Semiclassical Approach to the Boson-Fermion System and the Renormalization

by

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Abstract

We study the semiclassical method for the system in which fermions and bosons are strongly coupled together. This method is the 1/N expansion around "the classical solution" which takes account of the reaction from the Dirac sea and excited fermions, where N is the number of the equivalent fermion components.

On the other hand, in the ordinary semiclassical approximation the Planck constant is used as an expansion parameter. This is also reviewed for self-containedness and the explanation of the stationary phase approximation in the path integral formalism.

Taking the Gross-Neveu model as a theoretical laboratory, the renormalization program for the excited state is studied in the framework of the semiclassical 1/N expansion. For the system in which the subtraction scheme can not be applied, the multiplicative renormalization is performed.

Contents

§1. Introduction

In recent years, a piece of experimental data of the lepton-hadron deep inelastic scattering agrees with predictions of the Weinberg-Salam model even with those corrected by the QCD higher order calculation. This supports that the fundamental theory of the particle physics may be the gauge theory. While the perturbative approach to the gauge theory works well in the short distance region of particle reactions, non-perturbative methods seem to be necessary in treating the long distance behavior of the particle reactions. Up to this time non-perturbative approaches have not been enough developed to give reliable proof for the quark confinement and to calculate particle spectra. It is, therefore, an important problem to develope and study the details of non-perturbative methods. In this article we discuss two semiclassical approximations. One of them is the ordinary WKB method[1] in which $\hbar$ is used as an expansion parameter. This is refered to as WKB($\hbar$) in this paper. The other is the semiclassical method for the system in which fermions and bosons are strongly coupled together.[2,3] It is the stationary phase approximation (S.P.A.) around "the classical solution" which takes account of the reaction from the Dirac sea and the excited fermions. This is the 1/N expansion method and is refered to as SPA(1/N), where N is the number of the equivalent fermion components.

We will summarize the procedure and the characteristic feature of each method.

(1) The ordinary WKB method (WKB($\hbar$)) for the system with

many degrees of freedom was formulated by Dashen, Hasslacher and Neveu[1]. In this method, one finds the classical solution of the equation of motion as the first step of the approximation. The classical solution, in general, involves some arbitrary constants, s_a (a=1, ···, f), due to the continuous symmetry of the system. Let us denote this classical solution by $\phi_{cl}(t, x; s)$. In the second step, one changes the dynamical variables from the original one, ϕ, to the zero mode coordinates s_a and the quantum fluctuation around the classical solution, η. These s_a and η are defined by

$$\phi(t, \vec{x}) = \phi_{cl}(t, \vec{x}; s(t)) + \eta(t, \vec{x}; s(t)), \tag{1.1}$$

where η satisfies

$$\int d^D\vec{x} \left\{ \frac{\partial}{\partial s_a} \phi_{cl}(t, \vec{x}; s) \right\} \eta(t, \vec{x}; s) = 0 , \quad (a=1, \cdots, f). \tag{1.2}$$

The equation (1.2), since $\partial\phi_{cl}/\partial s_a$ is the eigenfunction associated with the zero mode s_a, implies that the quantum fluctuation is orthogonal to the zero modes. In the third step, one treats the zero modes as quantized coordinates disregarding the quantum fluctuation. This step is the zeroth order approximation of WKB($\hbar$) and leads us to the Bohr-Sommerfeld quantization condition. In the fourth step, the quantization of η is taken account of under the action (or the Hamiltonian) approximated up to the second order with respect to η. This is the first order approximation (the full WKB approximation).

The characteristic feature of WKB($\hbar$) is that the excited states of the system are those which correspond to the periodic classical solutions. It is the important difference between WKB($\hbar$) and the perturbation theory. In terms of the graph theory, the zeroth order of WKB($\hbar$) is the tree approximation for the proper graphs. The first order is the one-loop approximation. From these graphical interpretation, it is obvious that the renormalizable theory in the perturbation is also renormalizable in WKB($\hbar$).

In WKB($\hbar$) for the system consisting of bosons and fermions, the fermions are neglected in the zeroth order approximation and treated as the quantum fluctuations in the first order. If the fermions are strongly coupled with the bosons, WKB($\hbar$) seems to be a wrong approximation. In many practical models[4,5] there exists no interesting classical solution if the fermions are treated as the perturbation to the bosons. We will, therefore, consider the semiclassical approach in which the reaction of the fermions are taken into account in the zeroth order approximation.

(2) The semiclassical method for the boson-fermion system (SPA(1/N)) is the 1/N expansion method when the system has an O(N) symmetry. The basic idea of SPA(1/N) was first presented by Dashen, Hasslacher and Neveu (DHN)[2] and the full semiclassical program (WKB) was formulated by Kikkawa and the author.[3] In the path integral formalism, the fermion coordinates (anti-commuting coordinates) are first integrated out to obtain an effective action containing only boson coordinates (commuting coordinates).

This means that the fermions are quantized in the background fields of the bosons. In evaluating the path integral for the effective action of the bosons, S.P.A. is adopted. As the second step of SPA(1/N), one finds the stationary point of the effective action. The solution which provides the stationary point is called "the classical solution" (CS). This CS is the solution of coupled equations for bosons and fermions and takes account of the reaction from the Dirac sea and the excited fermions. In the zeroth order approximation of SPA(1/N) one integrates out (quantizes) the zero modes of CS to obtain the Bohr-Sommerfeld quantization condition. This step was given by DHN.[2] In order to complete the full semiclassical method (WKB), one has to evaluate the quantum fluctuations around CS up to the quadratic term in the effective action. There exists, however, a difficulty that the action for the quantum fluctuations always becomes non-local due to the fermion propagation which appears as a result of integrating the fermion field beforehand. In order to avoid the difficulty, we introduced new auxiliary fields in reference 3. With the help of the auxiliary field, the quantum correction term of the effective action can be made local and quadratic with respect to the boson fields and the auxiliary fields. Then the integration (quantization) of the quantum fluctuations can be done and provides the first order approximation of SPA(1/N). In SPA(1/N) excited states with the definite fermion number are obtained. In terms of the graph theory, the zeroth order of SPA(1/N) is the tree approximation of the boson fields together

with the one-loop approximation of the fermion fields for the proper graphs. The first order is the one-loop approximation of the boson fields while it includes the infinite fermion loops. The renormalization in the zeroth order of SPA(1/N) is, therefore, trivial. On the other hand, even if the system is renormalizable in the perturbation method, the renormalizability of the first order is not obvious in SPA(1/N). It is because the proper vertex in the first order of SPA(1/N) is the sum of the proper graphs with different number of loops.

The renormalization of SPA(1/N) is one of the main purposes in this article. In order to calculate the renormalization constants, one needs propagators and proper vertex functions. If one wishes to perform the consistent renormalization program in a non-perturbative method, one has to define proper vertices within the method. We will, therefore, calculate the generating functional for proper vertices, Γ, (proper Γ) in the semiclassical method (SPA(1/N)). Once the renormalization constants are determined by using proper Γ in the vacuum sector, the energy levels of excited states can be shown to be finite by the use of these renormalization constants.

In Sec. II we will review the semiclassical method for the boson system (WKB($\hbar$)) which was formulated by DHN.[1] In Sec. III the semiclassical method for the boson-fermion system (SPA(1/N)) will be studied. This is the summary of the works by DHN[2] and Kikkawa and the author.[3] In Sec. IV the renormalization program in SPA(1/N) will be studied in the Gross-Neveu model.[13] Although some were reported in the previous paper,[3] main part

of this section is new. In Sec. V we will discuss the questions about the renormalization in the 1/N expansion methods. As the special case we will study the system including $(\bar{\psi}\psi)^2/N$ term as well as $\sigma(\bar{\psi}\psi)$ term. The renormalization of this system in the zeroth order approximation of the 1/N expansion has to be performed by the multiplicative renormalization program, since the usual subtraction scheme can not be applied. This is also the original contribution of this paper.

§II. Semiclassical method for the boson system

A. Preliminary remarks

In this section we treat a system with r commuting coordinates q_i in the path integral (PI) formalism[6]. The generalization to field theory is straightforward. In order to compute energy spectra, it is useful to calculate the propagator G(E) defined by

$$G(E) \equiv \mathrm{Tr} \frac{1}{E - \hat{H}} = \sum_n \frac{1}{E - E_n} \quad , \tag{2.A.1}$$

where $\hat{H}$ is the Hamiltonian operator of the system and E_n is the energy of the n-th eigen state. From (2.A.1) one can confirm that the energy spectra are obtained by inspecting poles in the propagator G(E). In the PI formalism, G(E) is represented by

$$G(E) = \frac{1}{i\hbar} \int_0^\infty dT \cdot \mathrm{Tr}\left[\exp\left\{\frac{i}{\hbar}(E - \hat{H})T\right\}\right] \tag{2.A.2}$$

$$= \frac{1}{i\hbar} \int_0^\infty dT \cdot Z(T) \exp\left(\frac{i}{\hbar} E T\right) \quad , \tag{2.A.3}$$

where the partition function Z(T) is given by

$$Z(T) = \int \prod_{t=0}^{T} \left[\frac{d^r p(t)}{(2\pi\hbar)^r} d^r q(t)\right] \delta^r(q(T) - q(0)) \times \exp\left[\frac{i}{\hbar} \int_0^T dt\, (p\dot{q} - H)\right] \tag{2.A.4}$$

$$\equiv \lim_{\substack{K\to\infty \\ (\varepsilon\to 0)}} \int \left[\prod_{k=0}^{K} d^r q(k) \prod_{k=1}^{K} \frac{d^r p(k)}{(2\pi\hbar)^r}\right] \delta^r(q(K) - q(0)) \times \exp\left[\frac{i}{\hbar} \sum_{k=1}^{K} \varepsilon \{p(k)\, \dot{q}(k) - H(k)\}\right] . \tag{2.A.5}$$

In (2.A.5) the time interval T is devided into K bits ($\varepsilon \equiv T/K$), $q(t=k\varepsilon) \equiv q(k)$, $p(t=(k-\frac{1}{2})\varepsilon) \equiv p(k)$ and $\dot{q}(k) \equiv \{q(k)-q(k-1)\}/\varepsilon$. We will use the compact functional notation (2.A.4) for PI and perform formal integrations as if the integration variables are continuous in time. One has to, however, reconfirm the results by using the discrete PI defined on a finite mesh (2.A.5). In order that the procedure of functional calculations has precise agreement with that of the discrete PI, the Hamiltonian in (2.A.5) has to be a midpoint Hamiltonian[7], i.e.,

$$H(k) = H\left(p(k), \bar{q}(k)\right), \tag{2.A.6}$$

where $\bar{q}(k) \equiv \{q(k)+q(k-1)\}/2$. There occurs, however, no factor ordering problem within the semiclassical approximation because the reordering effects[8] are of order $\hbar^2$.

The wave function $\Psi_n(q)$ of the n-th energy eigen state can be obtained by inspecting the residue of pole in the propagator

$$\begin{aligned} G_T(q,q';E) &\equiv \langle q | \frac{1}{E-\hat{H}} | q' \rangle = \sum_n \frac{\Psi_n(q)\Psi_n^*(q')}{E-E_n} \\ &= \frac{1}{i\hbar}\int_0^\infty dT \, \langle q | e^{-\frac{i}{\hbar}\hat{H}T} | q' \rangle \, e^{\frac{i}{\hbar}ET}. \end{aligned} \tag{2.A.7}$$

This propagator is useful in both the Minkowski semiclassical method (WKB($\hbar$)) and the Euclidean semiclassical method[9] such as the one appears in the instanton business.

B. Transition amplitude

In this subsection we will apply the stationary phase approximation (S.P.A.) to the Feynman transition amplitude. This approach was first given by Keller, Gutzwiller and Maslov[10] and developed by DHN[1]. This section is the review of these works. The

transition amplitude is given by

$$F(q,q';T) \equiv \langle q| \exp\{-\frac{i}{\hbar}\hat{H}T\}|q'\rangle$$
$$= \int \prod_{t=0}^{T} \left[\frac{d^r p(t)}{(2\pi\hbar)^r} d^r q(t)\right] \delta^r(q(T)-q)\,\delta^r(q(0)-q') \qquad (2.B.1)$$
$$\times \exp\left[\frac{i}{\hbar}\int_0^T dt\,(p\dot{q} - H(p,q))\right].$$

In the classical limit ($\hbar \to 0$) S.P.A. to PI(2.B.1) will work well. One first looks for the stationary point of the exponent (action, $\int dt(p\dot{q}-H)$) in PI(2.B.1), and finds it to be the classical solution (p_{cl}, q_{cl}) of the Hamilton equations;

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad , \quad (i=1,\cdots,r) \, , \qquad (2.B.2)$$

which satisfies

$$q_i(T) = q_i \quad , \quad q_i(0) = q_i' \quad , \quad (i=1,\cdots,r) \, , \qquad (2.B.3)$$

because of the delta functions in (2.B.1). This is of order $\hbar^0$. Second, the quantum correction is calculated up to the second order in $(\tilde{p},\tilde{q})$ by substituting

$$q_i(t) = q_{cl,i}(t) + \tilde{q}_i(t) \quad , \quad p_i(t) = p_{cl,i}(t) + \tilde{p}_i(t) \, , \quad (i=1,\cdots,r) \qquad (2.B.4)$$

in the action. The Gauss integral is dominated by the region $|\tilde{p}|, |\tilde{q}| \leq \hbar^{1/2}$ so $\tilde{p}$ and $\tilde{q}$ are of order $\hbar^{1/2}$. This shows that S.P.A. to PI (2.B.1) is the $\hbar$ expansion.

By the substitution of (2.B.4) the transition amplitude F turns out to be

$$F(q,q';T) = \exp\left(\frac{i}{\hbar}S_{cl}\right)\int \prod_{t=0}^{T} \left[\frac{d^r \tilde{p}(t)}{(2\pi\hbar)^r} d^r \tilde{q}(t)\right] \delta^r(\tilde{q}(T))\,\delta^r(\tilde{q}(0))$$
$$\times \exp\left(\frac{i}{\hbar}S_Q\right), \qquad (2.B.5)$$

where

$$S_Q = \int_0^T dt \left[\tilde{p}\dot{\tilde{q}} - \frac{1}{2}\{ \tilde{p}(\partial_p \partial_p H)_{cl}\tilde{p} + 2\tilde{p}(\partial_p \partial_q H)_{cl}\tilde{q} + \tilde{q}(\partial_q \partial_q H)_{cl}\tilde{q} \} \right] \tag{2.B.6}$$

and

$$S_{cl} = \int_0^T dt \left[p_{cl}\dot{q}_{cl} - H(p_{cl}, q_{cl}) \right]. \tag{2.B.7}$$

The matrix notation is used in the above, e.g.,

$$\tilde{p}(\partial_p \partial_p H)_{cl}\tilde{p} \equiv \tilde{p}_i \frac{\partial^2 H}{\partial p_i \partial p_j}(p_{cl}, q_{cl})\tilde{p}_j \quad , \quad (i,j=1,\cdots,r) \, . \tag{2.B.8}$$

There is no linear term with respect to $(\tilde{P}, \tilde{q})$ in the exponent owing to the stationary condition (2.B.2). By the condition for the classical solution, (2.B.3), the restriction on q in (2.B.1) is reduced to the condition on $\tilde{q}$, i.e., $\tilde{q}(T)=\tilde{q}(0)=0$ in (2.B.5). The integration over $\tilde{p}$ is performed by completing the square. In order to rewrite the resulting PI in the configuration space, one should note the relations between Hamiltonian and Lagrangian formalisms;

$$L(q,\dot{q}) = p\dot{q} - H(p,q) \tag{2.B.9}$$

and

$$\begin{aligned} A_{ij} &\equiv (\partial_{\dot{q}_i}\partial_{\dot{q}_j}L)_{cl} = (\partial_{p_i}\partial_{p_j}H)^{-1}_{cl} = A_{ji} \, , \\ B_{ij} &\equiv (\partial_{\dot{q}_i}\partial_{q_j}L)_{cl} = -(\partial_{p_i}\partial_{p_k}H)^{-1}_{cl}(\partial_{p_k}\partial_{q_j}H)_{cl} \, , \\ C_{ij} &\equiv (\partial_{q_i}\partial_{q_j}L)_{cl} = C_{ji} \\ &= -(\partial_{q_i}\partial_{q_j}H)_{cl} + (\partial_{q_i}\partial_{p_k}H)_{cl}(\partial_{p_k}\partial_{p_l}H)^{-1}_{cl}(\partial_{p_l}\partial_{q_j}H)_{cl} \, , \\ &(i,j,k,l = 1,\cdots,r) \, . \end{aligned} \tag{2.B.10}$$

If one performs the $\tilde{p}$ integration, one obtains the following representation of F;

$$F = \exp\left(\frac{i}{\hbar} S_{cl}\right) \int \prod_{t=0}^{T} \left[d^r \tilde{\xi}(t)\right] \prod_{t=0}^{T} \left[\det A(t)/\Delta\right]^{1/2} \times \delta^r(\tilde{\xi}(T)) \delta^r(\tilde{\xi}(0)) \exp\left(\frac{i}{\hbar} S_Q\right), \tag{2.B.11}$$

where the quantum action S_Q is given by

$$S_Q = \int_0^T dt \left(\frac{1}{2} \dot{\tilde{\xi}} A \dot{\tilde{\xi}} + \dot{\tilde{\xi}} B \tilde{\xi} + \frac{1}{2} \tilde{\xi} C \tilde{\xi}\right) \tag{2.B.12}$$

and

$$S_{cl} = \int_0^T dt\, L(\xi_{cl}, \dot{\xi}_{cl}) . \tag{2.B.13}$$

The determinant factor in (2.B.11),

$$\prod_{t=0}^{T} \left[\det A(t)/\Delta\right]^{1/2} \equiv \lim_{\substack{K\to\infty \\ (\varepsilon\to 0)}} \prod_{k=1}^{K} \left[\det\left\{A(\bar{\xi}_{cl}(k), \dot{\xi}_{cl}(k))\right\}/(i2\pi\hbar\varepsilon)^r\right]^{1/2}, \tag{2.B.14}$$

should be kept in the configuration PI when the coefficient of the quadratic term with respect to the velocity in the Lagrangian is nontrivial. It should be noted that the action of the quantum correction (2.B.12) can be obtained by expanding the Lagrangian around the classical solution, too.

