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FUZZY AUTOMATA AND FUZZY GRAMMARS

by

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ABSTRACT

This thesis treats several problems associated with the theory of fuzzy automata and fuzzy grammars which was studied while the author was in the doctor course of Department of Electrical Engineering, Faculty of Engineering Sciences, Osaka University.

A fuzzy automaton was formulated by Wee and Fu by using the concept of fuzzy sets and fuzzy systems by Zadeh. It is shown that λ-fuzzy language by a fuzzy automaton is a regular language. The family of fuzzy languages defined by fuzzy automata forms a distributive lattice and the complement of the fuzzy language can be characterized by an optimistic fuzzy automata.

A new form of fuzzy grammar, which is called an n-fold fuzzy grammar, is defined and some of its properties are investigated. The n-fold fuzzy grammars are a generalization of fuzzy grammars defined by Lee and Zadeh, where the grade of application of the rule to be used next is conditioned by the n($\geqq$ 1) rules used before in a derivation. The n-fold fuzzy grammars with CF rules can be shown to generate CS languages, although fuzzy grammars with CF rules can not generate CS languages.

Finally, a general formulation of formal grammars is developed. A pseudo grammar is defined and from it various kinds of grammars which have or have not appeared in existing literatures are derived.

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CHAPTER 1

INTRODUCTION

Fuzzy set is a concept originated by Zadeh [1] in 1965. The ideas of fuzzysets are to cover the classification of objects encountered in the real life. In reality there are adjectives like good, appropriate, beautiful, rich etc. wherein, clearly, the classes of the type cited above do not constitute classes or sets in the usual mathematical sense as these terms, since they do not dichotomize all objects into those that belong to the class and those that do not. Fuzzy sets concept has certain properties and implication of use in dealing with such a class in a quantitative manner.

To characterize such a fuzzily defined class, Zadeh introduced the concept of a membership function which assigns to each object of the class a grade of membership ranging fullmembership (grade = 1) to nonmembership (grade = 0), and established many interesting mathematical structure of the fuzzy sets theorem.

Although research in this area is still somewhat tentative, it looks very promising. Papers have appeared on aspects of fuzzy sets (Zadeh [1,4,8] , Goguen [5] and Brown [6]), fuzzy automata (Zadeh [2,7,11] , Wee et al. [18,19] , Santos [20 ~ 24] , Mizumoto et al. [25,30] , Fu et al. [27] , Kitajima et al. [28,31] and Otsuki [29]), fuzzy languages (Zadeh et al. [16,38] and Mizumoto et al. [39 ~ 43, 48]), pattern recognition (Zadeh [1,4,14] , Bellman et al. [3] , Wee et al. [18,19] , Gitman et al. [12] and Tamura et al. [13]), fuzzy algorithms (Zadeh [7]), fuzzy topological space (Chang [9]), fuzzy logics (Marinos [10]), and decision-makings (Bellman et al. [15]).

One of the problems in computer science is the gap between natural languages for human beings and programming languages for digital computers, since the former are much more complex than the latter. This is due to the fact that natural languages are fuzzy in nature but programming languages are precise.

To reduce the gap between them, it is natural to introduce randomness into the structure of formal languages or automata, thus leading to the concept of probabilistic languages [49 ~ 52] or probabilistic automata [32 ~ 37] . Another possibility lies in the introduction of fuzziness. The first step in this direction was made by Wee and Fu [18,19] , who formulated fuzzy automata based on the concept of fuzzy sets and fuzzy systems by Zadeh [1,2] as a model of learning systems such as pattern recognition and automatic controls. And then Lee and Zadeh [38] defined fuzzy languages and fuzzy grammars as an extension of formal languages.

This thesis treats several problems concerning with fuzzy automata and fuzzy grammars.

In Chapter 2 we briefly review the concept of fuzzy sets originated by Zadeh [1] and fuzzy languages for the preparations of fuzzy automata and fuzzy grammars which will be discussed in the later chapters.

Chapter 3 treats the problems of a fuzzy automaton, which was formulated by Wee and Fu [18, 19] as a model of learning systems such as pattern recognition and automatic control systems. It is shown that the capability of fuzzy automata as acceptor is the same as that of finite automata, although fuzzy automata include the deterministic and non-deterministic finite automata as special cases, whose result was proved by Santos [21] independently. Moreover, the threshold of fuzzy automata can be changed arbitrarily. The fuzzy languages characterized by fuzzy automata constitute a distributive lattice, and the complement of the fuzzy language can be characterized by an optimistic fuzzy automaton.

In Chapter 4 fuzzy grammars by Lee and Zadeh [38] , and conditional fuzzy grammars (or $n(\geq 1)$-fold fuzzy grammars) are discussed. In n-fold fuzzy grammars the grade of application of the rule to be used next is conditioned by the n rules used before in a derivation. The n-fold fuzzy grammars whose rules are of context-free form can be shown to generate context-sensitive languages by setting a threshold appropriately. Fuzzy grammars with context-free rules, however, can not generate context-sensitive languages. As to n-fold fuzzy grammars, we focused our attention on n-fold fuzzy grammars with type 3 rules as a preliminary step.

In Chapter 5 we develop a general formulation of formal grammars by extracting the basic properties common to the formal grammars appeared in existing literatures. A pseudo grammar is defined and from it the well-known probabilistic grammars and fuzzy grammars are derived. Moreover, several interesting grammars such as $\cup *$ grammars, $\cup\cap$ grammars, $\cap\cup$ grammars, composite B-fuzzy grammars, and mixed fuzzy grammars, which have never appeared in any other paper before, are derived.

The pseudo grammar called a pseudo conditional grammar, whose weight of the application of a rule is conditioned by the rule used just before in a derivation, is also defined and from it several interesting conditional grammars are derived in the same manners as pseudo grammars.

CHAPTER 2

FUZZY SETS AND FUZZY LANGUAGES

2.1 Introduction

We shall briefly review some of the basic definitions relating to fuzzy sets originated by L. A. Zadeh [1] and define fuzzy languages by using the concept of fuzzy sets, which will be needed in later discussions.

2.2 Fuzzy Sets

Informally, a fuzzy set is a "class" with fuzzy boundaries, that is, a "class" of objects in which there is no sharp boundary between those objects that belong to the class and those that do not, e.g., the "class" of real numbers which are much larger than, say, 10. A more precise definition of fuzzy sets may be stated as follows.

Definition 2.1. Let $X = \{x\}$ be a collection of objects (points). Then a fuzzy set A in X is characterized by a membership (or fuzzy characteristic) function μ_A which is defined on X and takes values in the interval $[0, 1]$ i.e.,

$$\mu_A : X \to [0, 1] \tag{2.1}$$

The value of μ_A at x, $\mu_A(x)$, represents the grade of membership of x in A†. If A is a set in the usual sense, $\mu_A(x)$ is 1 or 0 according as x does or does not belong to A.

A fuzzy set A is represented as a set of ordered pairs:

$$A = \left\{ (x, \mu_A(x)) \right\}, \qquad x \in X \qquad (2.2)$$

Example 2.1. Let X be the set of integers from 0 to 100 representing the ages of individuals in a group, and let A be a fuzzy set of a "middle-aged" individuals. Then, such a set may be characterized, subjectively of course, by the membership function such as:

x(=age)	40	41	42	43	44	45	46	47	48	49	50	51	52	53
$\mu_A(x)$	0.3	0.5	0.8	0.9	1	1	1	1	1	0.9	0.8	0.7	0.5	0.3

where only those pairs $(x, \mu_A(x))$ in which $\mu_A(x)$ is positive are tabulated.

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† In a more general case, the range (or membership space) of the membership function can be taken to be a partially ordered set or, more particularly, a lattice [5], which will be discussed in Chapter 5.

We turn next to the several preliminary definitions of fuzzy sets which we shall need in later chapters.

Equality: Two fuzzy sets A and B are equal, written as $A = B$, if and only if $\mu_A(x) = \mu_B(x)$ for all x in X.

$$A = B \iff \mu_A(x) = \mu_B(x), \qquad x \in X \qquad (2.3)$$

Containment: A fuzzy set A is contained in, or is a subset of a fuzzy set B, written as $A \subseteq B$, if and only if $\mu_A(x) \leqq \mu_B(x)$ for all x in X.

$$A \subseteq B \iff \mu_A(x) \leq \mu_B(x), \qquad x \in X \qquad (2.4)$$

Complementation: The complement of a fuzzy set A is denoted by $\bar{A}$ and is defined by $\mu_{\bar{A}}(x) = 1 - \mu_A(x)$ for all x in X.

$$\bar{A} \iff \mu_{\bar{A}}(x) = 1 - \mu_A(x), \qquad x \in X \qquad (2.5)$$

Union: The union of two fuzzy sets A and B, written as $A \cup B$, is a fuzzy set $C = A \cup B$ characterized by $\mu_C(x) = \max\left[\mu_A(x),\ \mu_B(x)\right]$ for all x in X.

$$C = A \cup B \iff \mu_C(x) = \max\left[\mu_A(x),\ \mu_B(x)\right], \quad x \in X \qquad (2.6)$$

Note that $A \cup B$ is the smallest fuzzy set containing both A and B.

<u>Intersection</u>: The <u>intersection</u> of two fuzzy sets A and B is a fuzzy set C, written as $C = A \cap B$, defined by $\mu_C(x) = \min[\mu_A(x), \mu_B(x)]$, for all x in X.

$$C = A \cap B \Leftrightarrow \mu_C(x) = \min[\mu_A(x), \mu_B(x)], \; x \in X \tag{2.7}$$

By a dual argument to the above, we see that $A \cap B$ is the largest fuzzy set which is contained in both A and B.

<u>Empty Fuzzy Set</u>: A fuzzy set A is <u>empty</u> if and only if it is identically zero on X. The empty fuzzy set will be denoted by ϕ.

$$\phi \Leftrightarrow \mu_\phi(x) = 0, \quad x \in X \tag{2.8}$$

<u>Universal Fuzzy Set</u>: A fuzzy set A is <u>universal</u> if and only if it is identically unit on X. The universal fuzzy set is a space X.

$$X \Leftrightarrow \mu_X(x) = 1, \quad x \in X \tag{2.9}$$

The operations $\subseteq$, $\cup$, $\cap$, and $^{-}$ on fuzzy sets have a number of basic algebraic properties. Some of these are as follows:

(1) $A \subseteq A$ (reflexive law)

(2) $A \subseteq B, \quad B \subseteq A \Rightarrow A = B$ (anti-symmetric law)

(3) $A \subseteq B, \quad B \subseteq C \Rightarrow A \subseteq C$ (transitive law)

(4) $A \cup A = A, \qquad A \cap A = A$ (idempotent law)

(5) $A \cup B = B \cup A, \quad A \cap B = B \cap A$ (commutative law)

(6) $$\left.\begin{aligned}(A \cup B) \cup C &= A \cup (B \cup C)\\ (A \cap B) \cap C &= A \cap (B \cap C)\end{aligned}\right\}$$ (associative law)

(7) $$\left.\begin{aligned}A \cup (A \cap B) &= A\\ A \cap (A \cup B) &= A\end{aligned}\right\}$$ (absorption law)

(8) $$\left.\begin{aligned}A \cup (B \cap C) &= (A \cup B) \cap (A \cup C)\\ A \cap (B \cup C) &= (A \cap B) \cup (A \cap C)\end{aligned}\right\}$$ (distributive law)

(9) $\bar{\bar{A}} = A$ (involution law)

(10) $$\left.\begin{aligned}\overline{(A \cup B)} &= \bar{A} \cap \bar{B}\\ \overline{(A \cap B)} &= \bar{A} \cup \bar{B}\end{aligned}\right\}$$ (De Morgan's law)

(11) $A \cup X = X, \quad A \cap X = A$ (*identity law*).

$A \cup \Phi = A, \quad A \cap \Phi = \Phi$

(12) Generally,

$A \cup \bar{A} \neq X$ (*failure of complement law*).

$A \cap \bar{A} \neq \Phi$

From the properties concerning with fuzzy sets, we see that fuzzy sets form a distributive lattice, but do not form a Boolean lattice, because $\bar{A}$ is not the complement of A in the lattice sense.

In addition to the operations of union and intersection, we can define other operations to form combinations of fuzzy sets and to relate them to one another. Among these are the following.

Algebraic Product: The *algebraic product* of A and B is denoted by AB and is defined by $\mu_{AB}(x) = \mu_A(x)\mu_B(x)$ for all x in X.

$$AB \iff \mu_{AB}(x) = \mu_A(x)\mu_B(x), \quad x \in X \tag{2.10}$$

Algebraic Sum: The algebraic sum of A and B is denoted by $A \oplus B$ and is defined by

$$A \oplus B \Leftrightarrow \mu_{A \oplus B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), \quad x \in X \tag{2.11}$$

It is easy to verify that

$$A \oplus B = \overline{(\bar{A}\,\bar{B})} \tag{2.12}$$

Absolute Difference: The absolute difference of A and B is denoted by $|A - B|$ and is defined by

$$|A - B| \Leftrightarrow \mu_{|A-B|}(x) = |\mu_A(x) - \mu_B(x)|, \quad x \in X \tag{2.13}$$

Convex Combination: Let A, B, and Λ be arbitrary fuzzy sets. The convex combination of A, B, and Λ is denoted by $(A, B; \Lambda)$ and is defined by the relation:

$$(A, B; \Lambda) = \Lambda A + \bar{\Lambda} B \tag{2.14}$$

where $\bar{\Lambda}$ is the complement of Λ. Written out in terms of membership functions, (2.14) reads

$$\mu_{(A,B;\Lambda)}(x) = \mu_\Lambda(x)\mu_A(x) + [1 - \mu_\Lambda(x)]\mu_B(x), \quad x \in X \qquad (2.15)$$

A basic property of the convex combination of A, B, and Λ is expressed by

$$A \cap B \subseteq (A, B; \Lambda) \subseteq A \cup B, \quad \text{for all } \Lambda \qquad (2.16)$$

Relation: A fuzzy relation, R, in the product space $X \times Y = \{(x, y)\}$, $x \in X$, $y \in Y$, is a fuzzy set in $X \times Y$ characterized by a membership function μ_R which associates with each ordered pair (x, y) a grade of membership $\mu_R(x, y)$ in $[0, 1]$ More generally, an n-ary fuzzy relation in a product space $X = X_1 \times X_2 \times \cdots \times X_n$ is a fuzzy set in X characterized by an n-variate membership function $\mu_R(x_1, x_2, \ldots, x_n)$, $x_i \in X_i$, $i = 1, 2, \ldots, n$.

Composition of Relations: If R_1 and R_2 are two fuzzy relations in X^2, then by the composition (or product) of R_1 and R_2 is meant a fuzzy relation in X^2 which is denoted by R_1R_2 and is defined by

$$\mu_{R_1R_2}(x, z) = \sup_y \min[\mu_{R_1}(x, y), \mu_{R_2}(y, z)] \qquad (2.17)$$

where the supremum is taken over all y in X.

Comment: We can give other two different definitions concerning with the composition of fuzzy relations by the following:

$$\mu_{R_1R_2}(x, z) = \inf_y \max \left[\mu_{R_1}(x, y), \mu_{R_2}(y, z)\right] \qquad (2.18)$$

$$\mu_{R_1R_2}(x, z) = \sup_y \left[\mu_{R_1}(x, y)\cdot\mu_{R_2}(y, z)\right] \qquad (2.19)$$

where the operation "." is the product in the ordinary sense.

We may call (2.17) as max-min composition, (2.18) as min-max composition, and (2.19) as max-product composition.

In what follows, in order to avoid a confusing multiplicity of the composition, we shall be using (2.17) for the most part as our definition of the composition.

Example 2.2. Let X be the real line R^1. Then $x \gg y$ is a fuzzy relation in R^2. A subjective expression for μ_R in this case might be:

$$\mu_R(x, y) = \begin{cases} 0 & \cdots\cdots\cdots\cdots \quad x \leq y \\ \dfrac{1}{1 + \dfrac{100}{(x - y)^2}} & \cdots\cdots \quad x > y \end{cases}$$

And the composition (2.17), RR, of the above fuzzy relations R is a fuzzy relation such as $x \ggg y$ and is characterized by the following membership function:

$$\mu_{RR}(x, y) = \begin{cases} 0 & \cdots\cdots\cdots \quad x \leq y \\ \dfrac{1}{1 + \dfrac{100}{\left(\frac{x-y}{2}\right)^2}} & \cdots\cdots \quad x > y \end{cases}$$

<u>Conditioned Fuzzy Sets</u>: A fuzzy set $B(x)$ in $Y = \{ y \}$ is <u>conditioned</u> on x if its membership function depends on x as a parameter. This dependence is expressed by $\mu_B(y/x)$.

Suppose that the parameter x ranges over a space X. Then, the function $\mu_B(y/x)$ defines a mapping from X to the space of fuzzy sets defined on Y. Through this mapping, a fuzzy set A in X induces a fuzzy set B in Y which is defined by

$$\mu_B(y) = \sup_x \min \left[\mu_A(x), \mu_B(y/x) \right] \qquad (2.20)$$

where μ_A and μ_B denote the membership functions of A and B, respectively. In effect, (2.20) is a special case of the composition of fuzzy relations (2.17).

2.3 Fuzzy Languages

A fuzzy language is defined to be a fuzzy set of the set of strings over a finite alphabet. The notions of union, intersection, concatenation, and Kleene closure for such languages are defined as extensions of the corresponding notions in the theory of formal languages [54, 55].

Fuzzy Languages: A fuzzy language L is a fuzzy set in Σ^*. L can be written as the set of ordered pairs

$$L = \{(x, \mu_L(x))\}, \quad x \in \Sigma^* \tag{2.21}$$

where $\mu_L(x)$ is the grade of membership of x in L. We assume that $\mu_L(x)$ is a number in the interval $[0, 1]$†.

A trivial example of a fuzzy language is the set

$$L = \{(a, 1.0), (b, 1.0), (aa, 0.8), (ab, 0.7), (ba, 0.6), (bb, 0.5)\} \text{ in } \{a, b\}^*.$$

† More generally, we can define L-fuzzy languages as an extension of fuzzy languages, which will be denoted in Chapter 5.

It is understood that all strings in $\{a, b\}^*$ other than those listed have the grade of membership 0 in L.

The operations of fuzzy languages can be defined as an extension of those of ordinary languages.†

Let L_1 and L_2 be two fuzzy languages in Σ^*.

<u>Union</u>: The <u>union</u> of L_1 and L_2 is a fuzzy language denoted by $L_1 \cup L_2$ and defined by

$$\mu_{L_1 \cup L_2}(x) = \max\left[\mu_{L_1}(x),\ \mu_{L_2}(x)\right], \quad x \in \Sigma^* \qquad (2.22)$$

<u>Intersection</u>: The <u>intersection</u> of L_1 and L_2 is a fuzzy language denoted by $L_1 \cap L_2$ and defined by

$$\mu_{L_1 \cap L_2}(x) = \min\left[\mu_{L_1}(x),\ \mu_{L_2}(x)\right], \quad x \in \Sigma^* \qquad (2.23)$$

---

† In the ordinary formal languages, a language L over Σ is a subset of Σ^* and the operations of languages are defined as follows:

<u>Union</u>: $L_1 \cup L_2 = \{x \mid x \in L_1 \text{ or } x \in L_2\}$

<u>Intersection</u>: $L_1 \cap L_2 = \{x \mid x \in L_1 \text{ and } x \in L_2\}$

<u>Complement</u>: $\bar{L} = \Sigma^* - L = \{x \mid x \in \Sigma^*, x \notin L\}$

<u>Concatenation</u>: $L_1 L_2 = \{uv \mid u \in L_1,\ v \in L_2\}$

<u>Kleene Closure</u>: $L = L^0 \cup L \cup LL \cup LLL \cup \ldots\ldots$

Complement: The complement of a fuzzy language L is a fuzzy language denoted by $\bar{L}$ and defined by

$$\mu_{\bar{L}}(x) = 1 - \mu_L(x), \qquad x \in \Sigma^* \qquad (2.24)$$

Concatenation: The concatenation of L_1 and L_2 is a fuzzy language denoted by $L_1 \circ L_2$, or dually $L_1 \cdot L_2$ and defined as follows: Let a string x in Σ^* be expressed as a concatenation of a prefix string u and a suffix string v, that is, $x = uv$. Then

$$\mu_{L_1 \circ L_2}(x) = \sup_u \min \left[\mu_{L_1}(u), \ \mu_{L_2}(v) \right] \qquad (2.25)$$

dually

$$\mu_{L_1 \cdot L_2}(x) = \inf_u \max \left[\mu_{L_1}(u), \ \mu_{L_2}(v) \right] \qquad (2.26)$$

where the supremum of (2.25) and the infimum of (2.26) are taken over all prefixes u of x.

Kleene Closure: By using the concatenation $L_1 \circ L_2$ or $L_1 \cdot L_2$, Kleene closure of a fuzzy language L (written as L^* , or $\hat{L}$) is defined as

$$L^* = L^0 \cup L \cup L \circ L \cup L \circ L \circ L \cup \ldots. \qquad (2.27)$$

$$\hat{L} = L^0 \cap L \cap L \cdot L \cap L \cdot L \cdot L \cap \ldots. \qquad (2.28)$$

2.1 Conclusions

It should be noted that, although the concept of fuzzy sets (fuzziness) has some resemblance to that of probability (randomness) which treats an inexact concept, there are essential differences between these concepts. "Randomness" has to do with uncertainty concerning membership or nonmembership of objects in a non-fuzzy (or crisp) set. "Fuzziness", on the other hand, has to do with classes in which there may be grades of membership intermediate between fullmembership and nonmembership. In fact, the fuzzy sets theory is a calculus of vagueness, ambiguity and ambivalance rather than likelifood, and, therefore, the notion of fuzzy sets is completely non-statistical in nature.

