sigma}(Y)]Y$$

$$= \tilde{\sigma}(Y,\tilde{\sigma}(X,Y)) - \tilde{\sigma}(X,\tilde{\sigma}(Y,Y)) = 0.$$

This is a contradiction. Hence we have $r = 1$.

Moreover Corollary 2.2 implies that M is of compact type.

q.e.d.

Hereafter we assume that M is a simply connected compact irreducible symmetric space. Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$ and W the Weyl group of M relative to $\mathfrak{a}$. Denote by $S^3(\mathfrak{p})$ (resp. $S^3(\mathfrak{a})$) the vector space of all symmetric trilinear forms on $\mathfrak{p}$ (resp. on $\mathfrak{a}$). Then it is known that the vector subspace $\{ \tilde{\sigma} \in S^3(\mathfrak{p}) ; \mathfrak{k}\cdot\tilde{\sigma} = 0 \}$ is isomorphic to the vector subspace $\{ \tilde{\lambda} \in S^3(\mathfrak{a}) ; w\cdot\tilde{\lambda} = \tilde{\lambda}$ for all $w \in W \}$ by the restriction to the subspace $\mathfrak{a}$. Noting that the Weyl group W acts on $\mathfrak{p}$ irreducibly, we can see the following

Lemma 4.2. _Let_ M _be a simply connected compact irreducible symmetric space and set_ $d_M = \dim \{ \tilde{\sigma} \in S^3(\mathfrak{p}) ; \mathfrak{k}\cdot\tilde{\sigma} = 0 \}$. _Then_ $d_M = 1$ _if_ M _is one of the following spaces and_ $d_M = 0$ _otherwise_:

$SU(n)/SO(n)$ ($n \geq 3$), $SU(2n)/Sp(n)$ ($n \geq 3$), $SU(n)$ ($n \geq 3$), E_6/F_4.

Now we determine the set $\bar{\mathcal{H}}_M$.

Proposition 4.3. Let M^n be a simply connected compact irreducible symmetric space satisfying $d_M = 0$. Assume that the set $\bar{\mathcal{H}}_M$ is not empty. Then the riemannian manifold M^n is the sphere $S^n(c/4)$ with constant sectional curvatures $c/4$ and the set $\bar{\mathcal{H}}_M$ consists of one point. Moreover the unique element in $\bar{\mathcal{H}}_M$ corresponds to the natural totally geodesic isometric immersion $f : S^n(c/4) \to P^n(c)$.

Proof. Take $\tilde{\sigma} \in \mathcal{H}_M$. Then the assumption that $d_M = 0$ implies that $\tilde{\sigma} = 0$. Hence, by the condition (3) for $\tilde{\sigma}$, we have

$$R(X,Y)Z = (c/4)(\langle Y,Z\rangle X - \langle X,Z\rangle Y)$$

for all vectors $X,Y,Z \in \mathfrak{p}$. This implies that M^n has constant sectional curvatures $c/4$. The other assertions are obvious.

q.e.d.

Now we consider the case when $d_M = 1$. Then we have the following

Proposition 4.4. Let M^n be a simply connected compact irreducible symmetric space satisfying $d_M = 1$. Assume that the set $\bar{\mathcal{H}}_M$ is not empty. Then the metric of M^n is determined uniquely by the constant c and the set $\bar{\mathcal{H}}_M$ consists of one point.

Proof. Let $(M,\langle , \rangle_1)$ and $(M,\langle , \rangle_2)$ be symmetric spaces with the same underlying manifold M. Suppose that $\bar{\mathcal{H}}_{(M,<,>_1)}$ and $\bar{\mathcal{H}}_{(M,<,>_2)}$ are not empty, and take $\tilde{\sigma}_j \in \mathcal{H}_{(M,<,>_j)}$ for $j = 1,2$. Then, noting that M is not a sphere, we can see that each $\tilde{\sigma}_j$ is nonzero by the same proof as in Proposition 4.3. Since M is irreducible, we have $<,>_2 = \alpha<,>_1$ for some $\alpha > 0$. Moreover the assumption that $d_M = 1$ implies that $\tilde{\sigma}_2 = \beta\tilde{\sigma}_1$ for some β. By the condition (3) for $\tilde{\sigma}_j$ $(j=1,2)$, we have

$$(c/4)(<Y,Z>_j X - <X,Z>_j Y) = R(X,Y)Z - [\tilde{\sigma}_j(X),\tilde{\sigma}_j(Y)](Z)$$

and thus

$$(c/4)(\beta^2 - \alpha)(<Y,Z>_1 X - <X,Z>_1 Y) = (\beta^2 - 1)R(X,Y)Z$$

for all vectors $X,Y,Z \in \mathfrak{p}$. Since M is not a sphere, we have $\beta^2 = 1$ and $\alpha = 1$. Hence we have $<,>_1 = <,>_2$ and $\tilde{\sigma}_2 = \pm\tilde{\sigma}_1$. Note that the symmetry $\phi \in F_o(M)$ at o acts on the set $S^3(\mathfrak{p})$ by $\phi\cdot\tilde{\sigma} = -\tilde{\sigma}$ for any $\tilde{\sigma} \in S^3(\mathfrak{p})$. Then we can see that the set $\bar{\mathcal{H}}_{(M,<,>_1)} = \bar{\mathcal{H}}_{(M,<,>_2)}$ consists of one point.

q.e.d.

In the next section we shall construct a model of a totally real parallel isometric immersion of M^n into $P^n(c)$ for M^n satisfying $d_M = 1$. Hence, summing up Lemma 4.1 and Propositions 4.3,4.4, we have the following

Theorem 4.5. Let M^n be a simply connected symmetric space without Euclidean factor. Then the set $\bar{\mathcal{H}}_M$ is not empty if and only

<u>if the symmetric space</u> M^n <u>is one of the followings:</u>

$SU(n)/SO(n)$ ($n \geq 3$), $SU(2n)/Sp(n)$ ($n \geq 3$), $SU(n)$ ($n \geq 3$),

E_6/F_4, $SO(n+1)/SO(n)$ ($n \geq 2$).