To make the integral over $\tilde{q}$, we introduce the mapping

$$y(t) = \tilde{\xi}(t) - \int_0^t dt'\, \dot{R}(t') R^{-1}(t') \tilde{\xi}(t') \tag{2.B.15}$$

and its inverse

$$\tilde{\xi}(t) = y(t) - R(t) \int_0^t dt'\, R^{-1}(t') \dot{R}(t') y(t') . \tag{2.B.16}$$

The r×r matrix R has to satisfy the Euler equation of the quantum action (2.B.12),

$$A\ddot{R} + (\dot{A} + B - B^T)\dot{R} + (\dot{B} - C)R = 0, \tag{2.B.17}$$

and

$$R^{-1T}\dot{R}^T A - A\dot{R}R^{-1} = B - B^T, \tag{2.B.18}$$

where the superscript T represents the transposition. The end point conditions on y,

$$y(0) = 0, \tag{2.B.19}$$

$$R(T)\int_0^T dt\ R^{-1}(t)\dot{y}(t) = 0, \tag{2.B.20}$$

are derived from both the equation (2.B.16) and the end point conditions on $\tilde{q}$ in (2.B.11). One can avoid the restriction on y (2.B.20), which is non-local in time, by introducing Lagrange multipliers α_i (i=1, ··, r)[1]. The Jacobian of the transformation (2.B.15), which is a Volterra integral equation[11], is given by

$$\det\left(\frac{\delta y}{\delta \tilde{q}}\right) = \exp\left[-\frac{1}{2}\int_0^T dt\ \mathrm{tr}\{\dot{R}(t)R^{-1}(t)\}\right] = \left[\frac{\det R(T)}{\det R(0)}\right]^{1/2}. \tag{2.B.21}$$

The factor (1/2) is due to the midpoint prescription previously mentioned. Then F can be written in terms of y and α;

$$F = \left[\frac{\det R(T)}{\det R(0)}\right]^{1/2} e^{\frac{i}{\hbar}S_d}\int\prod_{t=0}^{T}\left[d^r y(t)\right]\prod_{t=0}^{T}\left[\det A(t)/\Delta\right]^{1/2}\delta^r(y(0)) \times\int\frac{d^r\alpha}{(2\pi\hbar)^r}\exp\left[\frac{i}{\hbar}\int_0^T\left\{\frac{1}{2}\dot{y}(t)A(t)\dot{y}(t) + \alpha R(T)R^{-1}(t)\dot{y}(t)\right\}dt\right]. \tag{2.B.22}$$

If one notes that y(T) has no restriction and shoud be integrated out, the integration in (2.B.22) can be performed as follows:

One first changes the variables from $y(k)$ $(k=1, \cdots, K)$ to $z(k)=\dot{y}(k)$ $(k=1, \cdots, k)$, then the integrations over z and α are performed by completing the square. The result is

$$F = \left[\frac{\det R(T)}{\det R(0)}\right]^{1/2} e^{\frac{i}{\hbar}S_{cl}} \left[\det\left\{R(T)\int_0^T dt\left(\tilde{R}^{-1}(t)\tilde{A}^{-1}(t)\tilde{R}^{-1T}(t)\right)R^{T}(T)\right\}\right]^{-1/2} / (2\pi i\hbar)^{r/2}$$

$$= \frac{\exp\left(\frac{i}{\hbar}S_{cl}\right)}{(2\pi i\hbar)^{r/2}}\left[\det\left(R(T)R(0)\right)\cdot\det\left\{\int_0^T dt\, R^{-1}(t)\tilde{A}^{-1}(t)\tilde{R}^{-1T}(t)\right\}\right]^{-1/2}. \qquad (2.B.23)$$

The determinant factor in (2.B.22), $\prod_{t=0}^{T}[\det A(t)/\Delta]^{1/2}$, cancels out with that induced by the z-integration. This can be checked by the precise calculation using the discrete PI.

Finaly, let us prove the equality

$$\frac{\partial^2 S_{cl}(q,q')}{\partial q_i \partial q'_j} = -\left[\tilde{R}^{-1T}(T)\left\{\int_0^T dt\, \tilde{R}^{-1}(t)\tilde{A}^{-1}(t)\tilde{R}^{-1T}(t)\right\}^{-1}R^{-1}(0)\right]_{ij}, \quad (i,j=1,\cdots,r), \qquad (2.B.24)$$

where S_{cl} is considered as the function of q and q' through q_{cl} as shown in the condition (2.B.3). Let us define the r×r matrix Q by

$$Q_{ij}(t) \equiv \frac{\partial q_{cl,i}(t)}{\partial q'_j}, \quad (i,j=1,\cdots,r). \qquad (2.B.25)$$

The m-th row of Q, $Q^{(m)}$, is the solution of the Euler equation for the quantum fluctuation $\tilde{q}$, (2.B.17), which can be proved by differentiating the Euler equation for the classical solution q_{cl} by q'. From the fact that $q_{cl}(T)=q$ and $q_{cl}(0)=q'$ it is obvious that

$$Q(t=T)=0, \quad Q(t=0)=1 \text{ (identity matrix)}. \qquad (2.B.26)$$

The ℓ-th row of R, $R^{(\ell)}$, is also the solution of (2.B.17). For two solutions of the Euler equation (2.B.17), $(\tilde{p}^{(1)}, \tilde{q}^{(1)})$ and

$(\tilde{p}^{(2)}, \tilde{q}^{(2)})$, there is a constant of motion,

$$\tilde{\xi}^{(1)} \tilde{p}^{(2)} - \tilde{p}^{(1)} \tilde{\xi}^{(2)}, \tag{2.B.27}$$

where $\tilde{p}$ is derived from the quantum action (2.B.12) as

$$\tilde{p} = A \dot{\tilde{\xi}} + B \tilde{\xi}. \tag{2.B.28}$$

Then the r×r matrix defined by

$$\begin{aligned} M_{lm} &\equiv R^{(l)} \{ A \dot{Q}^{(m)} + B Q^{(m)} \} - \{ \dot{R}^{(l)} A + R^{(l)} B^T \} Q^{(m)} \\ &= (R^T A \dot{Q} - \dot{R}^T A Q + R^T (B - B^T) Q)_{lm}, \quad (l,m = 1, \cdots, r), \end{aligned} \tag{2.B.29}$$

is a constant of motion. By the use of (2.B.18) M turns out to be

$$M = R^T(t)\, A(t)\, R(t)\, \frac{d}{dt} \left(R^{-1}(t)\, Q(t) \right). \tag{2.B.30}$$

One can integrate (2.B.30) over time to obtain

$$\begin{aligned} -R^{-1}(0) &= \left[\int_0^T dt\, R^{-1}(t)\, A^{-1}(t)\, R^{-1T}(t) \right] M \\ &= \left[\int_0^T dt\, R^{-1}(t)\, A^{-1}(t)\, R^{-1T}(t) \right] R^T(T)\, A(T)\, \dot{Q}(T), \end{aligned} \tag{2.B.31}$$

where the equations (2.B.26) have been used. If one notes that

$$\frac{\partial S_{cl}}{\partial \xi_i} = \frac{\partial L}{\partial \dot{\xi}_i} \left(\xi_{cl}(T), \dot{\xi}_{cl}(T) \right), \quad (i = 1, \cdots, r),$$

$\partial^2 S_{cl} / \partial q \partial q'$ can be written by

$$\begin{aligned} \frac{\partial^2 S_{cl}}{\partial \xi_i \partial \xi'_j} &= \frac{\partial \dot{\xi}_{cl,m}(T)}{\partial \xi'_j} \frac{\partial^2 L}{\partial \dot{\xi}_m \partial \dot{\xi}_i} \left(\xi_{cl}(T), \dot{\xi}_{cl}(T) \right) + \frac{\partial \xi_{cl,m}(T)}{\partial \xi'_j} \frac{\partial^2 L}{\partial \xi_m \partial \dot{\xi}_i} \left(\xi_{cl}(T), \dot{\xi}_{cl}(T) \right) \\ &= \left(A(T)\, \dot{Q}(T) \right)_{ij}, \quad (i,j,m = 1, \cdots, r). \end{aligned} \tag{2.B.32}$$

Thus the equality (2.B.24) is proved by using (2.B.31) and (2.B.32).

As the result of (2.B.24), F turns out to be

$$F = \left(\frac{i}{2\pi\hbar}\right)^{r/2} \exp\left(\frac{i}{\hbar} S_{cl}\right) \left[\det\left(\frac{\partial^2 S_{cl}}{\partial \xi \partial \xi'}\right)\right]^{1/2} . \qquad (2.B.33)$$

If there are critical points (turning points) along the classical trajectory, additional phases are needed in (2.B.33).[10),12)] They will not be important in what follows and will be disregarded from now on.

So far, we have evaluated PI (2.B.1) around one classical orbit satisfying the end point condition (2.B.3). In a system with many degrees of freedom there exists, in general, a discrete set of classical orbits satisfying the same end point condition. Since the quantum fluctuation around the classical orbit is of order $\hbar^{1/2}$ and the separation distance between classical orbits is of order $\hbar^0$, the transition amplitude F should be the summation over the contributions around all classical orbits, i.e.,

$$F = \sum_{\substack{\text{classical} \\ \text{orbits} \\ (\xi_{cl}(T)=\xi,\ \xi_{cl}(0)=\xi')}} \left(\frac{i}{2\pi\hbar}\right)^{r/2} \exp\left(\frac{i}{\hbar} S_{cl}\right) \left[\det\left(\frac{\partial^2 S_{cl}}{\partial \xi \partial \xi'}\right)\right]^{1/2} . \qquad (2.B.34)$$

C. Zero mode

In the following we will evaluate the partition function Z(T) which is written by the transition amplitude F as

$$Z(T) = \int d^r \xi \; F(\xi, \xi; T) . \qquad (2.C.1)$$

Substituting the approximated form for F in WKB(ħ) (2.B.34) into (2.C.1), one obtains that

$$Z(T)=\int d^r q \sum_{cl}\left(\frac{i}{2\pi\hbar}\right)^{r/2}\left|\det\frac{\partial^2 S_{cl}}{\partial q\,\partial q'}\Big|_{q'=q}\right|^{1/2}\exp\left[\frac{i}{\hbar}S_{cl}(q,q)\right]. \quad (2.C.2)$$

The sum $\sum_{cl}$ in (2.C.2) implies the sum over a discrete set of classical orbits satisfying the end point condition $q_{cl}(T)=q_{cl}(0)=q$. The q-intergration may be performed by S.P.A.. The stationary points are given by

$$0=\frac{\partial S_{cl}(q,q')}{\partial q_i}\Big|_{q'=q}+\frac{\partial S_{cl}(q,q')}{\partial q_i'}\Big|_{q'=q}=p_{cl,i}(T)-p_{cl,i}(0)\ ,(i=1,\cdots,r). \quad (2.C.3)$$

This stationary condition with the help of trace condition, $q_{cl}(T)=q_{cl}(0)=q$, selects classical periodic orbits with period T from the sum over a discrete set of classical orbits in (2.C.2). If the classical periodic solution is not invariant under any symmetry transformation of the system, the transformed orbit is an another periodic orbit with a different end point. It is convenient for the practical purpose to exchange the order of the summation and the q-integration in (2.C.2). In that case the sum should be taken over the discrete set of all the periodic orbits which can not be connected by the symmetry transformation, while the orbits connected with each other by this transformation should be regarded as the same orbit. For later convenience we will denote the parameters of the symmetry transformation by s_a (a=1, ··, f) and the transformed classical periodic solution by $\bar{q}_{cl}$ (t;s).

It is, in general, impossible to find all the periodic orbits, so some approximations are needed. We will therfore approximate

the sum in (2.C.2) by the sum over a family of periodic orbits including multiple traverses of the basic orbits.

Let us expand the classical action $S_{cl}(q,q')$ around a stationary point q^* up to the second order in $q-q^*$ and $q'-q^*$. For this purpose we will define $r\times r$ matrices G and H by

$$G_{ij} \equiv 2\left.\frac{\partial^2 S_{cl}(\xi,\xi')}{\partial(\xi_i+\xi_i')\partial(\xi_j+\xi_j')}\right|_{\xi=\xi'=\xi^*} = G_{ji}\ , \tag{2.C.4}$$

$$H_{ij} \equiv 2\left.\frac{\partial^2 S_{cl}(\xi,\xi')}{\partial(\xi_i-\xi_i')\partial(\xi_j-\xi_j')}\right|_{\xi=\xi'=\xi^*} = H_{ji}\ ,\ (i,j=1,\cdots,r). \tag{2.C.5}$$

By the use of G and H the classical action can be approximated as

$$S_{cl}(\xi,\xi') = S_{cl}(\xi^*,\xi^*)+\frac{1}{4}(\xi+\xi'-2\xi^*)G(\xi+\xi'-2\xi^*) + \frac{1}{4}(\xi-\xi')H(\xi-\xi'), \tag{2.C.6}$$

where linear terms and cross terms with respect to $(q+q')$ and $(q-q')$ do not appear because of the stationary condition (2.C.3) and the time reversal invariance of the system, $S_{cl}(q,q')=S_{cl}(q',q)$. In this approximation, $\partial^2 S_{cl}/\partial q\partial q'$ can be written by

$$\frac{\partial^2 S}{\partial\xi_i\partial\xi_j'} = \frac{1}{2}(G_{ij}-H_{ij}). \tag{2.C.7}$$

When the end points q and q' are near the stationary point q^*, the classical solution $q_{cl}(q,q')$ can be expanded as

$$\xi_{cl,i}(t;\xi,\xi') = \bar{\xi}_{cl,i}(t) + \tilde{\xi}_i(t), \tag{2.C.8}$$

where $\bar{q}_{cl}(t)=q_{cl}(q=q^*,\ q'=q^*)$. In (2.C.8) the quantum correction $\tilde{q}$ is the solution of the Euler equations for the quantum action S_Q (2.B.12) satisfying the end point condition

$$\tilde{q}_i(t=0) = q_i' - q_i^* \quad , \quad \tilde{q}_i(t=T) = q_i - q_i^* \quad , \quad (i=1,\cdots,r). \tag{2.C.9}$$

Using the quantum action S_Q and the quantum correction $\tilde{q}$, in the same approximation as (2.C.6) the classical action can be written by

$$S_{cl}(q,q') = S_{cl}(q^*,q^*) + S_Q[q]. \tag{2.C.10}$$

This equation implies that one can obtain the information about G and H from the quantum action S_Q.

In the quantum action around the periodic orbit the coefficients, A, B and C, are periodic functions of time. The Euler equations (2.B.17), which are linear in $\tilde{q}$, are second order differencial equations with the periodic coefficients. For these equations, one can find the 2r-independent solutions, $\xi^{(\alpha)}(t)$, with the property that

$$\xi_i^{(\alpha)}(t+T) = \exp(-i\nu_\alpha)\,\xi_i^{(\alpha)}(t) \quad , \quad (\alpha=1,\cdots,2r\ ;\ i=1,\cdots,r). \tag{2.C.11}$$

Since the classical periodic solution has to be stable, all the ν_α's should be real and are called stability angles. The stability angles appear in plus-minus pairs, ν and $-\nu$, because the Euler equations are real. The classical periodic orbit $\bar{q}_{cl}$ has the arbitrary parameters s_a due to the symmetry of the system and $\partial\bar{q}_{cl}/\partial s_a$ is a periodic solution of the Euler equation for quantum correction. This means that each freedom of the symmetry transformation makes a pair of ν's zero. From now on we will assume that all zero stability angles correspond to the symmetry and not accidental, i.e., f-independent pairs are zero. Generally, a time-independent quantity $\pi(q(t), \dot{q}(t))$ can be expanded up to the first order in $\tilde{q}$ as

$$\pi = \pi(\bar{q}_\ell, \dot{\bar{q}}_\ell) + \frac{\partial \pi}{\partial q_i}(\bar{q}_\ell, \dot{\bar{q}}_\ell)\tilde{q}_i + \frac{\partial \pi}{\partial \dot{q}_i}(\bar{q}_\ell, \dot{\bar{q}}_\ell)\dot{\tilde{q}}_i \quad , (i=1,\cdots,r) \ ,$$

where the coefficients of $\tilde{q}$ and $\dot{\tilde{q}}$ are periodic. If a solution $\xi^{(\alpha)}$ with $\nu_\alpha \neq 0$ contributes to π, the time-independent quantity π has a non-periodic term. Any solution with non zero stability angle, therefore, can not contribute to a conserved quantity, π, up to the first order in quantum correction.

Let us assume that real symmetric r×r matrix G has f'($\leq$r) zero eigenvalues. Making a local change of coordinates, G can be transformed into a block diagonal form;

$$G_{ij} = G_{ji} = 0 \quad , \ (i=1,\cdots,f' \ ; \ j=1,\cdots,r). \tag{2.C.12}$$

In this coordinate system, the f' canonical momenta p_i (i=1, ··, f') are conserved at least locally, i.e.,

$$p_i - p_i' = \frac{\partial S_\ell}{\partial q_i} + \frac{\partial S_\ell}{\partial q_i'} = \sum_{j=1}^{r} G_{ij}(q_j + q_j' - 2q_j^*) = 0 \quad , \ (i=1,\cdots,f'). \tag{2.C.13}$$

Conversely the invariance of the system under the continuous transformation leads that the canonical momentum for the symmetry coordinate is conserved and G has zero eigenvalue. We will assume that all zero eigenvalues of G correspond to the symmetry of the system. This is equivalent to the previous assumption that all zero stability angles are due to the symmetry of the system. We can, therefore, identify the zero eigenvalues of G with pairs of zero stability angles and f'=f. In the coordinate system where G is block diagonalized as (2.C.12), the coordinates q_i and q_i' for i=1, ··, f represent the symmetry coordinates, and

$\tilde{q}(t)$'s satisfying $\tilde{q}_i(T)=\tilde{q}_i(0)=0$ for i=f+1, ··, r span the stability angles $\nu=0$ manifold of the solutions of the Euler equations for quantum correction. Using H the conserved momenta p_i (i=1, ··, f) can be written by

$$p_i = \frac{1}{2}\left(\frac{\partial S_\ell}{\partial \xi_i} - \frac{\partial S_\ell}{\partial \xi_i'}\right) = \frac{1}{2}\sum_{j=1}^{r} H_{ij}(\xi_j - \xi_j') \quad , \quad (i=1,\cdots,f). \tag{2.C.14}$$

The coordinates q_i and q_i' for i=f+1, ··, r represent non-zero stability angle modes and these can not contribute to any conserved quantity up to the first order in the quantum fluctuation. Then,

$$H_{ij} = H_{ji} = 0 \quad , \quad (i=1,\cdots,f\,;\, j=f+1,\cdots,r). \tag{2.C.15}$$

From the above arguments one can find that the zero stability angle modes are completely decoupled from the non-zero stability angle modes, i.e.,

$$G_T = \left(\begin{array}{c|c} \overbrace{0}^{f} & \overbrace{0}^{r-f} \\ \hline 0 & \tilde{G} \end{array}\right) \begin{matrix} \}f \\ \}r-f \end{matrix} \quad , \quad H = \left(\begin{array}{c|c} \overbrace{\tilde{\tilde{H}}}^{f} & \overbrace{0}^{r-f} \\ \hline 0 & \tilde{H} \end{array}\right) \begin{matrix} \}f \\ \}r-f \end{matrix}, \tag{2.C.16}$$

where $\tilde{G}$ has no zero eigenvalues and

$$-\frac{1}{2}\tilde{\tilde{H}}_{ij} = \left.\frac{\partial^2 S_\ell}{\partial \xi_i \partial \xi_j'}\right|_{\xi=\xi'=\xi^*} \quad (i,j=1,\cdots,f). \tag{2.C.17}$$

The partition function Z(T), then, has the following form;

$$Z(T) = \sum_{\bar{\xi}_\ell} \exp\left[\frac{i}{\hbar} S_\ell(\xi^*,\xi^*)\right] \Delta_0 \cdot \Delta_Q \quad , \tag{2.C.18}$$

$$(\bar{\xi}_\ell(0) = \bar{\xi}_\ell(T) = \xi^*)$$

where

$$\Delta_0 \equiv \left(\frac{i}{2\pi\hbar}\right)^{f/2} \int d^f\xi \left[\det\left(\frac{\partial^2 S_\ell}{\partial\xi\,\partial\xi'}\right)\Big|_{\xi=\xi'=\xi^*}\right]^{1/2}, \tag{2.C.19}$$

$$\Delta_Q \equiv \left(\frac{i}{2\pi\hbar}\right)^{(r-f)/2} \int d^{(r-f)}\xi \left[\det\left(\frac{1}{2}(\tilde{G}-\tilde{H})\right)\right]^{1/2} \exp\left[\frac{i}{\hbar}(\xi-\xi^*)\tilde{G}(\xi-\xi^*)\right] \tag{2.C.20}$$

and $\sum_{\xi_\ell}$ means the sum over a discrete set of periodic orbits.

