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複素射影空間の全実 (i 平行部分为様体 Istally real parallel submanifolds in $P^{n}(\mathbb{C})$ 内藤博夫

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Totally real parallel submanifolds in Pⁿ(C)

By Hiroo Naitoh (*)

Introduction. In the study of submanifolds in symmetric spaces, parallel submanifolds often play an important role. For example, in the study of minimal submanifolds in the Euclidean sphere the symmetric R-spaces, which are parallel submanifolds, provide abundant examples for testing various conjectures. Hence it seems to be useful to classify the parallel submanifolds in a specific symmetric space. Actually, these submanifolds have been classified by D.Ferus [5],[6],[7] when the ambient space is the Euclidean space or the Euclidean sphere, and by M.Takeuchi [17] when the ambient space is the real hyperbolic space. Moreover H.Nakagawa and R.Takagi [10] and M.Takeuchi [16] have classified the parallel Kähler submanifolds in the complex projective space with constant holomorphic sectional curvatures.

In this paper we study n-dimensional complete totally real parallel submanifolds in the n-dimensional complex projective space $P^{n}(C)$ with constant holomorphic sectional curvatures. It is known that a riemannian manifold which admits a parallel isometric immersion into a symmetric space is a locally symmetric space. Fix an n-dimensional simply connected symmetric space M^{n} . Let $\overline{\mathcal{T}}_{M}$ (resp. $\overline{\mathcal{X}}_{M}$) be the set of all equivalence classes of totally real parallel isometric immersions of M^{n} into $P^{n}(C)$ (resp. of complete totally real parallel submanifolds in $P^{n}(C)$ with the universal riemannian cover-

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ing M^n). Moreover, in section 3 we define an equivalence relation among symmetric trilinear forms on a tangent space of M satisfying certain conditions, and denote by $\overline{\mathcal{A}}_M$ the set of all equivalence classes of these trilinear forms. In sections 2,3, we shall show that there are the natural correspondences among these sets $\overline{\mathcal{J}}_M$, $\overline{\mathcal{S}}_M$, $\overline{\mathcal{A}}_M$. In section 4,5, we shall determine the set $\overline{\mathcal{A}}_M$ for a symmetric space M without Euclidean factor. Moreover, in section 6, we shall study the set $\overline{\mathcal{A}}_M$ for a symmetric space M with Euclidean factor and an important example in the geometry of totally real surfaces in $P^2(C)$.

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1. Preliminaries

Let \overline{M}^{m} (resp. M^{n}) be an m-dimensional (resp. n-dimensional) connected riemannian manifold. Denote by $\overline{\nabla}$ (resp. ∇) the riemannian connection on \overline{M}^{m} (resp. M^{n}) and by \overline{R} (resp. R) the riemannian curvature tensor for $\overline{\nabla}$ (resp. ∇). Now let f be an isometric immersion of M^{n} into \overline{M}^{m} . We denote by the same notation <,> the riemannian metrics on the both riemannian manifolds. Moreover denote by σ_{f} the second fundamental form of M^{n} , by D the normal connection on the normal bundle N(M) of M^{n} and by R^{1} the curvature tensor for D. For a point p in M and a vector ζ in the normal space $N_{p}(M)$ at p, the shape operator A_{ζ} is defined

2

$$< A_r(X), Y > = < \sigma_f(X, Y), \zeta >$$

for all vectors $X, Y \in T_p(M)$. The shape operator A_{ζ} is a symmetric endomorphism on the tangent space $T_p(M)$ at p. It is also characterized by the equation that

$$\overline{\nabla}_{X} \zeta = - A_{\zeta} (X) + D_{X} \zeta$$

for any tangent vector field X of M and any normal vector field ζ of M.

Now we recall the following fundamental equations, called the equations of Gauss, Codazzi-Mainardi, and Ricci respectively.

(1.1)
$$\langle \overline{R}(X,Y)Z,W \rangle = \langle R(X,Y)Z,W \rangle + \langle \sigma_f(X,Z),\sigma_f(Y,W) \rangle$$

$$(1.2) \quad \{\overline{R}(X,Y)Z\}^{\perp} = (\nabla_{Y}^{*}\sigma_{f})(Y,Z) - (\nabla_{Y}^{*}\sigma_{f})(X,Z)$$

(1.3)
$$\langle \overline{R}(X,Y)\zeta,\eta\rangle = \langle R^{\perp}(X,Y)\zeta,\eta\rangle - \langle [A_{\gamma},A_{\eta}](X),Y\rangle$$

for all vectors $X, Y, Z, W \in T_p(M)$ and all vectors $\zeta, \eta \in N_p(M)$. Here we denote by $\{\star\}^{\perp}$ the normal component of \star and by $\nabla \star$ the covariant derivation associated to the isometric immersion $f: M + \overline{M}$, defined by

$$(\nabla_X^*\sigma_f)(Y,Z) = D_X(\sigma_f(Y,Z)) - \sigma_f(\nabla_X Y,Z) - \sigma_f(Y,\nabla_X Z)$$

by

for tangent vector fields X,Y,Z of M. The second fundamental form σ_{fc} as well as the isometric immersion f is said to be <u>parallel</u> if $\nabla * \sigma_{f} = 0$. Moreover when f is an imbedding, the submanifold f(M) is called a <u>parallel submanifold</u> in \overline{M} . If the second fundamental form σ_{f} is parallel, we have

(1.4)
$$D_X(\sigma_f(Y,Z)) = \sigma_f(\nabla_X Y,Z) + \sigma_f(Y,\nabla_X Z)$$

for all tangent vector fields X,Y,Z of M.

Now let $\overline{M}^{2r} = P^{r}(c)$ be the r-dimensional complex projective space with constant holomorphic sectional curvatures c (>0). The complex structure of $P^{r}(c)$ will be denoted by J. An isometric immersion $f: M^{n} + P^{r}(c)$ is called <u>totally real</u> if $JT_{p}(M) \subset N_{p}(M)$ for every point p in M. Moreover when f is an imbedding, the submanifold f(M) is called a <u>totally real submanifold</u> in $P^{r}(c)$. Then we have the following

Lemma 1.1 (cf. See [11]). Let f be a totally real isometric immersion of M^n into $P^r(c)$. Then

$$\langle \sigma_{f}(X,Y), JZ \rangle = \langle \sigma_{f}(X,Z), JY \rangle$$

for any point $p \in M$ and all vectors $X, Y, Z \in T_p(M)$.

From now on we assume that the complex dimension r equals n. For a totally real isometric immersion $f: M^n + p^n(c)$ we define the <u>associated tensor</u> \tilde{q}_f of M as follows:

$$\tilde{\sigma}_{f}(x, y) = J\sigma_{f}(x, y)$$

for vectors $X, Y \in T_p(M)$, $p \in M$. If we identify the tangent space $T_p(M)$ with the cotangent space $T_p^*(M)$ through the riemannian metric on M, the associated tensor $\tilde{\sigma}_f$ is a symmetric covariant tensor of degree 3 on M by Lemma 1.1. For a vector X in $T_p(M)$, we define a symmetric endomorphism $\tilde{\sigma}_f(X)$ of $T_p(M)$ by

$$\tilde{\sigma}_{f}(X)(Y) = \tilde{\sigma}_{f}(X,Y)$$

for a vector Y in $T_p(M)$. Since the isometric immersion f is totally real in $P^n(c)$, we have $\overline{R}(X,Y)Z \in T_p(M)$ for all vectors X,Y,Z $\in T_p(M)$ and hence the equation of Gauss reduces to

(1.5) $\overline{R}(X,Y)Z = R(X,Y)Z - [\tilde{\sigma}_{f}(X), \tilde{\sigma}_{f}(Y)](Z)$

for all vectors $X, Y, Z \in T_{p}(M)$. Moreover we have the following

Lemma 1.2. Let f be a totally real parallel isometric immersion of M^n into $P^n(c)$. Then $\nabla \tilde{\sigma}_f = 0$, that is,

$$\nabla_{\mathbf{x}} \left(\tilde{\sigma}_{f}(\mathbf{Y}, \mathbf{Z}) \right) = \tilde{\sigma}_{f}(\nabla_{\mathbf{x}} \mathbf{Y}, \mathbf{Z}) + \tilde{\sigma}_{f}(\mathbf{Y}, \nabla_{\mathbf{x}} \mathbf{Z})$$

for all tangent vector fields X,Y,Z of M.

Proof. Since JC is a tangent vector field of M for any

normal vector field ζ along M,

$$J\overline{\nabla}_{x}J\zeta = J\nabla_{x}J\zeta + J\sigma_{f}(X,J\zeta)$$

for every tangent vector field X of M, while

$$J\overline{\nabla}_{X}J\zeta = -\overline{\nabla}_{X}\zeta = A_{\zeta}(X) - D_{X}\zeta$$

since $J \circ \overline{\nabla}_X = \overline{\nabla}_X \circ J$. Hence, comparing normal components we get

$$JD_{x}\zeta = \nabla_{x}J\zeta .$$

Thus, substituting $\zeta = \sigma_f(Y,Z)$, together with (1.4) we have

$$\nabla_{\mathbf{X}}(\tilde{\sigma}_{\mathbf{f}}(\mathbb{Y},\mathbf{Z})) = \tilde{\sigma}_{\mathbf{f}}(\nabla_{\mathbf{X}}\mathbb{Y},\mathbf{Z}) + \tilde{\sigma}_{\mathbf{f}}(\mathbb{Y},\nabla_{\mathbf{X}}\mathbb{Z})$$

for all tangent vector fields X,Y,Z of M.

q.e.d.

Let $\mathfrak{O}(T_p(M))$ be the Lie algebra of all skew symmetric endomorphisms of $T_p(M)$ and $\mathfrak{K}(p)$ the Lie subalgebra in $\mathfrak{O}(T_p(M))$ generated by the set { $R_p(X,Y)$; $X,Y \in T_p(M)$ }. Since the isometric immersion f is parallel, the manifold M is a locally symmetric space and hence the Lie algebra $\mathfrak{K}(p)$ is spanned by the set { $R_p(X,Y)$; $X,Y \in T_p(M)$ } and coincides with the holonomy algebra of M at p. Thus, by Lemma 1.2, we have the following

Corollary 1.3. Let f be a totally real parallel isometric immersion of M^n into $P^n(c)$. Then $(k)(p) \cdot \tilde{\sigma}_f = 0$, that is,

$$\mathbf{T}(\tilde{\sigma}_{f}(\mathbf{X},\mathbf{Y})) = \tilde{\sigma}_{f}(\mathbf{T}(\mathbf{X}),\mathbf{Y}) + \tilde{\sigma}_{f}(\mathbf{X},\mathbf{T}(\mathbf{Y}))$$

for any endomorphism $T \in \mathbb{R}(p)$ and all vectors $X, Y \in T_p(M)$.

