Use of a Method of Coherent States for a Many-Boson System at High Density

by

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ABSTRACT

The ground-state energy and the momentum distribution of a many-boson system are obtained in the high-density limit by using a method of coherent states. Dealing with the problem in a lattice space, some difficulties encountered in previous collective variable theories have been overcome. In particular, it is shown that the objection about the non-existence of the canonical conjugate to the density is not fatal to the previous theory of the density and phase operator approach. The ground-state energy obtained here agrees with that given by Brueckner and Lee.

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CONTENTS

§1 Introduction

The theory of many-boson systems is yet a fascinating subject in the quantum mechanical many-body theory in relation to the superfluidity of liquid ^{4}He.[1] After the pioneering work of Landau titled "Theory of Liquid He II",[2] a number of papers have been presented to make a foundation of his theory, the so-called quantum hydrodynamics. These microscopic theories are common in considering that a many-boson system is a collection of quasi-particles instead of that of strongly interacting real atoms. Among these theories, we can name the following three classes. The first one is originated by Bogoliubov [3], assuming that a number of atoms occupy the single particle state of zero momentum. The second one is the theory of collective variables [4-7], considering the density fluctuation of the system as the fundamental quantity. To the third one belong the theories which intend to solve the Schrödinger equation under some assumptions such as the Bijl-Dingle-Jastrow approximation. One of these theories is known as the method of correlated basis functions of Feenberg.[8]

In this thesis, the main object is the second one, i.e. the theory of collective variables. It is not yet sufficiently clear what interrelations have the various theories in this class to each other and how to answer several objections raised by a number of authors against this theory. Recently, Rajagopal and Rama Rao (R-RR)[9] have invoked the coherent state representation,[10][11] frequently used in the theory of radiation,[12] to develop a collective variable

theory which is free from the objection that the conjugate variable to density assumed in previous collective variable theories does not exist in the Fock space (Fröhlich's objection).[13]

Coherent states were first discovered by Schrödinger[14] who was looking for wave packets for a harmonic oscillator which oscillate without spreading. They are the minimum uncertainty states for position and momentum and thus are the most classical states. In the second quantization representation, a coherent state $|\alpha\rangle_c$ appears as a right-eigenstate of the annihilation operator b with complex eigenvalue α, where b and its conjugate b^+ satisfy the bose commutation relation, $[b, b^+] = 1$.[10),11)] This definition of coherent states can be generalized to other systems. In the radiation theory, a coherent state can be defined as a right-eigenstate for the positive frequency part $\mathbf{E}^-(\mathbf{r}, t)$ of the quantized electric field $\mathbf{E}(\mathbf{r}, t)$, namely, it satisfies

$$\begin{aligned} \mathbf{E}^-(\mathbf{r}, t)\,|\mathcal{E}\rangle &= \mathcal{E}(\mathbf{r}, t)\,|\mathcal{E}\rangle \,, \\ \langle\mathcal{E}|\,\mathbf{E}^+(\mathbf{r}, t) &= \mathcal{E}^*(\mathbf{r}, t)\,\langle\mathcal{E}| \,, \end{aligned} \tag{1}$$

where $\mathbf{E}^+$ is the negative frequency part of $\mathbf{E}$. The name, coherent state, was introduced by Glauber[10] taking into the fact that the mutual coherence function

$$\gamma = \frac{\langle \mathbb{E}^{+}(\mathbf{r}_1, t_1)\, \mathbb{E}^{-}(\mathbf{r}_2, t_2) \rangle}{\left\{ \langle \mathbb{E}^{+}(\mathbf{r}_1, t_1)\, \mathbb{E}^{-}(\mathbf{r}_1, t_1) \rangle \langle \mathbb{E}^{+}(\mathbf{r}_2, t_2)\, \mathbb{E}^{-}(\mathbf{r}_2, t_2) \rangle \right\}^{\frac{1}{2}}} , \tag{2}$$

which measures the visibility of fringes in Young's experiment, takes the largest absolute value $|\gamma| = 1$ for a coherent state as is clear from eq. (1) ($0 \leq |\gamma| \leq 1$ from the Schwarz inequality)[12].

In the case of a many-boson system, coherent states $|\psi\rangle_c$ are defined as the right-eigenstates for a quantized boson field $\Psi(x)$ and satisfy

$$\Psi(x)\,|\psi\rangle_c = \psi(x)\,|\psi\rangle_c , \quad {}_c\langle\psi|\,\Psi^{\dagger}(x) = \psi^{*}(x)\,{}_c\langle\psi| , \tag{3}$$

$${}_c\langle\psi|\,\Psi(x)\,|\psi\rangle_c = \psi(x) \neq 0 , \tag{4}$$

$${}_c\langle\psi|\,\Psi^{\dagger}(x')\,\Psi(x)\,|\psi\rangle_c = \psi^{*}(x')\,\psi(x) , \tag{5}$$

$$[\,\Psi(x)\,,\,\Psi^{\dagger}(x')\,] = \delta(x - x') ,$$

among which eqs. (4) and (5) clearly show the relevance of coherent states to the theory of the superfluidity of liquid ^{4}He.[15] The factorization of the one-particle density matrix $\rho(x,x') = \langle \Psi^{\dagger}(x')\,\Psi(x) \rangle$ shown in eq. (5) reminds us of the Penrose-Onsager criterion for the Bose-Einstein condensation defined by the existence of non-zero asymptotic limit, $\lim_{|x-x'|\to\infty} \rho(x,x')$ [16] (the off-

diagonal-long-range-order). Non-vanishing of the average of the quantized field shown in eq.(4) allows us to define a macroscopic wave function (or order parameter) given as the local average of the quantized field[17], which provides us with a powerful method for investigating the superfluidity[18].

Coherent states have another aspect that they form a over-complete set and thus are the basis for a representation (the coherent state representation). The completeness is shown by the resolution of identity written in the form

$$\int \frac{\delta^2\psi}{\pi} |\psi\rangle_c \langle\psi| = 1 \quad , \tag{6}$$

where $\delta^2\psi$ is the element of functional integration over a complex function $\psi(x)$.[10] By means of eq. (6), the grand-canonical partition function $Z = \mathrm{Tr}\ \exp[-(H-\mu N)/kT]$ can be written as a functional integral of $\langle\psi|\exp[-(H-\mu N)/kT]|\psi\rangle_c \simeq \exp[-F(\psi)/kT]$, where $F(\psi)$ was shown by Langer[19] to have the same form as that of the free-energy functional assumed in the Landau theory of phase transition of second kind applied to liquid ^{4}He by Ginsburg and Pitaevskii.[20] Langer suggested that the order parameter for liquid ^{4}He can be identified with the most probable value of ψ. Using the coherent state representation, he has also derived the Gross-Pitaevskii equation for the order parameter.[21]

The use of coherent-state representation made by R-RR aimes at quite a different purpose to derive the excitation spectrum of

quasi-particles. Considering the intimate connection of the coherent-state representation to the order parameter mentioned above, this is very interesting. Unfortunately, the appearance of twice as many sets of conjugate variables as the degree of freedom of the system seems to prevent R-RR from obtaining any substantial results compared with other theories,[5)-7)] though their formalism is free from Fröhlich's objection to the use of canonical conjugate to density. It is the purpose of this thesis to present a formulation free from this kind of inconvenience by the aid of coherent states. Difficulties of the usual coherent state representation based on the completeness relation (6) for the treatment of a many boson system at absolute zero (or in a pure state) will be shown in §3. The formulation and its results given in §4 and 5 will shed some light on the relationships between previous collective variable theories and into points of ambiguity contained in them. We will give a brief account of these points in the next section.

§2 Surveys of Microscopic Theories

Assuming the two-body interaction, the Hamiltonian is given by

$$H = \sum_{i=1}^{N} \frac{-\hbar^2}{2m} \nabla_{x_i}^2 + \frac{1}{2} \sum_{i,j} V(x_i - x_j) \quad , \qquad (1)$$

where N is the number of bosons and V(x) are is interaction potential. In the second quantization representation, the Hamiltonian becomes

$$H = \frac{\hbar^2}{2m}\int dx\ \nabla\Psi^+(x)\cdot\nabla\Psi(x) \qquad (2)$$
$$+\frac{1}{2}\iint dx\,dy\ V(x-y)\ \Psi^+(x)\,\Psi^+(y)\,\Psi(y)\,\Psi(x)\ ,$$

where the quantized fields $\Psi^+(x)$ and $\Psi(x)$ satisfy the Bose commutation relations:

$$[\Psi(x),\Psi^+(y)] = \delta(x-y)\ ,\quad [\Psi(x),\Psi(y)] = [\Psi^+(x),\Psi^+(y)] = 0\ . \qquad (3)$$

Introducing the Fourier transform of $\Psi^+(x)$ and $\Psi(x)$ by

$$a_k = \frac{1}{\sqrt{V}}\int dx\ \Psi(x)\,e^{-ikx}\ ,\quad a_k^+ = \frac{1}{\sqrt{V}}\int dx\ \Psi^+(x)\,e^{+ikx}\ , \qquad (4)$$

where V is the volume of a box in which the system is enclosed with the periodic boundary condition, the Hamiltonian (2) can also be written as

$$H = \sum_k \frac{\hbar^2k^2}{2m}\,a_k^+a_k + \frac{1}{2}\sum_{k,k',q} V_q\ a_{k+q}^+\,a_{k'-q}^+\,a_{k'}\,a_k\ . \qquad (5)$$

In (5) V_q is the Fourier transform of the interaction potential defined by

$$V_q = \frac{1}{V}\int dx\ e^{-iqx}\ V(x) \quad . \qquad (6)$$

a_k and a_k^+ satisfy the commutation relations

$$[a_k, a_{k'}^+] = \delta_{k,k'} \quad , \quad [a_k, a_{k'}] = [a_k^+, a_{k'}^+] = 0 \quad . \qquad (7)$$

1. Exact Results

The fact that the linked-cluster expansion is impossible for a boson system makes the situation very difficult. Under some limited conditions, however, exact results have been obtained.

Lieb and Liniger[22] solved exactly the case of a one-dimensional Bose gas interacting via a repulsive δ-function potential (the LL model). Putting $V_k = 2c/L$ and $2m = \hbar = 1$ in (1) ($L(=V)$ is the length of the box), their results are given in terms of one dimensionless coupling-parameter, $\gamma = c/n$, where $n = N/L$, which is small in the weak-coupling and high-density limit. Takahashi[23] has solved their integral equation numerically and obtained the expansion of the ground-state energy E_0 for small γ written as

$$E_0^{LL} = L n^3 \left[\gamma - \frac{4}{3\pi}\gamma^{3/2} + 0.00654\,\gamma^2 + \cdots\right] \quad . \qquad (8)$$

The high-density expansion to second order of the ground-state energy for a charged Bose gas has been obtained rigorously from the modified Bogoliubov theory due to Brueckner[24] and Lee.[25]

Bogoliubov, in his treatment of a weakly interacting Bose gas, has argued that, as the average occupation number $\langle a_0^+ a_0 \rangle$ for the zero-momentum state remains to be a macroscopic number much larger than the commutator $[a_0, a_0^+] = 1$, the replacement of a_0 and a_0^+ by a c-number $\sqrt{N_0}$ is a reasonable approximation (the Bogoliubov approximation). Applying this approximation, the Hamiltonian (5) becomes

$$H(N_0) = H_0 + H_3 + H_4 \quad , \tag{9}$$

$$H_0 = \sum_{k\neq 0} \left\{ \frac{\hbar^2 k^2}{2m} a_k^+ a_{-k} + \frac{N_0 V_k}{2V} (2 a_k^+ a_k + a_k^+ a_{-k}^+ + a_k a_{-k}) \right\} + \frac{N_0^2}{2V} V_{k=0} \quad ,$$

where H_3 (H_4) contains the terms of the form $a^+ a^+ a$ and $a^+ a\, a$ ($a^+ a^+ a\, a$). If one neglects H_3 and H_4 as was done by Bogoliubov, the resulting Hamiltonian H_0 can be diagonalized by the Bogoliubov canonical transformation defined by

$$a_k = u_k \alpha_k^+ - v_k \alpha_{-k} \quad , \quad [\alpha_k, \alpha_{k'}^+] = \delta_{kk'} \quad , \tag{10}$$

to yield

$$H_0 = E_0^B + \sum_k \varepsilon_k^B \alpha_k^+ \alpha_k \quad , \tag{11}$$

$$E_0^B = \frac{N_0^2}{2V} V_{k=0} + \frac{1}{2}\sum_k \left(\varepsilon_k^B - \frac{\hbar^2 k^2}{2m} \right) - \frac{1}{2} N V(0) \quad ,$$

$$\varepsilon_k^B = \sqrt{\frac{\hbar^2 N_0}{m V} V_k \cdot k^2 + \left(\frac{\hbar^2 k^2}{2m}\right)^2} \quad . \tag{11}$$

The Bogoliubov spectrum ε_k^B becomes linear in k for small k (the phonon spectrum). To the same accuracy, N_0 can be replaced by N. In general, the neglect of the depletion $N' = N - N_0$ violates the sum rule and N_0 should be determined by

$$N' = N - N_0 = \left\langle \sum_{k \neq 0} a_k^+ a_k \right\rangle \quad . \tag{12}$$

With this prescription, the Bogoliubov approximation is able to yield exact result for the correlation functions.*

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* Using the coherent states, Ginible[26] formulated the Bogoliubov approximation as follows. The Fock space $\mathcal{H}$ can be written as a tensor product $\mathcal{H}_0 \otimes \mathcal{H}'$, where $\mathcal{H}_0$ is the Hilbert space of the zero-momentum oscillator. Denoting the coherent state for a_0 as $|\alpha_0\rangle \in \mathcal{H}_0$, the Bogoliubov approximation for an operator A on $\mathcal{H}$ is defined by the replacement of A by the operator $A(\alpha) = \langle \alpha_0 | A | \alpha_0 \rangle$ on $\mathcal{H}'$. Ginible showed that the pressure given in the Bogoliubov approximation, i.e. obtained from the grand canonical partition function $Z(\alpha) = T_r' \exp[-\beta H(\alpha)]$ is equal to the exact pressure obtained from $Z = T_r \exp[-\beta H]$ in the thermodynamic limit.

Treating the kinetic energy part in (9) as the unperturbed Hamiltonian, the leading term of E_0 in the high-density expansion is given by one-ring diagrams ($N_0 = O(N)$). This sum can be evaluated by the Bogoliubov canonical transformation applied to H_0. The next term is given by two-ring diagrams. This sum can be evaluated by the first- and second-order perturbation in the Bogoliubov representation. Determining N_0 by (12), Brueckner[24] obtained E_0 of a charged Bose gas at high density in the form

$$E_0/N = -0.8031 \cdot r_s^{-3/4} + 0.0280 + O(r_s^{3/4}) \quad , \qquad (13)$$

where r_s is the spacing per particles measured in units $\hbar^2/m e^2$ and the energy is measured in Ry.

Lee[25] extended the Brueckner result to a more general interaction by introducing a new parameter, $(N - N_0)/N$, regarded as the expansion parameter. Lee's result can be written in the form

$$E_0^{BL} = E_0^B + E_2(1\text{-ring}) + E_X + E_{Y\lambda} \quad , \qquad (14)$$

$$E_2(1\text{-ring}) = \frac{1}{16N} \sum_{k+l+m=0} \frac{\varepsilon_k^B}{\lambda_k \lambda_l} (1+\lambda_m)(1-\lambda_k)^2(1-\lambda_l)^2 \quad ,$$

$$E_X = \frac{1}{32N} \sum_{k+l+m=0} \frac{\varepsilon_k^B}{\lambda_k \lambda_l \lambda_m} (1-\lambda_k)(1-\lambda_l)(1+\lambda_k\lambda_l)(1-\lambda_m^2) \quad ,$$

$$E_{Y\lambda} = -\frac{1}{96N} \sum_{k+l+m=0} \frac{1}{\varepsilon_k^B + \varepsilon_l^B + \varepsilon_m^B} \cdot \frac{1}{\lambda_k \lambda_l \lambda_m} \times$$

$$\times \left[\varepsilon_k^B (1-\lambda_k^2)(1-\lambda_l\lambda_m) + \text{sym.} \right]^2 \quad ,$$

where ε_k^B is given by (11) except that N_0 is replaced by N and λ_k, the unperturbed structure factor, is given by

$$\lambda_k = \left\{ \frac{\hbar^2 k^2}{2m} \Big/ \left(\frac{\hbar^2 k^2}{2m} + 4m N V_k / V \right) \right\}^{1/2} . \qquad (15)$$

When applied to the LL model, E_0^{BL} given by (14) is shown by Takahashi to give the exact result (8).