To obtain further insight into the zero mode problem we will study the physical meaning of Δ_0 and Δ_Q. We introduce the change of variables by

$$\xi_i(t) = \bar{\xi}_{\ell,i}(t;\mathcal{A}(t)) + \tilde{\xi}_i(t;\mathcal{A}(t)) \quad ,(i=1,\cdots,r), \tag{2.C.21}$$

$$\sum_{i=1}^{r} \frac{\partial\bar{\xi}_{\ell,i}(t;\mathcal{A})}{\partial\mathcal{A}_a}\,\tilde{\xi}_i(t;\mathcal{A}) = 0 \quad ,(a=1,\cdots,f).$$

In (2.C.21) $\bar{q}_{cl}(t;s)$ and $\tilde{q}(t;s)$ are transformed from $\bar{q}_{cl}(t)$ and $\tilde{q}(t)$ by the symmetry transformation with the symmetry parameters s_a, respectively. The symmetry coordinate $s_a(t)$ corresponds to a pair of zero stability angles. The coordinate $\tilde{q}(t)$ orthogonal to $s_a(t)$ corresponds to the non-zero stability angle. The factor Δ_0(2.C.19) comes from zero stability angle modes (symmetry coordinates). We separate the symmetry coordinate into two parts. One of them is a time-dependent mode with the fixed end point. The integration over this mode can be approximated by S.P.A. and give the determinant factor

$$\left(\frac{i}{2\pi\hbar}\right)^{f/2}\left[\det\left(\frac{\partial^2 S_\ell}{\partial\xi\,\partial\xi'}\right)\Big|_{\xi=\xi'=\xi^*}\right]^{1/2}$$

in Δ_0(2.C.19). The other is a constant mode. The zero eigenvalue of G is due to this constant mode. The integration over this

constant zero mode is the trace integral over a classical orbit, $d^f q$ in (2.C.19), and can be performed exactly.

The integration over the quantum fluctuation $\tilde{q}$ orthogonal to the zero stability angle modes gives us the factor Δ_Q (2.C.20). The straightforward evaluations of $\tilde{G}$ and $\tilde{H}$ are difficult but the factor Δ_Q may be obtained by the following way. The product of Δ_0 and Δ_Q can be written in PI form by

$$\Delta_0 \cdot \Delta_Q = \int \prod_{t=0}^{T} \left[\frac{d^r \tilde{p}(t)}{(2\pi\hbar)^r} \, d^r \tilde{\xi}(t) \right] \delta^r(\tilde{\xi}(T) - \tilde{\xi}(0)) \exp\left(\frac{i}{\hbar} S_Q\right), \qquad (2.C.22)$$

where S_Q, which has been defined in (2.B.6), is the quantum action around the periodic orbit $\bar{q}_{cl}$. This PI is performed by a Gaussian integral and is proportional to the inverse square root of the determinant of the operator in S_Q. The inverse of the determinant is singular due to the zero stability angle mode. However, we already know how to extract these singular pieces. What we want to know is a contribution from the non-zero stability angle modes which is non-singular and this is the factor Δ_Q.

D. Quantum corrections (Stability angle method)

In this subsection we will evaluate the factor Δ_Q which is a contribution from the non-zero stability angle modes in (2.C.22).

The quantum action S_Q (2.B.6) in (2.C.22) is written in terms of the periodic functions A, B and C defined in (2.B.10) as

$$S_Q = \int_0^T dt \left[\tilde{p}(t) \dot{\tilde{\xi}}(t) - \frac{1}{2} \left\{ (\tilde{p} - \tilde{\xi} B^T) A^{-1} (\tilde{p} - B\tilde{\xi}) - \tilde{\xi} C \tilde{\xi} \right\} \right]. \qquad (2.D.1)$$

We will diagonalize the quantum action S_Q under the periodic boundary condition in (2.C.22). This is equivalent to solving

the eigenvalue problem for S_Q (2.D.1). For convenience sake we define a 2r×2r real matrix D and a 2r vector X by

$$D(t) \equiv \left(\begin{array}{c|c} A^{-1}B & -A^{-1} \\ \hline B^T A^{-1} B - C & -B^T A^{-1} \end{array} \right) = D(t+T) \quad , \tag{2.D.2}$$

$$X(t) \equiv \begin{pmatrix} \tilde{\xi}(t) \\ \tilde{p}(t) \end{pmatrix} . \tag{2.D.3}$$

The eigenvalue equation for S_Q is given by

$$\left[\frac{d}{dt} + D(t) \right] X_k(t) = i E_k X_k(t) \quad , \quad X_k(t+T) = X_k(t) . \tag{2.D.4}$$

Note that

$$\sigma_2 D = - D^T \sigma_2 \quad ,$$

where σ_2 is a Pauli matrix. This property of the real matrix D and the periodicity of X_k imply that the eigenvalue E_k is real and

$$\int_0^T dt \; X_l^\dagger \sigma_2 X_k = \int_0^T dt \cdot i \left(\tilde{p}_l^* \tilde{\xi}_k - \tilde{\xi}_l^* \tilde{p}_k \right) = 0 \quad , \quad \text{for } E_k \neq E_l .$$

If there is an eigenfunction X_k for the eigenvalue E_k ($\neq 0$), X_k^* is the eigenfunction for the eigenvalue $-E_k$. Let $\{X_k, X_k^*\}$ be a complete set of eigenfunctions. The orthogonality and completeness are given, respectively, as

$$\begin{cases} \int_0^T X_l^\dagger \sigma_2 X_k \, dt = - \int_0^T X_k^T \sigma_2 X_l^* \, dt = T \delta_{kl} \quad , \\ \int_0^T X_l^T \sigma_2 X_k \, dt = \int_0^T X_k^\dagger \sigma_2 X_l^* \, dt = 0 \end{cases} \tag{2.D.5}$$

and

$$\sum_k \frac{1}{T} \left[\left(X_k(t) \right) \left(X_k^\dagger(t') \sigma_2 \right) - \left(X_k^*(t) \right) \left(X_k^T(t') \sigma_2 \right) \right] = 1 \cdot \delta(t-t') \tag{2.D.6}$$

The quantum fluctuation X can be expanded as

$$X(t) = \sum_k \frac{1}{\sqrt{T}} \left[a_k X_k(t) + a_k^* X_k^*(t) \right] \tag{2.D.7}$$

and the quantum action S_Q (2.D.1) is diagonalized with respect to a_k and a_k^*. Then PI (2.C.22) is proportional to the inverse square root of the product of all eigenvalues E_k.

In order to obtain further information for the eigenvalue E_k, we will study the equations of motion for S_Q (2.D.1),

$$\left[\frac{d}{dt} + D(t) \right] X(t) = 0 . \tag{2.D.8}$$

These equations are equivalent to the Euler equations for S_Q (2.B.12). As previously stated one can find the 2r-independent solutions with the property that

$$X^{(\alpha)}(t+T) = \exp(-i\nu_\alpha) X^{(\alpha)}(t) . \tag{2.D.9}$$

These solutions satisfy the orthogonality and completeness;

$$\begin{cases} X^{(\alpha)\dagger}(t)\, \sigma_2\, X^{(\beta)}(t) = - X^{(\beta)T}(t)\, \sigma_2\, X^{(\alpha)*}(t) = \delta_{\alpha\beta} , \\ X^{(\alpha)T}(t)\, \sigma_2\, X^{(\beta)}(t) = X^{(\alpha)\dagger}(t)\, \sigma_2\, X^{(\beta)*}(t) = 0 \end{cases} \tag{2.D.10}$$

and

$$\sum_\alpha \left[\left(X^{(\alpha)}(t)\right)\left(X^{(\alpha)\dagger}(t)\,\sigma_2\right) - \left(X^{(\alpha)*}(t)\right)\left(X^{(\alpha)T}(t)\,\sigma_2\right) \right] = 1 . \tag{2.D.11}$$

From these solutions one can make the complete set of the eigenfunctions $\{X_k, X_k^*\}$ and eigenvalues E_k by

$$X_k(t) = \exp\left[i(\nu_\alpha + 2n\pi)t/T \right] X^{(\alpha)}(t) , \tag{2.D.12}$$

$$E_k = (\nu_\alpha + 2n\pi)/T , \quad (n = 0, \pm 1, \cdots) , \tag{2.D.13}$$

where the sufix k represents the set of indices (α, n). Then one can calculate the right hand side of (2.C.22) by using stability angles. The contribution from the non-zero stability angle modes is proportional to

$$\left[\prod_{\nu_\alpha \neq 0} \prod_{n=-\infty}^{\infty} E_k\right]^{-1/2} \propto \left[\prod_{\nu_\alpha \neq 0} \left\{ \frac{\nu_\alpha}{2} \prod_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \left(1+\frac{\nu_\alpha}{2n\pi}\right)\right\}\right]^{-1/2}$$
$$= \left[\prod_{\nu_\alpha \neq 0} \sin\left(\frac{\nu_\alpha}{2}\right)\right]^{-1/2}. \qquad (2.D.14)$$

Using the fact that the stability angles come in plus-minus pairs the factor Δ_Q amounts to be

$$\begin{aligned}\Delta_Q &= \text{const} \times \left[\prod_{\nu_\alpha > 0} \sin\left(\frac{\nu_\alpha}{2}\right)\right]^{-1} \\ &= \text{const} \times \exp\left\{-i \sum_{\nu_\alpha > 0} \frac{1}{2}\nu_\alpha\right\} \times \prod_{\nu_\alpha > 0} \left[1-\exp(-i\nu_\alpha)\right]^{-1} \\ &= \text{const} \times \sum_{\{n_\alpha\}} \exp\left\{-i \sum_{\nu_\alpha > 0} \frac{1}{2}\nu_\alpha\right\} \exp\left\{-i \sum_{\nu_\alpha > 0} n_\alpha \nu_\alpha\right\},\end{aligned} \qquad (2.D.15)$$

where n_α is zero or positive integer. The constant factor in (2.D.15) may be determined by comparing it with the result of harmonic oscillators and gives no dynamical effect. The physical meaning of (2.D.15) is as follows: The first factor is the analog of the zero point energy. The exponent of the second factor represents the energy of excited state modes with the occupation number $\{n_\alpha\}$. One can confirm the above interpretation by reminding that the stability angle ν_α equals the energy times the time interval T if the classical solution is time-independent. The fact that the occupation number takes any positive integer value or zero is the characteristic feature of bose systems.

E. Energy spectrum

Using previous results the partition function Z(T) can be written in the semiclassical approximation (WKB ($\hbar$)) by

$$Z(T)=\sum_{\bar{\xi}_d}\Delta_0[\bar{\xi}_d]\exp\left\{\frac{i}{\hbar}S_d[\bar{\xi}_d]\right\}\left[\sum_{\{n_\alpha\}}\exp\left\{-i\sum_{\substack{\alpha=1\\(\nu_\alpha>0)}}^{r-f}(n_\alpha+\tfrac{1}{2})\nu_\alpha[\bar{\xi}_d]\right\}\right], \tag{2.E.1}$$

where the zero mode factor Δ_0 is given by

$$\Delta_0[\bar{\xi}_d]=\int d^f\xi\left(\frac{i}{2\pi\hbar}\right)^{f/2}\left[\det\left(\frac{\partial^2 S_d}{\partial\xi\partial\xi'}\right)\Big|_{\xi=\xi'=\xi^*}\right]^{1/2}. \tag{2.E.2}$$

This should be compared with the partition function for a harmonic oscillator system with a angular frequency ω,

$$Z_{har}(T)=\sum_{n=0}^{\infty}\exp\{-i(n+\tfrac{1}{2})\omega T\}. \tag{2.E.3}$$

The partition function in WKB ($\hbar$) (2.E.1) has a large factor $\hbar^{-f/2}$. This comes from the quantum correction of the symmetry coordinates which correspond to the symmetry of the system violated in the classical periodic orbit. For the trivial orbit which is time-independent and invariant under the symmetry transformation, the partition function in the semiclassical approximation becomes

$$Z_{\text{trivial orbit}}(T)=\exp\left(-\frac{i}{\hbar}E_d T\right)\sum_{\{n_\alpha\}}\exp\left[-i\sum_{\substack{\alpha=1\\(\omega_\alpha>0)}}^{r}(n_\alpha+\tfrac{1}{2})\omega_\alpha T\right], \tag{2.E.4}$$

where E_{cl} is the classical energy for the trivial orbit and ω_α is the angular frequency of the α-th mode. In this case the quantum correction is simply an assembly of harmonic oscillators and

smaller by a factor of $\hbar^{f/2}$ than that for non-trivial orbit.

Now we can calculate the propagator G(E) defined by (2.A.1). For simplicity we assume that the symmetry violated in the classical solution is only the time translational invariance. According to (2.E.2), the zero mode factor Δ_0 is given by

$$\Delta_0 = \left(\frac{1}{i2\pi\hbar}\right)^{1/2} \tau \left(\frac{d^2 S_\ell}{dT^2}\right)^{1/2}, \qquad (2.E.5)$$

where τ is the basic period of the classical orbit $\bar{q}_{cl}(t)$, i.e., τ multiplied by a certain positive integer ℓ (revolution number) is the time interval T. In order to obtain a pole term in G(E) one has to sum over the multiple traverses of the basic orbits in (2.E.1). For the multiple traverse of the basic orbit, the action $S_{cl}(T)$ and the stability angle $\nu_\alpha(T)$ are simply the revolution number ℓ times those of the basic orbit; $S_{cl}(T)=\ell\, S_{cl}(\tau)$, $\nu_\alpha(T)=\ell\, \nu_\alpha(\tau)$. The propagator G(E) can then be obtained by substituting (2.E.1) into (2.A.3), and changing the integration parameter from T to the basic period τ,

$$G_T(E) = \frac{1}{i\hbar}\left(\frac{1}{i2\pi\hbar}\right)^{1/2} \sum_{\{n_\alpha\}} \sum_{\ell=1}^{\infty} \int_0^\infty d\tau \sqrt{\ell}\, \tau \left(\frac{d^2 S_\ell(\tau)}{d\tau^2}\right)^{1/2}$$
$$\times \exp\left[\frac{i}{\hbar}\ell\left\{S_\ell(\tau) + E\tau - \hbar\sum_{\alpha=1}^{n-1}\left(n_\alpha+\tfrac{1}{2}\right)\nu_\alpha(\tau)\right\}\right]. \qquad (2.E.6)$$

The exponent in (2.E.6) is proportional to a large number $\hbar^{-1}$. One can, therefore, perform the τ-integration by using the stationary phase method and make the summation over ℓ to obtain

$$G_T(E) = \frac{1}{i\hbar}\sum_{\{n_\alpha\}}\sum_{\ell=1}^{\infty} \tau_0(E)\exp\left[\frac{i}{\hbar}\ell\, W(E,\{n_\alpha\})\right]$$
$$= \frac{\tau_0(E)}{i\hbar}\sum_{\{n_\alpha\}} \frac{\exp\left[\frac{i}{\hbar}W(E,\{n_\alpha\})\right]}{1-\exp\left[\frac{i}{\hbar}W(E,\{n_\alpha\})\right]}. \qquad (2.E.7)$$

In (2.E.7) $\tau_0(E)$ is defined by the stationary condition

$$\frac{d}{d\tau} S_{\ell}(\tau)\Big|_{\tau_0(E)} = -E \qquad (2.E.8)$$

and

$$W(E) \equiv S_{\ell}(\tau_0(E)) + E\tau_0(E) - \hbar \sum_{\alpha=1}^{r-1} (n_\alpha + \tfrac{1}{2}) \nu_\alpha(\tau_0(E)) . \qquad (2.E.9)$$

From the pole in (2.E.7) the quantization condition of energies is given by

$$W(E, \{n_\alpha\}) = 2\pi\hbar m \quad , \quad (m = 0, 1, 2, \cdots) . \qquad (2.E.10)$$

In the zeroth order approximation of WKB ($\hbar$) (2.E.10) gives us the Bohr-Sommerfeld quantization condition

$$W_0(E) \equiv S_{\ell}(\tau_0(E)) + E\tau_0(E) = 2\pi\hbar m \quad , \quad (m = 0, 1, 2, \cdots) . \qquad (2.E.11)$$

This is valid only for a large integer m since the left hand side of (2.E.11) is of order $\hbar^0$. Even in the zeroth order of WKB ($\hbar$) one has to quantize the zero stability angle mode. Note that one can not obtain the pole terms in (2.E.7) if one disregards the quantum correction of the zero mode (2.E.5). In the first order approximation (the full WKB ($\hbar$)) one obtains (2.E.10) where the third term in (2.E.9) is the small quantum correction to the classical orbit. The quantization condition (2.E.10) is, therefore, valid only for small occupation numbers n_α .

For a static trivial classical orbit, it is obvious from (2.E.4) that the excited state energy for the configuration $\{n_\alpha\}$ is given by

$$E(\{n_\alpha\}) = E_{\ell} + \sum_{\alpha=1}^{r} (n_\alpha + \tfrac{1}{2}) \omega_\alpha . \qquad (2.E.12)$$

In closing this section we remark the following: In the semiclassical approximation one does not need the Hamiltonian and can calculate everything by using the Lagrangian. In practical models this simplifies the calculation.

§III. Semiclassical method for the boson-fermion system

A. Effective action

In this section we will formulate the semiclassical method for the system in which fermions and bosons are strongly coupled together. Our method is applicable to any field theoretical or quantum mechanical model if the Lagrangian is bilinear with respect to the fermion field. Even if the four-fermion coupling term is included, it can be easily decomposed into a bilinear form by using an auxiliary field as is well known.[13)14)] As remarked at the end of Sec.II, we will use the configuration space PI with the Lagrangian instead of the phase space PI with the Hamiltonian.

The model Lagrangian considered in the following is

$$L = \bar{\psi}\left[\frac{i}{2}\{\Gamma^{\mu}(\phi)\overrightarrow{\partial_{\mu}} - \overleftarrow{\partial_{\mu}}\Gamma^{\mu}(\phi)\} - V(\phi)\right]\psi + NL_B(\phi), \qquad (3.A.1)$$

where $NL_B(\phi)$ is the boson field Lagrangian including self-coupling terms and $\bar{\psi}=\psi^{\dagger}\gamma^0$. For the Hermiticity of the Lagrangian $V(\phi)$ is a real arbitrary function of the boson field ϕ and $\Gamma^{\mu}(\phi)$ should satisfy $\gamma^0\Gamma^{\mu\dagger}\gamma^0=\Gamma^{\mu}$. In order to clarify the expansion parameter, we assume that the Lagrangian of the boson field is proportinal to N and the fermion field ψ has N internal components in addition to the particle-anti-particle degrees of freedom, so that $\bar{\psi}\psi$ stands for $\sum_{k=1}^{N}\bar{\psi}^{(k)}\psi^{(k)}$. If the other quantities are independent of N, the expansion parameter of our semiclassical method (SPA(1/N)) is 1/N. The space-time dimension is assumed to be (n-1)+1 in general. The bag model[5)] and the quark-string[4)] model are considered to be special cases of (3.A.1).