Although the fuzzy sets theory is still in it infancy, it will be able to do at least what the probability theory has done and, moreover, will come to play an important role in a wide variety of problems relating to "soft" sciences such as social sciences, management sciences, economics, linguistics, etc. and to "hard" sciences which are too complex or too ill-defined to admit of precise analysis, say, large-scale systems, large-scale traffic control systems, pattern recognition, machine translations, artificial intelligence, information retrieval, etc.

Despite these arguments and promises, one must not expect too much of fuzzy sets. Ordinary set theory has been of greatest importance in providing a convenient language for mathematical thought. They have not made the exercise of creative intelligence unnecessary either in mathematics or its applications. Similarly we should not expect more of fuzzy sets than they facilitate the development and study of models in the inexact sciences, and that they be an interesting area for pure mathematical investigation.

CHAPTER 3

FUZZY AUTOMATA

3.1 Introduction

Among various types of automata, as is well-known, are deterministic, nondeterministic and probabilistic automata. Recently, W. G. Wee [18, 19] proposed another type of automata which he named fuzzy automata. The formulation of fuzzy automata is based on the concept of fuzzy sets and fuzzy systems defined by L. A. Zadeh [1, 2]. Fuzzy automata include deterministic and nondeterministic finite automata as special cases and also have some properties similar to those of probabilistic automata [32, 33, 34, 35, 36, 37]. In addition, fuzzy automata may be available, as its applications, to simulating learning systems such as pattern recognition and automatic control systems [15, 18, 19, 27, 28, 29, 30, 31].

In this chapter, it is shown that, although fuzzy automata include deterministic and nondeterministic finite automata, the capability of fuzzy automaton as an acceptor is equal to that of finite automaton, which was proved by E. S. Santos independently [21]. And the threshold of fuzzy automata can be changed arbitrarily by changing the values of each element of the fuzzy transition matrix and the initial state designator. Moreover, the family of the fuzzy sets (that is, fuzzy events) of input strings characterized by (pessimistic) fuzzy automata is closed under the operations of "union" and "intersection" in the sense of fuzzy sets, and the complement of the fuzzy event is characterized by an optimistic fuzzy automaton.

We show that the similar properties to those mentioned above also hold for optimistic fuzzy automata.

3.2 Fuzzy Automata

A fuzzy automaton was proposed by W. G. Wee [18, 19] as a model of pattern recognition and automatic control systems. An advantage of employing a fuzzy automaton as a learning model is its simplicity in design and computation. A learning fuzzy automaton is clearly nonstationary. In this chapter, however, we assume a fuzzy automaton to be stationary and extend the definition by Wee as follows: In Wee's paper, the initial state of a fuzzy automaton is given in deterministic way. But we will introduce the fuzzy distribution, that is, the initial distribution.

Let Σ be a finite non-empty alphabet. The set of all finite strings over Σ is denoted by Σ^*. The null string is denoted by ε and included in Σ^*. $\#(S)$ is the number of elements in the set S.

Definition 3.1. A finite fuzzy automaton over the alphabet Σ is a system

$$A = (\ S,\ \pi,\ \{F(\sigma) \mid \sigma \in \Sigma\},\ \delta^G\ ) \qquad (3.1)$$

where

(i) $S = \{s_1, s_2, \cdots, s_n\}$ is a non-empty finite set of internal states.

(ii) π is an n-dimensional fuzzy row vector, that is,

$\pi = (\pi_{s_1}, \pi_{s_2}, \cdots, \pi_{s_n})$, where $0 \leqq \pi_{s_i} \leqq 1$, $1 \leqq i \leqq n$, and is called the <u>initial state designator</u>.

(iii) G is a subset of S (the set of final states).

(iv) $\delta^G = (\delta_{s_1}, \delta_{s_2}, \cdots, \delta_{s_n})'$ is an n-dimensional column vector whose i-th component equals 1 if $s_i \in G$ and 0 otherwise, and is called the <u>final state designator</u>.

(v) For each $\sigma \in \Sigma$, $F(\sigma)$ is a fuzzy matrix of order n (the <u>fuzzy transition matrix</u> of A) such that

$$F(\sigma) = \| f_{s_i, s_j}(\sigma) \| \quad 1 \leqq i, j \leqq n\ . \qquad (3.2)$$

Let element $f_{s_i, s_j}(\sigma)$ of $F(\sigma)$ be $f_A(s_i, \sigma, s_j)$, where $s_i, s_j \in S$ and $\sigma \in \Sigma$. The function f_A is a membership function of a fuzzy set in $S \times \Sigma \times S$; i.e., $f_A : S \times \Sigma \times S \to [0,\ 1]$. f_A may be called the <u>fuzzy transition function</u>. That is to say, for $s, t \in S$ and $\sigma \in \Sigma$,

$f_A(s,\sigma,t)$ = the grade of transition from state s to state t when the input is σ.

The unity fuzzy transition function implies such a transition may exist definitely.

Remark. If f_A takes only two values 0 and 1, then a fuzzy automaton A is a nondeterministic finite automaton. In addition, only any one element of each row of matrix $F(\sigma)$, $\sigma \in \Sigma$, is "1" and the rest elements of each row are all equal to "0". Then a fuzzy automaton A is a deterministic finite automaton.

The grade of transition for an input string of length m is defined by an m-ary fuzzy relation. The fuzzy transition function is as follows: For input string $x = \sigma_1\sigma_2\cdots\sigma_m \in \Sigma^*$ and $s,\ t \in S$,

$$f_A(s,x,t) \tag{3.3}$$

$$= \max_{q_1,q_2,\cdots,q_{m-1} \in S} \min \left[f_A(s,\sigma_1,q_1), f_A(q_1,\sigma_2,q_2), \cdots, f_A(q_{m-1},\sigma_m,t)\right]$$

= the grade of transition from state s to state t when the input string is $x = \sigma_1\sigma_2\cdots\sigma_m$.

<u>Definition 3.2.</u> For ε, x, $y \in \Sigma^*$ and s, $t \in S$,

$$f_A(s,\varepsilon,t) = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{if } s \neq t, \end{cases} \tag{3.4}$$

$$f_A(s,xy,t) = \max_{q \in S} \min \left[f_A(s,x,q), f_A(q,y,t) \right]. \tag{3.5}$$

Especially, we call a fuzzy automaton with the grade of transition under the operation "max min" a <u>pessimistic fuzzy automaton</u> (pfa), and a fuzzy automaton under the operation "min max" an <u>optimistic fuzzy automaton</u> (ofa) [20] .

<u>Definition 3.3.</u> An <u>optimistic fuzzy automaton</u> over the alphabet Σ' is a system

$$B' = (S', \pi', \{F'(\sigma) \mid \sigma \in \Sigma'\}, \delta^{G'}), \tag{3.6}$$

where S' is a finite non-empty set (the internal states of B'), $\#(S') = n'$. π' is an n'-dimensional row vector (the initial state designator). A fuzzy transition function $f_{B'}$ is defined as follows: For ε, x, $y \in \Sigma'^*$ and s, $t \in S'$,

$$f'_{B'}(s,\varepsilon,t) = \begin{cases} 0 & \text{if } s = t \\ 1 & \text{if } s \neq t, \end{cases} \tag{3.7}$$

$$f'_{B'}(s,xy,t) = \min_{q \in S'} \max \left[f'_{B'}(s,x,q),\ f'_{B'}(q,y,t) \right]. \tag{3.8}$$

G' is a subset of S' (the set of final states), and an n'-dimensional column vector (the final state designator) $\delta^{G'} = (\delta s'_1, \delta s'_2, \ldots, \delta s'_{n'})'$ is defined such that $\delta s'_i = 0$ if $s_i \in G'$ and $\delta s'_i = 1$ otherwise.

Note that an element of zero in π' means the definite existence of such an initial state. In this paper, unless stated especially, by "fuzzy automaton" we shall mean a pessimistic fuzzy automaton.

Let us show the fundamental properties of fuzzy matrices.

We denote by a_{ij} the (i,j) th entry of a fuzzy matrix A, where $0 \leqq a_{ij} \leqq 1$. We define:

$$A \leqq B \iff a_{ij} \leqq b_{ij}$$

$$O = \| 0 \|, \qquad E = \| 1 \|$$

$$C = A \circ B$$

$$c_{ij} = \max_k \min(a_{ik}, b_{kj})$$

$$I = \| m_{ij} \| \text{ where } m_{ij} = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i \neq j \end{cases}$$

$$A^{m+1} = A^m \circ A, \quad A^0 = I$$

$$C = A * B$$

$$c_{ij} = \min_k \max(a_{ik}, b_{kj})$$

$$I' = \| m'_{ij} \| \text{ where } m'_{ij} = \begin{cases} 0 \text{ if } i=j \\ 1 \text{ if } i \neq j \end{cases}$$

$$B^{m+1} = B^m * B, \quad B^0 = I'$$

The following fundamental properties are derived immediately from the above definitions:

Fundamental Properties

(1) $O \leqq A \leqq E$

(2) $A \circ (B \circ C) = (A \circ B) \circ C$

(3) $A \circ I = I \circ A = A$

(4) $A \circ O = O \circ A = O$

(5) $A^p \circ A^q = A^{p+q}, \quad (A^p)^q = A^{pq}$

(6) if $A \leqq B$ and $C \leqq D$, then $A \circ C \leqq B \circ D$.

(1)' $O \leqq B \leqq E$

(2)' $A * (B * C) = (A * B) * C$

(3)' $B * I' = I' * B = B$

(4)' $B * E = E * B = E$

(5)' $B^p * B^q = B^{p+q}, \quad (B^p)^q = B^{pq}$

(6)' if $A \leqq B$ and $C \leqq D$, then $A * C \leqq B * D$.

The definitions and the properties shown on the left side of the tables given above relate to the operation " $\circ$ ", and on the right side to the operation " $*$ ". Moreover, the operations " $\circ$ " and " $*$ " correspond to a pfa and an ofa, respectively.

The domain of the fuzzy transition matrix F of a fuzzy automaton A can be extended from Σ to Σ^* as follows:

Definition 3.4. For $x = \sigma_1\sigma_2\cdots\sigma_m \in \Sigma^*$, $\sigma_i \in \Sigma \cup \{\varepsilon\}$, and $1 \leqq i \leqq m$, define n x n fuzzy transition matrices $F(x)$ by the following,

$$\begin{aligned} &(1) \quad F(\varepsilon) = I \quad (\text{ n x n identity matrix}), \\ &(2) \quad F(x) = F(\sigma_1)\circ F(\sigma_2)\circ \cdots\cdots\circ F(\sigma_m). \end{aligned} \qquad (3.9)$$

Let $F(x) = \| f_{s_i,s_j}(x) \|$, where $1 \leqq i,j \leqq n$, then obviously

$$f_{s_i,s_j}(x) = f_A(s_i,x,s_j). \qquad (3.10)$$

Now, for $A = (\ S,\ \pi,\ \{F(\sigma) \mid \sigma \in \Sigma\},\ \delta^G)$, define

$$f_A(x) = \pi \circ F(x) \circ \delta^G\ , \quad \text{for } x \in \Sigma^* \qquad (3.11)$$

$f_A(x)$ is designated as the <u>grade of transition</u> of A, when started with initial distribution π over S to enter into a state in G after scanning the input string x. Then an input string x is said to be <u>accepted</u> by A with grade $f_A(x)$.

Now, by using the fundamental properties mentioned above, we have the following theorems.

<u>Theorem 3.1</u>. Let $F(\sigma)$ be any n x n fuzzy transition matrix, then the sequence $F(\sigma), F(\sigma^2), F(\sigma^3), \ldots\ldots$ is ultimately periodic.

<u>Proof</u>: Let $T = \{f_1, f_2, \ldots, f_k\}$ be the set of all the elements which occur in the matrix $F(\sigma)$, then the number of different matrices which can be obtained by multipling $F(\sigma)$ is at most k^{n^2}, that is, finite.

<u>Theorem 3.2</u>. If $I \lneqq F(\sigma)$, then

$$I \lneqq F(\sigma) \lneqq F(\sigma^2) \lneqq \ldots \lneqq F(\sigma^{n-1}) = F(\sigma^n) = F(\sigma^{n+1}) = \ldots\ldots \tag{3.12}$$

<u>Proof</u>: We can prove our theorem in a similar way in a Boolean matrix [26].

3.3 λ-Fuzzy Languages

We show that the capability of fuzzy automata as an acceptor is the same as that of finite automata, though fuzzy automata include the deterministic and nondeterministic finite automata as special cases, which was proved by E. S. Santos independetly [21] . Furthermore, every fuzzy language can be represented in a fuzzy automaton with any threshold λ such that $0 \leqq \lambda < 1$.

Definition 3.5. Let $A = (S, \pi, \{F(\sigma) \mid \sigma \in \Sigma\}, \delta^G)$ be a fuzzy automaton and λ a real number $0 \leqq \lambda < 1$. The set of all input strings accepted by A with parameter λ is defined as

$$L(A, \circ, \lambda) = \{ x \in \Sigma^* \mid f_A(x) > \lambda \}. \tag{3.13}$$

λ is called a threshold of A and $L(A, \circ, \lambda)$ a λ-fuzzy language. For $0 \leqq \lambda < 1$, a language L is λ-fuzzy if and only if there exists a A such that $L = L(A, \circ, \lambda)$. A language L is fuzzy if and only if, for some λ, it is λ-fuzzy.

<u>Theorem 3.3.</u> λ-fuzzy language $L(A, \circ, \lambda)$ is a regular language.

<u>Proof:</u> For a fuzzy automaton $A = (S, \pi, \{F(\sigma) \mid \sigma \in \Sigma\}, \delta^G)$, let us define $F(\pi,x) = \pi \circ F(x)$, where $F(\pi,x)$ is an n-dimensional row vector, $x \in \Sigma^*$ and $n = \#(S)$.

Define now the relation R on Σ^* by the definition:

$$x \ R \ y \quad \text{iff} \quad F(\pi,x) = F(\pi,y) \tag{3.14}$$

for all $x, y \in \Sigma^*$. Then R is clearly an equivalence relation on Σ^*. Furthermore, for any $z \in \Sigma^*$,

$$F(\pi,xz) = F(F(\pi,x),z)$$

$$= F(F(\pi,y),z) = F(\pi,yz). \tag{3.15}$$

Therefore, $xz \ R \ yz$ holds.

Hence R is a right congruence relation on Σ^*.

As to the number of equivalence classes, let $T = \{f_1, f_2, \dots f_k\}$ be the set of all the elements which occur in the matrices $F(\sigma_i), 1 \leq i \leq \#(\Sigma)$, and in the vector π. Then the number of equivalence classes is at most k^n. Anyhow, R has finite rank. Moreover, it is easily verified that $L(A, \circ, \lambda)$ is the union of some of the equivalence classes.

The same theorem also holds for an optimistic fuzzy automaton.

Definition 3.6. For a fuzzy matrix $A = \| a_{ij} \|$, $0 \leq a_{ij} \leq 1$ and d a real number such as $0 \leq d \leq 1$, we define a fuzzy matrix $A' = \| a'_{ij} \|$ as follows:

$$a'_{ij} = \begin{cases} a_{ij} + d & \text{if } a_{ij} \leq 1 - d, \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 3.1. For two fuzzy matrices U and V of the same order, let the fuzzy matrices defined in Definition 3.6 be U' and V', respectively, then, for two fuzzy matrices $W = \| w_{ij} \|$ and $W' = \| w'_{ij} \|$ such that $W = U \circ V$ and $W' = U' \circ V'$, we have that

$$w'_{ij} = \begin{cases} w_{ij} + d & \text{if } w_{ij} \leq 1 - d, \\ 1 & \text{otherwise.} \end{cases}$$

Proof: It is clear from the property of the operation " $\circ$ ".

Likewise, for a fuzzy matrix $A = \| a_{ij} \|$ and d' a real number $0 \leq d' \leq 1$, define a fuzzy matrix $A'' = \| a''_{ij} \|$ as follows:

$$a''_{ij} = \begin{cases} a_{ij} - d' & \text{if } a_{ij} \geq d', \\ 0 & \text{otherwise.} \end{cases}$$

Then the similar result as in Lemma 3.1 holds.

Theorem 3.4. Every fuzzy language is λ -fuzzy for any λ such that $0 \leq \lambda < 1$.

Proof: Let $L = L(A, \circ, \mu)$ and let $A = (S, \pi, \{F(\sigma) \mid \sigma \in \Sigma\}, \delta^G)$ be a fuzzy automaton, where $F(\sigma) = \| f_{s_i,s_j}(\sigma) \|$, $\pi = (\pi_{s_i})$, $s_i, s_j \in S$, and $\sigma \in \Sigma$. Omitting the trivial case $\lambda = \mu$, we can assume that $\lambda \neq \mu$.

(1) In the case of $\lambda > \mu$:

Consider the fuzzy automaton $A' = (S', \pi', \{F'(\sigma) \mid \sigma \in \Sigma'\}, \delta^{G'})$, where $S' = S$, $\Sigma' = \Sigma$, $\delta^{G'} = \delta^G$. And the fuzzy transition matrices $F'(\sigma) = \| f'_{s_i,s_j}(\sigma) \|$, $\sigma \in \Sigma'$, and $\pi' = (\pi'_{s_i})$ are defined as follows:

$$f'_{s_i,s_j}(\sigma) = \begin{cases} f_{s_i,s_j}(\sigma) + (\lambda - \mu) & \text{if } f_{s_i,s_j}(\sigma) \leq 1 - \lambda + \mu, \\ 1 & \text{otherwise.} \end{cases}$$

$$\pi'_{s_i} = \begin{cases} \pi_{s_i} + (\lambda - \mu) & \text{if } \pi_{s_i} \leq 1 - \lambda + \mu, \\ 1 & \text{otherwise.} \end{cases}$$

Thus, according to Lemma 3.1, for $x \in \Sigma^*$,

$$\pi' \circ F'(x) \circ \delta^{G'} = \begin{cases} \pi \circ F(x) \circ \delta^G + (\lambda - \mu) \\ \qquad \text{if} \quad \pi \circ F(x) \circ \delta^G \leq 1 - \lambda + \mu, \\ 1 \qquad \text{otherwise.} \end{cases}$$

Therefore,

$$L(A', \circ, \lambda) = L(A, \circ, \mu) \quad \text{when } \lambda > \mu.$$

(2) In the case of $\lambda < \mu$:

Consider the fuzzy automaton $A'' = (S'', \pi'', \{F''(\sigma) \mid \sigma \in \Sigma''\}, \delta^{G''})$ where $S'' = S$, $\Sigma'' = \Sigma$, $\delta^{G''} = \delta^G$. And the fuzzy transition matrices $F''(\sigma) = \| f''_{s_i,s_j}(\sigma) \|$ where $\sigma \in \Sigma''$, and $\pi'' = (\pi''_{s_i})$ are defined as follows:

$$f''_{s_i,s_j}(\sigma) = \begin{cases} f_{s_i,s_j}(\sigma) - (\mu - \lambda) \\ \qquad \text{if} \quad f_{s_i,s_j}(\sigma) \geq \mu - \lambda, \\ 0 \qquad \text{otherwise.} \end{cases}$$

$$\pi^{*}_{s_1} = \begin{cases} \pi_{s_1} - (\mu - \lambda) & \text{if } \pi_{s_1} \geq \mu - \lambda , \\ \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for $x \in \Sigma^{*}$,

$$\pi^{*} \circ F^{*}(x) \circ \delta^{G^{*}} = \begin{cases} \pi \circ F(x) \circ \delta^{G} - (\mu - \lambda) \\ \qquad \text{if } \pi \circ F(x) \circ \delta^{G} \geq \mu - \lambda , \\ \\ 0 \qquad \text{otherwise} \end{cases}$$

Therefore,

$$L(A^{*}, \circ , \lambda) = L(A, \circ , \mu) \quad \text{when } \lambda < \mu .$$

Hence, it follows that in both cases $L = L(A', \circ , \lambda)$ or $L(A^{*}, \circ , \lambda)$, which implies our theorem.

It is easily shown that the same theorem holds for an optimistic fuzzy automaton.

3.4 Closure Properties of Fuzzy Automata

In this section, we use the concept of fuzzy sets instead of the set of input strings with threshold λ.

It is shown that a family of fuzzy events characterized by not only pessimistic fuzzy automata (pfa for short) but also optimistic fuzzy automata (ofa for short) is closed under the operations of intersection and union in the fuzzy sense. And the complement of the fuzzy event by a pfa (an ofa) is characterized by an ofa (a pfa).

Definition 3.7. For a pfa $A = (S, \pi, \{F(\sigma) | \sigma \in \Sigma\}, \delta^G)$, let a fuzzy event be the fuzzy set in Σ^* which is characterized by

$$f_A(x) = \pi \circ F(x) \circ \delta^G, \quad x \in \Sigma^*. \qquad (3.16)$$

We denote by $L(A,\circ)$ the fuzzy event by a pfa A and, similarly, by $L(B,*)$ the fuzzy event by an ofa B.

