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# FIELD DESCRIPTION
# WITH COMPLEX VARIABLES
# AND
# ITS APPLICATION

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## ABSTRACT

This work was done in the Electrical Engineering Ph.D. Course at Osaka University and some parts in this thesis have already been published in other articles.

When solving field problems ( boundary-value problems ) we will face in most cases the difficulty that is complication of wave functions in being dealt with analytically, and occasionally it will disturb the final developments in obtaining solutions. In some cases this may be avoided by employing numerical methods, however many unavoidable cases lie around us. We would be obliged to resolve this important subject rather in viewpoints of Engineering than Mathematics.

To this purpose, a method of using complex variables $z, \bar{z}$ available for two-dimensional problems, is proposed where $z = x + iy$ and $\bar{z}$ is the complex conjugate of z. Fields are expressed with these complex variables and are related to the corresponding regular functions of a single variable z through a new field-description.

The field-matching is performed at junctions of the regions, in which the sets of wave functions are given, by the analytic continuation of the corresponding regular functions instead of the fields, using the theory of regular function.

It is illustrated to the E plane taper in a rectangular waveguide.

To outer fields such as radiation fields, the field description with a contour integral representation in a complex plane is given. The complex variables z, $\bar{z}$ then are changed by a transforming function. This is applied to the boundary-value problems with arbitrary boundary. An approximate method is developed under the assumption of small boundary. The two dimensional scattering by the perfectly conducting obstacle with arbitrary cross section is solved.

Furthermore it is applied to the scattered field in the rectangular waveguide with an inductive post or a capacitive post. The transmission coefficient and the reflection coefficient are obtained in the general forms for arbitrarily shaped posts.

# Chapter I

# INTRODUCTION

Electromagnetic field problems today cover a vast area in the Electrical Engineering. In this area, the Boundary-Value Problems are the main subject matter. Its noticeable advances will be found now, nevertheless, it is still subjected to a troublesome problem. One of the difficulties is how to embody the solutions which have been specified by boundary conditions, either numerically or analytically. Particularly, it is demanded in the Engineering. Certainly, it is in the underground of field problems, but one need light up it all the more.

The modal expansion is one of the commonest techniques for solving the Helmholtz equation with boundary conditions. How to determine its coefficients will greatly depend on the sets of complete wave functions to be chosen. The labor of analytical manipulation will increase unless the orthogonality is established on the boundary surface. Consequently, the techniques of machine computation are employed on account of decreasing the analytical labor : the variational methods [1,2] , the point matching methods [3] , etc. .

Integral equations may be excellent only to represent the fields including the boundary conditions. However, they are disadvantageous to be exposed in the concrete form which is calculable, because of the complicated kernel functions. Hence, the integral equations are solved by numerical means [4,5] except for special cases.

It could be found that the numerical techniques except the method of nets [6] involve the analytical manipulation in leading to the final equations to be solved, and the complication of the wave functions disturbs the smoothness of the analytical development, as described above. Thus, it is reasonable to expect some fine techniques to reduce our analytical effort. The method of using a complex variable i.e. conformal mapping has remarkably succeeded in static field problems. The same idea for time-varying fields is found in the Wiener-Hopf techniques [7]. However, these are limited to the restricted boundary-shapes to which the Fourier analysis is applied.

The attempt to express the field with complex variables was initially made by Vekua [8]. In his work, a new field description was presented with a complex integral of a regular function, however, it is restricted in a closed region. Through his description, it can be pointed out that the regular function corresponds to the field, one to one, and in addition, the real part represents the corresponding static field. In this respect, he has extended the complex analysis for static fields to that for time varying fields, though it must be noted that it can not be applied to radiation fields.

This thesis follows applicably on the customary courses in the Electrical Engineering in that the treatment of using the complex variables reduces our analytical labor, therefore, is primarily concerned with an exposition of the mathematical tools, together with a series of examples.

Chapter II is mainly devoted to investigating the field description for the outer fields ( radiation fields ) which has never been found and to improving it so as to be convenient to actual shapes of boundary. The Vekua's description is concurrently described to utilize in the next chapter.

Chapter III is concerned with the field continuation in the two different regions which are partially common. If the field is given by a modal expansion, it is so-called " Mode-Matching " . This is applied to the E plane taper at the junction in a rectangular waveguide in viewpoints of a complex variable, much more to appreciate a mathematical device.

Chapter IV is devoted to apply the given formula to the scattering by an arbitrarily shaped obstacle, which is assumed to be a small size and a perfect conductor. The examples of waveguide problems are given in Chapter V to make the formula more widely applicable. These were dealt with by Miles [9], Schwinger [10,11] , Marcuvitz [12,13] , Collin [14], etc., using the variational methods. Lewin also showed another technique [15] in the case of a cylindrical post. The approximate procedure is similar to his one. It however is suggested that the method proposed gives general formulations for arbitrarily shaped posts. The specific posts are cylindrical posts, strips, square posts, and all of them are examined for two cases : the inductive posts and the capacitive posts.

## Chapter II

## FIELD DESCRIPTION

The fields satisfying the Maxwell's equations have been studied very well within the expressing by real variables, so the details are omitted, and we start from the following.

Let a relevant coordinate system be ( X,Y,Z ), and suppose that the field propagates along the Z axis with $e^{-j\beta Z}$, where $\beta$ is a constant. The field u then satisfies the Helmholtz equation:

$$\frac{\partial^2}{\partial X^2}u + \frac{\partial^2}{\partial Y^2}u + k^2 u = 0 \qquad (2.1)$$

where $k = \sqrt{k_o^2 - \beta^2}$ , $k_o$ = the wave number in free space. Equation (2.1) is a monochromatic, two dimensional equation, and is quite equivalent to purely two dimensional problems concerning its analysis. In the following, the analysis is developed with this assumption - which is also made to the boundary. For simplicity, the normalized coordinate system is employed:

$$x = \tfrac{1}{2} k X \quad , \quad y = \tfrac{1}{2} k Y \qquad (2.2)$$

Throughout the thesis, all the quantities in the normalized system are written in principle by small letters, and the corresponding ones in the original system by capital letters. The time factor is $e^{j\omega t}$ and "j" is called here " Temporal Imaginary Unit ". While

$$z = x + i\,y \qquad (2.3)$$

( z is not a normalized axis of Z ) is called " Spatial Complex Number ", and "i" is a spatial imaginary unit ( $i^2 = -1$, $j^2 = -1$ ). We should concentrate our attention rather on "i" than "j". The bar denotes the complex conjugate of the complex number including both the imaginary units, with respect to i .

The boundary in the normalized system is $\gamma$, and the " boundary " implies $\gamma$ unless explicitly specified ( see Fig. 2.1 ).

The relation

$$\frac{\partial}{\partial z} = \frac{1}{2}\left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \qquad (2.4)$$

and its complex conjugate lead to the hyperbolic equation, instead of (2.1) :

$$\frac{\partial^2}{\partial z\, \partial \bar{z}} u + u = 0 \qquad (2.5)$$

It is obvious that all the components $E_X, E_Y, E_Z, H_X, H_Y, H_Z$ of fields satisfy the above equation. The transverse fields can be expressed by the well-known, differential formulae*, and these become with (2.4)

$$E_X - iE_Y = -\frac{\partial}{\partial z} V_e$$
$$H_X - iH_Y = -\frac{\partial}{\partial z} V_m \qquad (2.6)$$

where $V_e, V_m$ are an electric complex potential, a magnetic complex potential respectively, and are functions of $z, \bar{z}$ :

$$V_e = j\left(\frac{\beta}{k} E_Z + i\frac{k_o}{k}\sqrt{\frac{\mu}{\varepsilon}}\, H_Z\right) \qquad (2.7)$$

$$V_m = j\left(\frac{\beta}{k} H_Z - i\frac{k_o}{k}\sqrt{\frac{\varepsilon}{\mu}}\, E_Z\right) \qquad (2.8)$$

here $\varepsilon$ = the permittivity, $\mu$ = the permeability, and $k_o = \omega\sqrt{\varepsilon\mu}$. The expression (2.6) is familiar to static fields [2] although the potentials are different: the static potentials are functions of z only, whereas (2.7)(2.8) involve two complex variables.

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* With vector representation, the transverse fields $E_t$, $H_t$ are

$$E_t = -\frac{j\beta}{2k}\nabla_t E_Z + \frac{jk_o}{2k}\sqrt{\frac{\mu}{\varepsilon}}\, I_Z \times \nabla_t H_Z$$
$$H_t = -\frac{j\beta}{2k}\nabla_t H_Z - \frac{jk_o}{2k}\sqrt{\frac{\varepsilon}{\mu}}\, I_Z \times \nabla_t E_Z$$

where $I_Z$ is a unit vector being directed to the Z axis.

It is readily seen from (2.5)(2.6) that

$$
\begin{aligned}
V_e &= \frac{\partial}{\partial \bar{z}} ( E_X - iE_Y ) \\
V_m &= \frac{\partial}{\partial \bar{z}} ( H_X - iH_Y )
\end{aligned}
\qquad (2.9)
$$

Static fields make the left hand side of (2.9) to zero, so the above relations do not hold for the static fields.

The partial derivatives with respect to $z, \bar{z}$ unfortunately make the physical interpretation obscue, instead permit us to deduce a great deal of useful mathematical tools. For instance, the normal component and the tangential component of the electric field on $\Gamma$ can be represented simultaneously by the real part and the imaginary part of

$$
( n_x + in_y )( E_X - iE_Y ) = -\mathbb{n}_z \frac{\partial}{\partial z} V_e \qquad (2.10)
$$

respectively, where $n_x, n_y$ are the normal components vs. the normalized coordinates, and $\mathbb{n}_z = n_x + in_y$ . Furthermore, we could expect that in the limit of $k = 0$, $V_e$ or $V_m$ tends to a transforming function ( a mapping function ) which is <u>regular</u> so that the expressions (2.6)(2.9) extend the complex analysis available for static fields to that for time varying fields, especially, on successive derivation from the statical treatment.

## 2.1 General Remarks

The theory of complex analysis was well studied in the field of mathematics and broadly used as a tool in that of the Engineering. Nevertheless, one can find that most parts of the theory which seem to be of interest in a way still remain to neglected ones; because the theory was built up under the assumption of a single variable z , — so the application to the cases of two variables $z, \bar{z}$ such as the fields described before becomes extremely difficult.

From this standpoint, we define a certain operator $\mathcal{H}$ with which the field u is related to the regular function $\Phi$ of z only, one to one.

$$u = \mathcal{H}\,\Phi \tag{2.11}$$

Evidently, the operator depends on what form of the regular function is to be chosen. If such an operator exists, all the behaviors of the field can be understood by studying the corresponding regular function. It is a straightfoward matter to seek out this operator in the present chapter, — it is described in the latter sections. In this section, the general properties of this operator are studied briefly.

It is apparent that the operator $\mathcal{H}$ involves all the characteristics of the wave and includes the variable $\bar{z}$ . Therefore, the operator defined at a certain point in the complex plane is different from that at another point, thereby so is the corresponding regular function, too.

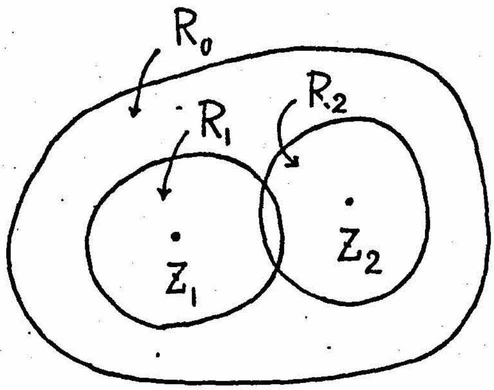



Fig.2.2

The different regions in the z plane where (2.11) is defined.

Suppose, the field u exists in the region $R_o$ ( see Fig.2.2 ), and the regions $R_1$, $R_2$ including the points $z_1$, $z_2$ respectively are constrained in $R_o$, and assume that any source of u does not lie in $R_o$. We dare denote the fields in $R_1$, $R_2$ by $u_1$, $u_2$ respectively and similarly the corresponding regular functions by $\Phi_1$, $\Phi_2$. Then,

$$u_1 = \mathcal{H}_1 \Phi_1 \tag{2.12}$$

$$u_2 = \mathcal{H}_2 \Phi_2 \tag{2.13}$$

where $\mathcal{H}_1$, $\mathcal{H}_2$ are the operators defined at the points $z_1$, $z_2$ respectively. Of course, $\Phi_1$ ( or $\Phi_2$ ) is regular in $R_1$ ( or $R_2$ ) and $\mathcal{H}_1$ ( or $\mathcal{H}_2$ ) is valid in $R_1$ ( or $R_2$ ).

Since $u_1$ is the same with $u_2$ in the first place, the expression (2.13) could be transformed into the expression (2.12) by the " transforming operator " $\mathcal{D}_{12}$ :

$$\mathcal{H}_2 = \mathcal{H}_1 \mathcal{D}_{12} \tag{2.14}$$

The right hand side may be considered an analytic continuation of $\mathcal{H}_1$ into $\mathcal{H}_2$. Thus, we can interpret as follows: the $\mathcal{D}_{12}$ is the operator which transforms the regular function defined in $R_2$ into that corresponding to $u_1$ in $R_1$ through <u>an analytic continuation</u> ( this is <u>not</u> a conventional one for regular functions of z only ). Let us define $\tilde{\Phi}_2$ as

$$\tilde{\Phi}_2 = \mathcal{D}_{12} \Phi_2 \tag{2.15}$$

Substituting (2.14) into (2.13) and comparing the result with (2.12), we find that $\tilde{\Phi}_2$ is a regular function and is equal to $\Phi_1$.

$$\tilde{\Phi}_2 = \Phi_1 \qquad \text{in } R_1 \tag{2.16}$$

It is noteworthy that $\Phi_1$ and $\Phi_2$ differ from each other, except for $z_1 = z_2$ .

## 2.2 Inner Field

The E field and the H field are fundamental solutions to Maxwell's equations with charge free and the combination constitutes the most actual field. One should remember the fact that the field behavior can not be discussed from <u>one</u> component of the field but from <u>two</u> components i.e. electric and magnetic field components to the same direction. This fact allows us to conclude that the regular function associated with the E field and that with the H field are undistinguishable. Therefore we examine henceforth the field descriptions separately for both cases: The symbol u represents either $E_z$ or $H_z$. In the former case, the field description is of the E field and in the latter case, is of the H field. In the following, the u is not specified explicitly.

The source free field in the closed region to $\gamma$ ( see also Fig.2.1 ) is named here " Inner Field ". For this one, the Vekua's field description [8] is quoteworthy; his derivation is based just on the method of solving the hyperbolic equation e.g. (2.5) [2,8] . We now explain his one from another point of view briefly, moreover, reasonably , to confirm the follow of the latter chapters.

Equations (2.6)(2.9) are consistent both for the E field and for the H field and therefore we can write down, in general for u, as

$$E = -\frac{\partial u}{\partial z}, \quad u = \frac{\partial E}{\partial \bar{z}} \qquad (2.17)$$

Note that u is a real function with respect to i, but forget it before the derivation, instead consider at a final step. First, we put

$$u = \Phi(z) + \bar{z}\,\Phi_1(z) + \bar{z}^2\,\Phi_2(z) + \ldots \qquad (2.18)$$

where $\Phi(z)$, $\Phi_1(z)$, ... are regular in the inner region. By substitution, E can be obtained. Further, the differentiation of E leads to u. Compare the resulting u with (2.18). We then have

$$\begin{aligned} &\Phi_1'(z) = -\Phi(z), \quad \Phi_2'(z) = -\tfrac{1}{2}\,\Phi_1(z), \\ &\Phi_3'(z) = -\tfrac{1}{3}\,\Phi_2(z), \;\ldots\ldots\ldots \end{aligned} \qquad (2.19)$$

where $\Phi_n'(z)$ denote the first derivatives. In the above, We shall put $\Phi_1(0) = \Phi_2(0) = \ldots = 0$. Instead, we must add a series of $\bar{z}$ only to (2.18) ( beginning from the term $\bar{z}$ ), —— such a series must fortunately vanish in view of (2.17). From (2.19), $\Phi_n(z)$ is derived successively: with exchanging the order of integration e.g. $\int_0^z dq \int_0^q dp = \int_0^z (z-p)dp$ ,

$$\Phi_n(z) = \frac{(-1)^n}{n!(n-1)!} \int_0^z (z-t)^{n-1} \Phi(t)dt$$

$$= - \int_0^z \frac{\partial}{\partial t} \left[ \frac{(-1)^n (z-t)^n}{(n!)^2} \right] \Phi(t)dt \tag{2.20}$$

$$n = 1, 2, \ldots\ldots$$

By (2.18) and (2.20), one can represent u, using the Bessel function $J_o$ [16] , as

$$\Phi(z) - \int_0^z \frac{\partial}{\partial t} J_o(2\sqrt{\bar{z}(z-t)}\ ) \Phi(t)dt \tag{2.21}$$

At this stage, bring to mind the real function condition of u. Evidently, the real and imaginary parts of (2.21) satisfy (2.1) or (2.5), so that we can confine ourselves to the real part to the purpose of expression of u.

$$u = (2.21) + \text{C.C.} \tag{2.22}$$

where C.C. designates the complex conjugate of (2.21) with respect to i. The above equation is just the Vekua's one [8].

The field description (2.22) bears the important statements that in the vicinity of the origin —— which is to say, actually, in a situation of quasi-static approximation, u is approximately $\Phi(z)$ + C.C. where the second term means the complex conjugate of the preceding term, and that if the variables $z,\bar{z}$ of $u(z,\bar{z})$ can be regarded in mind as individual, then $u(z,0)$ becomes $\Phi(z) + \overline{\Phi(0)}$. The former suggests utilization of conformal mapping to time varying fields. The latter produces a substantial mean for inverse operator $\mathcal{H}^{-1}$ : The $u(0,0)$, which is equal to $\Phi(0) + \overline{\Phi(0)}$ , is actually the value of u at the origin, and so the imaginary part of $\Phi(0)$ is meaningless. Thus, one can assume to take

$$\Phi(0) = \text{a real number with respect to } i \tag{2.23}$$

Then

$$\begin{aligned} \mathcal{H}^{-1} u &= \Phi(z) \\ &= u(z,0) - \tfrac{1}{2} u(0,0) \end{aligned} \tag{2.24}$$

Finally, it is added that Vekua has proved the existence of $\Phi$ assuming (2.23) and the Hölder condition of u on $\Upsilon$ [8], and besides that the uniqueness also can be proved [17] , thereby the field corresponds to the regular function, one to one.

## 2.3 Outer Field

We have, in the previous section, introduced the Vekua's field description for the inner field. We now here suppose the " Outer Field " - which is defined as an outgoing wave in the outer region ( see Fig.2.1 ); obviously it must specify the radiation conditions. We know that on account of its familiarity, the handling of it has been playing a major role as yet in the Microwave Engineering [18] . This would encourage us to examine a new field description for outer fields. To establish it, this section is mainly devoted.

First, focus attention on the previous successive derivation to ask why its simulation for this case may end in failure. The successive manner is certainly powerful and interesting. But, 1) the source term being nondescript in its complex representation must be added to the right hand side of (2.5), 2) (2.5) also does not involve, in itself, the radiation conditions. Thus, the present circumstances are quite different to the previous ones. From the reasons, it might be thought a short cut to seek out the field description by trial.

The field is absorbed at the infinity just like a transmission line wave at a matched load. The terminal, that is to say, the infinity is a singular point. Therefore, one should not anticipate a line integral , like (2.21), over the interval $[z, \infty]$.

Instead, a contour integration could be taken. Moreover, the Bessel function of zero order $J_o$ appearing in (2.21) should be exchanged by the Hankel function of the second kind of zero order $H_o^{(2)}$ in view of radiation. The corresponding function $\Psi_*(z)$ must be regular in the outer region even at the infinity. It is, in other words, bounded in the modulus. According to the " Liouville's theorem " [19] , all the singularities of $\Psi_*$ are located in the inner region, together on $\gamma$. Additionally, referring to the condition (2.23), the analogy

$$\Psi_*(\infty) \;=\; \text{a real number with respect to i} \qquad (2.25)$$

might be permissible. Now return to the previous section to investigate the statical relation in question between $\Psi_*$ and the outer field. The u in a quasi-static sense has become the twice of the real part of $\Phi$ as before, and the field behavior has agreed with that of $\Phi$ . In this case, however, there exists a difficulty on the way of analogy. Because, two dimensional static fields involve a logarithmic function - whose corresponding function is log z, which is a multi-valued function - not regular in a strict sense, and thus not suitable. On the other hand, the constant behavior of $\Psi_*$ at the infinity as mentioned above (= $\Psi_*(\infty)$ ) does not correspond to any one in actual cases. So one should relate it to the logarithmic field.

For our present purpose, these considerations, in fact, provide the following expression:

$$u_o(z,\bar{z}) = A_o K(z,\bar{z};0) + \frac{1}{2\pi i}\oint_{\gamma_c} \frac{\partial}{\partial t} K(z,\bar{z};t)\, \Psi_*(t)\,dt$$

$$+ \text{C.C.} \qquad (2.26)$$

where

$$K(z,\bar{z};t) = \pi j\, H_o^{(2)}\left(2\sqrt{\bar{z}(z-t)}\right) \qquad (2.27)$$

$$A_o = \Psi_*(\infty),$$

the suffix of $u_o$ implies the outer one, the origin is chosen in the interior, $\gamma_c$ is oriented in a positive sense as excluding the point z ( see Fig.2.3 ), and the integration is carried out over it. The function K of t is regular within the interior, so that the constant term of $\Psi_*$ ( $= \Psi_*(\infty)$ ) does not contribute to the integration at all. It is involved in the first term instead.

According to the Runge-Walsh's theorem [19] , $\Psi_*$ can be expressed by a Laurent series:

$$\Psi_*(z) = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{b_n(N)}{z^n} \tag{2.28}$$

where $b_n$ $(n \geq 1)$ are " spatial and temporal complex constants ", but $b_o$ $(= \Psi_*(\infty))$ a temporal one, and all of them depend on N. This is different slightly from the conventional one. Namely, the set of $b_n$ is altered in each summation. At any rate, however, $\Psi_*$ could be approximated by a finite series within the range of accuracy designated beforehand. In fact, the finite sum can only be realized. Therefore, in a situation of the realization of the summation, (2.28) can be understood as usual. Use of (2.28) in this sense, as all the singularities are poles, leads (2.26) to the " integration-less " evaluation. This is the well-known Residue Calculus [2,19] : indeed, the value of the integral

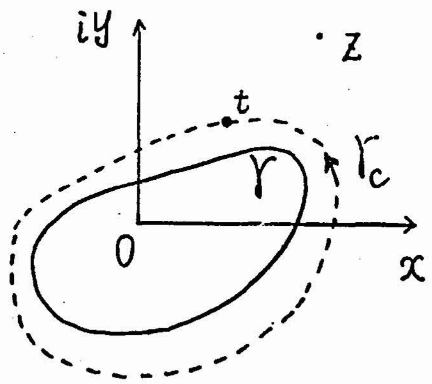



Fig.2.3

The outer region and $\gamma_c$ in the z plane.

is equal to the residue of the integrand. The residue can be obtained without too much effort by the Laurent expansion. This is really one of our aims and actually saves a considerable amount of analytical and computational labors.