The partition function of the system is given by

$$Z(T)=\int[d\psi^{\dagger}d\psi d\phi]\prod_{\vec{x}}\delta(\psi(\vec{x},0)+\psi(\vec{x},T))\,\delta(\phi(\vec{x},0)-\phi(\vec{x},T))$$
$$\times\prod_{\vec{x},t}\det[\gamma^{0}\Gamma^{0}(\phi)]^{-1}\exp\left[i\int_{0}^{T}d^{n}x\,L(\bar{\psi},\psi,\phi)\right], \qquad (3.A.2)$$

where

$$\int_{0}^{T}d^{n}x\equiv\int_{0}^{T}dt\int_{-\infty}^{\infty}d^{n-1}\vec{x} \qquad (3.A.3)$$

and the path integration of the fermion field (anti-commuting coordinate) is defined as the standard way.[15] Since we know how the Planck constant $\hbar$ appears in Z(T) and $\hbar$ is not expansion parameter, we take $\hbar=1$ in (3.A.2). It should be noted that if the coefficient of $\psi^{\dagger}\partial_{0}\psi$, $i\gamma^{0}\Gamma^{0}$, is field dependent, the determinant factor $\prod_{\vec{x},t}\det[\gamma^{0}\Gamma^{0}(\phi)]^{-1}$ is needed in the PI formula (3.A.2) in order that the PI formalism agrees with the Hamiltonian formalism. One can confirm this by comparing the perturbation expansion formulas derived from these two methods. The abbreviated notation $[d\psi^{\dagger}d\psi d\phi]$ implies the functional integration with the measure for the boson field which is derived from the phase space PI as in Sec.II. Since we know how to treat this measure we will disregard it in the following. The periodic condition for the boson field, $\phi(\vec{x},0)=\phi(\vec{x},T)$, and the anti-periodicity for the fermion field, $\psi(\vec{x},0)=-\psi(\vec{x},T)$, in (3.A.2) are due to the trace condition in the partition function.[15]

Before going into the semiclassical approximation we perform the fermion integration and get

$$Z(T) = \int [d\phi] \left[\det \left\{ (\Gamma^0)^{-1} \left(i\Gamma^\mu \partial_\mu + \frac{i}{2} \partial_\mu \Gamma^\mu - V \right) \right\} \right]^N$$

$$\times \prod_{\vec{x}} \delta(\phi(\vec{x},0) - \phi(\vec{x},T)) \exp\left[iN \int_0^T d^n x \, L_B(\phi) \right] . \qquad (3.A.4)$$

This procedure implies that the fermion field is quantized in the background field of the boson. The determinant factor in (3.A.4) equals the product of all the eigenvalues defined by the eigenvalue equation,

$$\left[i\Gamma^\mu \partial_\mu + \frac{i}{2} \partial_\mu \Gamma^\mu - V \right] \xi_{A,n}(\vec{x},t) = E_{A,n} \Gamma^0 \xi_{A,n}(\vec{x},n) , \qquad (3.A.5)$$

with the anti-periodic conditon

$$\xi_{A,n}(\vec{x}, t+T) = - \xi_{A,n}(\vec{x}, t) . \qquad (3.A.6)$$

The boson field in (3.A.5) is periodic with the period T due to the delta functions in (3.A.4). As was done in Sec.II.D the eigenvalues of (3.A.5) can be expressed by the stability angles ζ_A which are defined by the pseudo-periodicity

$$\Psi_A(\vec{x}, T) = \exp\{-i \zeta_A[\phi]\} \Psi_A(\vec{x}, 0) , \qquad (3.A.7)$$

where $\psi_A(\vec{x},t)$ is the solution of the Euler equation for the fermion field,

$$\left[i\Gamma^\mu \partial_\mu + \frac{i}{2} \partial_\mu \Gamma^\mu - V \right] \Psi_A(\vec{x}, t) = 0 . \qquad (3.A.8)$$

The eigenvalue $E_{A,n}$ and the eigenfunction $\xi_{A,n}$ of (3.A.5) are given by using the stability angle ζ_A and the solution of (3.A.8) as

$$\xi_{A,n}(\vec{x},t) = \exp(-iE_{A,n}t)\,\psi_A(\vec{x},t), \tag{3.A.9}$$

$$E_{A,n} = -\{(2n+1)\pi + \zeta_A[\phi]\}/T \quad , (n=0,\pm1,\cdots). \tag{3.A.10}$$

Note that $(2n+1)\pi$ in (3.A.10) is due to the anti-periodicity for the fermion field, while the periodicity for the boson coordinate gives us $2n\pi$ in (2.D.13). Let $\{\psi_A\}$ be the complete set of the solutions of (3.A.8) which satisfies the orthogonality and completeness as

$$\int_{-\infty}^{\infty} d^{n-1}\vec{x}\;\bar{\psi}_A(\vec{x},t)\,\Gamma^0(t)\,\psi_{A'}(\vec{x},t) = \delta_{A,A'}, \tag{3.A.11}$$

$$\sum_A \psi_A(\vec{x},t)\,\bar{\psi}_A(\vec{x}',t)\,\Gamma^0(t) = \delta(\vec{x}-\vec{x}'), \tag{3.A.12}$$

respectively. The complete set of the eigenfunctions of (3.A.5) is given by $\{\xi_{A,n}\}$ defined in (3.A.9). Its orthogonality and completeness are given, respectively, as

$$\int_0^T d^n x\;\bar{\xi}_{A,n}(\vec{x},t)\,\Gamma^0(t)\,\xi_{A',n'}(\vec{x},t) = T\,\delta_{A,A'}\,\delta_{n,n'}, \tag{3.A.13}$$

$$\frac{1}{T}\sum_{A,n}\xi_{A,n}(\vec{x},t)\,\bar{\xi}_{A,n}(\vec{x}',t')\,\Gamma^0(t') = \delta(t-t')\,\delta(\vec{x}-\vec{x}'). \tag{3.A.14}$$

The determinant factor in (3.A.4) is then given by

$$\left[\det\left\{(\Gamma^0)^{-1}\left(i\Gamma^\mu\partial_\mu+\frac{i}{2}\partial_\mu\Gamma^\mu-V\right)\right\}\right]^N$$
$$=\left[\prod_A\prod_{n=-\infty}^{\infty}E_{A,n}\right]^N\propto\left[\prod_A\prod_{n=-\infty}^{\infty}\left\{1+\frac{\xi_A}{(2n+1)\pi}\right\}\right]^N$$
$$=\left[\prod_A\cos(\xi_A/2)\right]^N$$
$$\propto\sum_{\{n_A\}}\rho(N,\{n_A\})\exp\left[iN\sum_{A>0}\xi_A-i\sum_{A>0}n_A\xi_A\right], \qquad (3.A.15)$$

where the sum is taken over all sets of integers n_A such that $0\le n_A\le 2N$. The degree of the degeneracy, ρ, for the configuration $\{n_A\}$ is given by

$$\rho(N,\{n_A\})=\prod_{A>0}\frac{(2N)!}{(2N-n_A)!\,n_A!}\,. \qquad (3.A.16)$$

Due to the charge conjugation invariance of the Dirac equation (3.A.8), the stability angles ζ_A appear in plus-minus pairs with equal magnitude but opposite signs. In the last expression of (3.A.15) we assumed that $\zeta_A=\zeta_{-A}>0$ for $A>0$. In what follows we will use the positive stability angle only, i.e., we will assume that ζ_A is always positive. Since there are equivalent N fermions and N anti-fermions in the Lagrangian (3.A.1), the occupation number for the A-th state, n_A, is restricted by the Pauli principle such that $0\le n_A\le 2N$ in (3.A.15). The physical meaning of (3.A.15) is obvious by the fact that the stability angle ζ_A equals the time interval T times the energy of the A-th eigenstate for a given potential $V(\phi)$ in (3.A.8) if ϕ is time-independent. The first term in the exponent in (3.A.15) represents the Dirac vacuum energy, while the second corresponds to the energy of the excited

fermions with the configuration $\{n_A\}$. Substituting (3.A.15) into (3.A.4) one obtains that

$$Z(T) = \sum_{\{n_A\}} \rho(N, \{n_A\}) \int [d\phi] \prod_{\vec{x}} \delta(\phi(\vec{x},0) - \phi(\vec{x},T)) \times \exp\{i I_{eff}[\phi; \{n_A\}, T]\}. \quad (3.A.17)$$

The effective action for the configuration $\{n_A\}$, $I_{eff}[\phi;\{n_A\},T]$, is given by

$$I_{eff}[\phi; \{n_A\}, T] = N\left\{\int_0^T d^n x L_B(\phi) + \sum_A \xi_A[\phi]\right\} - \sum_A n_A \xi_A[\phi]. \quad (3.A.18)$$

In (3.A.17) we omitted an irrelevant constant normalization factor. It should be noted that (3.A.17) is still exact.

B. Semiclassical approximation for the effective action

Our semiclassical method for the boson-fermion system, SPA(1/N), will be shown to be a 1/N expansion method and becomes a good approximation when N is a large number. In evaluating (3.A.17) one should note that the effective action I_{eff} (3.A.18) is proportional to a large number N if the occupation number n_A is zero or of order N. In these cases one can apply the stationary phase approximation (S.P.A.) to each term of the occupied fermion configurations $\{n_A\}$ as in Sec.II. In this method, 1/N obviously plays the role of $\hbar$ in WKB($\hbar$). This shows that our semiclassical method is a 1/N expansion.

As the first step of the approximation the stationary point of the effective action for a given configuration $\{n_A\}$ is given by

$$\delta I_{eff}[\phi;\{n_A\}]/\delta\phi = 0. \qquad (3.B.1)$$

The solution of (3.B.1) is called the classical solution and denoted as ϕ_{cl}. This classical solution, ϕ_{cl}, takes account of the reaction from the Dirac sea of the quantized fermion under the boson background field ϕ_{cl} as will be seen later. Since N can be factored out in (3.B.1) the classical solution ϕ_{cl} is of order N^0. In the second step the quantum correction around the classical solution is calculated up to the second order in η by substituting

$$\phi(\vec{x},t) = \phi_{cl}(\vec{x},t) + \frac{1}{\sqrt{N}}\eta(\vec{x},t), \qquad (3.B.2)$$

into the effective action $I_{eff}\{n_A\}$. Since the Gauss integral over the quantum fluctuation η is dominated by the region $(\eta/\sqrt{N})^2 \lesssim 1/N$, $\eta/\sqrt{N}$ is of order $1/\sqrt{N}$ (η is of order N^0).

The above procedure implies that the occupation number of the fermions dressed with the boson background field is not changed within the periodic motion of ϕ. We assume that the fermion number $\{n_A\}$ is a fairly good quantum number as one usually expects in the quark model.

In order to obtain the expansion of the stability angle ζ_A, we substitute

$$D(\partial,\phi) \equiv i\Gamma^\mu\partial_\mu + \frac{i}{2}\partial_\mu\Gamma^\mu - V = D^{(0)}(\phi_{cl}) + \frac{1}{\sqrt{N}}D^{(1)}(\phi_{cl},\eta) + \frac{1}{N}D^{(2)}(\phi_{cl},\eta),$$
$$\Gamma^\mu(\phi) = \Gamma^{\mu(0)}(\phi_{cl}) + \frac{1}{\sqrt{N}}\Gamma^{\mu(1)}(\phi_{cl},\eta) + \frac{1}{N}\Gamma^{\mu(2)}(\phi_{cl},\eta), \qquad (3.B.3)$$

$$E_{A,n}[\phi] = E^{(0)}_{A,n}[\phi_{cl}] + \frac{1}{\sqrt{N}} E^{(1)}_{A,n}[\phi_{cl},\eta] + \frac{1}{N} E^{(2)}_{A,n}[\phi_{cl},\eta],$$

$$\xi_{A,n}[\phi] = \xi^{(0)}_{A,n}[\phi_{cl}] + \frac{1}{\sqrt{N}} \xi^{(1)}_{A,n}[\phi_{cl},\eta] + \frac{1}{N} \xi^{(2)}_{A,n}[\phi_{cl},\eta], \qquad (3.B.3)$$

into (3.A.5). In (3.B.3) the superscript (i) denotes the power of $1/\sqrt{N}$ which equals the power in η. As in the usual perturbation method, $E^{(i)}_{A,n}$ and $\xi^{(i)}_{A,n}$ are calculated by comparing the same order terms in both sides of (3.A.5).

In the zeroth order, (3.A.5) provides

$$D^{(0)} \xi^{(0)}_{A,n} = E^{(0)}_{A,n} \Gamma^{(0)}_{0} \xi^{(0)}_{A,n} \qquad (3.B.4)$$

with the boundary condition

$$\xi^{(0)}_{A,n}(T) = - \xi^{(0)}_{A,n}(0). \qquad (3.B.5)$$

The orthogonality and completness are the same as (3.A.13) and (3.A.14), respectively, where the zeroth order quantities are substituted. In order to simplify the calculation we introduce the new function $\psi^{(0)}_{A}$ defined by

$$\Psi^{(0)}_{A}(\vec{x},t) \equiv \exp\left(i E^{(0)}_{A,n} t\right) \xi^{(0)}_{A,n}(\vec{x},t). \qquad (3.B.6)$$

These functions satisfy

$$D^{(0)} \Psi^{(0)}_{A} = 0, \qquad (3.B.7)$$

$$\Psi^{(0)}_{A}(\vec{x},T) = \exp\left(-i \zeta^{(0)}_{A}\right) \Psi^{(0)}_{A}(\vec{x},0), \qquad (3.B.8)$$

where the zeroth order stability angle $\zeta^{(0)}_{A}$ is given by

$$\zeta^{(0)}_{A} = -\left[(2n+1)\pi + E^{(0)}_{A,n} T\right], \qquad (3.B.9)$$

which is n-independent. The orthogonality and completeness for $\psi_A^{(0)}$ are obtained by substituting the zeroth order quantities into (3.A.11) and (3.A.12).

The first order term provides us with

$$T E^{(1)}_{A,n} = -\xi^{(1)}_A = \int_0^T d^n x\, \bar{\Psi}^{(0)}_A D^{(1)}(\eta) \Psi^{(0)}_A \tag{3.B.10}$$

and

$$\Psi^{(1)}_A = -\frac{1-P_A}{D^{(0)}(\phi_{cl})} D^{(1)}(\eta) \Psi^{(0)}_A , \tag{3.B.11}$$

where $\psi_A^{(1)}$ is defined by

$$\Psi^{(1)}_A \equiv \exp(i E^{(0)}_{A,n} t)\, \xi^{(1)}_{A,n} \tag{3.B.12}$$

and P_A is the projection operator onto the state $\psi_A^{(0)}$. The equations (3.B.10) and (3.B.11) imply that $E^{(1)}_{A,n}$ and $\psi_A^{(1)}$ are n-independent.

At this point, one can inspect the stationary condition (3.B.1) of the effective action I_{eff}. From (3.A.18) and (3.B.10), and reminding that η is the small variation around ϕ_{cl}, one obtains the stationary condition,

$$-\partial_\mu \frac{\partial L_B}{\partial(\partial_\mu \phi)} + \frac{\partial L_B}{\partial \phi} - \sum_A \bar{\Psi}^{(0)}_A \frac{\partial D^{(1)}}{\partial \eta} \Psi^{(0)}_A + \sum_A \left(\frac{n_A}{N}\right) \bar{\Psi}^{(0)}_A \frac{\partial D^{(1)}}{\partial \eta} \Psi^{(0)}_A = 0 , \tag{3.B.13}$$

with the Dirac equation (3.B.7) for $\psi_A^{(0)}$. This is the time-dependent Hartree-Fock equation which has been first given in field theory by DHN.[2] Note that if one neglects the third term and chooses $\{n_A\}$ such as $\sum_A n_A = N$, (3.B.13) with the Dirac equation (3.B.7) becomes the Euler equation for the Lagrangian (3.A.1). From this fact one can see that the third term in (3.B.13)

corresponds to the reaction from the Dirac sea and the fourth term to the reaction from the excited fermions. The solutions $\{\phi_{cl}, \psi_A^{(0)}\}$ of the coupled equations (3.B.7) and (3.B.13) are called the classical solutions for the given configuration $\{n_A\}$.

DHN has imposed the Bohr-Summerfeld condition on the classical solutions. The full semiclassical approximation (WKB) needs further approximation and this step was performed by Kikkawa and the author[3].

The second order term in (3.A.5) provides us with the second order stability angle,

$$TE^{(2)}_{A,n} = -\zeta_A^{(2)}$$

$$= -\int_0^T d^n x\, \bar{\Psi}_A^{(0)} D^{(1)}(\eta) \frac{1-P_A}{D^{(0)}} D^{(1)}(\eta) \Psi_A^{(0)} + \int_0^T d^n x\, \bar{\Psi}_A^{(0)} D^{(2)}(\eta) \Psi_A^{(0)} . \quad (3.B.14)$$

Summing up all terms calculated above, one obtains the approximated effective action,

$$I_{eff}\{n_A\} = N S_{cl}\{n_A\} + \int_0^T d^n x\, L_B^{(Q)}(\eta) + \sum_A \zeta_A^{(2)}[\phi_{cl}, \eta] - \sum_A \left(\frac{n_A}{N}\right) \zeta_A^{(2)}[\phi_{cl}, \eta] , \quad (3.B.15)$$

where

$$S_{cl}\{n_A\} \equiv \int_0^T d^n x\, L_B(\phi_{cl}) + \sum_A \left(1 - \frac{n_A}{N}\right) \zeta_A^{(0)}[\phi_{cl}, \eta] \quad (3.B.16)$$

and

$$L_B^{(Q)} \equiv \frac{1}{2} \frac{\partial^2 L_B}{\partial \phi^2}\bigg|_{\phi_{cl}} \eta^2 + \frac{\partial^2 L_B}{\partial(\partial_\mu \phi)\partial \phi}\bigg|_{\phi_{cl}} (\partial_\mu \eta)\eta + \frac{\partial^2 L_B}{\partial(\partial_\mu \phi)\partial(\partial_\nu \phi)}\bigg|_{\phi_{cl}} (\partial_\mu \eta)(\partial_\nu \eta) . \quad (3.B.17)$$

The effective action (3.B.15) is non-local due to the non-locality of the second order stability angles (3.B.14) with respect to η. Therefore, we do not know whether the integration over η provides the simple form, $\exp[-i\sum_\alpha(n_\alpha+\frac{1}{2})\nu_\alpha]$, which is expected by analogy with the discussion in Sec.II.D . It will be of great advantages if one is able to make I_{eff} local. In fact, using a set of infinite auxiliary fields $\{\chi_A\}$ the effective action I_{eff} can be rewritten as

$$I_{eff}\{n_A\} = N\, S_{cl}[\phi_{cl},\{n_A\}] + S_Q[\phi_{cl},\eta,\{n_A\}], \tag{3.B.18}$$

$$S_Q \equiv \int_0^T d^n x\left[L_B^{(Q)}(\eta) - \sum_A\left(1-\frac{n_A}{N}\right)L_{F,A}^{(Q)}(\eta,\chi_A)\right], \tag{3.B.19}$$

$$L_{F,A}^{(Q)} = \bar{\chi}_A\, D^{(0)}\chi_A + \bar{\psi}_A^{(0)}\, D^{(1)}(\eta)\chi_A + \bar{\chi}_A\, D^{(1)}(\eta)\psi_A^{(0)} + \bar{\psi}_A^{(0)}\, D^{(2)}(\eta)\psi_A^{(0)}, \tag{3.B.20}$$

where the subsidiary conditions for χ_A and η,

$$\int_0^T d^n x\, \bar{\psi}_A^{(0)}\, \Gamma_0^{(0)}\chi_A = 0,$$

$$\chi_A(\vec{x},T) = \exp(-i\zeta_A^{(0)})\chi_A(\vec{x},0), \tag{3.B.21}$$

$$\eta(\vec{x},T) = \eta(\vec{x},0),$$

are imposed. The equivalence of (3.B.18) with (3.B.21) to (3.B.15) can be shown by integrating out the PI formula with the action (3.B.18) over χ, i.e.,

$$\exp\left[iI_{eff}(3.B.15)\right]=\frac{\int[d\chi]\,\delta(3.B.21)\exp\left[iI_{eff}(3.B.18)\right]}{\int[d\chi]\,\delta(3.B.21)\exp\left[iI_{eff}(3.B.18)\big|_{\eta=0}\right]}\,. \tag{3.B.22}$$

The equivalence can be formally shown if χ's are assumed to be either commuting or anti-commuting coordinates. We emphasize that χ is assumed to be the commuting coordinate in what follows.