Definition 3.8. For two pfa A_1 and A_2

$$A_1 = (S_1, \pi_1, \{F_1(\sigma) | \sigma \in \Sigma\}, \delta^G{}_1)$$

$$A_2 = (S_2, \pi_2, \{F_2(\sigma) | \sigma \in \Sigma\}, \delta^G{}_2),$$

define a <u>min pfa</u> $A_1 \otimes A_2$ as follows:

$$A_1 \otimes A_2 = (S, \pi, \{F(\sigma) \mid \sigma \in \Sigma\}, \delta^G),$$

where

$$S = S_1 \times S_2 = \{(s_i, t_j) \mid s_i \in S_1,\ t_j \in S_2,\ 1 \leqq i \leqq m,\ 1 \leqq j \leqq n\}$$

$G = G_1 \times G_2$, $m = \#(S_1)$ and $n = \#(S_2)$. As to the fuzzy transition function $f_{A_1 \otimes A_2}$ of a min pfa $A_1 \otimes A_2$, define

$$f_{A_1 \otimes A_2}((s,t),\sigma,(q,r)) = \min\left[f_{A_1}(s,\sigma,q), f_{A_2}(t,\sigma,r) \right]$$

for $(s,t),(q,r) \in S$ and $\sigma \in \Sigma$.

Moreover, the mn-dimensional row vector π is defined as follows:

For $(s_i,t_j) \in S$, $1 \leqq i \leqq m$, and $1 \leqq j \leqq n$,

$$\pi = \pi_1 \otimes \pi_2 = (\xi_{(s_i,t_j)})$$

where

$$\xi_{(s_i,t_j)} = \min\left[\pi_{1s_i}, \pi_{2t_j} \right]$$

and $(s_i,t_j) = (s_1,t_1),(s_1,t_2),\ldots,(s_1,t_n),(s_2,t_1),\ldots,(s_m,t_n)$.

And the mn-dimensional column vector δ^G is also $\delta^G =$ $\delta^{G_1} \otimes \delta^{G_2}$.

Hence, the fuzzy transition matrices of order mn of $A_1 \otimes A_2$ is as follows:

For two pfa A_1 and A_2, let $F_1(\sigma) = \| f_{s_i,s_j}(\sigma) \|$ and $F_2(\sigma) = \| f_{t_k,t_l}(\sigma) \|$ be fuzzy transition matrices of A_1 and A_2, respectively, then fuzzy transition matrix $F(\sigma)$ of $A_1 \otimes A_2$ is defined by

$$F(\sigma) = F_1(\sigma) \otimes F_2(\sigma) = \| f_{(s_i,t_k),(s_j,t_l)}(\sigma) \| ,$$

where

$$f_{(s_i,t_k),(s_j,t_l)}(\sigma) = \min\left[f_{s_i,s_j}(\sigma),\ f_{t_k,t_l}(\sigma) \right]$$

$$= f_{A_1 \otimes A_2}((s_i,t_k),\sigma,(s_j,t_l)).$$

Note that the operation $\otimes$ of fuzzy matrices corresponds to the tensor product of ordinary matrices.

Lemma 3.2. For fuzzy matrices A_1, A_2, B_1, B_2, A and B, for row vectors π_1 and π_2, and for column vectors δ^{G_1} and δ^{G_2}, we have that

(1) $(A_1 \circ B_1) \otimes (A_2 \circ B_2) = (A_1 \otimes A_2) \circ (B_1 \otimes B_2)$.

(2) $(\pi_1 \circ A \circ \delta^{G_1}) \otimes (\pi_2 \circ B \circ \delta^{G_2})$

$$= (\pi_1 \otimes \pi_2) \circ (A \otimes B) \circ (\delta^{G_1} \otimes \delta^{G_2})$$

$$= \min \left[\pi_1 \circ A \circ \delta^{G_1},\ \pi_2 \circ B \circ \delta^{G_2} \right].$$

(3) $A_1 \otimes A_2$, $\pi_1 \otimes \pi_2, \cdots$ are fuzzy matrices.

Proof: Obvious.

This enables us to prove the following closure theorem.

Theorem 3.5. Let A_1, A_2 and $A_1 \otimes A_2$ be pfa as in Definition 3.8 and $L(A_1, \circ)$, $L(A_2, \circ)$ and $L(A_1 \otimes A_2, \circ)$ be the fuzzy events characterized by A_1, A_2 and $A_1 \otimes A_2$, respectively. Then, in the fuzzy sense,

$$L(A_1, \circ) \cap L(A_2, \circ) = L(A_1 \otimes A_2, \circ). \tag{3.17}$$

Proof: The membership functions of fuzzy events $L(A_1,\circ)$, $L(A_2,\circ)$ and $L(A_1\otimes A_2,\circ)$ are

$$f_{A_1}(x) = \pi_1\circ F_1(x)\circ\delta^G 1,$$

$$f_{A_2}(x) = \pi_2\circ F_2(x)\circ\delta^G 2,$$

and
$$f_{A_1\otimes A_2}(x) = \pi\circ F(x)\circ\delta^G,$$

respectively. From Lemma 3.2, we have:

$$\begin{aligned} f_{A_1\otimes A_2}(x) &= \pi\circ F(x)\circ\delta^G \\ &= (\pi_1\otimes\pi_2)\circ(F_1(x)\otimes F_2(x))\circ(\delta^G 1\otimes\delta^G 2) \\ &= \min\left[\pi_1\circ F_1(x)\circ\delta^G 1,\ \ \pi_2\circ F_2(x)\circ\delta^G 2\right] \\ &= \min\left[f_{A_1}(x),\ f_{A_2}(x)\right] \end{aligned}$$

for all $x\in\Sigma^*$.

Corollary 3.1. For two ofa B_1 and B_2, let $L(B_1,*)$ and $L(B_2,*)$ be the fuzzy events by B_1 and B_2, respectively, then, in the fuzzy sense, there exists an ofa B such that

$$L(B_1,*)\cup L(B_2,*) = L(B,*). \qquad (3.18)$$

Proof: In Definition 3.8, by replacing the operation "min" by the operation "max" and defining a max ofa, we can easily prove Corollary 3.1.

We have shown that the family of fuzzy events by pfa is closed under intersection, and the family by ofa is closed under union in the fuzzy sense.

Next, we will verify that the family of fuzzy events by pfa (ofa) is closed under union (intersection) in the fuzzy sense.

Theorem 3.6. For two pfa A_1 and A_2, let $L(A_1,\circ)$ and $L(A_2,\circ)$ be fuzzy events by A_1 and A_2, respectively, then, in the fuzzy sense, there exists a pfa A such that

$$L(A_1,\circ) \cup L(A_2,\circ) = L(A,\circ). \tag{3.19}$$

Proof: Let A_1 and A_2 be two pfa as follows:

$$A_1 = (\{s_1, s_2, \ldots, s_m\}, \pi_1, \{F_1(\sigma) | \sigma \in \Sigma\}, \delta^{G_1})$$

$$A_2 = (\{t_1, t_2, \ldots, t_n\}, \pi_2, \{F_2(\sigma) | \sigma \in \Sigma\}, \delta^{G_2}).$$

Now, consider a pfa A, that is,

$$A = (\{s_1,\dots,s_m,t_1,\dots,t_n\}, \pi, \{F(\sigma) \mid \sigma \in \Sigma\}, \delta^G),$$

where π, $F(\sigma)$ and δ^G are given as follws:

Let

$$\pi_1 = (\pi_{s_1},\pi_{s_2},\dots,\pi_{s_m}) \quad \text{and} \quad \pi_2 = (\pi_{t_1},\pi_{t_2},\dots,\pi_{t_n}),$$

then

$$\pi = (\pi_{s_1},\dots,\pi_{s_m},\pi_{t_1},\dots,\pi_{t_n}) = (\pi_1\pi_2).$$

Moreover,

$$F(\sigma) = \begin{pmatrix} F_1(\sigma) & 0 \\ 0 & F_2(\sigma) \end{pmatrix}$$

and

$$\delta^G = \begin{pmatrix} \delta^{G_1} \\ \delta^{G_2} \end{pmatrix}.$$

In general, in fuzzy matrices, we have that

$$(1) \qquad \begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix} \circ \begin{pmatrix} A_2 & 0 \\ 0 & B_2 \end{pmatrix} = \begin{pmatrix} A_1 \circ A_2 & 0 \\ 0 & B_1 \circ B_2 \end{pmatrix}$$

$$(2) \qquad (\pi_1 \pi_2) \circ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \circ \begin{pmatrix} \delta^{G_1} \\ \delta^{G_2} \end{pmatrix} = \max\left[\pi_1 \circ A \circ \delta^{G_1},\ \pi_2 \circ B \circ \delta^{G_2}\right] .$$

Therefore, let

$$f_{A_1}(x) = \pi_1 \circ F_1(x) \circ \delta^{G_1},$$

$$f_{A_2}(x) = \pi_2 \circ F_2(x) \circ \delta^{G_2},$$

and

$$f_A(x) = \pi \circ F(x) \circ \delta^G$$

be the membership functions which characterize fuzzy events $L(A_1, \circ)$, $L(A_2, \circ)$ and $L(A, \circ)$, respectively. Then, for $x \in \Sigma^*$, we have:

$$\begin{aligned} f_A(x) &= \pi \circ F(x) \circ \delta^G \\ &= (\pi_1 \pi_2) \circ \begin{pmatrix} F_1(x) & 0 \\ 0 & F_2(x) \end{pmatrix} \circ \begin{pmatrix} \delta^{G_1} \\ \delta^{G_2} \end{pmatrix} \\ &= \max\left[\pi_1 \circ F_1(x) \circ \delta^{G_1},\ \pi_2 \circ F_2(x) \circ \delta^{G_2}\right] \\ &= \max\left[f_{A_1}(x),\ f_{A_2}(x)\right]. \end{aligned}$$

<u>Corollary 3.2</u>. For two ofa B_1 and B_2, let $L(B_1,*)$ and $L(B_2,*)$ be the fuzzy events by B_1 and B_2, respectively. Then, in the fuzzy sense, there exists an ofa B such that

$$L(B_1,*) \cap L(B_2,*) = L(B,*). \tag{3.20}$$

<u>Proof</u>: For two ofa B_1 and B_2, that is,

$$B_1 = (S_1,\ \pi_1,\ \{F_1(\sigma) | \sigma \in \Sigma\},\ \delta^{G}1),$$

and

$$B_2 = (S_2,\ \pi_2,\ \{F_2(\sigma) | \sigma \in \Sigma\},\ \delta^{G}2),$$

let us define an ofa B as follows:

$$B = (S,\ \pi,\ \{F(\sigma) | \sigma \in \Sigma\},\ \delta^{G}),$$

where $S = S_1 \cup S_2$, $S_1 \cap S_2 = \phi$, $\pi = (\pi_1 \pi_2)$,

$$F(\sigma) = \begin{pmatrix} F_1(\sigma) & E \\ E & F_2(\sigma) \end{pmatrix} \quad \text{for all } \sigma \text{ in } \Sigma,$$

$$\delta^{G} = \begin{pmatrix} \delta^{G_1} \\ \delta^{G_2} \end{pmatrix}.$$

Then, we can prove our Corollary immediately in a similar way as in Theorem 3.6.

We shall show the inclusion property of pfa.

Theorem 3.7. Given two pfa A_1 and A_2 as follows:

$$A_1 = (S_1, \pi_1, \{F_1(\sigma) | \sigma \in \Sigma\}, \delta^{G}1),$$

$$A_2 = (S_2, \pi_2, \{F_2(\sigma) | \sigma \in \Sigma\}, \delta^{G}2).$$

If $\#(S_1) = \#(S_2)$, $F_1(\sigma) \lneq F_2(\sigma)$ for all σ in Σ,

$$\pi_1 \leqq \pi_2 \quad \text{,and} \quad \delta^{G}1 \leqq \delta^{G}2,$$

then, in the fuzzy sense,

$$L(A_1, \circ) \leqq L(A_2, \circ). \tag{3.21}$$

Proof: We can easily show that

$$f_{A_1}(x) \leqq f_{A_2}(x) \qquad \text{for} \quad x \in \Sigma^*$$

from the basic properties of fuzzy matrix described in Section 3.2.

Obviously, the same theorem holds for ofa.

Next, we shall show the complement of fuzzy event by a pfa (an ofa) is characterized by an ofa (a pfa).

Definition 3.9. If $A = (S, \pi, \{F(\sigma)|\sigma \in \Sigma\}, \delta^G)$ is a pfa, then a complementary ofa of A is defined as follows:

$$A' = (S', \pi', \{F'(\sigma)|\sigma \in \Sigma\}, \delta^{G'}),$$

where $S' = S$. As to the fuzzy transition function $f'_{A'}$ of A', for $\sigma \in \Sigma$, $x,y \in \Sigma^*$, $s,t \in S'$, and f_A of A, we define:

$$\begin{aligned} f'_{A'}(s,\sigma,t) &= 1 - f_A(s,\sigma,t), \\ f'_{A'}(s,xy,t) &= \min_q \max\left[f'_{A'}(s,x,q),\ f'_{A'}(q,y,t)\right] \\ &= 1 - f_A(s,xy,t), \end{aligned}$$

and the initial and final state vectors are

$$\pi' = (1,1,\ldots,1) - \pi, \quad \text{and} \quad \delta^{G'} = (1,1,\ldots,1)' - \delta^G.$$

Note that we can easily define a complementary pfa B' of an ofa B in a similar way.

Lemma 3.3. For a fuzzy matrix $U = \| u_{ij} \|$, let $U' = \| u'_{ij} \|$ be fuzzy matrix such that

$$u'_{ij} = 1 - u_{ij}.$$

For fuzzy matrices $U_1, U_2, \ldots, U_m$, let $U'_1, U'_2, \ldots, U'_m$ be fuzzy matrices as defined above, respectively, then

$$U_1 \circ U_2 \circ \cdots \circ U_m + U'_1 * U'_2 * \cdots * U'_m = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \cdots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}.$$

Theorem 3.8. Let A be a pfa and let A' be a complementary ofa of A, then, in the fuzzy sense,

$$\overline{L(A,\circ)} = L(A',*). \tag{3.22}$$

Proof: Let $A = (S, \pi, \{F(\sigma) | \sigma \in \Sigma\}, \delta^G)$ be a pfa and $A' = (S, \pi', \{F'(\sigma) | \sigma \in \Sigma\}, \delta^{G'})$ be a complementary ofa of A, then by Lemma 3.3, for $x \in \Sigma^*$,

$$\begin{aligned} f_A(x) &= \pi \circ F(x) \circ \delta^G \\ &= 1 - \pi' * F'(x) * \delta^{G'} \\ &= 1 - f'_{A'}(x). \end{aligned}$$

Therefore, we have $\overline{L(A,\circ)} = L(A',*)$.

Corollary 3.3. For an ofa B and a complementary pfa B' of B, in the fuzzy sense,

$$\overline{L(B,*)} = L(B',\circ). \tag{3.23}$$

Proof: Immediately.

We have shown that the family of fuzzy events characterized by pfa (ofa) constitutes a distributive lattice, but does not constitute a Boolean lattice clearly.

3.5 Conclusions

The pessimistic (optimistic) fuzzy automata are no more powerful than the finite automata as measured by sets of accepted input strings. The thresholds can be set arbitrarily by changing the values of each element of the fuzzy transition matrix and the initial state designator. Moreover, the family of fuzzy events characterized by pessimistic (optimistic) fuzzy automata is closed under the operations of union and intersection in the sense of fuzzy sets. And the complement of the fuzzy event by a pessimistic (an optimistic) fuzzy automaton is characterized by an optimistic (a pessimistic) fuzzy automaton.

CHAPTER 4

CONDITIONAL FUZZY GRAMMARS

4.1 Introduction

Natural languages such as English have incorrectness and ambiguity syntactically and semantically. It is natural to introduce randomness into the structure of formal languages in order to specify natural languages with ambiguity [49, 50, 51, 52]. Another way of extending the concepts of formal languages to those of natural languages is the introduction of fuzziness. The first step to this direction was made by Lee and Zadeh [38], who introduced fuzzy grammars as an extention of ordinary formal grammars by using the concept of fuzzy sets. The notion of fuzzy grammars was introduced by the author, independently [39].

In this chapter we shall discuss fuzzy grammars, and conditional fuzzy grammars (or $n(\geq 1)$-fold fuzzy grammars) which were defined by the author.

Ordinary formal grammars have the property that, after applying a rewriting rule to an intermediate string in a derivation, the next rewriting rule to be used can be chosen arbitrarily. This arbitrariness, however, is not sufficient for describing natural languages with fuzziness. For example, consider the following rewriting rules: (1) $S \longrightarrow A$ bites B, (2) $A \longrightarrow$ the dog, (3) $A \longrightarrow$ the boy, (4) $B \longrightarrow$ the boy, (5) $B \longrightarrow$ the dog. Then the sentences generated by these rewriting rules with an initial symbol S are as follows. {the dog bites the boy, the dog bites the dog, the boy bites the boy, the boy bites the dog}. Generally speaking, a sentence "the boy bites the dog" is rather doubtful semantically. We may say that, after applying the rewriting rule(3) to the intermediate string "A bites B", it is not rather proper to apply the rewriting rule (5).

To specify such a condition, we have defined conditional fuzzy grammars, more precisely, n-fold fuzzy grammars, in which the grade (or the propriety) of the application of the rewriting rule to be used next is conditioned by the n rules used before in a derivation. The grade approaches to unity nearer and nearer as the propriety becomes higher. In the case of applying several rewriting rules, we use the concept of composition of fuzzy relations.

We show that the set of all strings whose grades of the generation by fuzzy grammars with type i (i = 0, 1, 2, 3) rules are greater than a certain threshold is a type i language. But the set of all strings whose grades by fuzzy grammars with type 2 rules are between two thresholds is not necessary a type 2 language.

n-fold fuzzy grammars whose rules are of the type 2 form can be shown to generate type 1 languages by setting a threshold appropriately. As to n-fold fuzzy grammars, however, we focused our attention on n-fold fuzzy grammars with type 3 rules as a preliminary step.

4.2 Fuzzy Grammars

The notion of fuzzy grammars defined by Lee and Zadeh [38], and Mizumoto, et al. [39] is a natural generalization of the definition of formal grammars. In this section we shall show that the set of all strings whose grades of the generation obtained by fuzzy grammars with type i (i=0,1,2,3) rules are greater than a certain threshold is a type i language. But the set of all strings whose grades by fuzzy grammars with type 2 rules are between two thresholds is not necessary a type 2 language. The same holds for fuzzy grammars with type 0 rules.

Definition 4.1. A *fuzzy grammar* (FG for short) is a system

$$FG = (V_N, V_T, P, S, J, f) \tag{4.1}$$

where

(i) V_N is a nonterminal vocabulary.

(ii) V_T is a terminal vocabulary.

(iii) S is an initial symbol in V_N.

(iv) P is a finite set of productions such as

$$(r) \quad u \rightarrow v \qquad f(r) \tag{4.2}$$

where $r \in J$, $u \rightarrow v$ is an ordinary rewriting rule with $u \in V_N^* - \{\varepsilon\}$, $v \in (V_N \cup V_T)^*$, and $f(r)$ is the grade of the application of the production r, which will be denoted in (vi).

(v) J is a set of (rewriting rule) labels as shown in (iv). $J = \{r\}$.

(vi) f is a membership function such as

$$f : J \rightarrow [0, 1]. \tag{4.3}$$

f may be called a <u>fuzzy function</u> and the value $f(r)$, $r \in J$, is the grade of the application of a production r.†

We assume that, to each rewriting rule, there can correspond more than one label, but not conversely.

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† We often say the label r as the production r for convenience.

Next, we shall briefly explain a derivation chain with fuzzy grades (fuzzy derivation chain).

If (r) $u \rightarrow v$ $f(r)$ is in P, and α and β are any strings in $(V_N \cup V_T)^*$, then

$$\alpha u \beta \xrightarrow[r]{f(r)} \alpha v \beta \qquad (4.4)$$

and $\alpha v \beta$ is said to be directly derivable from $\alpha u \beta$ with the grade $f(r)$ by the production r. If $\alpha_1, \alpha_2, \cdots, \alpha_m$ are strings in $(V_N \cup V_T)^*$ and

$$\alpha_0 \xrightarrow[r_1]{f(r_1)} \alpha_1, \quad \alpha_1 \xrightarrow[r_2]{f(r_2)} \alpha_2, \cdots\cdots\cdots, \quad \alpha_{m-1} \xrightarrow[r_m]{f(r_m)} \alpha_m . \qquad (4.5)$$

Then α_m is said to be derivable from α_0 by the productions $r_1, r_2, \cdots, r_m$. The expression

$$\alpha_0 \xrightarrow[r_1]{f(r_1)} \alpha_1 \xrightarrow[r_2]{f(r_2)} \alpha_2 \longrightarrow \cdots\cdots\cdots \xrightarrow[r_m]{f(r_m)} \alpha_m \qquad (4.6)$$

will be referred to as a <u>fuzzy derivation chain</u> of length m from α_0 to α_m by the productions $r_1, r_2, \cdots, r_m$.

When $\alpha_0 = S$, $\alpha_m = x \ (\in V_T^*)$ in (4.6), i.e.,

$$S \xrightarrow[r_1]{f(r_1)} \alpha_1 \xrightarrow[r_2]{f(r_2)} \alpha_2 \longrightarrow \cdots\cdots \longrightarrow \alpha_{m-1} \xrightarrow[r_m]{f(r_m)} x , \tag{4.7}$$

S is said to generate a terminal string x by the productions $r_1, r_2, \cdots, r_m$. In general, there are more than one fuzzy derivation chain from S to x .