Recently, Rajagopal and Grest[27] have pointed out that the Bogoliubov approximation neglects the condensate fluctuation and proposed to replace a_0 and a_0^+ by

$$a_0 \cong a_0^+ \cong \left(N - \sum_{k \neq 0} a_k^+ a_k \right)^{1/2} . \qquad (16)$$

Making an expansion in powers of $1/N$ (the $1/N$ expansion), they obtained the same ground-state energy as that given by Lee. The meaning of the $1/N$ expansion, which is frequently used in the collective variable theories (see below), is not clear, but is sufficient for the calculation of the energy per particles, say, to the order of magnitude of $(V/N)^2$ formally.

Beliaev[28] analysed the Green's function for a many-boson system and found that the inapplicability of the linked-cluster expansion simply results in the replacement of a_0 and a_0^+ by $\sqrt{N_0}$, where N_0 should be determined by (12), and in the appearance of anormalous Green's functions. This matrix Green's function formalism was transformed into a more convenient form by Hugenholtz and Pines.[29] In the low-momentum limit, Gavoret and Nozières[30] have obtained exact

results by means of the Green's function method of Hugenholtz and Pines. To all orders of the perturbation series, they have found that a) the excitation spectrum ω_k has no gap, i.e. ω_k approaches the value ck [c is the true sound velocity] in the limit $k \to 0$, and b) the momentum distribution $n_p = \langle a_p^+ a_p \rangle$ approaches the value $(N_0/N)(mc/2\hbar p)$ for $p \to 0$. They have also shown that the poles of one-particle Green's functions and those of two-particle Green's functions coincide in the same limit, owing to the presence of condensate.

2. Collective Variable Theories

The relevance of the density fluctuation in describing the elementary excitations in a boson system have clearly been demonstrated by Feynmen.[31] He presented a very convincing physical argument for the shape of the low excited state and, by applying the variational method, could show that the state $\rho_k \phi_0$ ($\rho_k = \sum_i e^{ikx_i}$ and ϕ_0 is the true ground state) offers a good approximation for the excited states for all momenta with the excitation spectrum written as

$$\varepsilon_k^F = \hbar^2 k^2 / 2m\, S(k) \quad , \tag{17}$$

where S(k) is the structure factor. With S(k) obtained from the x-ray measurement, the Feynman spectrum (17) shows qualitative agreement with the experimental phonon-roton spectrum. Feynman and Cohen[32] improved this theory by introducing the back-flow

effect to the trial wave function. Their method culminated into the method of correlated basis functions(CBF) developed by Feenberg and co-workers.[8)]

An expression equivalent to E_o^{BL} is obtained also from the method of correlated basis functions, in the uniform limit defined by $\alpha = 1 - g(o) \ll 1$, where g(r) is the pair distribution function.[33)] The variational energy obtained by employing the Bijl-Dingle-Jastrow type trial function ϕ_o, together with the perturbational correction evaluated by including the three-phonon states, $\hat{\rho}_{k_1}\hat{\rho}_{k_2}\hat{\rho}_{k_3}\phi_o$, where $\hat{\rho}_k = \sum_i e^{ikx_i}$, yields an expression which can be written as[25),34)]

$$E_o^{CBF} = E_o^{BL} + \sum_{k,l} \frac{\hbar^2}{32m}(k\cdot l)\frac{(1-\lambda_k)^2(1-\lambda_l)^2}{\lambda_k \lambda_l} \qquad (18)$$

Soon after the work of Feynman, Bogoliubov and Zubarev (BZ)[4)] have presented a theory in which the density fluctuations, $\hat{\rho}_k = \sum_i e^{ikx_i}$, are introduced as (auxiliary) variables. BZ have noticed that the wave function $\phi(x_1, x_2 \cdots, x_N)$ for a many-boson system can always be expressed in a polynominal of $\hat{\rho}_k$ as $\varphi(\{\hat{\rho}_k\})$. Given the wave function written in this form, $H\phi(x) = E\phi(x)$ becomes

$$H_{BZ}\,\varphi(\hat{\rho}) = E\,\varphi(\hat{\rho}) \quad , \qquad (19)$$

$$H_{BZ} = \sum_k{}' \frac{\hbar^2 k^2}{2m}\left(\hat{\rho}_k \frac{\partial}{\partial \hat{\rho}_k} - N\frac{\partial}{\partial \hat{\rho}_k}\frac{\partial}{\partial \hat{\rho}_{-k}}\right)$$

$$+ \sum_{k+p\neq 0}{}' \frac{\hbar^2}{2m}(k\cdot p)\,\hat{\rho}_{k+p}\frac{\partial^2}{\partial \hat{\rho}_k \partial \hat{\rho}_p}$$

$$+\frac{1}{2}\sum_k V_k \hat{\rho}_k \hat{\rho}_{-k} + \frac{1}{2}N V(0) \quad ,$$

by means of the relations,

$$\hat{\rho}_0 = N \quad , \quad \text{and} \tag{20}$$

$$\frac{\partial}{\partial x_j} = \sum_k{}' \, ik \, e^{ikx_j} \, \frac{\partial}{\partial \hat{\rho}_k} \quad , \tag{21}$$

where $\sum'$ means the exclusion of k=0. Regarding the $\hat{\rho}_k$'s as independent variables ρ_k, (i.e. ρ_k is not related to x-variables by $\rho_k = \sum_j e^{ikx_j}$), we obtain the so-called ρ_k-representation. The solutions for the Shrödinger equation in the original x-representation are all contained in the ρ_k-representation (by replacing ρ_k by $\sum_j e^{ikx_j}$), but the inverse is not true.

The most striking feature of the BZ theory is that the Hamiltonian H_{BZ} is not hermitian in the sense that H^+_{BZ} defined by $\int d\rho \, \varphi^*(\rho) H_{BZ} \varphi(\rho) = \int d\rho \, (H^+_{BZ} \varphi(\rho))^* \varphi(\rho)$ is not equal to H_{BZ}. In connection to this, the variables ρ_k are superabundant, i.e. there are infinite number of them in spite of 3N degrees of freedom of the system. The latter point is not serious, when formulated in the second quantization representation.[7] The non-hermiticity, however, needs further investigations.

The non-hermiticity has not been taken into account properly by BZ. Their procedure is as follows. Introducing a new wave function $\Phi(\rho) = \exp\left[-\frac{1}{4N} \sum' \rho_k \rho_{-k}\right] \varphi(\rho)$, $H_{BZ} \varphi = E \varphi$ becomes

$$H'_{BZ} \Phi = E \Phi \quad , \tag{22}$$

$$H'_{BZ} = H_0 + H_1 \quad ,$$

$$H_0 = \sum_{k\neq 0}\left[\frac{1}{2N}\left(nV_k + \frac{\hbar^2k^2}{4m}\right)\rho_k\rho_{-k} - N\cdot\frac{\hbar^2k^2}{2m}\frac{\partial^2}{\partial\rho_k\,\partial\rho_{-k}}\right] \qquad (22)$$

$$-\frac{1}{2}\sum_{k\neq 0}\left(\frac{\hbar^2k^2}{2m} + nV_k\right) + \frac{1}{2}nNV_0 \quad ,$$

$$H_1 = \sum_{\substack{p,k,\\ p+k\neq 0}}\frac{\hbar^2}{2m}(k\cdot p)\,\rho_{k+p}\left(\frac{\partial}{\partial\rho_p} + \frac{1}{2N}\rho_{-p}\right)\left(\frac{\partial}{\partial\rho_k} + \frac{1}{2N}\rho_{-k}\right) \quad ,$$

$$(n = N/V) \quad .$$

H_0 in (22) can be diagonalized by introducing

$$B_k = \frac{1}{2\sqrt{N\lambda_k}}\rho_{-k} + \sqrt{N\lambda_k}\,\frac{\partial}{\partial\rho_k} \quad , \quad B_k^+ = \frac{1}{2\sqrt{N\lambda_k}}\rho_k - \sqrt{N\lambda_k}\,\frac{\partial}{\partial\rho_{-k}} \quad , \qquad (23)$$

$$\lambda_k^2 = \frac{\hbar^2k^2}{2m}\Big/\left(\frac{\hbar^2k^2}{2m} + 2nV_k\right) \quad ,$$

to give

$$H_0 = E_0^B + \sum_k \varepsilon_k^B\, B_k^+ B_k \quad , \qquad (24)$$

where E_0^B and ε_k^B have the same form as that given by Bogoliubov (eq. (11)), except that N_0 is replaced by N^*. BZ have shown that to lowest order, the wave function for the first excited-state has

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* Eq. (23) is the Bogoliubov canonical transformation. From (23), we have $\rho_k = 2\sqrt{N\lambda_k}\,(B_k^+ + B_{-k})$. On the other hand, in the Bogoliubov approximation, we have $\rho_k = a_0^+a_{-k} + a_k^+a_0 + \Sigma' a_{k+q}^+ a_q \cong \sqrt{N_0}(a_{-k} + a_k^+)$ $= \sqrt{N_0}(u_k - v_k)(\alpha_k^+ + \alpha_{-k})$, where α_k and α_k^+ are defined in (10). Thus in the weak-coupling limit, two expressions for ρ_k agree with each other ($N_0 \cong N$).

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the same form as that given by Feynman i.e. $\Phi = \rho_k \Phi_0$. BZ have obtained the first correction to E_0^B by applying the second-order perturbation to H_1 as if it were hermitian. The result of BZ can be written as[34]

$$E_0^{BZ} = E_0^{CBF} - \frac{1}{4N} \sum_{k,l} \frac{\hbar^2}{2m} (k \cdot l)(1 - \lambda_k)(1 - \lambda_l) \quad . \quad (25)$$

When applied to the LL model, E_0^{BZ} does not agree with the exact result (8) in the term of order γ^2.[23]

Hiroike[5] pointed out that the inner product should be defined with a weight factor. Accordingly, the energy should be calculated from

$$\langle H \rangle = \int (d\rho)\, e^{-W}\, \varphi^*(\rho)\, H(\rho)\, \varphi(\rho) \quad , \quad (26)$$

where e^{-W} is the weight factor and $\int (d\rho)$ stands for $\prod_{k_z>0} \int d(\mathrm{Re}\,\rho_k)\, d(\mathrm{Im}\,\rho_k)$ ($\rho_{-k} = \rho_k^*$) . In order to have a explicit form for e^{-W}, Hiroike introduced a cut-off wave number k_c. Then, assuming that $\varphi(\hat{\rho}_k)$ includes only those $\hat{\rho}_k$'s with $|k| < k_c$, he carried out the transformation of variables from x_i's to ρ_k's through the use of eq.(19) modified by the cut-off and of the relation written as

$$\int d^{3N}x\, F(x) = \int_{\substack{0<|k|<k_c, \\ k_z>0}} (d\rho) \int d^{3N}x\, F(x) \prod_{\substack{0<|k|<k_c, \\ k_z>0}} f_k\, g_k \quad (27)$$

$$f_k = \delta\left(\mathrm{Re}\,\rho_k - \frac{1}{\sqrt{N}} \mathrm{Re}\,\hat{\rho}_k\right) , \quad g_k = \delta\left(\mathrm{Im}\,\rho_k - \frac{1}{\sqrt{N}} \mathrm{Im}\,\hat{\rho}_k\right) ,$$

$$\left[\, \hat{\rho}_k = \sum_i e^{i k x_i} \,\right] .$$

Hiroike found that the weight factor is given by

$$e^{-W} = D(\rho) = \int d^{3N}x \prod_{\substack{|k|<k_c, \\ k_z>0}} f_k \, g_k \quad . \tag{28}$$

$H(\rho)$ agrees with H_{BZ} if the cut-off is neglected. Introducing a new wave function $\Phi(\rho)=\sqrt{D}\,\varphi(\rho)$, eq.(26) with (28) can be written as

$$\langle H \rangle = \int (d\rho)\, \Phi^*(\rho)\, \mathcal{H}_{BZH}\, \Phi(\rho) \quad , \tag{29}$$

where $\mathcal{H}_{BZH}$ is given by

$$\mathcal{H}_{BZH} = \{D(\rho)\}^{\frac{1}{2}}\, H(\rho)\, \{D(\rho)\}^{-\frac{1}{2}} \quad , \tag{30}$$

and is found to be hermitian. Thus $\sqrt{D}$ can be considered as to generate a non-unitary transformation. Treating the cut-off carefully, Hiroike obtained, in the $1/N$ expansion,

$$D(\rho) = \text{const.} \times \exp\Big[-\frac{1}{2} \sum_k \rho_k \rho_{-k} + \frac{1}{6\sqrt{N}} \sum_{k+l+m=0} \rho_k \rho_l \rho_m - \frac{1}{12N} \sum_{k+l+m+n=0} \rho_k \rho_l \rho_m \rho_n + \frac{1}{4N} \sum_{k,l} \rho_k \rho_{-k} \Big] \quad , \tag{31}$$

(We have taken the limit $k_c \to \infty$ in Hiroike's eq.(3·8))[5]. To lowest order the relation between Φ and φ are the same as in BZ theory. The principal part of H_{BZH} agrees with that of H'_{BZ} given in (22), but the interaction term is different and contains infinite series of order $N^{-\frac{n}{2}}$. The ground-state energy evaluated up to order $1/N$ is equivalent to E_0^{CBF} and not to E_0^{BZ}.