<u>In this case, the metric on the manifold</u> M^n <u>is determined uniquely by the constant</u> c <u>and the set</u> $\bar{\mathcal{H}}_M$ <u>consists of one point</u>.

5. Models of totally real parallel isometric immersions

Let V be an (n+1)-dimensional complex vector space furnished with a positive definite hermitian inner product (,). Then we can define the associated inner product $< , >_V$ on V as follows:

$$<X,Y>_V = \mathrm{Re}\,(X,Y)$$

for vectors $X,Y \in V$. Let P(V) be the complex projective space associated to V furnished with the Kähler metric < , > with constant holomorphic sectional curvatures c, and S the unit sphere in V furnished with the following riemannian metric $< , >_S$:

$$<X,Y>_S = (c/4)<X,Y>_V$$

for tangent vectors X,Y of S. Then the Hopf fibring $\pi : S \to P(V)$ is a riemannian submersion. For a point $p \in S$, the horizontal subspace H_p at p is given by

$$H_p = \{ X \in V ; \langle X,p\rangle_V = \langle X,\sqrt{-1}\cdot p\rangle_V = 0 \}.$$

Here note that the linear mapping $\pi_* : H_p \to T_{(p)}(P(V))$ is an Euclidean isometry satisfying $\pi_*(\sqrt{-1}X) = J(\pi_* X)$ for any $X \in H_p$. Let $\gamma(t)$ be a curve in S. Then a vector field Z_t along $\gamma(t)$ is called *horizontal* if $Z_t \in H_{\gamma(t)}$ for all t. The curve $\gamma(t)$ is called *horizontal* if $\dot{\gamma}(t)$ is a horizontal vector field along $\gamma(t)$. Moreover an isometric immersion $\hat{f}$ of a riemannian manifold M into S is called *horizontal* if $\hat{f}_*(T_p(M)) \subset H_{\hat{f}(p)}$ for any point p in M. And a horizontal isometric immersion $\hat{f}$ is called *totally real* if the subspaces $\hat{f}_*(T_p(M))$ and $\sqrt{-1}\hat{f}_*(T_p(M))$ are orthogonal. Let ∇^S be the riemannian connection on S for the riemannian metric $\langle\ ,\ \rangle_S$. Then we have the following

Lemma 5.1 (K.Nomizu [12] and B.O'Neill [13]). Let $\gamma(t)$ be a horizontal curve in S parametrized by arc-length. Then $(\nabla^S_t \dot{\gamma})(t)$ is a horizontal vector field along $\gamma(t)$. Moreover

$$\overline{\nabla}_t(\pi_* Z_t) = \pi_*(\nabla^S_t Z_t)$$

for any horizontal vector field Z_t along $\gamma(t)$.

Let $\hat{f}$ be a horizontal (resp. horizontal and totally real) isometric immersion of an n-dimensional riemannian manifold M^n into S. Then the mapping $f = \pi\circ\hat{f} : M^n \to P(V)$ is an isometric immersion (resp. a totally real isometric immersion). Now we have the following

Lemma 5.2. Let $\gamma(t)$ be a geodesic in M parametrized by arc-length. If the horizontal part of $(\nabla_t^S)^2\hat{f}_*(\dot{\gamma}(t))$ is contained in $\hat{f}_*(T_{\gamma(t)}(M))$, the normal vector $(\nabla_t^*\sigma_f)(\dot{\gamma}(t),\dot{\gamma}(t))$ at $f(\gamma(t))$ equals zero.

Proof. Since the vector field $\nabla_t^S\hat{f}_*(\dot{\gamma}(t))$ is horizontal and $\pi_*(\nabla_t^S\hat{f}_*(\dot{\gamma}(t))) = \overline{\nabla}_t(f_*(\dot{\gamma}(t))) = \sigma_f(\dot{\gamma}(t),\dot{\gamma}(t))$ by Lemma 5.1, we have by Lemma 5.1 again

$$(5.1)\quad \pi_*((\nabla_t^S)^2\hat{f}_*(\dot{\gamma}(t))) = \overline{\nabla}_t(\sigma_f(\dot{\gamma}(t),\dot{\gamma}(t))).$$

Note that

$$(\nabla_t^*\sigma_f)(\dot{\gamma}(t),\dot{\gamma}(t)) = D_t(\sigma_f(\dot{\gamma}(t),\dot{\gamma}(t)))$$

$$= \text{the normal component of } \overline{\nabla}_t(\sigma_f(\dot{\gamma}(t),\dot{\gamma}(t))).$$

By (5.1) and the assumption, the vector field $\overline{\nabla}_t(\sigma_f(\dot{\gamma}(t),\dot{\gamma}(t)))$ is a tangent vector field of M and thus $(\nabla_t^*\sigma_f)(\dot{\gamma}(t),\dot{\gamma}(t)) = 0$.

q.e.d.

Now we give the models of totally real parallel isometric immersions into $P^n(c)$ of irreducible compact simply connected symmetric spaces M satisfying $d_M = 1$.