<u>Definition 4.2</u>. The grade of the generation of terminal string $x \ (\in V_T^*)$ by a fuzzy grammar FG, which is denoted as $f_{FG}(x)$, is given as follows by using the concept of the composition of fuzzy relations of (2.17) and by the fuzzy derivation chain from S to x of (4.7). Clearly, $f_{FG}(x)$ is in $[0, 1]$.

$$f_{FG}(x) = \max \min \left[f(r_1), f(r_2), \ldots, f(r_m) \right] \tag{4.8}$$

where the maximum is taken over all the fuzzy derivation chains from S to x.

We shall define a language generated by a fuzzy grammar FG with the threshold λ , $0 \leqq \lambda < 1$.

Definition 4.3. Let $FG = (V_N, V_T, P, S, J, f)$ be a fuzzy grammar and λ a real number $0 \leqq \lambda < 1$, then a language generated by FG with the threshold λ is defined by

$$L(FG, \lambda) = \{ x \in V_T^* \mid f_{FG}(x) > \lambda \} \qquad (4.9)$$

Moreover, we can also define other languages as follows:

Definition 4.4. For two thresholds λ_1 and λ_2 with $0 \leqq \lambda_1 < \lambda_2 < 1$, a language $L(FG, \lambda_1, \lambda_2)$ is defined by the following:

$$L(FG, \lambda_1, \lambda_2) = \{ x \in V_T^* \mid \lambda_1 < f_{FG}(x) \leqq \lambda_2 \} \quad (4.10)$$

Besides, for a threshold λ , a language $L(FG, =, \lambda)$ is defined as

$$L(FG, =, \lambda) = \{ x \in V_T^* \mid f_{FG}(x) = \lambda \} \qquad (4.11)$$

where λ is $0 \leqq \lambda \leqq 1$.

<u>Example 4.1.</u> Let FG be the fuzzy grammar (V_N, V_T, P, S, J, f), where $V_N = \{S, A, B, C, D, E\}$, $V_T = \{a, b\}$, and the productions are

(1)	$S \longrightarrow ABC$	0.5 ,	(8)	$B \longrightarrow bB$	0.8
(2)	$S \longrightarrow ADC$	0.8 ,	(9)	$B \longrightarrow b$	0.9
(3)	$S \longrightarrow DBC$	0.7 ,	(10)	$C \longrightarrow aC$	0.8
(4)	$S \longrightarrow ABE$	0.6 ,	(11)	$C \longrightarrow a$	0.9
(5)	$S \longrightarrow AEC$	0.6 ,	(12)	$D \longrightarrow aDb$	0.8
(6)	$A \longrightarrow aA$	0.8 ,	(13)	$D \longrightarrow ab$	0.8
(7)	$A \longrightarrow a$	0.9 ,	(14)	$E \longrightarrow bEa$	0.8
			(15)	$E \longrightarrow ba$	0.8

A string, say, a^2b^2a is obtained by the following derivation.

$$\underline{S} \xrightarrow[1]{0.5} \underline{A}BC \xrightarrow[6]{0.8} a\underline{A}BC \xrightarrow[7]{0.9} a^2\underline{B}C \xrightarrow[8]{0.8} a^2b\underline{B}C \xrightarrow[9]{0.9} a^2b^2\underline{C}$$

$$\xrightarrow[11]{0.9} a^2b^2a \text{ ,}$$

where the underbar in the intermediate string represents the location where the next production was applied.

The grade of the generation of a^2b^2a by this derivation is the minimum value among the values indicated over the arrows, i.e.,

$$\min(0.5, 0.8, 0.9, 0.8, 0.9, 0.9) = 0.5$$

Similarly, for the same string a^2b^2a, the following derivation is also possible,

$$\underline{S} \xrightarrow[4]{0.6} \underline{A}BE \xrightarrow[6]{0.8} a\underline{A}BE \xrightarrow[7]{0.9} a^2\underline{B}E \xrightarrow[9]{0.9} a^2b\underline{E} \xrightarrow[15]{0.8} a^2b^2a \ .$$

In this case, we have 0.6. Furthermore, we can show other derivations of a^2b^2a, whose grades are shown to be less than 0.6. Thus we have $f_{FG}(a^2b^2a) = 0.6$ from (4.8).

Continuing in this manner, we can see that the languages generated by FG are, for example

(i) $L(FG, 0.45) = \{ a^i b^j a^k \mid i,j,k \geq 1 \}$.

(ii) $L(FG, 0.55) = \{ a^i b^j a^k \mid i,j,k \geq 1,\ i \neq j \text{ or } j \neq k \}$.

(iii) $L(FG, 0.75) = \{ a^{i+j} b^j a^k \mid i,j,k \geq 1 \}$.

(iv) $L(FG, 0.45, 0.55) = L(FG, =, 0.5)$

$$= \{ a^i b^i a^i \mid i \geq 1 \}.$$

It is interesting to note that languages (i), (ii), and (iii) given above are all context-free languages, but languages (iv) are context-sensitive languages.

Remark: In this example, the language $\{ a^i b^j a^k \mid i,j,k \geq 1 \}$ is generated by the rewriting rules whose labels are (1) and (6) ~ (11). And the language $\{ a^i b^j a^k \mid i \neq j \text{ or } j \neq k \}$ is by the rewriting rules of (2) ~ (15).

Definition 4.5. A fuzzy grammar FG in which the rewriting rules are of type i (i=0,1,2,3) is denoted as i-FG . The language by i-FG with the threshold λ is defined as $L(i\text{-}FG, \lambda)$.

Theorem 4.1. For any λ ($0 \leq \lambda < 1$), a language $L(i\text{-}FG, \lambda)$ is a type i language in the sense of Chomsky, where i = 0,1,2,3.

Proof: For the i-FG = (V_N, V_T, P, S, J, f) with the threshold λ $(0 \leqq \lambda < 1)$, let $J(\lambda)$ be the set of all labels such that $f(r) > \lambda$, where $r \in J$. More precisely, $J(r) = \{ r \mid f(r) > \lambda \}$. Then $L(i\text{-FG}, \lambda)$ is a language which was obtained from the only rewriting rules corresponding to the labels in $J(\lambda)$. $L(i\text{-FG}, \lambda)$ is, therefore, a type i language, where $i = 0,1,2,3$.

Theorem 4.2. For the i-FG, where i = 0, 2, the languages $L(i\text{-FG}, \lambda_1, \lambda_2)$ and $L(i\text{-FG}, =, \lambda)$ are not always type i languages.

Proof: The language $L(i\text{-FG}, \lambda_1, \lambda_2)$ is given by the difference $L(i\text{-FG}, \lambda_1) - L(i\text{-FG}, \lambda_2)$. We know that the type i (i = 0, 2) languages are not always closed under the difference. Thus, the languages $L(i\text{-FG}, \lambda_1)$ and $L(i\text{-FG}, \lambda_2)$ are type i languages, so $L(i\text{-FG}, \lambda_1, \lambda_2)$ is not always a type i language, where i = 0, 2. Similarly, $L(i\text{-FG}, =, \lambda)$ is given by $L(i\text{-FG}, \geqq, \lambda) - L(i\text{-FG}, \lambda)$, where $L(i\text{-FG}, \geqq, \lambda)$ is defined as $\{ x \in V_T^* \mid f_{FG}(x) \geqq \lambda \}$. Clearly, $L(i\text{-FG}, \geqq, \lambda)$ is a type i language, so $L(i\text{-FG}, =, \lambda)$ is not always type i language, where i = 0, 2.

Note: $L(3\text{-FG}, \lambda_1, \lambda_2)$ and $L(3\text{-FG}, =, \lambda)$ are also type 3 languages, since type 3 languages are closed under the difference. We can not, however, conclude whether L(1-FG, λ_1, λ_2) and L(1-FG, = , λ) are type 1 languages or not. For it is not known whether type 1 languages are closed under the difference or not.

4.3 N-Fold Fuzzy Grammars

In this section we shall define an n-fold fuzzy grammar in which the grade of the application of the rewriting rule to be used next is conditioned by the n rules used before in a derivation, where $n \geq 1$. And it is shown that n-fold fuzzy grammars with CF rules can generate CS languages.

Definition 4.6. An $n(\geq 1)$-fold fuzzy grammar (n-FG for short) is a system,

$$\text{n-FG} = (V_N, V_T, P, S, J, \{f_1, f_2, \dots, f_n\}) \qquad (4.12)$$

where V_N, V_T, S, and J have essentially the same meanings as those for the fuzzy grammars denoted in the previous section, and P is a finite set of rules with labels such as

$$(r) \quad u \longrightarrow v \qquad (4.13)$$

where $r \in J$, $u \rightarrow v$ is an ordinary rewriting rule with $u \in V_N^{*} - \{\varepsilon\}$ and $v \in (V_N \cup V_T)^{*}$. $J = \{r\}$ is a set of (rewriting rule) labels. f_i $(i = 0, 1, \cdots, n)$ is a (conditional) membership function of a fuzzy set in the label set J and is defined as follows:

(a) In the case of $i = 0$: f_0 is a membership function from J_S to $[0, 1]$, i.e.,

$$f_0 : J_S \longrightarrow [0, 1] \tag{4.14}$$

where J_S is the set of all labels whose rules are initial rules. The value $f_0(r)$ in $[0, 1]$ represents the grade of the application of an initial rule r in J_S.

(b) In the case of $1 \leqq i \leqq n$: f_i is a conditional membership function such as

$$f_i(r_1, r_2, \cdots, r_i ; r_{i+1}) \in [0, 1] \tag{4.15}$$

and represents the grade of membership of r_{i+1} in J given $r_1, r_2, \cdots, r_i$ in J. In other words, $f_i(r_1, r_2, \cdots, r_i; r_{i+1})$ is designated as the grade of the application of the rule r_{i+1} after the i rules $r_1, r_2, \cdots, r_i$ were applied sequentially to the intermediate string in a derivation.

In what follows, we shall call f_i $(i=0,1,\dots,n)$ as an <u>i-fold fuzzy function</u>.

We assumed that, to each rule, there may correspond more than one label, but not conversely.

<u>Remark</u>: For all r in J_S, all i $(i=1,2,\dots,n)$, and all $r_1, r_2, \dots, r_i$ in J, let

$$f_0(r) = f(r),$$

$$f_i(r_1, r_2, \dots, r_i; r_{i+1}) = f(r_{i+1}).$$

Then n-fold fuzzy grammar becomes a fuzzy grammar denoted in the previous section. Thus we may call a fuzzy grammar as a <u>0-fold fuzzy grammar</u>.

Now, we shall explain how to use i-fold fuzzy functions f_i, $i=0,1,\dots,n$, in a derivation.

The expression

$$S \xrightarrow[r_1]{} \alpha_1 \xrightarrow[r_2]{} \alpha_2 \xrightarrow[r_3]{} \cdots\cdots \xrightarrow[r_m]{} \alpha_m \qquad (4.16)$$

will be referred to as a derivation chain of length m by the rules $r_1, r_2, \cdots, r_m$ from S to α_m.

When the length m of a derivation chain is $m \leqq n$, the fuzzy functions $f_0, f_1, \cdots\cdot, f_{m-1}$ are employed as follows:

Let $f_0(r_1) = \mu_1$, $f_1(r_1;r_2) = \mu_2$, $f_2(r_1,r_2;r_3) = \mu_3, \cdots\cdot$, and $f_{m-1}(r_1,r_2,\cdots,r_{m-1};r_m) = \mu_m$, then we put each fuzzy grade $\mu_1, \mu_2, \cdots, \mu_m \in [0, 1]$ over the arrow by the following.

$$S \xrightarrow[r_1]{\mu_1} \alpha_1 \xrightarrow[r_2]{\mu_2} \alpha_2 \longrightarrow \cdots\cdot \xrightarrow[r_{m-1}]{\mu_{m-1}} \alpha_{m-1} \xrightarrow[r_m]{\mu_m} \alpha_m . \tag{4.17}$$

Moreover, when the length m is $m > n$, we let $m = n + j$, $j \geqq 1$. Then, in general, after the n rules $r_j, r_{j+1}, \cdots\cdot, r_{n+j-1}$, where $j \geqq 1$, were applied sequentially to the intermediate string, the grade of the application of the rule r_{n+j} is characterized by an n-fold fuzzy function f_n. Let $f_n(r_j,r_{j+1},\cdots\cdot,r_{n+j-1};r_{n+j}) = \mu_{n+j}$, $j \geqq 1$, then the grades $\mu_{n+1}, \mu_{n+2}, \cdots\cdot, \mu_{n+j}$ are expressed as follows, where $\mu_1, \mu_2, \cdots, \mu_n$ are dependent on the fuzzy functions $f_0, f_1, \cdots\cdot, f_{n-1}$ as mentioned before.

$$S \xrightarrow[r_1]{\mu_1} \alpha_1 \xrightarrow[r_2]{\mu_2} \cdots \xrightarrow[r_n]{\mu_n} \alpha_n \xrightarrow[r_{n+1}]{\mu_{n+1}} \alpha_{n+1} \longrightarrow \cdots \xrightarrow[r_{n+j}]{\mu_{n+j}} \alpha_{n+j} . \qquad (4.18)$$

We shall call this derivation chain with fuzzy grades as <u>fuzzy derivation chain</u>.

We shall next explain a fuzzy language characterized by an n-fold fuzzy grammar n-FG .

Let (4.19) be a fuzzy derivation chain from an initial symbol S to terminal string x in V_T^*, that is,

$$S \xrightarrow[r_1]{\mu_1} \alpha_1 \xrightarrow[r_2]{\mu_2} \alpha_2 \longrightarrow \cdots\cdot \xrightarrow[r_k]{\mu_k} \alpha_k \ (=x). \qquad (4.19)$$

Then the grade of the generation of x by this fuzzy derivation chain is defined as

$$\min \left[\mu_1, \mu_2, \cdots\cdot, \mu_k \right] \qquad (4.20)$$

By using the concept of composition of fuzzy relations (see (2.17) in Chapter 2), the grade of the generation of x in V_T^* by n-FG is given as follows.

$$\mu_{n\text{-}FG}(x) = \max\min\left[\mu_1, \mu_2, \cdots, \mu_k\right]. \qquad (4.21)$$

where the maximum is taken over all the fuzzy derivation chains from S to x.

Definition 4.7. A fuzzy language by n-FG is a fuzzy set in V_T^* characterized by the membership function $\mu_{n\text{-}FG}(x)$ as defined in (4.21) and may be called an n-fold fuzzy language which is denoted as $L(n\text{-}FG)$.

Especially, we call an n-fold fuzzy grammar with the grade of the generation of the terminal string under the operation "max min" as in (4.21) as an n-fold pessimistic fuzzy grammar (=n-PFG), and an n-fold fuzzy grammar under the operation "min max" which will be defined in (4.22) as an n-fold optimistic fuzzy grammar (=n-OFG).

$$\mu_{n\text{-}OFG}(x) = \min\max\left[\mu_1, \mu_2, \cdots, \mu_k\right]. \qquad (4.22)$$

A fuzzy language characterized by n-OFG is denoted as L(n-OFG).

In this paper, unless stated especially, by an "n-fold fuzzy grammar" we shall mean an n-fold pessimistic fuzzy grammar.

Example 4.2. Consider the following 2-fold fuzzy grammar,

$$2\text{-FG} = (V_N, V_T, P, S, J, \{f_0, f_1, f_2\})$$

where $V_N = \{S, A, B\}$, $V_T = \{a, b, c\}$, P consists of the followings.

(1) $S \rightarrow AB$, (4) $B \rightarrow cB$

(2) $A \rightarrow aAb$, (5) $B \rightarrow c$

(3) $A \rightarrow ab$,

And the 0, 1, 2-fold fuzzy functions are

$f_0(1) = 1$,

$f_1(1;3) = 0.9$, $f_1(1;2) = 0.8$

$f_2(1,3;5) = 0.9$, $f_2(1,2;2) = 0.8$

$f_2(2,2;4) = 0.7$, $f_2(2,4;4) = 0.7$

$f_2(4,4;2) = 0.7$, $f_2(4,2;2) = 0.7$

$f_2(4,4;3) = 0.6$, $f_2(4,3;5) = 0.6$

and all other f_1 and f_2 are 0.5.

Now, we shall obtain the grades of the generation of the terminal strings by this 2-FG. A string, say, $a^3b^3c^3$ is obtained by the following fuzzy derivation chain.

$$\underline{S} \xrightarrow[1]{1} \underline{A}B \xrightarrow[2]{0.8} a\underline{A}bB \xrightarrow[2]{0.8} a^2Ab^2\underline{B} \xrightarrow[4]{0.7} a^2Ab^2c\underline{B}$$

$$\xrightarrow[4]{0.7} a^2\underline{A}b^2c^2B \xrightarrow[3]{0.6} a^3b^3c^2\underline{B} \xrightarrow[5]{0.6} a^3b^3c^3.$$

The grade of the generation of $a^3b^3c^3$ by this derivation is given from (4.20) as follows.

$$\min[1,\ 0.8,\ 0.8,\ 0.7,\ 0.7,\ 0.6,\ 0.6] = 0.6.$$

The 0, 1, 2-fold fuzzy functions used sequentially in this derivation are

$f_0(1) = 1,\quad f_1(1;2) = 0.8,\quad f_2(1,2;2) = 0.8,$

$f_2(2,2;4) = 0.7,\quad f_2(2,4;4) = 0.7,\quad f_2(4,4;3) = 0.6,$

$f_2(4,3;5) = 0.6.$

Similarly, for the same string $a^3b^3c^3$, the following derivation is also possible.

$$\underline{S} \underset{1}{\overset{1}{\longrightarrow}} A\underline{B} \underset{4}{\overset{0.5}{\longrightarrow}} \underline{A}cB \underset{2}{\overset{0.5}{\longrightarrow}} aAbc\underline{B} \underset{4}{\overset{0.5}{\longrightarrow}} a\underline{A}bc^2B$$

$$\underset{4}{\overset{0.7}{\longrightarrow}} a^2\underline{A}b^2c^2B \underset{3}{\overset{0.6}{\longrightarrow}} a^3b^3c^2\underline{B} \underset{5}{\overset{0.6}{\longrightarrow}} a^3b^3c^3.$$

In this case, we have

$$\min\left[1, 0.5, 0.5, 0.5, 0.7, 0.6, 0.6 \right] = 0.5.$$

Furthermore, we can also show the different fuzzy derivation chains of $a^3b^3c^3$, whose all grades are shown to be 0.5 .

Hence, the grade of the generation of $a^3b^3c^3$ by 2-FG is given as thw maximum value among the grades for all the fuzzy derivation chains of $a^3b^3c^3$ (see (4.21)). Thus, we have $\mu_{2-FG}(a^3b^3c^3) = 0.6$.

Continuing in this manner, we can see that the fuzzy language characterized by 2-FG is

$$L(2\text{-FG}) = \{ (x, \mu_{2\text{-FG}}(x)) \}$$
$$= \{(abc, 0.9)\} \cup \{ (a^{2n+1}b^{2n+1}c^{2n+1}, 0.6) \mid n \geq 1\}$$
$$\cup \{(a^i b^i c^j, 0.5) \mid (i,j) \neq (2n-1,2n-1),\ i,j,n \geq 1\}.$$

Let $L(2\text{-FG}, \lambda) = \{x \in V_T^* \mid \mu_{2\text{-FG}}(x) > \lambda \}$, where $0 \leqq \lambda < 1$, and let $\lambda = 0.85,\ 0.55,\ 0.45$. Then

$$L(2\text{-FG},\ 0.85) = \{ abc \}.$$
$$L(2\text{-FG},\ 0.55) = \{ a^{2n-1}b^{2n-1}c^{2n-1} \mid n \geq 1\}.$$
$$L(2\text{-FG},\ 0.45) = \{ a^i b^i c^j \mid i,j \geq 1\}.$$

In the case of $n = 1$ in n-FG, 1-fold fuzzy function f_1 of $1\text{-FG} = (V_N, V_T, P, S, J, \{ f_0, f_1\})$ can be represented by the m fuzzy vectors whose dimension is m, where $m = \#(J)$. That is, let $J = \{ r_1, r_2, \cdots, r_m \}$, then for each $(r_i)\ u_i \rightarrow v_i$ in P,

$$(r_i) \quad u_i \rightarrow v_i \quad (f_{r_i r_1}, f_{r_i r_2}, \ldots, f_{r_i r_m}) \qquad (4.23)$$

where $f_{r_i r_j} = f_1(r_i; r_j)$ and $i,j = 1,2,\ldots,m.$

Example 4.3. Let 1-FG be $(V_N, V_T, P, S, J, \{f_0, f_1\})$, where $V_N = \{S, A, B, C\}$, $V_T = \{a, b, c\}$, $f_0(1) = 0.9$, and the rules with fuzzy vectors are

		1	2	3	4	5	6	7	8	9	10
(1)	$S \rightarrow ABC$		.7			.8			.9		
(2)	$A \rightarrow aA$			.7							
(3)	$B \rightarrow bB$				.7						
(4)	$C \rightarrow cC$		.7						.7		
(5)	$A \rightarrow aAa$						.8				
(6)	$B \rightarrow bBb$							.8			
(7)	$C \rightarrow cCc$					.8			.8		
(8)	$A \rightarrow a$									.9	
(9)	$B \rightarrow b$										.9
(10)	$C \rightarrow c$										

We assumed that the values of the blank portions of the fuzzy vectors are in the interval $(0, 0.65]$.