2. Equivariant immersions associated to trilinear forms

Assume that the manifold M^n is a simply connected symmetric space and fix a point o in M^n . Put

$$(p) = T_{O}(M)$$
, $(k) = (k)(O)$ and $(g) = (k) + (p)$

and define the bracket product [,] on (g) as follows:

$$[T,S] = T \circ S - S \circ T$$
, $[T,X] = - [X,T] = T(X)$,
 $[X,Y] = - R_0(X,Y)$

for endomorphisms T,S in (k) and vectors X,Y in (p). Then (G, [,]) is a Lie algebra over R and there exists a simply connected Lie group G acting on the symmetric space M isometrically and transitively, such that the Lie algebra of G is isomorphic to (g) and that the Lie subgroup $K = \{g \in G; g(o) = o\}$ is connected and has the Lie subalgebra (k) (cf. See [8]). Let \mathcal{M}_{M} be the set of all (p)-valued bilinear form $\tilde{\sigma}$ on (p) satisfying the following conditions: (1) $\tilde{\sigma}$ is a symmetric trilinear form on P under the canonical identification of $\textcircled{P}^* \otimes \textcircled{P}^* \otimes \textcircled{P}$ with $\textcircled{P}^* \otimes \textcircled{P}^* \otimes \textcircled{P}^*$ through the riemannian metric < , > on P,

(2) $(k) \cdot \tilde{\sigma} = 0$,

(3) $(c/4)(\langle Y,Z \rangle X - \langle X,Z \rangle Y) = R(X,Y)Z - [\tilde{\sigma}(X),\tilde{\sigma}(Y)](Z)$ for all vectors $X,Y,Z \in \mathbb{Q}$.

Let f be a totally real parallel isometric immersion of M^n into $P^n(c)$. Then

$$\overline{R}(X,Y)Z = (c/4)(\langle Y,Z \rangle X - \langle X,Z \rangle Y)$$

for all vectors $X, Y, Z \in \mathbb{D}$. Hence we have that $(\tilde{\sigma}_f)_o \in \mathcal{U}_M$ by Lemma 1.1, Corollary 1.3 and (1.5).

Now the riemannian manifold $P^{n}(c)$ is also a simply connected symmetric space. For $P^{n}(c)$, we use notations $\overline{o}, \overline{O}, \overline{K}, \overline{O}, \overline{G}, \overline{K}$ for corresponding objects $o, \overline{O}, \overline{K}, \overline{O}, G, K$. Note that \overline{G} (resp. \overline{O}) is isomorphic to the compact Lie group SU(n+1) (resp. the compact Lie algebra $\overline{SU}(n+1)$) and that \overline{K} is given by

 $\overline{(k)} = (\underline{b})(\overline{p}) = \{ T \in \mathbb{O}(\overline{p}) ; J \circ T = T \circ J \}.$

A linear subspace 0 in $\overline{\textcircled{D}}$ is called <u>totally real</u> if the subspaces 0 and J are orthogonal. Totally real subspaces in $\overline{\textcircled{D}}$ of the same dimension are conjugate cach other under the natural action of \overline{K} on $\overline{\textcircled{D}}$. Fix an n-dimensional totally real subspace 0 in $\overline{\textcircled{D}}$

8

and set

 $\overline{\mathbb{Q}}_{1} = \{ \mathbf{T} \in \overline{\mathbb{Q}} ; \mathbf{T}(\underline{0}) \subset \overline{\mathbb{Q}} \} \text{ and } \overline{\mathbb{Q}}_{2} = \{ \mathbf{T} \in \overline{\mathbb{Q}} ; \mathbf{T}(\underline{0}) \subset J_{\overline{0}} \}.$

Then $\overline{\otimes}_1$ (resp. $\overline{\otimes}_2$) is a Lie subalgebra (resp. linear subspace) in $\overline{\otimes}_1$ and $\overline{\otimes}$ is the direct sum of $\overline{\otimes}_1$ and $\overline{\otimes}_2$. In fact, take an orthonormal basis $\{e_1, \dots, e_n\}$ of $\overline{\bigcirc}$ and identify $\overline{\bigcirc}$ with \mathbf{C}^n by the correspondence:

$$\overline{\mathbb{O}} \ni (\Sigma \mathbf{x}_{j} \mathbf{e}_{j}) + \mathbf{J}(\Sigma \mathbf{y}_{j} \mathbf{e}_{j}) \longleftrightarrow (\mathbf{x}_{j} + \sqrt{-1} \mathbf{y}_{j}) \in \mathbb{C}^{n}.$$

Then (k), (k) and (k) are identified with the Lie algebra (k)(n) of all skew hermitian matrices of degree n, the Lie algebra (m) of all real skew symmetric matrices of degree n, and the linear space $\sqrt{-1} S^{n}(R) = \{ \sqrt{-1}A; A \text{ is a real symmetric matrix of degree n } \}$ respectively. This implies the assertion.

Let s be an Euclidean isometry of \bigcirc onto \bigcirc . We define an injective Lie homomorphism τ_s of SO(O) into \textcircled{B}_1 by

$$\tau_{s}(T)(s(X) + Js(Y)) = s(T(X)) + Js(T(Y))$$

for $T \in \mathfrak{O}(\mathbb{P})$ and vectors $X, Y \in \mathbb{P}$. Next, for an element $\tilde{\mathfrak{g}}$ in \mathfrak{M}_{M} , we define a linear mapping $\mu_{s,\tilde{\mathfrak{g}}}$ of \mathbb{P} into $\overline{\mathfrak{K}}_{2}$ by

$$\mu_{s,\tilde{\alpha}}(X)(s(Y) + Js(Z)) = s(\tilde{\alpha}(X,Z)) - Js(\tilde{\alpha}(X,Y))$$

for vectors $X, Y, Z \in \mathbb{D}$. Here note that the condition (1) for $\tilde{\sigma}$ implies that $\mu_{s,\tilde{\sigma}}(X) \in \overline{\mathbb{B}}$. Now we define a linear mapping $\rho_{s,\tilde{\sigma}}$ of $(\overline{\mathfrak{G}})$ into $\overline{\mathfrak{G}}$ by

$$\rho_{s,\tilde{\sigma}}(T+X) = \tau_{s}(T) + \mu_{s,\tilde{\sigma}}(X) + s(X)$$

for $T \in \mathbb{C}$ and $X \in \mathbb{C}$. Then we have the following

Lemma 2.1. <u>The linear mapping</u> $\rho_{s,\tilde{\sigma}}$ <u>of @ into @ is an</u> <u>injective Lie homomorphism</u>.

Proof. At first we shall prove the following three formulas:

(2.1)
$$[\tau_{s}(T), \mu_{s,\tilde{\sigma}}(X)] = \mu_{s,\tilde{\sigma}}(T(X))$$

(2.2)
$$[\mu_{s,\tilde{\sigma}}(X), \mu_{s,\tilde{\sigma}}(Y)] = \tau_{s}([\tilde{\sigma}(Y), \tilde{\sigma}(X)])$$

(2.3)
$$\overline{R}(s(X), s(Y)) = \tau_{s}(R(X, Y) - [\tilde{\sigma}(X), \tilde{\sigma}(Y)])$$

for any $T \in \mathbb{O}$ and all vectors $X, Y \in \mathbb{O}$. By the condition (2) for $\tilde{\sigma}$ we have

$$\begin{split} & [\tau_{s}(T), \mu_{s,\tilde{q}}(X)](s(Y) + Js(Z)) \\ &= s(T(\tilde{q}(X,Z))) - Js(T(\tilde{q}(X,Y))) + Js(\tilde{q}(X,T(Y))) - s(\tilde{q}(X,T(Z))) \\ &= s(\tilde{q}(T(X),Z)) - Js(\tilde{q}(T(X),Y)) \\ &= \mu_{s,\tilde{q}}(T(X))(s(Y) + Js(Z)) \end{split}$$

for all vectors Y,Z $\in \ensuremath{\widehat{p}}$, and hence (2.1) is proved. Next, by the definitions of $\tau_{\rm s}$ and $\mu_{{\rm s},\widetilde{\sigma}}$ we have

$$[\mu_{\mathbf{S}_{\ast}\widetilde{\boldsymbol{\alpha}}}(\mathbf{X}),\mu_{\mathbf{S}_{\ast}\widetilde{\boldsymbol{\alpha}}}(\mathbf{Y})](\mathbf{s}(\mathbf{Z}) + \mathbf{J}\mathbf{s}(\mathbf{W}))$$

 $= - Js(\tilde{\sigma}(X,\tilde{\sigma}(Y,W))) - s(\tilde{\sigma}(X,\tilde{\sigma}(Y,Z))) + Js(\tilde{\sigma}(Y,\tilde{\sigma}(X,W)))$ $+ s(\tilde{\sigma}(Y,\tilde{\sigma}(X,Z)))$

$$= s([\tilde{\sigma}(Y), \tilde{\sigma}(X)](Z)) + Js([\tilde{\sigma}(Y), \tilde{\sigma}(X)](W))$$

 $= \tau_{s}([\tilde{\sigma}(Y),\tilde{\sigma}(X)])(s(Z) + Js(W))$

for all vectors Z,W in \bigcirc , and hence (2.2) is proved. Since the subspace \bigcirc in \bigcirc is totally real, we have

$$\overline{R}(s(X), s(Y)) s(Z) = (c/4) (\langle Y, Z \rangle s(X) - \langle X, Z \rangle s(Y))$$

for all vectors $X, Y, Z \in \mathbb{O}$. By the condition (3) for $\tilde{\sigma}$ we have

$$\overline{R}(s(X), s(Y))(s(Z) + Js(W))$$

$$= \overline{R}(s(X), s(Y))s(Z) + J\overline{R}(s(X), s(Y))s(W)$$

$$= s((c/4)(\langle Y, Z \rangle X - \langle X, Z \rangle Y)) + Js((c/4)(\langle Y, W \rangle X - \langle X, W \rangle Y))$$

$$= s(R(X,Y)Z - [\tilde{\sigma}(X), \tilde{\sigma}(Y)]Z) + Js(R(X,Y)W - [\tilde{\sigma}(X), \tilde{\sigma}(Y)]W)$$

$$= \tau_{c}(R(X,Y) - [\tilde{\sigma}(X), \tilde{\sigma}(Y)])(s(Z) + Js(W))$$

for all vectors $.2, W \in \mathbb{Q}$. Hence (2.3) is proved.

Now by (2.1), (2.2) and (2.3) we have

$$[\rho_{s,\tilde{\sigma}}(T+X), \rho_{s,\tilde{\sigma}}(S+Y)]$$

$$= [\tau_{s}(T), \tau_{s}(S)] + [\tau_{s}(T), \mu_{s,\tilde{\sigma}}(Y)] + [\tau_{s}(T), s(Y)]$$

$$+ [\mu_{s,\tilde{\sigma}}(X), \tau_{s}(S)] + [\mu_{s,\tilde{\sigma}}(X), \mu_{s,\tilde{\sigma}}(Y)] + [\mu_{s,\tilde{\sigma}}(X), s(Y)]$$

$$+ [s(X), \tau_{s}(S)] + [s(X), \mu_{s,\tilde{\sigma}}(Y)] + [s(X), s(Y)]$$

$$= \tau_{s}([T,S]) + \mu_{s,\tilde{\sigma}}(T(Y)) + s(T(Y)) - \mu_{s,\tilde{\sigma}}(S(X))$$

$$+ \tau_{s}([\tilde{\sigma}(Y), \tilde{\sigma}(X)]) - Js(\tilde{\sigma}(X,Y)) - s(S(X)) + Js(\tilde{\sigma}(Y,X))$$

$$- \tau_{s}(R(X,Y) - [\tilde{\sigma}(X), \tilde{\sigma}(Y)])$$

$$= \tau_{s}([T,S] - R(X,Y)) + \mu_{s,\tilde{\sigma}}(T(Y) - S(X)) + s(T(Y) - S(X))$$

for all $T, S \in \mathbb{R}$ and all $X, Y \in \mathbb{Q}$, and hence $\rho_{S, \tilde{\sigma}}$ is a Lie homomorphism of \mathfrak{G} into $\overline{\mathfrak{G}}$. Moreover, since τ_{S} and s are injective, $\rho_{S, \tilde{\mathfrak{q}}}$ is injective.

q.e.d.