To make the relation between the field and the corresponding regular function clearer, substitute (2.28) into (2.26). Then, using the formula [16]

$$\frac{\partial}{\partial t} K = 2\pi j \bar{z} \left\{ \frac{H_1^{(2)}(2\sqrt{\bar{z}(z-t)}\,)}{2\sqrt{\bar{z}(z-t)}} \right\}$$

$$= \pi j \sum_{m=1}^{\infty} \frac{t^{m-1}}{(m-1)!} H_m^{(2)}(2\rho)\, e^{-im\theta} \qquad (2.29)$$

$$(\ |z| > |t|\ )$$

where $z = \rho e^{i\theta}$, we obtain immediately

$$u_o(z,\bar{z}) = \pi j\, b_o H_o^{(2)}(2\rho) + \lim_{N\to\infty} \pi j \sum_{n=1}^{N} \frac{b_n(N)}{(n-1)!} H_n^{(2)}(2\rho)\, e^{-in\theta}$$

$$+ \text{C.C.} \qquad (2.30)$$

where $H_n^{(2)} = J_n - jY_n$ ( be careful for "j" ); $Y_n$ = the Neumann function.

It is readily seen from the above that $u_o$ is, no doubt, an outgoing wave. We now turn to (2.28) and cofine ourselves to only one term - say, the n-th term. We then recognize that $z^{-n}$ is associated with the n-th cylindrical wave and in spite of the assumption $|z|>|t|$ this fact is valid in the whole outer region. Even though finitely summed up, it still remains unchanged. Consequently, after all, the expression of (2.30) must be preserved at every point of the outer region. This is the so-called Layleigh Hypothesis, which has been doubted regarding its convergence [20-23] . In this respect, the present statement replies to the question clearly.

In the statical limit, (2.26) must be able to agree with a conventional complex representation in statics. For the purpose of illustration, evaluate the derivative $\partial K/\partial t$ by a straightforward differentiation, taking into account of the singular term involved. We then find

$$\frac{\partial}{\partial t} K = \frac{1}{t-z} + \log(z-t) \frac{\partial}{\partial t} J_0(2\sqrt{\bar{z}(z-t)}\ ) + \text{regular terms} \qquad (2.31)$$

The second and third terms are, if the points z,t are located in the neighborhood of the origin, negligible compared to the first term. Accordingly, the derivative $\partial K/\partial t$ can be approximated by the function having a pole of order 1 at $t = z$. The second term of (2.26) then exhibits the Cauchy integral though it may be somewhat different: the Cauchy integral is presented in the interior and the present one is in the exterior, vice versa. Therefore, by deformation of $\gamma_C$ into the two closed contours surrounding the points $t = z, \infty$ closely in such a way that at no time a singularity of the integrand is crossed, the value of the integral can be computed directly. It is

$$\Psi_*(\infty) - \Psi_*(z) \tag{2.32}$$

On the one hand, the Hankel function $H_0^{(2)}$ becomes $(\pi j + 2C + 2 \log \rho)/\pi j$ approximately, where C is the Euler constant (= 0.5772... ) [16]. The function $2 \log \rho$ can be expressed by the complex form $\log z + \log \bar{z}$. Needless to say, the latter is the complex conjugate of the former. Accordingly, a glance of the C.C. in (2.26) shows that the first term can be replaced by $A_0(\pi j + 2C + 2 \log z)$. Thus,

$$u_0(z,\bar{z}) \simeq \Psi_*(\infty)(\pi j + 2C + 1 + 2 \log z) - \Psi_*(z) + \text{C.C.} \tag{2.33}$$

Evidently, at this time, $u_o$ does not satisfy the Helmholtz equation, but the Laplace equation. In such a limiting expression, insertion of (2.28) enables us to interpret about the coefficients $b_n$ on physical grounds. The $b_n$ are, in short, the multipole sources in the multipole field expansion [24] - although their signs or factors may, strictly speaking, be somewhat different. It should be noted that all the $b_n$ mentioned above imply the limiting values $b_n(\infty)$, and the values for a finite N are, in fact, the approximate ones. Of course, there exists an essential difference in expression:i.e. the one is with spatial complex numbers and the other is with vector ones.

As is well-known [2] , the Z component of the field can be regarded, in such a statical limit, as an electric potential or a magnetic potential. Suppose, for convenience, that $u_o$ is an electric potential and its boundary value is zero. Furthermore, remove the term ( $\pi j$ + 2C ) in the parenthesis. Then, (2.33) results in a purely statical expression. We will find at this stage that the function

$$2\,\Psi_*(\infty)\,\log z + \left\{ \Psi_*(\infty) - \Psi_*(z) \right\}$$

must, in effect, be a mapping function by which the outer region is transformed into an upper-half plane. From this result, it might be suggested that 1) the field ( wave) expression is continuously connected with the corresponding static field, so (2.26) is, in this regard, the <u>extended</u> Cauchy integral; 2) it is possible to utilize conformal mapping techniques positively in field manipulation, and in addition, the regular function obtained in the neighborhood of $\Gamma$ from the boundary conditions, can be combined with the far field by means of the Laurent expansion ( refer to (2.30) ).

## 2.4 Alterative Description for Outer Fields

What kind of regular functions is to be associated with the field, is influenced strongly by the choice of the operator $\mathcal{H}$ , as mentioned before. It should be chosen so as to be suited to practical cases as much as possible. It is doubtful whether the previous field description is applicable even to the cases of strange shapes of boundary or not. The purposes of the present section are first to point out its failing and secondly to improve it from standpoints of application.

We shall now reconsider about (2.33). It is, of course, valid in the neighborhood of the origin. Also, recall the assumptions that $\Gamma$ is a closed curve, the interior possesses non zero-area and the origin is chosen in it. The zero-area inner region is out of the preceding discussion. However, a thin plate boundary, like a strip - which is very interesting in the Engineering, belongs to such a case. Obviously, if disregarded and applied, the origin must be taken on the boundary curve, and in consequence, the field diverges at this point, contrary to its actual behavior ( refer to (2.33) ).This arises from the " badness " of the field description. In other words, the logarithmic singular terms being contained in (2.26) are isolated and are not manipulated through any integral disposal. Accordingly, we shall first integrate (2.26) by parts to put the first term in the integrand; we have

$$u_o(z,\bar{z}) = \frac{1}{2\pi i}\oint_{\gamma_c} K(z,\bar{z};t)\,\bar{\Psi}(t)\,dt + \text{C.C.} \qquad (2.34)$$

where

$$\Psi(z) = \frac{\Psi_*(\infty)}{z} - \frac{d}{dz}\Psi_*(z) \qquad (2.35)$$

The function $\Psi(z)$ is also regular in the outer region. As far as looking the above right hand side, $\Psi$ seems to behave as if it diverged at the origin, but surely there exists the bounded function, at the origin, which does not diverge ( this is touched later; its general form is (2.41) ).

The K involves the function $\log \left\{ \bar{z}(z-t) \right\} = \log \bar{z} + \log (z-t)$. Since the latter is to be integrated with respect to t, the singularity may fade away. But the former works only as a multiplication factor and its singularity still remains. It turns out, from this fact, that one must contrive to put out the singularity, for instance, by replacing $\bar{z}$ by $(\bar{z}-\bar{t})$ and integrating with respect to $\bar{t}$. In accordance with this idea, the improved field description is examined in the following.

Let us, for convenience, shift the z-plane only by $z_o$ $(= x_o + iy_o)$, and define the $\xi$-plane, newly, as

$$\xi = z - z_o \tag{2.36}$$

Also, put $\eta = t - z_o$. Evidently, any geometric figure in the $\xi$-plane is unchanged. Therefore, Fig.2.3 can be quoted in this case, too, of course, mentally by interchanging z with $\xi$ and t with $\eta$. This is always implied, henceforth, unless explicitly specified. We shall here repeat again that $\xi$ and z, also $\eta$ and t can be regarded as identical in the following development.

To avoid the inevitable difficulty, described before, which one is to encounter necessarily, as it is, when employing the field expression with z or $\xi$, the variable $\xi$ is transferred to the variable w through a conformal transformation.

Let the transforming function be $\xi(w)$. Suppose, $\xi(w)$ maps the exterior in the $\xi$-plane onto the exterior $|w|>1$. Of course, $\gamma$ corresponds to $|w|=1$ i.e. $\gamma_w$ ( see Fig.2.4 ). Needless to say, the infinity in the $\xi$-plane is mapped into the infinity in the w-plane. The $\xi(w)$ at infinity must behave in proportion to w linearly. Its coefficient can, since the geometric picture in the w-plane is unchanged by any rotation, be made a positive real number by an appropriate rotation. We have therefore [1]

$$\xi = \alpha w + \frac{\beta_1}{w} + \frac{\beta_2}{w^2} + \frac{\beta_3}{w^3} + \cdots \qquad (2.37)$$

$$( \alpha > 0 )$$

where $\beta_n$ are spatial complex constants. Specifically, remember the fact that no constant term is contained in (2.37), though the shifting factor $z_o$ is, in fact, chosen in such a way.

For the moment, we shall interpret about $\beta_n$, on geometrical bases. Proceed with the argument, constraining all points on $\gamma$ or $\gamma_w$. The point $\xi$ in the $\xi$-plane draws the boundary curve with a motion of w on $\gamma_w$. Obviously, we can write $w = e^{i\psi}$. Imagine the equation obtained by dividing the both sides of (2.37) by $\alpha$ w. This resulting series may be regarded as a Fourier series with the initial term unity,- the left hand side exhibits the normalized amplitude of $\xi$ versus $\alpha$ w, although it may be a complex number, and the expansion coefficients $\beta_{n-1}/\alpha$ for the powers $e^{-in\psi}$ shows how much the normalized amplitude deviates from the unity. In fact, if $\beta_1 = \beta_2 = \ldots\ldots = 0$, $\gamma$ is a circle with radius $\alpha$, and if $\beta_2 = \beta_3 = \ldots\ldots = 0$, $\gamma$ is an ellipse, and in general, if an m-regular polygon is indicated, only $\beta_{(m-1+mn)}$ ( n = 0,1,2,... ) dominate.

Now, we turn to the main discourse. Let u be a corresponding point of $\eta$, in the w-plane. The $\eta$ is restricted to $\Gamma_c$, and u is to the corresponding contour $\gamma_{cw}$ ( see also Fig.2.4 ). Using the variables w,u, and considering the previous suggestion, we reach

$$u_o(\xi,\bar{\xi}) = \frac{1}{2\pi i}\oint_{\Gamma_c} \tilde{G}_o(\xi,\bar{\xi};\eta)\,\Psi(\eta)\,d\eta + \text{C.C.} \tag{2.38}$$

where

$$\tilde{G}_o(\xi,\bar{\xi};\eta) = \overline{\frac{1}{2\pi i}\oint_{\gamma_{cw}} \overline{G_o(\xi,\bar{\xi};\eta,\bar{\eta})}\,\frac{du}{u}} \tag{2.39}$$

$$G_o(\xi,\bar{\xi};\eta,\bar{\eta}) = -\pi j\, H_o^{(2)}\left(2\sqrt{(\xi-\eta)(\bar{\xi}-\bar{\eta})}\right) \tag{2.40}$$

and $H_o^{(2)} = J_o - jY_o$, as before ( be careful for "j", again ).
It is worth remarking that the integration of (2.39) is performed for eliminating the singularity at $\bar{\xi} = 0$ and is to be carried out in the following manners:

1) Among the variables $\eta$ , $\bar{\eta}$ of $G_o$, always devoting attention to $\bar{\eta}$ - which is a function of $\bar{u}$, because $\bar{\eta}$ ( or $\bar{u}$ ) is transferred to $\eta$ ( or u ) by taking the complex conjugate of $G_o$ (= $\bar{G}_o$ ),

2) Evaluating the residue of $\bar{G}_o$ i.e. the value of the constant term of $\bar{G}_o$ with respect to $\bar{u}$,

3) Taking the complex conjugate of the resulting value.

Consequently, the function $\tilde{G}_o$ can be regarded as the improved one of the function K.

The function $\Psi$ which has been regular in the exterior and has vanished at infinity, can be expressed by a series of $w^{-1}$. On the other hand, it may be convenient in computing the integral of (2.38), by the residue calculus, to multiply $\Psi$ by the regular function $d\xi/dw$ - which tends to a constant at infinity. Therefore, considering the above fact, we write

$$\Psi(\xi)\frac{d\xi}{dw} = \frac{a_o}{w} + \frac{a_1}{w^2} + \frac{a_2}{w^3} + \cdots \qquad (2.41)$$

where $a_1$, $a_2$, ... are spatial and temporal complex constants, but $a_o$, analogous to $\Psi_*(\infty)$, is , from the behavior of $\Psi(\xi)$ at infinity ( refer to (2.35) ),

$$a_o = \text{a real number with respect to } i \qquad (2.42)$$

Evidently, these coefficients contain all the informations of the boundary conditions, and in a manner of speaking, determining them is equivalent to knowing the field behavior. We shall avoid further discussions to describe the details in Chapter IV.

## Chapter III

## FIELD CONTINUATION

Discontinuous changes of geometric boundary structure provide at once necessity of the so-called " Field Continuation". Various mathematical means have been, or are yet, proposed and developed by many mathematicians and technicians [2,10,15,25–28,etc.], in particular, in the field of applied mathematics.

One may still be able to recognize their glorious contributions, in the texts, to radiation from waveguides, diaphragms or bifurcations in waveguides, diffraction or scattering by apertures, etc.. The most typical ones among them are: the variational methods [1,2,10] , the point matching methods[3], the methods of integral equations [15] , the Wiener-Hopf techniques [2,7,25] . It may, however, safely be said that actual demands exceed their abilities. In fact, those, except the last one, are based on the treatment with real variables, so that the " complication " of wave functions, probably, reduces their capabilities.

Fortunately, Section 2.1 suggests that the above difficulty may be overcome by dealing with the corresponding regular function- which is, in fact, with a simple form. The analytic continuation of regular functions signifies that two functions, both real and imaginary parts, are, at the same time, continuously connected. As to the field, this corresponds just to the fact that both tangential electric and magnetic field components must be smooth. Therefore, the use of regular functions facilitates the mathematical manipulation.

### 3.1 General Treatments

The general theory has been discussed in Section 2.1, and has been proceeded without any detailed explanation. We shall, in the present section, give more concrete expressions, restricting discussion to the field continuation.

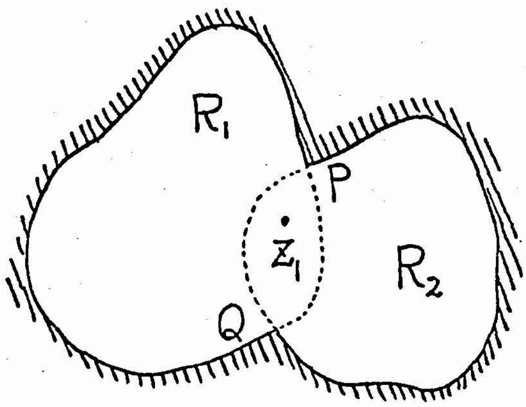



Fig.3.1

Field continuation in partially overlapped regions.

We first consider the two regions $R_1$ and $R_2$ shown in Fig.3.1. The circumferences cross each other at the points P and Q: i.e. $R_1$ and $R_2$ possess a common region. Let $u_1$ and $u_2$ be the fields in $R_1$ and $R_2$, respectively. Furthermore, assume that those have already satisfied any given conditions on the circumferences, except on the broken arcs P-Q -

where the boundary values are unknown.

Obviously, in the common region, $u_1$ and $u_2$ must be identical. Consequently, the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ are, if their reference points are situated at the same point $z_1$ in the common region, also identical; besides, the corresponding regular functions $\Phi_1$ and $\Phi_2$ are equal to each other. According to the theory of complex analysis [19], fortunately, identity of those functions in the neighborhood of $z_1$ establishes that in the whole region $R_1 + R_2$. On the other hand, in a small disk including $z_1$, $\Phi_1$ and $\Phi_2$ are given, from (2.24), by

$$\Phi_1(z) = u_1(z,\bar{z}_1) - \tfrac{1}{2}\, u_1(z_1,\bar{z}_1)$$

$$= \sum_{n=0}^{\infty} \frac{\epsilon_n c_n}{n!} (z - z_1)^n \qquad (3.1)$$

$$\Phi_2(z) = u_2(z,\bar{z}_1) - \tfrac{1}{2}\, u_2(z_1,\bar{z}_1)$$

$$= \sum_{n=0}^{\infty} \frac{\epsilon_n d_n}{n!} (z - z_1)^n \qquad (3.2)$$

where $\epsilon_0 = \frac{1}{2}$, $\epsilon_n = 1$ ( $n \geq 1$ ), and

$$c_n = \frac{\partial^n}{\partial z^n} u_1(z,\bar{z}_1) \Big|_{z=z_1}$$

$$d_n = \frac{\partial^n}{\partial z^n} u_2(z,\bar{z}_1) \Big|_{z=z_1}$$

$$( n = 0,1,2,\ldots )$$

Hence, the above $c_n$ and $d_n$ can be evaluated directly by differentiation, after achieving the complex variables representation for the given fields by substitution of $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/2i$ . Using them , one can attain the field continuation, in other words, the field matching , by

$$c_n = d_n \tag{3.3}$$

$$( n = 0,1,2,\ldots )$$

It must be confessed here that in usual cases, $c_n$ and $d_n$ involve unknown factors to be solved; for example,if $u_1$ and $u_2$ are given in each region by a modal expansion, those indicate the expansion coefficients.

In actual calculation, the number of the equations of (3.3) should be the same to that of the unknown factors.

Occasionally, it provides an algebraic equation– a simultaneous equation. It should, in addition, be remarked that one need not take account of usual two continuity conditions for tangential components.

Secondly, suppose that $R_1$ and $R_2$ have no common region but their circumferences are in contact with each other at the points P and Q ( see Fig.3.2 ). Also, establish the same assumptions with the previous case for $u_1, u_2, \Phi_1$, and $\Phi_2$. The field u exists in the whole region as assumed, and $\Phi_1$ or $\Phi_2$ can, in addition, be determined in the whole region, in view of analytic continuation. Therefore, $\mathcal{H}_1$ or $\mathcal{H}_2$ defined in $R_1$ or $R_2$ must be continued analytically in the whole region. It follows that the suffices of those operators represent only the reference points, but not the regions. This is the reason why the operation of $\mathcal{H}_1^{-1}$ to $u_1$ is possible– ;this question may arise in the following.

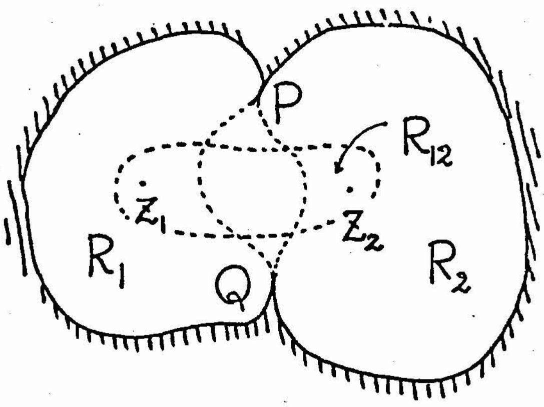



Fig.3.2

Field continuation in the region $R_{12}$ bridging the gap between $R_1$ and $R_2$ .

We shall define the function $\Phi_2^c$ in $R_{12}$ ( see Fig.3.2 ) as an analytically continued function to $\Phi_2$ . This could be achieved, as is well-known, by the idea of a chain of desks- each of them being the region of convergence of a Taylor series of the function [19]. If , in (2.13), $\Phi_2$ was replaced by $\Phi_2^C$, such an expression was valid in $R_{12}$. Obviously, such a field represents both $u_1$ and $u_2$ in the only $R_{12}$. We shall thus write it as $u_1^C$ in a sense of analytic continuation. The $u_1^C$ can, really, be obtained from (2.22):

$$u_1^C(z,\bar{z}) = \Phi_2^C(z) - \int_{z_2}^{z} \frac{\partial}{\partial t} J_0(2\sqrt{(\bar{z}-\bar{z}_2)(z-t)}\,)\, \Phi_2^C(t)dt + \text{c.c.} \qquad (3.4)$$

By transforming the above equation with $\mathcal{H}_1^{-1}$, we have a function of z associated with $u_1^C$ and at least regular at the point $z_1$. We shall repeat the fact $u_1^C = \mathcal{H}_2\, \Phi_2^C$. Accordingly, from (2.14), $\mathcal{H}_1^{-1}u_1^C = \mathcal{H}_1^{-1}\mathcal{H}_2\, \Phi_2^C = \mathcal{D}_{12}\, \Phi_2^C$. Compare this result with (2.15). Then, we have to recall the matter that the large domain $R_0$ defined in Section 2.1 stands comparison with the smaller domain $R_{12}$ in this case. It should be remarked that in the previous case, all the singularities were removed from $R_0$, so the expression of $\Phi_2$ defined in $R_2$ held in $R_1$, by virtue of analytic continuation,- that is to say, $\Phi_2$ was substituted for $\Phi_2^C$. In considering the definition (2.15) strictly, it would be reasonable to replace $\Phi_2$ in (2.15) by $\Phi_2^C$. Consequently, in the present case, $\mathcal{D}_{12}\,\Phi_2^C$ is $\tilde{\Phi}_2$ . This can be obtained actually by taking the inversion operation $\mathcal{H}_1^{-1}$: i.e. $u_1^C(z,\bar{z}_1) - \frac{1}{2}u_1^C(z_1,\bar{z}_1)$.

$$\tilde{\Phi}_2(z) = -\tilde{\Phi}_2(z_1)$$
$$+\ \Phi_2^C(z) - \int_{z_2}^{z} \frac{\partial}{\partial t} J_o\left(2\sqrt{(\bar{z}_1-\bar{z}_2)(z-t)}\right) \Phi_2^C(t)dt$$
$$+\ \overline{\Phi_2^C(z_1) - \int_{\bar{z}_2}^{\bar{z}_1} \frac{\partial}{\partial \bar{t}} J_o\left(2\sqrt{(z-z_2)(\bar{z}_1-\bar{t})}\right) \Phi_2^C(t)d\bar{t}}$$

(3.5)

where

$$\tilde{\Phi}_2(z_1) = \Phi_2^C(z_1) - \int_{z_2}^{z_1} \frac{\partial}{\partial t} J_o\left(2\sqrt{(\bar{z}_1-\bar{z}_2)(z_1-t)}\right) \Phi_2^C(t)dt$$

$$= \frac{1}{2} u_1(z_1,\bar{z}_1)$$

= a real number with respect to i (3.6)

Equations (3.5)(3.6) can be carried out, at least, by a direct computation, with the use of Bessel's identity. In short, $\tilde{\Phi}_2$ obtained by a possible mean must be expressed by

$$\tilde{\Phi}_2(z) = \sum_{n=0}^{\infty} \frac{\epsilon_n\, e_n}{n!}(z-z_1)^n \qquad (3.7)$$

Clearly, the equality between $\tilde{\Phi}_2$ and $\Phi_1$ leads to

$$c_n = e_n \qquad (3.8)$$

$$n = 0,1,2,\ldots$$

It is needless to say that this is the field continuation condition.

## 3.2 Application to the E plane Taper at the Junction

The theory described in the previous section may be somewhat vague to imagine its substance. It is now our task to clarify the concept furthermore with application to a simple structure of boundary. This application is limited to the geometrical configuration of Fig.3.2. Its interpretation will cover that of the case of Fig.3.1.