A string, say, $a^3b^3c^3$ is obtained by the following fuzzy derivation chain.

$$\underline{S} \xrightarrow[1]{0.9} \underline{A}BC \xrightarrow[2]{0.7} aA\underline{B}C \xrightarrow[3]{0.7} aAbB\underline{C} \xrightarrow[4]{0.7} a\underline{A}bBcC \xrightarrow[2]{0.7} a^2Ab\underline{B}cC$$

$$\xrightarrow[3]{0.7} a^2Ab^2Bc\underline{C} \xrightarrow[4]{0.7} a^2\underline{A}b^2Bc^2C \xrightarrow[8]{0.7} a^3b^2\underline{B}c^2C \xrightarrow[9]{0.9} a^3b^3c^2\underline{C}$$

$$\xrightarrow[10]{0.9} a^3b^3c^3.$$

The grade of the generation of $a^3b^3c^3$ by this derivation is from (4.20) as follows.

$$\min \{ 0.9, 0.7, 0.7, \cdots , 0.7, 0.9, 0.9 \} = 0.7$$

Similarly, other derivation chains of $a^3b^3c^3$ are considered.

$$\underline{S} \xrightarrow[1]{0.9} \underline{A}BC \xrightarrow[5]{0.8} aAa\underline{B}C \xrightarrow[6]{0.8} aAabBb\underline{C} \xrightarrow[7]{0.8} a\underline{A}abBbcCc$$

$$\xrightarrow[8]{0.8} a^3b\underline{B}bcCc \xrightarrow[9]{0.9} a^3b^3c\underline{C}c \xrightarrow[10]{0.9} a^3b^3c^3.$$

In the above case, we have 0.8 . Furthermore, we can also show the different derivations of $a^3b^3c^3$, the grades of which are all less than or equal 0.65 . Thus, $f_{1-FG}(a^3b^3c^3) = 0.8$.

Continuing in this manner, we can see that the languages with the thresholds generated by this 1-FG are, for example,

$$L(1\text{-FG}, 0.95) = \phi.$$
$$L(1\text{-FG}, 0.85) = \{abc\}.$$
$$L(1\text{-FG}, 0.75) = \{a^{2n-1}b^{2n-1}c^{2n-1} \mid n \geqq 1\}.$$
$$L(1\text{-FG}, 0.65) = \{a^nb^nc^n \mid n \geqq 1\}.$$
$$L(1\text{-FG}, 0) = \{a^pb^qc^r \mid p,q,r \geqq 1\}.$$
$$L(1\text{-FG}, 0.65, 0.75) = \{a^{2n}b^{2n}c^{2n} \mid n \geqq 1\}.$$
$$L(1\text{-FG}, =, 0.8) = \{a^{2n+1}b^{2n+1}c^{2n+1} \mid n \geqq 1\}.$$

where the language $L(1\text{-FG}, 0.65, 0.75)$ is defined as $\{x \in V_T^* \mid 0.65 < \mu_{1-FG}(x) \leqq 0.75\}$ and the language $L(1\text{-FG}, =, 0.8)$ is as $\{x \in V_T^* \mid \mu_{1-FG}(x) = 0.8\}$ (see Definition 4.4).

It is interesting to note that, as shown in the above examples 4.2 and 4.3 , the languages L(n-FG,λ) by n($\geq$ 1)-fold fuzzy grammars n-FG with the rules of CF form can be CS languages. But L(FG,λ) by fuzzy grammars FG (or 0-fold fuzzy grammars) with CF rules denoted in the previous section are CF languages.

4.4 N-Fold Type 3 Fuzzy Grammars

In this section we discuss n-fold fuzzy grammars with type 3 rules only. It is shown that there exist n-fold fuzzy grammars which realize "union", "intersection", "concatenation", and "Kleene closure" of fuzzy languages characterized by n-fold type 3 fuzzy grammars. And fuzzy languages defined by n-fold type 3 fuzzy grammars can be characterized by (n-1)-fold type 3 fuzzy grammars and vice versa.

Definition 4.8. An n-fold type 3 fuzzy grammar (abbreviated n-FG) is an n-fold fuzzy grammar (V_N, V_T, P, S, J, $\{f_0, f_1, \ldots, f_n\}$) in which each rule in P is of the form:

$$(r) \quad A \rightarrow aB \qquad \text{or} \qquad (r) \quad A \rightarrow a$$

where $r \in J$, $A, B \in V_N$, and $a \in V_T$.

Similarly, we can define an $\underbrace{\text{n-fold type 3 fuzzy grammar}}_{\text{optimistic}}$ (n-OFG for short).

Now, we shall prepare for the following notations in order to put restrictions on the domains of 0,1,...,n-fold fuzzy functions without loss of generality.

Let J_{AB} be a set of all the labels such that the nonterminal symbols of the left and the right hand sides of the nonterminal rule in P are A and B $(\in V_N)$, respectively. And let J_A be a set of all the labels such that the left hand side of the rule in P is A $(\in V_N)$. Moreover, for non-empty i label sets $J_{A_0A_1}, J_{A_1A_2}, \ldots\ldots, J_{A_{i-1}A_i}$, let us define the set of label strings of length i as follows.

(4.24)

$$J_{A_0A_1\cdots\cdot A_i} = J_{A_0A_1}J_{A_1A_2}\cdots\cdots J_{A_{i-1}A_i}$$

$$= \left\{ r_1r_2\cdots\cdot r_i \mid r_k \in J_{A_{k-1}A_k},\ k = 1, 2, \ldots, i \right\},$$

which shows that, after the rule r_k $(k=1,2,\dots,i-1)$ was used in a derivation, the next rule r_{k+1} is applicable.

Now, we shall define the i-fold fuzzy function f_i, $i=0,1,\dots,n$, using $J_{A_0A_1\dots A_i}$ and J_{A_i} defined above.

(i) As to the 0-fold fuzzy function f_0, let $A_i = S$ in J_{A_i} and define

$$f_0(r) \in [0, 1] \tag{4.25}$$

for each $r \in J_S$.

(ii) As to the i-fold fuzzy function f_i, $i=1,2,\dots,n-1$, let $A_0 = S$ in $J_{A_0A_1\dots A_i}$ and define

$$f_i(r_1,r_2,\dots,r_i;r_{i+1}) \in [0, 1] \tag{4.26}$$

for each $r_1r_2\dots r_i \in J_{SA_1\dots A_i}$ and $r_{i+1} \in J_{A_i}$.

This is carried out for all the non-empty sets $J_{SA_0'\dots A_i'}$ and $J_{A_i'}$.

(iii) As to the n-fold fuzzy function f_n, define

$$f_n(r_1, r_2, \ldots, r_n; r_{n+1}) \in [0, 1] \tag{4.27}$$

for each $r_1 r_2 \cdots r_n \in J_{A_0 A_1 \cdots A_n}$ and $r_{n+1} \in J_{A_n}$.

This is carried out for all the non-empty sets $J_{A_0' A_1' \cdots A_n'}$ and $J_{A_n'}$.

<u>Theorem 4.3</u>. For two n-fold type 3 fuzzy grammars $n\text{-FG}^{(1)}$ and $n\text{-FG}^{(2)}$, let $L(n\text{-FG}^{(1)})$ and $L(n\text{-FG}^{(2)})$ be the fuzzy languages by $n\text{-FG}^{(1)}$ and $n\text{-FG}^{(2)}$, respectively. Then, in the fuzzy sense, there exists an n-FG such that

$$L(n\text{-FG}) = L(n\text{-FG}^{(1)}) \cup L(n\text{-FG}^{(2)}). \tag{4.28}$$

<u>Proof</u>: Let $n\text{-FG}^{(1)}$ and $n\text{-FG}^{(2)}$ be as follows.

$$n\text{-FG}^{(1)} = (V_N^{(1)}, V_T^{(1)}, P^{(1)}, S_1, J^{(1)}, \{f_0, f_1, \ldots, f_n\})$$

$$n\text{-FG}^{(2)} = (V_N^{(2)}, V_T^{(2)}, P^{(2)}, S_2, J^{(2)}, \{g_0, g_1, \ldots, g_n\})$$

where it is assumed that $V_N^{(1)} \cap V_N^{(2)} = \phi$ and $J^{(1)} \cap J^{(2)} = \phi$.

Now, consider an n-FG, that is,

$$\text{n-FG} = (V_N, V_T, P, S, J, \{h_0, h_1, \cdots, h_n\})$$

where V_N, V_T, P, J, and $h_0, h_1, \cdots, h_n$ are given as follows.

$$V_N = V_N^{(1)} \cup V_N^{(2)} \cup \{S\},$$

$$V_T = V_T^{(1)} \cup V_T^{(2)},$$

$$P = P^{(1)} \cup P^{(2)} \cup P_I \cup P_{II} ,$$

$$J = J^{(1)} \cup J^{(2)} \cup J_I \cup J_{II} ,$$

where $J^{(1)} \cap J^{(2)} \cap J_I \cap J_{II} = \phi$.

P_I, J_I, P_{II}, and J_{II} are defined as follows:

[I] : For each initial rule (r) $S_1 \to w$ in $P^{(1)}$, where $r \in J_{S_1}^{(1)}$, we construct a new initial rule of P such as $(\tau_1(r))$ $S \to w$, where $\tau_1(r)$ is a different new label, and S is a new initial symbol of n-FG . Let P_I and J_I be the set of all new initial rules obtained above and the set of labels corresponding to these initial rules, respectively. Formally,

$$P_I = \{ (\tau_1(r)) \ S \to w \mid (r) \ S_1 \to w \in P^{(1)}, \ r \in J_{S_1}^{(1)} \},$$

$$J_I = \{ \tau_1(r) \mid r \in J_{S_1}^{(1)} \}.$$

[II] : We can get P_{II} and J_{II} for the initial rules in $P^{(2)}$ in a similar way as [I]. That is,

$$P_{II} = \{ (\tau_2(r)) \ S \to w \mid (r) \ S_2 \to w \in P^{(2)}, \ r \in J_{S_2}^{(2)} \},$$

$$J_{II} = \{ \tau_2(r) \mid r \in J_{S_2}^{(2)} \}.$$

Finally, the i-fold fuzzy function h_i, i=0,1,..,n, of n-FG is defined by the following.

(a) 0-fold fuzzy function h_0 is given as

$$h_0(p) = \begin{cases} f_0(r) & \cdots\cdots\cdots \quad \text{if} \quad p = \tau_1(r) \in J_I, \\ \\ g_0(r) & \cdots\cdots\cdots \quad \text{if} \quad p = \tau_2(r) \in J_{II}, \end{cases}$$

where h_0 is not defined for other $p \in J$.

(b) i-fold fuzzy function h_i, $i=1,2,\dots,n-1$, is given as follows:

$$h_i(p_1,p_2,\dots,p_i;p_{i+1})$$

$$= \begin{cases} f_i(r_1,p_2,\dots,p_i;p_{i+1}) \\ \qquad \cdots\cdots \text{ if } p_1 = \tau_1(r_1) \in J_I \text{ and } p_2,\dots,p_{i+1} \in J^{(1)}, \\ g_i(r_1,p_2,\dots,p_i;p_{i+1}) \\ \qquad \cdots\cdots \text{ if } p_1 = \tau_2(r_1) \in J_{II} \text{ and } p_2,\dots,p_{i+1} \in J^{(2)}, \end{cases}$$

where h_i is not defined for other $p_1,p_2,\dots,p_{i+1} \in J$.

(c) n-fold fuzzy function h_n is

$$h_n(p_1,p_2,\dots,p_n;p_{n+1})$$

$$
= \begin{cases}
f_n(r_1,p_2,\dots\dots,p_n;p_{n+1}) \\
\qquad \dots\dots\dots \text{ if } \; p_1 = \tau_1(r_1) \in J_I \text{ and } p_2,\dots,p_{n+1} \in J^{(1)} \\
f_n(p_1,p_2,\dots\dots,p_n;p_{n+1}) \quad \dots\dots\dots \text{ if } p_1,p_2,\dots,p_{n+1} \in J^{(1)} \\
g_n(r_1,p_2,\dots\dots,p_n;p_{n+1}) \\
\qquad \dots\dots \text{ if } \; p_1 = \tau_2(r_1) \in J_{II} \text{ and } p_2,\dots,p_{n+1} \in J^{(2)} \\
g_n(p_1,p_2,\dots\dots,p_n;p_{n+1}) \quad \dots\dots \text{ if } \; p_1,p_2,\dots,p_{n+1} \in J^{(2)}
\end{cases}
$$

where h_n is not defined for other $p_1,p_2,\dots,p_{n+1} \in J$.

Example 4.4. Let us consider two $2\text{-}FG^{(1)}$ and $2\text{-}FG^{(2)}$ such that

$$2\text{-}FG^{(1)} = (\{S_1, A\} , \{a, b\} , P^{(1)}, S_1, \{1,2,3\} ,\{f_0,f_1,f_2\})$$

where $P^{(1)}$ consists of the followings.

(1) $S_1 \to aA$, (2) $A \to bA$

(3) $A \to a$

and 0, 1, 2-fold fuzzy functions f_0, f_1, f_2 are

$f_0(1) = 0.1$,

$f_1(1;2) = 0.2$, $f_1(1;3) = 0.3$

$f_2(1,2;2) = 0.4$, $f_2(1,2;3) = 0.5$

$f_2(2,2;2) = 0.6$, $f_2(2,2;3) = 0.7$

And

$$2\text{-}FG^{(2)} = (\{S_2\}, \{a, b\}, P^{(2)}, S_2, \{4, 5\}, \{g_0, g_1, g_2\})$$

where $P^{(2)}$ consists of

(4) $S_2 \to bS_2$, (5) $S_2 \to a$

and 0, 1, 2-fold fuzzy functions g_0, g_1, g_2 are

$g_0(4) = 0.9$, $g_0(5) = 0.8$

$g_1(4;4) = 0.7$, $g_1(4;5) = 0.6$

$g_2(4,4;4) = 0.5$, $g_2(4,4;5) = 0.4$

Then the 2-FG which realizes

$$L(2\text{-}FG) = L(2\text{-}FG^{(1)}) \cup L(2\text{-}FG^{(2)})$$

is given as follows.

2-FG

$$= (\{S, S_1, S_2, A\}, \{a, b\}, P, S, \{1,2,3,4,5,6,7,8\}, \{h_0, h_1, h_2\})$$

where P consists of the followings.

(1) $S_1 \rightarrow aA$, (5) $S_2 \rightarrow a$

(2) $A \rightarrow bA$, (6) $S \rightarrow aA$

(3) $A \rightarrow a$, (7) $S \rightarrow bS_2$

(4) $S_2 \rightarrow bS_2$, (8) $S \rightarrow a$

where $\tau_1(1) = 6$, $\tau_2(4) = 7$, and $\tau_2(5) = 8$.

And 0, 1, 2-fold fuzzy functions h_0, h_1, h_2 are

$$h_0(6) = f_0(1) = 0.1$$

$$h_0(7) = g_0(4) = 0.9$$

$$h_0(8) = g_0(5) = 0.8$$

$$h_1(6;2) = f_1(1;2) = 0.2$$

$$h_1(6;3) = f_1(1;3) = 0.3$$

$$h_1(7;4) = g_1(4;4) = 0.7$$

$$h_1(7;5) = g_1(4;5) = 0.6$$

$$h_2(6,2;2) = f_2(1,2;2) = 0.4$$

$$h_2(6,2;3) = f_2(1,2;3) = 0.5$$

$$h_2(7,4;4) = g_2(4,4;4) = 0.5$$

$$h_2(7,4;5) = g_2(4,4;5) = 0.4$$

$$h_2(2,2;2) = f_2(2,2;2) = 0.6$$

$$h_2(2,2;3) = f_2(2,2;3) = 0.7$$

$$h_2(4,4;4) = g_2(4,4;4) = 0.5$$

$$h_2(4,4;5) = g_2(4,4;5) = 0.4$$

<u>Theorem 4.4.</u> For two fuzzy languages $L(n\text{-FG}^{(1)})$ and $L(n\text{-FG}^{(2)})$ by $n\text{-FG}^{(1)}$ and $n\text{-FG}^{(2)}$, respectively, there exists an $n\text{-FG}$ such that

$$L(n\text{-FG}) = L(n\text{-FG}^{(1)}) \cap L(n\text{-FG}^{(2)}) \tag{4.29}$$

<u>Proof</u>: For two n-FG$^{(1)}$ and n-FG$^{(2)}$, that is,

$$\text{n-FG}^{(1)} = (V_N^{(1)},\ V_T,\ P^{(1)},\ S_1,\ J^{(1)},\ \{f_0,f_1,\ldots,f_n\}),$$

$$\text{n-FG}^{(2)} = (V_N^{(2)},\ V_T,\ P^{(2)},\ S_2,\ J^{(2)},\ \{g_0,g_1,\ldots,g_n\}),$$

let us define an n-FG as follows:

$$\text{n-FG} = (V_N,\ V_T,\ P,\ S,\ J,\ \{h_0,h_1,\ldots,h_n\})$$

where the rules in P are given by the following:

[I] : For two nonterminal rules in $P^{(1)}$ and $P^{(2)}$ such that the terminal symbols of the right hand side of these two rules are common, say, $a \in V_T$. That is, for

$$(r)\quad A_1 \to \underset{\sim}{a}A_2 \in P^{(1)} \qquad \text{and} \qquad (p)\quad B_1 \to \underset{\sim}{a}B_2 \in P^{(2)},$$

let a new nonterminal rule in P be defined as follows.

$$(r,p)\quad \langle A_1,B_1\rangle \to a\langle A_2,B_2\rangle .$$

[II] : For the two terminal rules in $P^{(1)}$ and $P^{(2)}$ such as

$$(r) \quad A_1 \rightarrow \underset{\sim}{a} \in P^{(1)} \qquad \text{and} \qquad (p) \quad B_1 \rightarrow \underset{\sim}{a} \in P^{(2)},$$

where the terminal symbols are common, define a new terminal rule in P as follows:

$$(r,p) \qquad \langle A_1, B_1 \rangle \rightarrow a \, .$$

Let P_a be the set of new rules obtained in [I] and [II] and J_a be the set of labels corresponding to these new rules. Then P and J in n-FG are given as follows.

$$P = \bigcup_{a \in V_T} P_a \, , \text{ and } \qquad J = \bigcup_{a \in V_T} J_a \, .$$

Moreover, V_N is the set of pairs $\langle A_1, B_1 \rangle$ obtained in [I] and [II], and $S = \langle S_1, S_2 \rangle$. Clearly, we have

$$V_N \subseteq V_N^{(1)} \times V_N^{(2)} \qquad \text{and} \qquad J \subseteq J^{(1)} \times J^{(2)}.$$

Now, we shall obtain i-fold fuzzy function h_i.

(a) 0-fold fuzzy function h_0 is given as

$$h_0((r,p)) = \min\left[f_0(r),\ g_0(p)\right] \ \cdots\cdots \ \text{if } (r,p) = S .$$

(b) i-fold fuzzy function h_i, $i=1,2,\ldots,n$, is

$$h_i((r_1,p_1),(r_2,p_2),\ldots,(r_i,p_i);(r_{i+1},p_{i+1}))$$

$$= \min\left[f_i(r_1,r_2,\ldots,r_i;r_{i+1}),\ g_i(p_1,p_2,\ldots,p_i;p_{i+1})\right]$$

where $(r_k,p_k) \in J$ and $k=1,2,\ldots,i+1$.

<u>Theorem 4.5</u>. For the two fuzzy languages $L(n\text{-}FG^{(1)})$ and $L(n\text{-}FG^{(1)})$ by $n\text{-}FG^{(1)}$ and $n\text{-}FG^{(2)}$, respectively, there exists an n-FG which realizes the concatenation of $L(n\text{-}FG^{(1)})$ and $L(n\text{-}FG^{(2)})$ (see (2.25)).

$$L(n\text{-}FG) = L(n\text{-}FG^{(1)}) \circ L(n\text{-}FG^{(2)}). \tag{4.30}$$

Proof: Let

$$n\text{-}FG^{(1)} = (V_N^{(1)},\ V_T^{(1)},\ P^{(1)},\ S_1,\ J^{(1)}, \{f_0, f_1, \ldots, f_n\})$$

$$n\text{-}FG^{(2)} = (V_N^{(2)},\ V_T^{(2)},\ P^{(2)},\ S_2,\ J^{(2)}, \{g_0, g_1, \ldots, g_n\})$$

where $V_N^{(1)} \cap V_N^{(2)} = \phi$ and $J^{(1)} \cap J^{(2)} = \phi$.

We construct a new n-FG,

$$n\text{-}FG = (V_N,\ V_T,\ P,\ S,\ J, \{h_0, h_1, \ldots, h_n\})$$

where $S = S_1$, $V_N = V_N^{(1)} \cup V_N^{(2)}$, $V_T = V_T^{(1)} \cup V_T^{(2)}$, and $J = J^{(1)} \cup J^{(2)}$.

We introduce the following notations in order to get P.

Let $P_{nr}^{(i)}$ and $P_{tr}^{(i)}$ be the sets of nonterminal rules and terminal rules, respectively, in $P^{(i)}$, $i=1,2$. And let $J_{nr}^{(i)}$ and $J_{tr}^{(i)}$ be the sets of the labels corresponding to the rules in $P_{nr}^{(i)}$ and $P_{tr}^{(i)}$, respectively. Then, clearly, we have $P^{(i)} = P_{nr}^{(i)} \cup P_{tr}^{(i)}$ and $J^{(i)} = J_{nr}^{(i)} \cup J_{tr}^{(i)}$ for each $i=1,2$.