Since $\overline{\textcircled{O}}$ is a compact Lie algebra, we have the following

Corollary 2.2. If the set \mathcal{M}_{M} is not empty, the Lie algebra (g) is a compact Lie algebra.

We call $\rho_{s,\tilde{\sigma}}$ the <u>Lie homomorphism associated to</u> s <u>and</u> $\tilde{\sigma}$.

Since G is a simply connected Lie group, there exists the unique Lie homomorphism $\hat{\rho}_{s,\tilde{\sigma}}$ of G into \overline{G} such that the differential $d\hat{\rho}_{s,\tilde{\sigma}}$ is $\rho_{s,\tilde{\sigma}}$. The associated homomorphism $\rho_{s,\tilde{\sigma}}$ maps the Lie subalgebra (k) into the Lie subalgebra (c) and the isotropy subgroup K is connected. Hence we can define a Gequivariant C[∞]-mapping $f_{s,\tilde{\sigma}}$ of Mⁿ into Pⁿ(c) by

 $f_{s,\tilde{\alpha}}(g(o)) = \hat{\rho}_{s,\tilde{\alpha}}(g)(\overline{o})$

for $g \in G$. Then we have the following

Theorem 2.3. Let M^n be a simply connected symmetric space. <u>Then, for any Euclidean isometry</u> s and any $\tilde{\sigma} \in \mathcal{H}_M$, the associated G-equivariant mapping $f_{s,\tilde{\sigma}}$ of M^n into $P^n(c)$ is a totally real parallel isometric immersion such that

$$(f_{s,\tilde{\sigma}})_{*o} = s \text{ and } (\tilde{\sigma}_{f_{s,\tilde{\sigma}}})_{o} = \tilde{\sigma}.$$

Proof. Note that \overline{G} divided by the center is the group of all holomorphic isometries of $P^{n}(c)$. The claim $(f_{s,\tilde{\sigma}})_{*o} = s$ is obvious by the definition of $f_{s,\tilde{\sigma}}$. Now we show that $f_{s,\tilde{\sigma}}$ is a totally real parallel isometric immersion. Since $f_{s,\tilde{\sigma}}$ is G-equivariant, it is sufficient to see our claim at o. The linear mapping s is a isometry and the image \bigcirc of s is a totally real subspace in \bigcirc . Hence $f_{s,\tilde{\sigma}}$ is a totally real and isometric immersion at o. Moreover, to show that $f_{s,\tilde{\sigma}}$ is parallel, it is sufficient to see that

(2.4)
$$[\rho_{s,\tilde{q}}(x)_{\overline{K}}, [\rho_{s,\tilde{\sigma}}(x)_{\overline{K}}, \rho_{s,\tilde{q}}(x)_{\overline{D}}]] \in \mathbb{Q}$$

for any vector X in O (See Proposition 5.2 in [11]). Here the suffix O (resp. O) means the O-component (resp. O-component) with respect to the decomposition O = O + O. In fact, since

$$\rho_{s,\tilde{\sigma}}(x)_{\overline{\otimes}} = \mu_{s,\tilde{\sigma}}(x) \text{ and } \rho_{s,\tilde{\sigma}}(x)_{\overline{\otimes}} = s(x) ,$$

the left hand of (2.4) equals $-s(\tilde{\sigma}(X,\tilde{\sigma}(X,X))) \in \mathbb{Q}$. Now the second fundamental form at o of the G-equivariant immersion $f_{s,\tilde{\sigma}}$ is given by

(2.5)
$$(\tilde{\sigma}_{f_{s,\tilde{\sigma}}}) \circ (X,Y) = [(\rho_{s,\tilde{\sigma}})(X)_{\bar{\mathcal{B}}}, (\rho_{s,\tilde{\sigma}})(Y)_{\bar{\mathcal{D}}}]_{J_{\bar{\mathcal{C}}}}$$

for all vectors X,Y in O (See Proposition 5.1 in [11]). Here the suffix JO means the JO-component with respect to the decomposition O = O + J O. Hence we have $(\overset{\circ}{\sigma}_{f_{s,\widetilde{\sigma}}})_{o} = -J_{s}(\overset{\circ}{\sigma}(X,Y))$. This implies $(\overset{\circ}{\sigma}_{f_{s,\widetilde{\sigma}}})_{o} = \overset{\circ}{\sigma}$.

q.e.d.

3. Frenet curves and rigidity problems

Let \overline{M} be a riemannian manifold and c(t) be a C^{∞} -curve in \overline{M} defined on an open interval I containing 0 which is parametrized by arc-length. The curve c(t) is called a <u>Frenet curve</u> in \overline{M} of osculating rank $r (\geq 1)$ if for all $t \in I$ its higher order derivatives

$$\dot{\mathbf{c}}(t) = (\overline{\nabla}^{0}_{\frac{\partial}{\partial t}} \dot{\mathbf{c}})(t), (\overline{\nabla}_{\frac{\partial}{\partial t}} \dot{\mathbf{c}})(t), \cdots, (\overline{\nabla}_{\frac{\partial}{\partial t}}^{r-1} \dot{\mathbf{c}})(t)$$

are linearly independent but

$$\dot{\mathbf{c}}(t) = (\overline{\nabla}^{0}_{\frac{\partial}{\partial t}} \dot{\mathbf{c}})(t), \quad (\overline{\nabla}_{\frac{\partial}{\partial t}} \dot{\mathbf{c}})(t), \cdots, (\overline{\nabla}_{\frac{\partial}{\partial t}}^{r} \dot{\mathbf{c}})(t)$$

are linearly dependent in $T_{c(t)}(\overline{M})$. Then there exist the unique C^{∞} -positive functions $\kappa_1(t), \cdots, \kappa_{r-1}(t)$ on I and the unique C^{∞} -orthonormal vector fields $V_1(t), \cdots, V_r(t)$ along the curve c(t) such that

$$(3.1) \begin{cases} \dot{\mathbf{c}}(t) = \mathbf{v}_{1}(t) \\ (\overline{\mathbf{v}}_{\frac{\partial}{\partial t}} \mathbf{v}_{1})(t) = \kappa_{1}(t)\mathbf{v}_{2}(t) \\ (\overline{\mathbf{v}}_{\frac{\partial}{\partial t}} \mathbf{v}_{2})(t) = -\kappa_{1}(t)\mathbf{v}_{1}(t) + \kappa_{2}(t)\mathbf{v}_{3}(t) \\ \vdots \\ (\overline{\mathbf{v}}_{\frac{\partial}{\partial t}} \mathbf{v}_{j})(t) = -\kappa_{j-1}(t)\mathbf{v}_{j-1}(t) + \kappa_{j}(t)\mathbf{v}_{j+1}(t) \\ \vdots \\ (\overline{\mathbf{v}}_{\frac{\partial}{\partial t}} \mathbf{v}_{r-1})(t) = -\kappa_{r-2}(t)\mathbf{v}_{r-2}(t) + \kappa_{r-1}(t)\mathbf{v}_{r}(t) \\ (\overline{\mathbf{v}}_{\frac{\partial}{\partial t}} \mathbf{v}_{r})(t) = -\kappa_{r-1}(t)\mathbf{v}_{r-1}(t). \end{cases}$$

Here we call $\kappa_j(t)$ $(1 \le j \le r-1)$ the <u>Frenet curvature functions</u> on I, the vector fields { $V_j(t)$; $1 \le j \le r$ } the <u>Frenet r-frame</u> along c(t), and the equations (3.1) the <u>Frenet formulas</u>. For a given integer r (:21) and given C^{∞} -positive functions $\kappa_1(t), \dots, \kappa_{r-1}(t)$ on I, the Frenet formulas (3.1) may be regarded as a system of differential equations with variables c, V_1, \dots, V_r . It is known that this system of differential equations has the unique local solution for given initial conditions; a point $c(0) = p \in \overline{M}$ and an orthonormal r-frame { $V_1(0) =$ $V_1, \dots, V_r(0) = V_r$ } of $T_p(\overline{M})$. If the riemannian manifold \overline{M} is complete, the Frenet curve c(t) is defined for $-\infty < t < +\infty$ (cf. See [4] and [15]). Now we have the following

Lemma 3.1 (W.Strübing [15]). Let M and \overline{M} be riemannian manifolds and f a parallel isometric immersion of M into \overline{M} . Suppose that a curve c(t) defined on I containing 0 is a geodesic in M parametrized by arc-length. Then

a) the curve (foc)(t) on I is a Frenet curve in \overline{M} ,

b) the Frenet curvature functions $\kappa_1(t), \dots, \kappa_{r-1}(t)$ are constant (and positive), where r denotes the osculating rank of (foc)(t),

c) the integer r (≥ 1), the constant positive numbers $\kappa_1, \cdots, \kappa_{r-1}$ and the orthonormal vectors $V_1 = V_1(0), \cdots, V_r = V_r(0)$ are determined only by the initial point p = c(0) of c(t), the initial tangent vector $X = \dot{c}(0)$ of c(t), the differential $(f_*)_p$ at p, and the second fundamental form $(\sigma_f)_p$ at p.

Now, by Lemma 3.1, we have the following fundamental lemma.

Lemma 3.2. Let g and f be parallel isometric immersions of <u>a complete riemannian manifold</u> M into another riemannian manifold M.

16

If there exists a point o in M such that

 $g(o) = f(o) = \overline{o}, \ (g_{\ddagger})_{o} = (f_{\ast})_{o} : T_{o}(M) \rightarrow T_{\overline{o}}(\overline{M}), \ (\sigma_{g})_{o} = (\sigma_{f})_{o},$

then the mapping g and f coincide on M.

Proof. For any point p in M, there exists a geodesic c(t)in M parametrized by arc-length, such that c(0) = o and c(l) = p. Then $(g \circ c)(t)$ and $(f \circ c)(t)$ are Frenet curves in \overline{M} by Lemma 3.1,a). By Lemma 3.1,c), the above assumption implies that the Frenet curves $(f \circ c)(t)$ and $(g \circ c)(t)$ are solutions of same Frenet formulas for the same initial conditions. Hence, by the uniqueness for solutions of the system of differential equations, we have $(f \circ c)(t) = (g \circ c)(t)$ and paticularly f(p) = g(p).

q.e.d.