Figure 3.3 shows the two dimensional E plane taper connected with a parallel plate waveguide. All the quantities written in the Figure are, as a matter of course, normalized ones. The O is an origin, and the O" is a point of intersection to the x axis and the extension line of a taper plane. The O' is restricted to the x axis and also lies in the sickle-shaped part with two arcs- the one being a part of the circle with the center O" ( the radius $\frac{1}{2}b$ cosec $\theta_o$ ), the other being a part of the circle with the center O ( the radius $\frac{1}{2}b$ ).

We now define the region $x \leq 0$ as " Waveguide Region ". Both the field and the corresponding regular function in this region are distinguished from the others with use of the subscript "W". While, the region $\rho'' \geq \frac{1}{2}b$ cosec $\theta_o$ is named " Taper Region", and the corresponding subscript is "T". The O and O' are reference points pertaining to these regions.

Suppose, a fundamental mode with the magnitude unity is incident from the left side, and thereby radiation occurs in the taper region. We write the fields approximately, using large integers $N_W$ and $N_T$, as

$$u_W = e^{-\gamma_0 x} + \sum_{\ell=0}^{N_W} R_\ell \cos \frac{2\ell\pi}{b} y \; e^{\gamma_\ell x}$$

$$( x \leq 0 ) \qquad (3.9)$$

$$u_T = \sum_{\ell=0}^{N_T} T_\ell \, H^{(2)}_{\nu_\ell}(2\rho'') \cos \nu_\ell \theta''$$

$$( \rho'' \geq \tfrac{1}{2}b \text{ cosec } \theta_o ) \qquad (3.10)$$

where $\gamma_\ell^2 = (2\ell\pi/b)^2 - 4, \quad \gamma_0 = 2j, \quad \nu_\ell = \ell\pi/\theta_o$ .

We shall consider, in mind, that the waveguide region and the taper region correspond to $R_1$ and $R_2$ in the previous theory, respectively. Namely, $z_1 = 0$ and $z_2 = d$ . The corresponding regular functions $\Phi_W$ and $\Phi_T$ are, therefore, identical to $\Phi_1$ and $\Phi_2$, respectively. Thus, from (3.1), we have

$$\Phi_W(z) = e^{-\frac{1}{2}\gamma_0 z} - \frac{1}{2} + \sum_{\ell=0}^{N_W} R_\ell \left\{ \cosh \frac{\ell\pi}{b} z \;\; e^{\frac{1}{2}\gamma_\ell z} - \frac{1}{2} \right\}$$

$$= \sum_{m=0}^{\infty} \frac{\epsilon_m c_m}{m!} z^m \qquad (3.11)$$

where

$$c_m = (-\tfrac{1}{2}\gamma_0)^m + \tfrac{1}{2} \sum_{\ell=0}^{N_W} R_\ell \left\{ (\tfrac{1}{2}\gamma_\ell + \frac{\ell\pi}{b})^m + (\tfrac{1}{2}\gamma_\ell - \frac{\ell\pi}{b})^m \right\}$$

$$(3.12)$$

It should be emphasized that the radius of convergence of the series (3.11) is unknown, but exists surely, may be extremely small, by virtue of the choice of the finite number $N_W$. With increasing $N_W$, the radius probably becomes small gradually, but never vanishes. The existence sustained supports a possibility of the " regular function matching " at the origin. If, instead, infinite numbers were taken at the beginning, matters would be different at all.

Let us next evaluate $\Phi_T$ as a series about O'. Apply the addition formulae of the Hankel functions [16]

$$H^{(2)}_{\nu_\ell}(2\rho'')\cos\nu_\ell\theta'' = \sum_{m=-\infty}^{\infty} H^{(2)}_{\nu_\ell-m}(2d')J_m(2\rho')\cos m\theta'$$

$$( d' = \tfrac{1}{2}b\cot\theta_o + d )$$

to (3.10) and expand $u_T$ around O' in a sense of a series of the Bessel functions. Further, represent the result as a function of complex variables z' ($=\rho' e^{i\theta'} = z-d$ ) and $\bar{z}'$, using

$$\rho' = \sqrt{z'\bar{z}'}$$

$$\cos m\theta' = \tfrac{1}{2}\left\{\sqrt{(z'/\bar{z}')^m} + \sqrt{(\bar{z}'/z')^m}\right\}$$

In the same manner to (3.11), on putting $\bar{z}' = 0$ , after all, we obtain

$$\Phi_T(z') = \sum_{m=0}^{\infty} \frac{\epsilon_m d_m}{m!} z'^m \quad ,( z' = z-d ) \qquad (3.13)$$

where

$$d_m = \frac{1}{2} \sum_{n=0}^{N_T} T_n \left\{ H^{(2)}_{\gamma_n - m}(2d') + (-1)^m H^{(2)}_{\gamma_n + m}(2d') \right\} \tag{3.14}$$

Questions regarding the convergence of (3.13) which ought to occur here will be solved immediately by thinking of the following statements: 1) as mentioned before, there must exist the corresponding regular function in the whole " guiding region "; 2) the regular function possesses a Taylor expansion whose circle of convergence is determined by the position of the nearest singular point of the function [19] ,- indeed, it lies in the exceptional region. These facts allow us to extend the circle of convergence of (3.13) at least up to the inscribed circle of the broken parallel plates. It will be clearly understood on geometrical situations that such a circle contains the point O. This means that (3.13) is also valid at O. Accordingly, we need not take any artificial complication occurring in computing the function $\Phi^C_T$ analytically continued from O' to O. Therefore, by substituting (3.13) instead of $\Phi^C_2$ into (3.5)(3.6), we obtain $\tilde{\Phi}_T$ ($\equiv \tilde{\Phi}_2$ ). The integrations involved is, this time expressing the Bessel function $J_o$ as a series, carried out term by term, and afterwords the result is rearranged, with the use of the Bessel's identity, such that

$$\tilde{\Phi}_T(z) = \sum_{m=0}^{\infty} \frac{\epsilon_m e_m}{m!} z^m \tag{3.15}$$

where

$$e_m = \sum_{n=0}^{\infty} \epsilon_n \left\{ J_{m-n}(2d) + (-1)^n J_{m+n}(2d) \right\} d_n \tag{3.16}$$

It should be kept in mind that the number of the unknown coefficients is $N_W + N_T + 2$ . All of them are involved in $c_m$ and $e_m$, and are determined by the same number of equations - say, the algebraic equation being obtained by substituting $c_m$ and $e_m$ prepared above into (3.8).

$$\frac{1}{2} \sum_{\ell=0}^{N_T} T_\ell \sum_{n=0}^{\infty} \epsilon_n \left\{ J_{m-n} + (-1)^n J_{m+n} \right\} \left\{ H_{\gamma_\ell - n} + (-1)^n H_{\gamma_\ell + n} \right\}$$

$$- \frac{1}{2} \sum_{\ell=0}^{N_W} R_\ell \left\{ \left(\frac{1}{2}\gamma_\ell + \frac{\ell\pi}{b}\right)^m + \left(\frac{1}{2}\gamma_\ell - \frac{\ell\pi}{b}\right)^m \right\}$$

$$= \left(-\frac{1}{2}\gamma_0\right)^m \tag{3.17}$$

$$m = 0, 1, 2, \ldots, N_W + N_T + 1$$

where $J_n$ and $H_\gamma$ denote $J_n(2d)$ and $H_\gamma^{(2)}(2d')$, respectively. This simultaneous equation will be solved with actually using an inversion process. We shall now assume that $\theta_o$ is small i.e. the incident field almost passes through and thereby only $T_o$, $R_o$, and $R_1$ dominate. This assumption, then reducing (3.17) to a set of 3 equations, will make it possible to evaluate the unknowns by hand. The result is

$$T_o = \frac{1}{\sum \epsilon_n H_n \left\{ (J_n - jJ_n') - j\left(\frac{b}{2\pi}\right)^2 \left(\frac{\gamma_1 - \gamma_0}{2}\right)(J_{n+2} + 2J_n + J_{n-2}) \right\}}$$

$$R_o = T_o \sum \epsilon_n H_n \left\{ (J_n + jJ_n') + j\left(\frac{b}{2\pi}\right)^2 \left(\frac{\gamma_1 + \gamma_0}{2}\right)(J_{n+2} + 2J_n + J_{n-2}) \right\}$$

$$R_1 = 2T_o \left(\frac{b}{2\pi}\right)^2 \sum \epsilon_n H_n \quad (J_{n+2} + 2J_n + J_{n-2})$$

(3.18)

where $J_n'$ is the derivative of $J_n$ and $\sum$ denotes $\sum_{n=0}^{\infty}$ .

The assumption of a small $\theta_o$ is equivalent to taking a large d'. The value of d is, in consequence, permitted to be small. Therefore, with approximation of

$$J_o(2d) \simeq 1, \; J_{\pm 1}(2d) \simeq \pm d, \; J_n(2d) \simeq 0 \text{ otherwise,}$$

$$H_\gamma^{(2)}(2d') \simeq \frac{1}{\sqrt{\pi d'}} \left\{ 1 - j\frac{(4\gamma^2 - 1)}{16d'} \right\} e^{-j(2d' - \frac{1}{4}\pi - \frac{1}{2}\pi\gamma)}$$

we have

$$T_o \simeq \frac{1}{H_o^{(2)}(2d')}\left\{1 - j2d + \frac{j}{8d'}\left[1 + j\left(\frac{b}{\pi}\right)^2(\gamma_1 - \gamma_0)\right]\right\}$$

$$R_o \simeq \frac{j}{8d'}\left\{1 + j\left(\frac{b}{\pi}\right)^2(\gamma_1 + \gamma_0)\right\}$$

$$R_1 \simeq \frac{j}{2d'}\left(\frac{b}{\pi}\right)^2 \tag{3.19}$$

If the term $(b/\pi)^2$ is ignored in view of more rough approximation, (3.19) coincides with the Lewin's result [15] .
Finally, it should be emphasized again that in (3.17), no integral term is contained. This fact is desired in order that simultaneous equations for field matching may be handled ingeniously in much more complicated cases, rather than being manipulated in simpler cases in which conventional techniques are available.

# Chapter IV

# APPLICATION TO SCATTERING AND WAVEGUIDE PROBLEMS

Field scatterings have ever been studied since the discovery of " waves ". Physical insight also has been provided into the behavior of scattering strongly depending on the shape of the scatterer, together with a complete comprehension of the physical meaning. They have been analyzed as boundary value problems, but individually and differently according to the boundary shapes.

Recently, we have the tendency to force the mathematical situation to fit the " arbitrariness " of the boundary shape, with the progress of machine computing techniques [4,5,6,29,30]. One should desire it to be so even if the development of computers was disturbed. It would seem that this unification could be found faintly in the variational methods [1,2,10]. Certainly, it is achieved in the expression and there is no question in itself. However, on the stage of performance of the calculation, a fatal difficulty- that is to say, the disposal of complicated integral terms involved , dependent on the choice of trial functions- may arise.

The aim of this chapter is to show that use of the field description in Section 2.4 makes us unify the manifold scattering analyses for boundary shapes. First, the field scattering by a perfectly conducting scatterer with arbitrary cross section is analyzed in free space. Secondly, the same thing is done in a rectangular waveguide. For the sake of simplicity, the boundary size of the scatterers is assumed to be small. In fact, the E field scattering is approximated within the range of accuracy of order 2 with respect to the boundary size, and the H field one is within the range of accuracy of order 4.

Throughout this chapter, the auxiliary equations derived in the way of mathematical deformation of equations are listed up together in the rear of this thesis. To those equation numbers, the "A" is affixed.

## 4.1 Scattering Problems

In this section, a monochromatic, two dimensional scattering in free space is dealt with under the assumption of a perfectly conducting, small scatterer. The E field scattering is described in the first part, and the H field one is in the later part.

We shall imagine the boundary curve $\gamma$ of cross section of a scatterer ( obstacle ) shown in Fig.2.3, and devote our attention to the manipulation of (2.38). Then, we assume that $u_o$ is a Z component of the scattered field, and the wave number k is

$$k = \sqrt{k_o^2 - \beta^2} \tag{4.1}$$

( refer to Page 5 ). It goes without saying that all the quantities associated with the coordinates are normalized with this value.

We shall first describe the scattered field in a far zone; the modulus $|\xi|$ is larger than $|\eta|$ .

To this purpose, replace in the mathematical formula (IV-1) ( Appendix IV ) a,z by $-\eta/\xi$ , $2\sqrt{\bar{\xi}(\xi-\eta)}$, and in (IV-2) z, $\zeta$ by $4\,\xi\bar{\xi}$, $-4\,\bar{\xi}\eta$ , respectively. From both the resulting ones, we have

$$H_o^{(2)}\left(2\sqrt{(\xi-\eta)(\bar{\xi}-\bar{\eta})}\right)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^m}{m!\;\;n!}\,\eta^m\,\bar{\eta}^n \left(\frac{\xi}{\bar{\xi}}\right)^{\frac{n-m}{2}} H_{n-m}^{(2)}\left(2\sqrt{\xi\bar{\xi}}\right) \tag{4.2}$$

The $G_o$ is obtained by multiplying the above one by $-\pi j$ . It is permissible to regard $\eta$ as a point near $\Upsilon$ and therefore as a small value in the modulus. We shall here approximate $G_o$ with an accuracy of order 3 regarding the boundary size. This will enable us, in (4.2), to take account of only four terms in each sum. The powers of $\eta$ can, from (2.37), be expressed by

$$\begin{aligned}
\eta^0 &= 1 \\
\eta &= \alpha u + \ldots. \\
\eta^2 &= \alpha^2 u^2 + 2\alpha\beta_1 + \ldots.. \\
\eta^3 &= \alpha^3 u^3 + 3\alpha^2\beta_1 u + 3\alpha^2\beta_2 + \ldots.
\end{aligned} \tag{4.3}$$

where the above omitted terms consist of the powers of $u^{-1}$ and those do not contribute to the following calculations.

Equation (2.39) says, according to the residue theorem, that if $G_o$ is expanded into a series of $\bar{u}$ , the value of the constant term ( the initial term ) of this series just becomes $\tilde{G}_o$ . Therefore, substituting the complex conjugate of (4.3) into the terms $\bar{\eta}^n$ of (4.2), rearranging to a series of $\bar{u}$, and multiplying the constant term by $-\pi j$, we have

$$\tilde{G}_o(\xi,\bar{\xi};\eta) = -\pi j \sum_{m=0}^{\infty} \frac{\eta^m}{m!} \Bigg\{ \left(\frac{\bar{\xi}}{\xi}\right)^{\frac{1}{2}m} H_m^{(2)}(2\sqrt{\xi\bar{\xi}})$$

$$+ \frac{2\alpha\bar{\beta}_1}{2!} \left(\frac{\bar{\xi}}{\xi}\right)^{\frac{m-2}{2}} H_{m-2}^{(2)}(2\sqrt{\xi\bar{\xi}})$$

$$- \frac{3\alpha^2\bar{\beta}_2}{3!} \left(\frac{\bar{\xi}}{\xi}\right)^{\frac{m-3}{2}} H_{m-3}^{(2)}(2\sqrt{\xi\bar{\xi}}) + \ldots \Bigg\} \tag{4.4}$$

It should be recalled that the quantities $\alpha$ , $\beta_n$, and $\eta$ are of order 1 . Further, calculate (2.38) in a similar manner and within the same accuracy. Note that $\Psi(\eta)d\eta$ is replaced by

$$( a_o/u + a_1/u^2 + a_2/u^3 + \ldots\ldots )du$$

( refer to (2.41) ). Of course, the contour $\gamma_c$ also must be replaced by $\gamma_{cw}$ . Computation of the residue of the resulting integrand ( the coefficient of the term $u^{-1}$ ) leads to the far field $u_o$:

$$\begin{aligned} u_o(\xi,\bar{\xi}) = & -\pi j\, H_o^{(2)}(2\rho)\, a_o \\ & -\pi j\, H_1^{(2)}(2\rho)\, e^{-i\theta}( \alpha a_1 - \alpha^2 \beta_1 \bar{a}_1 ) \\ & -\pi j\, H_2^{(2)}(2\rho)\, e^{-2i\theta}( 2\alpha \beta_1 a_o + \alpha^2 a_2/2 ) \\ & -\pi j\, H_3^{(2)}(2\rho)\, e^{-3i\theta}( \alpha^2 \beta_2 a_o + \alpha^2 \beta_1 a_1/2 + \alpha^3 a_3/6 ) \\ & + \text{C.C.} \end{aligned} \tag{4.5}$$

where $a_o$ is a real number, as assumed before, and $\xi = \rho e^{i\theta}$ ( this differs from the definition in Section 2.3 ).

Evidently, this represents an outgoing wave. Needless to say, the positive angular dependence of $u_o$ i.e. the behavior with $e^{i\theta}$, $e^{2i\theta}$,... , is involved in the term C.C. . It should be remarked that $a_n$ ( $n \geq 4$ ) have no contribution to the far field of order 3 -, indeed this number "3" may be able to be altered to "4" or "5" when the order of $a_n$ is taken into account ( this is touched later in the case of the H field scattering ).

To determine the coefficients $a_n$ from the near field, evaluate $G_o$ in the neighborhood of $\Gamma$ with the mathematical formula (IV-3) ( $x \longrightarrow 2\sqrt{(\xi-\eta)(\bar{\xi}-\bar{\eta})}$ ). The $\widetilde{G}_o$ is given by the value of the constant term of $G_o$ with respect to $\bar{u}$, as mentioned before:( see Appendix III )

$$\widetilde{G}_o(\xi,\bar{\xi};\eta) =$$

$$-\left\{\pi j + 2C + \log(\xi-\eta) + \log(\alpha\bar{w})\right\}\left\{1 - \bar{\xi}(\xi-\eta)\right\}$$
$$-(\xi-\eta)(\alpha\bar{w} + \bar{\xi} - \bar{\beta}_1/\bar{w} - \bar{\beta}_2/2\bar{w}^2 - \bar{\beta}_3/3\bar{w}^3 - \dots ) \tag{4.6}$$

The (4.6) is expressible in terms of u. Substitute it into (2.38). The integration can be carried out in the same manner with the previous one. We write the result ( the near field ) only :

$$u_o(\xi,\bar{\xi}) =$$

$$- a_o \Big\{ (1 - \xi\bar{\xi})(\pi j + 2C + 2\log\alpha + 2\log|w|) + 2\bar{\xi}(\alpha w - \beta_1/w - \beta_2/2w^2 - \beta_3/3w^3 - \dots) \Big\}$$
$$+ a_1 \Big\{ (1 - \xi\bar{\xi})/w - \alpha\bar{\xi}(\pi j + 2C - 1 + 2\log\alpha + 2\log|w|) + \alpha(\alpha\bar{w} - \bar{\beta}_1/\bar{w} - \bar{\beta}_2/2\bar{w}^2 - \bar{\beta}_3/3\bar{w}^3 - \dots) + \bar{\xi}(\beta_1/2w^2 + \beta_2/3w^3 + \dots) \Big\}$$
$$+ a_2 \Big\{ (1 - \xi\bar{\xi})/2w^2 + \bar{\xi}(\alpha/w + \beta_1/3w^3 + \beta_2/4w^4 + \dots) \Big\}$$
$$+ a_3 \Big\{ (1 - \xi\bar{\xi})/3w^3 + \bar{\xi}(\alpha/2w^2 + \beta_1/4w^4 + \beta_2/5w^5 + \dots) \Big\}$$
$$+ \dots\dots + C.C. \qquad (4.7)$$

It should be kept in mind that since (IV-3) is expanded in order 3, the approximation of (4.7) is also retained in this order, - although it is valid under the assumption that the orders of $a_n$ are not taken into account i.e. they are of order zero.

We have not touched so far on the orders of $a_n$. The $a_n$ will, in fact, be determined by the boundary conditions and therefore contain the " information " of the boundary size. If all $a_n$ retain their magnitudes as the size decreases, in other words, those are of order zero, then no question arises in the above development. However, if all $a_n$ have the orders at least larger than zero, matters will be altered: i.e. the accuracy of the preceding approximation will be improved.

To go on with this consideration, we shall write $a_n$ as follows:

$$a_n = a_n^{(0)} + a_n^{(1)} + a_n^{(2)} + \dots \qquad (4.8)$$
$$n = 0,1,2,\dots.$$

where the shoulder number of $a_n^{(m)}$ indicates the order number, - the term being of order m .

Henceforth, we shall restrict the analysis to the E scattered field. Let $f(\xi,\bar{\xi})$ be the boundary value of $u_o$. Obviously, we can represent it as a function of w and $\bar{w}$ . At this time, as a matter of course, w is $e^{i\psi}$ and $\bar{w}$ is $e^{-i\psi}$ . Accordingly, we shall, for simplicity, rewrite $f(\xi,\bar{\xi})$ as $f(\psi)$.
The boundary value is always expressible in the following form:

$$f(\psi) = f_o + f_1 w + f_2 w^2 + \dots \quad + \text{C.C.} \qquad (4.9)$$
$$( w = e^{i\psi} )$$

The imaginary part of $f_o$ is meaningless, so that for convenience, $f_o$ is assumed to be a real number ( with respect to i ).
We shall also write $f_n$ ( given coefficients ) in the same fashion

with (4.8):

$$f_n = f_n^{(0)} + f_n^{(1)} + f_n^{(2)} + \dots \qquad (4.10)$$

$$n = 0,1,2,\dots .$$

Here, it is assumed for $n = 0$ that all of $f_o^{(m)}$ are real numbers individually.

Since the obstacle is a perfect conductor, the boundary value $f(\psi)$ is given by the negative value of the incident field $u_o^i$ . Further, $f(\psi)$ can be expressed by a Taylor expansion:

$$- u_o^i(0,0) - \left[ \frac{\partial u_o^i(\xi,0)}{\partial \xi} \right]_{\xi=0} \xi \; - \; \left[ \frac{\partial u_o^i(0,\bar{\xi})}{\partial \bar{\xi}} \right]_{\bar{\xi}=0} \bar{\xi} \; - \dots .$$

The first term is a constant of order zero, and the second and third terms are of order 1 . We should recall that evidently, from (2.37), $\xi$ and $\bar{\xi}$ have no constant term. These facts lead at once to

$$f_n^{(0)} = 0 \;\; ( n \geq 1 ), \;\; f_o^{(1)} = 0 \qquad (4.11)$$

One will take notice of the fact that these relations relieve the complication of equations. We shall explain this in detail. To this purpose, we first consider the zero order approximation: $a_n$ and $f_n$ are approximated only by $a_n^{(0)}$ and $f_n^{(0)}$, respectively. In addition, the higher order terms of (4.7) are removed. As the result, (4.7) is expressed simply in the zero order form. ( Keep in mind that $\alpha$, $\beta_n$, $\xi$ are of order 1, but w, $\log\alpha$ , $\log|w|$ are of order zero ! ) Such an expression is related on $\gamma$ to the same order boundary-value:

$$f_o^{(0)} + f_1^{(0)} w + f_2^{(0)} w^2 + \ldots + C.C.$$

$$= -(\pi j + 2C + 2\log\alpha)\, a_o^{(0)} + \bar{a}_1^{(0)} w + \bar{a}_2^{(0)} w^2/2 + \ldots + C.C. \qquad (4.12)$$

where, of course, $w = e^{i\psi}$, $|w| = 1$, and the bar denotes the complex conjugate the same as the C.C. does. The (4.12) can be regarded as a Fourier expansion in the interval $[0, 2\pi]$,– one cycle of the polar coordinate in the w-plane. Thus, by virtue of (4.11),

$$a_o^{(0)} = -f_o^{(0)}/(\pi j + 2C + 2\log\alpha)$$

$$a_n^{(0)} = 0 \quad (n \geq 1) \qquad (4.13)$$

It is seen with substitution of (4.13) into (4.7) that the lowest (zero) order term is involved in the only first brace and the remains are of the orders at least larger than zero. In a similar manner, we shall pick out the first order field from (4.7), considering (4.13). Notice that the same neglect of the higher order terms in each brace is still performed. Accordingly, we have, from (4.11),

$$a_o^{(1)} = 0$$
$$a_n^{(1)} = n\,\bar{f}_n^{(1)} \quad ( n \geq 1 ) \qquad (4.14)$$

Equations (4.13)(4.14) permit us, in order 2, to ignore, in each brace of (4.7) except for the first one, the second order terms. In short, the second order field exhibits rather a simple feature. For convenience of the calculation of the second order terms involved in the first brace of (4.7), we shall introduce the second order quantities $P_n$ and $Q_n$ ( see (A-1)(A-2) ).