The need for a weight factor was also noted by Chan and Valatin.[35]

However, their result written as

$$e^{-W} = \exp\left[-\int dx\,(\rho(x)\ln\rho(x) - \rho(x))\right] \qquad (32)$$

$$= \text{const.} \times \exp\left[-\frac{1}{2}\sum_k \rho_k \rho_{-k} + \frac{1}{6\sqrt{N}}\sum_{k+l+m=0}\rho_k\rho_l\rho_m - \frac{1}{12N}\sum_{k+l+m+n=0}\rho_k\rho_l\rho_m\rho_n + o(N^{-1})\right],$$

differs from D by the absence of the term,

$$\Delta W = \frac{1}{4N}\sum_{k,l}\rho_k\rho_{-k} \quad , \qquad (33)$$

, which leads to the appearance of absolutely divergent terms of the form

$$\sum_{k,l}(k\cdot l)\,\lambda_k\lambda_l \quad , \qquad (34)$$

in the expression for E_0.[6]

In the second quantized representation, the super-abundance of the ρ_k-variables is not serious as mentioned before, and the use of the density operator $\rho(x) = \Psi^+(x)\Psi(x)$ as the basic variable together with its conjugate variable has been made by many authors. These collective variable theories can be looked upon as the microscopic version of the quantum hydrodynamics presented by Landau.[2]

Prior to the work of BZ, Nishiyama [6),36)] presented a theory, now called the density and phase operator approach (DPO), by introducing the phase operator $\phi(x)$ through

$$\Psi(x) = e^{i\phi(x)}\,\rho(x)^{\frac{1}{2}} \quad , \quad \Psi^{+}(x) = \rho(x)^{\frac{1}{2}}\,e^{-i\phi(x)} \quad , \qquad (35)$$

$$\rho(x) = \Psi^{+}(x)\,\Psi(x) \quad ,$$

where $\phi(x)$ and $\rho(x)$ satisfy the commutation relations

$$[\rho(x), \phi(x')] = i\,\delta(x-x') \quad , \quad [\rho(x), \rho(x')] = [\phi(x), \phi(x')] = 0 \quad . \qquad (36)$$

The assumption of the existence of the canonical conjugate to density (such as $\phi(x)$) typical to the collective variable theories has been criticized by several authors, and we will turn to this point later. Using (35) in the Hamiltonian (2), we obtain H_{DPOC} given by[6]

$$H_{DPOC} = T_{DPOC} + V \quad , \qquad (37)$$

$$T_{DPOC} = \int dx \left\{ \frac{m}{2}\,\rho^{\frac{1}{2}}\,v^2\,\rho^{\frac{1}{2}} + \frac{\hbar^2}{2m}\,|\nabla\rho^{\frac{1}{2}}|^2 + \frac{\hbar}{2i}\left[\rho^{\frac{1}{2}}\,v\cdot\nabla\rho^{\frac{1}{2}} - \nabla\rho^{\frac{1}{2}}\cdot v\,\rho^{\frac{1}{2}}\right] \right\} ,$$

$$V = \frac{1}{2}\iint dx\,dy\; V(x-y)\left[\rho(x) - \delta(x-y)\right]\rho(y) \quad ,$$

where the velocity operator $v(x)$ is defined by

$$v(x) = \frac{\hbar}{m}\,\nabla\phi(x) \quad , \qquad (38)$$

and satisfies

$$[\rho(x), v(x')] = \frac{i\hbar}{m}\,\nabla\delta(x-x') \quad . \qquad (39)$$

The kinetic energy T_{DPOC} can also be written as

$$T_{DPOC} = T_{MPO} + T' \quad , \tag{40}$$

$$T_{MPO} = \int dx \left\{ \frac{m}{2} v \rho v + \frac{\hbar^2}{8m} (\nabla \rho)^2 \rho^{-1} + \frac{\hbar}{4i} [v, \nabla \rho] \right\} ,$$

$$T' = \int dx \left\{ -\frac{m}{8} |[\rho, v]|^2 \rho^{-1} + \frac{\hbar}{4i} [\rho, v] \cdot \nabla \rho \cdot \rho^{-1} \right\} ,$$

(where MPO stands for the modified phase operator approach). Here, the Hamiltonian of MPO defined by $H_{MPO} = T_{MPO} + V$ is related to the BZ Hamiltonian by a non-unitary transformation of the form

$$S = \exp \left\{ -\frac{1}{2} \int dx \left[\rho(x) \ln \rho(x) - \rho(x) \right] \right\} \quad , \tag{41}$$

which is identical to the root of Chan and Valatin's weight factor (33). By means of the relation

$$S^{-1} \phi(x) S = \phi(x) - \frac{1}{2i} \ln \rho(x) \quad , \tag{42}$$

we obtain

$$S^{-1} T_{MPO} S = \int dx \frac{\hbar^2}{2m} \left[\nabla \phi \cdot \rho \nabla \phi - \frac{1}{i} \nabla \rho \cdot \nabla \phi \right] = T_{BZ} \quad . \tag{43}$$

Now introducing the Fourier transform of $\rho(x)$ and $\phi(x)$ by

$$\rho(x) = \frac{1}{V} \sum_k \rho_k e^{ikx} \quad , \quad \phi(x) = \sum_k \phi_k e^{-ikx} \quad , \tag{44}$$

$$[\rho_k, \phi_\ell] = i \delta_{k\ell} \quad , \tag{45}$$

we rewrite (44) in the form

$$T_{BZ} = \frac{\hbar^2}{2m}\left[-\sum_{k,l}(k\cdot l)\,\phi_k\,\rho_{k+l}\,\phi_l - \frac{1}{\lambda}\sum_k k^2\,\rho_k\,\phi_k \right] . \quad (46)$$

Noting that the commutation relation (45) is realized by setting $\phi_k = -\lambda\,\partial/\partial\rho_k$ (ie. in the representation which diagonalizes ρ_k), we see that $S^{-1}H_{MPO}S$ is identical to H_{BZ} given in (19) (as ϕ_0 is a cyclic coordinate, ρ_0 can be replaced by c-number N).

The kinetic energy given in (43) has been obtained by Berdahl[37], who noticed that the commutation relations given in (3) are realized by a set of quantized fields, $\Psi_1(x)$ and $\Psi_2(x)$, defined by

$$\Psi_1(x) = e^{\lambda\phi(x)} \quad \text{and} \quad \Psi_2(x) = \rho(x)\,e^{-\lambda\phi(x)} , \quad (47)$$

respectively, where ϕ and ρ satisfy the commutation relation (36), and replaced $\Psi(x)$ and $\Psi^+(x)$ by $\Psi_1(x)$ and $\Psi_2(x)$, respectively, to obtain T_{BZ} and the non-unitary transformation S. Arguing that $[\rho(x), \nabla\phi(x)] = \sum k = +\lambda\nabla\delta(0)$ vanish from the symmetry, Berdahl neglected T' and concluded that H_{DPOC} is equivalent to H_{BZ}. Grest and Rajagopal[38] have shown that H_{BZ} and the Hamiltonian H_S given by the old Sunakawa theory, in which a velocity operator conjugate to density is introduced, are only different version of the method of the current algebra. Thus the equivalence of the BZ theory, the old Sunakawa theory and the method of DPO seemed to be confirmed.

The neglect of T', however, is not permissible, as pointed out by Nishiyama.[6] Setting $\rho_0 = N$ and regarding $\rho_k\,(k\neq 0)$ is of order

$\sqrt{N}$, we can expand H_{DPOC} in powers of $1/\sqrt{N}$ (the $1/N$ expansion). The leading term (for both H_{DPOC} and H_{MPO}) is identical to the principal part of H'_{BZ} given in (22) and we can introduce the Bogoliubov operator B_k^+ and B_k through (23) ($i\phi_k \leftrightarrow \partial/\partial\rho_k$). The ground state can be evaluated by applying the Rayleigh-Schrödinger perturbation theory and, up to order $1/N$, H_{MPO} yields

$$E_0' = E_0^{BZ} - \frac{\hbar^2}{8mN} \sum_{k,l} (k\cdot l)\, \lambda_k \lambda_l \quad . \tag{48}$$

The expectation value of T', which can be written, up to order $1/N$, as

$$T' = \frac{-1}{8mN} \sum_{k,l} (k\cdot l)\left(1 + \frac{2}{N}\rho_k \rho_{-k}\right) \quad , \tag{49}$$

is given by

$$\langle T' \rangle = -\frac{\hbar^2}{8mN} \sum_{k,l} (k\cdot l)(1 - 2\lambda_k) \quad . \tag{50}$$

Following Berdahl, both the additional term in (48) and $\langle T' \rangle$ is equal to zero and we obtain E_0^{BZ}. However, these terms are absolutely divergent and cannot be simply put equal to zero. Nishiyama has shown that these divergent terms add to give a convergent result E_0^{CBF} with an appropriate choice of the sign of them. Thus the conclusion of Berdahl that H_{BZ} is equivalent to H_{DPOC} does not apply at least to the LL model, in which E_0^{BZ} and E_0^{CBF} are different.

Recently, Yamasaki, Kebukawa and Sunakawa (YKS)[39] have modified the definition of their velocity operator and obtained E_0^{CBF}.

More recently, Nishiyama has calculated the kinetic energy through the momentum distribution function n_p given by DPO(DPOM)[40] and found that in this case the ground-state energy is equal to E_0^{BL}. Though the difference between E_0^{BL} and E_0^{CBF} is negligible numerically (at least for the LL model) only E_0^{BL} yield the correct n_p which satisfies the Gavoret-Nozières condition, when the adiabatic theorem is applied. (Following the adiabatic theorem applied by Berdahl[37], n_p is obtainable from

$$n_p = \frac{\partial}{\partial \varepsilon_{p\eta}^0} E_0 \Big|_{\eta=1} , \tag{51}$$

regarding the energy parameter $\varepsilon_{p\eta}^0 = \frac{\hbar^2 p^2}{2m}\eta$ as the adiabatic parameter). Thus we conclude that though previous collective variable theories, the DPO and the theory of YKS, present a very powerful way to deal with a many-boson system at high density, there remain some points left to be discussed.

Apart from these points, the collective variable theories have faced several objections. Mathematically speaking, the canonical conjugate to the density (or number) operator in any form does not exist in the Fock space, due to the fact that the density operator is bounded from below.[13)41)] In the case of the phase operator, the situation is as follows.[11)42)] For simplicity, we consider the case of a system with one degrees of freedom, i.e. a harmonic oscillator described by a set of Bose operators b and b^+ satisfying $[b, b^+]=1$ etc. Now, we define the lowering and raising operator, χ and χ^+ by the relation

$$b = \chi\sqrt{b^+b} \quad , \quad b^+ = \sqrt{b^+b}\,\chi^+ \ . \tag{52}$$

Then the matrix elements of χ between the number states $|n\rangle$ ($b^+b\,|n\rangle = n\,|n\rangle$) are given by

$$\langle n|\chi|m\rangle = \begin{cases} \delta_{n+1,m} & (m>0) \\ 0 & (m=0) \end{cases} \ , \tag{53}$$

$$\langle n|\chi^+|m\rangle = \delta_{n,m+1} \qquad (m \geqq 0) \ .$$

For m>0, eq. (53) is obvious from

$$\begin{aligned} \langle n|b|m\rangle &= \sqrt{m}\ \delta_{n+1,m} = \sqrt{m}\ \langle n|\chi|m\rangle \ , \\ \langle n|b^+|m\rangle &= \sqrt{n}\ \delta_{n,m+1} = \sqrt{n}\ \langle n|\chi^+|m\rangle \ . \end{aligned} \tag{54}$$

This relation (54) does not determine $\langle 1|\chi|0\rangle$ and we expand the state $\chi|0\rangle$ as

$$\chi|0\rangle = \sum_{n=0}^{\infty} a_n\,|n\rangle \ . \tag{55}$$

Then noticing $\chi^+\chi\,|m\rangle = |m\rangle$ for m>0, we have

$$0 = \langle m|\chi^+\chi|0\rangle = \sum_n a_n\,\langle m|\chi^+|n\rangle = a_{m-1} \qquad (m>0) \ , \tag{56}$$

which leads to $\chi|0\rangle = 0$ i.e. to eq. (53). From eq. (53), we have

$$\chi^+\chi = 1 - P_0 \quad , \quad \chi\chi^+ = 1 \quad , \tag{57}$$

where $P_0 = |0\rangle\langle 0|$. Eq. (57) shows that χ and χ^+ are not unitary

and any hermite phase operator cannot be introduced. However, as far as the states $|n\rangle$ with $n > 0$ are concerned, we have $\langle n|\chi^\dagger\chi|n\rangle = \langle n|\chi\chi^\dagger|n\rangle = 1$ and the prescription (35) of DPO is expected to be a reasonable approximation at high-density. (Recently, the use of a larger space than the Fock space has been suggested by Kobe and Coomer[43], YKS, and Rajagopal.[44])

There are other objections in connection to the use of super-abundant variables in a collective variable theory, for example BZ, formulated in the N-th configuration space[45] and to the assumption of existence of a continuum density $\rho(x)$.[13] The first point is not serious in principle in the second quantization representation, as mentioned before, and for the second point, we only mention that the expectation value for density is really continuous. In the following sections, we will develop a formulation utilizing the coherent states and the lattice space, which will shed some light on the problems to be answered in this thesis.

§3 Coherent State Representation

Non-existence of the phase operator in the Fock space has led Rajagopal and Rama Rao (R-RR)[9] to use the coherent-state representation (C.S.R.)[10][11] or the phase-space method for the evaluation of the ground-state energy and the excitation spectrum. The phase-space method developed by Cahill and Glauber[46], Sudarshan[47], and Agarwal and Wolf (AW)[48] is a generalization of the method of the Wigner distribution function[49] and contains the C.S.R. as a special case.

For simplicity, we consider a one-dimensional oscillator described by using the annihilation and creation operator, b and b^+, which satisfy the Bose commutation relations:

$$[b, b^+] = 1 \quad , \quad [b, b] = [b^+, b^+] = 0 \quad . \tag{1}$$

A coherent state $|\alpha\rangle_c$ is defined as a right eigenstate of b with an eigenvalue $\alpha = \sqrt{J}\, e^{i\phi}$, namely, it satisfies

$$b\,|\alpha\rangle_c = \alpha\,|\alpha\rangle_c \quad , \quad {}_c\langle\alpha|\, b^+ = \alpha^*\, {}_c\langle\alpha| \quad . \tag{2}$$

In terms of the normalized Fock-state $|n\rangle$ defined by

$$|n\rangle = (n!)^{-\frac{1}{2}}\,(b^+)^n\,|0\rangle \quad , \tag{3}$$

where $|0\rangle$ is the vacuum state satisfying $b|0\rangle = 0$, the coherent state $|\alpha\rangle_c$ has the form

$$|\alpha\rangle_c = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} (n!)^{-\frac{1}{2}}\,\alpha^n\,|n\rangle = e^{-\frac{|\alpha|^2}{2}}\, e^{\alpha b^+}\,|0\rangle \quad , \tag{4}$$

and the scalar product of a coherent state $|\alpha\rangle_c$ and a Fock state $|n\rangle$ is given by

$$\langle n|\alpha\rangle_c = e^{-\frac{|\alpha|^2}{2}} \cdot \frac{\alpha^n}{\sqrt{n!}} = \left({}_c\langle\alpha|\, n\rangle\right)^* \quad . \tag{5}$$

From eq.(4), we have

$$\langle\beta|\alpha\rangle_c = \exp\left[-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \alpha\beta^*\right] \quad , \quad \langle\alpha|\alpha\rangle_c = 1 \quad , \tag{6}$$

and the resolution of identity of the form

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle_c \langle\alpha| = \sum_n |n\rangle\langle n| = 1 \quad , \tag{7}$$

where $\int \frac{d^2\alpha}{\pi}$ stands for $\iint_{-\infty}^{\infty} d(\mathrm{Re}\,\alpha)\, d(\mathrm{Im}\,\alpha) = \int_0^\infty dJ \int_{-\pi}^{\pi} \frac{d\phi}{2\pi}$. Thus, the coherent states are not orthogonal to each other, but form a complete set (the over-completeness). Using eq. (7), the trace of an operator A can be obtainable from the formula

$$\mathrm{Tr}\, A = \int \frac{d^2\alpha}{\pi} \langle\alpha| A |\alpha\rangle_c \quad . \tag{8}$$

A coherent state $|\alpha\rangle_c$ can be generated from the vacuum state $|0\rangle$ by means of a unitary operator $U(\alpha)$ defined by

$$U(\alpha)|0\rangle = |\alpha\rangle_c \quad , \tag{9}$$

$$U(\alpha) = \exp\left[\alpha b^+ - \alpha^* b\right]$$

$$= e^{-\frac{|\alpha|^2}{2}} e^{\alpha b^+} e^{-\alpha^* b} = e^{\frac{|\alpha|^2}{2}} e^{-\alpha^* b} e^{\alpha b^+} \quad .$$

The operators $\{U(\alpha)\}$ are complete in the same sense that any physical quantity can be expressed in terms of b and b^+, i.e. a physical quantity can be represented completely by using $U(\alpha)$. From this completeness, various kinds of one-to-one correspondence between c-number functions and functions of Bose operators can be

set up. Introducing a weighted Fourier transform of $U(\alpha)$ defined with weight $\Omega(\alpha, \alpha^*)$ as

$$\Delta^{\Omega}(\alpha - b, \alpha^* - b^+) = \int \frac{d^2\xi}{\pi^2} \Omega(\xi, \xi^*) e^{\alpha\xi^* - \alpha^*\xi} U(\xi) \quad , \quad (10)$$

AW have shown that an arbitrary operator $A(b,b^+)$ given in powers of b and b^+ can be expanded uniquely in the form

$$A(b, b^+) = \int d^2\alpha \; F_A^{\Omega}(\alpha, \alpha^*) \; \Delta^{\Omega}(\alpha - b, \alpha^* - b^+) \quad . \quad (11)$$

In the case with $\Omega(\alpha,\alpha^*) = \exp[\alpha^*\alpha/2] \equiv N$, they found that $F_A^N(\alpha,\alpha^*)$ is given by $\langle\alpha|A(b,b^+)|\alpha\rangle_c$. Then, from the uniqueness of the expansion (11), we have a one-to-one correspondence

$$A(b, b^+) \quad \longleftrightarrow \quad F_A^N(\alpha, \alpha^*) = \langle\alpha|A|\alpha\rangle_c \quad , \quad (12)$$

between c-number functions and operators. Further, the c-number function corresponding to the product $A_1 A_2 = A$ of two operators $A_1(b, b^+)$ and $A_2(b, b^+)$ can be shown to be expressed in terms of $F_{A_1}^N$ and $F_{A_2}^N$ as

$$F_{A_1A_2}^N(\alpha,\alpha^*) = F_{A_1}^N\left(\alpha_1 + \frac{\partial}{\partial\alpha^*} \; , \alpha_1^*\right) F_{A_2}^N(\alpha, \alpha^*) \Big|_{\substack{\alpha_1 = \alpha \\ \alpha_1^* = \alpha^*}} \quad , (13a)$$

or alternatively as

$$F_{A_1A_2}^N(\alpha,\alpha^*) = F_{A_2}^N\left(\alpha_1 \; , \alpha_1^* + \frac{\partial}{\partial\alpha}\right) F_{A_1}^N(\alpha, \alpha^*) \Big|_{\substack{\alpha_1 = \alpha \\ \alpha_1^* = \alpha}} \quad , (13b)$$

where the differentiation is carried out regarding α and α^* as being independent of each other*. In other words, $\frac{\partial}{\partial\alpha} f(\alpha^*) = 0$ and $\frac{\partial}{\partial\alpha^*} f(\alpha) = 0$ hold, where

$$\frac{\partial}{\partial\alpha} = J^{\frac{1}{2}} e^{-i\phi}\left[\left(\frac{\partial}{\partial J}\right) - \frac{i}{2J}\left(\frac{\partial}{\partial\phi}\right)\right],$$
$$\frac{\partial}{\partial\alpha^*} = J^{\frac{1}{2}} e^{i\phi}\left[\left(\frac{\partial}{\partial J} + \frac{i}{2J}\left(\frac{\partial}{\partial\phi}\right)\right]\right., \tag{14}$$

and $\alpha = J^{\frac{1}{2}} e^{i\phi}$.