Now let \mathcal{T}_{M} be the set of all totally real parallel isometric immersions of a simply connected symmetric space M^{n} into the riemannian manifold $P^{n}(c)$, I(M) the group of all isometries of M, and \overline{G} the group of all holomorphic isometries of $P^{n}(c)$. Then we can define an action of $\overline{G} \times I(M)$ on \mathcal{T}_{M} by

$$(\overline{g},g) \cdot f = \overline{g} \circ f \circ g^{-1}$$

for $\overline{g} \in \overline{G}$, $g \in I(M)$ and $f \in \mathcal{J}_M$. Let $\overline{\mathcal{J}}_M$ be the set of all orbits of the $\overline{G} \times I(M)$ -action on \mathcal{J}_M . The orbit $[f]_{\mathcal{J}}$ of f in \mathcal{J}_M is

called the equivalence class of f.

Secondly, let \mathscr{S}_{M} be the set of all complete totally real parallel submanifolds with the universal riemannian covering M^{n} . Then we can define an action of \overline{G} on \mathscr{S}_{M} by

$$\overline{g} \cdot N = \overline{g}(N)$$

for $\overline{g} \in \overline{G}$ and $N \in \mathscr{S}_{M}^{M}$. Let $\overline{\mathscr{S}}_{M}^{M}$ be the set of all orbits of the \overline{G} -action on \mathscr{S}_{M}^{M} . The orbit $[N]_{\mathscr{S}}^{M}$ of N in \mathscr{S}_{M}^{M} is called the <u>equivalence class</u> of N.

Lastly, set

 $F_{O}(M) = \{ g \in I(M) ; g(O) = O \}.$

Then we can define an action of $F_{O}(M)$ on \mathcal{M}_{M} by

$$(k \cdot \tilde{\sigma}) (X, Y) = (k_*)_{o} (\tilde{\sigma} ((k_*)_{o}^{-1}X, (k_*)_{o}^{-1}Y))$$

for $k \in F_{O}(M)$, $\tilde{\sigma} \in \mathcal{H}_{M}$ and $X, Y \in \mathbb{Q}$. Let \mathcal{H}_{M} be the set of all orbits of the $F_{O}(M)$ -action on \mathcal{H}_{M} . The orbit $[\tilde{\sigma}]_{\mathcal{H}}$ of $\tilde{\sigma}$ in \mathcal{H}_{M} is called the <u>equivalence class</u> of $\tilde{\sigma}$.

Now we study the relations among three kinds of equivalences. At first we have the following

Lemma 3.3. For any $\overline{g} \in \overline{G}$, $g \in I(M)$ and $f \in \mathcal{J}_M$, there exists some $k \in F_O(M)$ such that

$$(\tilde{\sigma}_{\overline{g}\circ f\circ g}^{-1})_{o} = k \cdot (\tilde{\sigma}_{f})_{o}$$

Moreover, if $g \in F_0(M)$, the very same element g can be taken as the above element k.

Proof. Since \overline{g}_* and J are comutative, we have

$$(3.2) \quad (\tilde{\sigma}_{\overline{g}\circ f\circ g}^{-1})_{o}(X,Y) = (\tilde{\sigma}_{f\circ g}^{-1})_{o}(X,Y)$$
$$= g_{*}((\tilde{\sigma}_{f})_{g}^{-1}(o)((g_{*})^{-1}X,(g_{*})^{-1}Y))$$

for all vectors $X, Y \in \mathbb{D}$. Let $\gamma(t)$ be a geodesic joining o to $g^{-1}(o)$. Since M is a symmetric space, there exists some $h \in I(M)$ such that $h(o) = g^{-1}(o)$ and that $h^{-1} \cdot (\tilde{\sigma}_f)_{h(o)}$ is the parallel translate of $(\tilde{\sigma}_f)_{h(o)}$ along the geodesic $\gamma(t)$, where

$$h^{-1} \cdot (\tilde{a}_{f})_{h(o)} (x, y) = h_{\star}^{-1} ((\tilde{a}_{f})_{h(o)} (h_{\star} x, h_{\star} y))$$

for all vectors $X, Y \in \mathbb{D}$ (cf. See [8]). Putting $k = g \circ h$, we have $k \in F_0(M)$. Since $\tilde{\sigma}_f$ is parallel by Lemma 1.2, we have

the last term of (3.2)

$$= k_{\star} (h_{\star}^{-1} ((\tilde{\sigma}_{f})_{h(o)} (h_{\star} (k_{\star}^{-1} x), h_{\star} (k_{\star}^{-1} x)))$$

$$= k_{\star} ((\tilde{\sigma}_{f})_{o} (k_{\star}^{-1} x, k_{\star}^{-1} x)) = (k \cdot (\tilde{\sigma}_{f})_{o}) (x, x).$$

The second assertion is clear from the above proof.

q.e.d.

Now we define a mapping $i_M^{}$ of $\overline{\mathcal{J}}_M^{}$ into $\overline{\mathcal{A}}_M^{}$ by

$$i_{M}([f]_{\mathcal{J}}) = [(\tilde{\sigma}_{f})_{O}]_{\mathcal{H}}$$

for f in \mathcal{T}_{M} . By Lemma 3.3 the mapping i_{M} is well-defined. Then we have the following

Theorem 3.4. The mapping i_M of $\overline{\mathcal{J}}_M$ into $\overline{\mathcal{A}}_M$ is bijective.

Proof. By Theorem 2.3 it is obvious that i_M is onto. We show that the mapping i_M is injective. Take two mappings f_1, f_2 in \mathcal{T}_M and suppose that $(\tilde{\sigma}_{f_1})_0 = k \cdot (\tilde{\sigma}_{f_2})_0$ for some $k \in F_0(M)$. Then, putting $f_3 = f_2 \cdot k^{-1}$, we have $(\tilde{\sigma}_{f_1})_0 = (\tilde{\sigma}_{f_3})_0$ by Lemma 3.3. Since f_1 and f_3 are totally real, there exists some $\overline{g} \in \overline{G}$ such that

$$(\overline{g} \circ f_3)(o) = f_1(o) = \overline{o}$$
 and $(\overline{g} \circ f_3)_*(T_o(M)) = (f_1)_*(T_o(M)) = \overline{g}$.

Moreover, since any Euclidean isometry of the totally real subspace (G) is the differential at \overline{o} of some holomorphic isometry of $P^{n}(c)$, we may assume that $(\overline{g} \circ f_{3})_{*0} = (f_{1})_{*0}$. Here note that $(\tilde{\sigma}_{\overline{g}} \circ f_{3})_{0} = (\tilde{\sigma}_{f_{3}})_{0}$ by Lemma 3.3. Hence, by Lemma 3.2, we have $\overline{g} \circ f_{3} = f_{1}$ on M and thus $[f_{1}]_{\mathcal{T}} = [f_{3}]_{\mathcal{T}} = [f_{2}]_{\mathcal{T}}$.

q.e.d.

Theorem 3.5. Any totally real parallel isometric immersion of M^n into $P^n(c)$ is G-equivariant.

Proof. Let f be a totally real parallel isometric immersion and put $f(o) = \overline{o}$. Then we have $f = f_{(f_*)_{O}, (\tilde{\sigma}_f)_{O}}$ by Theorem 2.3 and Lemma 3.2. This implies the theorem.

q.e.d.

Now let j_M be a mapping of $\overline{\mathcal{T}}_M$ into $\overline{\mathcal{S}}_M$ defined by

 $j_{M}([f]_{\mathcal{T}}) = [f(M)]_{g}$

for $f \in \mathcal{T}_M$. Here note that the image f(M) is a submanifold in $P^n(c)$ by Theorem 3.5. Then we have the following

Theorem 3.6. The mapping j_M of $\overline{\mathcal{J}}_M$ into $\overline{\mathcal{S}}_M$ is bijective.

Proof. It is obvious that j_M is onto. We show that the mapping j_M is injective. Take two mappings $f_1, f_2 \in \mathcal{J}_M$ and suppose that $f_1(M) = \overline{g}(f_2(M))$ for some $\overline{g} \in \overline{G}$. Put $\overline{o} = f_1(o)$ and $N = f_1(M)$. Taking some $g \in I(M)$ and putting $f_3 = \overline{g} \circ f_2 \circ g$, we have

$$f_1(o) = f_3(o) = \overline{o}$$
 and $f_1(M) = f_3(M) = N$.

Let $(\sigma_N)_{\overline{O}}$ be the second fundamental form at \overline{O} of the submanifold N. Then we have

$$(\sigma_{N})_{\overline{o}}(\overline{X},\overline{Y}) = (\sigma_{f_{1}})_{o}((f_{1})_{*}^{-1}\overline{X},(f_{1})_{*}^{-1}\overline{Y})$$
$$= (\sigma_{f_{3}})_{o}((f_{3})_{*}^{-1}\overline{X},(f_{3})_{*}^{-1}\overline{Y})$$

for all vectors $X, Y \in T_{\overline{O}}(N)$. Hence we have

$$(\tilde{\sigma}_{f_3})_{o}(X,Y) = ((f_3)_{*}^{-1} \circ (f_1)_{*}) ((\tilde{\sigma}_{f_1})_{o}((f_1)_{*}^{-1} \circ (f_3)_{*}X, (f_1)_{*}^{-1} \circ (f_3)_{*}Y))$$

for all vectors $X, Y \in T_0(M)$. Note that $f_3^{-1} \circ f_1$ defines a local isometry of M around o. Since M is a simply connected symmetric space, there exists a unique element $k \in F_0(M)$ that coincides with $f_3^{-1} \circ f_1$ around o. Hence we have $(\tilde{\sigma}_{f_3})_0 = k \cdot (\tilde{\sigma}_{f_1})_0$. By Theorem 3.4 we have $[f_3]_{\mathcal{J}} = [f_1]_{\mathcal{J}}$ and thus $[f_2]_{\mathcal{J}} = [f_1]_{\mathcal{J}}$.

q.e.d.

4. The set $\overline{\mathcal{H}}_{M}$ for a simply connected symmetric space M without Euclidean factor

In this section we assume that Mⁿ is a simply connected symmetric space without Euclidean factor, thus, M is decomposed as a riemannian manifold as follows:

$$M^{n} = M_{1}^{n} 1 \times \cdots \times M_{r}^{n} r$$
 ($n = \Sigma_{j=1}^{r} n_{j}$)

where $M_j^n j$ is an n_j -dimensional irreducible simply connected symmetric space for each j. Then the tangent space $T_O(M) = \bigcirc$ (resp. the holonomy algebra \bigotimes) is decomposed as follows:

where the subspace $(\mathfrak{P}_{j} \subset \mathfrak{P})$ (resp. the subalgebra $(\mathfrak{R}_{j} \subset \mathfrak{R})$) denotes the tangent space $T_{0}(M_{j})$ (resp. the holonomy algebra of M_{j}). For a \mathfrak{P} -valued symmetric bilinear form $\tilde{\sigma}$ on \mathfrak{P} and any ordered triple {i,j,k} (l \leq i,j,k \leq r), a mapping $\tilde{\sigma}_{ij}^{k} : \mathfrak{P}_{i} \times \mathfrak{P}_{j} \to \mathfrak{P}_{k}$ is defined by

$$\tilde{\sigma}_{ij}^{k}(x_{i}, y_{j}) = \text{the } \mathcal{P}_{k} \text{-component of } \tilde{\sigma}(x_{i}, y_{j})$$

for $X_i \in \hat{\mathbb{Q}}_i$ and $Y_i \in \hat{\mathbb{Q}}_i$. Then we may write symbolically as

$$\tilde{\sigma} = \Sigma_{i,j,k=1}^{r} \tilde{\sigma}_{ij}^{k}$$
.