It is seen easily that the second order field contains the zero order coefficient $a_o^{(0)}$. On the boundary, we have

$$f_o^{(2)} + f_1^{(2)} w + f_2^{(2)} w^2 + \dots + C.C.$$

$$= \quad - ( \pi j + 2C + 2 \log \alpha ) \, a_o^{(2)}$$

$$+ 2 \Big\{ ( \pi j + 2C + 2 \log\alpha )( P_o/2 + P_1 w + P_2 w^2 + \dots ) + Q_o + Q_1 w + Q_2 w^2 + \dots \Big\} a_o^{(0)}$$
$$+ \bar{a}_1^{(2)} w + \bar{a}_2^{(2)} w^2/2 + \bar{a}_3^{(2)} w^3/3 + \dots$$
$$+ C.C. \tag{4.15}$$

By substituting (4.13) for $a_o^{(0)}$, we get

$$a_o^{(2)} = - \Big\{ f_o^{(2)} + P_o f_o^{(0)} + 2 Q_o f_o^{(0)} / ( \pi j + 2C + 2 \log\alpha ) \Big\} / ( \pi j + 2C + 2 \log\alpha )$$
$$a_n^{(2)} = n \bar{f}_n^{(2)} + 2 n \bar{P}_n f_o^{(0)} + 2 n \bar{Q}_n f_o^{(0)} / ( \pi j + 2C + 2 \log\alpha )$$
$$( n \geq 1 ) \tag{4.16}$$

where $P_n$ and $Q_n$ are defined in (A-1) and (A-2), respectively. With use of (4.13)(4.14)(4.16), the far field of order 2 can be written as

$$u_o(\xi, \bar{\xi}) = -\pi j H_o^{(2)}(2\rho)( a_o^{(0)} + a_o^{(2)} )$$
$$-\pi j H_1^{(2)}(2\rho) e^{-i\theta} \alpha a_1^{(1)}$$
$$-\pi j H_2^{(2)}(2\rho) e^{-2i\theta} 2 \alpha \beta_1 a_o^{(0)}$$
$$+ C.C. \tag{4.17}$$

If we give the constants $\alpha$ and $\beta_n$ instead of indication of the boundary shape, and express the boundary-value as the fashion of (4.9) with knowledge of the incident field, we can obtain the above coefficients with a straightfoward computation.

As seen, $f_o^{(0)}$ is the half of the negative value of the incident field at the origin. In a zero order approximation, $f(\psi)$ is $2 f_o^{(0)}$. Then, the value of $\alpha$ is very small, so that $a_o^{(0)}$ can, furthermore, be approximated by $-f_o^{(0)}/ 2 \log\alpha$ and the other coefficients are negligible. The zero order far field, therefore, results in the expression of quasi-static approximation [2,31,32]. We shall give physical interpretations a few more to (4.17). The (4.17) is separated into three parts. The first part is a quasi-static field and its correction term. The second part shows a dipole field produced by a uniform gradient of the incident field ( a constant transverse electric field ). The third one is a quadrupole field,- indeed, which arises in order that the field induced by the only monopole source may be corrected on the boundary surface,- the determination of the magnitude is therefore independent of that of the dipole field strength, and this source is induced only when the boundary curve is distorted from a circle, more strictly speaking, $\beta_1$ does not vanish.

We shall next focus our attention to the H field scattering. Henceforth, regard $u_o$ as a Z component of the H scattered field ( = $H_z$ ). It may be convenient in the following to define a complex normal in the $\xi$-plane, such as the $\mathfrak{n}_z$ in (2.10):

$$\mathfrak{n}_\xi = n_{\xi x} + i\, n_{\xi y} \tag{4.18}$$

where $n_{\xi x}$ and $n_{\xi y}$ are two fundamental components of the outward normal to $\gamma$ in the $\xi$-plane. It is readily seen from (2.4) that the complex representation of the normal derivative is

$$\frac{\partial}{\partial n} = \mathfrak{n}_\xi \frac{\partial}{\partial \xi} + \bar{\mathfrak{n}}_\xi \frac{\partial}{\partial \bar{\xi}} \tag{4.19}$$

On the one hand, the small displacement $\Delta\xi$ restricted to $\gamma$ indicates the tangential direction. Thus, $\mathfrak{n}_\xi = -i\Delta\xi / |\Delta\xi|$. On the other hand, the corresponding displacement $\Delta w$ in the w-plane is $i\,w\,|\Delta w|$. By substitution, we find $\mathfrak{n}_\xi = (w\Delta\xi/\Delta w) \cdot |\Delta w/\Delta\xi|$. Therefore, in the limit,

$$\mathfrak{n}_\xi = w \frac{d\xi}{dw} \left| \frac{dw}{d\xi} \right|$$

Employment of the polar coordinate $\underline{w = \rho_w e^{i\psi}}$ yields

$$\left|\frac{d\xi}{dw}\right| \frac{\partial}{\partial n} = w \frac{\partial}{\partial w} + \bar{w} \frac{\partial}{\partial \bar{w}}$$

$$= \rho_w \frac{\partial}{\partial \rho_w} \Bigg|_{\rho_w = 1} \qquad (4.20)$$

As a matter of course, (4.20) is defined on $\gamma_{cw}$ .

A glance of (4.20) shows that the normal derivative in the $\xi$-plane is transformed into that in the w-plane,- being familiar as a polar coordinate. If we remove the restriction $\rho_w = 1$ , (4.20) will be extended to the whole outer region. The differentiation with respect to $\rho_w$ is associated with that along the curve in the $\xi$ -plane corresponding to the radial line in the w-plane. The $\partial u_o/\partial n$ at an arbitrary point is therefore related to the tangential electric field on the closed curve which passes through this point and is perpendicular to the corr-esponding radial line. We should bring to mind the fact that the $|dw/d\xi|$ is a weight function for the transformation and contains, if the boundary possesses edges, the properties of singularities of static fields at these points [2]. The behaviors of time varying fields at the edges are the same as those of static fields [25] . Accordingly, the extended normal derivative (4.20),- being divided by this factor,- possesses no singularity, i.e., it is a smooth derivative. This fact may be advantageous in analytical viewpoints. Furthermore, this extended normal derivative makes it possible to let $\rho_w$ approach unity after the handling of equations at a far point. This is just an analytic continuation. This manipulation will make us avoid mathematical, troublesome questions e.g. convergence of the series.

Let us differentiate (4.7) with respect to $\rho_w$. In practice, it is carried out with use of the first operator written at the right hand side of (4.20). It is noteworthy that the term C.C. contains not only $\bar{w}$, but also w. For convenience, we shall introduce the second order quantities $I_n$ and $K_n$ ( see (A-5) ). Using them, we obtain

$$\begin{aligned}\rho_w \frac{\partial u_o}{\partial \rho_w} = & - a_o \Big\{ 2 - ( \pi j + 2C + 2 \log\alpha )( I_o \\ & + \bar{I}_o ) - I_1 - \bar{I}_1 \Big\} \\ & - a_1 \Big\{ w^{-1} + \alpha \bar{w} ( d \xi / d\bar{w} )( \pi j \\ & + 2C - 1 + 2 \log\alpha ) + ( I_o + \bar{I}_o )/w \\ & - K_o \Big\} \\ & - a_2 \Big\{ w^{-2} + ( I_o + \bar{I}_o )/2w^2 - K_1 \Big\} \\ & - a_3 \Big\{ w^{-3} + ( I_o + \bar{I}_o )/3w^3 - K_2 \Big\} \\ & - a_4 \Big\{ w^{-4} + ( I_o + \bar{I}_o )/4w^4 - K_3 \Big\} \\ & - \ldots\ldots + C.C. \end{aligned} \tag{4.21}$$

The (4.21) exhibits, in the limit of $\rho_w = 1$, a Fourier series. The $I_n$ and $K_n$, at this time, can be expressed by a series of $e^{i\psi}$ ( $= w$ ) and $e^{-i\psi}$ ( $= \bar{w}$ ) ( see (A-6) ). We shall turn to (4.21), again. The first term in each brace is of order zero, and the rests are of order 2. The determination of the coefficients are achieved by the similar manner to the one for the E field scattering. Specifically, we write the boundary-value of $\partial u_o/\partial \rho_w$ as

$$g(\psi) = -\frac{\partial u_o^i}{\partial \rho_w}$$

$$= g_o + g_1 w + g_2 w^2 + \dots + C.C. \qquad (4.22)$$

$$( w = e^{i\psi} )$$

Also,

$$g_n = g_n^{(0)} + g_n^{(1)} + g_n^{(2)} + \dots \qquad (4.23)$$

$$n = o,1,2,\dots$$

where $g_o^{(m)}$ is a real number, for all m, with respect to i.

We shall imagine a Taylor expansion of the incident field $u_o^i$ ( refer to Page 54, the middle ), and remember the fact that the order of (4.20) is zero. It will be found that with operation of (4.20), the constant term being of order zero vanishes and the first order terms behave with $d\xi/dw$ and $d\bar{\xi}/d\bar{w}$ . However, $w\,(d\xi/dw)$,- which is one part of the first order boundary-value,- does not contain a constant term in itself. Accordingly, from both the facts,

$$g_n^{(0)} = 0 \qquad \text{for all } n \tag{4.24}$$

$$g_o^{(1)} = 0 \tag{4.25}$$

The (4.24) suggests that the order of the boundary-value is at least equal to, or larger than, unity. The $a_n$ should have, therefore, the same order:

$$a_n^{(0)} = 0 \qquad \text{for all } n \tag{4.26}$$

This will be used in the following without specification.

Substitute (4.8) into (4.21) and rearrange it with respect to the orders. We shall omit the details. To order 1, the following equality is established on $\Upsilon$ :

$$g_o^{(1)} + g_1^{(1)}w + g_2^{(1)}w^2 + \ldots\ldots + \text{C.C.}$$

$$= -2\,a_o^{(1)} - \bar{a}_1^{(1)}w - \bar{a}_2^{(1)}w^2 - \ldots + \text{C.C.} \tag{4.27}$$

$$( w = e^{i\psi} )$$

Thus, with (4.25),

$$a_o^{(1)} = 0 \tag{4.28}$$

$$\overline{a_n^{(1)}} = - \bar{g}_n^{(1)} \quad ( n \geq 1 ) \tag{4.29}$$

To order 2, we find the same fashion with (4.27), permitting difference of the superscript-numbers. Thus,

$$\begin{aligned} a_o^{(2)} &= - g_o^{(2)}/2 \\ a_n^{(2)} &= - \bar{g}_n^{(2)} \quad ( n \geq 1 ) \end{aligned} \tag{4.30}$$

One will be able to construct the higher order equations including (4.28)-(4.30), and to get the higher order coefficients with a successive process. We shall proceed within the range of accuracy of order 4. It is readily seen from (4.5) that the far field with this accuracy demands the additional computations of $a_o^{(3)}$, $a_1^{(3)}$, and $a_o^{(4)}$. The computations may be somewhat complicated; because, the third and fourth order equations of (4.21) contain the first and second order coefficients which have already been prepared above, respectively, and in fact, the arrangement of $I_n$ and $K_n$ as a Fourier expansion exhibits rather complicated forms. The Fourier series of $I_n, K_n$ are, with use of the zero order quantities $R_\ell$, $S_\ell$, $U_{\ell n}$, and $V_{\ell n}$ ( see (A-7) ), given in (A-6). Omitting the details, we write the results only.

$$a_o^{(3)} = -g_o^{(3)}/4 - \alpha^2 \sum_{n=1}^{\infty} ( R_n - \bar{\beta}_{n-1}/\alpha ) \bar{g}_n^{(1)}/2n + \text{C.C.} \quad (4.31)$$

$$\begin{aligned} a_1^{(3)} = & -\bar{g}_1^{(3)} + \alpha^2 \Big\{ ( \pi j + 2C - 1 + 2 \log\alpha )( \bar{g}_1^{(1)} - \beta_1 g_1^{(1)}/\alpha ) \\ & + ( 2 - 2R_o + U_{01} ) \bar{g}_1^{(1)} \\ & - ( \bar{R}_2 + \bar{S}_2 - \bar{V}_{21} ) g_1^{(1)} \Big\} \\ & + \alpha^2 \sum_{n=2}^{\infty} \Big\{ V_{n-1\,n} - ( R_{n-1} + S_{n-1} )/n \\ & + ( \bar{\beta}_{n-2}/\alpha ) \left[ 3/n - 1/(n-1) \right] \Big\} \bar{g}_n^{(1)} \\ & + \alpha^2 \sum_{n=2}^{\infty} \Big\{ \bar{V}_{n+1\,n} - ( \bar{R}_{n+1} + \bar{S}_{n+1} )/n \\ & + ( \beta_n/\alpha ) \left[ 1/n + 1/(n-1) \right] \Big\} g_n^{(1)} \end{aligned} \quad (4.32)$$

$$a_o^{(4)} = -g_o^{(4)}/4$$

$$- \left\{ \alpha^2 (1 - R_o)(\pi j + 2C + 2\log\alpha) - P_o \right\} g_o^{(2)}/4$$

$$- \alpha^2 \sum_{n=1}^{\infty} (R_n - \bar{\beta}_{n-1}/\alpha)\, \bar{g}_n^{(2)}/2n$$

$$+ \text{C.C.} \tag{4.33}$$

These seem to be somewhat complicated, compared to $a_n^{(1)}$ and $a_n^{(2)}$. However, the infinite sums involved can, occasionally, be replaced by the finite sums, provided the series (2.37) is approximated by several terms ( this approximation is often permissible in practise, for example, refer to Chapter V ).

Using (4.28)-(4.33), we have the H scattered, far field of order 4 such that

$$u_o(\xi,\bar{\xi}) =$$

$$-\pi j\, H_o^{(2)}(2\rho)( a_o^{(2)} + a_o^{(3)} + a_o^{(4)} )$$

$$-\pi j\, H_1^{(2)}(2\rho) e^{-i\theta} \left\{ \alpha( a_1^{(1)} + a_1^{(2)} + a_1^{(3)} ) - \alpha^2 \beta_1 \bar{a}_1^{(1)} \right\}$$

$$-\pi j\, H_2^{(2)}(2\rho) e^{-2i\theta} \left\{ 2\alpha\beta_1 a_o^{(2)} + \alpha^2 ( a_2^{(1)} + a_2^{(2)} )/2 \right\}$$

$$- \pi j H_3^{(2)}(2\rho) e^{-3i\theta} ( \alpha^2 \beta_1 a_1^{(1)}/2$$

$$+ \alpha^3 a_3^{(1)}/6 )$$

$$+ C.C. \qquad (4.34)$$

where the Hankel functions are expressible in the asymptotic form: $H_m^{(2)}(2\rho) \sim (\pi\rho)^{-\frac{1}{2}} \exp\left[- 2j\rho + j(2m+1)\pi /4\right]$ .

To the lowest degree of accuracy, only the monopole field with the amplitude $-\pi j a_o^{(2)}$ and the dipole field with $-\pi j \alpha\ a_1^{(1)}$, are retained. The $a_1^{(1)}$ has been obtained, in the expansion of the incident field, from the first order terms,- which produce a constant transverse electric field. This means, as is well-known, that the source of the dipole field is an electric polarization induced by the incident electric field, so $a_1^{(1)}$ is related to this polarization in a " spatial complex representation ". On the other hand, the magnetic charge which is the source of the monopole field, is induced inside the obstacle so as to cancel a circulating electric field around the obstacle,- being caused by the incident magnetic field passing through the cross section. If the cross section possessed no area such as a thin plate, this charge did not arise. These results agree with those of Bladel [32]. The above mention allows us to conclude that the electro- and magneto-static approximations ,- introducing electric and magnetic potentials,- can be done in order to approximate the near field, and in order to determine the coefficients of the far field with an accuracy of order 2. It should be noticed, from (4.29)(4.30), that both the types of approximation are accomplished independently.

## 4.2 Waveguide Problems

Suppose that the $H_{no}$ mode is incident on a perfectly conducting post in a rectangular waveguide. The transverse electric and magnetic fields are orthogonal to each other. When the post is parallel to the electric ( the magnetic ) field, it is called " Inductive ( Capacitive ) Post ". We shall now take the post axis Z, the propagation axis x, and the position of the walls Z = 0,D and Y = 0,A. The field varies with $\cos(n\pi Z/D)$ or $\sin(n\pi Z/D)$, so that the wave number in the X-Y plane is

$$k = \sqrt{k_o^2 - (n\pi/D)^2} \qquad (4.35)$$

For unification of coordinate systems, the present coordinates are also normalized likewise: i.e. A and D correspond to a and d, respectively. The arrangement in the z-plane is depicted in Fig.4.1.

It is possible to remove the walls if, instead, the concept of " Image Posts " is conceived. We shall separate the Green's function into the two parts:

$$G_{green} = G_{o\ green} + G_{1\ green} \tag{4.36}$$

where the first part is the Green's function in free space.

$$G_{o\ green} = \frac{1}{4j} H_o^{(2)}\left(2\sqrt{(\xi-\eta)(\bar{\xi}-\bar{\eta})}\right) \tag{4.37}$$

The second part is the function for expressing the " Image Field ". Evidently, we have the relation between $G_{o\ green}$ and the previous function $G_o$ such that $G_o = 4\pi G_{o\ green}$ . Similarly, we define $G_1 = 4\pi G_{1\ green}$ , $G = 4\pi G_{green}$ . Then, the scattered field can be written, on the analogy of (2.38), as

$$u(\xi,\bar{\xi}) = \frac{1}{2\pi i}\oint_{\gamma_c} \widetilde{G}(\xi,\bar{\xi};\eta)\,\Psi(\eta)\,d\eta + \text{C.C.} \tag{4.38}$$

where, needless to say, u denotes the scattered field in the rectangular waveguide, and represents the Z component of the electric ( the magnetic ) field when the incident field is the E ( the H ) field, regarding the Z axis,- that is to say, when the post is parallel ( perpendicular ) to the shorter side of the waveguide, and finally, $\tilde{G}$ is

$$\tilde{G}(\xi,\bar{\xi};\eta) = \overline{\frac{1}{2\pi i}\oint_{\gamma_{cw}} \overline{G(\xi,\bar{\xi};\eta,\bar{\eta})}\,\frac{du}{u}} \tag{4.39}$$

As found in the fundamental texts ( for example, [14,15] ), the Green's functions for the E and H fields are given by

$$G_{green} = \begin{cases} \sum_{n=1}^{\infty} (1/\gamma_n a)\sin(n\pi y/a)\sin(n\pi y'/a)e^{-\gamma_n|x-x'|} \\ \sum_{n=0}^{\infty} (\epsilon_n/\gamma_n a)\cos(n\pi y/a)\cos(n\pi y'/a)e^{-\gamma_n|x-x'|} \end{cases} \tag{4.40}$$

respectively, where $\gamma_n^2 = (n\pi/a)^2 - 4$, and $\gamma_o = 2j$. To the far field description, it is permissible to replace the Green's function by the propagating terms involved in (4.40).

Now, consider that the $H_{10}$ mode is incident and is travelling to the left with the magnitude unity.

$$u^i = \begin{cases} \sin(\pi y/a)\, e^{\gamma_1 (x-x_o)} & (= E_Z^i) \quad (4.41) \\ e^{\gamma_0 (x-x_o)} & (= H_Z^i) \quad (4.42) \end{cases}$$

Since (4.42) is constant in the Z direction, k is equal to $k_o$, and $\gamma_1$ becomes a pure imaginary ( plus sign ) with respect to j. Whereas, k for (4.41) is $\sqrt{k_o^2 - (\pi/D)^2}$, and $\gamma_n$ ( $n \geq 1$ ) are real numbers. Accordingly, $\gamma_n$ are different in both the cases.

To evaluate the reflection coefficient R and the transmission coefficient T, substitute the propagating terms of (4.40) into (4.39). Note that $|x-x'| = \pm(x-x_o) \mp (x'-x_o)$, $x'-x_o = (\eta + \bar{\eta})/2$. Further, notice the fact that $\sin(\pi y/a)\exp[\pm\gamma_1(x-x_o)]$ and $\exp[\pm\gamma_0(x-x_o)]$ express the behaviors of the propagating mode. We shall neglect them on the calculations of R and T. We get in this way

$$\left.\begin{matrix} R \\ T-1 \end{matrix}\right\} = \frac{1}{2\pi i} \oint_{\gamma_c} K^{\pm}(\eta)\,\Psi(\eta)\,d\eta \; + \text{C.C.} \qquad (4.43)$$

( for inductive posts )

$$\left.\begin{matrix} -R \\ T-1 \end{matrix}\right\} = \frac{1}{2\pi i} \oint_{\gamma_c} L^{\pm}(\eta)\,\Psi(\eta)\,d\eta \; + \text{C.C.} \qquad (4.44)$$

( for capacitive posts )

where

$$K^{\pm}(\eta) = \frac{}{\dfrac{1}{2\pi i}\oint_{\Gamma_{cw}} (4\pi/\Gamma_1 a)\sin\left[\pi(2y_o i + \eta - \bar{\eta})/2ai\right] \cdot e^{\pm\Gamma_1(\eta+\bar{\eta})/2}\, du/u} \qquad (4.45)$$

$$L^{\pm}(\eta) = \frac{}{\dfrac{1}{2\pi i}\oint_{\Gamma_{cw}} (2\pi/\Gamma_o a)\, e^{\pm\Gamma_o(\eta+\bar{\eta})/2}\, du/u} \qquad (4.46)$$

The evanescent mode-amplitudes can be obtained in a similar manner.

The regular function $\Psi(\xi)$ is given by (2.41). The boundary conditions decide the coefficients. Let $u_1$ be the image field. It is seen that the interchanges of G and $\tilde{G}$ with $G_1$ and $\tilde{G}_1$, in (4.38) and (4.39), lead to the expression of $u_1$ . Of course, the scattered field is the sum of $u_o$ and $u_1$. It is also a matter of course that both regular functions involved in the expressions of $u_o$ and $u_1$ are the same.