Thus, in the case with $\Omega(\alpha,\alpha^*) = N$, we have the C.S.R. (If we choose $\Omega(\alpha,\alpha^*) = I = W(\alpha,\alpha^*)$, we have another one-to-one correspondence such that the density matrix corresponds to the Wigner distribution function.).

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* In the case with $A_1 = b^+b$ and $A_2 = b^+$, we have $F^N_{A_1} = \langle\alpha|b^+b|\alpha\rangle_c = \alpha^*\alpha$, $F^N_{A_2} = \alpha^*$ and the r.h.s. of eqs. (13a) and (13b) become $\{\alpha^*(\alpha + \partial/\partial\alpha^*)\}\alpha^*$ and $(\alpha^* + \partial/\partial\alpha)(\alpha^*\alpha)$, respectively, which are both equal to $\alpha^*\alpha^*\alpha + \alpha^* = \langle\alpha|(b^+b^+b + b^+)|\alpha\rangle_c = \langle\alpha|b^+b\,b^+|\alpha\rangle_c$. As another example, consider the case of two operators $H=b^+b$ and ρ (the density matrix). In this case, we have

$$F^N_{H\rho}(\alpha,\alpha^*) = \alpha^*(\alpha + \frac{\partial}{\partial\alpha^*})\, F^N_\rho(\alpha,\alpha^*) = (J + J\frac{\partial}{\partial J} + \frac{i}{2}\frac{\partial}{\partial\phi})\,\rho(\phi,J)$$

by using (13a) and

$$F^N_{\rho H}(\alpha,\alpha^*) = \alpha(\alpha^* + \frac{\partial}{\partial\alpha})\, F^N_\rho(\alpha,\alpha^*) = (J + J\frac{\partial}{\partial J} - \frac{i}{2}\frac{\partial}{\partial\phi})\,\rho(\phi,J)$$

by using (13b), where $\rho(\phi,J) = F^N_\rho(\alpha,\alpha^*)$.

R-RR have noticed that the existence of one-to-one correspondence enables one to avoid an ad hoc use of canonical conjugate to density. Their argument starts from replacing the Schrödinger equation $H|\psi\rangle = E|\psi\rangle$ into an equivalent set of equations for the density matrix $\rho = |\psi\rangle\langle\psi|$ written as

$$H\rho + \rho H = 2E\rho \quad , \quad H\rho - \rho H = 0 \quad . \qquad (15)$$

Then, in the case of one degree of freedom, two operator equations in (15) can be transformed into two c-number equations for $\mathcal{H}_\rho^N(\alpha, \alpha^*)$ by using eqs. (13 a), (13 b) and (14).

In the case of a many-boson system described by the Hamiltonian (2-2), a functional $\rho[\psi] = {}_c\langle\psi|\rho|\psi\rangle_c$, takes the place of ${}_c\langle\alpha|\rho|\alpha\rangle_c$, where $|\psi\rangle_c$ satisfies (1-3), i.e. $\Psi(x)|\psi\rangle_c = \psi(x)|\psi\rangle_c$. Accordingly, there appear the functional derivatives $\delta/\delta\psi(x)$ and $\delta/\delta\psi^*(x)$ in place of $\partial/\partial\alpha$ and $\partial/\partial\alpha^*$. R-RR have noticed that the c-number functional corresponding to the density $\Psi^\dagger(x)\Psi(x)$ is given by ${}_c\langle\psi|\Psi^\dagger(x)\Psi(x)|\psi\rangle_c = \psi^*(x)\psi(x)$. Then, introducing the polar form $\psi(x) = \sqrt{\rho(x)}\, e^{i\theta(x)}$, they considered that $\rho(x)$ and $\theta(x)$ take place of the density and phase operator used in the collective variable theories. Regarding the Fourier transform ρ_k (θ_k) of $\rho(x)$ ($\theta(x)$) as quantities of order $\sqrt{N}$ ($1/\sqrt{N}$), R-RR applied the the $1/N$ expansion to solve the resulting set of c-number functional equations and, up to the lowest order, obtained E_α^B and ε_k^B.

Thus, the formalism of R-RR is free from Fröhlich's objection. Unfortunately, they did not succeed in obtaining correction terms,

due to the appearance of the superabundant variables, $\{ P_k , \partial/\partial P_k \}$ and $\{ q_k, \partial/\partial q_k \}$ *, which are supposed to be reduced owing to the presence of two equations in (15).

Here we present an argument, which suggests that this kind of inconvenience can be removed if only those coherent states with $|\alpha| = J = 1$ are used. In so doing, we note that the expectation value of an operator $A(b,b^+)$ can be obtained from

$$\langle A \rangle = \mathrm{Tr}\, A\rho = \int \frac{d^2\alpha}{\pi} \langle \alpha | A \rho | \alpha \rangle_c \tag{16}$$

$$= \int \frac{d^2\alpha}{\pi} F_A^N \left(\alpha + \frac{\partial}{\partial \alpha^*} , \alpha^* \right) F_\rho^N (\alpha, \alpha^*) \quad ,$$

where we have used eqs. (8), (12) and (13a) and in F_A^N all α^*'s are placed to the left of $\partial/\partial\alpha^*$ given by (14).

For a pure state, $F_\rho^N(\alpha,\alpha^*)$ has an important property that it factorizes with respect to α and α^*. Since a Fock state $|\varphi\rangle$ can be written in the form $|\varphi\rangle = \sum_n \varphi_n |n\rangle$, using eq. (5), we obtain

$$F_\rho^N(\alpha,\alpha^*) = \langle \alpha | \rho | \alpha \rangle_c = \langle \alpha | \varphi \rangle \langle \varphi | \alpha \rangle_c \tag{17}$$

$$= e^{-|\alpha|^2} \left\{ \sum_n \varphi_n (n!)^{-\frac{1}{2}} (\alpha^*)^n \right\} \cdot \left\{ \sum_m \varphi_m^* (m!)^{-\frac{1}{2}} \alpha^m \right\}$$

$$\equiv e^{-|\alpha|^2} \varphi(\alpha^*) \tilde{\varphi}(\alpha) \quad .$$

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* In the case of one degree of freedom, we have two sets of conjugate variables, $\{ J, \frac{\partial}{\partial J} \}$ and $\{ \phi, \frac{\partial}{\partial \phi} \}$, which is easily seen from the equations given in the foot note to eq. (13).

Noting that the differentiation with respect to α^* (i.e. $\partial/\partial\alpha^*$) is to be carried out regarding α and α^* as independent variables in (16), we have

$$\langle A\rangle = \int \frac{d^2\alpha}{\pi}\ \widehat{\varphi}(\alpha)\ F_A^N(\alpha + \partial/\partial\alpha^*, \alpha^*)\ e^{-\alpha^*\alpha}\ \varphi(\alpha^*) \qquad (18)$$

$$= \int \frac{d^2\alpha}{\pi}\ e^{-\alpha^*\alpha}\ \widehat{\varphi}(\alpha)\ F_A^N(\partial/\partial\alpha^*, \alpha^*)\ \varphi(\alpha^*)\ ,$$

where use has been made of the relation

$$\frac{\partial}{\partial\alpha^*}\ e^{-\alpha^*\alpha} = -\alpha\, e^{-\alpha^*\alpha} + e^{-\alpha^*\alpha}\frac{\partial}{\partial\alpha^*}\ . \qquad (19)$$

Here, we note that a function of α^* alone appears to the right of $\partial/\partial\alpha^*$ in (18). Then, by means of a formula

$$\frac{\partial}{\partial\alpha^*}\ f(\alpha^*) = i\, J^{-\frac{1}{2}}\ e^{i\phi}\ \frac{\partial}{\partial\phi}\ f(J^{\frac{1}{2}}\ e^{-i\phi})\ , \qquad (20)$$

which holds for any function of $\alpha^* = J^{\frac{1}{2}} e^{-i\phi}$ alone and is derived from $\partial/\partial\alpha\ f(\alpha^*) = 0$ and the definition (14), (18) can be transformed into

$$\langle A\rangle = \int_0^\infty dJ \int_{-\pi}^{\pi}\frac{d\phi}{2\pi}\ e^{-J}\ \Big\{\sum_m \varphi_m^*\,(m!)^{-\frac{1}{2}}\,(J^{\frac{1}{2}}\ e^{i\phi})^m\Big\}$$

$$\times\ F_A^N(i\,J^{-\frac{1}{2}}\ e^{i\phi}\frac{\partial}{\partial\phi}\ ,\ J^{\frac{1}{2}}\ e^{-i\phi})\ \Big\{\sum_n \varphi_n\,(n!)^{-\frac{1}{2}}\,(J^{\frac{1}{2}}\ e^{-i\phi})^n\Big\}\ .$$

In this form the integration over J can be carried out under quite a general condition. Writing F_A^N as

$$F_A^N = \sum_{k,\ell} a_{k\ell} \left(J^{\frac{1}{2}} e^{-i\phi} \right)^k \left(i J^{-\frac{1}{2}} e^{i\phi} \frac{\partial}{\partial \phi} \right)^\ell ,$$

the integration over ϕ yields a factor $\delta_{m-k+\ell-n,0}$ and the integration over J becomes

$$\int_0^\infty dJ \; e^{-J} J^{(m+k-\ell+n)/2} = \int_0^\infty dJ \; e^{-J} J^m = m! \; .$$

Thus, we have

$$\langle A \rangle = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \left\{ \sum_m m! \; \varphi_m^* (m!)^{-\frac{1}{2}} (e^{i\phi})^m \right\} \tag{21}$$

$$\times \left\{ \sum_{k,\ell} a_{k\ell} (e^{-i\phi})^k \left(i e^{i\phi} \frac{\partial}{\partial \phi} \right)^\ell \right\} \left\{ \sum_n \varphi_n (n!)^{-\frac{1}{2}} (e^{-i\phi})^n \right\}$$

$$= \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \langle \varphi | \Gamma | e^{i\phi} \rangle_c \; F_A^N \left(i e^{i\phi} \frac{\partial}{\partial \phi} , e^{-i\phi} \right) \langle e^{i\phi} | \varphi \rangle ,$$

where $\Gamma = \Gamma(b^\dagger b + 1)$ (Γ is the gamma function). The integration over J results in the appearance of the Γ-factor and in the reduction of $|\alpha\rangle_c$ to $|e^{i\phi}\rangle_c$.

§4 Formulation

A many-boson system with two-body interaction can be described by using a Hamiltonian expressed in terms of a field operator $\Psi(x)$ and its Hermite conjugate $\Psi^\dagger(x)$, satisfying the Bose commutation relations, $[\Psi(x), \Psi^\dagger(y)] = \delta(x-y)$, etc. In the previous theories worked in the continuous space, the second order perturbation terms

involve several absolutely divergent terms which should not be put equal to zero.[23, 6] Although we obtain a convergent result by summing these divergent terms, there appear some ambiguities in obtaining the final result (see §2). No ambiguities of these kinds do not appear in this theory of lattice space as shown in the following section. In this section, a representation based on coherent states is formulated in a lattice space with the lattice constant a.

We first replace $\Psi(x)$ and $\Psi^{\dagger}(x)$ by a set of operators $\Psi(i)$ and $\Psi^{\dagger}(i)$, satisfying the Bose commutation relations, $[\Psi(i), \Psi^{\dagger}(j)] = a^{-d}\delta_{ij}$ etc. where d is the dimension of the lattice space. Then, $\Psi(i)$ and $\Psi^{\dagger}(i)$ are replaced by $b_i = a^{d/2}\Psi(i)$ and $b_i^{\dagger} = a^{d/2}\Psi^{\dagger}(i)$ satisfying $[b_i, b_j^{\dagger}] = \delta_{ij}$, etc. For simlicity of notations, we consider a one-dimensional lattice space (d=1). Then, in the case of the density-fluctuation operator, the correspondence among the three sets of operators becomes

$$\int dx\, \Psi^{\dagger}(x)\Psi(x)\, e^{-ikx} \leftrightarrow \sum_i a\, \Psi^{\dagger}(i)\Psi(i)\, e^{-ikx_i} \leftrightarrow \sum_i b_i^{\dagger} b_i\, e^{-ikx_i} \quad , \tag{1}$$

where $k = 2n\pi/L$ (L is the length in which the system is enclosed.)

In this one-dimensional lattice space, the Hamiltonian (2-2) assumes the form[50]

$$H = \left(\frac{\hbar^2}{2ma^2}\right) \sum_i (b_{i+1}^{\dagger} - b_i)(b_{i+1} - b_i) \tag{2}$$

$$+ \frac{1}{2} \sum_{i,j} V(i-j)\, b_i^{\dagger} b_j^{\dagger} b_j b_i \quad ,$$

where $V(i-j)=V(x_i-x_j)$ is the two-body interaction potential, and the current operator becomes

$$J_i = \left(\frac{i\hbar}{2ma}\right)\left(b_{i+1}^{+} b_i - b_i^{+} b_{i+1}\right) \quad . \tag{3}$$

The extension to the three-dimensional case is straightforward, and is discussed briefly in Appendix C.

As an illustrative example of the present coherent state theory, we consider the problem of a single harmonic oscillator. A coherent state $|\alpha\rangle_c$ is defined as a right eigenstate of the annihilation operator b with an eigenvalue $\alpha=\sqrt{J}\,e^{i\phi}$ (see §3). The representation based on the usual completeness relation (3-7) however, gives two sets of conjugate variables, $\{\phi\,,\ i\frac{\partial}{\partial\phi}\}$ and $\{J\,,\ i\frac{\partial}{\partial J}\}$, as mentioned before, and we would find a different form of the completeness relation from eq.(3-7), which is more suitable for our present purpose, taking the argument in §3 into account.