Assume that $\tilde{\sigma} \in \mathcal{H}_{M}$. Since each holonomy algebra \mathfrak{R}_{j} ($l \leq j \leq r$) acts on the subspace \mathfrak{Q}_{j} irreducibly and on the other subspaces \mathfrak{Q}_{k} ($j \neq k$) trivially, the condition (2) for $\tilde{\sigma}$ implies that

(4.1)
$$\tilde{\sigma} = \sum_{j=1}^{r} \tilde{\sigma}_{jj}^{j}$$
.

Now we have the following

Lemma 4.1. Assume that the set \mathcal{H}_{M} is not empty. Then the simply connected symmetric space M without Euclidean factor is irreducible and of compact type.

Proof. Suppose that $r \ge 2$ and $\tilde{\sigma} \in \mathcal{H}_{M}$. In the condition (3)

for \tilde{g} , let X be a nonzero vector in \bigoplus_{j} and Y = Z a nonzero vector in \bigoplus_{k} with $j \neq k$. Then, by (4.1), we have

$$\begin{aligned} \langle \mathbf{c}/4 \rangle \langle \mathbf{Y}, \mathbf{Y} \rangle \mathbf{X} &= \mathbf{R}(\mathbf{X}, \mathbf{Y}) \mathbf{Y} - [\tilde{\sigma}(\mathbf{X}), \tilde{\sigma}(\mathbf{Y})] \mathbf{Y} &= - [\tilde{\sigma}(\mathbf{X}), \tilde{\sigma}(\mathbf{Y})] \mathbf{Y} \\ &= \tilde{\sigma}(\mathbf{Y}, \tilde{\sigma}(\mathbf{X}, \mathbf{Y})) - \tilde{\sigma}(\mathbf{X}, \tilde{\sigma}(\mathbf{Y}, \mathbf{Y})) = 0. \end{aligned}$$

This is a contradiction. Hence we have r = 1.

Moreover Corollary 2.2 implies that M is of compact type.

q.e.d.

Hereafter we assume that M is a simply connected compact irreducible symmetric space. Let (a) be a maximal abelian subspace in (b) and W the Weyl group of M relative to (a). Denote by $S^{3}(p)$ (resp. $S^{3}(a)$) the vector space of all symmetric trilinear forms on (p) (resp. on (a)). Then it is known that the vector subspace { $\tilde{q} \in S^{3}(p)$; $\hat{W} \cdot \tilde{q} = 0$ } is isomorphic to the vector subspace { $\tilde{\lambda} \in S^{3}(a)$; $W \cdot \tilde{\lambda} = \tilde{\lambda}$ for all $W \in W$ } by the restriction to the subspace (a). Noting that the Weyl group W acts on (b) irreducibly, we can see the following

Lemma 4.2. Let M be a simply connected compact irreducible symmetric space and set $d_M = \dim \{ \tilde{\sigma} \in S^3(\mathbb{O}) ; (\mathbb{R} \cdot \tilde{\sigma} = 0 \}$. Then $d_M = 1$ if M is one of the following spaces and $d_M = 0$ otherwise:

SU(n)/SO(n) ($n \ge 3$), SU(2n)/Sp(n) ($n \ge 3$), SU(n) ($n \ge 3$), E_6/F_4 .

Now we determine the set $\overline{\mathcal{H}}_{M}$.

Proposition 4.3. Let M^n be a simply connected compact irreducible symmetric space satisfying $d_M = 0$. Assume that the set $\overline{\mathcal{A}}_M$ is not empty. Then the riemannian manifold M^n is the sphere $S^n(c/4)$ with constant sectional curvatures c/4 and the set $\overline{\mathcal{A}}_M$ consists of one point. Moreover the unique element in $\overline{\mathcal{A}}_M$ corresponds to the natural totally geodesic isometric immersion $f: S^n(c/4) \rightarrow P^n(c)$.

Proof. Take $\tilde{q} \in \mathcal{A}_{M}$. Then the assumption that $d_{M} = 0$ implies that $\tilde{q} = 0$. Hence, by the condition (3) for \tilde{q} , we have

 $R(X,Y)Z = (c/4)(\langle Y,Z \rangle X - \langle X,Z \rangle Y)$

for all vectors $X, Y, Z \in \mathbb{Q}$. This implies that M^n has constant sectional curvatures c/4. The other assertions are obvious.

q.e.d.

Now we consider the case when $d_{M} = 1$. Then we have the following

Proposition 4.4. Let M^n be a simply connected compact irreduc-<u>ible symmetric space satisfying</u> $d_M = 1$. Assume that the set $\overline{\mathcal{H}}_M$ <u>is not empty</u>. Then the metric of M^n is determined uniquely by the <u>constant</u> c <u>and the set</u> $\overline{\mathcal{H}}_M$ <u>consists of one point</u>.

Proof. Let $(M, <, >_1)$ and $(M, <, >_2)$ be symmetric spaces with

the same underlying manifold M. Suppose that $\overline{\mathcal{H}}_{(M,<,>_1)}$ and $\overline{\mathcal{H}}_{(M,<,>_2)}$ are not empty, and take $\tilde{\sigma}_j \in \mathcal{H}_{(M,<,>_j)}$ for j = 1,2. Then, noting that M is not a sphere, we can see that each $\tilde{\sigma}_j$ is nonzero by the same proof as in Proposition 4.3. Since M is irreducible, we have $<,>_2 = \alpha <,>_1$ for some $\alpha > 0$. Moreover the assumption that $d_M = 1$ implies that $\tilde{\sigma}_2 = \beta \tilde{\sigma}_1$ for some β . By the condition (3) for $\tilde{\sigma}_j$ (j=1,2), we have

 $(c/4) (\langle Y, Z \rangle_{i} X - \langle X, Z \rangle_{i} Y) = R(X, Y) Z - [\tilde{\sigma}_{i}(X), \tilde{\sigma}_{i}(Y)] (Z)$

and thus

$$(c/4) (\beta^2 - \alpha) (\langle Y, Z \rangle_1 X - \langle X, Z \rangle_1 Y) = (\beta^2 - 1) R(X, Y) Z$$

for all vectors X,Y,Z $\in \mathbb{Q}$. Since M is not a sphere, we have $\beta^2 = 1$ and $\alpha = 1$. Hence we have $\langle , \rangle_1 = \langle , \rangle_2$ and $\tilde{\sigma}_2 = \pm \tilde{\sigma}_1$. Note that the symmetry $\phi \in F_0(M)$ at o acts on the set $S^3(\Phi)$ by $\phi \cdot \tilde{\sigma} = -\tilde{\sigma}$ for any $\tilde{\sigma} \in S^3(\Phi)$. Then we can see that the set $\overline{\mathcal{H}}_{(M, <, >_1)}$ $= \overline{\mathcal{H}}_{(M, <, >_2)}$ consists of one point.

In the next section we shall construct a model of a totally real parallel isometric immersion of M^n into $P^n(c)$ for M^n satisfying $d_M = 1$. Hence, summing up Lemma 4.1 and Propositions 4.3,4.4, we have the following

Theorem 4.5. Let M^n be a simply connected symmetric space without Euclidean factor. Then the set $\overline{\mathcal{H}}_M$ is not empty if and only if the symmetric space Mⁿ is one of the followings:

SU(n)/SO(n) ($n \ge 3$), SU(2n)/Sp(n) ($n \ge 3$), SU(n) ($n \ge 3$),

 E_{6}/F_{4} , SO(n+1)/SO(n) (n ≥ 2).

In this case, the metric on the manifold M^n is determined uniquely by the constant c and the set $\overline{\mathcal{A}}_M$ consists of one point.

5. Models of totally real parallel isometric immersions

Let V be an (n+1)-dimensional complex vector space furnished with a positive definite hermitian inner product (,). Then we can define the associated inner product < , >_v on V as follows:

 $\langle X, Y \rangle_{V} = \text{Re}(X, Y)$

for vectors $X, Y \in Y$. Let P(V) be the complex projective space associated to Y furnished with the Kähler metric < , > with constant holomorphic sectional curvatures c, and S the unit sphere in Y furnished with the following riemannian metric < , >_S:

$$< X, Y >_{S} = (c/4) < X, Y >_{V}$$

for tangent vectors X,Y of S. Then the Hopf fibring $\pi: S + P(V)$ is a riemannian submersion. For a point $p \in S$, the horizontal subspace H_p at p is given by

$$H_{p} = \{ X \in V ; \langle X, p \rangle_{V} = \langle X, \sqrt{-1} \cdot p \rangle_{V} = 0 \}.$$

Here note that the linear mapping $\pi_*: H_p + T_{(p)}(P(V))$ is an Euclidean isometry satisfying $\pi_*(\sqrt{-IX}) = J(\pi_*X)$ for any $X \in H_p$. Let $\gamma(t)$ be a curve in S. Then a vector field Z_t along $\gamma(t)$ is called *horizontal* if $Z_t \in H_{\gamma(t)}$ for all t. The curve $\gamma(t)$ is called *horizontal* if $\dot{\gamma}(t)$ is a horizontal vector field along $\gamma(t)$. Moreover an isometric immersion \hat{f} of a riemannian manifold M into S is called *horizontal* if $\hat{f}_*(T_p(M)) \subset H_{\hat{f}(p)}$ for any point p in M. And a horizontal isometric immersion \hat{f} is called *totally real* if the subspaces $\hat{f}_*(T_p(M))$ and $\sqrt{-I\hat{f}_*(T_p(M))}$ are orthogonal. Let ∇^S be the riemannian connection on S for the riemannian metric < , >_S. Then we have the following

Lemma 5.1 (K.Nomizu [12] and B.O'Neill [13]). Let $\gamma(t)$ be a horizontal curve in S parametrized by arc-length. Then $(\nabla_t^S \dot{\gamma})(t)$ is a horizontal vector field along $\gamma(t)$. Moreover

$$\overline{\nabla}_{\mathsf{t}}(\pi_{\star}\mathbf{Z}_{\mathsf{t}}) = \pi_{\star}(\nabla_{\mathsf{t}}^{\mathsf{S}}\mathbf{Z}_{\mathsf{t}})$$

for any horizontal vector field Z_{+} along $\gamma(t)$.

Let \hat{f} be a horizontal (resp. horizontal and totally real) isometric immersion of an n-dimensional riemannian manifold M^n into S. Then the mapping $f = \pi \cdot \hat{f} : M^n + P(V)$ is an isometric immersion (resp. a totally real isometric immersion). Now we have the following

28

Lemma 5.2. Let $\gamma(t)$ be a geodesic in M parametrized by arclength. If the horizontal part of $(\nabla_t^S)^2 \hat{f}_*(\dot{\gamma}(t))$ is contained in $\hat{f}_*(T_{\gamma(t)}(M))$, the normal vector $(\nabla_t^*\sigma_f)(\dot{\gamma}(t),\dot{\gamma}(t))$ at $f(\gamma(t))$ equals zero.