The near field of $u_o$ has already been obtained in the previous section. Provided that $-( u^i + u_1 )$ is regarded as an " effective incident field in free space ", in determining the coefficients, the previous results are, therefore, applicable. Specifically, we shall write the boundary-values of $u_1$ with prime, such that

$$f'(\psi) = - u_1 \quad = f'_o + f'_1 w + f'_2 w^2 + \dots + C.C. \tag{4.47}$$

$$g'(\psi) = - \frac{\partial}{\partial \rho_w} u_1 = g'_o + g'_1 w + g'_2 w^2 + \dots + C.C. \tag{4.48}$$

$$( \rho_w = 1, \ w = e^{i\psi})$$

and

$$f'_n = f'^{(0)}_n + f'^{(1)}_n + f'^{(2)}_n + \dots\dots \tag{4.49}$$

$$g'_n = g'^{(0)}_n + g'^{(1)}_n + g'^{(2)}_n + \dots\dots \tag{4.50}$$

$$( n = 0,1,2,\dots )$$

Evidently, $f(\psi)$ and $g(\psi)$ given in the previous section must be replaced by $f(\psi) + f'(\psi)$ and $g(\psi) + g'(\psi)$, respectively. Namely, we must exchange $f^{(m)}_n$ in (4.9)-(4.16) for $f^{(m)}_n + f'^{(m)}_n$, and similarly, $g^{(m)}_n$ in (4.23)-(4.33) for $g^{(m)}_n + g'^{(m)}_n$.

It is noteworthy that the " provisional coefficients ", - being obtained by merely replacing the boundary-value,- contain the prime boundary-value components $f_n'^{(m)}$ ( or $g_n'^{(m)}$ ) of the same or smaller orders. This fact will, later, make us to employ a successive process.

It would be a straightfoward matter to calculate the prime boundary-value components. We shall, therefore, represent the image field with spatial complex variables. The real variables-expression is well-known [14,15] . Therefore, only by the rewriting, we have

$$G_{1\,\mathrm{green}} = \frac{1}{4j} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} H_0^{(2)}\left(2\left| \xi - \eta - 2ina \right|\right)$$
$$\mp \frac{1}{4j} \sum_{n=-\infty}^{\infty} H_0^{(2)}\left(2\left| \xi - \bar{\eta} + 2i(y_0 - na) \right|\right) \tag{4.51}$$

where the minus and plus signs are associated with the inductive and capacitive posts, respectively. Note that the arguments of the Hankel functions in the first sum and the second sum are equivalent to $2\sqrt{(x-x')^2 + (y-y'-2na)^2}$ and $2\sqrt{(x-x')^2 + (y+y'-2na)^2}$ , respectively.

By the help of the well-known addition formulae of the

Hankel functions [16], we attain the following deformation ( refer to Appendix I ):

$$G_1(\xi,\bar{\xi};\eta,\bar{\eta}) =$$

$$\sum_{n=0}^{\infty} \epsilon_n A_n\, i^n \left\{ (\bar{\xi}-\bar{\eta})^n + (\eta-\xi)^n \right\} J_n(2|\xi-\eta|)/|\xi-\eta|^n$$

$$\mp \sum_{n=0}^{\infty} \epsilon_n B_n\, i^n \left\{ (\bar{\xi}-\eta)^n + (\bar{\eta}-\xi)^n \right\} J_n(2|\xi-\bar{\eta}|)/|\xi-\bar{\eta}|^n \tag{4.52}$$

where, as defined in (A-9), $A_n$ are constants, but $B_n$ are functions of the position $y_o$ of the post. The definitions of (A-9) may, in general, be disadvantageous on actual calculations. Hence, $A_o$, $B_o$, $B_1$, $A_2$, and $B_2$,- only which we need in the final results,- are reformed, in Appendix II, into the available fashions; the display of the results is in (A-10).

Application of the previous section will force us to approximate $G_1$ with an accuracy of order 3. Under the assumption of the small moduli of $\xi$, $\bar{\xi}$, $\eta$, and $\bar{\eta}$, and with use of the mathematical formulae (IV-7), (4.52) reaches (A-11), approximately. As has been mentioned repeatedly, its constant term with respect to $\bar{u}$,- being involved in $\bar{\eta}$,- results in $\widetilde{G}_1$. Thus,

$$\tilde{G}_1(\xi,\bar{\xi};\eta) =$$

$$A_o\left\{1 - \bar{\xi}(\xi-\eta)\right\} \mp B_o\left\{1 - \xi(\bar{\xi}-\eta)\right\}$$

$$\mp iB_1\left\{\alpha\bar{\beta}_1\xi - (\xi - \bar{\xi} + \eta)(1 + \alpha\bar{\beta}_1 - \xi\bar{\xi}/2 + \bar{\xi}\eta/2)\right\}$$

$$-(A_2/2)\left\{(\xi-\eta)^2 + \bar{\xi}^2 + 2\alpha\bar{\beta}_1\right\}$$

$$\pm(B_2/2)\left\{(\bar{\xi}-\eta)^2 + \xi^2 + 2\alpha\bar{\beta}_1\right\}$$

$$\pm i(B_3/6)\left\{(\bar{\xi}-\eta)^3 - \xi^3 - 6\alpha\bar{\beta}_1\xi + 3\alpha^2\bar{\beta}_2\right\}$$

(4.53)

Referring to the evaluation of $u_o$ in the previous section, we have the image field for the case of the inductive post, within the range of order 2, such that

$$\begin{aligned} u_1(\xi,\bar{\xi}) = {} & (A_o-B_o)(a_o^{(0)} + a_o^{(2)} - \xi\bar{\xi}\,a_o^{(0)}) \\ & + iB_1\,(2\,\xi\,a_o^{(0)} + \alpha\,a_1^{(1)}) \\ & - (A_2-B_2)(\xi^2 + 2\alpha\beta_1)\,a_o^{(0)} \\ & + C.C. \end{aligned} \qquad (4.54)$$

The Fourier expansion of $\xi\bar{\xi}$ and $\xi^2$ in the above are given in (A-12)(A-13). By substitution, we obtain $f'(\psi)$:

$$f_o'^{(0)} = -(A_o - B_o)\, a_o^{(0)}$$

$$f_o'^{(1)} = 0$$

$$f_o'^{(2)} = -(A_o - B_o)\, a_o^{(2)} + \Big\{ (A_o - B_o)(1 - P_o) + 2\alpha (A_2 - B_2)(\beta_1 + \bar{\beta}_1) \Big\} a_o^{(0)} + \alpha B_1 (a_1^{(1)} - \bar{a}_1^{(1)})/2i$$

$$f_1'^{(1)} = -2iB_1\, (\alpha - \bar{\beta}_1)\, a_o^{(0)}$$

(4.55)

For the moment, we shall confine ourselves to the computation of the reflection coefficient and the transmission coefficient for the case of the inductive post. Substitution of (4.55) into the " provisional coefficients " mentioned above, leads to the simultaneous equations,- being solvable in a successive manner. The solutions are:

$$a_o^{(0)} = -f_o^{(0)} / (\pi j + 2C + 2 \log \alpha - A_o + B_o)$$

$$a_1^{(1)} = \bar{f}_1^{(1)} + 2iB_1\, (\alpha - \beta_1)\, a_o^{(0)}$$

$$a_o^{(2)} = -a_o^{(0)} - \Big\{ f_o^{(0)} + f_o^{(2)} - \alpha B_1\, (f_1^{(1)} - \bar{f}_1^{(1)})/2i \Big\} \Big/ \Big\{ (1 - P_o)(\pi j + 2C + 2 \log \alpha - A_o$$

$$+ B_o) - 2Q_o + 2\alpha(A_2 - B_2)(\beta_1 + \bar{\beta}_1)$$
$$+ 2\alpha^2 B_1^2 \left[1 - (\beta_1 + \bar{\beta}_1)/2\alpha\right] \Big\} \tag{4.56}$$

Note that $a_o^{(1)}$ vanishes because of $f_o^{(1)} = f_o'^{(1)} = 0$. It is clear from a glance of (4.5) that the above three coefficients make it possible to express the far field with an accuracy of order 2,- in this case, which is a scattered, travelling mode. To the present purpose, therefore, we can ignore the terms higher than $a_3$. Employment of the residue calculus to (4.45) and expansion of the integrand yield, within the range of order 2,

$$K^{\pm}(\eta) =$$
$$(4\pi/\gamma_1 a)\Big\{\left[1 - \alpha\bar{\beta}_1 + (\pi/a)^2\alpha\bar{\beta}_1/2\right]\sin(\pi y_o/a)$$
$$\mp \gamma_1(\pi/a)(\alpha\bar{\beta}_1/2i)\cos(\pi y_o/a)$$
$$+ (\pi/2ai)\,\eta\,(1 \pm \gamma_1\eta/2)\cos(\pi y_o/a)$$
$$+ \left[\pm\gamma_1\eta/2 - \eta^2/2 + (\pi\eta/2a)^2\right]\sin(\pi y_o/a)\Big\} \tag{4.57}$$

Substituting (4.3) into (4.57), and using the residue calculus in (4.43), we obtain

$$\left.\begin{array}{l} R \\ T-1 \end{array}\right\} = (8\pi/\gamma_1 a)\Big\{\Big[1-(1-\pi^2/2a^2)\alpha(\beta_1 + \bar{\beta}_1)\Big]\sin(\pi y_o/a)$$

$$\pm\gamma_1(\pi/2ai)\alpha(\beta_1-\bar{\beta}_1)\cos(\pi y_o/a)\Big\}$$

$$\cdot a_o$$

$$+(4\pi/\gamma_1 a)\Big\{(\pi\alpha/2ai)(a_1-\bar{a}_1)$$

$$\cdot\cos(\pi y_o/a)\pm(\gamma_1\alpha/2)(a_1+\bar{a}_1)$$

$$\cdot\sin(\pi y_o/a)\Big\} \qquad (4.58)$$

where $a_o = a_o^{(0)} + a_o^{(2)}$, and $a_1 = a_1^{(1)}$. This is a general expression for the inductive post with arbitrary cross section. At this stage, however, calculation of the boundary-value still remains. This is achieved by making use of (A-12)(A-13)(A-14):

$$f_o^{(0)} = -\tfrac{1}{2}\sin(\pi y_o/a)$$

$$f_1^{(1)} = -(\pi/2ai)(\alpha-\bar{\beta}_1)\cos(\pi y_o/a)$$

$$-(\gamma_1/2)(\alpha+\bar{\beta}_1)\sin(\pi y_o/a)$$

$$f_o^{(2)} = -(\gamma_1\alpha/2)(\pi/2ai)(\beta_1-\bar{\beta}_1)$$

$$\cdot\cos(\pi y_o/a)$$

$$-\Big\{(\pi^2/2a^2-1)\alpha(\beta_1+\bar{\beta}_1)/2$$

$$- P_o/2 \quad \Big\} \sin(\pi y_o/a)$$

(4.59)

It should be noticed that (4.58) does not contain the parameters $\beta_2, \beta_3, \ldots$, explicitly,- daringly to say, those are involved in $P_o$ and $Q_o$. In other words, $\alpha$ and $\beta_1$ play a main role of the shape dependence. It is readily seen that the neglect of $\beta_2, \beta_3, \ldots$, in (2.37) is equivalent to the replacement of the boundary curve by an ellipse, - having the major axis 2 ($\alpha + |\beta_1|$) and the minor axis 2 ($\alpha - |\beta_1|$). Therefore, the concept of an " Effective Elliptic Post " is quite reasonable in the second order theory, although $P_o$ and $Q_o$ must be corrected. When $\beta_n = 0$ i.e. a cylindrical post is assumed, (4.58) just coincides with the Lewin's results [15]. When the given cross-sectional geometry of $\gamma$ has no component of an ellipse in a sense of Fourier expansion, such as a square, a hexagon, etc. , $\beta_1$ vanishes, so (4.58) becomes the same one that is obtained in the case of a cylindrical post, - except for $P_o$ and $Q_o$. This fact means, therefore, that for such posts, the replacement by the corresponding effective radii is available.

We shall next direct our attention towards the capacitive post. The analysis is identical to that of the inductive post. It should, however, be kept in mind that to order 4, the expression of the far field demands the nine coefficients $a_o^{(2)}$, $a_o^{(3)}, a_o^{(4)}, a_1^{(1)}, a_1^{(2)}, a_1^{(3)}, a_2^{(1)}, a_2^{(2)}$, and $a_3^{(1)}$. Accordingly, we shall take account of only the first four terms in (2.41).

The function $L^{\pm}(\eta)$ is, to order 4,

$$L^{\pm}(\eta) = (2\pi/\gamma_o a)\left\{ 1 \pm \gamma_o\eta/2 + \gamma_o^2(\eta^2 + 2\alpha\bar{\beta}_1)/8 \pm \gamma_o^3(\eta^3 + 6\alpha\bar{\beta}_1\eta + 3\alpha^2\bar{\beta}_2)/48 \right\} \tag{4.60}$$

Accordingly, from (4.44), we obtain

$$\left.\begin{array}{c} -R \\ T-1 \end{array}\right\} = (4\pi/\gamma_o a)\left\{ 1 + \gamma_o^2\alpha(\beta_1 + \bar{\beta}_1)/4 \pm \gamma_o^3\alpha^2(\beta_2 + \bar{\beta}_2)/16 \right\} a_o$$
$$\pm (2\pi/\gamma_o a)\left\{ \gamma_o\alpha\left[1 + \gamma_o^2\alpha(\beta_1 + \bar{\beta}_1)/8\right](a_1 + \bar{a}_1)/2 + \gamma_o^3\alpha^2(\bar{\beta}_1 a_1 + \beta_1\bar{a}_1)/16 \right\}$$
$$+ (2\pi/\gamma_o a)\gamma_o^2\alpha^2(a_2 + \bar{a}_2)/8$$
$$\pm (2\pi/\gamma_o a)\gamma_o^3\alpha^3(a_3 + \bar{a}_3)/48 \tag{4.61}$$

where

$$
\begin{aligned}
a_o &= a_o^{(2)} + a_o^{(3)} + a_o^{(4)} \\
a_1 &= a_1^{(1)} + a_1^{(2)} + a_1^{(3)} \\
a_2 &= a_2^{(1)} + a_2^{(2)} \\
a_3 &= a_3^{(1)}
\end{aligned}
$$

The (4.61) is, needless to say, a general expression for the capacitive post with arbitrary cross section.

The following task is, in sequence, to determine the coefficients. For this purpose, we shall, beforehand, approximate $u_1$ within the range of order 4. Recalling $a_n^{(0)} = 0$ and $a_o^{(1)} = 0$, we can obtain $u_1$ as follows:

$$
\begin{aligned}
u_1(\xi, \bar{\xi}) \ = \ & \\
& (A_o+B_o)(a_o - \xi\bar{\xi} a_o^{(2)}) \\
& + \alpha \left\{ (A_o+B_2)\bar{\xi} + (A_2+B_o)\xi \right\} (a_1^{(1)} + a_1^{(2)}) \\
& - (A_2+B_2) \left\{ (\xi^2 + 2\alpha\beta_1)\, a_o^{(2)} + \alpha^2 a_2/2 \right\} \\
& - iB_1 \left\{ 2\xi (a_o^{(2)} + a_o^{(3)}) + \alpha a_1 - \alpha\xi\bar{\xi} a_1^{(1)} \right. \\
& \left. + \alpha^2 \xi a_2^{(1)}/2 + \alpha \xi^2 a_1^{(1)}/2 + \alpha^2 \bar{\beta}_1 a_1^{(1)} \right\} \\
& + i(B_3/6) \left\{ 3\alpha \bar{\xi}^2 a_1^{(1)} + 3\alpha^2 \beta_1 a_1^{(1)} \right.
\end{aligned}
$$

$$+ \alpha^3 a_3^{(1)} - 3\alpha^2 \bar{\xi} a_2^{(1)} \Big\} + \text{C.C.}$$

(4.62)

where $a_o, a_1$, and $a_2$ are the same ones as those in (4.61). It is noteworthy that in (4.62), the terms all of whose orders are less than, or equal to, 2, are constants, and in consequence, vanish through the normal derivative. This fact implies that the effect of the presence of the walls is negligible in the second order theory. The value $\rho_w \partial u_1/\partial \rho_w$ can be expressed by (A-15). Then, by letting the observation point $\xi$ approach a point on $\gamma$, – that is, $\rho_w \longrightarrow 1$, and by substituting (2.37), (A-6)(= $I_o$), (A-16), and (A-17), we can obtain

$$\begin{aligned}
g_n'^{(1)} &= g_n'^{(2)} = 0 \quad ( n \geq 0 ) \\
g_o'^{(3)} &= 0 \\
g_1'^{(3)} &= 2iB_1 (\alpha + \bar{\beta}_1)\, a_o^{(2)} - (A_o + B_2) \\
&\quad \cdot \alpha (\alpha \bar{a}_1^{(1)} - \bar{\beta}_1 a_1^{(1)}) \\
&\quad - (A_2 + B_o) \alpha (\alpha a_1^{(1)} - \bar{\beta}_1 \bar{a}_1^{(1)}) \\
g_o'^{(4)} &= 2\alpha^2 (1 - R_o)(A_o + B_o)\, a_o^{(2)} \\
&\quad - iB_1 \alpha^3 (1 - R_o)(a_1^{(1)} - \bar{a}_1^{(1)})
\end{aligned}$$

(4.63)

Replacement of $g_n^{(m)}$ by $g_n^{(m)} + g_n'^{(m)}$ in (4.29)-(4.33) leads to the simultaneous equations for $a_n^{(m)}$. Indeed, they are solved by a successive manner. We shall show only the final results:

$$
\begin{aligned}
a_o^{(1)} &= (4.28), \quad a_n^{(1)} = (4.29) \quad ( n \geq 1 ) \\
a_n^{(2)} &= (4.30) \quad ( n \geq 0 ) \\
a_o^{(3)} &= (4.31) \\
a_1^{(3)} &= - iB_1 (\alpha + \beta_1) g_o^{(2)} \\
&\quad - (A_o+B_2)\alpha (\alpha \bar{g}_1^{(1)} - \beta_1 g_1^{(1)}) \\
&\quad - (A_2+B_o)\alpha (\alpha g_1^{(1)} - \beta_1 \bar{g}_1^{(1)}) \\
&\quad + (4.32) \\
a_o^{(4)} &= \alpha^2(1 - R_o)(A_o+B_o) g_o^{(2)}/2 \\
&\quad -\alpha^3(1 - R_o) B_1 (g_1^{(1)} - \bar{g}_1^{(1)})/2i \\
&\quad + (4.33)
\end{aligned}
\tag{4.64}
$$

where the numbers (...) denote the right hand side terms of the corresponding equations.

We shall now repeat the statement mentioned in the previous section ( Page 67 ): In the second order theory of the H field scattering, the two coefficients $a_o^{(2)}$ and $a_1^{(1)}$ can be determined by the magneto- and electro-static approximations. This statement is also applicable to this case; because it is, as mentioned, permissible to remove the walls, and to discuss the relevant matters in free space. Let us consider this fact in a standpoint of the T-network, ( see Fig.5.2 ). Ignore the higher terms in (4.61). A little calculation leads to

$$1 + R - T \simeq -(8\pi/\gamma_o a)\, a_o^{(2)}$$

$$1 - R - T \simeq (2\pi\alpha/a)\,(a_1^{(1)} + \bar{a}_1^{(1)})$$

Each of them determines the series arm impedance $jX_1$ and the shunt arm impedance $jX_2$ , individually ( refer to (5.11)(5.12) ). As has been described, $a_o^{(2)}$ ( or $a_1^{(1)}$) can be obtained by the magneto- ( or the electro- ) static approximation. Therefore, $jX_1$ ( or $jX_2$ ) can be computed with the magneto- ( or the electro- ) static approximation.

Calculation of $g_n^{(m)}$ still remains. To order 4, $g(\psi)$ is given by

$$g(\psi) = -w\frac{\partial}{\partial w} e^{\gamma_o(\xi+\bar{\xi})/2} + \text{c.c.}$$

$$= -(\gamma_o/2)(w d\xi/dw + \bar{w} d\bar{\xi}/d\bar{w})$$

$$\cdot \Big\{ 1 + (\gamma_o/2)(\xi+\bar{\xi}) + (\gamma_o^2/8)(\xi+\bar{\xi})^2$$

$$+ (\gamma_o^3/48)(\xi+\bar{\xi})^3 \Big\} \tag{4.65}$$

Consequently, we have

$$g_n^{(m)} = -\frac{1}{2\pi}\int_0^{2\pi} \frac{\epsilon_n}{(m-1)!}(\gamma_o/2)^m(\xi+\bar{\xi})^{m-1}$$

$$\cdot\ (w d\xi/dw + \bar{w} d\bar{\xi}/d\bar{w})\ e^{-in\psi} d\psi \tag{4.66}$$

where $w = e^{i\psi}$, and the point $\xi$ is restricted to $\gamma$.

Particularly,

$$g_1^{(1)} = -\gamma_o(\alpha - \bar{\beta}_1)/2$$
$$g_n^{(1)} = n\gamma_o\bar{\beta}_n/2 \qquad (n \geq 2)$$
$$g_o^{(2)} = -\gamma_o^2\alpha^2(1 - R_o)/4 \tag{4.67}$$

## Chapter V

## EXAMPLES

The results we have obtained in the previous chapter meet our actual needs by indication of the boundary shape i.e. inspection of the boundary parameters $\alpha$ and $\beta_n$ . The previous chapter, therefore, makes us to concentrate our whole mind upon the study of obtaining the boundary parameters with knowledge of costructional requirements. This may be achieved by constructing the conformal mapping functions. It is therefore worthwhile to look up them in the texts or the dictionaries ( e.g. [33] ) beforehand whether the desired functions has already been studied. Even if those were not found, we could, fortunately, ask for help to the numerical techniques: Kantrovitch and Krylov [1] proposed various techniques for conformal mapping.

This chapter is not concerned with those techniques, but quotes the results. We devote ourselves to the illustration and the understanding of our method as a mathematical tool. The simpler, comprehensible examples are examined in order that the concept of the Fourier expansion of the boundary shape may be able to stand on physical situations. Particularly, the affection of the Fourier components to the far field might be understood in these examples.