Denoting a coherent state with the amplitude J=1 as $|\phi\rangle\!\rangle$, we define D by

$$D = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi}\ |\phi\rangle\!\rangle\langle\!\langle\phi| \quad , \tag{4}$$

$$|\phi\rangle\!\rangle = \exp\left[e^{i\phi} b^{+}\right]|0\rangle = \sum_n (n!)^{-\frac{1}{2}}\, e^{in\phi}\,|n\rangle \quad ,$$

where the normalization is changed to $\langle 0|\phi\rangle\!\rangle = 1$. Of course, $|\phi\rangle\!\rangle$ satisfies eq.(3-2), which can be written explicitly as

$$b\,|\phi\rangle\!\rangle = e^{i\phi}\,|\phi\rangle\!\rangle \quad , \quad \langle\!\langle\phi|\,b^{+} = e^{-i\phi}\,\langle\!\langle\phi| \quad . \tag{5}$$

Performing the integration over ϕ, D becomes

$$D = \sum_{n=0}^{\infty} \frac{1}{n!} |n\rangle\langle n| \quad . \tag{6}$$

Using this expression D, we obtain

$$\sqrt{\Gamma}\, D \sqrt{\Gamma} = \sqrt{\Gamma} \left\{ \int \frac{d\phi}{2\pi} |\phi\rangle\!\rangle\langle\!\langle \phi| \right\} \sqrt{\Gamma} = 1 \quad , \tag{7}$$

$$\Gamma = \Gamma(b^+ b + 1) \quad ,$$

where $\Gamma(z)$ is the gamma function. Eq.(7) is the desired completeness relation (C.R.).

Noting eq.(7), we introduce the notation $\varphi(\phi)$ by

$$\varphi(\phi) \equiv \langle\!\langle \phi | \sqrt{\Gamma} | \varphi \rangle = \sum_{n=0}^{\infty} \varphi_n e^{-in\phi} \quad , \tag{8}$$

for a state $|\varphi\rangle = \sum_n \varphi_n |n\rangle$ in the Fock space. Then, by means of eq.(7) the inner product of states $|\varphi_1\rangle$ and $|\varphi_2\rangle$ becomes

$$\langle \varphi_1 | \varphi_2 \rangle = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \varphi_1^*(\phi)\, \varphi_2(\phi) \quad . \tag{9}$$

We would call $\varphi(\phi)$ the representative of $|\varphi\rangle$, and more in general $\langle\!\langle \phi | \sqrt{\Gamma}\, A(b, b^+) | \varphi \rangle$ the representative of $A(b, b^+)|\varphi\rangle$ where $A(b, b^+)$ is an operator given in terms of b and b^+. For the number state $|\varphi\rangle = |n\rangle$, $\varphi_{n'} = \delta_{n,n'}$ we have $\varphi_n(\phi) = e^{-in\phi}$. Noting the relation $b|n\rangle = \sqrt{n}\,|n-1\rangle$ and $b^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$, we see that the representatives of the state $b|n\rangle$ and $b^+|n\rangle$ are given respectively by

$$\langle\langle\phi|\sqrt{\Gamma}\, b\,|n\rangle = \sqrt{n}\,\exp[-i(n-1)\phi] = e^{i\phi}\sqrt{i\frac{\partial}{\partial\phi}}\;\varphi_n(\phi)\ ,$$

(10)

$$\text{and}\quad \langle\langle\phi|\sqrt{\Gamma}\, b^+\,|n\rangle = \sqrt{n+1}\,\exp[-i(n+1)\phi] = \sqrt{i\frac{\partial}{\partial\phi}}\;e^{-i\phi}\,\varphi_n(\phi)\ ,$$

where $\sqrt{i\partial/\partial\phi}$ is defined by the equation $\sqrt{i\partial/\partial\phi}\; e^{-in\phi} = \sqrt{n}\, e^{-in\phi}$. More generally, the representative of $A(b,b^+)|\varphi\rangle$ is given by

$$\langle\langle\phi|\sqrt{\Gamma}\, A(b,b^+)|\varphi\rangle = A\left(e^{i\phi}\sqrt{i\frac{\partial}{\partial\phi}}\ ,\ \sqrt{i\frac{\partial}{\partial\phi}}\;e^{-i\phi}\right)\varphi(\phi)\ , \qquad (11)$$

where we need not expand A in the normal order (see Appendix A). We shall call $A(e^{i\phi}\sqrt{i\partial/\partial\phi}\ ,\cdots\cdots)$ the representation of $A(b,b^+)$, or say that $A(b,b^+)$ is represented by $A(e^{i\phi}\sqrt{i\partial/\partial\phi}\ ,\cdots\cdots)$. In particular, the number operator b^+b is represented by $i\partial/\partial\phi$. In the following, the notation $\vec{A}$ will also be used to denote the representation of an operator A.

Further, for the raising operator χ^+ and the lowering operator χ defined in (2-§2), namely by

$$\chi|n\rangle = |n-1\rangle\ ,\quad \chi^+|n\rangle = |n+1\rangle\ , \qquad (12)$$

we have $\chi|0\rangle=0$, and the representative of the state $\chi|n\rangle$ becomes $(1-\delta_{n,0})e^{i\phi}e^{-in\phi}$. Noting $\delta_{n,0}\,e^{-in\phi} = \vec{P}_0\, e^{-in\phi}$, where $\vec{P}_0$ is defined by

$$\vec{P}_0\,\varphi(\phi) = \int_{-\pi}^{\pi}\frac{d\phi'}{2\pi}\,\varphi(\phi') = \varphi_0\ , \qquad (13)$$

(and so is a projection operator satisfying $\vec{P}_0^{\,2} = \vec{P}_0$), we see that

the representatives are given by

$$
\begin{aligned}
&\langle\langle \phi | \sqrt{P}\, \chi | \varphi \rangle = e^{i\phi} (1 - \vec{P}_0)\, \varphi(\phi) \ , \\
&\langle\langle \phi | \sqrt{P}\, \chi^+ | \varphi \rangle = (1 - \vec{P}_0)\, e^{-i\phi}\, \varphi(\phi) \ , \\
&\langle\langle \phi | \sqrt{P}\, \chi \chi^+ | \varphi \rangle = \sum_n \varphi_n e^{-in\phi} = \varphi(\phi) \ , \\
&\langle\langle \phi | \sqrt{P}\, \chi^+ \chi | \varphi \rangle = \sum_{n \geq 1} \varphi_n e^{-in\phi} = (1 - \vec{P}_0)\, \varphi(\phi) \ .
\end{aligned}
\tag{14}
$$

By means of eqs.(9) and (11), we can calculate the expectation value of a quantity A by the formula

$$
\langle A \rangle = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi}\ \varphi^*(\phi)\ A\left(e^{i\phi} \sqrt{i \frac{\partial}{\partial \phi}}\ , \sqrt{i \frac{\partial}{\partial \phi}}\ e^{-i\phi} \right) \varphi(\phi) \ . \tag{15}
$$

This particular representation used here is different from the usual one, but is sufficient to consider our problems at the zero temperature, for which we have only to deal with pure states (see § 3)*.

---

* In order to treat a system at finite temperatures (or in a mix state) by the present formulation, we will need the off-diagonal matrix element of the density matrix $\langle\langle \phi | \rho | \phi' \rangle\rangle$, which is related to $F_\rho^N(\alpha, \alpha^*) = \langle \alpha | \rho | \alpha \rangle_c$ used in the usual C.S.R. by

$$
e^{-J} \int_{-\pi}^{\pi} dr \left(\sum_l J^l e^{-ilr} \right) \langle\langle R + \tfrac{r}{2} | \rho | R - \tfrac{r}{2} \rangle\rangle = \langle \alpha | \rho | \alpha \rangle_c \ ,
$$

where $R + \frac{r}{2} = \phi$, $R - \frac{r}{2} = \phi'$, $\alpha = \sqrt{J}\, e^{iR}$ and we have used the fact that $\langle\langle \phi | \rho | \phi' \rangle\rangle$ has the form $\sum (m!\, n!)^{-\frac{1}{2}} \rho_{mn} e^{-im\phi} e^{in\phi'}$.

In order to gain a familiarity with the collective variable theories, we define the Fourier transform of the representative $\varphi(\phi)$ by

$$\Phi(\rho) = \lim_{\varphi\to\infty} (2\pi\sqrt{G})^{-1}\int_{-\varphi}^{\varphi} d\phi\; e^{i\rho\phi}\; \varphi(\phi) \quad . \tag{16}$$

Then, the operators, $i\phi$ and $i\,\partial/\partial\phi$, are transformed into $\partial/\partial\rho$ and ρ, respectively, and the representation of the operator $A(b, b^{+})$ is given by

$$\vec{A}(b, b^{+})\,\Phi(\rho) = A\left(\exp\left[\frac{\partial}{\partial\rho}\right]\sqrt{\rho}\;,\; \sqrt{\rho}\,\exp\left[-\frac{\partial}{\partial\rho}\right]\right)\Phi(\rho) \quad . \tag{17}$$

The new representative $\Phi(\rho)$ (eq.(16)) is normalized as

$$\int_{-\infty}^{\infty} d\rho\;\; \Phi^{*}(\rho)\;\Phi(\rho) \;=\; 1 \quad . \tag{18}$$

Now, the coherent state representation of the lattice space problem is easily obtained by applying eq.(17) to each set of operators, b_i and b_i^{+}. The Hamiltonian and the current operator (eqs.(2) and (3)) are represented by

$$\vec{H} = \left(\frac{\hbar^2}{2ma^2}\right)\sum_i \left\{ \rho_{i+1} + \rho_i - \sqrt{\rho_{i+1}}\, e^{-i(\phi_{i+1}-\phi_i)}\sqrt{\rho_i} - \sqrt{\rho_i}\, e^{i(\phi_{i+1}-\phi_i)}\sqrt{\rho_{i+1}} \right\}$$

$$+ \frac{1}{2}\sum_{i,j} V(i-j)\,\sqrt{\rho_i}\, e^{-i\phi_i}\sqrt{\rho_j}\, e^{-i\phi_j} e^{i\phi_j}\sqrt{\rho_j}\, e^{i\phi_i}\sqrt{\rho_i} \quad , \tag{19}$$

$$\vec{J}_i = \left(\frac{i\hbar}{2ma}\right)\left(\sqrt{\rho_{i+1}}\, e^{-i(\phi_{i+1}-\phi_i)}\sqrt{\rho_i} - \sqrt{\rho_i}\, e^{i(\phi_{i+1}-\phi_i)}\sqrt{\rho_{i+1}}\right) \quad , \tag{20}$$

respectively, where $\partial/\partial p_i$ is denoted as $i\phi_i$ for simplicity of notations. Eqs.(19) and (20) are precisely of the form as expected from the method of DPO applied to the lattice space problem. It should be emphasized that the representation of the lowering operator χ is not $e^{i\phi}$ but $e^{i\phi}(1-\vec{P}_0)$ as given in eq.(14). The final results of the ground-state energy and the momentum distribution, however, are not changed by omitting $\vec{P}_0$ in the course of calculation, because these operators χ and χ^+ appears only in such products as $\sqrt{n}\,\chi^+(=b^+)$ or $\chi\sqrt{n}\,(=b)$. In other words, it is not necessary that all the representations are completely equivalent to those of physical quantities defined in the Fock space (the Fock representation). However, from our rather conservative point of view, we prefer to make the representation equivalent to the Fock representation imposing the following subsidiary condition on the representative $\varphi(\phi)$, which is a periodic function with period 2π and allows negative values n=-1,-2,.... for the representation of the number operator $i\partial/\partial\phi$. Noting that the inverse of the gamma function $\Gamma^{-1}(z)$ has zero's at z=0,-1,-2,.... and only there, we impose the subsidiary condition written as

$$\Gamma^{-1}\left(-i\frac{\partial}{\partial\phi}\right)\varphi(\phi)=0 \quad , \text{ or } \quad \Gamma^{-1}(-p)\,\Phi(p)=0 \quad , \qquad (21)$$

which prevents us from treading out of the representation equivalent to the Fock representation.

---

* We can show that also in the R-RR theory some subsidiary condition should be imposed on $\rho(\alpha,\alpha^*)$, in order that $\rho(\alpha,\alpha^*)$ corresponds to a Fock state.

In our coherent-state theory, we are allowed to use formal calculation $i\partial/\partial\rho$, bearing in mind that the projection operator is involved in the representation of the lowering operator χ . In this lattice theory, the subsidiary condition (21) assures that every ρ_i takes on zero or positive integers. In this way, we have overcome Fröhlich's objection that the canonical conjugate to the number operator does not exist.

Before we go over to the continuum limit, we would introduce the lattice Fourier transform of ρ_j . Noting the correspondence (1), we define the density fluctuation ρ_k by

$$\rho_k = \sum_i \rho_i e^{-ikx_i} \quad , \quad \rho_i = \frac{a}{L} \sum_k e^{ikx_i} \rho_k , \tag{22}$$

where L is the length in which the system is enclosed with periodic boundary condition, and the wave number k runs over the first Brillouin zone only. Then, $\phi_i = -i\partial/\partial\rho_i$ is given by

$$\phi_i = \sum_k \phi_k e^{-ikx_i} = \sum_k (-i\partial/\partial\rho_k) e^{-ikx_i} \quad . \tag{23}$$

From eq.(23), $\phi_{i+1} - \phi_i$ becomes $\sum_k (\exp[-ika] - 1)\phi_k \exp[-ikx_i]$, which is of order a. We also introduce v_i given by

$$v_i = \frac{\hbar}{ma}(\phi_{i+1} - \phi_i) = -\left(\frac{\hbar}{ma}\right) \sum_k f_k \phi_k e^{-ikx_i} \quad , \tag{24}$$

where the factor f_k is given by

$$f_k \equiv 1 - e^{-ika} \quad . \tag{25}$$

This factor f_k plays a crucial role in our lattice theory and relations it satisfies are given in Appendix B. f_k appears in the lattice Fourier transform of an operator, for example v_i, defined with respect to two neighbouring lattice points.

By means of the correspondence given in eq.(7), the density in the continuum limit $a \to 0$ is given by $\rho(x) \leftrightarrow a^{-d} \rho_i$. To complete the connection with the collective variable theories formulated in the continuous space, we introduce the phase operator in the continuum limit $\phi(x)$ by the correspondence $\phi(x) \leftrightarrow \phi_i = -i \partial/\partial \rho_i$. Introducing the projection operator and imposing the subsidiary condition (21), we can arrive at a correct result even if we used the commutation relation $[\rho(x), \phi(x')] = i \delta(x-x')$ $(\leftrightarrow i a^{-d} \delta_{ij})$ in the course of calculation. Eq.(17) reduces to

$$\Psi(x) \leftrightarrow e^{i\phi(x)} \sqrt{\rho(x)} \quad , \quad \Psi^{+}(x) \leftrightarrow \sqrt{\rho(x)}\, e^{-i\phi(x)} \quad , \qquad (26)$$

which agrees with the prescription of DPO, because the lowering operator χ_i stands always in the l.h.s. of $\sqrt{\rho_i}$ and the raising operator χ_i^{+} does in the r.h.s. of $\sqrt{\rho_i}$ in DPO, as long as we consider the problem in the lattice space.