Model 1. Let M be the manifold $SU(n)/SO(n)$ $(n \geq 3)$ and V the complex vector space $S^n(\mathbb{C})$ of all complex symmetric matrices of degree n furnished with the hermitian inner product:

$$(X,Y) = \mathrm{Tr}\, X Y^*$$

for $X, Y \in V$. An imbedding $\hat{f} : M \to S$ is defined by

$$\hat{f}(g \cdot SO(n)) = (1/\sqrt{n})\, {}^t g \cdot g$$

for $g \in SU(n)$ and thus the manifold M is furnished with the riemannian metric induced from that of S. Let e_n be the identity element of $SU(n)$ and put $o = e_n \cdot SO(n) \in M$. Now we can see easily the following facts:

(1) The tangent space $T_o(M)$ at o is identified with the space $\mathfrak{p} = \{ \sqrt{-1}A \,;\, A \in S^n(\mathbb{R}),\ \mathrm{Tr}\, A = 0 \}$ and the following set $\mathfrak{a}$ is a maximal abelian subspace in $\mathfrak{p}$:

$$\mathfrak{a} = \left\{ \sqrt{-1} \begin{pmatrix} -\Sigma x_j & & & 0 \\ & x_1 & & \\ & & \ddots & \\ 0 & & & x_{n-1} \end{pmatrix} ;\ x_j \in \mathbb{R} \right\}.$$

(2) The isometric imbedding $\hat{f}$ is equivariant relative to the representation $\rho : SU(n) \to SU(V)$ defined by

$$\rho(g)(X) = {}^t g\, X\, g$$

for $g \in SU(n)$ and $X \in V$.

(3) $\hat{f}(o) = (1/\sqrt{n})\, e_n$ and $(\hat{f}_*)_o(\mathfrak{p}) = \mathfrak{p}$. Hence $\hat{f}$ is horizontal and totally real at o.

Then the riemannian metric of M is invariant under $SU(n)$ by (2) and hence M is a symmetric space, and the isometric imbedding $\hat{f}$ is horizontal and totally real by (2) and (3). Hence $f = \pi \circ \hat{f}$ is a totally real isometric immersion.

Now we show that the isometric immersion f has the parallel second fundamental form. Since f is totally real in $P(V)$, the equation of Codazzi-Mainardi implies that $\nabla^* \sigma_f$ is a normal bundle valued symmetric tensor of degree 3. Note that f is equivariant by (2), and that maximal abelian subspaces in $\mathfrak{p}$ are conjugate to each other under the natural action of $K = SO(n)$ on $\mathfrak{p}$. Hence it is sufficient for our claim to see that $(\nabla^*_X \sigma_f)(X,X) = 0$ for any unit vector

$$X = \sqrt{-1} \cdot \begin{pmatrix} -\Sigma x_j & & & 0 \\ & x_1 & & \\ & & \ddots & \\ 0 & & & x_{n-1} \end{pmatrix}$$

in $\mathfrak{a}$. Let $\gamma(t)$ be the geodesic in M such that $\gamma(0) = o$ and $\dot{\gamma}(0) = X$. Then we have

$$\hat{f}(\gamma(t)) = (1/\sqrt{n}) \cdot \begin{pmatrix} e^{-2t(\Sigma x_j)\sqrt{-1}} & & & 0 \\ & e^{2tx_1\sqrt{-1}} & & \\ & & \ddots & \\ 0 & & & e^{2tx_{n-1}\sqrt{-1}} \end{pmatrix}$$

and

$$\hat{f}_*(\dot{\gamma}(t)) = (1/\sqrt{n}) \cdot \begin{pmatrix} -2\sqrt{-1}(\Sigma x_j)e^{-2t(\Sigma x_j)\sqrt{-1}} & & & 0 \\ & 2x_1\sqrt{-1}e^{2tx_1\sqrt{-1}} & & \\ & & \ddots & \\ 0 & & & 2x_{n-1}\sqrt{-1}e^{2tx_{n-1}\sqrt{-1}} \end{pmatrix}$$

Note that $\nabla_t^S Z_t = \frac{d}{dt}(Z_t) + (c/4)\langle \hat{f}_*(\dot{\gamma}(t)), Z_t \rangle_S \hat{f}(\gamma(t))$ for any vector field Z_t along $f(\gamma(t))$. Thus we have

$$\nabla_t^S \hat{f}_*(\dot{\gamma}(t)) = (1/\sqrt{n}) \cdot \begin{pmatrix} (c/4 - 4(\Sigma x_j)^2) e^{-2t(\Sigma x_j)\sqrt{-1}} & & 0 \\ & (c/4 - 4x_1{}^2) e^{2tx_1\sqrt{-1}} & \\ 0 & \ddots & (c/4 - 4x_{n-1}{}^2) e^{2tx_{n-1}\sqrt{-1}} \end{pmatrix}$$

and

$$(\nabla_t^S)^2 \hat{f}_*(\dot{\gamma}(t))|_{t=0} = (2\sqrt{-1}/\sqrt{n}) \cdot \begin{pmatrix} -(c/4 - 4(\Sigma x_j)^2)(\Sigma x_j) & & 0 \\ & (c/4 - 4x_1{}^2) x_1 & \\ 0 & \ddots & (c/4 - 4x_{n-1}{}^2) x_{n-1} \end{pmatrix}$$

Hence the horizontal part of $(\nabla_t^S)^2 \hat{f}_*(\dot{\gamma}(t))|_{t=0}$ is given by

$$(\nabla_t^S)^2 \hat{f}_*(\dot{\gamma}(t))|_{t=0} - \frac{\langle (\nabla_t^S)^2 \hat{f}_*(\dot{\gamma}(t))|_{t=0}, \sqrt{-1}\hat{f}(\gamma(0)) \rangle_S}{|\sqrt{-1}\hat{f}(\gamma(0))|_S^2} \cdot \sqrt{-1}\hat{f}(\gamma(0))$$

$$= (\sqrt{-1}/\sqrt{n}) \cdot \begin{pmatrix} -2(\Sigma x_j)(c/4 - 4(\Sigma x_j)^2) - \lambda\sqrt{c}/2 & & 0 \\ & 2x_1(c/4 - 4x_1{}^2) - \lambda\sqrt{c}/2 & \\ 0 & \ddots & 2x_{n-1}(c/4 - 4x_{n-1}{}^2) - \lambda\sqrt{c}/2 \end{pmatrix}$$

where $\lambda = (16/n\sqrt{c})((\Sigma x_j)^3 - (\Sigma x_j{}^3))$. Here note that the trace of the above matrix equals zero. Hence the horizontal part of $(\nabla_t^S)^2 \hat{f}_*(\dot{\gamma}(t))|_{t=0}$ is contained in Ⓟ. This implies that $(\nabla^* \sigma_f)(\dot{\gamma}(0), \dot{\gamma}(0), \dot{\gamma}(0)) = 0$ by Lemma 5.2. Hence f is a totally real parallel isometric immersion of M into $P(V)$.