The quantum effective action $S_Q\{n_A\}$ (3.B.19) is now local and quadratic in the multicomponent commuting field $H\equiv\{\eta,\chi_1,\chi_2,..\}$ and one can apply the stability angle method as in Sec.II.D. If the new stability angles $\nu_\alpha(\{n_A\},T)$ to the Euler equations of S_Q for the quantum fluctuation H are obtained, the partition function turns out to be

$$Z(T)=\sum_{\{n_\alpha\}}\sum_{\{p_\alpha\}}\sum_{\ell=1}^{\infty}\tau\left[C_\ell(\tau,T)\right]^{1/2}\rho(N,\{n_A\})\times\exp\left[i\ell I_{eff}(\{n_A\},\tau)\right], \tag{3.B.23}$$

where τ is the basic period of the periodic classical solution ϕ_{cl} and $T/\tau=\ell$ (positive integer). The effective action $I_{eff}(\{n_A\},\tau)$ in (3.B.23) is given by

$$I_{eff}(\{n_A\},\tau)=N\cdot S_\ell(\{n_A\},\tau)-\sum_\alpha\left(p_\alpha+\frac{1}{2}\right)\nu_\alpha(\{n_A\},\tau). \tag{3.B.24}$$

In (3.B.23) we used that $S_{cl}(\{n_A\},T)=\ell\cdot S_{cl}(\{n_A\},\tau)$ and $\nu_\alpha(\{n_A\},T)=\ell\nu_\alpha(\{n_A\},\tau)$. The occupation number p_α of the α-th mode takes zero or an arbitrary positive integer. The factor τ comes from the trace of the zero stability angle mode and $[C_{cl}(\tau,T)]^{1/2}$ represents the quantum correction of the zero stability angle

coordinate like (2.E.5). This factor may be calculated by the collective coordinate method and satisfy $[C_{cl}(\tau,T)]^{1/2}=[C_{cl}(\tau)]^{1/2}/\sqrt{\ell}$.

C. Energy spectrum

From the expression for Z(T), (3.B.23), one can easily see that the propagator G(E) can be obtained through the same way as in Sec.II.E, where $\hbar$ is replaced by 1/N. By inspecting poles in G(E) the energy spectra are given by

$$W(E,\{n_A\},\{p_\alpha\}) = 2\pi m \quad , (m=0,1,2,\cdots), \tag{3.C.1}$$

where $W(E,\{n_A\},\{p_\alpha\})$ is defined as

$$W(E,\{n_A\},\{p_\alpha\}) \equiv N S_\ell(\{n_A\},\tau_0(E)) + E\,\tau_0(E) - \sum_\alpha (p_\alpha+\tfrac{1}{2})\,\nu_\alpha(\{n_A\},\tau_0(E)) . \tag{3.C.2}$$

In (3.C.2), $\tau_0(E)$ is the solution of

$$\frac{d}{d\tau} S_\ell(\{n_A\},\tau) = -\frac{E}{N} . \tag{3.C.3}$$

In the zeroth order approximation, (3.C.1) provides us with the Bohr-Sommerfeld condition proposed by DHN,

$$S_\ell(\{n_A\},\tau_0(E)) + \frac{E}{N}\,\tau_0(E) = 2\pi\,\frac{m}{N} . \tag{3.C.4}$$

This condition is valid if m is of order N. Note that n_A is also of order N or zero. In this case E becomes of order N, too. We emphasize that even in this approximation the energy includes the quantum effect of the fermion under the boson background field.

The quantization condition of energy in the first order is the equation (3.C.1). The third term in (3.C.2) gives the quantum correction to the energy for the configuration $(\{n_A\},\{p_\alpha\})$ and the occupation number p_α of the α-th mode should be of order N^0. For a static classical solution ϕ_{cl}, the excited state energy for the configuration $(\{n_A\},\{p_\alpha\})$ turns out to be

$$-E(\{n_A\},\{p_\alpha\}) = N\frac{dS_{cl}(\{n_A\},T)}{dT} - \sum_\alpha (p_\alpha+\tfrac{1}{2})\frac{d\nu_\alpha(\{n_A\},T)}{dT}, \quad (3.C.5)$$

which is T-independent.

So far we have not considered the renormalization problem. It will be discussed in the following sections.

§IV. Renormalization in the semiclassical method

A. Semiclassical method for the Gross-Neveu model

Taking the Gross-Neveu model as a theoretical laboratory, we will study the renormalization program in the semiclassical method. The Gross-Neveu model is a good example of applying our method, since the non-trivial classical solution does not appear if the coupling between the fermion and the boson is switched off.

The model is given by the Lagrangian in the 1+1 dimensional space-time,

$$L_\psi = \bar{\psi}_u i \partial\!\!\!/ \psi_u + \frac{g_u^2}{2N} (\bar{\psi}_u \psi_u)^2 , \tag{4.A.1}$$

where $\bar{\psi}_u\psi_u$ stands for $\sum_{k=1}^{N} \bar{\psi}_u^{(k)}\psi_u^{(k)}$, $\partial\!\!\!/ = \gamma_\mu \partial^\mu$ and the suffix u means the unrenormalized quantity. It is well known[13),14)] that with the help of a subsidiary boson field σ_u for the composite field $\bar{\psi}_u\psi_u$ the Lagrangian (4.A.1) can be rewritten as the bilinear form

$$L_{\psi\sigma} = \bar{\psi}_u (i\partial\!\!\!/ - g_u\sigma_u)\psi_u - \frac{N}{2}\sigma_u^2 . \tag{4.A.2}$$

Since this Lagrangian is a special case of (3.A.1), one can apply our SPA (1/N) in Sec.III to this model.

In terms of the renormalized quantities, (4.A.2) becomes

$$L_{\psi\sigma} = \bar{\psi}(i\partial\!\!\!/ - Z_1 g\sigma)\psi - \frac{N}{2} Z_3 \sigma^2 . \tag{4.A.3}$$

Since we are interested in the renormalization in SPA (1/N), the two constants Z_1 and Z_3 are needed. In general the wave function renormalization constant Z_2 and the counter term for four-fermion vertex $(\bar{\psi}\psi)^2$ are necessary. In these cases although the subsidiary

field σ seems to be a superfluous variable if the four-fermion term is needed, σ should be still incorporated as a representative of the dynamical variable for $\bar{\psi}\psi$ because the propagator of $\bar{\psi}\psi$ has a pole in the 1/N expansion[13]. We will discuss these problems in Sec.V.

Since the renormalization in SPA (1/N) should be performed order by order in the 1/N expansion, we expand the renormalization constants Z_1 and Z_3 as

$$\begin{aligned} Z_1 &= 1 + \frac{1}{N} z_1^{(1)} + O(1/N^2), \\ Z_3 &= z_3^{(0)} + \frac{1}{N} z_3^{(1)} + O(1/N^2). \end{aligned} \tag{4.A.4}$$

The stationary condition (3.B.13) for the Gross-Neveu model turns out to be

$$-z_3^{(0)} \sigma_c = -g \sum_A \bar{\psi}_A^{(0)} \psi_A^{(0)} + g \sum_A \left(\frac{n_A}{N}\right) \bar{\psi}_A^{(0)} \psi_A^{(0)}, \tag{4.A.5}$$

which should be solved together with the Dirac equation,

$$(i\partial\!\!\!/ - g\sigma_c)\, \psi_A^{(0)} = 0, \tag{4.A.6}$$

and the normalization conditon for $\psi_A^{(0)}$,

$$\int_{-\infty}^{\infty} dx\, \psi_A^{(0)\dagger} \psi_{A'}^{(0)} = \delta_{A,A'}. \tag{4.A.7}$$

The zeroth order stability angle $\zeta_A^{(0)}$ is given by the pseudo-periodicity

$$\psi_A^{(0)}(x,T) = \exp(-i\zeta_A^{(0)})\, \psi_A^{(0)}(x,0). \tag{4.A.8}$$

Once the classical solutions $(\sigma_{cl}, \psi_A^{(0)})$ are determined by (4.A.5)~(4.A.7), the effective action I_{eff} in SPA (1/N) is given by

$$I_{eff}(\{n_A\},T) = N S_{cl}(\{n_A\},T) + S_Q(\{n_A\},T) + S_{count}(\{n_A\},T), \quad (4.A.9)$$

where

$$S_{cl} \equiv \int_0^T d^2x \left(-\frac{1}{2}\beta_3^{(0)}\sigma_{cl}^2\right) + \sum_A \left(1-\frac{n_A}{N}\right)\zeta_A^{(0)}[\sigma_{cl}], \quad (4.A.10)$$

$$S_Q \equiv \int_0^T d^2x \left\{-\frac{1}{2}\beta_3^{(0)}\eta^2 - \sum_A \left(1-\frac{n_A}{N}\right) L_{F,A}^{(Q)}(\chi_A,\eta)\right\}, \quad (4.A.11)$$

$$S_{count} \equiv \int_0^T d^2x \left(-\frac{1}{2}\beta_3^{(1)}\sigma_{cl}^2\right) + \sum_A \left(1-\frac{n_A}{N}\right)\beta_1^{(1)} g \frac{\partial}{\partial g}\zeta_A^{(0)} \quad (4.A.12)$$

and

$$L_{F,A}^{(Q)} \equiv \bar{\chi}_A (i\not{\partial} - g\sigma_{cl})\chi_A - g(\bar{\psi}_A^{(0)}\chi_A + \bar{\chi}_A\psi_A^{(0)})\eta. \quad (4.A.13)$$

The renormalization program is performed by the following steps. In the first step the renormalization constants are determined in the vacuum sector ($\{n_A=0\}$). In the second step using these renormalization constants one can show that the physical quantities, for example the difference between the vacuum energy and the excited state energy, become finite in the other sector ($\{n_A\}\neq\{0\}$).

B. Renormalization of the vaccum sector

In order to determine the renormalization constants, one needs propagators and proper vertex functions. In the perturbation method the proper vertices are defined by using the Feynman graphs. Since the semiclassical method has no reference to the graph theory,

one has to define the proper vertices in the framework of the semiclassical method. We will, therefore, calculate the generating functional for the proper vertices[16),17)] Γ (proper Γ) in the semiclassical 1/N expansion method.

Let us introduce the generating functional $\mathcal{W}$ with the source functions J, β and $\bar{\beta}$, respectively, for the field operators $\hat{\Sigma}$, $\hat{\bar{\Psi}}$ and $\hat{\Psi}$, whose vacuum expectation values are associated with σ, $\bar{\psi}$ and ψ. This functional $\mathcal{W}$ generates connected Green's functions and is defined by

$$\exp\{i\mathcal{W}[J,\beta,\bar{\beta}]\} \equiv \int [d\Sigma\, d\bar{\Psi}\, d\Psi] \exp\left\{i\int_{-\infty}^{\infty} d^2x \left(L_{\Psi\Sigma} + N\Sigma J + \bar{\beta}\Psi + \bar{\Psi}\beta\right)\right\}, \tag{4.B.1}$$

where J is commuting function, and β and $\bar{\beta}$ are anti-commuting functions. Since we are interested in the vacuum sector only in this subsection, the time interval T is taken to be infinite from the outset in (4.B.1)[16)]. The justification of this procedure can be checked by comparing $\mathcal{W}$ in (4.B.1) with the contribution from the vacuum sector for the finite time interval T, which is taken to be infinite at the end step. By using $\mathcal{W}$ the proper Γ is defined by

$$\Gamma[\sigma,\bar{\Psi},\Psi] \equiv \mathcal{W}[J,\beta,\bar{\beta}] - \int_{-\infty}^{\infty} d^2x \left(N\sigma J + \bar{\beta}\Psi + \bar{\Psi}\beta\right). \tag{4.B.2}$$

The vacuum expectation values σ, $\bar{\psi}$ and ψ in the system with the non-vanishing source terms are given by

$$\sigma(x) = \frac{1}{N}\frac{\delta\mathcal{W}}{\delta J(x)}, \quad \bar{\Psi}(x) = -\frac{\delta\mathcal{W}}{\delta\beta(x)}, \quad \Psi(x) = \frac{\delta\mathcal{W}}{\delta\bar{\beta}(x)}. \tag{4.B.3}$$

Note that there are the following relations

$$\frac{\delta\Gamma}{\delta\sigma} = -NJ \ , \ \frac{\delta\Gamma}{\delta\bar{\psi}} = -\beta \ , \ \frac{\delta\Gamma}{\delta\psi} = \bar{\beta} . \tag{4.B.4}$$

In (4.B.3) and (4.B.4) the derivative with respect to the anti-commuting field is defined by the left derivative.

As the first step of calculating the proper Γ, we evaluate $\mathcal{W}$[18] by the same method as in Sec. III. The integration over Ψ fields provides us with

$$\mathcal{W}[J,\beta,\bar{\beta}] = -i\ln\left\{\int[d\Sigma]\exp\left(iN I_{\Sigma}[\Sigma;J,\beta,\bar{\beta}] + iI_{count}[\Sigma;J,\beta,\bar{\beta}]\right)\right\}, \tag{4.B.5}$$

where

$$I_{\Sigma}[\Sigma;J,\beta,\bar{\beta}] \equiv \int_{-\infty}^{\infty} d^2x \left(-\frac{1}{2}\mathcal{Z}_3^{(0)}\Sigma^2 + \Sigma J\right) - i\,\mathrm{Tr}\ln\left(\not{\partial} + ig\Sigma\right)$$
$$+ \frac{i}{N}\int_{-\infty}^{\infty} d^2x_1 d^2x_2\ \bar{\beta}(x_1) S_{\Sigma}(x_1,x_2)\beta(x_2) \tag{4.B.6}$$

and, up to the zeroth order in 1/N (the leading order is of order N),

$$I_{count}[\Sigma;J,\beta,\bar{\beta}] = \int_{-\infty}^{\infty} d^2x \left(-\frac{1}{2}\mathcal{Z}_3^{(1)}\Sigma^2\right) + \mathcal{Z}_1^{(1)} g\int_{-\infty}^{\infty} d^2x\,\Sigma(x)\,\mathrm{tr}\,S_{\Sigma}(x,x)$$
$$+ \frac{1}{N}\mathcal{Z}_1^{(1)} g\int_{-\infty}^{\infty} d^2x_1 d^2x_2 d^2x_3\ \bar{\beta}(x_1) S_{\Sigma}(x_1,x_2)\Sigma(x_2) S_{\Sigma}(x_2,x_3)\beta(x_3) . \tag{4.B.7}$$

The fermion propagator S_{Σ} under the background field of Σ is defined by

$$S_{\Sigma}(x_1,x_2) \equiv \left(\not{\partial} + ig\Sigma\right)^{-1}(x_1,x_2) . \tag{4.B.8}$$

The notations tr and Tr in (4.B.6) and (4.B.7) represent, respectively, the traces with respect to γ-matrices and space-time coordinates together with γ-matrices. Since excited states

do not contribute to (4.B.5) due to the infinite time interval in the usual iε prescription[16], $(NI_\Sigma+I_{count})$ is supposed to equal I_{eff} ($\{n_A=0\}$, $T\to\infty$) in (3.A.18) if the source terms are switched off. Note that the Dirac vacuum energy multiplied by the time interval T, $N\sum_A \zeta_A[\Sigma]$ in (3.A.18), becomes $(-iN)$ Tr $(\not\partial + iZ_1 g\Sigma)$ as T goes infinity. We assume that $\bar\beta\beta$ and $\bar\psi\psi$ are of order N because the fermion field ψ has N internal components.

Since I_Σ is multiplied by N in the exponent of (4.B.5), we can apply S.P.A. to (4.B.5) as in Sec. III. Then we make an expansion of the exponent in (4.B.5) around the stationary point Σ_c up to the zeroth order in 1/N by substituting $\Sigma(x)=\Sigma_c(x)+\eta(x)/\sqrt{N}$. Note that the leading order is of order N. By performing the integration over η one obtains

$$\mathcal{W} = N I_\Sigma[\Sigma_c] + \frac{i}{2}\mathrm{Tr}\ln\frac{\delta^2 I_\Sigma}{\delta\Sigma\delta\Sigma}[\Sigma_c] + I_{count}[\Sigma_c] + O(1/N), \tag{4.B.9}$$

where the stationary point Σ_c is defined by

$$\left.\frac{\delta I_\Sigma}{\delta\Sigma}\right|_{\Sigma=\Sigma_c} = 0. \tag{4.B.10}$$

Substituting (4.B.9) into (4.B.3), β, $\bar\beta$ and the stationary point Σ_c, which is functional of J, β and $\bar\beta$ through (4.B.10), can be solved in terms of σ, $\bar\psi$ and ψ as

$$\begin{aligned}
\beta(x) &= -i(\not\partial + ig\sigma(x))\psi(x) + \frac{1}{N}\beta^{(1)}[\sigma,\bar\psi,\psi;x],\\
\bar\beta(x) &= -i\bar\psi(x)(-\overleftarrow{\not\partial} + ig\sigma(x)) + \frac{1}{N}\bar\beta^{(1)}[\sigma,\bar\psi,\psi;x],\\
\Sigma_c(x) &= \sigma(x) + \frac{1}{N}\Sigma^{(1)}[\sigma,\bar\psi,\psi;x].
\end{aligned} \tag{4.B.11}$$

The proper Γ is calculated from (4.B.2) if (4.B.9) and (4.B.11) are substituted;

$$\Gamma[\sigma,\bar{\Psi},\Psi] = N I_{\Sigma}[\Sigma=\sigma, \beta=\frac{1}{i}S_{\sigma}^{-1}\Psi, \bar{\beta}=\frac{1}{i}\bar{\Psi}S_{\sigma}^{-1}] - N\int_{-\infty}^{\infty} d^2x\,\sigma]$$
$$+2i\int_{-\infty}^{\infty} d^2x\,\bar{\Psi}S_{\sigma}^{-1}\Psi + \frac{i}{2}Tr\ln\frac{\delta^2 I_{\Sigma}}{\delta\Sigma\delta\Sigma}[\Sigma=\sigma, \beta=\frac{1}{i}S_{\sigma}^{-1}\Psi, \bar{\beta}=\frac{1}{i}\bar{\Psi}S_{\sigma}^{-1}]$$
$$+I_{count}[\Sigma=\sigma, \beta=\frac{1}{i}S_{\sigma}^{-1}\Psi, \bar{\beta}=\frac{1}{i}\bar{\Psi}S_{\sigma}^{-1}], \qquad (4.B.12)$$

where

$$S_{\sigma}(x_1,x_2) \equiv (\not{\partial} + ig\sigma)^{-1}(x_1,x_2). \qquad (4.B.13)$$

In obtaining (4.B.12) $\Sigma^{(1)}$, $\beta^{(1)}$ and $\bar{\beta}^{(1)}$ were cancelled out up to the zeroth order in 1/N. Some algebraic computation leads us to

$$\Gamma[\sigma,\bar{\Psi},\Psi] = N\Gamma^{(0)}[\sigma,\bar{\Psi},\Psi] + \Gamma^{(1)}[\sigma,\bar{\Psi},\Psi] + \Gamma_{count}[\sigma,\bar{\Psi},\Psi], \qquad (4.B.14)$$

where

$$\Gamma^{(0)} = \int_{-\infty}^{\infty} d^2x\left\{-\frac{1}{2}\mathfrak{z}_3^{(0)}\sigma^2 + \frac{1}{N}\bar{\Psi}(i\not{\partial} - g\sigma)\Psi\right\} - iTr\ln(\not{\partial} + ig\sigma), \qquad (4.B.15)$$

$$\Gamma^{(1)} = \frac{i}{2}Tr\ln\Big[\mathfrak{z}_3^{(0)}\delta(x-y) + ig^2 tr\,S_{\sigma}(x,y)S_{\sigma}(y,x)$$
$$+\frac{i}{N}g^2\{\bar{\Psi}(x)S_{\sigma}(x,y)\Psi(y) + \bar{\Psi}(y)S_{\sigma}(y,x)\Psi(x)\}\Big], \qquad (4.B.16)$$

$$\Gamma_{count} = \int_{-\infty}^{\infty} d^2x\left\{-\frac{1}{2}\mathfrak{z}_3^{(1)}\sigma^2 - \frac{1}{N}\mathfrak{z}_1^{(1)}g\bar{\Psi}\Psi\sigma + \mathfrak{z}_1^{(1)}g\,tr\,S_{\sigma}(x,x)\sigma(x)\right\}. \qquad (4.B.17)$$

It is worthwhile to note that when $\psi=0$ in (4.B.15)~(4.B.17) $\Gamma^{(0)}$, $\Gamma^{(1)}$ and Γ_{count} correspond to S_{cl}, S_Q and S_{count} with $\{n_A=0\}$ in (4.A.10)~(4.A.12) respectively.