Proof. Since the vector field $\nabla_t^S \hat{f}_*(\dot{\gamma}(t))$ is horizontal and $\pi_*(\nabla_t^S \hat{f}_*(\dot{\gamma}(t))) = \overline{\nabla}_t(f_*(\dot{\gamma}(t))) = \sigma_f(\dot{\gamma}(t),\dot{\gamma}(t))$ by Lemma 5.1, we have by Lemma 5.1 again

(5.1)
$$\pi_*(\langle \nabla_t^S \rangle^2 \hat{f}_*(\dot{\gamma}(t))) = \overline{\nabla}_t(\sigma_f(\dot{\gamma}(t),\dot{\gamma}(t))).$$

Note that

$$(\nabla^{*}_{+}\sigma_{f})(\dot{\gamma}(t),\dot{\gamma}(t)) = D_{+}(\sigma_{f}(\dot{\gamma}(t),\dot{\gamma}(t)))$$

= the normal component of $\overline{\nabla}_{t}(\sigma_{f}(\dot{\gamma}(t), \dot{\gamma}(t)))$.

By (5.1) and the assumption, the vector field $\overline{\nabla}_{t}(\sigma_{f}(\dot{\gamma}(t),\dot{\gamma}(t)))$ is a tangent vector field of M and thus $(\nabla_{t}^{*}\sigma_{f})(\dot{\gamma}(t),\dot{\gamma}(t)) = 0$. q.e.d.

Now we give the models of totally real parallel isometric immersions into $P^{n}(c)$ of irreducible compact simply connected symmetric spaces M satisfying $d_{M} = 1$.

Model 1. Let M be the manifold SU(n)/SO(n) ($n \ge 3$) and V the complex vector space $S^{n}(C)$ of all complex symmetric matrices of degree n furnished with the hermitian inner product:

$$(X,Y) = Tr X Y^*$$

for $X, Y \in V$. An imbedding $\hat{f} : M \rightarrow S$ is defined by

$$\hat{f}(g \cdot SO(n)) = (1/\sqrt{n})^{t} g \cdot g$$

for $g \in SU(n)$ and thus the manifold M is furnished with the riemannian metric induced from that of S. Let e_n be the identity element of SU(n) and put $o = e_n \cdot SO(n) \in M$. Now we can see easily the following facts:

(1) The tangent space $T_0(M)$ at o is identified with the space $\mathfrak{P} = \{ \sqrt{-1}A; A \in S^n(\mathbb{R}), Tr A = 0 \}$ and the following set (a) is a maximal abelian subspace in (p:

$$(a) = \left\{ \sqrt{-1} \begin{pmatrix} -\Sigma \mathbf{x}_{\mathbf{j}_{\mathbf{x}_{1}}} & \mathbf{0} \\ \mathbf{x}_{1} \\ \mathbf{0} & \mathbf{x}_{n-1} \end{pmatrix}; \mathbf{x}_{\mathbf{j}} \in \mathbb{R} \right\}.$$

(2) The isometric imbedding \hat{f} is equivariant relative to the representation $\rho: SU(n) \rightarrow SU(V)$ defined by

$$\rho(g)(x) = t^{t}g x g$$

for $g \in SU(n)$ and $X \in V$.

(3) $\hat{f}(o) = (1/\sqrt{n})e_n$ and $(\hat{f}_*)_o(0) = 0$. Hence \hat{f} is horizontal and totally real at o.

Then the riemannian metric of M is invariant under SU(n) by (2) and hence M is a symmetric space, and the isometric imbedding \hat{f} is horizontal and totally real by (2) and (3). Hence $f = \pi \circ \hat{f}$ is a totally real isometric immersion.

Now we show that the isometric immersion f has the parallel second fundamental form. Since f is totally real in P(V), the equation of Codazzi-Mainardi implies that $\nabla *_{\sigma_f}$ is a normal bundle valued symmetric tensor of degree 3. Note that f is equivariant by (2), and that maximal abelian subspaces in \bigcirc are conjugate^{to}_Aeach other under the natural action of K = SO(n) on \bigcirc . Hence it is sufficient for our claim to see that $(\nabla^*_{X^{\sigma_f}})(X,X) = 0$ for any unit vector

$$\mathbf{x} = \sqrt{-1} \cdot \begin{pmatrix} -\Sigma \mathbf{x}_{\mathbf{j}_{\mathbf{x}_{1}}} & \mathbf{0} \\ \mathbf{0} & \ddots \\ \mathbf{0} & \ddots \\ \mathbf{x}_{n-1} \end{pmatrix}$$

in (a). Let $\gamma(t)$ be the geodesic in M such that $\gamma(0) = 0$ and $\dot{\gamma}(0) = X$. Then we have

$$\hat{f}(\gamma(t)) = (1/\sqrt{n}) \cdot \begin{pmatrix} e^{-2t(\Sigma x_j)/-I} \\ e^{2tx_1/-I} \\ 0 \\ \cdot e^{2tx_{n-1}/-I} \end{pmatrix}$$

and

$$\hat{f}_{*}(\dot{\gamma}(t)) = (1/\sqrt{n}) \cdot \begin{pmatrix} -2\sqrt{-1}(\Sigma x_{j})e^{-2t}(\Sigma x_{j})\sqrt{-1} \\ 2x_{1}\sqrt{-1}e^{2tx_{1}}\sqrt{-1} \\ \vdots \\ 0 \\ 2x_{n-1}\sqrt{-1}e^{2tx_{n-1}}\sqrt{-1} \end{pmatrix}$$

Note that $\nabla_t^S Z_t = \frac{d}{dt}(Z_t) + (c/4) < \hat{f}_*(\dot{\gamma}(t)), Z_t > S^{\hat{f}}(\gamma(t))$ for any vector field Z_t along $f(\gamma(t))$. Thus we have

$$\nabla_{t}^{\hat{s}_{t}}(\dot{\gamma}(t)) = (1/\sqrt{n}) \cdot \begin{pmatrix} (c/4 - 4(\Sigma x_{j})^{2})e^{-2t(\Sigma x_{j})\sqrt{-1}} \\ (c/4 - 4x_{1}^{2})e^{2tx_{1}}\sqrt{-1} \\ (c/4 - 4x_{1}^{2})e^{2tx_{1}}\sqrt{-1} \end{pmatrix}$$

and

$$(\nabla_{t}^{S})^{2} \hat{f}_{*}(\dot{\gamma}(t))|_{t=0} = (2\sqrt{-1}/\sqrt{n}) \cdot \begin{pmatrix} -(c/4 - 4(\Sigma x_{j})^{2})(\Sigma x_{j}) & 0 \\ (c/4 - 4x_{1}^{2})x_{1} \\ 0 & \cdot (c/4 - 4x_{n-1}^{2})x_{n-1} \end{pmatrix}$$

Hence the horizontal part of $(\nabla_t^S)^2 \hat{f}_*(\dot{\gamma}(t))|_{t=0}$ is given by

$$(\nabla_{t}^{S})^{2} \hat{f}_{*}(\dot{\gamma}(t))|_{t=0} - \frac{\langle (\nabla_{t}^{S})^{2} \hat{f}_{*}(\dot{\gamma}(t))|_{t=0}, \sqrt{-1} \hat{f}(\gamma(0)) \rangle_{S}}{|\sqrt{-1} \hat{f}(\gamma(0))|_{S}^{2}} \cdot \sqrt{-1} \hat{f}(\gamma(0))$$

$$= (\sqrt{-1}/\sqrt{n}) \cdot \begin{pmatrix} -2(\Sigma x_{j})(c/4 - 4(\Sigma x_{j})^{2}) - \lambda\sqrt{c}/2 & 0 \\ 2x_{1}(c/4 - 4x_{1}^{2}) - \lambda\sqrt{c}/2 & 0 \\ 2x_{1}(c/4 - 4x_{1}^{2}) - \lambda\sqrt{c}/2 \\ 0 & 2x_{n-1}(c/4 - 4x_{n-1}^{2}) - \lambda\sqrt{c}/2 \end{pmatrix}$$

where $\lambda = (16/n\sqrt{c})((\Sigma x_j)^3 - (\Sigma x_j^3))$. Here note that the trace of the above matrix equals zero. Hence the horizontal part of $(\nabla_t^S)^2 \hat{f}_*(\dot{\gamma}(t))|_{t=0}$ is contained in O. This implies that $(\nabla^* q_f)(\dot{\gamma}(0), \dot{\gamma}(0), \dot{\gamma}(0)) = 0$ by Lemma 5.2. Hence f is a totally real parallel isometric immersion of M into P(V).

Model 2. Let M be the manifold $SU(2n)/Sp(n) (n \ge 3)$ and V the complex vector space $\mathfrak{O}(2n; \mathbb{C})$ of all complex skew symmetric matrices of degree 2n furnished with the hermitian inner product:

$$(X,Y) = Tr X Y^*$$

for yectors $X, Y \in V$. An imbedding $\hat{f} : M \rightarrow S$ is defined by

$$\hat{f}(g \cdot Sp(n)) = (1/\sqrt{2n})^{t} g J_{n} g$$

for $g \in SU(2n)$, where $J_n = \begin{pmatrix} 0 & -e_n \\ e_n & 0 \end{pmatrix} \in V$, and thus the manifold M: is

furnished with the riemannian metric induced from that of S. Put o = e_{2n}·Sp(n) ∈ M. Now we can see easily the following facts: (1) The tangent space T_o(M) at o is identified with the

space

and the following set (a) is a maximal abelian subspace in (b):

$$(a) = \left\{ \begin{array}{c} \sqrt{-1} \cdot \begin{pmatrix} -(\Sigma x_{j}) & 0 \\ x_{1} & 0 \\ \cdot & \ddots \\ & \ddots \\ & \ddots \\ & -(\Sigma x_{j}) \\ 0 & \ddots \\ & x_{n-1} \\ 0 & \ddots \\ & & x_{n-1} \\ \end{pmatrix} ; x_{j} \in \mathbb{R} \right\}.$$

33

(2) The isometric imbedding \hat{f} is equivariant relative to the representstion ρ : SU(2n) \rightarrow SU(V) defined by

$$\rho(g)(X) = f g X g$$

for $g \in SU(2n)$ and $X \in V$. (3) $\hat{f}(o) = (1/\sqrt{2n})J_n$ and $(\hat{f}_*)_O(Q) = \{ \begin{pmatrix} -\overline{W} & -t_Z \\ Z & W \end{pmatrix}; Z \in SU(n), \}$ $W \in \mathfrak{O}(n; \mathbb{C})$ }. Hence \hat{f} is horizontal and totally real at o.

Then, by the same way as in Model 1, we can see that $f = \pi \circ \hat{f}$ is a totally real parallel isometric immersion.