Model 2. Let M be the manifold $SU(2n)/Sp(n)$ ($n \geq 3$) and V the complex vector space $\mathfrak{so}(2n;\mathbb{C})$ of all complex skew symmetric matrices of degree 2n furnished with the hermitian inner product:

$$(X,Y) = \mathrm{Tr}\, X Y^*$$

for vectors $X, Y \in V$. An imbedding $\hat{f} : M \to S$ is defined by

$$\hat{f}(g \cdot Sp(n)) = (1/\sqrt{2n})\, {}^t g\, J_n\, g$$

for $g \in SU(2n)$, where $J_n = \begin{pmatrix} 0 & -e_n \\ e_n & 0 \end{pmatrix} \in V$, and thus the manifold M is furnished with the riemannian metric induced from that of S. Put $o = e_{2n} \cdot Sp(n) \in M$. Now we can see easily the following facts:

(1) The tangent space $T_o(M)$ at o is identified with the space

$$\mathfrak{p} = \left\{ \begin{pmatrix} Z & W \\ \overline{W} & {}^t Z \end{pmatrix} ; Z \in \mathfrak{su}(n),\ W \in \mathfrak{so}(n;\mathbb{C}) \right\}$$

and the following set $\mathfrak{a}$ is a maximal abelian subspace in $\mathfrak{p}$:

$$\mathfrak{a} = \left\{ \sqrt{-1} \cdot \begin{pmatrix} -(\Sigma x_j) & & & & & & & \\ & x_1 & & & & & 0 & \\ & & \ddots & & & & & \\ & & & x_{n-1} & & & & \\ & & & & -(\Sigma x_j) & & & \\ & & & & & x_1 & & \\ & 0 & & & & & \ddots & \\ & & & & & & & x_{n-1} \end{pmatrix} ; x_j \in \mathbb{R} \right\}.$$

(2) The isometric imbedding $\hat{f}$ is equivariant relative to the representstion $\rho : SU(2n) \to SU(V)$ defined by

$$\rho(g)(X) = {}^t g X g$$

for $g \in SU(2n)$ and $X \in V$.

(3) $\hat{f}(o) = (1/\sqrt{2n}) J_n$ and $(\hat{f}_*)_o(\mathfrak{p}) = \{ \begin{pmatrix} -\overline{W} & -{}^t Z \\ Z & W \end{pmatrix} ; Z \in \mathfrak{su}(n)$, $W \in \mathfrak{so}(n;\mathbb{C}) \}$. Hence $\hat{f}$ is horizontal and totally real at o.

Then, by the same way as in Model 1, we can see that $f = \pi \circ \hat{f}$ is a totally real parallel isometric immersion.

Model 3. Let M be the manifold $SU(n) \times SU(n)/SU(n)$ ($n \geq 3$) and V the complex vector space $M_n(\mathbb{C})$ of all complex matrices of degree n furnished with the hermitian inner product:

$$(X, Y) = \mathrm{Tr}\, X Y^*$$

for vectors $X, Y \in V$. An imbedding $\hat{f} : M \to S$ is defined by

$$\hat{f}((g,h) \cdot SU(n)) = (1/\sqrt{n}) g h^{-1}$$

for $g, h \in SU(n)$ and thus the manifold M is furnished with the riemannian metric induced from that of S. Put $o = (e_n, e_n) \cdot SU(n) \in M$. Now we can see easily the following facts:

(1) The tangent space $T_o(M)$ at o is identified with the space $\mathfrak{p} = \{ (X, -X) ; X \in \mathfrak{su}(n) \}$ and the following set $\mathfrak{a}$ is a maximal abelian subspace in $\mathfrak{p}$:

$$\mathfrak{a} = \{ (X,-X) \in \mathfrak{p} ; \ X \text{ is diagonal} \}.$$

(2) The isometric imbedding $\hat{f}$ is equivariant relative to the representation $\rho : SU(n) \times SU(n) \to SU(V)$ defined by

$$\rho((g,h))(X) = g X h^{-1}$$

for $g,h \in SU(n)$ and $X \in V$.

(3) $\hat{f}(o) = (1/\sqrt{n})e_n$ and $(\hat{f}_*)_o(\mathfrak{p}) = \mathfrak{su}(n)$. Hence $\hat{f}$ is horizontal and totally real at o.

Then, by the same way as in Model 1, we can see that $f = \pi \circ \hat{f}$ is a totally real parallel isometric immersion.

Model 4. Let $\mathbb{C}$ be the Cayley algebra over $\mathbb{R}$ furnished with the caninical conjugation $-$, and set $\mathcal{J} = \{ X \in M_3(\mathbb{C}) ; {}^t\overline{X} = X \}$. On the real vector space $\mathcal{J}$, we define the Jordan product $\circ$, the inner product $((\, , \,))$, the cross product $\times$, and the determinant det as follows respectively:

$$X \circ Y = (1/2)(XY + YX), \quad ((X,Y)) = \mathrm{Tr}(X \circ Y),$$

$$X \times Y = (1/2)(2X \circ Y - \mathrm{Tr}(X)Y - \mathrm{Tr}(Y)X + (\mathrm{Tr}(X)\mathrm{Tr}(Y) - \mathrm{Tr}(X \circ Y))e_3),$$

$$\det(X) = (1/3)((X \times X, X))$$

for $X,Y \in \mathcal{J}$. Let V be the complexification of the real vector space $\mathcal{J}$ and extend these $\circ$, $((\, , \,))$, $\times$, det $\mathbb{C}$-linearly and naturally on V.