From (4.B.14)~(4.B.17) one can understand what SPA (1/N) is if interpreted in the language of the graph theory[19]. The proper Γ in the zeroth order approximation of SPA (1/N), $\Gamma^{(0)}$ (4.B.15), consists of the proper tree graphs and the proper graphs with

fermion one-loops (see Fig.1). This implies that the first term in S_{cl} (4.A.10) corresponds to the proper tree vacuum graph and the second term which is the sum of all the stability angles for the fermion eigen modes, $\sum_A \zeta_A^{(0)}$, stands for the sum of the proper vacuum graphs with fermion one-loops. The first order proper Γ, $\Gamma^{(1)}$ (4.B.16), consists of the proper graphs with the boson one-loops, while infinite fermion loops are included in $\Gamma^{(1)}$ (see Fig.2). This corresponds to the sum of all the stability angles for the quantum fluctuation, $-\sum_\alpha \frac{1}{2}\nu_\alpha$ ($\{n_A=0\}$, $T\to\infty$). The first order counter terms in Γ_{count} (4.B.17) are shown in Fig.3. The above results can be understood from the fact that the σ-propagator is proportional to 1/N and one fermion loop provides a factor N due to N equivalent fermion components.

The renormalization in the zeroth order of SPA (1/N) is, therefore, performed by the usual one-loop renormalization program. On the other hand, the first order of SPA (1/N) is not a simple two-loop approximation. Note that the proper vertex in the first order of SPA (1/N) includes the proper graphs with different number of loops. This means that even if the system is renormalizable in the perturbation method, the renormalizability of the first order is not obvious in SPA (1/N).

In the Gross-Neveu model one can show that the renormalization in SPA (1/N) can be performed order by order in the parameter 1/N.

The vacuum expectation values σ_0, $\bar{\psi}_0$ and ψ_0 of the field operators $\hat{\Sigma}$, $\hat{\bar{\Psi}}$ and $\hat{\Psi}$ in the original system without source terms are given by (4.B.4) as the solution of the stationary conditions of the proper Γ,

$$\frac{\delta\Gamma}{\delta\sigma} = \frac{\delta\Gamma}{\delta\bar{\psi}} = \frac{\delta\Gamma}{\delta\psi} = 0. \qquad (4.B.18)$$

From (4.B.14)~(4.B.17) it is obvious that $\psi_0 = \bar{\psi}_0 = 0$ satisfies (4.B.18), so ψ_0 and $\bar{\psi}_0$ are supposed to be zero. If we assume that the vacuum in this model is invariant under the space-time translation, the vacuum expectation value σ_0 should be a constant.

Before solving (4.B.18) one has to perform the renormatization program at a certain renormalization point μ to determine the renormalization constants by some renormalization conditions. After these procedure σ_0 is represented in terms of μ, the coupling constant g and the expansion parameter 1/N. Conversely μ is written in terms of σ_0, g and 1/N by solving this relation and so the renormalization constants are. At this point the stationary condition,

$$\left.\frac{\delta\Gamma}{\delta\sigma(x)}\right|_{\sigma=\sigma_0,\ \psi=\bar{\psi}=0} = 0, \qquad (4.B.19)$$

can be interpreted as a renormalization condition for Z_3. It should be noted that σ_0 has to be restricted such that the proper Γ is maximum at the point $\sigma=\sigma_0$[16]. This condition implies that the σ field has no tachyon. In this paper we adopt (4.B.19) for Z_3 and the following renormalization conditon for Z_1,

$$\left.\frac{\delta^3\Gamma}{\delta\sigma(P_1=0)\delta\psi(P_2)\delta\bar{\psi}(P_3)}\right|_{\sigma=\sigma_0,\ \psi=\bar{\psi}=0} = -g, \qquad (4.B.20)$$

where $\sigma(P_1)$ and others are the Fourier transforms of $\sigma(x)$, etc.

Before going to practical calculations we comment on the regularization of divergences. In order to perform the consistent

renormalization, one has to use a common regularization method throughout calculations. In particular one has to compute the sum of the stability angles for the fermion excitation $\sum_A \zeta_A^{(0)}$, the sum of the stability angles for the quantum fluctuation $\sum_\alpha \frac{1}{2}\nu_\alpha(\{n_A\})$ and the proper Γ (4.B.14) by the same method.

Since the time interval is taken to be infinite from the outset in calculating the proper Γ it is useful to integrate over the space-time coordinates in the infinite volume. On the other hand, in making the sum over the stability angles it is convenient to quantize waves in the finite box and the finite time interval. The continuum limit (infinite volume limit) should be taken after finding the integration measure for the quantum numbers which are assigned to each stability angle mode.

In the leading order of 1/N the renormalization constant $z_3^{(0)}$ is determined by

$$\left.\frac{\delta\Gamma^{(0)}}{\delta\sigma}\right|_{\sigma=\sigma_0,\,\psi=\bar{\psi}=0} = 0. \qquad (4.B.21)$$

The result is

$$z_3^{(0)} = \frac{ig^2}{2\pi^2}\int d^2k\,\frac{1}{k^2-g^2\sigma_0^2} = \frac{g^2}{2\pi}\ln\left(\Lambda^2/g^2\sigma_0^2\right), \qquad (4.B.22)$$

where the straight cut off regularization is adopted. It is obvious that the expansion of Z_1 in (4.A.4) agrees with the renormalization condition (4.B.20) in the leading order of 1/N.

The renormalization conditions of the next to the leading order in 1/N turn out to be

$$\left.\frac{\delta(\Gamma^{(1)}+\Gamma_{count})}{\delta\sigma}\right|_{\sigma=\sigma_0,\,\psi=\bar{\psi}=0}=0\,,$$

$$\left.\frac{\delta(\Gamma^{(1)}+\Gamma_{count})}{\delta\sigma\,\delta\psi\,\delta\bar{\psi}}\right|_{\sigma=\sigma_0,\,\psi=\bar{\psi}=0}=0\,, \tag{4.B.23}$$

where $z_3^{(0)}$ is substituted so that the renormalization constants are determined order by order in 1/N. From (4.B.23) one obtaines

$$\begin{aligned}
z_1^{(1)} &= \frac{i}{2\pi}\int d^2k\,\frac{k^2+g^2\sigma_0^2}{(k^2-g^2\sigma_0^2)^2}\,\frac{1}{B(-k^2,g^2\sigma_0^2)} \\
&= \frac{1}{2}\ln\ln(\Lambda^2/g^2\sigma_0^2)+\text{finite}\,, \\
z_3^{(1)} &= \frac{-i}{2\pi^2}\int d^2k\left[\frac{g^2}{k^2-g^2\sigma_0^2}-2z_1^{(1)}g^2\frac{k^2}{(k^2-g^2\sigma_0^2)^2}\right] \\
&= -\frac{g^2}{2\pi}\left[\ln(\Lambda^2/4g^2\sigma_0^2)+2z_1^{(1)}\{1-\ln(\Lambda^2/g^2\sigma_0^2)\}\right],
\end{aligned} \tag{4.B.24}$$

where

$$B(-k^2,g^2\sigma_0^2)\equiv\sqrt{\frac{4g^2\sigma_0^2-k^2}{-k^2}}\ln\left[\frac{\sqrt{-k^2+4g^2\sigma_0^2}+\sqrt{-k^2}}{\sqrt{-k^2+4g^2\sigma_0^2}-\sqrt{-k^2}}\right]. \tag{4.B.25}$$

If $z_3^{(0)}$, $z_3^{(1)}$ and $z_1^{(1)}$ so obtained are substituted into (4.B.14) the proper Γ becomes finite up to the next to the leading order in 1/N except a constant divergent number which can be swept away by the subtraction of the vacuum energy.

It should be emphasized that although $\Gamma^{(1)}$ (4.B.16) includes the infinite number of loops, Γ_{count} (4.B.17) has finite numbers of counter terms which are linear with respect to the renormalization constants. This fact enables us to perform the renormalization by the subtraction scheme even in SPA (1/N).

C. Excited fermion states

In general, it is difficult to solve the stationary condition (3.B.13) with the Dirac equation (3.B.7). To do so, one has to solve the Dirac equation in an arbitrary boson background field to obtain the complete set of the fermion wave functions which are functionals of the boson field. After substituting these fermion wave functions into (3.B.13), the classical solution for the boson field is determined by (3.B.13). For static classical solutions in the Gross-Neveu model, DHN[2] succeeded in solving the stationary condition (4.A.5) with (4.A.6) by using the inverse scattering method. In order to obtain time-dependent solutions, some speculative consideration is needed. One approach is given by restricting configurations of the boson field into a certain set of functions.

In the case of σ_{cl} having a single discrete level and which is occupied by n_0 excited fermions, the classical solution for (4.A.5)~(4.A.7) is given by

$$\sigma_{cl}(x) = \sigma_0 + \frac{K}{g}\left[\tanh K(x-\delta_0) - \tanh K(x+\delta_0)\right], \tag{4.C.1}$$

where

$$K = g\sigma_0 \sin\left(\frac{\pi}{2}\frac{n_0}{N}\right), \quad \delta_0 = \frac{1}{4K}\ln\left(\frac{g\sigma_0+K}{g\sigma_0-K}\right). \tag{4.C.2}$$

The fermion wave function for the discrete level is

$$\psi_0^{(0)}(x) = \frac{\sqrt{K}}{2}\begin{pmatrix} \operatorname{sech} K(x+\delta_0) \\ -\operatorname{sech} K(x-\delta_0) \end{pmatrix} \tag{4.C.3}$$

and for continuum levels

$$\psi^{(0)}_{(k,x)} = \frac{1}{\sqrt{2N(k)}} \frac{e^{ikx}}{k+iK} \begin{pmatrix} k + iK\tanh K(x+\delta_0) \\ \frac{(ik-g\sigma_0)}{\sqrt{k^2+g^2\sigma_0^2}}\{k+iK\tanh K(x-\delta_0)\} \end{pmatrix}, \quad (4.C.4)$$

where the normalization constant N(k) is given by

$$N(k) = L - 2K/(k^2+K^2) \quad (4.C.5)$$

and L is the length of the quantization box. In this case, the energy levels of the leading order in 1/N are determined by using (3.C.5);

$$E_{n_0} = -\frac{dS_{cl}}{dT} = \frac{2}{\pi} g\sigma_0 N \sin\left(\frac{\pi}{2}\frac{n_0}{N}\right), \quad (4.C.6)$$

where the fermion occupation number n_0 of the discrete level is restricted to $0 \le n_0 \le N-1$ because there is no solution for $n_0 \ge N$. The divergence in the energy is cancelled away by that in the renormalization constant $z_3^{(0)}$ which was determined in the vacuum sector. The degeneracy number of the state $\{n_0\}$ is

$$(2N)!/n_0!\,(2N-n_0)!,$$

which is derived from (3.A.16).

Quantum corrections are calculated from (4.A.11). Since the above classical solution is static, the stability angle of the quantum fluctuation becomes the angular frequency times the time interval T. Then we will find the eigenfrequencies of the quantum fluctuations χ_A and η which obey

$$(i\partial\!\!\!/ - g\sigma_{cl})\chi_A + g\eta\psi^{(0)}_A = 0, \quad (4.C.7)$$

$$-z_3^{(0)}\eta + \sum_A g(\bar{\psi}^{(0)}_A\chi_A + \bar{\chi}_A\psi^{(0)}_A) - g\frac{n_0}{N}(\bar{\psi}^{(0)}_0\chi_0 + \bar{\chi}_0\psi^{(0)}_0) = 0. \quad (4.C.8)$$

Eliminating χ_A, one obtains the homogeneous integral equation for the eigenfrequency ω,

$$\beta_3^{(0)} \eta_\omega(x) = -g^2 \int dy\, K_\omega(\{n_0\}; x, y)\, \eta_\omega(y), \tag{4.C.9}$$

where

$$\eta(x,t) = \int \frac{d\omega}{2\pi}\, \eta_\omega(x)\, e^{-i\omega t}. \tag{4.C.10}$$

The kernel function to the configuration $\{n_0\}$ is given by

$$K_\omega(\{n_0\}; x, y) = \sum_A \left(\frac{n_A}{N} - 1\right)\Big[\bar{\Psi}_A^{(0)}(x)\, G_{\omega_A+\omega}(\sigma_{cl}; x, y)(1-P_A)\Psi_A^{(0)}(y) + \bar{\Psi}_A^{(0)}(y)\gamma^0 G^\dagger_{\omega_A-\omega}(\sigma_{cl}; y, x)\gamma^0(1-P_A)\Psi_A^{(0)}(x)\Big]\Big|_{\{n_A\}=\{n_0\}} \tag{4.C.11}$$

$$= -\Big[\Big\{\int \frac{dk}{2\pi} N(k)\, \bar{\Psi}^{(0)}_{(k,x)}\, G_{\omega(k)+\omega}(\sigma_{cl}; x, y)(1-P_k)\Psi^{(0)}_{(k,y)} + \left(1 - \frac{n_0}{N}\right)\bar{\Psi}_0^{(0)}(x)\, G_{\omega_0+\omega}(\sigma_{cl}; x, y)(1-P_0)\Psi_0^{(0)}(y)\Big\} + \{(x, y, \omega) \to (y, x, -\omega)\}^*\Big], \tag{4.C.12}$$

where the energy of the zeroth order fermion state ω_A is defined by $\omega(k)=\sqrt{g^2\sigma_0^2+k^2}$ for the continuum state and $\omega_0=\sqrt{g^2\sigma_0^2-K^2}$ for the discrete level state. The fermion propagator in the boson background field σ_{cl} for the configuration $\{n_0\}$ is obtained by substituting the classical solution σ_{cl} into (4.B.13);

$$G_E(\sigma_{cl}; x, y) = \int dt\, e^{iE(t-t')}\, S_{\sigma=\sigma_{cl}}(x, t; y, t')$$

$$= i\frac{EN(k)}{k}\Big[\Psi^{(0)}_{(k,x)}\bar{\Psi}^{(0)}_{(k,y)}\theta(x-y) + \Psi^{(0)}_{(-k,x)}\bar{\Psi}^{(0)}_{(-k,y)}\theta(y-x)\Big], \tag{4.C.13}$$

where $k=\sqrt{E^2-g^2\sigma_0^2}$.

It may be instructive to give an alternative expression for

(4.C.9),

$$\int\Big[-\mathfrak{z}_3^{(0)}\delta(x-y)-i\frac{g^2}{N}\Big\{(N-n_0)\int_C\frac{dE}{2\pi}\,\mathrm{tr}\,G_E(\sigma_{cl};x,y)\,G_{E-\omega}(\sigma_{cl};y,x)$$

$$+n_0\int_{C_0}\frac{dE}{2\pi}\,\mathrm{tr}\,G_E(\sigma_{cl};x,y)\,G_{E-\omega}(\sigma_{cl};y,x)\Big\}\Big]\eta_\omega(y)\,dy=0. \qquad (4.C.14)$$

The crucial point in (4.C.14) consists in the E-integration. The contour C, which corresponds to the usual i ε prescription, provides us with the vacuum polarization amplitude (Fig.4). The contour C_0 takes the round path avoiding the discrete eigenstate pole which is occupied by the excited fermion (Fig.4). In general, for each component of the fermion field which has N components, the contour $C_{\{n_A\}}$ must be introduced. This contour avoids the eigen state poles which are occupied by the excited fermions or antifermions (Fig.5). In this representation n_A is restricted to $0\le n_A\le 2$, which implies that each component of the fermion field has only particle-anti-particle degrees of freedom. By the use of the contours $C_{\{n_A\}}$ the kernel function (4.C.11) for the configuration $\{n_A\}$ is given by

$$K_\omega(\{n_A\};x,y)=\frac{i}{N}\sum_{k=1}^{N}\int_{C_{\{n_A^{(k)}\}}}\frac{dE}{2\pi}\,\mathrm{tr}\,G_E(\sigma_{cl};x,y)\,G_{E-\omega}(\sigma_{cl};y,x), \qquad (4.C.15)$$

where $n_A=\sum_{k=1}^{N}n_A^{(k)}$. Note that (4.C.14) with $n_0=0$ and $\sigma_{cl}=\sigma_0$ equals the term inside the square brackets in $\Gamma^{(1)}$ (4.B.16) with $\psi=\bar\psi=0$ and $\sigma=\sigma_0$. From this fact one can confirm that $\Gamma^{(1)}$ (4.B.16) equals $\sum_\alpha(-\frac{1}{2})\nu_\alpha(\{n_A=0\},T\to\infty)$ in (3.B.24) as previously commented. In the usual i ε prescription excited states can not contribute to the proper Γ because of the infinite time interval. The expression of the kernel function (4.C.15), however, implies

that if the contour $C_{\{n_A\}}$ is adopted in the energy integration of the Fourier transform of $\Gamma^{(1)}$ instead of the usual i ε prescription, $\Gamma^{(1)}$ can provide us with the excited state energy. At the same time one has to use an appropriate contour for Tr ln $(\not{\partial}+ig\sigma)$ in $\Gamma^{(0)}$ (4.B.15) when one finds the classical solution from the stationary condition $\delta\Gamma^{(0)}/\delta\sigma(x)=0$.

The diverging term in the r. h. s. of (4.C.9) can be eliminated by $z_3^{(0)}$ in the l. h. s. which was determined in the vacuum sector.

For the static classical solution, the energy of excited state $(\{n_A\}, \{p_\alpha\})$ is given by

$$
\begin{aligned}
E(\{n_A\},\{p_\alpha\}) = N\Big\{ & \int dx \tfrac{1}{2} z_3^{(0)} \sigma_{cl}^2(\{n_A\};x) - \sum_A \left(1-\frac{n_A}{N}\right)\frac{1}{T}\zeta_A^{(0)}[\sigma_{cl}]\Big\} \\
& + \Big\{ \int dx \tfrac{1}{2} z_3^{(1)} \sigma_{cl}^2(\{n_A\};x) - z_1^{(1)} g \sum_A \left(1-\frac{n_A}{N}\right)\frac{1}{T}\frac{\partial}{\partial g}\zeta_A^{(0)}[\sigma_{cl}]\Big\} \\
& + \sum_\alpha \left(p_\alpha+\tfrac{1}{2}\right)\omega_\alpha(\{n_A\}),
\end{aligned}
\qquad (4.C.16)
$$

where $\omega_\alpha(\{n_A\})$ is the eigenfrequency of (4.C.9). The divergencies in (4.C.16) are eliminated by substitution of the renormalization constants determined in the vacuum sector except for the divergence which is independent of $(\{n_A\}, \{p_\alpha\})$. However, the energy measured from the vacuum energy,

$$
M(\{n_A\},\{p_\alpha\}) = E(\{n_A\},\{p_\alpha\}) - E(\{0\},\{0\}), \qquad (4.C.17)
$$

is finite.