We shall next obtain the rules of n-FG.

For each terminal rule (r) $A \to a$ in $P_{tr}^{(1)}$ and an initial symbol S_2 of n-FG$^{(2)}$, we construct a new rule (r) $A \to aS_2$, where the label is not changed. Let P' be the set of such rules, then P of n-FG is given as

$$P = P_{nr}^{(1)} \cup P' \cup P^{(2)}.$$

Let us obtain i-fold fuzzy function h_i, $i=1,2,\ldots,n$.

(a) 0-fold fuzzy function h_0 is

$$h_0(r) = f_0(r) \quad \ldots\ldots \quad \text{if} \quad r \in J_S \; (= J_{S_1}^{(1)})$$

(b) i-fold fuzzy function h_i, $i=1,2,\ldots,n-1$, is given as follows.

$$h_i(r_1,r_2,\cdots,r_i;r_{i+1})$$

$$= \begin{cases} f_i(r_1,r_2,\cdots,r_i;r_{i+1}) \quad \cdots\cdots \quad \text{if} \quad r_1,r_2,\cdots,r_{i+1} \in J^{(1)} \\ \qquad\qquad\qquad\qquad\qquad (\text{where} \quad r_1 \in J_{S_1}^{(1)}) \\ \\ g_0(r_{i+1}) \quad \cdots\cdots \quad \text{if} \quad r_1,r_2,\cdots,r_i \in J^{(1)} \text{ and } r_{i+1} \in J^{(2)} \\ \qquad\qquad\qquad (\text{where} \quad r_1 \in J_{S_1}^{(1)}, \quad r_{i+1} \in J_{S_2}^{(2)}) \\ \\ g_{i-j}(r_{j+1},\cdots,r_i;r_{i+1}) \\ \quad \cdots\cdots \quad \text{if} \quad r_1,r_2,\cdots,r_j \in J^{(1)} \text{ and } r_{j+1},\cdots,r_{i+1} \in J^{(2)} \\ \qquad\qquad (\text{where} \quad r_1 \in J_{S_1}^{(1)}, \quad r_{j+1} \in J_{S_2}^{(2)}, \quad 1 \leq j < i\ ) \end{cases}$$

where h_i are not defined for other $r_1,r_2,\cdots,r_{i+1} \in J$.

(c) n-fold fuzzy function h_n is given as follows.

For j $(1 \leqq j < n)$,

$$h_n(r_1,r_2,\cdots,r_n;r_{n+1})$$

$$= \begin{cases} f_n(r_1,r_2,\cdots,r_n;r_{n+1}) \quad \cdots\cdots \quad \text{if } r_1,r_2,\cdots,r_{n+1} \in J^{(1)} \\ g_0(r_{n+1}) \quad \cdots\cdots\cdots \quad \text{if } r_1,r_2,\cdots,r_n \in J^{(1)} \text{ and } r_{n+1} \in J_{S_2}^{(2)} \\ g_{n-j}(r_{j+1},r_{j+2},\cdots,r_n;r_{n+1}) \\ \qquad \cdots\cdots \quad \text{if } r_1,r_2,\cdots,r_j \in J^{(1)} \text{ and } r_{j+1},\cdots,r_{n+1} \in J^{(2)} \\ \qquad\qquad (\text{where } r_{j+1} \in J_{S_2}^{(2)}) \\ g_n(r_1,r_2,\cdots,r_n;r_{n+1}) \quad \cdots\cdots \quad \text{if } r_1,r_2,\cdots,r_{n+1} \in J^{(2)} \end{cases}$$

where h_n is not defined for other $r_1,r_2,\cdots,r_{n+1} \in J$.

<u>Theorem 4.6.</u> For a fuzzy language L(n-FG) by n-FG, there exists an n-FG$'$ which realizes Kleene closure (see (2.27))

$$L(n\text{-}FG') = L(n\text{-}FG)^{*}. \qquad (4.31)$$

<u>Proof:</u> For the n-FG = $(V_N, V_T, P, S, J, \{f_0, f_1, \dots, f_n\})$, let an n-FG$'$ be $(V_N', V_T, P', S', J', \{h_0, h_1, \dots, h_n\})$ where $V_N' = \{S'\} \cup V_N$. P' is obtained from [I], [II], and [III] denoted later. It is assumed that the mappings τ_1, τ_2, and τ_3 in [I], [II], and [III], respectively, are all one to one mappings from labels to new labels, and the obtained new labels are all different from each other.

[I] : For each initial rule (r) $S \rightarrow w$ in P, we construct a new initial rule $(\tau_1(r))$ $S' \rightarrow w$ in P'. Let P_I be the set of such new initial rules and J_I be the set of the labels corresponding to these initial rules. Formally,

$$P_I = \{(\tau_1(r)) \ S' \rightarrow w \mid (r) \ S \rightarrow w \in P, \ r \in J_S\}.$$

$$J_I = \{\tau_1(r)\}.$$

[II] : For each terminal initial rule (r) $S \rightarrow a$ in P, define a new rule $(\tau_2(r))$ $S' \rightarrow aS$. Then, let

$$P_{II} = \{ (\tau_2(r)) \ \ S' \rightarrow aS \mid (r) \ \ S \rightarrow a \in P, \ \ r \in J_S \},$$

$$J_{II} = \{\tau_2(r)\}.$$

[III] : For each terminal rule (r) $A \rightarrow a$ in P, construct a new rule $(\tau_3(r))$ $A \rightarrow aS$ in P' and let

$$P_{III} = \{ (\tau_3(r)) \ \ A \rightarrow aS \mid (r) \ \ A \rightarrow a \in P \},$$

$$J_{III} = \{\tau_3(r)\} .$$

Then, P' and J' in $n\text{-}FG'$ are given from [I] , [II] , and [III] as follows.

$$P' = P \cup P_I \cup P_{II} \cup P_{III} \cup \{ (p) \ \ S' \rightarrow \varepsilon \},$$

$$J' = J \cup J_I \cup J_{II} \cup J_{III} \cup \{ p \},$$

where all labels in J' are different from each other.

It is noted that in any derivation we start with an initial rule in P_I or P_{II} (not in P, P_{III}), and then rules in P or P_{III} are used throughout in the derivation (see Fig. 4.1).

Now, we shall obtain i-fold fuzzy function h_i.

[A] 0-fold fuzzy function h_0 is as follows.

$$h_0(p) = \begin{cases} f_0(r) & \cdots\cdots \text{ if } p = \tau_1(r) \in J_I \\ f_0(r) & \cdots\cdots \text{ if } p = \tau_2(r) \in J_{II} \\ 1 & \cdots\cdots \text{ if } (p)\ S' \rightarrow \varepsilon \end{cases}$$

where h_0 is not defined for other $p \in J'$.

[B] i-fold fuzzy function h_i, i=1,2,...,n-1, is given by the followings. Note that let h_i, i=1,2,...,n-1, be $h_i(p_1, p_2, \ldots, p_i; p_{i+1})$, then $p_1 \in J_I \cup J_{II}$ and $p_2, \ldots, p_{i+1} \in J_{III} \cup J$ from Fig. 4.1.

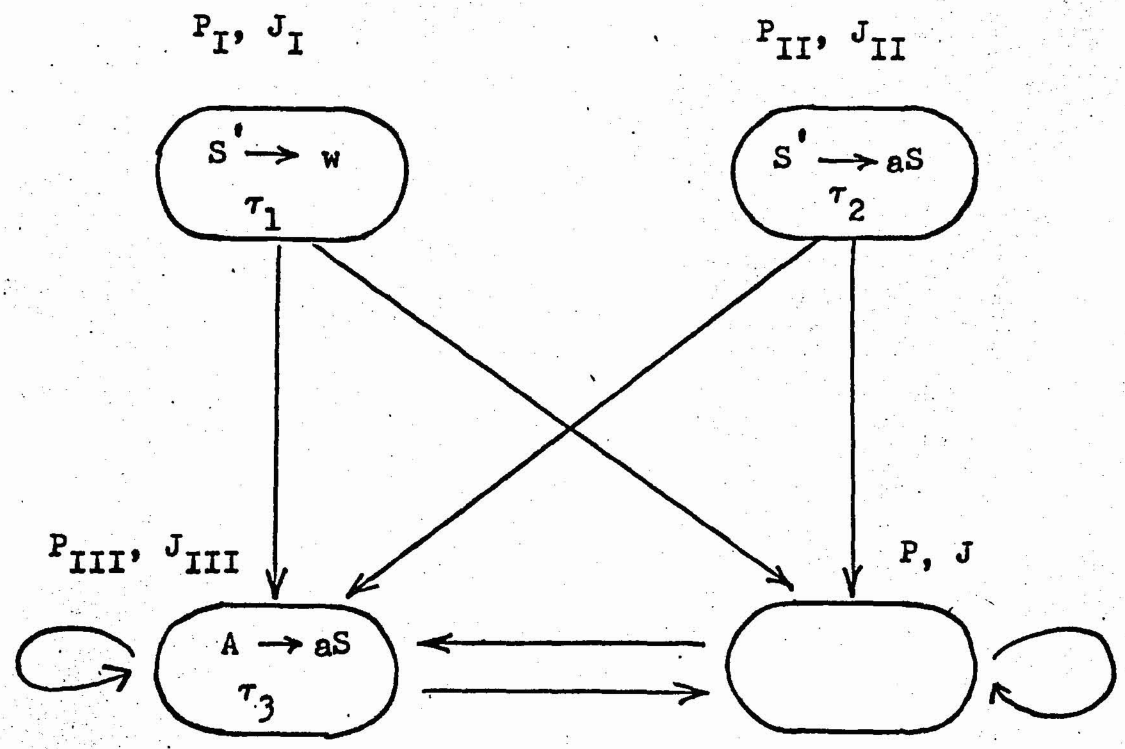

Fig. 4.1. Derivation flows of rules in n-FG'

(b-1) The cases of $p_2, \cdots, p_i \in J$ and $p_{i+1} \in J \cup J_{III}$:

2 cases arise.

((b-1-1)) When $p_1 = \tau_1(r_1) \in J_I$;

$$h_i(p_1, p_2, \cdots, p_i; p_{i+1})$$

$$= \begin{cases} f_i(r_1, p_2, \cdots, p_i; p_{i+1}) & \cdots \quad \text{if} \quad p_{i+1} \in J, \\ f_i(r_1, p_2, \cdots, p_i; r_{i+1}) & \cdots \quad \text{if} \quad p_{i+1} = \tau_3(r_{i+1}) \in J_{III}. \end{cases}$$

((b-1-2)) When $p_1 = \tau_2(r_1) \in J_{II}$;

$$h_i(p_1, p_2, \cdots, p_i; p_{i+1})$$

$$= \begin{cases} f_{i-1}(p_2, \cdots, p_i; p_{i+1}) & \cdots \quad \text{if} \quad p_{i+1} \in J, \\ f_{i-1}(p_2, \cdots, p_i; r_{i+1}) & \cdots \quad \text{if} \quad p_{i+1} = \tau_3(r_{i+1}) \in J_{III}. \end{cases}$$

(b-2) The case of $p_1 \in J_{III}$:

$$h_i(p_1,\dots,p_i;p_{i+1})$$

$$= \begin{cases} f_0(p_{i+1}) & \cdots \quad \text{if} \quad p_{i+1} \in J, \\ f_0(r_{i+1}) & \cdots \quad \text{if} \quad p_{i+1} = \tau_3(r_{i+1}) \in J_{III}, \end{cases}$$

where $p_1 \in J_I \cup J_{II}$ and $p_2,\dots,p_{i-1} \in J \cup J_{III}$.

(b-3) The case that there exists j $(2 \leqq j \leqq i-1)$, and $p_j \in J_{III}$, $p_{j+1},\dots,p_i \in J$:

$$h_i(p_1,\dots,p_j,p_{j+1},\dots,p_i;p_{i+1})$$

$$= \begin{cases} f_{i-j}(p_{j+1},\dots,p_i;p_{i+1}) & \cdots \quad \text{if} \quad p_{i+1} \in J \\ f_{i-j}(p_{i+1},\dots,p_i;r_{i+1}) & \cdots \quad \text{if} \quad p_{i+1} = \tau_3(r_{i+1}) \in J_{III}, \end{cases}$$

where $p_1 \in J_I \cup J_{II}$ and $p_2,\dots,p_{j-1} \in J \cup J_{III}$.

Finally, we shall give n-fold fuzzy function h_n .

[C] Let h_n be $h_n(p_1,p_2,\dots,p_n;p_{n+1})$, then $p_1 \in J'$, and $p_2,\dots,p_{n+1} \in J_{III} \cup J$.

(c-1) The case of $p_2,\dots,p_i \in J$ and $p_{i+1} \in J \cup J_{III}$:

4 cases arise.

((c-1-1)) When $p_1 \in J_I$;

Let $p_1 = \tau_1(r_1)$, then

$$h_n(p_1,p_2,\dots,p_n;p_{n+1})$$

$$\begin{cases} f_n(r_1,p_2,\dots,p_n;p_{n+1}) & \cdots\cdots \text{ if } p_{n+1} \in J, \\ f_n(r_1,p_2,\dots,p_n;r_{n+1}) & \cdots \text{ if } p_{n+1} = \tau_3(r_{n+1}) \in J_{III} . \end{cases}$$

((c-1-2)) When $p_1 \in J_{II}$;

$$h_n(p_1,p_2,\dots,p_n;p_{n+1})$$

$$\begin{cases} f_{n-1}(p_2,p_3,\dots,p_n;p_{n+1}) & \cdots\cdots \text{ if } p_{n+1} \in J , \\ f_{n-1}(p_2,p_3,\dots,p_n;r_{n+1}) & \cdots \text{ if } p_{n+1} = \tau_3(r_{n+1}) \in J_{III} . \end{cases}$$

((c-1-3)) When $p_1 \in J_{III}$: See (c-3) defined later.

((c-1-4)) When $p_1 \in J$;

$$h_n(p_1,p_2,\dots,p_n;p_{n+1})$$

$$= \begin{cases} f_n(p_1,p_2,\dots,p_n;p_{n+1}) & \cdots \quad \text{if} \quad p_{n+1} \in J, \\ f_n(p_1,p_2,\dots,p_n;r_{n+1}) & \text{if} \quad p_{n+1} = \tau_3(r_{n+1}) \in J_{III}. \end{cases}$$

(c-2) The case of $p_n \in J_{III}$:

$$h_n(p_1,p_2,\dots,p_n;p_{n+1})$$

$$= \begin{cases} f_0(p_{n+1}) & \cdots \quad \text{if } p_{n+1} \in J, \\ f_0(r_{n+1}) & \cdots \quad \text{if} \quad p_{n+1} = \tau_3(r_{n+1}) \in J_{III}, \end{cases}$$

where $p_1 \in J'$ and $p_2,\dots,p_{n-1} \in J \cup J_{III}$.

(c-3) The case that there exists some j $(1 \leq j \leq n-1)$, and $p_j \in J_{III}$, $p_{j+1},\dots,p_n \in J$:

$$h_n(p_1,\dots,p_j,p_{j+1},\dots,p_n;p_{n+1})$$

$$= \begin{cases} f_{n-j}(p_{j+1},\dots,p_n;p_{n+1}) & \dots \text{ if } p_{n+1} \in J, \\ f_{n-j}(p_{j+1},\dots,p_n;r_{n+1}) & \dots \text{ if } p_{n+1} = \tau_3(r_{n+1}) \in J_{III}, \end{cases}$$

where $p_1 \in J'$ and $p_2,\dots,p_{j-1} \in J \cup J_{III}$.

This concludes the way how to obtain h_i, $i=0,1,\dots,n$.

Theorem 4.7. For a fuzzy language L(n-FG) by n-FG, there exists an n-fold type 3 optimistic fuzzy grammar n-OFG which realizes the complement of L(n-FG).

$$L(n\text{-OFG}) = \overline{L(n\text{-FG})}. \qquad (4.32)$$

Proof: For an n-FG = $(V_N, V_T, P, S, J, \{f_0,f_1,\dots,f_n\})$, let n-OFG be $(V_N, V_T, P'', S, J', \{h_0,h_1,\dots,h_n\})$. The rules which generate V_T^* , where $V_T = \{a_1,a_2,\dots,a_m\}$, are given as follows.

$$
\begin{array}{llll}
(p_1) & S \rightarrow a_1S\ , & (p_{m+1}) & S \rightarrow a_1 \\
(p_2) & S \rightarrow a_2S\ , & (p_{m+2}) & S \rightarrow a_2 \\
 & \vdots & & \vdots \\
(p_m) & S \rightarrow a_mS\ , & (p_{2m}) & S \rightarrow a_m
\end{array}
$$

Let p_I be the set of such rules and let J_I be the set of labels corresponding to these rules, that is, $J_I = \{p_1, p_2, \cdots, p_{2m}\}$. Then the rule set P' and label set J' in n-OFG are given as follows.

$$P' = P \cup P_I \qquad \text{and} \qquad J' = J \cup J_I$$

where $J \cap J_I = \phi$.

Let us give i-fold fuzzy function h_i, $i=0,1,\cdots,n$.

(a) 0-fold fuzzy function h_0 is

$$
h_0(p_i') = \begin{cases} 1 - f_0(p_i') & \cdots\cdots \text{ if } p_i' \in J\ , \\ 1 & \cdots\cdots \text{ if } p_i' \in J_I\ . \end{cases}
$$

(b) i-fold fuzzy function h_i, $i=1,2,\cdots,n$, is

$$h_i(p_1,p_2,\cdots,p_i;p_{i+1})$$

$$= \begin{cases} 1 - f_i(p_1,\cdots,p_i;p_{i+1}) & \cdots \text{ if } p_1,p_2,\cdots,p_{i+1} \in J, \\ 1 & \cdots \text{ otherwise.} \end{cases}$$

It can be proved from the fact, in general,

$$\min \max \left[1-\mu_1,\ 1-\mu_2,\ \cdots\ ,1-\mu_k \right]$$

$$= 1- \max \min \left[\mu_1,\ \mu_2,\ \cdots,\mu_k \right].$$

Moreover, it is noted that, if there exists at least one element equal to 1 among the grades $\mu_1,\mu_2,\cdots,\mu_m$, then we have

$$\max \left[\mu_1,\mu_2,\cdots\cdots,\mu_m\right] = 1 .$$

Next, we shall show that $n(\geqq 1)$-FG can be transformed to $(n+1)$-FG, and $n(\geqq 2)$-FG to $(n-1)$-FG.

__Theorem 4.8.__ For an n-FG, $n \geqq 1$, there exists an $(n+1)$-FG such that

$$L((n+1)\text{-FG}) = L(n\text{-FG}). \tag{4.33}$$

__Proof:__ For n-FG $= (V_N, V_T, P, S, J, \{f_0, f_1, \ldots, f_n\})$, let $(n+1)$-FG be $(V_N, V_T, P, S, J, \{h_0, h_1, \ldots, h_n, h_{n+1}\})$, where i-fold fuzzy function h_i, $i=0,1,\ldots,n+1$, are as follows,

(a) The case of $0 \leqq i \leqq n$; h_i is

$$h_i = f_i, \quad i=0,1,\ldots,n.$$

(b) The case of $i=n+1$; h_{n+1} is given by the following.

Let n-fold fuzzy function f_n of n-FG be $f_n(r_1, r_2, \ldots, r_n; r_{n+1})$ and its label string $r_1 r_2 \cdots r_n$ be an element of $J_{A_0 A_1 \cdots A_n}$ (see (4.24)). Still more, let $J(A_0)$ be the set of labels such that the nonterminal symbol of the right-hand side of the corresponding rule in P is $A_0 \in V_N$. Then, h_{n+1} of $(n+1)$-FG is as follows. Let

$$h_{n+1}(r, r_1, \ldots, r_n; r_{n+1})$$

$$= f_n(r_1, \ldots, r_n; r_{n+1})$$

for every $r \in J(A_0)$.

Theorem 4.9. For $n(\geq 2)$-FG, there exists an $(n-1)$-FG such that

$$L((n-1)\text{-FG}) = L(n\text{-FG}). \qquad (4.34)$$

Proof: For $n\text{-FG} = (V_N, V_T, P, S, J, \{f_0, f_1, \ldots, f_n\})$, let $(n-1)$-FG be $(V_N', V_T, P', S', J', \{h_0, h_1, \ldots, h_{n-1}\})$ where $V_N' = \{\langle 0\rangle\} \cup \{\langle r\rangle \mid r \in J\}$ and $S' = \langle 0\rangle$. P' is obtained by the followings.

(1) For each initial rule in P, that is,

$$(r) \quad S \rightarrow aA \qquad \text{and} \qquad (r) \quad S \rightarrow a,$$

let us construct new initial rules in P' as follows.

$$(0,r) \quad \langle 0 \rangle \longrightarrow a \langle r \rangle$$

and
$$(0,r) \quad \langle 0 \rangle \longrightarrow a$$

(2) For two nonterminal rules in P such that the non-terminal symbol of the right hand side of the one rule is coincident with that of the left hand side of the other rule, that is, for the following two rules

$$(r_1) \quad A_1 \longrightarrow aA_2 \quad , \qquad\qquad (r_2) \quad A_2 \longrightarrow a'A_3 \, ,$$

define a new nonterminal rule such as

$$(r_1, r_2) \quad \langle r_1 \rangle \longrightarrow a' \langle r_2 \rangle .$$

(3) For a nonterminal rule and a terminal rule in P such that the nonterminal symbol of the right hand of the nonterminal rule is coincident with that of the left hand of the terminal rule, that is, for the following two rules

$$(r_1) \quad A_1 \longrightarrow a_1 A_2 \quad , \qquad\qquad (r_2) \quad A_2 \longrightarrow a_2 \; ,$$

let us construct a new terminal rule such as

$$(r_1,r_2) \quad \langle r_1\rangle \longrightarrow a_2 .$$

Let P' be the set of new rules which were obtained in (1), (2) and (3), and let J' be the set of labels corresponding to these rules.

i-fold fuzzy function h_i $(i=0,1,\cdots,n-1)$ is given as follows.