## 5.1 Scattering by a Strip

We shall consider a perfectly conducting strip situated in free space, with the width b and the angle $\phi$ to the x axis, in the normalized coordinate system ( see Fig.5.1 ). For the case $\phi = 0$, the rigorous transforming ( mapping ) function is given by $z = \xi = (b/4)( w + w^{-1} )$ ( [2] p.451 ). The transforming function for the case of an arbitrary angle, can be obtained by an appropriate rotation in such a way that $\alpha$ becomes a positive real number. A few considerations lead to the replacement of $\xi$ by $\xi e^{-i\phi}$ and w by $w \cdot e^{-i\phi}$.

$$\xi = (b/4) ( w + e^{2i\phi}/w ) , \quad z_o = 0 \qquad (5.1)$$

Evidently, the edge points in the $\xi (= z)$-plane are mapped at the points $\pm e^{i\phi}$ in the w-plane. In addition, assume that the plane wave propagating along the x axis is incident on the strip.

$$u_o^i = e^{-jkX} = e^{-2jx} \qquad (5.2)$$

We shall deal with both the E field scattering and the H field scattering, together. Insert (5.1) into (5.2), and expand it into a series around the orign. Derivation of (4.9) and (4.22) from those equations would be made easily. As the results, we have .

$$f_o^{(0)} = -\tfrac{1}{2}, \qquad f_1^{(1)} = j(b/2)\cos\phi\, e^{-i\phi},$$
$$f_o^{(0)} = (b^2/8)\cos^2\phi, \qquad f_2^{(2)} = (b^2/8)\cos^2\phi\, e^{-2i\phi},$$
$$\text{all the others } f_n^{(0)}, f_n^{(1)}, f_n^{(2)} = 0$$

$$g_1^{(1)} = -(jb/2i)\sin\phi\, e^{-i\phi}, \quad g_2^{(2)} = -(b^2/4i)\sin\phi\cos\phi\, e^{-2i\phi},$$
$$g_1^{(3)} = (jb^3/16i)\sin\phi\cos^2\phi\, e^{-i\phi}, \qquad g_o^{(3)} = 0,$$
$$g_3^{(3)} = (jb^3/16i)\sin\phi\cos^2\phi\, e^{-3i\phi}, \qquad g_o^{(4)} = 0,$$
$$\text{all the others } g_n^{(1)}, g_n^{(2)} = 0 \tag{5.3}$$

Also,

$$\alpha = b/4, \qquad \beta_1 = b\, e^{2i\phi}/4,$$
$$P_o = b^2/8, \qquad P_2 = b^2\, e^{-2i\phi}/16,$$
$$R_o = S_o = 0, \; Q_o = 0,$$
$$U_{on} = V_{on} = 1/(n+1),$$
$$\text{all the others } \beta_n, P_n, Q_n, R_n, S_n, U_{\ell n}, V_{\ell n} = 0 \tag{5.4}$$

Use of these quantities enables us to evaluate the coefficients in a straightfoward manner, with (4.14)-(4.16) and (4.28)-(4.33). Further, by substituting the coefficients into (4.17) and (4.34), we obtain,

for the scattered E field to order 2 :

$$
\begin{aligned}
u_o = & -\pi j\, H_o^{(2)}(2\rho)(1 - b^2\cos^2\phi/4) \Big/ \left\{ (1 - b^2/8)(\pi j \right. \\
& \left. + 2C + 2\log b/4) \right\} \\
& - \pi j\, H_1^{(2)}(2\rho)(jb^2/4)\cos\phi\cos(\theta - \phi) \\
& - \pi j\, H_2^{(2)}(2\rho)(b^2/8)\cos 2(\theta - \phi)/(\pi j + 2C \\
& + 2\log b/4)
\end{aligned}
\tag{5.5}
$$

for the scattered H field to order 4 :

$$
\begin{aligned}
u_o = & \pi H_1^{(2)}(2\rho)(b^2/4)\left\{ 1 - (b^2/8)(\pi j + 2C - 5/4 \right. \\
& \left. + \cos^2\phi + 2\log b/4) \right\} \sin\phi\sin(\theta - \phi) \\
& - \pi j\, H_2^{(2)}(2\rho)(b^4/64)\sin\phi\cos\phi\sin 2(\theta - \phi) \\
& + \pi H_3^{(2)}(2\rho)(b^4/128)\sin\phi\sin 3(\theta - \phi)
\end{aligned}
\tag{5.6}
$$

5.2 Cylindrical Capacitive Post

Figure 5.3 shows a perfectly conducting, cylindrical capacitive post in the rectangular waveguide guiding a $TE_{10}$ mode. The post axis Z is perpendicular to the shorter side. The post radius measured in the normalized coordinates is $\rho_o$ . The wave number in the perpendicular plane to the post axis is, as a matter of course,

$$k = \sqrt{k_o^2 - (\pi/D)^2} \tag{5.7}$$

Such a post is the most typical one to which an arbitrary cross-sectional post is reduced with the concept of the effective radius, so we can see the analysis and the result in the commonest texts. Even in our analysis, this is the most special, the easiest type. This section is, therefore, devoted to comparison with the Lewin's results [15] and those of the Waveguide Handbook [13].

In this case, things are simpler; the transforming function is $\xi = \rho_o w$ . Thus,

$$\alpha = \rho_o , \quad \beta_n = 0 \quad ( n \geq 1 ) \tag{5.8}$$

$$z_o = iy_o \tag{5.9}$$

By (5.8), we find

$$P_o = -Q_o = \rho_o^2 , \quad P_n = Q_n = 0 \ ( n \geq 1 )$$
$$R_n = S_n = 0 \quad ( n \geq 0 )$$
$$U_{\ell n} = V_{\ell n} = 0 \quad ( n, \ell \geq 0 )$$
$$(5.10)$$

A direct insertion of (5.8) into (4.65) leads to $g(\psi)$ or $g_n^{(m)}$. Furthermore, substituting (4.64) into (4.61), we arrive at the conclusion. As a traditional expression, we shall represent their properties by the T-network shown in Fig.5.2. The series arm impedance $jX_1$ and the shunt arm impedance $jX_2$ are related to R and T with [13,14,15]

$$jX_1 = \frac{1 + R - T}{2 - (1 + R - T)} \tag{5.11}$$

$$jX_2 = \frac{1}{1 - R - T} - \frac{1}{2} - \frac{1}{2} jX_1 \tag{5.12}$$

Only by substitution, we obtain

$$jX_1 = -\frac{1}{2}\left(\frac{\pi}{a}\right) \gamma_o \rho_o^2 \left\{ 1 + \frac{3}{8}( \gamma_o \rho_o)^2 + \rho_o^2 (\pi j - 1 + 2C + 2 \log \rho_o - A_o - B_o + \frac{2\pi}{\gamma_o a} ) \right\} \tag{5.13}$$

$$jX_2 = \frac{1}{2}\left(\frac{a}{\pi}\right)\frac{1}{\gamma_o \rho_o^2}\Big\{1 - \frac{3}{8}(\gamma_o \rho_o)^2 + \rho_o^2(\pi j$$

$$+ 2C + 1 + 2\log \rho_o - A_o - B_o - A_2 - B_2$$

$$+ \frac{4\pi}{\gamma_o a})\Big\} \qquad (5.14)$$

Note that both the values of the second parentheses in the braces are real numbers with respect to j . As seen in (A-10), the $A_o$, $B_o$,$A_2$, and $B_2$ involve the rapid convergent infinite-sums.
Lewin has obtained $X_1$ and $X_2$, neglecting those sums. Such an approximation in this case leads to a coincidence. It must, however, be confessed that the Lewin's $X_1$ differs only by the term -1 involved in the parenthesis of (5.13). This difference is caused by his rough approximation [15, p.40, (2.56)] .
It is possible to say that with more precise approximation, Lewin's one becomes coincident. It should be noted additionally that our results and the Lewin's corrected ones differ from those of the Waveguide Handbook, together.

## 5.3 Cylindrical Inductive Post

We shall consider a perfectly conducting, inductive post parallel to the shorter side of the waveguide ( see Fig.5.4). The post radius is also $\rho_o$ . The wave number in the perpendicular plane to the post is

$$k = k_o \tag{5.15}$$

To this case, the T-network of Fig.5.2 is also applicable. The quantities of (5.8)-(5.10) depend only on the cross-sectional geometry of the post, but not on the position $y_o$ and the arrangement of the waveguide, so that those can be used in this case, too. Omitting the details, we write the final results:

$$jX_1 = -(2\pi/\gamma_1 a)(\gamma_1 \rho_o)^2 \sin^2(\pi y_o/a) \tag{5.16}$$

$$\begin{aligned} jX_2 = {} & (\gamma_1 a/4\pi) \operatorname{cosec}^2(\pi y_o/a) \Big\{ S_R - \rho_o^2 \\ & - (\pi \rho_o/a)^2 \Big[ S_R' - S_R \cot(\pi y_o/a) \Big]^2 \Big\} \\ & - \tfrac{1}{2} jX_1 \end{aligned} \tag{5.17}$$

where $S_R$ and $S_R'$ are real quantities ( with respect to j ) such that

$$S_R = -\log \alpha - C - \pi j/2 + (A_o - B_o)/2 - (2\pi /$$

$$\gamma_1 a) \sin^2(\pi y_o/a) \tag{5.18}$$

$$S_R' = -(a/\pi)\ B_1 - (\pi/\gamma_1 a) \sin(2\pi y_o/a) \tag{5.19}$$

The more available forms of $S_R$ and $S_R'$ are given by (A-19). It should be added that our results coincide with those of the Waveguide Handbook.

## 5.4 Capacitive Strip

A perfectly conducting capacitive strip is shown in Fig. 5.5. The angle $\phi$ to the x axis is chosen arbitrarily. The width of the strip in the normarized coordinates is b. The wave number is, of course, (5.7). The quantities (5.4) are also usable in this case. We shall cut down all the explanations of derivation to avoid the duplication. The impedances of the T-network are:

$$jX_1 = -(\pi/\gamma_o a)(\gamma_o a)(\gamma_o b/4)^4 \sin^2\phi \cos^2\phi \tag{5.20}$$

$$jX_2 = (a/\pi b)(4/\gamma_o b) \operatorname{cosec}^2\phi \Big\{ 1 - (\gamma_o b/4)^2 \cos 2\phi /2$$
$$- (\gamma_o b/4)^2/8 + (b^2/8) \Big[ \pi j + 2C - 3/4$$

$$+ 2 \log b/4 - A_o - B_2 + 2\pi/\gamma_o a$$

$$+ (A_2 + B_o - 2\pi/\gamma_o a) \cos 2\phi \Big] \Big\} \qquad (5.21)$$

Note that the term of the bracket of (5.21) is a real number with respect to j. As $\phi$ tends to zero, $X_1$ and $X_2$ go to zero and infinity, respectively. This fact agrees with the physical situations. In addition, note that the replacement of $\phi$ by $\pi - \phi$ does not change the fashions of (5.20) and (5.21). Indeed, this is the reason why the symmetric T-network representation is possible, although the constructional geometry is asymmetric. If $\phi = \pi/2$, then $X_1 = 0$. To this case, Lewin has obtained $X_2$ by the method of integral equations. However, by reason of an essential difference in both the manipulations between the present one and the Lewin's one, those can not be borne comparison with each other; for instance, we are counting the order of log b/4 as zero. Fortunately, the numerical comparison bears a good coincidence.

## 5.5 Inductive Strip

Figure 5.6 shows a perfectly conducting inductive strip. Figure 5.2 is also held in the cases $\phi = 0$ and $\phi = \pi/2$. Repeating to say, the quantities (5.4) are usable ( $z_o = iy_o$ ). It is needless to say the $TE_{10}$ incidence.

For the case $\phi = 0$:

$$jX_1 = -(4\pi/\gamma_1 a)(\gamma_1 b/4)^2 \sin^2(\pi y_o/a) \tag{5.22}$$

$$jX_2 = (\gamma_1 a/4\pi)\operatorname{cosec}^2(\pi y_o/a) \Big\{ S_R - (b^2/8)\, S_R'' - \Big[(\pi b/4a)^2 + (\gamma_1 b/4)^2\Big] S_R \Big\} - jX_1/2 \tag{5.23}$$

For the case $\phi = \pi/2$:

$$jX_1 = 0$$

$$jX_2 = (\gamma_1 a/4\ )\operatorname{cosec}^2(\pi y_o/a) \Big\{ S_R + (b^2/8)\, S_R'' + \Big[(\pi b/4a)^2 + (\gamma_1 b/4)^2\Big] S_R - 2\,(\pi b/4a)^2 \Big[ S_R' - S_R \cot(\pi y_o/a)\Big]^2 \Big\} \tag{5.25}$$

The $S_R$ and $S_R'$ are defined in (5.18) and (5.19), respectively ( in this case, $\alpha = b/4$ ). The $S_R''$ is a real quantity with respect to j, such that

$$S_R'' = A_2 - B_2 + (\pi/\gamma_1 a) \left[ (\pi/a)^2 + \gamma_1^2 \right] \sin^2(\pi y_o/a) \tag{5.26}$$

The available forms of $S_R$, $S_R'$, and $S_R''$ are in (A-19).

5.6 Inductive Square Post

A perfectly conducting, inductive square post is depicted in Fig.5.7. Also, the $TE_{10}$ mode is incident. The post is inserted with the angle $\phi$ to the x axis.

We shall quote the transforming function which transforms the square region enclosed by the straight lines $\xi = \pm 1$, $\pm i$, into the outer region to a unit circle in the w-plane. This has been obtained by successive means [1]: $\xi = (1125/1024)\, w - (203/2048)\, w^{-3} + (1/2048)\, w^{-7}$ . Thus, by $\xi \rightarrow (2/b)\, \xi\, e^{-i\phi}$ and $w \longrightarrow w\, e^{-i\phi}$ ( refer to Page 88 ),

$$\xi = \alpha w + \beta_3/w^3 + \beta_7/w^7 \tag{5.27}$$

$$z = i\, y_o \tag{5.28}$$

where

$$\alpha = 0.549316\ b$$
$$\beta_3 = -\,0.090222\,\alpha\, e^{4i\phi}$$
$$\beta_7 = 0.000444\,\alpha\, e^{8i\phi}$$
$$\text{the others} = 0 \tag{5.29}$$

Accordingly,

$$P_o = 1.008140\,\alpha^2 \quad , \quad Q_o = -\,0.997287\,\alpha^2 \quad ,$$
$$P_4 = -\,0.090262\,\alpha^2 e^{-4i\phi}, \quad Q_4 = -\,0.060129\,\alpha^2 e^{-4i\phi} \quad ,$$
$$P_8 = 0.000444\,\alpha^2 e^{-8i\phi}, \quad Q_8 = -\,0.000381\,\alpha^2 e^{-8i\phi} \quad ,$$
$$R_o = S_o = 0.024421, \quad W_2 = \alpha^2(1 - 0.180444\ e^{-4i\phi}),$$
$$R_4 = -\,0.000120\ e^{-4i\phi} \quad , \quad W_6 = 0.009029\,\alpha^2 e^{-8i\phi} \quad ,$$
$$S_4 = -\,0.000281\ e^{-4i\phi} \quad , \quad W_{10} = -\,0.000080\,\alpha^2 e^{-12i\phi} \quad ,$$
$$\text{all the others } P_n, Q_n, R_n, S_n, W_n = 0\ . \tag{5.30}$$

$$U_{on} = V_{on} = 0.02442/(n+3)$$

$$U_{4n} = R_4/(n+7)$$

$$V_{4n} = S_4/(n+3)$$

$$\text{all the others } U_{\ell n}, V_{\ell n} = 0 \tag{5.31}$$

As mentioned before, to calculation of $jX_1$ and $jX_2$, the other $\beta_n$ except $\beta_1$ ( in this case, $\beta_1 = 0$ ) do not contribute. Therefore, the calculation is simplified very much:

$$jX_1 = -(2\pi/\gamma_1 a)(\gamma_1 \alpha)^2 \sin^2(\pi y_o/a) \tag{5.32}$$

$$\begin{aligned} jX_2 = {} & (\gamma_1 a/4\pi) \operatorname{cosec}^2(\pi y_o/a) \Big\{ S_R + Q_o \\ & - (\pi\alpha/a)^2 \left[ S_R' - S_R \cot(\pi y_o/a) \right]^2 \Big\} \\ & - jX_1/2 \end{aligned} \tag{5.33}$$

where $S_R$ and $S_R'$ are given in (A-9) as available fashions. The above expressions are very similar to those for the cylindrical inductive post ( see (5.16) and (5.17) ) except for $Q_o$. The replacement of $Q_o$ by $-\alpha^2$ permits the error only 0.3 per cent. To the inductive square post, therefore, the concept of an effective radius (= $\alpha$ ) is very profitable. The Waveguide Handbook

[13] exhibits another form. It should, however, be confessed that such a form,- being obtained as a general expression of rectangular posts with the concept of an effective ellipse,- does not tend to the form for a cylindrical post, in the limit.

## 5.7 Capacitive Square Post

Finally, we shall consider a perfectly conducting, capacitive square post shown in Fig.5.8. The matters concerning the fields and the post are the same as the previous ones. The quantities of (5.28)-(5.31) can be used in this case, too. The evaluation of $g_n^{(m)}$,- may be somewhat laborious,- are achieved by a straightfoward computation of (4.65). We abbreviate every explanation of the details and make haste to the results. The impedances of the T-network are:

$$jX_1 = -\frac{1}{2}\frac{\pi\gamma_o\alpha^2}{a}\Big\{0.97558 + \frac{3}{8}(\gamma_o\alpha)^2(0.98907 + 0.12030\cos 4\phi) + \alpha^2\Big[0.95176\,(\pi j + 2C + 2\log\alpha - A_o - B_o + \frac{2\pi}{\gamma_o a}) - 0.97537\Big]\Big\} \tag{5.34}$$

$$
\begin{aligned}
jX_2 = \frac{1}{2}\frac{a}{\pi \gamma_0 \alpha^2} \Big\{ 1 - \frac{3}{8}(\gamma_0 \alpha)^2 (0.97285 \\
+ 0.06015 \cos 4\phi \,) + \alpha^2 (\pi j + 2C + 0.93694 \\
+ 2 \log \alpha - A_0 - B_0 - A_2 - B_2 + \frac{4\pi}{\gamma_0 a} ) \Big\}
\end{aligned}
\tag{5.35}
$$

If the terms involving cos $4\phi$ were neglected, the above expressions became the approximate ones for the cylindrical capacitive post ( refer to (5.13) and (5.14) ). It is important to notice that the rotation of the post affects the fourth order terms within the error ten per cent.

From geometrical considerations, it is readily seen that substitution of $\phi$ for $\pi - \phi$ must be equivalent to replacement of $y_0$ by $a - y_0$ . We can easily ascertain this fact in (5.34) and (5.35). Further, we can recognize the fact that this system is mirror symmetric,- although we represented this system as the symmetric T-network after recognition of this fact.

## Chapter IV

## CONCLUSION

The aim of this thesis was to save our analytical effort, as much as possible, in the handling of complicated wave functions which were occasionally encountered in field problems- in particular, in the boundary-value problems. Machine computation seems to be powerful in this regard since one may be able to leave every trouble occurring in- indeed, only in- computation to a machine. In the other sided view, additionally for the purpose of prompting the numerical techniques, a pure analytical approach to such a troublesome matter is desirable and may be, perhaps, instructive.

As an experiential fact, we know that as far as the conventional wave functions and manipulations are used, the thing mentioned above still remains unchanged. In short,if desired so, those must be- may be impossible - radicalized in a suitable way.

The Vekua's excellent idea, as has been described, is of interest in a mathematical standpoint. It seems, however, that his field description is slightly apart from our aim grounded in the Engineering. Of course, its benifical points were accepted here and successfully applied to the field continuation in a waveguide. At that time, we observed that the expression of the corresponding regular function as a Taylor expansion played a special role i.e. the "flattenization" of exponential functions varying steeply. We should emphasize this respect rather than the employment of complex variables.

The outer field description obtained by trial may be considered an extension of the Cauchy Integral which is familiar in a complex analysis and is a worthwhile, mathematical tool in static field problems. In fact, it was ascertained that both coincided with each other in the limit of low-frequency. Such an expression may be suited to discussion of behaviors of the outer fields in viewpoints of mathematical rigor, but seems to be out of interest in view of mathematical tools . The improved description is fit for this purpose, vice versa. However, it was somewhat complicated in the fashion, so that we ended in the approximate use. Nevertheless, the concept of the Fourier expansion of boundary- unlike the conventional analyses - carried a general expression for an arbitrarily shaped obstacle. Besides, all we had proceeded was discussed in a mapped domain whose boundary was a unit circle- being a smooth curve. As the result, we dealt with an edged curve the same as a smooth curve, without too much effort. So to speak, such a convenience bears on the seeking of the transforming function. This can be regarded as the "simplification" of a complicated model, or the "transposition" into a simple model, and it is therefore just our aim at the beginning. The latter question will be solved immediately by employment of the successive techniques proposed by Kantrovitch and Krylov [1] .

The field near the obstacle was not represented here. If desired, it can be achieved by a few additional calculations. The near field is, as described, the corrected one of the quasi-statically approximated field; the most rough solution is obtained by a conformal mapping. Therefore, this field obtained will cover the near field behaviors almost.

## AUXILIARY EQUATIONS

(A-1) $P_o = \alpha^2 + |\beta_1|^2 + |\beta_2|^2 + \cdots\cdots$

$P_n = \alpha\bar{\beta}_{n-1} + \sum_{m=1}^{\infty}\beta_m\bar{\beta}_{m+n} \quad (n \geq 1), \quad \beta_o \equiv 0$

(A-2) $Q_o = -\alpha^2 + |\beta_1|^2 + |\beta_2|^2/2 + |\beta_3|^2/3 + \cdots\cdots$

$Q_n = (\frac{1}{n-1} - 1)\alpha\bar{\beta}_{n-1} + \sum_{m=1}^{\infty}(\frac{1}{m} + \frac{1}{m+n})\beta_m\bar{\beta}_{m+n}$

( $n \geq 1$ ), the first term = 0 for the case n=1 .