§V. Multiplicative renormalization

A. Renormalization by the subtraction scheme

As pointed out in Sec. IV. B, the renormalization of the Gross-Neveu model can be performed by the subtraction scheme even in SPA(1/N) if the Lagrangian (4.A.3) is adopted. Up to the first order of SPA(1/N) the four-fermion vertex is finite in the particular model Lagrangian (4.A.3), although it can be divergent as far as the power counting is concerned. In the higher order the cancellation of the devergence in the four-fermion vertex has not be known. If the four-fermion vertex is divergent its counter term, $g^2 Z_4 (\bar{\psi}\psi)^2/2N$, is needed in (4.A.3).

For the O(N) symmetric ϕ^4 model,[20] SPA(1/N) can be also applied. In this model, even if the subsidiary field B for the composite field ϕ^2 is introduced, the ϕ^4 vertex is still divergent in the 4-dimensional space-time. The counter term $Z_5 g^2 \phi^4/2N$ is necessary in addition to the $B\phi^2$ vertex.

We will, therefore, study the Lagrangian including the $g^2 Z_4 (\bar{\psi}\psi)^2/2N$ term as well as the $gZ_1 \sigma(\bar{\psi}\psi)$ term in the following. If the renormalization constant Z_4 does not include the zeroth order term in 1/N, $Z_4 = z_4^{(1)}/N + O(1/N^2)$, the additional counter term,

$$\frac{g^2}{2} Z_4^{(1)} \int d^2x \left\{ \mathrm{tr}\, S_\sigma(x,x) - \frac{1}{N} \bar{\Psi}(x)\Psi(x) \right\}^2, \qquad (5.A.1)$$

appears in Γ_{count} (4.B.17). In this case, the renormalization can be performed by the subtraction scheme. In the O(N)

symmetric ϕ^4 theory, the renormalization can be performed by the same way as in the Gross-Neveu model.

The $\sigma(\bar{\psi}\psi)$ term generates the $(\bar{\psi}\psi)^2$ term in the Lagrangian (4.A.1) if σ is integrated out. Therefore, one has to prove that the separation of the $\sigma(\bar{\psi}\psi)$ term from the $(\bar{\psi}\psi)^2$ term gives no effect to the physical observable quantities.[19] This is a future problem.

B. Multiplicative renormalization in the 1/N expansion method

In this subsection, the Lagrangian defined by

$$L = \bar{\psi} i \partial\!\!\!/ \psi - \frac{N}{2} Z_3 \sigma^2 - Z_1 g \sigma \bar{\psi}\psi + \frac{Z_4}{2N} g^2 (\bar{\psi}\psi)^2 , \qquad (5.B.1)$$

is adopted. This system is equivalent to the Gross-Neveu model, but the subsidiary field σ is incorporated in a different way. The renormalization constant Z_4 is supposed to include the zeroth order in 1/N; $Z_4 = z_4^{(0)} + z_4^{(1)}/N + O(1/N^2)$. In this case the four-fermion vertex is proportional to a factor 1/N in the leading order of 1/N. On the other hand, if the loopwise contraction over spinor indices is taken, each fermion loop provides a factor N due to N fermion components. Therefore, all proper vertices include infinite loops and terms with any higher power of the renormalization constants in the leading order of 1/N as shown in Fig. 6 and Fig. 7. One can not separate counter (subtraction) terms from the proper vertices and the renormalization in the leading order of 1/N should be performed by the multiplicative renormalization. Since we are studying the renormalization program of this model only in the

leading order of 1/N, we will omit the superscript (0) in the following.

In the multiplicative renormalization program, one first arranges the renormalization constants by using integral equations in which a finite number of the renormalization constants appear. Then one can determine the renormalization constants under appropriate renormalization conditions. Since all proper vertices are divergent, one has to substitute the renormalization constants determined above into these proper vertices and to examine whether they become finite.

In the vacuum sector of this system, the proper $(\bar{\psi}\psi)$ vertex function turns out to be

$$\langle(\bar{\psi}\psi)\rangle_{proper} = -iM, \tag{5.B.2}$$

where M is the fermion mass in this approximation and determined by the integral equation (Fig. 6),

$$M = Z_4 g^2 \int \frac{d^2k}{(2\pi)^2} \, tr \frac{i}{k-M} + Z_1 g \sigma_0 . \tag{5.B.3}$$

The vacuum expectation value σ_0 of the σ field must satisfy that

$$\sigma_0 = \frac{Z_1}{Z_3 Z_4} \frac{1}{g} M - \frac{(Z_1)^2}{Z_3 Z_4} \sigma_0 , \tag{5.B.4}$$

which corresponds to the stationary condition $\delta\Gamma/\delta\sigma|_{\sigma=\sigma_0}=0$. The proper $(\bar{\psi}\psi)^2$ and $\sigma(\bar{\psi}\psi)$ vertices with zero external momenta (Fig. 7) are given by

$$\langle(\bar{\Psi}\Psi)^2\rangle_{proper} = -\frac{1}{N}\left(\frac{Z_4}{Z_1}\right)\langle\sigma(\bar{\Psi}\Psi)\rangle_{proper}$$
$$= i\frac{g^2}{N}\left[Z_4^{-1} + ig^2\int\frac{d^2k}{(2\pi)^2}\,tr\left(\frac{i}{\not k - M}\right)^2\right]^{-1}. \qquad (5.B.5)$$

From Fig. 6 and Fig. 7 one findsthat the proper vertices consist of infinite loops and the renormalization constants while the numbers of the renormalization constants in (5.B.2)~(5.B.5) are finite due to the integral equation (5.B.3). This fact enables us to calculate the renormalization constants under some renormalization conditions.

We will adopt the following renormalization conditions,

$$\langle\sigma(\bar{\Psi}\Psi)\rangle_{proper}\big|_{\text{zero external momenta}} = -ig\,, \qquad (5.B.6)$$

$$\langle(\bar{\Psi}\Psi)^2\rangle_{proper}\big|_{\text{zero external momenta}} = \frac{ig^2}{N} \qquad (5.B.7)$$

and (5.B.4). The renormalization constatns are, then, determined from these renormalization conditions;

$$Z_1 = Z_4 = -Z_3 + \frac{M}{g\sigma_0} = \left[1 + \frac{g^2}{2\pi}\ln(\Lambda^2/M^2) - \frac{g^2}{\pi}\right]^{-1}, \qquad (5.B.8)$$

where the fermion mass is given by

$$M = g\sigma_0\left(1 - \frac{g^2}{\pi}\right)^{-1}. \qquad (5.B.9)$$

These renormalization constants make the proper Γ to be finite in the leading order of 1/N.

It is more complicated to perform the renormalization of this system in the higher order of 1/N. We have not known how to prove the renormalizability in the multiplicative renormalization in general. This analysis is an interesting problem.

§VI. Conclusions

In this article we have studied the semiclassical method (SPA(1/N)) for the boson-fermion system in which N is the number of equivalent fermions and used as an expansion parameter. In particular, since SPA (1/N) is not the loop expansion, it is necessary to check whether the renormalization program can be performed consistently in the framework of SPA (1/N). We have formulated the renormalization program from the general viewpoint. This analysis will be useful for the renormalization in other non-perturbative methods which will be developed in future. It has been shown that the multiplicative renormalization is necessary and possible in the leading order of 1/N expansion method for the particular Lagrangian,

$$\mathcal{L} = \bar{\psi} i\partial\!\!\!/ \psi - \frac{N}{2} Z_3 \sigma^2 - Z_1 g \sigma \bar{\psi}\psi + \frac{Z_4}{2N} g^2 (\bar{\psi}\psi)^2 . \qquad (6.1)$$

Finally, we would like to point out that the following problems should be investigated as next steps.

(1) The treatment of the zero mode in the usual semiclassical method has been discussed in Sec. II. C, according to DHN[1]. It has been found that the mapping (2.C.21) may separate out the zero mode from the non-zero stability angle modes. By this mapping, the intuitive meaning of the zero mode becomes more clear. It is also necessary to show explicitly that the zero mode factor in SPA (1/N) can be separated out in the similar way.

(2) The multiplicative renormalization method has been applied in the leading order of 1/N for the Lagrangian (6.1). It is interesting to study whether or not the multiplicative renormal-

ization program can be constructed for the O(N) symmetric ϕ^4 theory in the 4-dimensional space-time.

(3) Our final purpose is to compute hadron spectra by using SPA (1/N). To do so one has to find "the classical solution" for a 4-dimensional model and it is an open problem.

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Figure captions

Fig. 1

(a) Feynman rules for the Gross-Neveu model with the Lagrangian (4.A.3): Wavy and straight lines indicate the boson and the fermion propagation, respectively.

(b) The graphical representation of the leading order proper Γ (4.B.15): Lines with σ and ψ indicate the external fields of the boson and the fermion, respectively.

Fig. 2 The graphical representation of the next leading order Γ (4.B.16).

Fig. 3 The graphical representation of the counter terms Γ_{count} (4.B.17).

Fig. 4

(a) The analyticity of the integral function $G_E\, G_{E-\omega}$ in (4.C.14): Bold and dashed lines indicate the cuts of $G_{E-\omega}$ and G_E, respectively. A circle means the discrete pole for the fermion from G_E, a bold circle for the fermion from $G_{E-\omega}$, a circle with a cross for the anti-fermion from G_E, and a bold circle with a cross for the anti-fermion from $G_{E-\omega}$.

(b) The contours C and C_0: The contour C, which corresponds to the usual $i\varepsilon$ prescription, provides the vacuum state. The contour C_0 provides the state in which the discrete level is occupied by one excited fermion.

Fig. 5 The contours $C_{\{n_A\}}$: $C_{\{n_0=0\}}$ and $C_{\{n_0=1\}}$ equal C and C_0 in Fig. 4, respectively. $C'_{\{n_0=1\}}$ provides the state

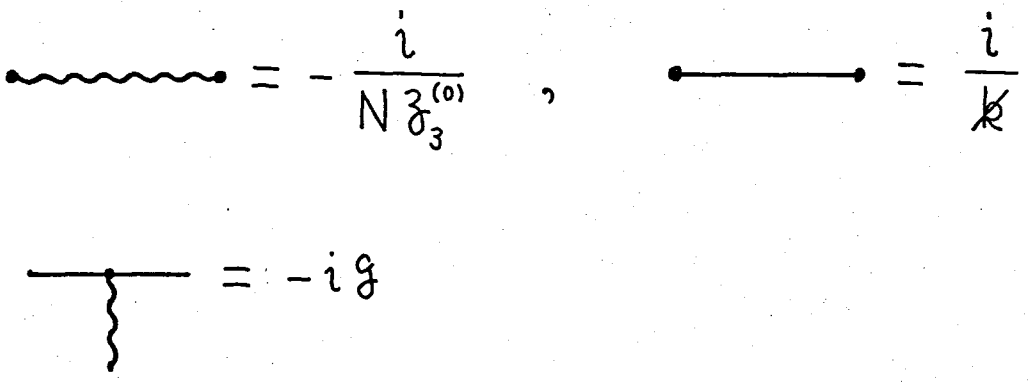
in which the discrete level is occupied by one anti-fermion and $C_{\{n_0=2\}}$ corresponds to the state occupied by one fermion and one anti-fermion.

Fig. 6

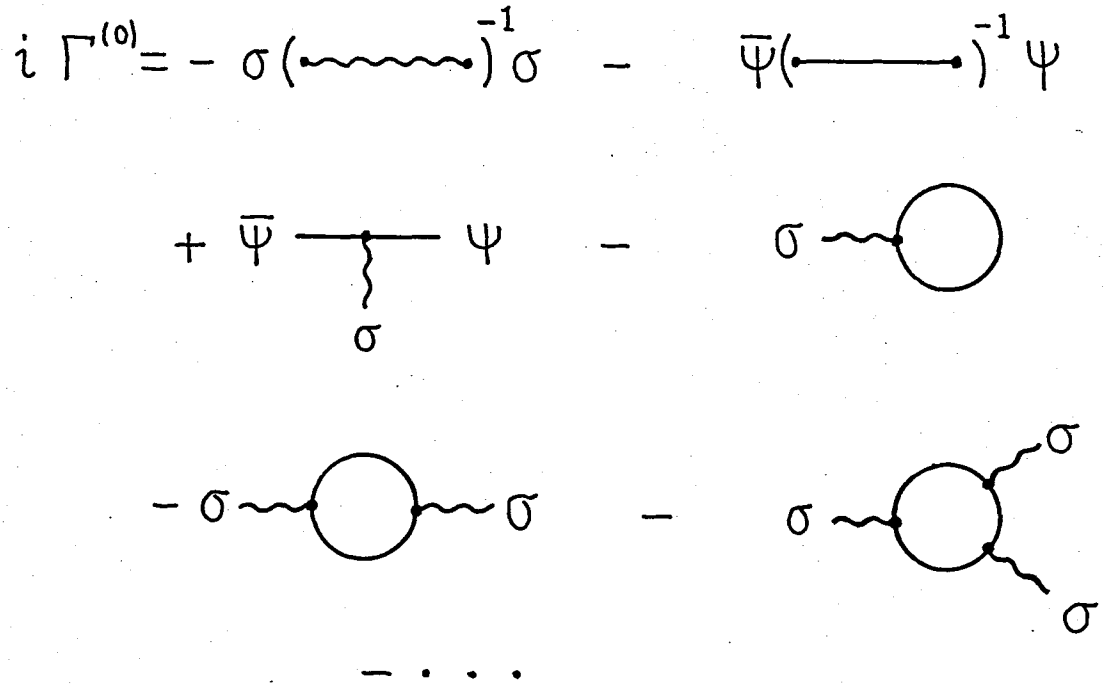
(a) Feynman rules for the Gross-Neveu model with the Lagrangian (5.B.1): Wavy and straight lines indicate the boson and the fermion propagation, respectively.

(b) The fermion mass M: The loopwise contraction over spinor indices is taken in this figure and Fig. 7.

Fig. 7 The proper vertices.

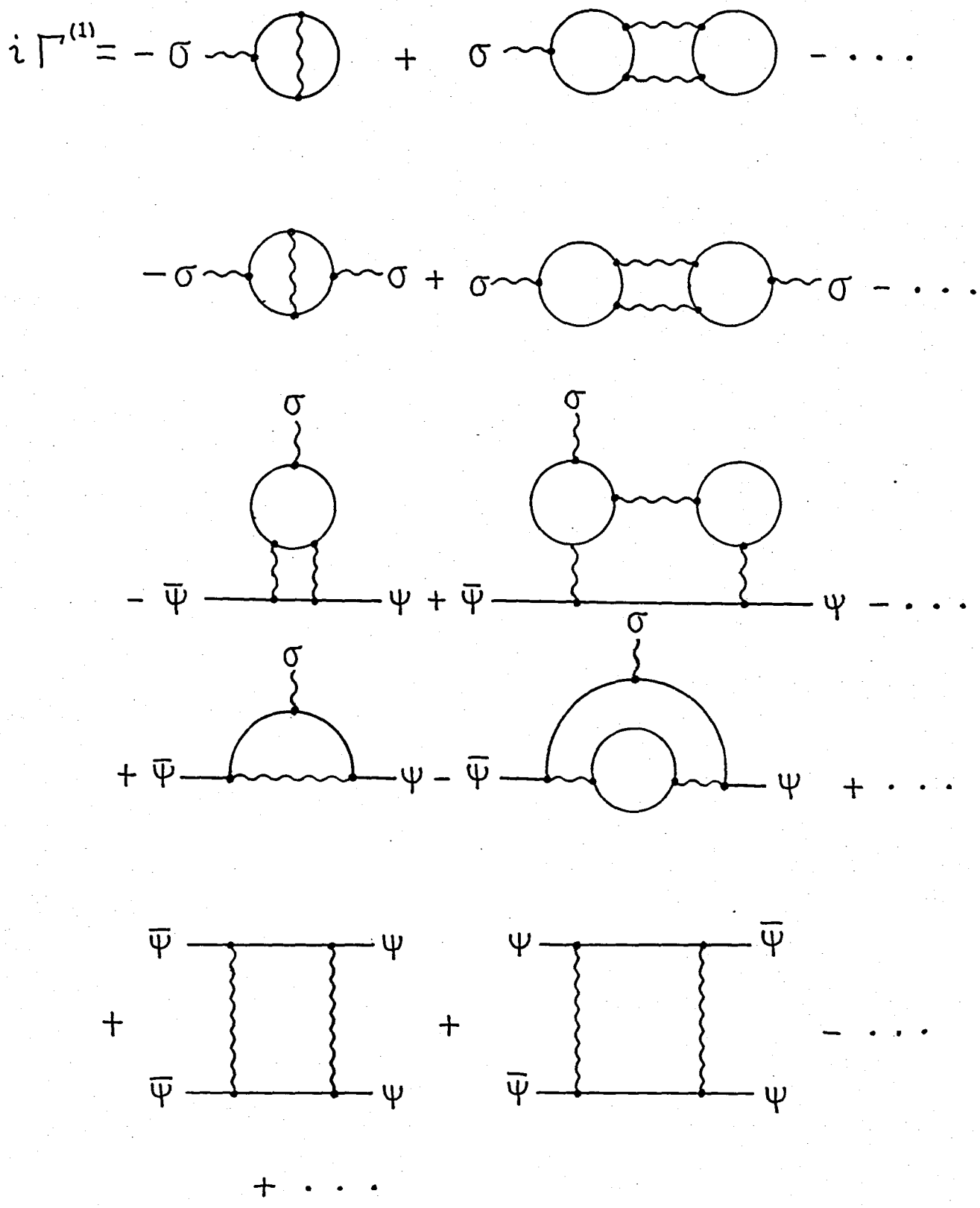


(a)



(b)

Fig. 1


$i\Gamma^{(1)} = -$
σ
+
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- . . .
$-$
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σ
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σ
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ψ
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+ . . .