(a) o-fold fuzzy function h_0 is

$$h_0((0,r)) = f_0(r) , \qquad r \in J_S.$$

(b) i-fold fuzzy function h_i, $i=1,2,\cdots,n-2$, is

$$h_i((0,r_1),(r_1,r_2),\cdots,(r_{i-1},r_i);(r_i,r_{i+1}))$$

$$= f_i(r_1,r_2,\cdots,r_n;r_{n+1}).$$

(c) (n-1)-fold fuzzy function h_{n-1} is

$$h_{n-1}((0,r_1), \ldots\ldots , (r_{n-2},r_{n-1});(r_{n-1},r_n))$$

$$= f_{n-1}(r_1,r_2,\ldots\ldots,r_{n-1};r_n),$$

and

$$h_{n-1}((r_1,r_2), \ldots\ldots , (r_{n-1},r_n);(r_n,r_{n+1}))$$

$$= f_n(r_1,r_2,\ldots\ldots,r_n;r_{n+1}).$$

From the above two theorems 4.8 and 4.9 , we can transform $n(\geq 2)$-FG into 1-FG and, conversely, 1-FG into $n(\geq 2)$-FG.

Next, we shall show that 1-FG can be transformed into 0-FG (or fuzzy automaton) and also 0-FG can be transformed into 1-FG.

<u>Theorem 4.10</u>. Given an 1-FG, there exists a fuzzy automaton A such that

$$L(A) = L(1\text{-FG}) \tag{4.35}$$

and vice versa.

<u>Proof:</u> ($\Rightarrow$) Let $1\text{-FG} = (V_N, V_T, P, \sigma, J, \{f_0, f_1\})$, then a fuzzy automaton $A = (S, s_1, \{F(a) \mid a \in V_T\}, G)$ is defined as follows.

The set of states is $S = \{\langle 0 \rangle\} \cup \{\langle r \rangle \mid r \in J\}$, the initial state s_1 is $\langle 0 \rangle$, and the set of final states G is $\{\langle r \rangle \mid r \in J_{tr}\}$, where J_{tr} is a set of all labels whose rules in 1-FG are terminal rules, i.e.,

$$J_{tr} = \{r \mid (r) \quad A \rightarrow a\}.$$

Before we obtain the fuzzy transition matrices $F(a)$ with $a \in V_T$, let us introduce a label set J^a. For each $a \in V_T$, define J^a as the set of all labels such that the terminal symbol which appears on the right-hand side of the rule is a $(\in V_T)$. More precisely, we define

$$J^a = \{r \mid (r) \quad A \rightarrow \underline{a}B\} \cup \{r \mid (r) \quad A \rightarrow \underline{a}\}$$

for each $a \in V_T$. Clearly we have that, for $a, b \in V_T$

$$\text{(i)} \qquad J^a \cap J^b = \phi \qquad \ldots\ldots \text{ if } a \neq b,$$

$$\text{(ii)} \qquad \bigcup_{a \in V_T} J^a = J .$$

Now, we shall obtain the fuzzy transition matrices $F(a)$, $a \in V_T$, of a fuzzy automaton A by the followings.

Let $F(a) = \| f_A(\langle r \rangle , a , \langle p \rangle) \|$, where $\langle r \rangle$, $\langle p \rangle \in S$, and $r, p \in J \cup \{0\}$.

4 cases arise:

(1) For each rule of the form (r) $A \longrightarrow aB$ with 1-fold fuzzy function $f_1(r;p)$, where $a \in V_T$ is given, $A, B \in V_N$ are arbitrary, and $p \in J_B$, let

$$f_A(\langle r \rangle , a , \langle p \rangle) = \begin{cases} f_1(r;p) & \ldots\ldots \text{ if } p \in J_B \cap J^a, \\ 0 & \ldots\ldots \text{ otherwise.} \end{cases}$$

(2) For each terminal rule (r) $A \longrightarrow a$, let

$$f_A(\langle r\rangle, a, \langle p\rangle) = 0$$

for all $p \in J$.

(3) For each 0-fold fuzzy function $f_0(p)$, $p \in J_\sigma$, let $\langle r\rangle = \langle 0\rangle$ and

$$f_A(\langle 0\rangle, a, \langle p\rangle) = \begin{cases} f_0(p) & \dots \text{ if } p \in J_\sigma \cap J^a, \\ 0 & \dots \text{ otherwise}. \end{cases}$$

(4) When $\langle p\rangle = \langle 0\rangle$, let

$$f_A(\langle r\rangle, a, \langle 0\rangle) = 0$$

for all $r \in J$.

($\Leftarrow$) For a fuzzy automaton $A = (S, s_1, \{F(a_k) \mid a_k \in V_T\}, G)$, let 1-FG be $(V_N, V_T, P, \sigma, J, \{f_0, f_1\})$, where $V_N = \{\langle s_i \rangle \mid s_i \in S\}$, and $\sigma = \langle s_1 \rangle$. The rules of 1-FG are given as follows. To the element $f_A(s_i, a_k, s_j)$ of the fuzzy transition matrix $F(a_k)$, correspond the rule such as $\langle s_i \rangle \to a_k \langle s_j \rangle$, where $1 \leq i, j \leq n$, $1 \leqq k \leqq h$, $n = \#(S)$, and $h = \#(V_T)$. Then the number of the corresponding rules, that is, the number of their labels is $n^2 h$.

Finally, the terminal rules are given as follows.

For each rule $\langle s_i \rangle \to a_k \langle s_j \rangle$ obtained above, if $s_j \in G$, that is, s_j is a final state, then we give a terminal rule as $\langle s_i \rangle \to a_k$. Thus, the number of labels of the terminal rules is hnq, where $q = \#(G)$.

Hence, the total number of labels, i.e., $\#(J)$ is $hn^2 + hnq$ $(=t)$. We can appropriately attach the labels to the rules obtained above without overlapping. In this paper, the label r of the rule whose form is (r) $\langle s_i \rangle \to a_k \langle s_j \rangle$ is in $\{1, 2, \cdots, hn^2\}$, and the label r of the terminal rule (r) $\langle s_i \rangle \to a_k$ is in $\{hn^2+1, \cdots, t\}$.

Next, we shall obtain 0,1-fold fuzzy function f_0 and f_1.

[I] 0-fold fuzzy function f_0 is as follows.

(i) If the initial rule is of the form $(r)\ \langle s_1\rangle \rightarrow a\langle s\rangle$ where $r \in J_{\langle s_1\rangle}$, and s_1 is an initial state of fuzzy automaton A, then

$$f_0(r) = f_A(s_1,\ a,\ s).$$

(ii) If the initial rule is of the form $(r)\ \langle s_1\rangle \rightarrow a$, and let this rule $\langle s_1\rangle \rightarrow a$ be obtained from the rule $\langle s_1\rangle \rightarrow a\langle s_f\rangle$ for some $s_f \in G$, then

$$f_0(r) = f_A(s_1,\ a,\ s_f).$$

[II] 1-fold fuzzy function $f_1(r;p)$ is given as follows, where r-th rule is of the form $(r)\ \langle s_1\rangle \rightarrow a_k\langle s_j\rangle$, $1 \leqq r \leqq hn^2$, and $1 \leqq p \leqq t$.

(i) In the case of $1 \leqq p \leqq hn^2$:

If the p-th rule is $(p)\ \langle s\rangle \rightarrow a\langle s'\rangle$, then

$$f_1(r;p) = \begin{cases} f_A(s,\ a,\ s') & \cdots\cdots \text{ if } s_j = s, \\ \\ 0 & \cdots\cdots \text{ if } s_j \neq s. \end{cases}$$

(ii) In the case of $hn^2+1 \leq p \leq t$:

Let the terminal rule (p) $\langle s \rangle \rightarrow a$ be obtained from the rule $\langle s \rangle \rightarrow a \langle s_f \rangle$ for some $s_f \in G$, then

$$f_1(r;p) = \begin{cases} f_A(s,\ a,\ s_f) & \cdots\cdots \text{ if } s_j = s, \\ \\ 0 & \cdots\cdots \text{ if } s_j \neq s. \end{cases}$$

Example 4.5. Let 1-FG = $(V_N, V_T, P, \sigma, J, \{f_0, f_1\})$, where $V_N = \{\sigma, A\}$, $V_T = \{a, b\}$, $f_0(1) = 0.9$, $f_0(2) = 0.6$, and the rules with fuzzy vector are

		1	2	3	4	5
(1)	$\sigma \rightarrow aA$	(0	0	.2	.45	.1)
(2)	$\sigma \rightarrow b\sigma$	(.5	.8	0	0	0)
(3)	$A \rightarrow aA$	(0	0	.3	.35	.7)
(4)	$A \rightarrow a\sigma$	(.4	.9	0	0	0)
(5)	$A \rightarrow b$	(1	1	1	1	1)

Then we can construct a fuzzy automaton $A = (S, \langle 0 \rangle, \{F(a), F(b)\}, G)$, where $S = \{\langle 0\rangle, \langle 1\rangle, \langle 2\rangle, \langle 3\rangle, \langle 4\rangle, \langle 5\rangle\}$, $G = \{\langle 5 \rangle\}$, and the fuzzy transition matrices $F(a)$ and $F(b)$ are given as follows:

$$F(a) = \begin{array}{cc} & \begin{array}{cccccc} \langle 0\rangle & \langle 1\rangle & \langle 2\rangle & \langle 3\rangle & \langle 4\rangle & \langle 5\rangle \end{array} \\ \begin{array}{c} \langle 0\rangle \\ \langle 1\rangle \\ \langle 2\rangle \\ \langle 3\rangle \\ \langle 4\rangle \\ \langle 5\rangle \end{array} & \left[\begin{array}{cccccc} & .9 & & & & \\ & & & .2 & .45 & \\ & .5 & & & & \\ & & & .3 & .55 & \\ & .4 & & & & \\ & & & & & \end{array} \right] \end{array}$$

$$F(b) = \begin{array}{cc} & \begin{array}{cccccc} \langle 0\rangle & \langle 1\rangle & \langle 2\rangle & \langle 3\rangle & \langle 4\rangle & \langle 5\rangle \end{array} \\ \begin{array}{c} \langle 0\rangle \\ \langle 1\rangle \\ \langle 2\rangle \\ \langle 3\rangle \\ \langle 4\rangle \\ \langle 5\rangle \end{array} & \left[\begin{array}{cccccc} & & .6 & & & \\ & & & & & .1 \\ & & .8 & & & \\ & & & & & .7 \\ & & .9 & & & \\ & & & & & \end{array} \right] \end{array}$$

Note that the values of the blank portions of $F(a)$ and $F(b)$ are equal to 0 .

Example 4.6. Let $A = (S, s_1, \{F(a) \mid a \in V_T\}, G)$ be a fuzzy automaton, where $S = \{s_1, s_2\}$, $V_T = \{a, b\}$, $G = \{s_1, s_2\}$, and the fuzzy transition matrices are

$$F(a) = \begin{array}{c} \\ s_1 \\ s_2 \end{array} \begin{array}{c} \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{bmatrix} .5 & .4 \\ .3 & .8 \end{bmatrix} \end{array}, \qquad F(b) = \begin{array}{c} \\ s_1 \\ s_2 \end{array} \begin{array}{c} \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{bmatrix} .2 & .7 \\ .9 & .6 \end{bmatrix} \end{array}$$

Define $1\text{-FG} = (V_N, V_T, P, \sigma, J, \{f_0, f_1\})$ as follows. $V_N = \{\langle s_1\rangle \langle s_2\rangle\}$, $V_T = \{a, b\}$, $\sigma = \langle s_1\rangle$, and the 0-fold fuzzy function f_0 is

$$f_0(1) = 0.5, \quad f_0(2) = 0.4, \quad f_0(3) = 0.2, \quad f_0(4) = 0.7,$$

$$f_0(9) = 0.5, \quad f_0(10) = 0.4, \quad f_0(11) = 0.2, \quad f_0(12) = 0.7.$$

And the rules with fuzzy vector are

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
(1) $\langle s_1\rangle \to a\langle s_1\rangle$	.5	.4	.2	.7					.5	.4	.2	.7				
(2) $\langle s_1\rangle \to a\langle s_2\rangle$					.3	.8	.9	.6					.3	.8	.9	.6
(3) $\langle s_1\rangle \to b\langle s_1\rangle$	.5	.4	.2	.7					.5	.4	.2	.7				
(4) $\langle s_1\rangle \to b\langle s_2\rangle$					.3	.8	.9	.6					.3	.8	.9	.6
(5) $\langle s_2\rangle \to a\langle s_1\rangle$	.5	.4	.2	.7					.5	.4	.2	.7				
(6) $\langle s_2\rangle \to a\langle s_2\rangle$					.3	.8	.9	.6					.3	.8	.9	.6
(7) $\langle s_2\rangle \to b\langle s_1\rangle$	.5	.4	.2	.7					.5	.4	.2	.7				
(8) $\langle s_2\rangle \to b\langle s_2\rangle$					.3	.8	.9	.6					.3	.8	.9	.6
(9) $\langle s_1\rangle \to a$								1								
(10) $\langle s_1\rangle \to a$								1								
(11) $\langle s_1\rangle \to b$								1								
(12) $\langle s_1\rangle \to b$								1								
(13) $\langle s_2\rangle \to a$								1								
(14) $\langle s_2\rangle \to a$								1								
(15) $\langle s_2\rangle \to b$								1								
(16) $\langle s_2\rangle \to b$								1								

It is noted that, in the fuzzy vectors from (1) to (8), the values of the blank portions are 0, and the rewriting rule, say, $\langle s_1\rangle \to a$ in (10) is obtained from the rule $\langle s_1\rangle \to a\langle s_2\rangle$ with $s_2 \in G$.

<u>Theorem 4.11</u>. Fuzzy languages L(n-FG) characterized by n-fold type 3 fuzzy grammars n-FG form a distributive lattice.

<u>Proof</u>: It is clear from Theorem 4.9 and the theorm that fuzzy events L(A) defined by fuzzy automata A form a distributive lattice (see Chapter 3).

4.5 <u>Conclusions</u>

As the reader can see, the concepts of n-fold fuzzy grammars and, especially, fuzzy grammars can be discussed readily as extensions of ordinary formal grammars. The proofs, however, are generally somewhat longer since they involve not just the positivity of fuzzy functions but their value is in the interval $[0, 1]$.

The theory of fuzzy languages offers what appears to be a fertile field for further study. It may prove to be of use in the construction of better models for natural languages and may contribute to a better understanding of the role of fuzzy algorithms and fuzzy automata in decision making, pattern recognition and learning process of languages, and other processes involving the manipulation of fuzzy data.

CHAPTER 5

GENERAL FORMULATION OF FORMAL GRAMMARS

5.1 Introduction

By introducing the concepts of randomness and fuzziness into the structure of formal grammars, some interesting grammars such as probabilistic (or stochastic) grammars and fuzzy grammars have been formulated [38, 39, 40, 41, 42, 43, 49, 50 51, 52].

In this chapter, we develop a general formulation of formal grammars by extracting the basic properties common to the formal grammars appeared in existing literatures. By corresponding the element of the appropriate algebra, say, the complete distributive lattice, to each rule of a pseudo grammar, the evaluation (or weight) of the application of the rule is given. We evaluate a sentence by performing the operations of the corresponding algebra to the weight of the rules used in a generation of the sentence.

We derived from the pseudo grammars with various types of algebras the well-known phrase-structure grammars, probabilistic grammars, and fuzzy grammars. Still more, the grammars, which have never appeared before, say, $\sqcup *$ grammars, $\sqcup \sqcap$ grammars, $\sqcap \sqcup$ grammars, composite B-fuzzy grammars, mixed fuzzy grammars, and label string grammars, are also derived.

It can be shown that there are max-weighted grammars, max-probabilistic grammars and label string grammars as special cases of $\sqcup *$ grammars, (pessimistic) fuzzy grammars and phrase structure grammars as special cases of $\sqcup\sqcap$ grammars, and optimistic fuzzy grammars as special cases of $\sqcap\sqcup$ grammars.

The pseudo grammar called a pseudo conditional grammar, whose weight of the application of a rule is conditioned by the rule used just before in a derivation, is also defined and from it several interesting conditional grammars are derived in the same manners as the pseudo grammars.

5.2 L-Fuzzy Sets

We shall briefly review L-fuzzy sets by J. A. Goguen [5] for the purpose of $\sqcup *$ grammars, $\sqcup\sqcap$ grammars, $\sqcap\sqcup$ grammars, and fuzzy grammars which will be defined later.

L-Fuzzy Sets: A *L-fuzzy set* A in a space $X = \{x\}$ is characterized by a membership function μ_A such as

$$\mu_A : X \rightarrow L, \tag{5.1}$$

where L is called a *membership space* and the value $\mu_A(x) \in L$ represents the *grade of membership* of x in A.

A membership space L may be assumed to be a partially ordered set or, more particulary, a lattice.

When L is the unit interval $[0, 1]$, A is a fuzzy set defined by L. A. Zadeh [1]. Moreover, when L contains only two points 0 and 1, A is a non-fuzzy set and its membership function μ_A reduces to the conventional characteristic function of a non-fuzzy set.

The notions of containment, equality, union, and intersection of L-fuzzy sets are defined as extensions of the corresponding notions in the ordinary non-fuzzy sets.

Let A and B be two L-fuzzy sets in X, and let μ_A and μ_B be membership functions of A and B, respectively, then, for all x in X,

<u>Containment</u>: $$A \subseteq B \Leftrightarrow \mu_A(x) \leqq \mu_B(x), \tag{5.2}$$

<u>Equality</u>: $$A = B \Leftrightarrow \mu_A(x) = \mu_B(x), \tag{5.3}$$

<u>Union</u>: $$A \cup B \Leftrightarrow \mu_{A \cup B}(x) = \mu_A(x) \sqcup \mu_B(x), \tag{5.4}$$

<u>Intersection</u>: $$A \cap B \Leftrightarrow \mu_{A \cap B}(x) = \mu_A(x) \sqcap \mu_B(x), \tag{5.5}$$

where the operations $\leqq$, $\sqcup$ and $\sqcap$ represent an order relation, lub, glb in L, respectively.

In the case of $L = [0, 1]$, that is, fuzzy sets by Zadeh, the operation $\sqcup$ reduces to max, and $\sqcap$ to min. In addition, the complement of a fuzzy set A is defined as

<u>Complement</u>: $\bar{A} \Leftrightarrow \mu_{\bar{A}}(x) = 1 - \mu_A(x)$. (5.6)

In this paper, the structure of the membership space L is assumed to be the <u>complete distributive lattice</u> (or, more generally, the <u>complete lattice ordered semigroup</u>) on account of L-fuzzy relations denoted hereafter [5]†.

<u>L-Fuzzy Relation</u>: A <u>L-fuzzy relation</u> R in the product space $X \times Y = \{(x,y) \mid x \in X,\ y \in Y\}$ is a L-fuzzy set in $X \times Y$ characterized by a membership function μ_R, i.e.,

$$\mu_R : X \times Y \rightarrow L. \qquad (5.7)$$

---

† A complete lattice which is a semigroup with identity under $*$ and also satisfies the distributive law; for $x, y, x_i, y_i \in L$,

$$x * (\bigsqcup_i y_i) = \bigsqcup_i (x * y_i)$$

and

$$(\bigsqcup_i x_i) * y = \bigsqcup_i (x_i * y),$$

is a <u>complete lattice ordered semigroup</u> (=<u>closg</u>). Still more, if $*$ is replaced by $\sqcap$ in closg L, L becomes a <u>complete distributive lattice</u>.

Product of L-Fuzzy Relations: If R_1 and R_2 are two L-fuzzy relations in X x X, then by the product (or composition) of R_1 and R_2 is meant a L-fuzzy relation in X x X which is denoted by R_1R_2 and is defined as follows:

If L is a closg, then

$$\mu_{R_1R_2}(x, z) = \bigsqcup_y \left[\mu_{R_1}(x, y) * \mu_{R_2}(y, z) \right], \qquad (5.8)$$

where $\sqcup$ and $*$ are the operations of lub and semigroup in L, respectively.