Denote by τ the complex conjugate on V with respect to $\mathcal{J}$. Then $(X,Y) = ((\tau X,Y))$ is a positive definite hermitian inner product on V. We define

$$E_6 \equiv \{ g \in GL_{\mathbb{C}}(V) \; ; \begin{array}{l} \det(g(X)) = \det(X), \ (gX,gY) = (X,Y) \\ \text{for any } X,Y \in V \end{array} \}$$

and

$$F_4 = \{ g \in E_6 \; ; g(e_3) = e_3 \}.$$

Then E_6 (resp. F_4) is a simply connected compact simple Lie group of type E_6 (resp. of type F_4).(cf. O.Shukugawa-I.Yokota [14])

Let M be the manifold E_6/F_4. An imbedding $\hat{f} : M \to S$ is defined by

$$\hat{f}(g \cdot F_4) = (1/\sqrt{3}) g(e_3)$$

for $g \in E_6$ and thus the manifold M is furnished with the riemannian metric induced from that of S. Put $o = e_3 \cdot F_4 \in M$ and set $\mathcal{J}_0 = \{ X \in \mathcal{J} \; ; \mathrm{Tr}\, X = 0 \}$. Now we can see easily the following facts:

(1) Define the right translation R_X on $\mathcal{J}$ for $X \in \mathcal{J}$ by $R_X(Y) = Y \circ X$ for $Y \in \mathcal{J}$. The tangent space $T_o(M)$ at o is identified with the space $\mathfrak{p} = \{ \sqrt{-1} R_X \in \mathfrak{gl}(V) \; ; X \in \mathcal{J}_0 \}$ and the following set $\mathfrak{a}$ is a maximal abelian subspace in $\mathfrak{p}$:

$$\mathfrak{a} = \{ \sqrt{-1} R_X \in \mathfrak{gl}(V) \; ; X \in \mathcal{J}_0, \ X \text{ is diagonal} \}.$$

(2) The isometric imbedding $\hat{f}$ is equivariant relative to the representation $\rho : E_6 \to SU(V)$ defined by

for $g \in E_6$ and $X \in V$.

(3) $\hat{f}(o) = (1/\sqrt{3})e_3$ and $(\hat{f}_*)_o(p) = \sqrt{-1}\mathcal{J}_0$. Hence $\hat{f}$ is horizontal and totally real at o.

Then, by the same way as in Model 1, we can see that $f = \pi \circ \hat{f}$ is totally real parallel isometric immersion.

Remark 5.3. It is known that the isometric imbeddings $\hat{f} : M \to S$ in the above models are minimal. Since the imbeddings $\hat{f}$ are horizontal, the isometric immersions f are minimal.

Remark 5.4. We can see easily that the above isometric immersion $f : M \to P(V)$ is $(\sqrt{c}/2\sqrt{2})$-isotropic (that is, $|\sigma_f(X,X)| = \sqrt{c}/2\sqrt{2}$ for any unit tangent vector X of M) if the symmetric space M is of rank two. Hence these isometric immersions f are examples of Theorem 4.13 in [11].

6. The set $\mathcal{K}_M$ for a simply connected symmetric space M with Euclidean factor

In this section we assume that M^n is a simply connected symmetric space with Euclidean factor, thus, M is decomposed as a riemannian manifold as follows:

$$M^n = \mathbb{R}^{n_0} \times M_1^{n_1} \times \cdots \times M_r^{n_r} \quad (n = \Sigma_{j=0}^{r} n_j , \ n_0 > 0)$$

where $M_j^{n_j}$ is an n_j-dimensional irreducible simply connected symmetric space for each j. Then the tangent space $T_o(M) = \mathfrak{p}$ (resp. the holonomy algebra $\mathfrak{k}$) is decomposed as follows:

$$\mathfrak{p} = \mathfrak{p}_0 + \Sigma_{j=1}^{r} \mathfrak{p}_j \quad (\text{ resp. } \mathfrak{k} = \Sigma_{j=1}^{r} \mathfrak{k}_j)$$

where the subspaces $\mathfrak{p}_j$ and $\mathfrak{p}_0$ in $\mathfrak{p}$ (resp. the subalgebra $\mathfrak{k}_j$ in $\mathfrak{k}$) denote the tangent spaces $T_o(M_j)$ and $T_o(R^n 0)$ (resp. the holonomy algebra of M_j). For a $\mathfrak{p}$-valued symmetric bilinear form $\tilde{\sigma}$ on $\mathfrak{p}$ and any ordered triple $\{i,j,k\}$ ($0 \leq i,j,k \leq r$), a mapping $\tilde{\sigma}_{ij}^{k} : \mathfrak{p}_i \times \mathfrak{p}_j \to \mathfrak{p}_k$ is defined as in the section 4. Assume that $\tilde{\sigma} \in \mathcal{H}_M$. Since each holonomy algebra $\mathfrak{k}_j$ ($1 \leq j \leq r$) acts on the subspace $\mathfrak{p}_j$ irreducibly and on the other spaces $\mathfrak{p}_k$ ($j \neq k$) trivially, the condition (2) for $\tilde{\sigma}$ implies that

$$(6.1) \quad \tilde{\sigma} = \Sigma_{j=0}^{r} \tilde{\sigma}_{jj}^{j} + \Sigma_{j=1}^{r} \tilde{\sigma}_{jj}^{0} + \Sigma_{j=1}^{r} \tilde{\sigma}_{0j}^{j} + \Sigma_{j=1}^{r} \tilde{\sigma}_{j0}^{j} .$$

Now we define the *Euclidean j-th mean curvature vector* H_j ($1 \leq j \leq r$) in $\mathfrak{p}_0$ by

$$H_j = (1/n_j)\mathrm{Tr}\, \tilde{\sigma}_{jj}^{0} = (1/n_j)\Sigma_{k=1}^{n_j} \tilde{\sigma}_{jj}^{0}(e_{jk}, e_{jk})$$

where $\{e_{jk}\}_{k=1}^{n_j}$ denotes an orthonormal basis of $\mathfrak{p}_j$, and call the length h_j of the vector H_j the *Euclidean j-th mean curvature*. Then we have the following

Lemma 6.1. Let $\tilde{\sigma}$ in $\mathcal{H}_M$. Then

$$\tilde{\sigma}_{jj}^{\ 0}(X_j,Y_j) = \langle X_j,Y_j\rangle H_j$$

$$\tilde{\sigma}_{j0}^{\ j}(X_j,Z_0) = \tilde{\sigma}_{0j}^{\ j}(Z_0,X_j) = \langle Z_0,H_j\rangle X_j$$

<u>for any</u> j $(1 \leq j \leq r)$ <u>and</u> $Z_0 \in \mathfrak{p}_0$, $X_j, Y_j \in \mathfrak{p}_j$.