Model 3. Let M be the manifold $SU(n) \times SU(n)/SU(n)$ (n > 3) and V the complex vector space $M_n(C)$ of all complex matrices of degree n furnished with the hermitian inner product:

 $(X, Y) = Tr X Y^*$

for vectors X, Y \in Y. An imbedding $\hat{f} : M \rightarrow S$ is defined by

 $f((g,h) \cdot SU(n)) = (1/\sqrt{n})gh^{-1}$

for $g,h \in SU(n)$ and thus the manifold M is furnished with the riemannian metric induced from that of S. Put $o = (e_n, e_n) \cdot SU(n) \in M$. Now we can see easily the following facts:

(1) The tangent space $T_{O}(M)$ at o is identified with the space $(p) = \{ (X, -X) ; X \in \mathfrak{S}(n) \}$ and the following set (a) is a maximal abelian subspace in (p:

$$a = \{ (X, -X) \in p; X \text{ is diagonal } \}.$$

(2) The isometric imbedding \hat{f} is equivariant relative to the representation ρ : SU(n) × SU(n) → SU(V) defined by

$$\rho((g,h))(X) = gXh^{-1}$$

for $g,h \in SU(n)$ and $X \in V$.

(3) $\hat{f}(o) = (1/\sqrt{n})e_n$ and $(\hat{f}_*)_o(p) = \mathfrak{W}(n)$. Hence \hat{f} is horizontal and totally real at o.

Then, by the same way as in Model 1, we can see that $f = \pi \circ \hat{f}$ is a totally real parallel isometric immersion.

Model 4. Let G be the Cayley algebra over \mathbb{R} furnished with the caninical conjugation -, and set $\mathcal{F} = \{ X \in M_3(G) ; {}^t\overline{X} = X \}$. On the real vector space \mathcal{F} , we define the Jordan product \circ , the inner product ((,)), the cross product \times , and the determinant det as follows respectively:

 $X \circ Y = (1/2) (XY + YX), ((X,Y)) = Tr(X \circ Y),$

 $X \times Y = (1/2) (2X \circ Y - Tr(X)Y - Tr(Y)X + (Tr(X)Tr(Y) - Tr(X \circ Y))e_3),$

 $det(X) = (1/3)((X \times X, X))$

for $X, Y \in \mathcal{F}$. Let V be the complexification of the real vector space \mathcal{F} and extend these \circ , ((,)), ×, det C-linearly and naturally on V.

Denote by τ the complex conjugate on V with respect to \mathcal{F} . Then (X,Y) = ((τ X,Y)) is a positive definite hermitian inner product on V. We define

$$E_6 \equiv \{ g \in GL_{\mathbb{C}}(\mathbb{V}) ; \text{ for any } X, Y \in \mathbb{V} \}$$

and

$$F_4 = \{ g \in E_6 ; g(e_3) = e_3 \}.$$

Then E_6 (resp. F_4) is a simply connected compact simple Lie group of type E_6 (resp. of type F_4).(cf. O.Shukugawa-I.Yokota [14])

Let M be the manifold E_6/F_4 . An imbedding $\hat{f}: M \rightarrow S$ is defined by

$$\hat{f}(g \cdot F_{4}) = (1/\sqrt{3})g(e_{3})$$

for $g \in E_6$ and thus the manifold M is furnished with the riemannian metric induced from that of S. Put $o = e_3 \cdot F_4 \in M$ and set $\mathcal{F}_0 = \{x \in \mathcal{F}; Tr X = 0\}$. Now we can see easily the following facts:

(1) Define the right translation R_X on \mathcal{F} for $X \in \mathcal{F}$ by $R_X(Y) = Y \circ X$ for $Y \in \mathcal{F}$. The tangent space $T_0(M)$ at o is identified with the space $\bigcirc = \{ \sqrt{-1}R_X \in \bigcirc (V) ; X \in \mathcal{F}_0 \}$ and the following set (a) is a maximal abelian subspace in \bigcirc :

$$= \{ \sqrt{-IR_x} \in \mathbb{GD}(V) ; X \in \mathcal{F}_0, X \text{ is diagonal } \}.$$

(2) The isometric imbedding \hat{f} is equivariant relative to the representation $\rho: E_6 \rightarrow SU(V)$ defined by

for $g \in E_6$ and $X \in V$.

(3) $\hat{f}(o) = (1/\sqrt{3})e_3$ and $(\hat{f}_*)_o(p) = \sqrt{-1}\mathcal{F}_0$. Hence \hat{f} is horizontal and totally real at o.

Then, by the same way as in Model 1, we can see that $f = \pi \circ \hat{f}$ is totally real parallel isometric immersion.

Remark 5.3. It is known that the isometric imbeddings $\hat{f}: M \rightarrow S$ in the above models are minimal. Since the imbeddings \hat{f} are horizontal, the isometric immersions f are minimal.

Remark 5.4. We can see easily that the above isometric immersion $f: M \neq P(V)$ is $(\sqrt{c}/2\sqrt{2})$ -isotropic (that is, $|\sigma_f(X,X)| = \sqrt{c}/2\sqrt{2}$ for any unit tangent vector X of M) if the symmetric space M is of rank two. Hence these isometric immersions f are examples of Theorem 4.13 in [11].

6. The set $\overline{\mathcal{H}}_{M}$ for a simply connected symmetric space M with Euclidean factor

In this section we assume that Mⁿ is a simply connected symmetric space with Euclidean factor, thus, M is decomposed as a riemannian manifold as follows:

$$M^{n} = \mathbb{R}^{n} 0 \times M_{1}^{n} 1 \times \cdots \times M_{r}^{n} r \quad (n = \Sigma_{j=0}^{r} n_{j}, n_{0} > 0)$$

where $M_j^n j$ is an n_j -dimensional irreducible simply connected sym-

metric space for each j. Then the tangent space $T_0(M) = O(resp.)$ the holonomy algebra (k) is decomposed as follows:

where the subspaces \bigoplus_j and \bigoplus_0 in \bigoplus (resp. the subalgebra \bigotimes_j in \bigotimes) denote the tangent spaces $T_o(M_j)$ and $T_o(R^n 0)$ (resp. the holonomy algebra of M_j). For a \bigotimes -valued symmetric bilinear form $\tilde{\sigma}$ on \bigotimes and any ordered triple {i,j,k} ($0 \le i,j,k \le r$), a mapping $\tilde{\sigma}_{ij}^k : \bigoplus_j : \bigoplus_j : \bigoplus_k$ is defined as in the section 4. Assume that $\tilde{\sigma} \in \mathcal{M}_M$. Since each holonomy algebra $\bigotimes_j (1 \le j \le r)$ acts on the subspace \bigotimes_j irreducibly and on the other spaces $\bigotimes_k (j \ne k)$ trivially, the condition (2) for $\tilde{\sigma}$ implies that

(6.1)
$$\tilde{\sigma} = \Sigma_{j=0}^{r} \tilde{\sigma}_{jj}^{j} + \Sigma_{j=1}^{r} \tilde{\sigma}_{jj}^{0} + \Sigma_{j=1}^{r} \tilde{\sigma}_{0j}^{j} + \Sigma_{j=1}^{r} \tilde{\sigma}_{j0}^{j}$$
.

Now we define the <u>Euclidean j-th mean curvature vector</u> H_j ($l \le j \le r$) in \bigoplus_0 by

$$H_{j} = (1/n_{j}) \operatorname{Tr} \tilde{\sigma}_{jj}^{0} = (1/n_{j}) \Sigma_{k=1}^{n_{j}} \tilde{\sigma}_{jj}^{0} (e_{jk}, e_{jk})$$

where $\{e_{jk}\}_{k=1}^{n_j}$ denotes an orthonormal basis of \bigoplus_j , and call the length h_j of the vector H_j the <u>Euclidean j-th mean curvature</u>. Then we have the following

Lemma 6.1. Let $\tilde{\sigma}$ in \mathcal{H}_{M} . Then

38

$$\tilde{\sigma}_{jj}^{0}(X_{j},Y_{j}) = \langle X_{j},Y_{j} \rangle^{H}_{j}$$

$$\tilde{\sigma}_{j0}^{j}(X_{j},Z_{0}) = \tilde{\sigma}_{0j}^{j}(Z_{0},X_{j}) = \langle Z_{0},H_{j} \rangle X_{j}$$

for any $j(l_{\leq}j_{\leq}r)$ and $Z_0 \in \mathbb{Q}_0, X_j, Y_j \in \mathbb{Q}_j$.

Proof. Since $\bigotimes_{j} \cdot \tilde{\sigma} = 0$, we have

(6.2)
$$\tilde{\sigma}_{jj}^{0}(T_{j}X_{j},Y_{j}) + \tilde{\sigma}_{jj}^{0}(X_{j},T_{j}Y_{j}) = 0$$

and

$$(6.3) \quad \tilde{\sigma}_{jj}^{j}(T_{j}X_{j},Y_{j}) + \tilde{\sigma}_{jj}^{j}(X_{j},T_{j}Y_{j}) = T_{j}(\tilde{\sigma}_{jj}^{j}(X_{j},Y_{j}))$$

for any $T_j \in \mathbb{R}_j$ and all vectors $X_j, Y_j \in \mathbb{Q}_j$. Let $\{e_a\}_{a=1}^{n_0}$ be an orthonormal basis of \mathbb{Q}_0 . Since M_j is irreducible, the condition (6.2) implies that

$$\langle \tilde{\sigma}_{jj}^{0}(X_{j},Y_{j}), e_{a} \rangle = c_{j}^{a} \langle X_{j},Y_{j} \rangle$$

for some $c_j^a \in \mathbb{R}$ and thus

$$\tilde{\sigma}_{jj}^{0}(x_{j}, y_{j}) = \langle x_{j}, y_{j} \rangle (\Sigma_{a=1}^{n_{0}} c_{j}^{a} e_{a}) = \langle x_{j}, y_{j} \rangle H_{j}$$

for all vectors $X_j, Y_j \in \mathbb{Q}_j$.

The second equality is obtained by the symmetry condition (1) for $\tilde{\sigma}$ and the first equality.

q.e.d.

We denote by \mathcal{H}_{M}^{d} the set defined in the same way as \mathcal{H}_{M} by replacing the number c/4 in the condition (3) with the number d. Then we have the following

Lemma 6.2. Let $\tilde{\sigma}$ in \mathcal{M}_{M} . Then $\tilde{\sigma}_{jj} \in \mathcal{M}_{Mj}^{c/4 + h_{j}^{2}}$ for each j. Proof. The conditions (1) and (2) for $\mathcal{M}_{Mj}^{c/4 + h_{j}^{2}}$ is obvious by the condition (1) for $\tilde{\sigma}$ and (6.3). We show that $\tilde{\sigma}_{jj}^{j}$ satisfies the condition (3) for $\mathcal{M}_{Mj}^{c/4 + h_{j}^{2}}$. Denote by \mathbb{R}^{M} the curvature tensor of M_{j} . Then, by the condition (3) for $\tilde{\sigma}$,

$$(c/4)(\langle Y_{j}, Z_{j} \rangle X_{j} - \langle X_{j}, Z_{j} \rangle Y_{j}) = R^{M}j(X_{j}, Y_{j})Z_{j} - [\tilde{\sigma}(X_{j}), \tilde{\sigma}(Y_{j})]Z_{j}$$

for all vectors $X_{j}, Y_{j}, Z_{j} \in \mathcal{O}_{j}$. By (6.1) and Lemma 6.1, the second term of the right hand side is calculated as follows:

$$[\tilde{\alpha}(x_{j}), \tilde{\alpha}(y_{j})]z_{j} = [\tilde{\alpha}_{jj}(x_{j}), \tilde{\alpha}_{jj}(y_{j})]z_{j} + h_{j}^{2}(\langle y_{j}, z_{j} \rangle x_{j} - \langle x_{j}, z_{j} \rangle y_{j}).$$

Hence \tilde{q}_{jj}^{j} satisfies the condition (3) for $\mathcal{H}_{M_{j}}^{c/4 + h_{j}^{2}}$.

q.e.d.