So far, our discussion is based on the C.R. (7). As $\sqrt{\Gamma(b^{+}b+1)}$ and D, which are diagonal in the number representation , are commutative with each other, the C.R. (7) can also be transformed into

$$D\,\Gamma(b^{+}b+1) = 1 \quad , \qquad (27)$$

and the inner product of a state becomes

$$\langle\varphi|\varphi\rangle = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \langle\varphi|\phi\rangle\!\rangle\langle\!\langle\phi|\Gamma|\varphi\rangle \quad . \tag{28}$$

Thus, we have two representatives of a state the r-representative $\varphi_r(\phi) = \langle\!\langle\phi|\Gamma|\varphi\rangle$ and the $\mathcal{L}$-representative $\varphi_{\mathcal{L}}^*(\phi) = \langle\varphi|\phi\rangle\!\rangle$. Here, $\varphi_r(\phi)$ and $\varphi_{\mathcal{L}}^*(\phi)$ can be regarded as the representatives of the states $\Gamma^{\frac{1}{2}}|\varphi\rangle$ and $\langle\varphi|\Gamma^{-\frac{1}{2}}$, respectively, which are obtained by a non-unitary transformation:

$$\begin{aligned} &|\varphi\rangle \to \Gamma^{\frac{1}{2}}|\varphi\rangle \quad , \quad \langle\varphi| \to \langle\varphi|\Gamma^{-\frac{1}{2}} \quad , \\ &A(b,b^+) \longrightarrow \Gamma^{\frac{1}{2}} A(b,b^+)\Gamma^{-\frac{1}{2}} \quad , \end{aligned} \tag{29}$$

where $\Gamma = \Gamma(b^+b+1)$. By similar arguments as given below eq.(9) the r-representative of the state $A(b,b^+)|\varphi\rangle$ becomes

$$\langle\!\langle\phi|\Gamma A(b^+,b)|\varphi\rangle = A\left(e^{i\phi}(1-\vec{P}_0),\ i\frac{\partial}{\partial\phi}\ e^{-i\phi}\right)\varphi_r(\phi) \quad , \tag{30}$$

which is a non-hermitian representation. The r-representation $e^{i\phi}(1-\vec{P}_0)$ is nothing but the representation of $\Gamma^{\frac{1}{2}} b\, \Gamma^{-\frac{1}{2}}$ and connected with $e^{i\phi}\sqrt{i\partial/\partial\phi}$ by a non-unitary transformation S given by

$$S = \exp\left[\frac{1}{2}\ln\Gamma\left(i\frac{\partial}{\partial\phi}+1\right)\right] = \sqrt{\Gamma\left(i\frac{\partial}{\partial\phi}+1\right)} \quad , \tag{31}$$

in such a manner as

$$S^{-1} e^{i\phi}(1-\vec{P}_0)\, S = e^{i\phi}\sqrt{i\frac{\partial}{\partial\phi}} \quad . \tag{32}$$

The more detailed account will be given in Appendix A, where the relation $S^{-1} e^{i\phi} \vec{P}_0 S = 0$ is also shown. So, as far as the calculation is made after the hermitization process as introduced previously,[51] $(1-\vec{P}_0)$ factor in the representation (30) can be safely omitted. Then, in the continuum limit, the representation (30) reduces to

$$\Psi(x) \leftrightarrow e^{i\phi(x)} \quad , \quad \Psi^{+}(x) \leftrightarrow \rho(x)\, e^{-i\phi(x)} \quad , \tag{33}$$

with the same $\rho(x)$ and $\phi(x)$ as in eq.(26). Eq.(33) coincides with the non-hermitian prescription given by Berdahl,[37] from which a Hamiltonian equivalent to that given by BZ is obtained. (In his paper $\phi(x)$ is denoted as $\sigma(x)$.)

In this way, the equivalence of BZ and DPO can be proved. In showing this equivalence, it must be noticed that the expectation values of physical quantities should be taken before going over to the continuum limit.

§5 The ground-state energy and the momentum distribution

On the basis of the formulation described in §4, we now evaluate the ground-state energy E_0 and the momentum distribution n_p, up to order 1/N (N is the total number of bosons) in the 1/N perturbation expansion scheme.[5-7] The subsidiary condition is neglected.

First, we consider the kinetic energy part in eq. (4-19). Using V_i given in eq. (4-24), we expand the exponential factor in power series to obtain

$$\vec{T} = \frac{\hbar^2}{2ma^2} \sum_i \Big\{ (\rho_{i+1} + \rho_i - 2\sqrt{\rho_{i+1}}\sqrt{\rho_i}) \qquad (1)$$

$$- \sum_{n=1}^{\infty} (n!)^{-1} \left(\frac{ma}{\hbar}\right)^n \left(\sqrt{\rho_{i+1}}\,(-i V_i)^n \sqrt{\rho_i} + \sqrt{\rho_i}\,(i V_i)^n \sqrt{\rho_{i+1}}\right) \Big\} .$$

Taking the continuum limit $a \to 0$ in eq.(1), terms with $n \geq 3$ vanish and we obtain T_{DPOC} given by DPO in the continuous space. However, it should be mentioned that the factor f_k/a in V_i becomes of order 1/a for such k's as $ka \approx \pi$. These k's always exist and contribute to E_0. This seems to be what Hiroike's cut-off procedure[5] means, and we would retain these terms with $n \geq 3$ in the following calculation.

For a N-boson system, $\rho_{k=0}$ given by eq.(4-22) is equal to N. As $\phi_{k=0} = -i\,\partial/\partial\rho_{k=0}$ does not appear in eq.(4-24) and so in the Hamiltonian, we can safely put $\rho_{k=0} = N$. The 1/N expansion is attained by setting

$$\rho_k = \sqrt{N}\,\rho_k' \quad , \qquad \phi_k \left(= -i\frac{\partial}{\partial\rho_k}\right) = \phi_k'/\sqrt{N} \quad , \qquad (2)$$

and expanding $\sqrt{\rho_i}$ in the ascending power of $1/\sqrt{N}$ as

$$\sqrt{\rho_i} = \left(\frac{aN}{L}\right)^{\frac{1}{2}} \left\{1 + \frac{1}{2\sqrt{N}}\rho_i' - \frac{1}{8N}\rho_i'^2 + \frac{1}{16N^{3/2}}\rho_i'^3 - \frac{1}{128N^2}\rho_i'^4 + \cdots\right\} , \qquad (3)$$

$$\rho_i = \left(\frac{aN}{L}\right)\left(1 + N^{-\frac{1}{2}}\rho_i'\right) \quad , \quad \rho_i' = \sum_{k\neq 0} \rho_k' e^{ikx_i} .$$

As V_i (eq.(4-24)) is of order $1/\sqrt{N}$, we see that the terms with n=3 and 4 in eq.(1) also contribute terms of order 1/N. For instance,

the second term in the n=3 term becomes

$$\sum_i (3!)^{-1} \left(\frac{ma}{\hbar}\right)^3 \sqrt{\rho_i}\, (i v_i)^3 \sqrt{\rho_{i+1}} \qquad (4)$$

$$= \frac{a}{6\sqrt{N}L} \sum_i (i v_i')^3 + \frac{a}{12NL} \sum_i \left(\rho_i' (i v_i')^3 + (i v_i')^3 \rho_{i+1}' \right) ,$$

$$V_i = \frac{\hbar}{ma\sqrt{N}} v_i' \quad , \quad v_i' = -\sum_{k\neq 0} f_k \phi_k' e^{-ikx_i} .$$

Using the equality

$$\frac{a}{L} \sum_i e^{-ikx_i} e^{-il x_i} e^{-im x_i} e^{in x_{i+1}} = e^{ina}\, \delta_{k+l+m,n} \quad , \qquad (5)$$

and similar ones, where $\delta_{k,l}$ is the Kronecker's delta and wave numbers differing by a reciprocal lattice vector are regarded as equal, eq.(4) becomes

$$(\text{r.h.s. of eq.}(4)) = \frac{i^3}{6\sqrt{N}} \sum_{k+l+m=0} f_k f_l f_m \phi_k \phi_l \phi_m \qquad (6)$$

$$+ \frac{i^3}{12N} \sum_{k+l+m+n=0} \left\{ f_l f_m f_n \rho_{-k} \phi_l \phi_m \phi_n + e^{-ika} f_l f_m f_n \phi_l \phi_m \phi_n \rho_{-k} \right\} ,$$

where dashes are omitted. Other terms can be evaluated in a similar manner. Then, using the relation $[i\phi_l , \rho_k] = \delta_{k,l}$ and the property of f_k given in Appendix B such as $f_k f_l = f_k + f_l - f_{k+l}$, the final expression for $\vec{\vec{T}}$ becomes up to order 1/N

$$\vec{\vec{T}} = \vec{\vec{T}}_0 + \vec{\vec{T}}_1 + \vec{\vec{T}}_2 + \vec{\vec{T}}_3 \quad , \qquad (7)$$

$$\vec{T}_0 = \frac{\hbar^2}{2ma^2}\sum_k \langle k,-k\rangle\left(\phi_k\phi_{-k} + \frac{1}{4}\rho_k\rho_{-k} - \frac{1}{2}\right), \qquad (7)$$

$$\vec{T}_1 = -\frac{\hbar^2}{2ma^2\sqrt{N}}\sum_{k+l+m=0}\langle k,l\rangle\left(\frac{1}{4}\rho_{-k}\rho_{-l}\rho_{-m} - \rho_{-m}\phi_k\phi_l\right),$$

$$\vec{T}_2 = \frac{\hbar^2}{2ma^2N}\sum_{k+l+m+n=0}\frac{1}{12}\langle k,-k\rangle\rho_k\rho_l\rho_m\rho_n,$$

$$\vec{T}_3 = -\frac{\hbar^2}{2ma^2N}\sum_{k+l+m+n=0}\frac{1}{12}\langle k,l,m,n\rangle \times$$

$$\times\Big\{\frac{1}{16}\rho_{-l}\rho_{-m}\rho_{-n}\rho_{-k} + \phi_l\phi_m\phi_n\phi_k + \frac{3}{4}\delta_{m,-k}\delta_{n,-l}$$

$$+ 3\delta_{l,-k}\left(\frac{1}{4}\rho_{-m}\rho_{-n} - \phi_m\phi_n\right) + \frac{3}{4}\left(\rho_{-m}\rho_{-n}\phi_l\phi_k + \phi_k\phi_l\rho_{-n}\rho_{-m}\right)\Big\},$$

where $\langle\cdots\cdot\rangle$ is defined by

$$\langle k,l,\cdots,m\rangle = \left(f_k f_l\times\cdots\times f_m + f_{-k}f_{-l}\times\cdots\times f_{-m}\right)/2, \qquad (8)$$

and by means of the definition of f_k given in eq.(4-25), the expressions $\langle k,l\rangle$ and $\langle k,l,m,n\rangle$ reduce to $-a^2k\cdot l$ and $a^4k\cdot l\cdot m\cdot n$, respectively, in the limit $a\to 0$. The $\phi_l\phi_m\phi_n$ term in eq.(6) does not appear in eq.(7) due to the relation (B·5). The total Hamiltonian (4-19) becomes

$$H = \vec{T} + \vec{V}, \quad \vec{V} = \frac{N^2}{2L}V_0 - \frac{N}{2}V(0) + \frac{N}{2L}\sum_{k\neq 0}V_k\rho_k\rho_{-k}, \qquad (9)$$

$$V_k = L^{-1}\sum_j V(i-j)\,e^{ik(x_i-x_j)},$$

where $\vec{T}$ is given by eq.(7).

Here, we would introduce the annihilation and creation operators B_k and B_k^+ for the Bogoliubov excitation defined by

$$\phi_k = \frac{i}{2\sqrt{\lambda_k}} (B_{-k} - B_k^+) \quad , \quad \rho_k = \sqrt{\lambda_k} (B_k + B_{-k}^+) \quad , \tag{10}$$

$$\lambda_k^2 = \varepsilon_k^0 / (\varepsilon_k^0 + 2NV_k/L) \quad , \quad \varepsilon_k^0 = \frac{\hbar^2 \langle k, -k \rangle}{2m a^2} \quad .$$

Then, the lowest order Hamiltonian $\vec{H}_0 = \vec{T}_0 + V$ can be diagonalized to give

$$\vec{H}_0 = E_0^B + \sum_k \varepsilon_k^B B_k^+ B_k \quad , \tag{11}$$

$$E_0^B = \frac{N^2}{2L} V_0 - \frac{N}{2} V(0) + \sum_k \frac{1}{2} (\varepsilon_k^B - \varepsilon_k^0) \quad , \quad \varepsilon_k^B = \frac{\varepsilon_k^0}{\lambda_k} \quad .$$

On applying these operators B_k and B_k^+ in place of ρ_k and ϕ_k, we denote the representative $\Phi(\rho)$ as $|\Phi)$. In particular, the unperturbed ground-state is denoted as $|0)$, which satisfies $B_k|0) = 0$ and $\vec{H}_0|0) = E_0^B|0)$. In this notation, eq.(4-15) becomes $E = (\Phi|\vec{H}|\Phi)$. Then, treating $\vec{T}_1 + \vec{T}_2 + \vec{T}_3$ as the perturbation in the usual Rayleigh-Schrödinger perturbation theory, the ground-state energy up to order 1/N is obtained from

$$E_0 = E_0^B + (0|\vec{T}_1 \frac{Q}{E_0^B - \vec{H}_0} \vec{T}_1|0) + (0|(\vec{T}_2 + \vec{T}_3)|0) + o(N^{-1}) \quad , \tag{12}$$

where Q is a projection operator defined by $Q = |-|0)(0|$.
Using the property of f_k given in Appendix B, the evaluation of (12) is straightforward, and we obtain the expression for E_0 given by

$$E_0 = E_0^B + E_1^{Bz} + E_1^b + E_1^c \quad , \tag{13}$$

$$E_1^{Bz} = -\frac{1}{N}\left(\frac{\hbar^2}{2ma^2}\right)^2 \sum_{k+l+m=0} \frac{\lambda_k \lambda_l \lambda_m}{8(\varepsilon_k^B + \varepsilon_l^B + \varepsilon_m^B)} \times$$

$$\times \left\{ \langle k,l\rangle^2 \left(1+\frac{1}{\lambda_k\lambda_l}\right)^2 + 2\langle k,l\rangle\langle l,m\rangle\left(1+\frac{1}{\lambda_k\lambda_l}\right)\left(1+\frac{1}{\lambda_l\lambda_m}\right)\right\}$$

$$+ \frac{1}{N}\cdot\frac{\hbar^2}{16ma^2} \sum_{k+l+m=0} \langle m,-m\rangle \lambda_k \lambda_l \quad ,$$

$$E_1^b = \frac{\hbar^2}{24ma^2N} \sum_{k+l+m=0} \left[\langle k,l\rangle (1-\lambda_k)(1-\lambda_l) + \text{sym.}\right] \quad ,$$

$$E_1^c = -\frac{\hbar^2}{96ma^2N} \sum_{k+l+m=0} \left[\langle k,l\rangle \frac{(1-\lambda_k)^2(1-\lambda_l)^2}{\lambda_k\lambda_l} + \text{sym.}\right] \quad .$$

As $\langle k,l\rangle/a^2$ reduces to $-k\cdot l$ in the continuum limit $a\to 0$, E_0 obtained in eq. (13) completely coincides with E_0^{BL} obtained from the modified Bogoliubov theory due to Breuckner and Lee, in the high-density limit. $E_0^B + E_1^{Bz}$ is equivalent to the expression given by Bogoliubov and Zubarev, and $E_0^B + E_1^{Bz} + E_1^b$ gives E_0^{CBF} obtained by the method of correlated basis function, which proves to be exact in the high-density limit for the Lieb-Liniger model. The difference $E_0^{BL} - E_0^{CBF}$ is given by E_1^c (see eq. (2-18)). Here, the sign of the expressions E_1^b and E_1^c can be determined uniquely. In the continuous space, manipulations like $\sum_{k,l} k\cdot l \cdots = -\sum_{k,l} k\cdot l \cdots$ are admissible, which raise some ambiguity about the sign of the expressions E_1^b and E_1^c. Although both $\langle k,-l\rangle$ and $-\langle k,l\rangle$ reduce to $a^2 k\cdot l$, $\langle k,-l\rangle$ is not equal to $-\langle k,l\rangle$ and

manipulations like $\sum_{k,\ell} \langle k,\ell\rangle \cdots = \sum_{k,\ell} \langle k,-\ell\rangle \cdots = \sum_{k,\ell} -\langle k,\ell\rangle \cdots$ are not admissible in our lattice treatment. $\langle k,\ell\rangle$ can also be rewritten as $(\langle k,-k\rangle + \langle \ell,-\ell\rangle - \langle m,-m\rangle)/2$ in eq. (13). (See eq. (18).)