Proof. Since $\mathfrak{k}_j \cdot \tilde{\sigma} = 0$, we have

$$(6.2) \quad \tilde{\sigma}_{jj}^{\ 0}(T_jX_j,Y_j) + \tilde{\sigma}_{jj}^{\ 0}(X_j,T_jY_j) = 0$$

and

$$(6.3) \quad \tilde{\sigma}_{jj}^{\ j}(T_jX_j,Y_j) + \tilde{\sigma}_{jj}^{\ j}(X_j,T_jY_j) = T_j(\tilde{\sigma}_{jj}^{\ j}(X_j,Y_j))$$

for any $T_j \in \mathfrak{k}_j$ and all vectors $X_j, Y_j \in \mathfrak{p}_j$. Let $\{e_a\}_{a=1}^{n_0}$ be an orthonormal basis of $\mathfrak{p}_0$. Since M_j is irreducible, the condition (6.2) implies that

$$\langle \tilde{\sigma}_{jj}^{\ 0}(X_j,Y_j), e_a\rangle = c_j^a \langle X_j,Y_j\rangle$$

for some $c_j^a \in \mathbb{R}$ and thus

$$\tilde{\sigma}_{jj}^{\ 0}(X_j,Y_j) = \langle X_j,Y_j\rangle(\Sigma_{a=1}^{n_0} c_j^a e_a) = \langle X_j,Y_j\rangle H_j$$

for all vectors $X_j, Y_j \in \mathfrak{p}_j$.

The second equality is obtained by the symmetry condition (1) for $\tilde{\sigma}$ and the first equality.

q.e.d.

We denote by $\mathcal{H}_M^d$ the set defined in the same way as $\mathcal{H}_M$ by replacing the number $c/4$ in the condition (3) with the number d. Then we have the following

Lemma 6.2. <u>Let</u> $\tilde{\sigma}$ <u>in</u> $\mathcal{H}_M$. <u>Then</u> $\tilde{\sigma}_{jj}^{j} \in \mathcal{H}_{M_j}^{c/4 + h_j^2}$ <u>for each</u> j.

Proof. The conditions (1) and (2) for $\mathcal{H}_{M_j}^{c/4 + h_j^2}$ is obvious by the condition (1) for $\tilde{\sigma}$ and (6.3). We show that $\tilde{\sigma}_{jj}^{j}$ satisfies the condition (3) for $\mathcal{H}_{M_j}^{c/4 + h_j^2}$. Denote by R^{M_j} the curvature tensor of M_j. Then, by the condition (3) for $\tilde{\sigma}$,

$$(c/4)(\langle Y_j, Z_j\rangle X_j - \langle X_j, Z_j\rangle Y_j) = R^{M_j}(X_j, Y_j)Z_j - [\tilde{\sigma}(X_j), \tilde{\sigma}(Y_j)]Z_j$$

for all vectors $X_j, Y_j, Z_j \in \mathfrak{P}_j$. By (6.1) and Lemma 6.1, the second term of the right hand side is calculated as follows:

$$[\tilde{\sigma}(X_j), \tilde{\sigma}(Y_j)]Z_j = [\tilde{\sigma}_{jj}^{j}(X_j), \tilde{\sigma}_{jj}^{j}(Y_j)]Z_j + h_j^2(\langle Y_j, Z_j\rangle X_j - \langle X_j, Z_j\rangle Y_j).$$

Hence $\tilde{\sigma}_{jj}^{j}$ satisfies the condition (3) for $\mathcal{H}_{M_j}^{c/4 + h_j^2}$.

q.e.d.

Lemma 6.3. <u>Let</u> $\tilde{\sigma}$ <u>in</u> $\mathcal{H}_M$. <u>Then</u> $\tilde{\sigma}_{00}^{0} \in \mathcal{H}_{\mathbb{R}^{n_0}}$ <u>and</u>

$$\tilde{\sigma}_{00}^{0}(X_0, H_j) = \langle X_0, H_j\rangle H_j - (c/4)X_0$$

for any $X_0 \in \mathfrak{p}_0$. *Moreover* $\langle H_j, H_k \rangle = -c/4$ *for distinct indeces* j,k $(1 \leq j,k \leq r)$.

Proof. Note that the condition (2) for $\mathcal{H}_{\mathbb{R}^{n_0}}$ is obvious since $\mathbb{R}^{n_0}$ is flat. Moreover by the conditions (1) and (3) for $\tilde{\sigma}$ we can see easily that $\tilde{\sigma}^0_{00}$ satisfies the conditions (1) and (3) for $\mathcal{H}_{\mathbb{R}^{n_0}}$. Put $X = X_0 \in \mathfrak{p}_0$, $Y = Y_j$, $Z = Z_j \in \mathfrak{p}_j$ in the condition (3) for $\tilde{\sigma}$. Then we have

$$(c/4)\langle Y_j, Z_j\rangle X_0 = -[\tilde{\sigma}(X_0), \tilde{\sigma}(Y_j)]Z_j .$$