Fig. 2

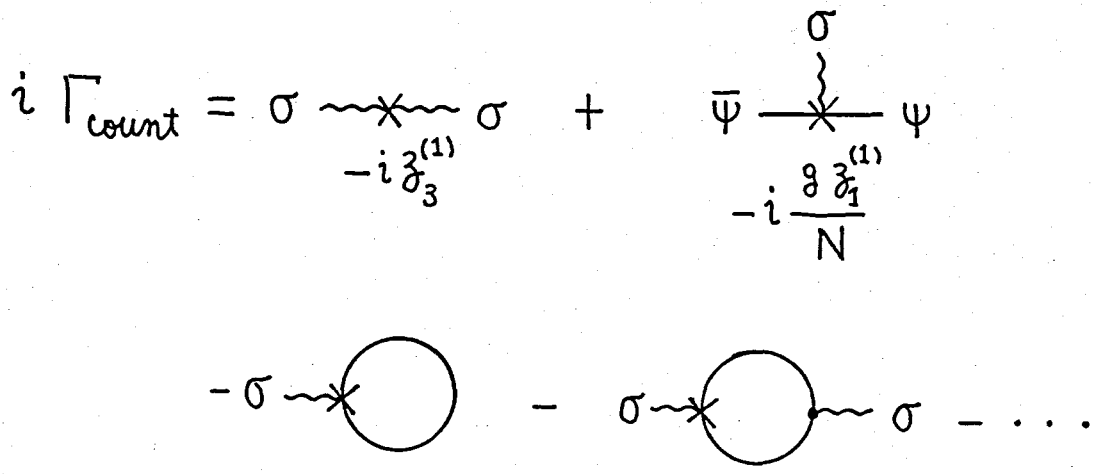
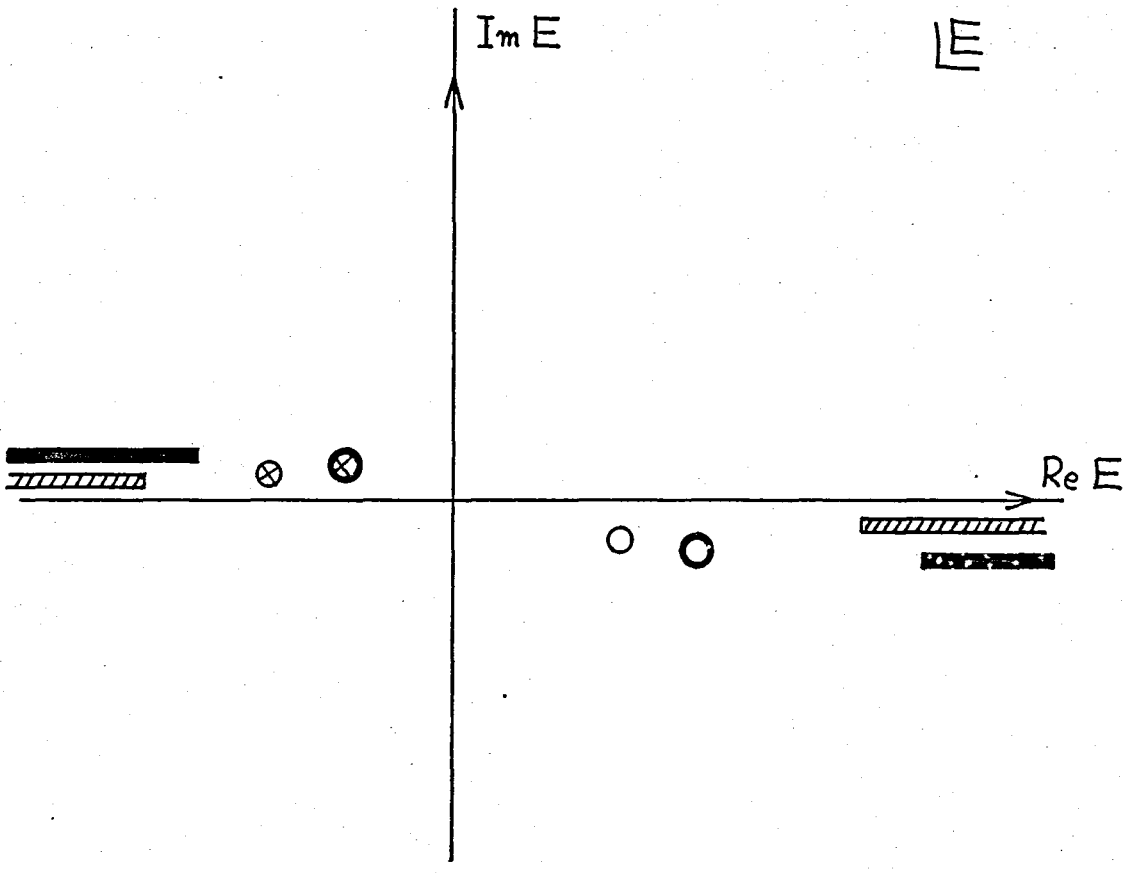

$i\Gamma_{count} = \sigma$
σ
$-i z_3^{(1)}$
$+$
$\bar{\psi}$
ψ
σ
$-i \frac{g z_1^{(1)}}{N}$
$-\sigma$
$-$
σ
σ
$- \ldots$

Fig. 3


Im E
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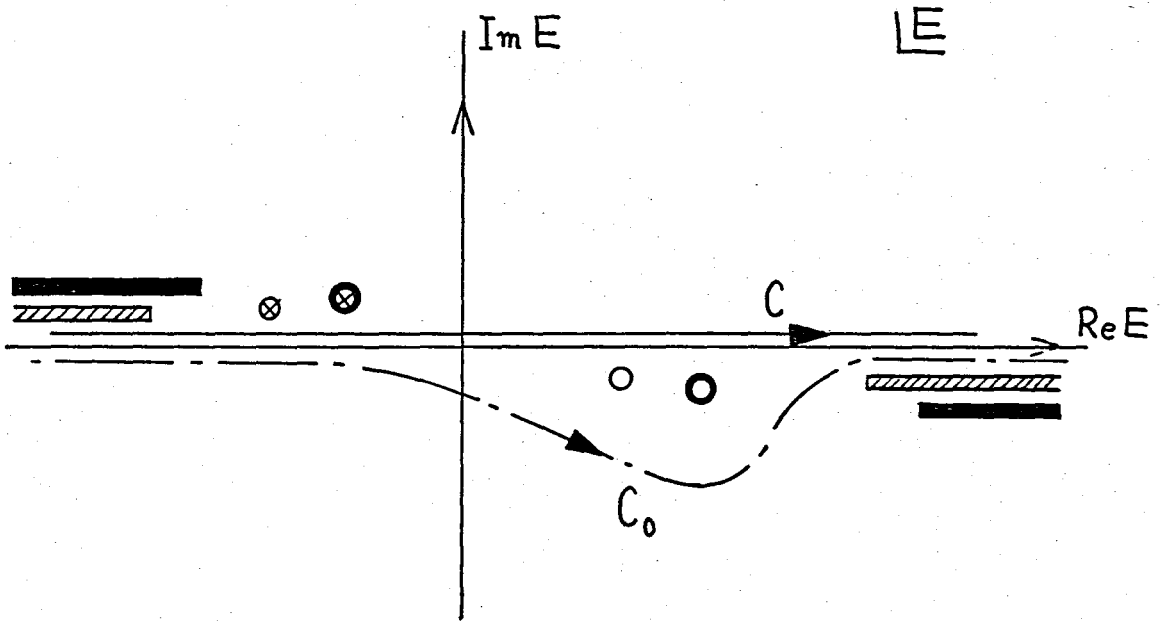
(a)

Fig. 4



(b)

Fig. 4

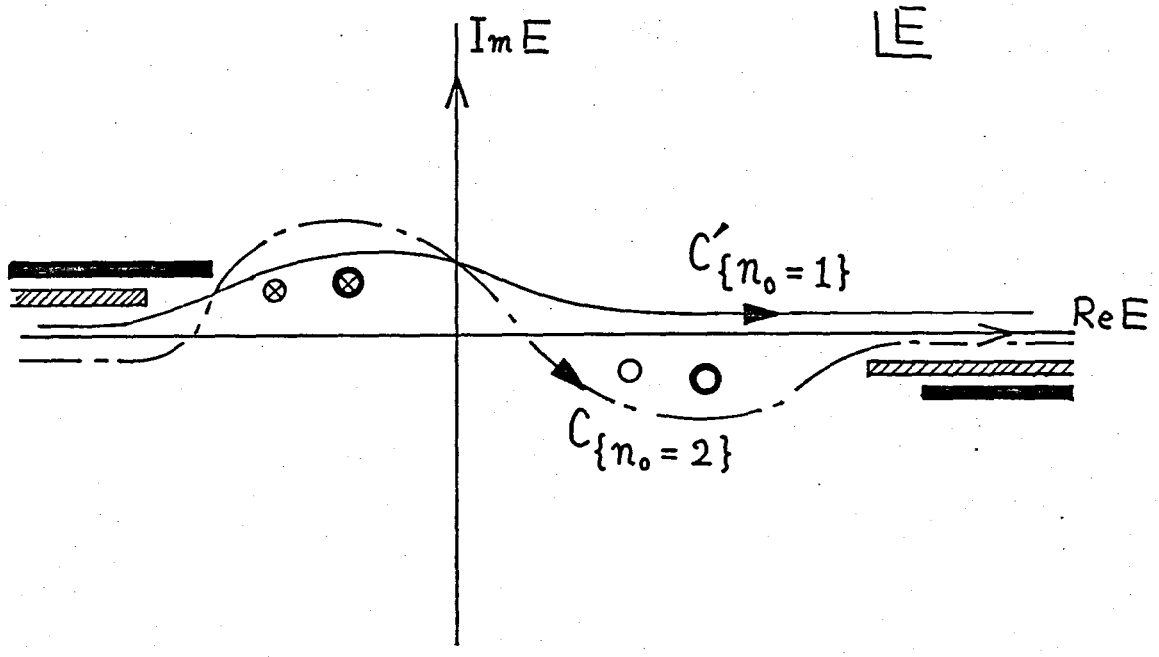
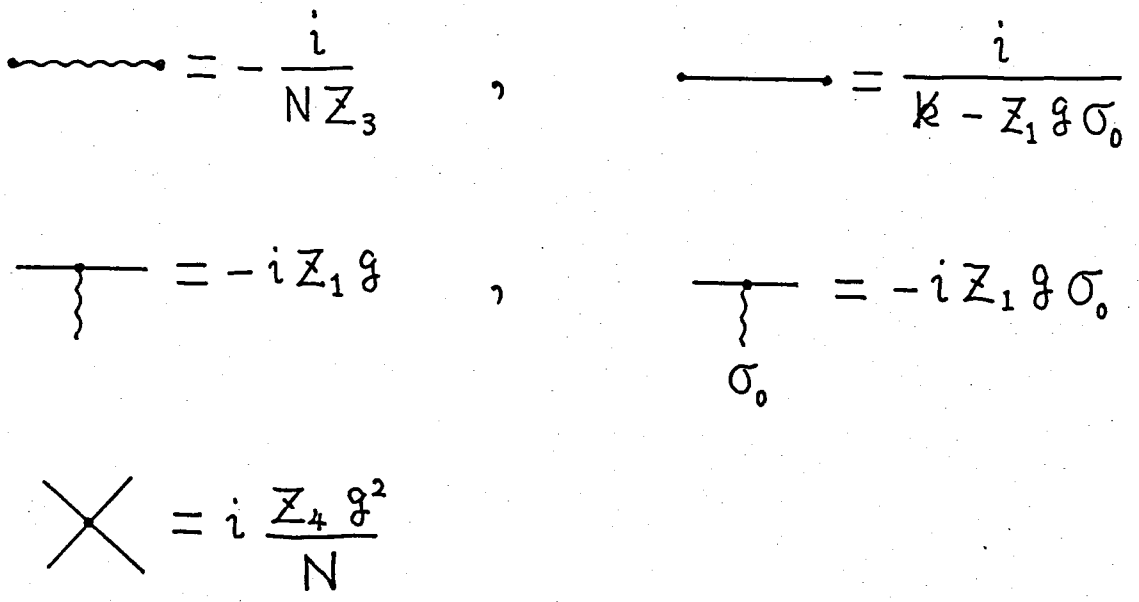
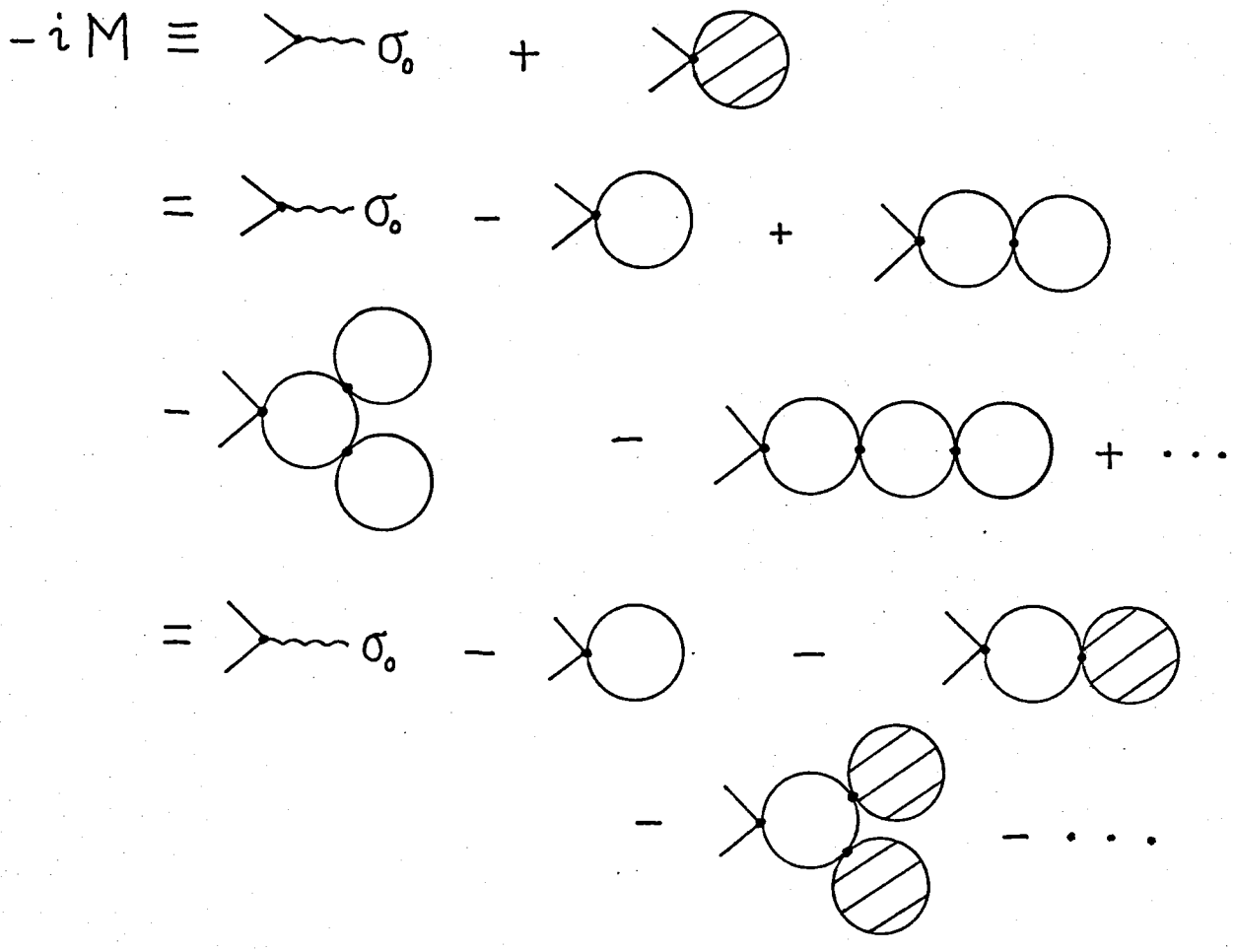


Fig. 5


$= -\frac{i}{N Z_3}$,
$= \frac{i}{\not{k} - Z_1 g \sigma_0}$
$= -i Z_1 g$,
$= -i Z_1 g \sigma_0$
σ_0
$= i \frac{Z_4 g^2}{N}$

(a)


$-iM \equiv$
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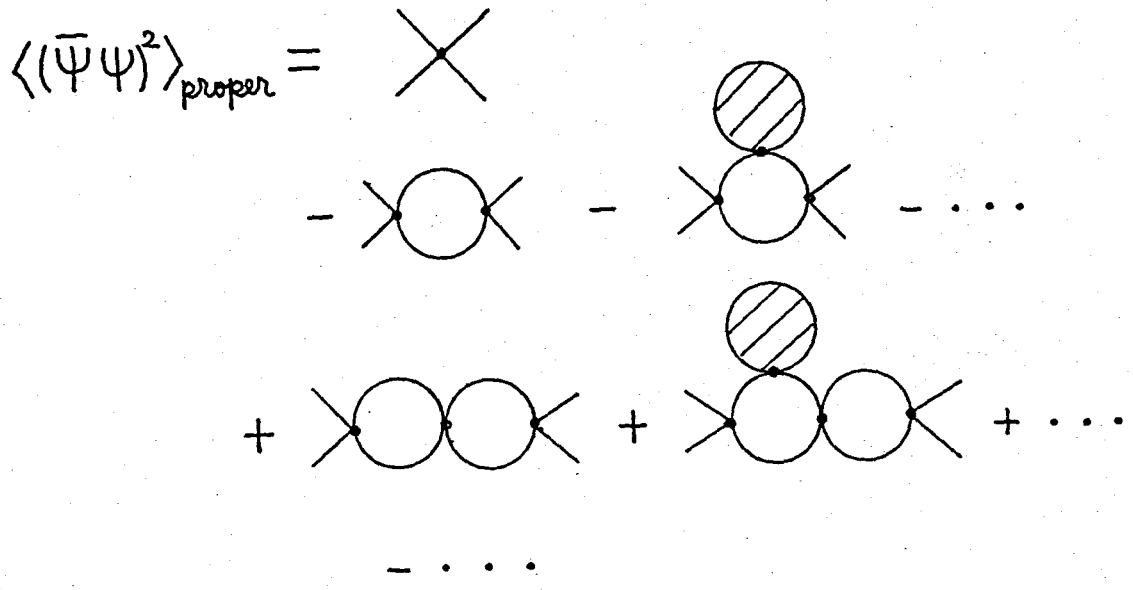
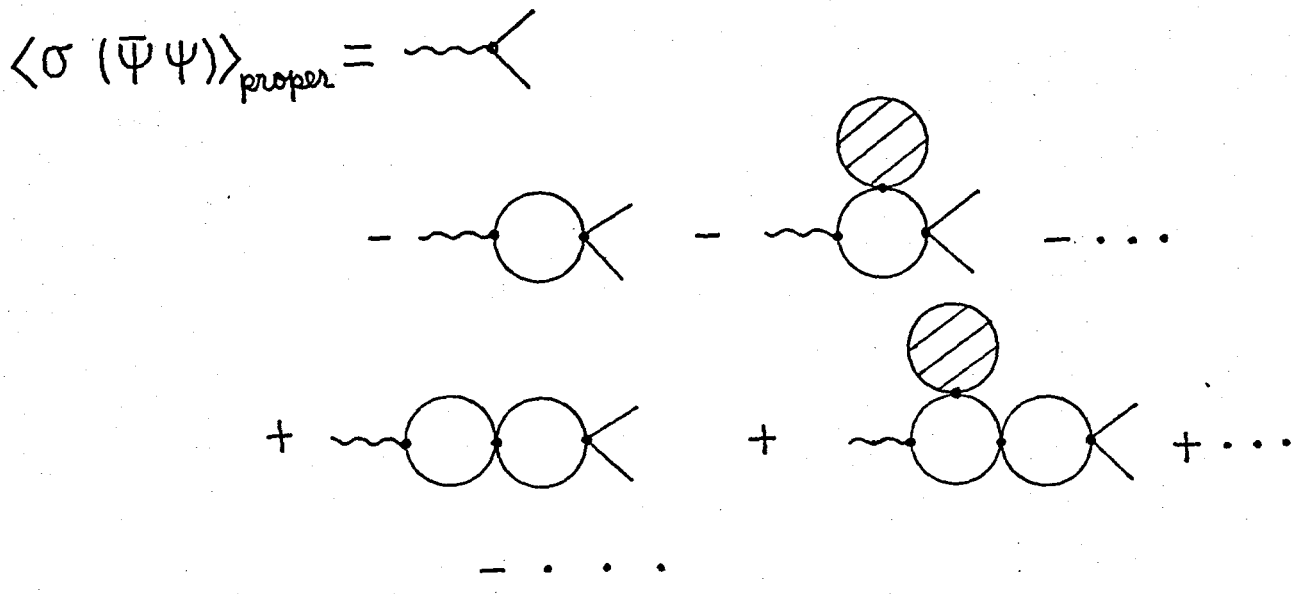
+
=
σ_0
−
+
−
−
+ ···
=
σ_0
−
−
−
− ···

(b)

Fig. 6


$\langle(\bar{\Psi}\Psi)^2\rangle_{proper} =$

(a)


$\langle\sigma\,(\bar{\Psi}\Psi)\rangle_{proper} =$

(b)

Fig. 7