Next, we consider the momentum distribution n_p. In the continuous space, n_p is given by the expectation value of $a_p^+ a_p$, where $a_p = L^{-\frac{1}{2}} \int dx\, \Psi(x)\, e^{-ipx}$. In our lattice space, a_p corresponds to $b_p = (a/L)^{\frac{1}{2}} \sum_i b_i\, e^{-ipx_i}$, whose representation is given by

$$\vec{b}_p = (a/L)^{\frac{1}{2}} \sum_i e^{i\phi_i} \sqrt{\rho_i}\, e^{-ipx_i} \quad . \tag{14}$$

By means of eq's. (2), (3) and (4-23), $\vec{b}_p$ and $\vec{b}_p^+$ can be expanded in powers of $1/\sqrt{N}$, and the representation of $\vec{b}_p^+ \vec{b}_p$ takes the form

$$\vec{b}_p^+ \vec{b}_p = \vec{n}_p^0 + \vec{n}_p^1 + \vec{n}_p^2 + o(N^{-1}) \quad , \quad \vec{n}_p^0 = (\tfrac{1}{2}\rho_{-p} - i\phi_p)(\tfrac{1}{2}\rho_p + i\phi_{-p}) \quad , \tag{15}$$

where $\vec{n}_p^1$ ($\vec{n}_p^2$) is a complicated expressions of order $1/\sqrt{N}(1/N)$ composed of three (four) - fold products of ρ_k and ϕ_k, and so of B_k and B_k^+ through eq. (10). As a_p in the continuous space is defined without the gradient of $\Psi(x)$, its counterpart b_p in the lattice space does not contain the factor f_k. Therefore, its expression can be obtained from that in the continuous space only by regarding the summations $\sum_{k+\ell+m=o}$ and so on as those of lattice space, and we would refer the reader to ref. 40 for the explicit form of $\vec{b}_p$ and $\vec{b}_p^+ \vec{b}_p$.

Thus, in our lattice theory, the momentum distribution n_p up to order 1/N can be obtained from $(\Phi_o|\vec{b}_p^+ \vec{b}_p|\Phi_o)/(\Phi_o|\Phi_o)$, where $|\Phi_o)$ is the perturbed ground-state. The unperturbed value n_p^o is easily obtained by means of eqs. (15) and (10), and becomes

$$n_p^0 = (0|\vec{n}_p^0|0) = (1-\lambda_p)^2/4\lambda_p \quad . \tag{16}$$

The evaluation of the correction terms is cumbersome but straightforward, and we obtain

$$n_p = \langle n_p^{DPO} \rangle + \frac{(1-\lambda_p^2)^2}{16N\lambda_p \varepsilon_p^0} \sum_k \frac{\hbar^2}{2ma^2} \langle p,k \rangle (1-\lambda_k)^2 \lambda_k^{-1} \quad , \tag{17}$$

where $\langle n_p^{DPO} \rangle$ is a expression obtained by replacing the factors like $-p\cdot k$ by $\langle p,k \rangle$ in the expression for n_p given by Nishiyama from the method of DPO.[40] The difference from $\langle n_p^{DPO} \rangle$ is a single summation over wave number, say k, and essentially vanishes after the integration.* This situation is compatible with the fact that the double summations encountered in the expression for the ground-state energy (13), such as $\sum_{k,l} a^{-2} \langle k,l \rangle (1-\lambda_k)(1-\lambda_l)$, cannot be put equal to zero, as emphasized by Takahashi[23] and others.[6),34),39] Thus, the additional term to $\langle n_p^{DPO} \rangle$ can safely be discarded and reduces to n_p^{DPO} , which have been shown to satisfy the Gavoret-Nozières condition,[30] namely, n_p approaches the value $(N_0/N)(mc/2\hbar p)$ as the wave number p tends to zero, where N_0 and c are the number of the condensed particles of zero momentum and the true sound velocity, respectively.

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* Indeed, owing to the fact that pa becomes much less than π for $a \to 0$, we can safely approximate $\langle p,k \rangle$ by $i pa (f_k - f_{-k})$ and the single summation vanishes due to its symmetry.

Following the adiabatic theorem applied by Berdahl[37] in obtaining n_p, n_p is obtainable from $(\partial/\partial\varepsilon_p^0)E_0$, regarding the energy parameter ε_p^0 as the adiabatic parameter. Rewriting E_1^c, by means of the identity $\langle k,l\rangle = \frac{1}{2}(\langle k,-k\rangle + \langle l,-l\rangle - \langle m,-m\rangle)$ (eq. (B.4)) as

$$E_1^c = -\sum_{k+l+m=0} \frac{(\varepsilon_k^0 + \varepsilon_l^0 - \varepsilon_m^0)(1-\lambda_k)^2(1-\lambda_l)^2}{16N\lambda_k\lambda_l}, \qquad (18)$$

and using the equality $(\partial/\partial\varepsilon_p^0)\lambda_p = \lambda_p(1-\lambda_p^2)/2\varepsilon_p^0$, the contribution from E_1^c is obtained as

$$\frac{\partial}{\partial\varepsilon_p^0}E_1^c = \frac{1}{16N}\sum_{p+l+m=0}\left\{-\frac{(1-\lambda_p)^2(1-\lambda_l)^2}{\lambda_p\lambda_l} + \frac{(1-\lambda_l)^2(1-\lambda_m)^2}{2\lambda_l\lambda_m}\right\} + \Delta n_p, \qquad (19)$$

where Δn_p is the difference $n_p - \langle n_p^{DPO}\rangle$ in eq. (17). Here it should be noted that E_1^c contribute not only to Δn_p, but also terms contained in $(0|\vec{n}_p^2|0)$, which is independent of the way how the limit $a\to 0$ is taken. So, in order to have a result consistent with the adiabatic theorem, E_1^c is indespensable as pointed out previously by Nishiyama.[40] Indeed, E_1^c is directly obtainable in the lattice theory, in which the continuum limit is taken after the final expression for E_0 is obtained. The expression E_0^{BL} can be obtained also from[40]

$$E_0 = \frac{\hbar^2}{2m}\sum_p p^2 n_p + \frac{N}{2}\sum_p V_p(S_p - 1) = \langle T\rangle + \langle V\rangle, \qquad (20)$$

where S_p is the structure factor. Thus the present lattice theory is consistent with the adiabatic theory. The inconsistency between DPOC and DPOM is related to the fact that the order of taking the continuum limit and taking the expectation value cannot be interchanged. Further discussion will be given in §6. Since the structure factor

and the excitation energy are equivalent to those given by DPO, as is clear from the argument below(17), we do not discuss them.

§6 Summary and discussion

The discussion of §5 is based on the representation given in (4-17). However, the same result can also be obtainable from the representation given in (4-30). In this case, neglecting the factor $\vec{P}_0$, a non-hermitian Hamiltonian which reduces to that given by BZ is obtained. Working in the lattice space, this Hamiltonian $\vec{H}_{n.h.}$ can be transformed into the Hamiltonian (5-9) by non-unitary transformation S given by*

$$\ln S = \sum_k \frac{\rho_k \rho_{-k}}{4} - \sum_{k+l+m=0} \frac{\rho_k \rho_l \rho_m}{12\sqrt{N}} + \sum_{k+l+m+n=0} \frac{\rho_k \rho_l \rho_m \rho_n}{24N} - \sum_{k,l} \frac{\rho_k \rho_{-k}}{8N}, \quad (1)$$

which coincides the weight function $\sqrt{D}$ given by Hiroike (eq. (2-31)). S given in (1) is nothing but a 1/N expansion of the Stirling-formula approximation $\frac{1}{2}\sum_\lambda \{(\rho_\lambda+\frac{1}{2})\ln\rho_\lambda - \rho_\lambda\}$ of $\frac{1}{2}\sum_\lambda \ln\Gamma(\rho_\lambda+1)$.

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* In a previous paper[51], we have presented a coherent-state theory in which the occupation number n_i of the i-th lattice point is restricted to 0 or 1, and obtained the same result for the ground-state energy E_0 and the transformation S as given in this thesis. The restriction n_i=0 of 1, however, is neglected in the course of calculation and this expression E_0 cannot be applied to a one-dimensional system at low density. By taking account of this restriction, it is expected that this theory is useful for investigating a system at low density.

If we approximate $\ln \Gamma(P_i+1)$ by $P_i \ln P_i$, the Chan-Valatin transformation [35] is obtained. This approximation, however, yields the divergent terms as encountered in the old collective variable theories.[6] Thus, the meaning of Hiroike's cut-off procedure has been elucidated by introducing a limiting procedure as given in §5.

The relations of the present lattice theory described in §4 and 5 to other collective variable theories can be summarized as in the figure.

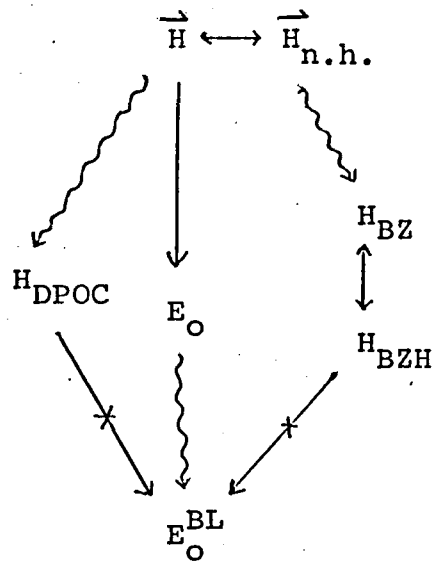

Figure:
$\rightsquigarrow$ implying that the continuum limit is taken, $\longleftrightarrow$ connected by non-unitary transformation $\sqrt{\Gamma}(\sqrt{D})$ with each other, and $\longrightarrow$ ($\not\longrightarrow$) expectation value is (isn't) derived. The importance of taking a proper continuum limit becomes clear from this figure.

H_{DPOC} and H_{BZ} are obtained by taking the continuum limit of $\vec{H}$ and $\vec{H}_{n.h.}$ expanded as in eq. (5-1), respectively. Only $\vec{H}$ yields E_o^{BL} obtained from the Bogoliubov theory reformulated by Breuckner and Lee and is therefore consistent with the adiabatic theorem applied by Berdahl in obtaining n_p. H_{BZ} gives E_o^{BZ} and H_{DPOC} yields E_o^{CBF}. The Hamiltonian H_{BZH} given by the improved BZ theory due to Hiroike yields E_o^{CBF}.

In order to obtain E_0^{BL} , $\vec{T}_3$ in eq. (5-7), which is formally higher order in a, is indispensable. As pointed out previously by Nishiyama[6], $\vec{T}_2$ gives absolutely divergent sum, which in our formulation is written as $N^{-1}\sum_{k,l}(\hbar^2/16ma^2)\langle k,l\rangle\lambda_k\lambda_l$ in addition to the contribution to E_0^{BZ} . This, together with the contribution from $\vec{T}_3$, gives the difference between $E_0 = E_0^{BL}$ and E_0^{BZ} . However, the ground-state energy obtained from DPOC is E_0^{CBF} and not E_0^{BL}. This is because the terms of the form $\phi\phi\phi\phi$ and $\delta_{k,-l}\phi\phi$ in $\vec{T}_3$ comes from the terms with n=3 and 4 in eq. (5-1), and therefore is missing in T_{DPOC}, although these terms do not affect the exact values of the ground-state energy for the one-dimensional L-L model and for the charged Bose gas in the high-density limit. These terms are also missing in H_2 given by Hiroike (his eq. (3.15)),[5] which corresponds to $\vec{T}_3$.

Though DPO (described in the coordinate space (DPOC))[6] gives E_0^{CBF}, it gives E_0^{BL} when the kinetic energy is calculated by using the expression for the momentum distribution function (DPOM)[40]. This situation can be interpreted by the present formulation as follows. The lattice Hamiltonian (4-2) can be rewritten as

$$H = \sum_k \frac{\hbar^2}{2ma^2}\langle k,-k\rangle n_k + (\text{int. term}) \qquad (2)$$

$$n_k = b_k^+ b_k \quad , \quad b_k = \left(\frac{a}{L}\right)^{\frac{1}{2}} \sum_\lambda b_\lambda\, e^{-ikx_\lambda} \, .$$

Contrary to the representation $\vec{H}$ given in eq. (5-9), the representation obtained from (2) does not involve terms which vanish in the continuum limit, because the representation of b_k (or $b_k^+ b_k$) does not contain the factor f_k (see discussions below (4-25)). This is the reason why DPOM yields the ground-state energy E_0^{BL} instead of E_0^{CBF}.

In general, the continuum limit should be made after taking the expectation values.

The relation to the velocity field approach of Yamasaki, Kebukawa and Sunakawa (TKS)[39] is not so clear as in the case of the BZ theory and the method of DPO. Recently, Nishiyama has shown that the Hamiltonian of DPOC is unitarily equivalent to the Hamiltonian given by the new Sunakawa theory of YKS and expressed in the so called small-b operator of Yamasaki up to order 1/N.

It has also been shown that introducing the canonical conjugate to density causes no trouble, because the representations of the raising and lowering operators X^+ and X include a projection operator as shown by (4-14) and the quantity ϕ (the phase operator) appears only in such forms as $\exp[i\phi]\sqrt{P}$ and $\sqrt{P}\exp[-i\phi]$. Although the method of calculation taking the subsidiary condition into account is left as a future problem, the ground-state energy obtained by neglecting this condition coincides with the exact result obtained for the one-dimensional Lieb-Liniger model and for the charged Bose gas in the high-density limit. This situation suggests that the subsidiary condition can be safely neglected in the high-density limit.

Finally we would like to make a remark on the superfluidity of this system. As mentioned in §5, the zero-th Fourier transform of the phase operator ϕ_0 is a cyclic coordinate. Let us consider two systems with the total number of particles N_1 and N_2 and the chemical potential μ_1 and μ_2, respectively which are allowed to

transfer a small number of particles to each other through a narrow channel. We consider that each of the two systems is sufficiently small compared with the macroscopic scale but large compared with the atomic scale so that the bulk property of the system can be preserved. The zero-th Fourier transforms of the phases of the two systems are donoted by ϕ_1 and ϕ_2, respectively and the total numbers N_1 and N_2 can be written as $\lambda^{\dagger}\partial/\partial\phi_1$ and $\lambda^{\dagger}\partial/\partial\phi_2$ in our coherent state representation, respectively.

Here we assume a symbolic expression for the Hamiltonian of the two systems in the form

$$H_{II} = \mu_1 N_1 + \mu_2 N_2 + V_{1,2} \quad , \tag{3}$$

where the interaction Hamiltonian $V_{1,2}$ is considered as commutative with the phase difference $\phi = \phi_1 - \phi_2$ and may depend upon ϕ as in the case with the problem of the ac-Josephson effect[52]. Hence the temporal change of the phase difference is given by

$$\frac{\partial}{\partial t}\phi = \frac{\lambda^{\dagger}}{\hbar}[H, \phi] = -\mu/\hbar \quad , \tag{4}$$

where μ is the difference of the chemical potential; $\mu = \mu_1 - \mu_2$. On the other hand, according to the two fluid theory,[53] the driving force of the superfluid is given by the spatial gradient of the Gibbs'free energy per unit mass; G and the superfluid velocity satisfies

$$\frac{\partial}{\partial t} v_s = -\,\mathrm{grad}\ G \quad . \tag{5}$$

When the curl-free superfluid velocity has the velocity potential of the form $\hbar\phi_s/m$, the difference of the velocity potential at the two ends of the channel should be equal to the time derivative of the difference of the Gibbs free energy, i.e. $\hbar(\dot{\phi}_{s,1} - \dot{\phi}_{s,2})/m = -G_1 + G_2$.

Since the chemical potential is given by mG, we can identify the phase difference ϕ to the difference of the velocity potential (multiplied by $m/\hbar$) as

$$\phi = \phi_{s,1} - \phi_{s,2} \quad . \tag{6}$$

In this paper we have considered the homogeneous system, therefore the phases are homogeneous in each system. In the above consideration we have found that the phase difference is closely related to the superfluid velocity potential $\hbar\phi_s/m$. It is promissing to approach to the phenomenological two fluid theory from our microscopic point of view by considering several average quantities, together with such constants of motion as the total number of particles and the total momentum, as local quantities which vary spatially from one place to the other.

Appendix A Proof of eqs. (4-11), (4-30) and the relation $S^{-1} e^{i\phi} \vec{P}_0 S = 0$

In this appendix, we first show eq. (4-11) in its general form and then derive the representation (4-30) together with another representation. We also show the relation $S^{-1} e^{i\phi} \vec{P}_0 S = 0$.