The right hand side is calculated by (6.1) and Lemma 6.2 as follows:

$$-[\tilde{\sigma}(X_0), \tilde{\sigma}(Y_j)]Z_j = \langle X_0, H_j\rangle\langle Y_j, Z_j\rangle H_j - \langle Y_j, Z_j\rangle \tilde{\sigma}^0_{00}(X_0, H_j) .$$

Hence we have

$$(c/4)X_0 = \langle X_0, H_j\rangle H_j - \tilde{\sigma}^0_{00}(X_0, H_j).$$

Now, putting $X = X_j \in \mathfrak{p}_j$ and $Y = Y_k$, $Z = Z_k \in \mathfrak{p}_k$ $(1 \leq j \neq k \leq r)$ in the condition (3) for $\tilde{\sigma}$, we have

$$(c/4)\langle Y_k, Z_k\rangle X_j = -\langle Y_k, Z_k\rangle\langle H_j, H_k\rangle X_j$$

by (6.1) and Lemma 6.2, and thus $\langle H_j, H_k\rangle = -c/4$.

q.e.d.

Summing up Lemmas 6.1, 6.2 and 6.3, we have the claim (A) in the following

Theorem 6.4. Let M^n be a simply connected symmetric space with Euclidean factor decomposed as $M^n = \mathbb{R}^{n_0} \times \Pi_{j=1}^{r} M_j^{n_j}$ and $n = \Sigma_{j=0}^{r} n_j$. Then the following claims are true:

(A) Let $\tilde{\sigma} \in \mathcal{H}_M$. Then

(1) $\tilde{\sigma} = \Sigma_{j=0}^{r} \tilde{\sigma}_{jj}^{j} + \Sigma_{j=1}^{r} \tilde{\sigma}_{jj}^{0} + \Sigma_{j=1}^{r} \tilde{\sigma}_{j0}^{j} + \Sigma_{j=1}^{r} \tilde{\sigma}_{0j}^{j}$

(2) $\tilde{\sigma}_{jj}^{j} \in \mathcal{H}_{M_j}^{c/4 + h_j^2}$

(3) $\tilde{\sigma}_{00}^{0} \in \mathcal{H}_{\mathbb{R}^{n_0}}$, $\langle H_j, H_k \rangle = -c/4 \ (1 \leq j \neq k \leq r)$,

$\tilde{\sigma}_{00}^{0}(Z_0, H_j) = \langle Z_0, H_j \rangle H_j - (c/4) Z_0$

(4) $\tilde{\sigma}_{j0}^{j}(X_j, Z_0) = \tilde{\sigma}_{0j}^{j}(Z_0, X_j) = \langle Z_0, H_j \rangle X_j$,

$\tilde{\sigma}_{jj}^{0}(X_j, Y_j) = \langle X_j, Y_j \rangle H_j$

for any $Z_0 \in \mathfrak{p}_0$ and all vectors $X_j, Y_j \in \mathfrak{p}_j$.

(B) Conversely any p-valued bilinear form $\tilde{\sigma}$ on $\mathfrak{p}$ satisfying the conditions (1),(2),(3),(4) of (A) is an element in $\mathcal{H}_M$.

Here the proof of our claim (B) is omitted since it is straightforward.

Remark 6.5. Let M^n be a simply connected symmetric space with Euclidean factor. Changing the metric on M^n componentwise, we can construct infinitely many elements in $\mathcal{H}_M$. In fact, decompose M as above and suppose that $n_0 = r \geq 1$. First we shall show that there exist a basis $\{H_i\}_{i=1}^r$ of $\mathbb{R}^r$ and an $\mathbb{R}^r$-valued bilinear form $\tilde{\sigma}_{00}^0$ on $\mathbb{R}^r$ satisfying the condition (3) of (A). If there exist such basis and $\mathbb{R}^r$-valued form, by (B) of Theorem 6.4, an element in $\mathcal{A}_M$ can be constructed. Let $\{e_j\}_{j=1}^r$ be an orthonormal basis of $\mathbb{R}^r$ and set $H_i = \Sigma_{j=1}^r a_i^j e_j$, $A = (a_i^j)$. Moreover, for positive real numbers $h_1, \cdots, h_r$, we set

$$S(h_1,\cdots,h_r) = \begin{pmatrix} {h_1}^2 & -c/4 & \cdots & -c/4 \\ -c/4 & {h_2}^2 & & \vdots \\ \vdots & & \ddots & -c/4 \\ -c/4 & \cdots & -c/4 & {h_r}^2 \end{pmatrix}.$$

Then the condition for that $\{H_i\}$ is a basis of $\mathbb{R}^r$ such that $|H_j| = h_j$ $(1 \leq j \leq r)$ and $\langle H_j, H_k\rangle = -c/4$ $(j \neq k)$ is written as follows:

$$(6.4) \quad \det A \neq 0, \ A^tA = S(h_1,\cdots,h_r).$$

Since the matrix $S(h_1,\cdots,h_r)$ is symmetric, for sufficiently large numbers $h_1,\cdots,h_r$, there exists a positive definite symmetric matrix A satisfying the condition (6.4). Then we define an $\mathbb{R}^r$-valued bilinear form $\tilde{\sigma}_{00}^0$ on $\mathbb{R}^r$ as follows:

$$\tilde{\sigma}_{00}^0(H_j,H_k) = \langle H_j,H_k\rangle H_k - (c/4)H_j .$$

By easy calculations, we can see that the $\mathbb{R}^r$-valued bilinear form $\tilde{\sigma}_{00}^0$ on $\mathbb{R}^r$ satisfies the condition (3) of (A). Thus we get infinitely many elements in $\mathcal{H}_M$ by taking suitable metrics on M_j ($1 \leq j \leq r$).

Now, in the case when $M = \mathbb{R}^2$, we have the following

Theorem 6.6. <u>There exists a unique complete totally real parallel flat minimal surface</u> M^2 <u>in</u> $P^2(c)$ (<u>up to holomorphic isometries of</u> $P^2(c)$). <u>The norm</u> $|\sigma|$ <u>of the second fundamental form</u> σ <u>of</u> M^2 <u>is given by</u> $|\sigma|^2 = (1/2)c$.