(A-3) $\xi\bar{\xi} = P_o/2 + P_1 w + P_2 w^2 + \cdots\cdots + \text{C.C.} \quad (w = e^{i\psi})$

(A-4) $\xi(\alpha w - \beta_1/w - \beta_2/2w^2 - \beta_3/3w^3 - \cdots) + \text{C.C.}$

$= -(Q_o + Q_1 w + Q_2 w^2 + \cdots) + \text{C.C.} \quad (w = e^{i\psi})$

(A-5) $I_o = \bar{\xi}\, w\, d\xi/dw$

$I_1 = -w\,(d\xi/dw)(\alpha\bar{w} - \bar{\beta}_1/\bar{w} - \bar{\beta}_2/2\bar{w}^2 - \bar{\beta}_3/3\bar{w}^3 - \cdots)$

$K_o = \bar{w}\,(d\bar{\xi}/d\bar{w})(\beta_1/2w^2 + \beta_2/3w^3 + \beta_3/4w^4 + \cdots)$

$K_m = \bar{w}\,(d\bar{\xi}/d\bar{w})\{\alpha/mw^m + \beta_1/(m+2)w^{m+2} + \beta_2/(m+3)w^{m+3}$

$+ \cdots\} \quad (m \geq 1)$

(A-6) The values on $\gamma$ : $w = e^{i\psi}$ ,

$I_o = \alpha^2(1 - R_o) - \alpha^2\sum_{\ell=1}^{\infty}(R_\ell - \bar{\beta}_{\ell-1}/\alpha)w^\ell - \alpha^2\sum_{\ell=1}^{\infty}\{\bar{S}_\ell$

$+ (\ell - 1)\beta_{\ell-1}/\alpha\}/w^\ell$

$$I_1 = - P_o - \alpha^2 \sum_{\ell=1}^{\infty} \left\{ U_{\ell 0} - \bar{\beta}_{\ell-1}/(\ell - 1)\alpha \right\} w^{\ell}$$

$$- \alpha^2 \sum_{\ell=1}^{\infty} \left\{ \bar{V}_{\ell 0} - (\ell - 1) \beta_{\ell-1}/\alpha \right\} / w^{\ell}$$

$$K_o = - \alpha^2 U_{01}/w - \alpha^2 \sum_{\ell=1}^{\infty} \left\{ \bar{U}_{\ell 1} - \beta_{\ell-1}/\alpha \ell \right\} / w^{\ell+1}$$

$$- \alpha^2 \sum_{\ell=1}^{\infty} V_{\ell 1} \; w^{\ell-1}$$

$$K_m = \alpha^2 \left( \frac{1}{m} - U_{o\; m+1} \right) / w^{m+1} - \alpha^2 \sum_{\ell=1}^{\infty} \left\{ \bar{U}_{\ell\; m+1} - \right.$$

$$\left. \beta_{\ell-1}/(m + \ell)\alpha \right\} / w^{m+\ell+1} - \alpha^2 \sum_{\ell=1}^{\infty} \left\{ V_{\ell\; m+1} + \right.$$

$$\left. + (\ell - 1) \bar{\beta}_{\ell-1}/m\alpha \right\} / w^{m-\ell+1}$$

$$( \beta_o \equiv 0 \;, \; m \geq 1 )$$

where

(A-7)
$$R_{\ell} = \sum_{m=1}^{\infty} m \beta_m \bar{\beta}_{m+\ell} / \alpha^2$$

$$S_{\ell} = \sum_{m=1}^{\infty} (m + \ell) \beta_m \bar{\beta}_{m+\ell} / \alpha^2$$

$$U_{\ell n} = \sum_{m=1}^{\infty} m \beta_m \bar{\beta}_{m+\ell} / (n + m + \ell)\alpha^2$$

$$V_{\ell n} = \sum_{m=1}^{\infty} (m + \ell) \beta_m \bar{\beta}_{m+\ell} / (m + n)\alpha^2$$

$$( \ell , n \geq 0 )$$

(A-8)
$$\Delta(w,u) = ( \beta_1/\alpha\, w + \beta_2/\alpha\, w^2 + \ldots )/u + ( \beta_2/\alpha\, w + \beta_3/\alpha\, w^2 + \ldots )/u^2 + ( \beta_3/\alpha\, w + \beta_4/\alpha\, w^2 + \ldots )/u^3 + \ldots$$

(A-9) $$A_n = -\pi j \sum_{m=1}^{\infty} \left\{ H_n^{(2)}(4ma) + H_{-n}^{(2)}(4ma) \right\}$$

$$A_1 = A_3 = A_5 = \dots = 0$$

$$B_n = -\pi j\, H_{-n}^{(2)}(4y_o) - \pi j \sum_{m=1}^{\infty} \left\{ H_n^{(2)}(4ma - 4y_o) + H_{-n}^{(2)}(4ma + 4y_o) \right\}$$

$$( n \geq 0 )$$

(A-10) $$A_o = 2C + \pi j + (\pi / \gamma_o a) - 2\log(\pi / a) + \sum_{n=1}^{\infty} (8\pi / a) / \left\{ \gamma_n (n\pi / a)(n\pi / a + \gamma_n) \right\}$$

$$B_o = -2\log\left\{ 2\sin(\pi y_o / a) \right\} + (\pi / \gamma_o a) + \sum_{n=1}^{\infty} (8\pi / a)\cos(2n\pi y_o / a) / \left\{ \gamma_n (n\pi / a)(n\pi / a + \gamma_n) \right\}$$

$$B_1 = -(\pi / 2a)\cot(\pi y_o / a) - \sum_{n=1}^{\infty} (4\pi / a)\sin(2n\pi y_o / a) / \left\{ \gamma_n (n\pi / a + \gamma_n) \right\}$$

$$A_2 = (\pi / a)^2 / 12 - 1 + (\pi / \gamma_o a) - \sum_{n=1}^{\infty} (8\pi / a) / \left\{ \gamma_n (n\pi / a + \gamma_n)^2 \right\}$$

$$B_2 = (\pi/2a)^2 \operatorname{cosec}^2(\pi y_o/a) + (\pi/\gamma_o a)$$

$$- \sum_{n=1}^{\infty} (8\pi/a)\ \cos(2n\pi y_o/a)/\left\{\gamma_n(n\pi/a + \gamma_n)^2\right\}$$

(A-11) $$G_1 = A_o\left\{1 - (\xi-\eta)(\bar{\xi}-\bar{\eta})\right\} \mp B_o\left\{1 - (\xi-\bar{\eta})(\bar{\xi}-\eta)\right\}$$
$$- \tfrac{1}{2} A_2\left\{(\xi-\eta)^2 + (\bar{\xi}-\bar{\eta})^2\right\} \pm \tfrac{1}{2} B_2\left\{(\xi-\bar{\eta})^2\right.$$
$$\left. + (\bar{\xi}-\eta)^2\right\} \mp i\, B_1\, (\bar{\xi}-\eta+\bar{\eta}-\xi)\left\{1 - \tfrac{1}{2}(\xi-\bar{\eta})(\bar{\xi}\right.$$
$$\left.-\eta)\right\} \pm \tfrac{1}{6}\, i\, B_3\left\{(\bar{\xi}-\eta)^3 + (\bar{\eta}-\xi)^3\right\}$$

(A-12) $$\xi^2 + \bar{\xi}^2 = W_o/2 + W_1 w + W_2 w^2 + \ldots\ldots + \text{C.C.}$$
$$(\ w = e^{i\psi}\ )$$

where

$$W_o = 2\alpha(\beta_1 + \bar{\beta}_1)$$
$$W_1 = 2\alpha\bar{\beta}_2$$
$$W_2 = \alpha^2 + 2\alpha\bar{\beta}_3 + \bar{\beta}_1^2$$
$$W_n = 2\alpha\bar{\beta}_{n+1} + \sum_{m=1}^{n-1} \bar{\beta}_m\bar{\beta}_{n-m} \qquad (\ n \geq 3\ )$$

(A-13) $$\xi^2 = 2\alpha\beta_1 + 2\alpha\beta_2/w + \alpha^2 w^2 + (\beta_1^2 + 2\alpha\beta_3)/w^2$$
$$+ \bar{W}_3/w^3 + \bar{W}_4/w^4 + \ldots\ldots$$
$$(\ w = e^{i\psi})$$

(A-14) $$f(\psi) = - \sin \frac{\pi}{a}(y_o + \frac{\xi-\bar{\xi}}{2\,i})\ e^{\frac{1}{2}\gamma_1(\xi+\bar{\xi})} =$$

$$- \sin(\pi L y_o/a) - (\pi L/a)\cos(\pi L y_o/a)\,(\xi - \bar{\xi})/2i$$
$$- \gamma_1 \sin(\pi L y_o/a)\,(\xi + \bar{\xi})/2 - (\gamma_1/4i)(\pi L/a)$$
$$\cos(\pi L y_o/a)\,(\xi^2 - \bar{\xi}^2) - \sin(\pi L y_o/a)\Big\{(\pi L^2/a^2$$
$$- 2)(\xi^2 + \bar{\xi}^2) - 4\xi\bar{\xi}\Big\}/4$$

(A-15) $$\rho_w \partial u_1/\partial \rho_w = -2w\,(d\xi/dw)\Big\{(A_o + B_o)\bar{\xi}$$
$$+ (A_2 + B_2)\xi\Big\}\, a_o^{(2)}$$
$$+ \Big\{(A_o + B_2)(\bar{w}\, d\bar{\xi}/d\bar{w}) + (A_2 + B_o)(w\, d\xi/dw)\Big\}\alpha\,(a_1^{(1)}$$
$$+ a_1^{(2)}) - i B_1 (w\, d\xi/dw)\Big\{2\,(a_o^{(2)} + a_o^{(3)})$$
$$- \alpha\bar{\xi}(a_1^{(1)} - \bar{a}_1^{(1)}) + \alpha\xi\, a_1^{(1)} + \alpha^2 a_2^{(1)}/2\Big\}$$
$$+ i B_3 (\bar{w}\, d\bar{\xi}/d\bar{w})\,\alpha\,(\bar{\xi}\, a_1^{(1)} - \alpha\, a_2^{(1)}/2) + \text{C.C.}$$

(A-16) $$w\, d\xi/dw = \alpha w - \beta_1/w - 2\beta_2/w^2 - 3\beta_3/w^3 - \cdots\cdots$$

(A-17) $$\xi(w\, d\xi/dw) = \alpha^2 w^2 - \alpha\beta_2\bar{w} - (\beta_1^2 + 2\alpha\beta_3)\bar{w}^2 - \cdots\cdots$$
$$(w = e^{i\psi})$$

(A-18) $$u_1 = (A_o - B_o)\, a_o^{(0)} + 2\, i\, B_1\Big\{(\alpha - \bar{\beta}_1)w - \bar{\beta}_2 w^2 - \bar{\beta}_3 w^3 - \cdots$$
$$\Big\}\, a_o^{(0)} - \Big\{(A_o - B_o)(P_o + 2P_1 w + 2P_2 w^2 + \cdots)$$
$$+ (A_2 - B_2)(W_o + W_1 w + W_2 w^2 + \cdots\cdots)\Big\}\, a_o^{(0)}$$
$$+ (A_o - B_o)\, a_o^{(2)} + i B_1 \alpha\, a_1^{(1)} + \text{C.C.}$$

( $w = e^{i\psi}$ , refer to $W_n$ in (A-12). )

$$\text{(A-19)}\quad S_R = -\log\alpha + \log\left\{(2a/\pi)\sin(\pi y_o/a)\right\} - 2\sin^2(\pi y_o/a)$$

$$+ (8\pi/a)\sum_{n=2}^{\infty}\sin^2(n\pi y_o/a)/\left\{\gamma_n(n\pi/a)(n\pi/a+\gamma_n)\right\}$$

$$S_R' = \tfrac{1}{2}\cot(\pi y_o/a) - \sin(2\pi y_o/a)$$

$$+ 4\sum_{n=2}^{\infty}\sin(2n\pi y_o/a)/\left\{\gamma_n(n\pi/a+\gamma_n)\right\}$$

$$S_R'' = (\pi/a)^2/12 - 1 + 2(\pi/a)^2\sin^2(\pi y_o/a)$$

$$-(\pi/2a)^2\operatorname{cosec}^2(\pi y_o/a)$$

$$-(16\pi/a)\sum_{n=2}^{\infty}\sin^2(n\pi y_o/a)/\left\{\gamma_n(n\pi/a+\gamma_n)^2\right\}$$

$$(\gamma_n^2 = (n\pi/a)^2 - 4)$$

<u>APPENDIX I</u>

Evaluation of (4.52)

In Fig.I, let the segments $\overline{AB},\overline{PQ},\overline{BP},\overline{AQ},\overline{BQ}$ be $d_{AB}, r_{PQ}$, $\rho_1, \rho_2, \rho_o$, respectively. The $d_{AB}$ is assumed to be large compared to $\rho_1$ and $\rho_2$. The angles $\theta_1, \theta_2$, and $\theta_o$ are measured versus the horizontal lines perpendicular to the segment $\overline{AB}$. Let us define the spatial complex variables $\xi_1$ and $\xi_2$ as $\rho_1 e^{i\theta_1}$ and $\rho_2 e^{i\theta_2}$, respectively.

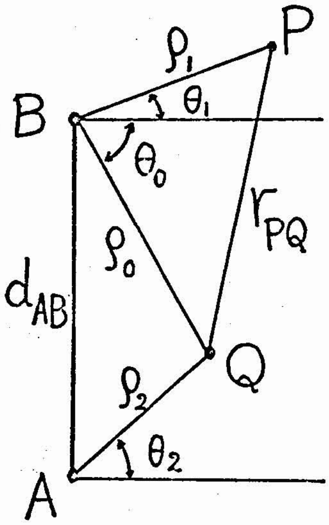



Fig.I

Construction of image points

Use of the mathematical formulae (IV-6) leads to

$$H_o^{(2)}(2r_{PQ}) = \sum_{n=-\infty}^{\infty} H_n^{(2)}(2\rho_o) J_n(2\rho_1)\, e^{-in(\theta_o+\theta_1)} \qquad \text{(I-1)}$$

$$H_n^{(2)}(2\rho_o) = e^{in(\theta_o+\theta_2)} \sum_{m=-\infty}^{\infty} (-i)^m H_m^{(2)}(2d_{AB}) J_{m+n}(2\rho_2)\, e^{im\theta_2} \qquad \text{(I-2)}$$

$$\left(\frac{\xi_2 - \xi_1}{\bar{\xi}_2 - \bar{\xi}_1}\right)^{\frac{n}{2}} e^{-in\theta_2} J_n(2\left|\xi_2 - \xi_1\right|)$$

$$= \sum_{m=-\infty}^{\infty} J_m(2\rho_1)\, J_{m+n}(2\rho_2)\, e^{im(\theta_2-\theta_1)} \qquad \text{(I-3)}$$

$$(\rho_2 > \rho_1)$$

Substituting (I-2) into (I-1), changing the order of sum, and using (I-3), we find

$$H^{(2)}(2r_{PQ}) =$$

$$\sum_{m=-\infty}^{\infty} (-i)^m H_m^{(2)}(2d_{AB})\, (\xi_2 - \xi_1)^m J_m(2\left|\xi_2 - \xi_1\right|)/\left|\xi_2 - \xi_1\right|^m$$

(I-4)

In the same way, we can obtain the same result in the case $\rho_2 < \rho_1$. We shall first assume that A is an origin of the $\xi$-plane and B is restricted to the imaginary axis ( the point i2na , $n > 0$ ). This means that P is a source point $\eta$ on the image post, and Q is an observation point $\xi$. Hence, we have $r_{PQ} = \left|\xi - \eta - i\ 2na\right|$. Conversely, if B is an origin and A is situated at the point $-\ i2na$ ( $n > 0$ ) on the imaginary axis, the points P and Q are an observation point and a source point, respectively. In this time, we have $r_{PQ} = \left| \xi - \eta + i\ 2na \right|$. Thus

$$H_0^{(2)}\left(2\left|\xi-\eta - i\ 2na\right|\right) =$$

$$\sum_{m=-\infty}^{\infty} (-i)^m\, \bar{H}_{\pm m}^{(2)}(4|n|a)(\xi-\eta)^m\, J_m(2|\xi-\eta|)/|\xi-\eta|^m$$

$$\left(n \gtrless 0\right) \tag{I-5}$$

Similarly,

$$H_0^{(2)}\left(2\left|\xi-\bar{\eta} + 2i\,(y_0 - na)\right|\right) =$$

$$\sum_{m=-\infty}^{\infty} (-i)^m\, H_{\pm m}^{(2)}(4|na - y_0|)(\xi-\bar{\eta})^m\, J_m(2|\xi-\bar{\eta}|)/|\xi-\bar{\eta}|^m$$

$$\left(n \gtrless 0\right) \tag{I-6}$$

Substitute (I-5) and (I-6) into (4.51), and see (A-9). Then, recalling $G_1 = 4\pi G_{1\ green}$, we reach at once (4.52).

# APPENDIX II

Evaluation of $A_o, B_o, B_1, A_2, B_2$

Equations (4.40) are two types of the Green's functions demanded in the expressions of the E field scattering and the H field scattering. For convenience, we shall rewrite both with use of the mathematical formulae (IV-10), such that the one differs from the other only by the sign. Then, it is possible to compare both the resulting ones with the fashions in which (4.37) is added to (4.51). In this way, we obtain

$$-\pi j \sum_{n=-\infty}^{\infty} H_o^{(2)}(2r_n) = (2\pi/a) \sum_{n=0}^{\infty} (\epsilon_n/\gamma_n) \cos\left[(n\pi/a) \cdot (y - y')\right] e^{-\gamma_n |x-x'|} \qquad \text{(II-1)}$$

where $\gamma_n = \sqrt{(x - x')^2 + (y - y' - 2na)^2}$ . Put $x = x'$, $y' = 0$.

A glance of the definition of $A_o$ shows that

$$A_o = \left\{ \pi j \, H_o^{(2)}(2y) + (2\pi/a) \sum_{n=0}^{\infty} (\epsilon_n/\gamma_n) \cos(n\pi y/a) \right\}_{y=0}$$

(II-2)

Referring to (IV-3), we can replace the first term of (II-2) by $\pi j + 2C + 2\log y$ . On the other hand, $\Upsilon_n^{-1}$ involved in the second term can be deformed as $(n\pi/a)^{-1} + 4\left[\Upsilon_n(n\pi/a)(n\pi/a + \Upsilon_n)\right]^{-1}$. In this deformation, the series of (II-2) concerning the terms $(n\pi/a)^{-1}$ can be readily summed up with the help of (IV-8). This is so-called "dominant series" ( such a manipulation is described in detail in [14, p.341] ). The dominant series becomes $-2\log\left[2\sin(\pi y/2a)\right] \simeq -2\log(\pi/a) - 2\log y$ . The singularity of the first term of (II-2) is therefore cancelled. We thus obtain the fashion of $A_o$ of (A-10).

We shall next consider $B_o$. Putting $x = x'$ and $2y_o = y - y'$ in (II-1) , we arrive at

$$B_o = (2\pi/a)\sum_{n=0}^{\infty}(\epsilon_n/\Upsilon_n)\cos(2n\pi y_o/a) \qquad \text{(II-3)}$$

This becomes (A-10) in the same manner as in $A_o$.

The definitions of $B_o$ and $B_1$ make us to notice that

$$B_1 = \tfrac{1}{4}(d B_o/dy_o) \qquad \text{(II-4)}$$

Accordingly, a straightfoward differentiation of $B_o$ of (A-10) leads to the available fashion of $B_1$ ( see (A-10) ).

By substitution of (IV-5)( m=1 ), we find

$$B_2 = -B_o - \pi j \sum_{n=-\infty}^{\infty} H_1^{(2)}(4|y_o - na|)/\{2|y_o - na|\} \tag{II-5}$$

Let us, in turn, deform the above expression in the following manners: 1) Putting x'= 0, y'= 0, y= 2 $y_o$, and x>0 in (II-1). 2) Differentiating both the sides with respect to x. 3) Dividing both the sides by x. 4) Letting x approach zero.

In so doing, the second term of (II-5) becomes

$$(\pi/a) \sum_{n=0}^{\infty} \epsilon_n \left. (e^{-\gamma_n x}/x) \right|_{x \to 0} \cdot \cos(2n\pi y_o/a) \tag{II-6}$$

We shall decompose $e^{-\gamma_n x}$ as $e^{-(n\pi/a)x} + e^{-(n\pi/a)x} \left[ e^{(n\pi/a - \gamma_n)x} - 1 \right]$. Note that the second term is approximately equal to $(n\pi/a - \gamma_n)x$. We thus find

$$e^{-\gamma_n x}/x \simeq e^{-(n\pi/a)x}/x + (n\pi/a - \gamma_n) \tag{II-7}$$

Insertion of the first term of (II-7) instead of $e^{-\gamma_n x}/x$ in (II-6) leads to $(\pi/2a)^2 \operatorname{cosec}^2(\pi y_o/a)$ ( refer to (IV-9) ).

We shall, in sequence, use the identity $n\pi/a - \gamma_n = (2a/n\pi) + 8/\left\{(n\pi/a)(n\pi/a + \gamma_n)^2\right\}$, except for $n=0$, in order to sum up the series regarding the second part of (II-7). Then, the term for $n=0$ ($= 2\pi/\gamma_0 a$) is to be taken into account after the manipulation. The above decomposition makes it possible to use (IV-8) again in the series involving $(2a/n\pi)$. Namely, the use yields $-2\log\left[2\sin(\pi y_0/a)\right]$. In such a way, we have

$$B_2 = -B_0 + (\pi/2a)^2 \operatorname{cosec}^2(\pi y_0/a) + 2\pi/\gamma_0 a - 2\log\left[2\sin(\pi y_0/a)\right] + (8\pi/a)\sum_{n=1}^{\infty} \cos(2n\pi y_0/a)/\left\{(n\pi/a)(n\pi/a + \gamma_n)^2\right\} \tag{II-8}$$

Substitution for $B_0$ leads to the final result (A-10).

Looking the definition of $A_2$, we notice

$$A_2 = \left\{ B_2 + \pi j\, H_2^{(2)}(4y_0) \right\}_{y_0 = 0} \tag{II-9}$$

Note that the second term is approximately equal to $-1-(4y_0^2)^{-1}$. Whereas the singularity of $B_2$ is $(\pi/2a)^2\operatorname{cosec}^2(\pi y_0/a) \simeq (4y_0^2)^{-1} + (\pi/a)^2/12$. Therefore, the summation cancels the singularities. The limiting value of $A_2$ in (II-9) is given in (A-10).

## APPENDIX III

Performance of the Residue Calculus

"A function f(z) which has an isolated singularity can be expanded into a Laurent series, and the value of a contour integral whose integrand is f(z) is given by the $2\pi i$ times of the residue i.e. the coefficient of $z^{-1}$ of such a Laurent series". This is called the Residue Theorem [19]. In applying this theorem to (2.38) or (2.39), we should pay attention to the Laurent expansion of the integrand with respect to u . However, a logarithmic function $\log(\xi - \eta)$ involved in $G_0$ may disturb its performance. The other terms being the powers of $\eta$ can, easily, be expanded into a series of u by only rearrengement. Briefly speaking, the Laurent expansion of $\log(\xi - \eta)$ is our present purpose.

First, we shall assume that $|w| > |u| \gg 1$ i.e. $\Upsilon_{cw}$ is a largely closed curve. Then, $\log(\xi - \eta)$ can be expressed by

$$\begin{aligned} \log(\xi - \eta) &= \log\left[\alpha(w - u)(1 - \Delta)\right] \\ &= \log(\alpha w) - \left(\frac{u}{w}\right) - \frac{1}{2}\left(\frac{u}{w}\right)^2 - \frac{1}{3}\left(\frac{u}{w}\right)^3 - \dots \\ &\quad - \Delta - \frac{1}{2}\Delta^2 - \frac{1}{3}\Delta^3 - \dots \end{aligned} \tag{III-1}$$

where $\Delta$ is - defined in (A-8) - a regular function of w and u, and in addition, tends to zero as either of w and u $\longrightarrow \infty$ . Certainly, (III-1) is a Laurent series. We always use this series regardless of that assumption. Its validity can be said by the interpretation of "Analytic Continuation"; because, it is surely valid when $|\xi|$ is sufficiently large, and the function is continued analytically as $|\xi|$ becomes small. In the way of the continuation, one may conceive a question of divergence of the series. This question will be solved immediately by considering the fact that there is no source in the outer region- say, there belongs no singularity to the outer region, and the fact that the function can be continued until a singularity of the function is encountered. Whereas, all the singularities lie in the disk $|w|<1$ ( $|u|<1$ ). Therefore, such a procedure is reasonable in the whole outer region.

## APPENDIX IV

Mathematical Formulae [16]

(IV-1)
$$H_o^{(2)}(\sqrt{1+a}\; z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\; 2^n}\, a^n z^n H_n^{(2)}(z)$$
$$( |a| < 1 )$$

(IV-2)
$$(z + \zeta)^{\frac{1}{2}n} H_n^{(2)}(\sqrt{z+\zeta}) =$$
$$\sum_{m=0}^{\infty} \frac{1}{m!\; 2^m}\, \zeta^m z^{\frac{1}{2}(n-m)} H_{n-m}^{(2)}(\sqrt{z}), \qquad (|z|>|\zeta|)$$

(IV-3)
$$H_o^{(2)}(x) \simeq (-j/\pi) \left\{\pi j + 2C + 2\log(x/2)\right\} (1 - x^2/4)$$
$$- (j/2\pi)\, x^2, \qquad C = 0.5772156649\ldots$$
$$( \text{to order } 3 )$$

(IV-4)
$$H_m^{(2)} = (-1)^m H_{-m}^{(2)}, \quad J_m = (-1)^m J_{-m}$$

(IV-5)
$$H_{m-1}^{(2)}(x) + H_{m+1}^{(2)}(x) = 2\, m\, H_m^{(2)}(x)/x$$

(IV-6)
$$Z_m( \sqrt{x^2 + y^2 - 2\,xy\cos\theta} )$$
$$= \left\{(x - y\, e^{i\theta})/(x - y\, e^{-i\theta})\right\}^{\frac{1}{2}m} \sum_{n=-\infty}^{\infty} Z_{m+n}(x)\, J_n(y)\, e^{in\theta}$$
$$= e^{-im\varphi} \sum_{n=-\infty}^{\infty} Z_{m+n}(x)\, J_n(y)\, e^{in\theta}, \qquad (|x|>|y|)$$

where $Z_m = J_m$ or $H_m^{(2)}$ , and $\varphi$ is the angle which corresponds to the side y of a triangle,$-\theta$ is defined as the angle between the sides x and y .