We first consider a state written as $b\, B(b, b^+)|\varphi\rangle$, where B is an operator given in powers of b and b^+ and $|\varphi\rangle$ is given below eq. (4-8). As the state $B(b, b^+)|\varphi\rangle$ can also be written as $\sum_n \varphi_n' |n\rangle$, from arguments given below eq. (4-9), we obtain

$$\ll\phi|\sqrt{\Gamma}\, b\, B|\varphi\rangle = \sum_n \varphi_n' \ll\phi|\sqrt{\Gamma}\, b|n\rangle = e^{i\phi\sqrt{i\partial/\partial\phi}} \sum_n \varphi_n' \ll\phi|\sqrt{\Gamma}|n\rangle$$

$$= e^{i\phi\sqrt{i\frac{\partial}{\partial\phi}}} \ll\phi|\sqrt{\Gamma}\, B|\varphi\rangle = e^{i\phi\sqrt{i\frac{\partial}{\partial\phi}}}\ \vec{B}\ \varphi(\phi) \qquad \text{(A.1)}$$

, where $\vec{B}$ stands for the representation of $B = B(b, b^+)$. similarly, we have

$$\ll\phi|\sqrt{\Gamma}\, b^+ B|\varphi\rangle = \sqrt{i\frac{\partial}{\partial\phi}}\ e^{-i\phi}\ \vec{B}\ \varphi(\phi) \qquad \text{(A.2)}$$

As an operator $A(b, b^+)$ given in powers of b and b^+ can always be written as $b\, B_1(b, b^+) + b^+ B_2(b, b^+)$, from (A.1) and (A.2), eq. (4-11) follows immediately. It is clear form the transformation in eq. (A.1) that eq. (4-11) can be generalized to include the operators χ, χ^+ and P_0. In particular, the number operator b^+b is represented by $i\partial/\partial\phi$.

Next, we consider the representation (4-30). From (4-7), we obtain

$$\Gamma^s \sqrt{\Gamma}\, D\, \sqrt{\Gamma}\, \Gamma^{-s} = 1 \quad , \quad \Gamma = \Gamma(b^+b + 1) \quad , \qquad \text{(A.3)}$$

and the inner product of a state can be expressed as

$$\langle\varphi|\varphi\rangle = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \langle\varphi|\Gamma^{s}\sqrt{\Gamma}|\phi\rangle\rangle\langle\langle\phi|\sqrt{\Gamma}\,\Gamma^{-s}|\varphi\rangle \,. \tag{A.4}$$

Thus, we have the r-representative $\langle\langle\phi|\sqrt{\Gamma}\,\Gamma^{-s}|\varphi\rangle$ and the l-representative $\langle\varphi|\Gamma^{s}\sqrt{\Gamma}|\phi\rangle\rangle$. Putting $s=\frac{-1}{2}$, we obtain eqs. (4-27) and (4-28). Using the representation (4-11) given in the text and proved in the above, the r-representative of a state $|\varphi\rangle$ given in eq. (A.4) becomes

$$\varphi_r(\phi) = \langle\langle\phi|\sqrt{\Gamma}\,\Gamma^{-s}|\varphi\rangle = \left\{\Gamma\left(i\frac{\partial}{\partial\phi}+1\right)\right\}^{-s}\varphi(\phi) \equiv \left(\vec{\Gamma}\right)^{-s}\varphi(\phi) \,, \tag{A.5}$$

Where $\varphi(\phi)$ is given by eq. (4-8) and the r-representative of a state $b|\varphi\rangle$ becomes

$$\langle\langle\phi|\sqrt{\Gamma}\,\Gamma^{-s}b|\varphi\rangle = \langle\langle\phi|\sqrt{\Gamma}\,\Gamma^{-s}b\Gamma^{s}\Gamma^{-s}|\varphi\rangle \tag{A.6}$$

$$= \vec{\Gamma}^{-s}e^{i\phi}\sqrt{i\frac{\partial}{\partial\phi}}\,\vec{\Gamma}^{s}\langle\langle\phi|\sqrt{\Gamma}\,\Gamma^{-s}|\varphi\rangle = e^{i\phi}\left(1-\vec{P}_0\right)\left(i\frac{\partial}{\partial\phi}\right)^{\frac{1}{2}+s}\varphi_r(\phi) \,,$$

where the last equality follows from

$$\vec{\Gamma}^{-s}\,e^{i\phi}\sqrt{i\frac{\partial}{\partial\phi}}\,\vec{\Gamma}^{s}\,e^{-in\phi} = \vec{\Gamma}^{-s}\,e^{i\phi}\sqrt{n}\,\left\{\Gamma(n+1)\right\}^{s}e^{-in\phi} \tag{A.7}$$

$$= \begin{cases} \left\{\Gamma(n)\right\}^{-s}\sqrt{n}\left\{\Gamma(n+1)\right\}^{s}e^{-i(n-1)\phi} & (n\neq 0) \\ 0 & (n=0) \end{cases}$$

$$= e^{i\phi}\,n^{\frac{1}{2}+s}\left(1-\delta_{n,0}\right)e^{-in\phi} \,,$$

and $\vec{P}_0$ is defined by eq. (4-13). Similarly, we obtain

$$\langle\langle\phi|\sqrt{\Gamma}\,\Gamma^{-s}b^{+}|\varphi\rangle = \left(i\frac{\partial}{\partial\phi}\right)^{\frac{1}{2}-s}\left(1-\vec{P}_0\right)e^{-i\phi}\,\varphi_r(\phi) \,, \tag{A.8}$$

where the factor $(1-\vec{P}_0)$ is inserted considering that $i\partial/\partial\phi$ has negative eigenvalues. Putting $S=-\frac{1}{2}$, we obtain eq. (4-30), because in this cace $(1-\vec{P}_0)$ factor in (A.8) can be safely omitted. In this way, the relation to the non-unitary transformation (4-31) has become clear.

The relation $S^{-1}e^{i\phi}\vec{P}_0 S=0$ can be proved in a similar manner as in eq. (A.7), namely

$$S^{-1}e^{i\phi}\vec{P}_0 S e^{-in\phi} = \vec{\Gamma}^{-\frac{1}{2}} e^{i\phi}\vec{P}_0\vec{\Gamma}^{\frac{1}{2}} e^{-in\phi} \qquad (A.9)$$

$$= \vec{\Gamma}^{-\frac{1}{2}} e^{i\phi} S_{n,0}\{\Gamma(n+1)\}^{\frac{1}{2}} e^{-in\phi} = \{\Gamma(0)\}^{-\frac{1}{2}} e^{i\phi}\delta_{n,0} = 0 \ ,$$

which follows from the fact that the inverse of the gamma function have zero at z = 0.

Appendix B Property of f_k and $\langle k,l,\cdots,m\rangle$

In this appendix, we will show several relations satisfied by f_k and $\langle k,l,\cdots,m\rangle$, which are frequently used.

They are defined in eq. (4-25) and in eq. (5-8). From the definition, we readily obtain

$$f_{k=0} = 0 \quad , \quad \sum_{k\neq 0} f_k = \sum_k 1 \quad , \qquad (B.1)$$

$$f_k f_l = (1-e^{-ika})(1-e^{-ila}) = 1 - e^{-ika} - e^{-ila} + e^{-i(k+l)a}$$

$$= f_k + f_l - f_{-m} \qquad (k+l+m=0) \ . \qquad (B.2)$$

The relation (B.2) is most important and clearly shows that f_k should not be simply regarded as of order a. Other relations can be obtained by using eq. (B.2). In particular, we have

$$\langle k,-k\rangle = f_k + f_{-k} \simeq f_k f_{-k} \qquad \text{(B.3)}$$

$$\langle k,\ell\rangle = (f_k f_\ell + f_{-k} f_{-\ell})/2 \simeq (f_k + f_\ell - f_{-m} + f_{-k} + f_{-\ell} - f_m)/2$$

$$= (\langle k,-k\rangle + \langle \ell,-\ell\rangle - \langle m,-m\rangle)/2 \quad . \qquad \text{(B.4)}$$

$$(k+\ell+m=0)$$

We also obtain, by a simple calculation, the following relations

$$\langle k,\ell,m\rangle = 0 \qquad (k+\ell+m=0)\ , \qquad \text{(B.5)}$$

$$\langle k,\ell,m,n\rangle = 2\langle k,\ell,m\rangle \qquad (k+\ell+m+n=0)\ , \qquad \text{(B.6)}$$

$$\langle k,\ell,m,n\rangle = 2\{\langle k,-k\rangle + \langle \ell,-\ell\rangle + \langle m,-m\rangle + \langle n,-n\rangle$$

$$-\langle k+\ell,-k-\ell\rangle - \langle k+m,-k-m\rangle - \langle k+n,-k-n\rangle\} \qquad \text{(B.7)}$$

$$(k+\ell+m+n=0)\ .$$

In the continuum limit $a\to 0$, f_k/a reduces to ik and $\langle k,\ell\rangle/a^2$ (or $\langle k,-k\rangle/a^2$) to $-k\cdot\ell$ (or k^2). Thus, the relation (B.4) reduces to $-k\cdot\ell = (k^2+\ell^2-m^2)/2$

Appendix C Extension to 3-dimensional case

In the three-dimensional lattice space, the kinetic energy assumes the form

$$\sum_{\vec{x}}\sum_{\alpha} a^3(\hbar^2/2ma^2)\{\Psi^+(\vec{x}_i+\vec{e}_\alpha)-\Psi^+(\vec{x}_i)\}\{\Psi(\vec{x}_i+\vec{e}_\alpha)-\Psi(\vec{x}_i)\}\ , \qquad \text{(C.1)}$$

where α runs over x, y and z and $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ is given by (a,0,0), (0,a,0,) and (0,0,a), respectively. Correspondingly,

there appears three f_k's, namely, f_k^x, f_k^y and f_k^z given respectively by

$$f_k^\alpha = 1 - e^{-i\vec{k}\vec{e}_\alpha} \quad . \tag{C.2}$$

$\langle k, \ell, \cdots, m \rangle$ in the one-dimension is replaced by

$$\langle k, \ell, \cdots, m \rangle = \sum_\alpha \left(f_k^\alpha f_\ell^\alpha \times \cdots \times f_m^\alpha + f_{-k}^\alpha f_{-\ell}^\alpha \times \cdots \times f_{-m}^\alpha \right) / 2 \quad . \tag{C.3}$$

From (C.3), we see that the relations in Appendix B hold also in the three-dimensional case. The interaction term does not contain derivatives and so does not change. Thus, the resulting expressions remain the same as in the one-dimensional case.

Appendix D The phonon Hamiltonian

From the Hamiltonian (5-9) of bosons written in terms of the density fluctuation ρ_k and the phase operator ϕ_k, we obtain the phonon Hamiltonian given in terms of the phonon field-operators B_k and B_k^+ (see eq.(5-10)) having the following form

$$H = H_0 + H_{1,1} + H_{1,2} + H_{2,1} + H_{2,2} + H_{3,a} + H_{3,c} , \qquad (D.1)$$

$$H_0 = (\hbar^2/2M) \sum_k (k^2/\lambda_k) B_k^\dagger B_k + E_0^{(0)} + E_0^{(1)} , \qquad (D.2)$$

$$E_0^{(0)} = (1/2)N^2 V_0 - (\hbar^2/8M) \sum_k k^2 (1 - \lambda_k^{-1})^2 , \qquad (D.3)$$

$$E_0^{(1)} = (\hbar^2/16MN) \sum_{k,l} \{k^2(1 - 2\lambda_k + 2\lambda_k\lambda_l) + [(k+l)^2 - k^2 - l^2](1-\lambda_k)^2(1-\lambda_l)^2/(4\lambda_k\lambda_l)\} , \qquad (D.4)$$

$$H_{1,1} = \sum_{k,l} \Gamma_{1,1}(k,l) B_k^\dagger B_k , \qquad (D.5)$$

$$H_{1,2} = \sum_{k,l} \Gamma_{1,2}(k,l) B_k^\dagger B_l^\dagger B_l B_k , \qquad (D.6)$$

$$H_{2,1} = \sum_{k,l} \Gamma_{2,1}(k,l) B_k^\dagger B_{-k}^\dagger + h.c. , \qquad (D.7)$$

$$H_{2,2} = \sum_{k,l} \Gamma_{2,2}(k,l) B_k^\dagger B_{-k}^\dagger B_{-l} B_l , \qquad (D.8)$$

$$H_{3,a} = \sum_{k,l,m} \delta_{k+l+m,0} \Gamma_{3,a}(k,l,m) B_k^\dagger B_l^\dagger B_m^\dagger + h.c. , \qquad (D.9)$$

$$H_{3,c} = \sum_{k,l,m} \delta_{k+l+m,0} \Gamma_{3,c}(k,l,m) B_{-k}^\dagger B_l B_m + h.c. , \qquad (D.10)$$

where M, V_k and λ_k are the particle mass, the Fourier transform of the interaction potential and the unperturbed structure factor, respectively; $\lambda_k = k/(k^2 + 4MNV_k/\hbar^2)^{1/2}$ and

$$\Gamma_{1,1}(k,l) = (\hbar^2/4MN)[(k^2+l^2)\lambda_k\lambda_l - k^2\lambda_k] \quad , \qquad (D.11)$$

$$\Gamma_{1,2}(k,l) = (\hbar^2/8MN)[2(k+l)^2\lambda_k\lambda_l + k\cdot l(\lambda_k^{-1}\lambda_l^{-1} + \lambda_k/\lambda_l + \lambda_l/\lambda_k - 3\lambda_k\lambda_l)] \quad , \qquad (D.12)$$

$$\Gamma_{2,1}(k,l) = \Gamma_{1,1}(k,l)/2 \quad , \qquad (D.13)$$

$$\Gamma_{2,2}(k,l) = (\hbar^2/16MN)[2(k+l)^2\lambda_k\lambda_l + k\cdot l(\lambda_k^{-1}\lambda_l^{-1} - \lambda_k/\lambda_l + 4 - \lambda_l/\lambda_k - 3\lambda_k\lambda_l)] \quad , \qquad (D.14)$$

$$\Gamma_{3,a}(k,l,m) = (\hbar^2/24M)(N\lambda_k\lambda_l\lambda_m)^{-1/2}a(k,l,m) \quad , \qquad (D.15)$$

$$a(k,l,m) = k\cdot l(\lambda_k\lambda_l + 1)\lambda_m + l\cdot m(\lambda_l\lambda_m + 1)\lambda_k + m\cdot k(\lambda_m\lambda_k + 1)\lambda_l \quad , \qquad (D.16)$$

$$\Gamma_{3,c}(k,l,m) = (\hbar^2/8M)(N\lambda_k\lambda_l\lambda_m)^{-1/2}c(k,l,m) \quad , \qquad (D.17)$$

$$c(k,l,m) = k\cdot l(\lambda_k\lambda_l - 1)\lambda_m + l\cdot m(\lambda_l\lambda_m + 1)\lambda_k + m\cdot k(\lambda_m\lambda_k - 1)\lambda_l \quad , \qquad (D.18)$$

These vertex functions characteristic of DPO are related to those characteristic of CBF by

$$\alpha(k,l,m) = a(k,l,m) + b(k,l,m)$$
$$= k\cdot l(\lambda_k - 1)(\lambda_l - 1)\lambda_m + l\cdot m(\lambda_l - 1)(\lambda_m - 1)\lambda_k + m\cdot k(\lambda_m - 1)(\lambda_k - 1)\lambda_l \quad , \qquad (D.19)$$

$$\gamma(k,l,m) = c(k,l,m) + d(k,l,m)$$
$$= k\cdot l(\lambda_k + 1)(\lambda_l - 1)\lambda_m + l\cdot m(\lambda_l - 1)(\lambda_m - 1)\lambda_k + m\cdot k(\lambda_m - 1)(\lambda_k + 1)\lambda_l \quad , \qquad (D.20)$$

where $b(k,l,m) = k^2\lambda_l\lambda_m + l^2\lambda_m\lambda_k + m^2\lambda_k\lambda_l$ and $d(k,l,m) = -k^2\lambda_l\lambda_m + l^2\lambda_m\lambda_k + m^2\lambda_k\lambda_l$ [In the above the continuum limit is taken properly].

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