Proof. Let $\{e_1, e_2\}$ be an orthonormal basis of $\mathbb{R}^2$. Then the condition $\tilde{\sigma} \in \mathcal{H}_{\mathbb{R}^2}$ is equivalent to the condition that

$$(6.5) \quad \left\{ \begin{aligned} \tilde{\sigma}(e_1, e_1) &= \alpha e_1 + \beta e_2 \\ \tilde{\sigma}(e_1, e_2) &= \beta e_1 + \gamma e_2 \\ \tilde{\sigma}(e_2, e_2) &= \gamma e_1 + \delta e_2 \end{aligned} \right\}, \text{ and } \quad c/4 = \beta^2 + \gamma^2 - \alpha\gamma - \beta\delta.$$

Suppose that the totally real parallel immersion of $\mathbb{R}^2$ corresponding to $\tilde{\sigma}$ is minimal. Then $\alpha + \gamma = \beta + \delta = 0$ and thus $\beta^2 + \gamma^2 = c/8$ by the second equality of (6.5). Put $\beta = \sqrt{c/8}\cos\theta$ and $\gamma = \sqrt{c/8}\sin\theta$ for some θ and define a linear isometry g of $\mathbb{R}^2$ by

$$(g(e_1), g(e_2)) = (e_1, e_2) \begin{pmatrix} \cos(\theta/3) & \sin(\theta/3) \\ -\sin(\theta/3) & \cos(\theta/3) \end{pmatrix}.$$

Then we have

$$(g\cdot\tilde{\sigma})(e_1,e_1) = -(g\cdot\tilde{\sigma})(e_2,e_2) = \sqrt{c/8}\, e_2 , \quad (g\cdot\tilde{\sigma})(e_1,e_2) = \sqrt{c/8}\, e_1 .$$

Hence all elements in $\mathcal{H}_R 2$ corresponding to minimal immersions belong to the same equivalence class. Now by Theorem 3.4 and 3.6 we get our first claim. The second claim follows from $|g\cdot\tilde{\sigma}|^2 = (1/2)c$.

q.e.d.

Remark 6.7. S.T.Yau [18] has shown that if M^2 is a complete non-negative curved totally real minimal surface in $P^2(c)$, M^2 is totally geodesic or flat, and moreover in the last case the second fundamental form is parallel. The minimal surface of Theorem 6.6 gives a unique example of such surfaces in the flat case. This has been constructed concretely in my previous paper [11] and it is compact.

Remark 6.8. B.Y.Chen and K.Ogiue [3] has shown that if M^n is a compact totally real minimal submanifold in $P^n(c)$ such that $|\sigma_p|^2 < (n(n+1)/4(2n-1))c$ for any point p in M, then M^n is totally geodesic. Suppose that $|\sigma_p|^2 = (n(n+1)/4(2n-1))c$ for any point p in M. Then, along their proof, the second fundamental form is parallel. In the case when $n = 2$ (then $(n(n+1)/4(2n-1))c = (1/2)c$), the universal covering of the compact totally real parallel minimal surface M^2 has Euclidean factor and thus is flat. Hence our minimal surface in $P^2(c)$ of Theorem 6.6 is a unique compact totally real minimal surface M^2 in $P^2(c)$ such that $|\sigma_p|^2 = (1/2)c$

for any point p in M^2.

Remark 6.9. In the next paper together with M.Takeuchi the complete classification of n-dimensional complete totally real parallel submanifolds in $P^n(c)$ shall be given by a different way.

References

[1] S.Araki : On root systems and an infinitesimal classification of irreducible symmetric spaces, J. Math. Osaka City Univ, 13 (1962), 1-34.

[2] N.Bourbaki : Elements de Mathematique Groupes et Algebres de Lie, Chap. 4-6, 1968.

[3] B.Y.Chen and K.Ogiue : On totally real submanifolds, Transactions of the Amer. Math. Soc, 193(1974), 257-266.

[4] P.Dombrowski : Differentiable maps into riemannian manifolds of constant stable osculating rank I, J. Reine Angew. Math, 274/275 (1975), 310-341.

[5] D.Ferus : Product-zerlegung von Immersionen mit paralleler zweiter Fundamentalform, Math. Ann, 211(1974), 1-5.

[6] ——— : Immersions with parallel second fundamental form, Math. Z, 140(1974), 87-93.

[7] ——— : Symmetric submanifolds of euclidean space, to appear.

[8] S.Helgason : Differential Geometry, Lie groups and Symmetric spaces, Academic Press, 1978.

[9] S.Kobayashi and K.Nomizu : Foundations of Differential geometry I,II, Wiley(Interscience), 1963 and 1969.

[10] H.Nakagawa and R.Takagi : On locally symmetric Kaehler submanifolds in a complex projective space, J. Math. Soc. Japan, 28 (1976), 638-667.

[11] H.Naitoh : Isotropic submanifolds with parallel second fundamental forms in $P^m(c)$, to appear.

[12] K.Nomizu : A characterization of the Veronese varieties, Nagoya Math. J, 60(1976), 181-188.

[13] B.O'Neill : The fundamental equations of a submersion, Michigan Math. J, 13(1966), 459-469.

[14] O.Shukugawa and I.Yokota : Non-compact simple Lie group $E_{6(6)}$ of Type E_6, J. Fac. Sci. Shinshu Univ, 14(1979), 1-13.

[15] W.Strübing : Symmetric Submanifolds of Riemannian Manifolds, Math. Ann, 245(1979), 37-44.

[16] M.Takeuchi : Homogeneous Kähler submanifolds in complex projective spaces, Japanese J. Math, 4(1978), 171-219.

[17] ———— : Parallel submanifolds of space forms, to appear.

[18] S.T.Yau : Submanifolds with constant mean curvature I, Amer. J. of Math, 96(1974), 346-366.

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