Lemma 6.3. Let $\tilde{\sigma}$ in \mathcal{M}_{M} . Then $\tilde{\sigma}_{00}^{0} \in \mathcal{M}_{R}^{n}$ and

$$\tilde{\sigma}_{00}^{0}(X_{0},H_{j}) = \langle X_{0},H_{j} \rangle H_{j} - (c/4)X_{0}$$

for any $X_0 \in \mathbb{P}_0$. Moreover $\langle H_j, H_k \rangle = -c/4$ for distinct indeces j,k ($1 \le j, k \le r$).

Proof. Note that the condition (2) for $\mathcal{H}_{\mathbb{R}}^{n_{0}}$ is obvious since $\mathbb{R}^{n_{0}}$ is flat. Moreover by the conditions (1) and (3) for $\tilde{\sigma}$ we can see easily that $\tilde{\sigma}_{00}^{0}$ satisfies the conditions (1) and (3) for $\mathcal{H}_{\mathbb{R}}^{n_{0}}$. Put $X = X_{0} \in \mathbb{P}_{0}$, $Y = Y_{j}$, $Z = Z_{j} \in \mathbb{P}_{j}$ in the condition (3) for $\tilde{\sigma}$. Then we have

$$(c/4) < Y_{j}, Z_{j} > X_{0} = - [\tilde{\sigma}(X_{0}), \tilde{\sigma}(Y_{j})] Z_{j}$$

The right hand side is calculated by (6.1) and Lemma 6.2 as follows:

$$- [\tilde{\sigma}(x_0), \tilde{\sigma}(y_j)]z_j = \langle x_0, H_j \rangle \langle y_j, z_j \rangle H_j - \langle y_j, z_j \rangle \tilde{\sigma}_{00}^0(x_0, H_j)$$

Hence we have

$$(c/4) X_0 = \langle X_0, H_j \rangle H_j - \tilde{\sigma}_{00}^0 (X_0, H_j).$$

Now, putting $X = X_j \in \mathcal{D}_j$ and $Y = Y_k$, $Z = Z_k \in \mathcal{D}_k$ $(l \le j \ne k \le r)$ in the condition (3) for $\tilde{\sigma}$, we have

$$(c/4) < Y_k, Z_k > X_j = - < Y_k, Z_k > < H_j, H_k > X_j$$

by (6.1) and Lemma 6.2, and thus $\langle H_j, H_k \rangle = -c/4$.

q.e.d.

Summing up Lemmas 6.1, 6.2 and 6.3, we have the claim (A) in the following

Theorem 6.4. Let M^n be a simply connected symmetric space with Euclidean factor decomposed as $M^n = \mathbb{R}^n 0 \times \Pi_{j=1}^r M_j^{n_j}$ and $n = \sum_{j=0}^r n_j$. Then the following claims are true:

(A) Let $\tilde{\sigma} \in \mathcal{H}_{M}$. Then

(1)
$$\tilde{\sigma} = \Sigma_{j=0}^{r} \tilde{\sigma}_{jj}^{j} + \Sigma_{j=1}^{r} \tilde{\sigma}_{jj}^{0} + \Sigma_{j=1}^{r} \tilde{\sigma}_{j0}^{j} + \Sigma_{j=1}^{r} \tilde{\sigma}_{0j}^{j}$$

(2)
$$\tilde{\sigma}_{jj}^{j} \in \mathcal{M}_{M_{j}}^{c/4}$$

(3)
$$\tilde{\sigma}_{00}^{0} \in \mathcal{H}_{\mathbb{R}}^{n}$$
, $\langle H_{j}, H_{k} \rangle = -c/4 \ (l \leq j \neq k \leq r),$

$$\tilde{\sigma}_{00}(z_0, H_j) = \langle z_0, H_j \rangle H_j - (c/4) z_0.$$

(4)
$$\tilde{\sigma}_{j0}^{j}(x_{j}, z_{0}) = \tilde{\sigma}_{0j}^{j}(z_{0}, x_{j}) = \langle z_{0}, H_{j} \rangle x_{j},$$

 $\tilde{\sigma}_{jj}^{0}(x_{j}, y_{j}) = \langle x_{j}, y_{j} \rangle H_{j}$

for any $Z_0 \in \mathbb{Q}_0$ and all vectors $X_j, Y_j \in \mathbb{Q}_j$. (B) Conversely any p-valued bilinear form \tilde{o} on \mathbb{Q} satisfying the conditions (1),(2),(3),(4) of (A) is an element in \mathcal{H}_M .

Here the proof of our claim (B) is omitted since it is straightforward.

Remark 6.5. Let M^n be a simply connected symmetric space with Euclidean factor. Changing the metric on M^n componentwise, we can construct infinitely many elements in \mathcal{H}_M . In fact, decompose M as above and suppose that $n_0 = r \ge 1$. First we shall show that there exist a basis $\{H_i\}_{i=1}^r$ of \mathbb{R}^r and an \mathbb{R}^r -valued bilinear form $\tilde{\sigma}_{00}^0$ on \mathbb{R}^r satisfying the condition (3) of (A). If there exist such basis and \mathbb{R}^r -valued form, by (B) of Theorem 6.4, an element in \mathcal{A}_M can be constructed. Let $\{e_j\}_{j=1}^r$ be an orthonormal basis of \mathbb{R}^r and set $H_i = \sum_{j=1}^r a_i^j e_j$, $A = (a_i^j)$. Moreover, for positive real numbers h_1, \dots, h_r , we set

$$S(h_{1}, \dots, h_{r}) = \begin{pmatrix} h_{1}^{2} & -c/4 & \cdots & -c/4 \\ -c/4 & h_{2}^{2} & \vdots \\ \vdots & & \ddots & -c/4 \\ -c/4 & \cdots & -c/4 & h_{r}^{2} \end{pmatrix}$$

Then the condition for that $\{H_i\}$ is a basis of \mathbb{R}^r such that $|H_i| = h_i (1 \le j \le r)$ and $\langle H_i, H_k > = -c/4 (j \ne k)$ is written as follows:

(6.4) det
$$A \neq 0$$
, $A^{L}A = S(h_1, \dots, h_r)$.

Since the matrix $S(h_1, \dots, h_r)$ is symmetric, for sufficiently large numbers h_1, \dots, h_r , there exists a positive definite symmetric matrix A satisfying the condition (6.4). Then we define an \mathbb{R}^r -valued bilinear form $\tilde{\mathfrak{g}}_{00}^0$ on \mathbb{R}^r as follows:

$$\tilde{\sigma}_{00}^{0}(H_{j},H_{k}) = \langle H_{j},H_{k} \rangle H_{k} - (c/4)H_{j}$$

By easy calculations, we can see that the \mathbb{R}^{r} -valued bilinear form \tilde{q}_{00}^{0} on \mathbb{R}^{r} satisfies the condition (3) of (A). Thus we get infinitely

many elements in \mathcal{K}_{M} by taking suitable metrics on M_{j} ($l \leq j \leq r$).

Now, in the case when $M = \mathbb{R}^2$, we have the following

Theorem 6.6. <u>There exists a unique complete totally real</u> <u>parallel flat minimal surface</u> M^2 <u>in</u> $P^2(c)$ (<u>up to holomorphic</u> <u>isometries of</u> $P^2(c)$). <u>The norm</u> $|\sigma|$ <u>of the second fundamental</u> <u>form</u> σ <u>of</u> M^2 <u>is given by</u> $|\sigma|^2 = (1/2)c$.

Proof. Let $\{e_1, e_2\}$ be an orthonormal basis of \mathbb{R}^2 . Then the condition $\tilde{\sigma} \in \mathcal{A}_{\mathbb{R}}^2$ is equivalent to the condition that

(6.5)
$$\begin{cases} \tilde{\sigma}(e_1, e_1) = \alpha e_1 + \beta e_2 \\ \tilde{\sigma}(e_1, e_2) = \beta e_1 + \gamma e_2 \\ \tilde{\sigma}(e_2, e_2) = \gamma e_1 + \delta e_2 \end{cases} , \text{ and } c/4 = \beta^2 + \gamma^2 - \alpha\gamma - \beta\delta.$$

Suppose that the totally real parallel immersion of \mathbb{R}^2 corresponding to $\tilde{\sigma}$ is minimal. Then $\alpha + \gamma = \beta + \delta = 0$ and thus $\beta^2 + \gamma^2 = c/8$ by the second equality of (6.5). Put $\beta = \sqrt{c/8} \cos \theta$ and $\gamma = \sqrt{c/8} \sin \theta$ for some θ and define a linear isometry g of \mathbb{R}^2 by

$$(g(e_1),g(e_2)) = (e_1,e_2) \begin{pmatrix} \cos(\theta/3) & \sin(\theta/3) \\ -\sin(\theta/3) & \cos(\theta/3) \end{pmatrix}.$$

44

Then we have

$$(g \cdot \tilde{g}) (e_1, e_1) = -(g \cdot \tilde{g}) (e_2, e_2) = \sqrt{c/8} e_2, (g \cdot \tilde{g}) (e_1, e_2) = \sqrt{c/8} e_1.$$

Hence all elements in \mathcal{H}_R^2 corresponding to minimal immersions belong to the same equivalence class. Now by Theorem 3.4 and 3.6 we get our first claim. The second claim follows from $|g \cdot \tilde{\sigma}|^2 = (1/2)c$. q.e.d.

Remark 6.7. S.T.Yau [18] has shown that if M^2 is a complete non-negative curved totally real minimal surface in $P^2(c)$, M^2 is totally geodesic or flat, and moreover in the last case the second fundamental form is parallel. The minimal surface of Theorem 6.6 gives a unique example of such surfaces in the flat case. This has been constructed concretely in my previous paper [11] and it is compact.

Remark 6.8. B.Y.Chen and K.Ogiue [3] has shown that if M^n is a compact totally real minimal submanifold in $P^n(c)$ such that $|q_p|^2 < (n(n+1)/4(2n-1))c$ for any point p in M, then M^n is totally geodesic. Suppose that $|\sigma_p|^2 = (n(n+1)/4(2n-1))c$ for any point p in M. Then, along their proof, the second fundamental form is parallel. In the case when n = 2 (then (n(n+1)/4(2n-1))c= (1/2)c), the universal covering of the compact totally real parallel minimal surface M^2 has Euclidean factor and thus is flat. Hence our minimal surface in $P^2(c)$ of Theorem 6.6 is a unique compact totally real minimal surface M^2 in $P^2(c)$ such that $|\sigma_p|^2 = (1/2)c$

45

for any point p in M^2 .

Remark 6.9. In the next paper together with M.Takeuchi the complete classification of n-dimensional complete totally real parallel submanifolds in $P^{n}(c)$ shall be given by a different way.

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