(IV-7) $$J_0(2x) \simeq 1 - x^2,\ J_1(2x) \simeq x - \tfrac{1}{2} x^3,$$

$$J_2(2x) \simeq \tfrac{1}{2} x^2, \quad J_3(2x) \simeq \tfrac{1}{6} x^3, \qquad (\text{ to order } 3\ )$$

(IV-8) $$\sum_{n=1}^{\infty} \frac{1}{n} \cos(nx) = -\log(2 \sin \frac{x}{2}\ ), \quad (\ x > 0\ )$$

(IV-9) $$\sum_{n=0}^{\infty} \epsilon_n \cos(ny)\ e^{-nx} = \frac{1}{2} \frac{\sinh x}{\cosh x - \cos y}$$

$$(\ x > 0,\ \epsilon_0 = \tfrac{1}{2},\ \epsilon_n = 1\ (\ n > 1\ )\ )$$

(IV-10) $$\cos x \cos y = \tfrac{1}{2} \left\{ \cos(x + y) + \cos(x - y) \right\}$$

$$\sin x \sin y = \tfrac{1}{2} \left\{ \cos(x - y) - \cos(x + y) \right\}$$

LIST OF FIGURES

Fig.2.1

The boundary in the normalized coordinate system.

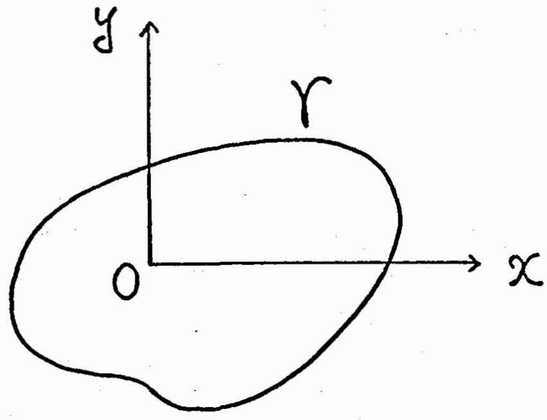
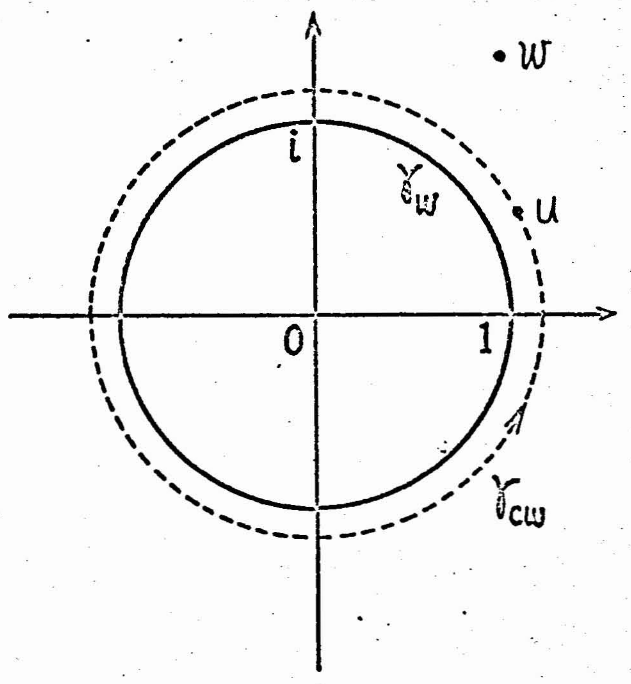



Fig.2.4

The boundary curve $\Gamma_w$ and the contour $\Gamma_{cw}$ in a mapped domain ( the w-plane ). The $\Gamma_w$ is a unit circle.



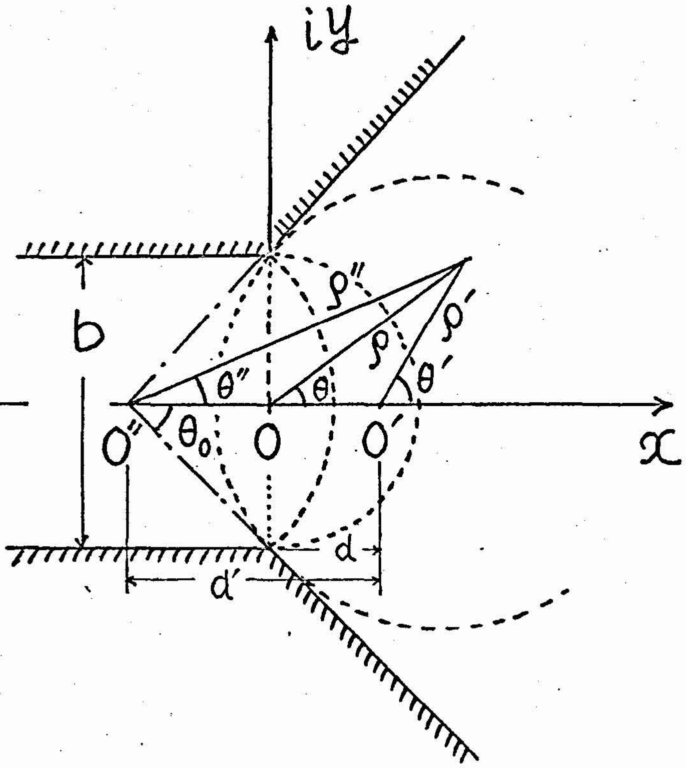



Fig.3.3

Configuration of the E plane taper at the mouth of a waveguide.

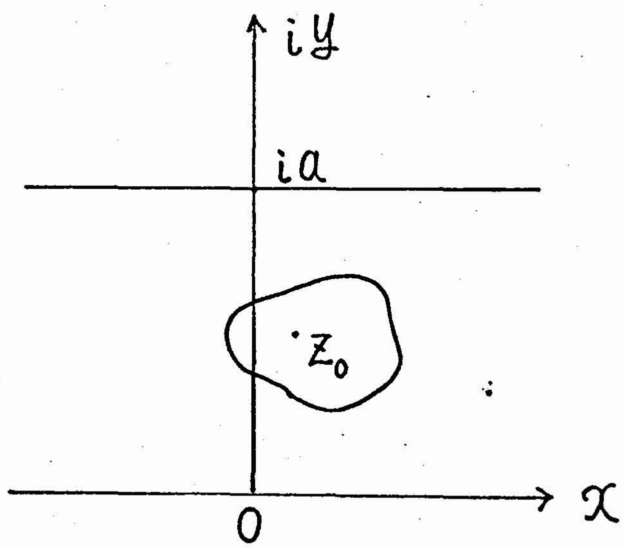



Fig.4.1

The scatterer in a rectangular waveguide.

Fig.5.1

A perfectly conducting strip in the normalized coordinate system.

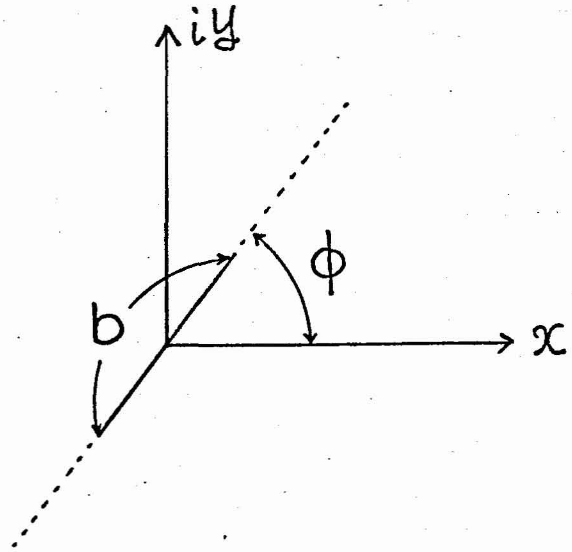



Fig.5.2

The T-network.

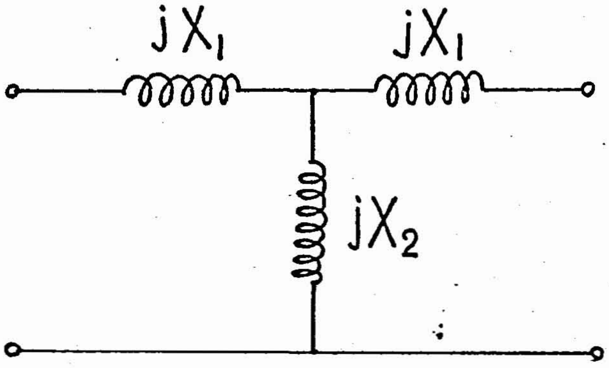


Fig.5.3

The cylindrical capacitive post in the normalized system.

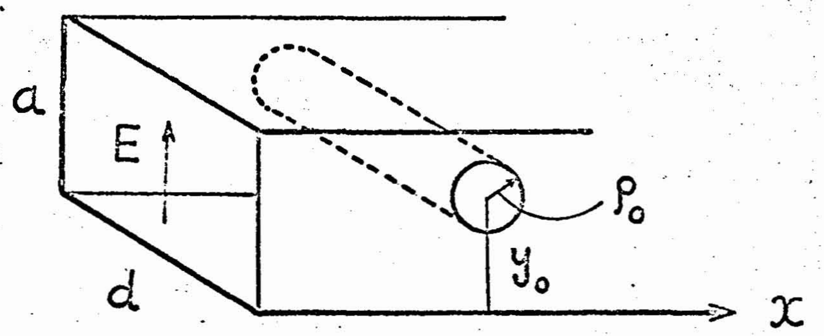
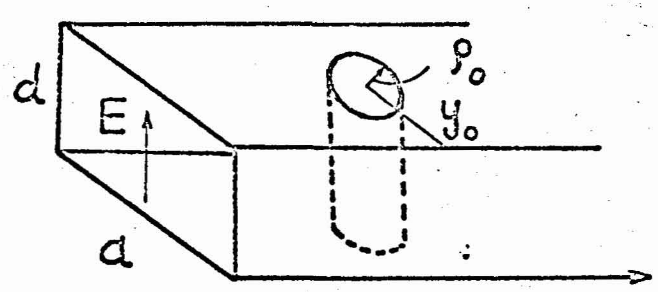



Fig.5.4

The cylindrical inductive post in the normalized system.



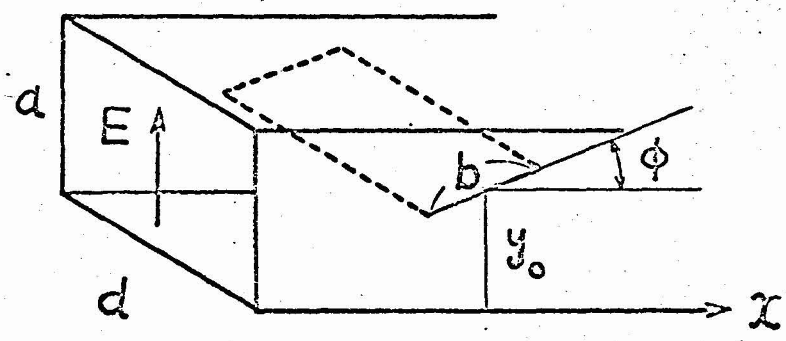



Fig.5.5

The capacitive strip in the normalized system.

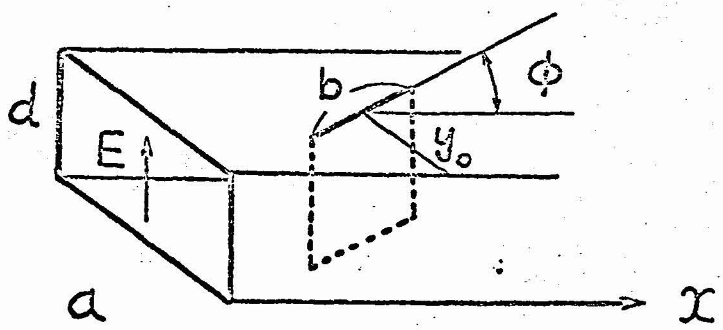



Fig.5.6

The inductive strip in the normalized system.

Fig.5.7

The inductive square post in the normalized system.

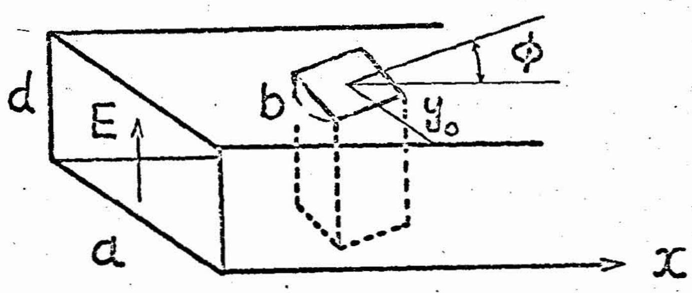
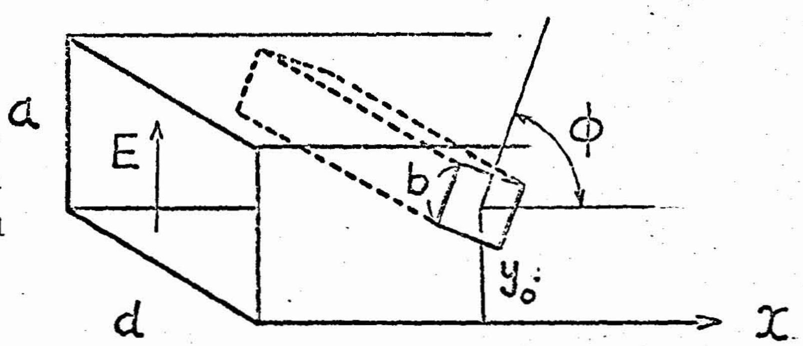



Fig.5.8

The capacitive square post in the normalized system.



## SYMBOLS

| | | |
|---|---|---|
| $i$ | : | a spatial imaginary unit. |
| $j$ | : | a temporal imaginary unit. |
| bar | : | a complex conjugate with respect to i. |
| C.C. | : | the complex conjugate of preceding terms with respect to i. |
| X,Y,Z | : | relevant coordinates. |
| x,y | : | normalized coordinates. |
| $z$ | : | $x + iy$. |
| $\beta$ | : | the propagation constant in the Z direction. |
| $k_o$ | : | the wave number in free space. |
| $k$ | : | the wave number in the X-Y plane, $\sqrt{k_o^2 - \beta^2}$ or $\sqrt{k_o^2-(n\pi/D)^2}$. |
| $w$ | : | a complex variable. |
| $u$ | : | a field or a point in the w-plane. |
| $\mathcal{H}$ | : | an operator. |
| $\Phi$ | : | a regular function in the inner region. |
| $\Psi_*$ | : | a regular function in the outer region. |
| K | : | see (2.27). |
| $b_n$ | : | the coefficients of $\Psi_*$ . |
| $\bar{\Psi}$ | : | a regular function in the outer region. |
| $z_o$ | : | a complex constant, a shifting factor. |
| $\xi$ | : | $z - z_o$ . |

| | | |
|---|---|---|
| $\gamma$ | : | the boundary curve in the z and $\xi$-planes. |
| $\gamma_c$ | : | a closed curve in the z and $\xi$-planes. |
| $\eta$ | : | a point in the $\xi$-plane. |
| $\alpha, \beta_n$ | : | see (2.37). |
| $\gamma_w$ | : | the corresponding boundary curve ( unit circle ) in the w-plane. |
| $\gamma_{cw}$ | : | the corresponding contour in the w-plane. |
| C | : | the Euler's constant (= 0.5772156... ). |
| $u_o$ | : | a radiation field. |
| $u_1$ | : | an image field. |
| x',y' | : | the variables of the Green's function ( $\eta = x' + iy' - z_o$ ). |
| $R_\ell$ | : | the mode amplitudes in a waveguide in Section 3.2, or the parameters in Chapters IV, V. |
| $T_\ell$ | : | the coefficients of radiation modes. |
| R | : | the reflection coefficient. |
| T | : | the transmission coefficient. |
| $u_W$ | : | the field in the waveguide region. |
| $u_T$ | : | the field in the taper region. |
| $\gamma_n$ | : | $\sqrt{(2n\pi/b)^2 - 4}$ in Section 3.2, $\sqrt{(n\pi/a)^2 - 4}$ in Chapter IV. |
| $G_{o\ green}$ | : | the Green's function in free space. |
| $G_{1\ green}$ | : | the Green's function for the image field. |
| $G_{green}$ | : | $G_{o\ green} + G_{1\ green}$. |

$\tilde{G}_o$ : see (2.40).

$\tilde{G}_1$ : see Page 72.

$\tilde{G}$ : $\tilde{G}_o + \tilde{G}_1$ , see (4.39).

$G_o$ : $4\pi G_{o\ green}$.

$G_1$ : $4\pi G_{1\ green}$.

$G$ : $4\pi G_{green}$.

$f(\psi)$: the boundary value for the E field.

$f_n^{(m)}$ : the coefficients of $f(\psi)$.

$g(\psi)$: the boundary value for the H field.

$g_n^{(m)}$ : the coefficients of $g(\psi)$.

$g'(\psi), f'(\psi), g'^{(m)}_n, f'^{(m)}_n$: see (4.47)-(4.50).

$\mathfrak{n}_z, \mathfrak{n}_\xi$ : the complex normals, see (2.10) or (4.18).

$\rho, \theta$ : $z = \rho e^{i\theta}$ in Section 2.3,
$\xi = \rho e^{i\theta}$ in Chapter IV.

$\epsilon_n$ : $\epsilon_o = \frac{1}{2}$, $\epsilon_n = 1$ ( $n \geq 1$ ).

$a_n$ : the coefficients of $\Psi$ .

$a_n^{(m)}$ : see (4.8).

$P_n, Q_n, S_n, U_{\ell n}, V_{\ell n}, I_n, K_n$ : the parameters defined in (A-1), (A-2), (A-5), (A-6), (A-7).

## REFERENCES


[1] L.V.Kantrovitch and V.I.Krylov, Approximate Methods of Higher Analysis. Groninger, Netherland: Noordhoff Ltd, 1958, pp.358-478.

[2] P.M.Morse and H.Feshbach, Methods of Theoretical Physics. McGraw-Hill, 1953, pp.443-453.

[3] R.H.T.Bates,"The theory of the point-matching method for perfectly conducting waveguides and transmission lines", IEEE Trans. MTT, vol.17, pp.294-301, June 1969.

[4] K.K.Mei and J.G.Bladel,"Scattering by perfectly-conducting rectangular cylinders", IEEE Trans. AP, vol.11, pp.185-192, March 1963.

[5] M.Hashimoto and K.Fujisawa,"Considerations on matrix methods and estimation of their errors", IEEE Trans. MTT, vol.18, pp.352-359, July 1970.

[6] H.E.Green,"The numerical solution of some important transmission line problems", IEEE Trans. MTT, vol.13, pp.676-692, September 1965.

[7] B.Noble, Methods Based on the Wiener-Hopf Techniques for Solution of Partial Differential Equations. Pergamon Press, 1958.

[8] I.N.Vekua, New Methods for Solving Elliptic Equations. North-Holland Pub., 1967.

[9] J.W.Miles,"The equivalent circuit for plane discontinuity in a cylindrical waveguide", Proc.IRE, vol.34, pp.728-742, October 1946.

[10] J.Schwinger and D.Saxon, Discontinuities in Waveguides. Notes on Lectures by J.Schwinger, Gordon and Breach Science Pub., 1968.

[11] N.Marcuvitz and J.Schwinger,"On the representation of the electric and magnetic fields produced by currents and discontinuities in waveguide.I", J.Appl.Phys., vol.22, pp.806-819, June 1951.

[12] N.Marcuvitz,"On the representation and measurement of waveguide discontinuities", Proc.Inst.Radio Engrs, vol.36, pp. 728-735, June 1958.

[13] N.Marcuvitz, Waveguide Handbook. McGraw-Hill, 1951.

[14] R.E.Collin, Field Theory of Guided Waves. McGraw-Hill, 1960.

[15] L.Lewin, Advanced Theory of Waveguides, Iliffe and Sons, 1951.

[16] G.N.Watson, A Treatise on the Theory of Bessel Functions. Cambridge at the University Press, 1922, pp.140.141,359,361.

[17] M.Hashimoto and K.Fujisawa,"A numerical technique for the waveguide with fin", private communication at the committee of Electro-magnetic field theory (Tokyo), April 1970.

[18] G.L.Ragan, Microwave Transmission Circuits. McGraw-Hill, 1948.

[19] M.Tsuji, The Theory of Complex Functions. Maki Pub. (Japanese) 1968.
or Z.Nehari, Introduction to Complex Analysis. Marusen, 1961.

[20] M.L.Burrows,"Equivalence of the Rayleigh solution and the extended boundary-condition solution for scattering problems", Electron Lett., vol.5, pp.227-278, 1969.

[21] R.F.Millar,"Rayleigh hypothesis in scattering problems", Electron Lett., vol.5, pp.416-417, 1969.

[22] R.H.T.Bates,"Rayleigh hypothesis, the extended boundary condition and point-matching", Electron Lett., vol.5, pp.694-695, 1969.

[23] M.L.Burrows,"Example of the generalised-function validity of the Rayleigh hypothesis", Electron Lett., vol.5, pp.694-695, 1969.

[24] J.D.Jackson, Classical Electrodynamics. John Wiley and Sons, 1962.

[25] D.S.Jones, The Theory of Electromagnetism. Pergamon Press, 1964.

[26] D.G.Bodnar and D.T.Paris,"New variational principle in electromagnetics", IEEE Trans. AP, vol.18, pp.216-233, March 1970.

[27] T.Kato,"Upper and lower bounds of scattering phases", Progress of Theoretical Physics (Kyoto Univ.), vol.6, pp.394-407, May - June 1951.

[28] N.A.Gal'chenko and V.S.Mikhalevsky,"Application of Swaltz's method to the calculation of the electrical parameters of single ridge & double ridge wavegudes", Radio Engineering and Electronic Physics, vol.15, pp.38-44, 1970.

[29] R.F.Harrington,"Matrix methods for field problems",Proc. IEEE, vol.55, pp.136-149, February 1967.

[30] R.F.Harrington, Field Computation by Moment Methods. Macmillan, 1968.

[31] G.G.Macfarlane,"Quasi-stationary field theory and its application to diaphragms and junctions in transmission lines and waveguides", Jour. IEE, vol.93, pp.703-719, 1946.

[32] J.Bladel, Electromagnetic Fields. McGraw-Hill, 1964, pp.393-397.

[33] H.Kober, Dictionary of Conformal Representations. Dover, 1957.