If L is a complete distributive lattice, then

$$\mu_{R_1R_2}(x, z) = \bigsqcup_y \left[\mu_{R_1}(x, y) \sqcap \mu_{R_2}(y, z) \right], \qquad (5.9)$$

$$\mu_{R_1R_2}(x, z) = \bigsqcap_y \left[\mu_{R_1}(x, y) \sqcup \mu_{R_2}(y, z) \right]. \qquad (5.10)$$

If L-fuzzy relation R is a fuzzy relation by Zadeh, that is, R is characterized by a membership function

$$\mu_R : X \times Y \rightarrow [0, 1], \qquad (5.11)$$

then the product of fuzzy relations R_1 and R_2 is defined as special cases of (5.9) and (5.10), that is,

$$\mu_{R_1R_2}(x, z) = \sup_y \min \left[\mu_{R_1}(x, y), \mu_{R_2}(y, z) \right], \quad (5.12)$$

$$\mu_{R_1R_2}(x, z) = \inf_y \max \left[\mu_{R_1}(x, y), \mu_{R_2}(y, z) \right]. \quad (5.13)$$

Note that the operation of the product of (L-) fuzzy relations has the associative property, i.e.,

$$R_1(R_2R_3) = (R_1R_2)R_3 . \quad (5.14)$$

Hence, let $R_1, R_2, \ldots, R_n$ be the (L-) fuzzy relations on X, then the product $R_1R_2\ldots R_n$, say, in the case of (5.8), is defined as

$$\mu_{R_1R_2\ldots R_n}(x_1, x_{n+1}) \quad (5.15)$$

$$= \bigsqcup_{x_2,\ldots,x_n} \left[\mu_{R_1}(x_1, x_2) * \mu_{R_2}(x_2, x_3) * \ldots * \mu_{R_n}(x_n, x_{n+1}) \right].$$

Let the membership space L be the Boolean lattice B, then the convex combination of B-fuzzy sets is defined as follows:

<u>Convex Combination</u>: Let A, C, and Λ be B-fuzzy sets. The <u>convex combination</u> of A, C, and Λ is denoted by $(A,C;\Lambda)$ and is defined by the relation:

$$(A,C;\Lambda) = (\Lambda \cap A) \cup (\bar{\Lambda} \cap C), \tag{5.16}$$

where $\bar{\Lambda}$ is the complement of Λ.

It is easy to verify that, for all B-fuzzy sets Λ,

$$A \cap C \subseteq (A,C;\Lambda) \subseteq A \cup C. \tag{5.17}$$

<u>Note</u>: In the case of $L = [0, 1]$, that is, fuzzy sets by Zadeh, the convex combination of fuzzy sets A, C, and Λ is given by (2.14), that is,

$$(A,C;\Lambda) = \Lambda A + \bar{\Lambda} C. \tag{5.18}$$

Next, by using the concept of L-fuzzy sets, we shall define L-fuzzy languages. For simplicity, we call L-fuzzy languages as fuzzy languages hereafter.

Let Σ be a finite non-empty alphabet. The set of all finite strings over Σ is denoted by Σ^*. The null string is denoted by ε and included in Σ^*.

<u>Fuzzy Languages</u>: A <u>fuzzy language</u> FL is a L-fuzzy set in Σ^* characterized by a membership function such as $\mu_{FL}: \Sigma^* \to L$.

The operations such as containment, equality, union, and intersection of fuzzy languages are the same as those of L-fuzzy sets mentioned previously (see (5.2) $\sim$ (5.5)). Moreover, the notions of concatenation and Kleene closure of ordinary languages can be extended to fuzzy languages by the following:

Let L_1 and L_2 be two fuzzy languages in Σ^*, and μ_{L_1} and μ_{L_2} be membership functions of L_1 and L_2, respectively.

<u>Concatenation</u>: The <u>concatenation</u> of L_1 and L_2 is a fuzzy language denoted by $L_1 \circ L_2$ or $L_1 \cdot L_2$ and defined as follows: Let a string x in Σ^* be expressed as a concatenation of a prefix string u and a suffix string v, that is, $x = uv$. Then

$$\mu_{L_1 \circ L_2}(x) = \bigsqcup_u \left[\mu_{L_1}(u) \sqcap \mu_{L_2}(v) \right], \qquad (5.19)$$

$$\mu_{L_1 \cdot L_2}(x) = \bigsqcap_u \left[\mu_{L_1}(u) \sqcup \mu_{L_2}(v) \right], \qquad (5.20)$$

where $\sqcup$ in (5.19) and $\sqcap$ in (5.20) are taken over all prefixes u of x.

Note that the concatenation $L_1 \circ L_2$ in (5.19) is related as $\sqcup\sqcap$ grammars and $L_1 \cdot L_2$ in (5.20) is related as $\sqcap\sqcup$ grammars which will be defined later.

Kleene Closure: By using the concatenation $L_1 \circ L_2$ or $L_1 \bullet L_2$, Kleene closure of a fuzzy language L (written as L^*, or $\hat{L}$) is defined as

$$L^* = \varepsilon \cup L \cup L \circ L \cup L \circ L \circ L \cup \ldots . \tag{5.21}$$

$$\hat{L} = \varepsilon \cap L \cap L \cdot L \cap L \cdot L \cdot L \cap \ldots . \tag{5.22}$$

5.3 Various Kinds of Grammars

In this section we define a pseudo grammar each production of which has a label, an ordinary rewriting rule, and weight $\mu(r)$ as in (5.24) and derive from it various kinds of grammars, which have, or have not appeared in the existing literatures, by employing an appropriate algebra system as a weighting space and performing the corresponding operations to weights $\mu(r)$'s.

Definition 5.1. A pseudo grammar (PSG for short) is a system:

$$PSG = (V_N, V_T, P, S, J, M, \mu) \tag{5.23}$$

where

(i) V_N is a nonterminal vocabulary.

(ii) V_T is a terminal vocabulary.

(iii) S is an initial symbol in V_N.

(iv) P is a finite set of productions such as

$$(r) \qquad u \rightarrow v \qquad \mu(r), \tag{5.24}$$

where $r \in J$, $u \rightarrow v$ is an ordinary rewriting rule with $u \in V_N^* - \{\varepsilon\}$ and $v \in (V_N \cup V_T)^*$, and $\mu(r)$ is a weight of the application of the production r, which will be denoted in (vii)†.

(v) J is a set of (rewriting rule) labels as shown in (iv). $J = \{r\}$.

(vi) M is a <u>weighting space</u>.

(vii) μ is a function such that

$$\mu : J \rightarrow M . \tag{5.25}$$

---

† In this paper, we often say label r as production r for convenience.

μ may be called a <u>weighting function</u> and the value $\mu(r)$ is a <u>weight</u> of the application of a production r.

The expression

$$\alpha_o \xrightarrow[r_1]{\mu(r_1)} \alpha_1 \xrightarrow[r_2]{\mu(r_2)} \alpha_2 \xrightarrow[r_3]{\mu(r_3)} \cdots\cdot \xrightarrow[r_m]{\mu(r_m)} \alpha_m \tag{5.26}$$

will be referred to as a <u>weighted derivation chain</u> of length m from α_o to α_m by the productions $r_1, r_2, r_3, \ldots, r_m$, where $\alpha_o, \alpha_1, \ldots, \alpha_m \in (V_N \cup V_T)^*$. The meanings of weights $\mu(r)$ denoted over the arrow $\rightarrow$ in a derivation chain will be stated in each grammar defined later.

When $\alpha_o = S$, $\alpha_m = x$ ($\in V_T^*$) in (5.26), i.e.,

$$S \xrightarrow[r_1]{\mu(r_1)} \alpha_1 \xrightarrow[r_2]{\mu(r_2)} \alpha_2 \longrightarrow \cdots\cdot \longrightarrow \alpha_{m-1} \xrightarrow[r_m]{\mu(r_m)} x, \tag{5.27}$$

S is said to generate a terminal string x by the productions $r_1, r_2, \ldots, r_m$. In general, there are more than one weighted derivation chain from S to x.

Now, we shall obtain various kinds of grammars by adopting the appropriate algebra systems as the weighting spaces M of a weighting function $\mu: J \to M$ of a pseudo grammar PSG, and by performingthe corresponding operations to $\mu(r)$'s.

[I] $\sqcup *$ GRAMMAR (= $\sqcup * G$)†

(i-a): Let the weighting space M in PSG be the complete lattice ordered semigroup L. Namely, the weighting function μ is

$$\mu : J \to L .$$

In this case, μ can be regarded as the membership function of a L-fuzzy set in J.

(i-b): The grade of the generation of x in V_T^* by $\sqcup * G$, which is denoted as $\mu_{\sqcup * G}(x)$, is given by using the concept of the product of L-fuzzy relations of (5.8) and by the weighted derivation chain from S to x of (5.27). Clearly, $\mu_{\sqcup * G}(x)$ is in M (=L).

† As special cases of $\sqcup * G$, there are [II] $\sqcup \sqcap$ Grammar, [XI] Max-Weighted Grammar and [XII] Max-Probabilistic Grammar which will be defined before long.

$$\mu_{\sqcup * G}(x) = \sqcup \left[\mu(r_1) * \mu(r_2) * \cdots * \mu(r_m) \right], \quad (5.28)$$

where the lub $\sqcup$ is taken over all the weighted derivation chains from S to x.

[II] $\sqcup\sqcap$ GRAMMAR ($= \sqcup\sqcap$ G)†

(ii-a): The weighting space M is the complete distributive lattice L', that is, μ is

$$\mu : J \to L'.$$

(ii-b): The grade $\mu_{\sqcup\sqcap G}(x)$ of the generation of x in V_T^* by $\sqcup\sqcap$ G is given by using the product of L-fuzzy relations of (5.9).

$$\mu_{\sqcup\sqcap G}(x) = \sqcup \left[\mu(r_1) \sqcap \mu(r_2) \sqcap \cdots \sqcap \mu(r_m) \right], \quad (5.29)$$

where $\sqcup$ is taken over all the weighted derivation chains from S to x.

---

† [V] (Pessimistic) Fuzzy Grammar and [VIII] Ordinary Phrase Structure Grammar are considered as special cases of $\sqcup\sqcap$ Grammar.

Example 5.1. Consider the following $\sqcup\sqcap G$,

$$\sqcup\sqcap G = (V_N, V_T, P, S, J, L', \mu)$$

where $V_N = \{S, A, B, C, D, E\}$, $V_T = \{a, b, c\}$, L' is the Boolean lattice in Figure 5.1, and P is

(1)	$S \to ABC$	0,
(2)	$S \to ADC$	x_1,
(3)	$S \to DBC$	x_1,
(4)	$S \to ABE$	x_2,
(5)	$S \to AEC$	x_2,
(6)	$A \to aA$	I
(7)	$A \to a$	I
(8)	$B \to bB$	I
(9)	$B \to b$	I
(10)	$C \to cC$	I
(11)	$C \to c$	I
(12)	$D \to aDb$	I
(13)	$D \to ab$	I
(14)	$E \to bEc$	I
(15)	$E \to bc$	I

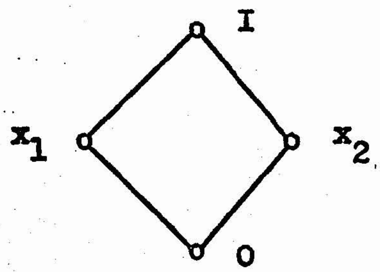

Fig. 5.1. Structure of L'

Then a L'-fuzzy language characterized by $\sqcup\sqcap G$ is

$$L(\sqcup\sqcap G) = \{(a^i b^j c^k, I) \mid i \neq j,\ j \neq k\} \cup \{(a^i b^j c^k, x_1) \mid i \neq j,\ j = k\}$$
$$\cup \{(a^i b^j c^k, x_2) \mid i = j,\ j \neq k\} \cup \{(a^i b^i c^i, 0) \mid i \geq 1\},$$

where $i, j, k \geq 1$.

It is interesting to note that the set of all the strings x in V_T^* such that $\mu_{\sqcup\sqcap G}(x) = 0$ is $\{a^i b^i c^i \mid i \geq 1\}$ and this language is a context-sensitive language.

[III] $\sqcap\sqcup$ GRAMMAR $(= \sqcap\sqcup\, G)^{\dagger}$

(iii-a): It is the same as (ii-a).

(iii-b): The grade $\mu_{\sqcap\sqcup G}(x)$ of the generation of x is given from the product of L-fuzzy relations of (5.10).

$$\mu_{\sqcap\sqcup G}(x) = \sqcap \left[\mu(r_1) \sqcup \mu(r_2) \sqcup \ldots \sqcup \mu(r_m) \right], \qquad (5.30)$$

where $\sqcap$ is taken over all the weighted derivation chains from S to x.

Let $L' = B$ (complete Boolean lattice) in (ii-a) and (iii-a), then we can define $\sqcup\sqcap\, G$ and $\sqcap\sqcup\, G$ on B, which may be written as $\sqcup\sqcap\, BG$ and $\sqcap\sqcup\, BG$, respectively. We will denote the grades of the generation of x by $\sqcup\sqcap\, BG$ and $\sqcap\sqcup\, BG$ as $\mu_{\sqcup\sqcap BG}(x)$ and $\mu_{\sqcap\sqcup BG}(x)$, respectively.

[IV] COMPOSITE B-FUZZY GRAMMAR (=CBFG)

(iv-a): The weighting space M is the complete Boolean lattice B.

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† As a special case of $\sqcap\sqcup$ Grammar, there is [VI] Optimistic Fuzzy Grammar.

(iv-b): The grade $\mu_{CBFG}(x)$ of the generation of x is defined from (5.16) as

$$\mu_{CBFG}(x) = (\alpha \sqcap \mu_{\sqcup\sqcap BG}(x)) \sqcup (\bar{\alpha} \sqcap \mu_{\sqcap\sqcup BG}(x)), \quad (5.31)$$

where $\alpha \in B$ and $\bar{\alpha}$ $(\in B)$ is the complement of α.

[V] (<u>PESSIMISTIC) FUZZY GRAMMAR</u> (=PFG), or <u>MAXIMIN GRAMMAR</u> [38, 39]

(v-a): Let $L' = [0, 1]$ in (ii-a).

(v-b): The grade $\mu_{PFG}(x)$ of the generation of x by PFG is given as follows by using the product of fuzzy relations of (5.12), in other words, by replacing $\sqcup$ by max and $\sqcap$ by min in (ii-b):

$$\mu_{PFG}(x) = \max \min [\mu(r_1), \mu(r_2), \ldots, \mu(r_m)], \quad (5.32)$$

where the maximum is taken over all the derivation chains from S to x.

[VI] <u>OPTIMISTIC FUZZY GRAMMAR</u> (=OFG), or <u>MINIMAX GRAMMAR</u>

(vi-a): It is the same as (v-a).

(vi-b): $\mu_{OFG}(x)$ is given as follows by using the product of fuzzy relations of (5.13), that is, by replacing $\sqcap$ by min and $\sqcup$ by max in (iii-b).

$$\mu_{OFG}(x) = \min \max \left[\mu(r_1), \mu(r_2), \ldots , \mu(r_m) \right], \qquad (5.33)$$

where the minimum is taken over all the derivation chains from S to x.

[VII] MIXED FUZZY GRAMMAR (=MFG)

(vii-a): It is the same as (v-a).

(vii-b): $\mu_{MFG}(x)$ is given as follows:

$$\mu_{MFG}(x) = a\, \mu_{PFG}(x) + b\, \mu_{OFG}(x), \qquad (5.34)$$

where a and b are real numbers such that a+b=1, and the subscripts PFG and OFG denote [V] (Pessimistic) Fuzzy Grammar and [VI] Optimistic Fuzzy Grammar, respectively.

[VIII] PHRASE STRUCTURE GRAMMAR (=G)

(viii-a): $L' = \{0, 1\}$ in (ii-a) or (v-a).

(viii-b): $\mu_G(x)$ is obtained in the same manner as

$\mu_{PFG}(x)$ in (v-b).

<u>Note</u>: In this case the language $L(G)$ generated by G is defined as

$$L(G) = \left\{ x \in V_T^* \mid \mu_G(x) = 1 \right\}.$$

[IX] <u>WEIGHTED GRAMMAR</u> (=WG)

(ix-a): The weighting space M is the set of non-negative real numbers.

(ix-b): $\mu_{WG}(x)$ is given as follows:

$$\mu_{WG}(x) = \sum \mu(r_1) \cdot \mu(r_2) \cdot \; \cdots \cdot \mu(r_m), \tag{5.35}$$

where the operations " Σ " and "$\cdot$" are sum and product in the ordinary sense, respectively.

[X] <u>PROBABILISTIC</u> (or <u>STOCHASTIC</u>) <u>GRAMMAR</u> (=PG)[49,50,51]

(x-a): $M = [0, 1]$, i.e., $\mu(r) \in [0, 1]$, and, in addition, $\mu(r)$ satisfies the following constraint:

For each J_u,

$$\sum_{r \in J_u} \mu(r) = 1$$

where J_u is the set of all labels such that the left hand side of the rewriting rule in the production of the pseudo grammar PSG is u ($\in V_N^* - \{\varepsilon\}$).

(x-b): $\mu_{PG}(x)$ is defined in the same manner as $\mu_{WG}(x)$ in (ix-b) and can be regarded as the probability of the gene- of x by PG.

Comment: It is assumed that the rewriting rules are of context-free form and the derivation is a left-most derivation.

[XI] MAX-WEIGHTED GRAMMAR (=MWG)

(xi-a): It is the same as (ix-a).

(xi-b): We take the maximum in stead of taking Σ in (ix-b), i.e.,

$$\mu_{MWG}(x) = \max \left[\mu(r_1) \cdot \mu(r_2) \cdot \; \ldots \cdot \mu(r_m) \right]. \qquad (5.36)$$

It is noted that the expression above can be obtained by replacing $\sqcup$ by max and $*$ by $\cdot$ in $\sqcup * G$ of [I].

[XII] MAX-PROBABILISTIC GRAMMAR (=MPG)

(xii-a): It is the same as (x-a).

(xii-b): $\mu_{MPG}(x)$ is obtained in the same manner as $\mu_{MWG}(x)$ in (xi-b).

[XIII] LABEL STRING GRAMMAR (=LSG)

(xiii-a): The weighting space M is J^*, where J is the set of labels. The weight $\mu(r)$, $r \in J$, is defined as

$$\mu(r) = r \qquad \text{for each } r \in J.$$

(xiii-b): $\mu_{LSG}(x)$, $x \in V_T^*$, is given as

$$\begin{aligned} \mu_{LSG}(x) &= \vee \left[\mu(r_1) \cdot \mu(r_2) \cdot \ldots \cdot \mu(r_m) \right] \\ &= \vee \left[r_1 \cdot r_2 \cdot \ldots \cdot r_m \right] \end{aligned} \qquad (3.37)$$

where the operations "$\vee$" and "." are union and concatenation of (label) strings, respectively. This expression (3.37) can be obtained by replacing $\sqcup$ by $\vee$ and $*$ by $\cdot$ in $\sqcup * G$ of [I].

Note: We could regard $\mu_{LSG}(x)$ as the set of all the label strings from S to x. Let C be the subset of J^*, then the language

$$L_C = \{ x \in V_T^* | \mu_{LSG}(x) \cap C \neq \phi \}$$

can be regarded as the languages controlled by control language C [53].

5.4 Various kinds of conditional grammars

In this section we define a pseudo conditional grammar (PSCG for short) as an extension of a pseudo grammar PSG denoted in previous section and derive from it several interesting conditional grammars, which have or have not appeared in the existing papers, in the same ways as we have derived from PSG various kinds of grammars in section 5.3.

Definition 5.2. A pseudo conditional grammar (PSCG for short) is a system

$$PSCG = (V_N, V_T, P, S, J, M, \{ \mu_1, \mu_2 \}), \qquad (5.38)$$

where V_N, V_T, S, J and M have essentially the same meanings as those for the PSG in the previous section. P is a set of the rules with labels as follows:

$$(r) \quad u \rightarrow v, \tag{5.39}$$

μ_1 is a weighting function which is called an <u>initial rule designating function</u> such that

$$\mu_1 : \quad J_S \longrightarrow M, \tag{5.40}$$

where J_S is the set of all labels whose rules are initial rules. μ_2 is a <u>conditional weighting function</u> as follows:

$$\mu_2(r/r') \in M, \tag{5.41}$$

where r, $r' \in J$. $\mu_2(r/r')$ represents the weight of the application of the rule r given the rule r' used just before in a derivation.

It is noted that the notion of a conditional weighting function is similar to that of a conditional probability function. In what follows, we shall write μ for μ_1 and μ_2 if there occurs no confusion.

If the derivation chain from S to x ($\in V_T^*$) is

$$S \xrightarrow[r_1]{} \alpha_1 \xrightarrow[r_2]{} \alpha_2 \rightarrow \cdots\cdot \rightarrow \alpha_{m-1} \xrightarrow[r_m]{} x, \qquad (5.41)$$

then the weights μ are put over the arrows as follows:

$$S \xrightarrow[r_1]{\mu(r_1)} \alpha_1 \xrightarrow[r_2]{\mu(r_2/r_1)} \alpha_2 \rightarrow \cdots\cdots \rightarrow \alpha_{m-1} \xrightarrow[r_m]{\mu(r_m/r_{m-1})} x. \qquad (5.42)$$

Now, let us define various kinds of conditional grammars.

[A] <u>CONDITIONAL $\sqcup *$ GRAMMAR</u> (= C $\sqcup *$ G)

(a-1): The weighting space M in $C \sqcup * G$ is the complete lattice ordered semigroup L.

(a-2): The grade of the generation of x in V_T^* by $C \sqcup * G$, is given as follows by using the concept of the product of L-fuzzy relations of (5.8) and from the weighted derivation chain from S to x of (5.42)

$$\mu_{C\sqcup * G}(x)$$

$$= \sqcup \left[\mu(r_1) * \mu(r_2/r_1) * \mu(r_3/r_2) * \cdots * \mu(r_m/r_{m-1}) \right] \quad (5.43)$$

where the lub $\sqcup$ is taken over all the weighted derivation chains from S to x.

[B] <u>CONDITIONAL $\sqcup\sqcap$ GRAMMAR</u> (= $C\sqcup\sqcap G$)

(b-1): The weighting space M is the complete distributive lattice L'.

(b-2): $\mu_{C\sqcup\sqcap G}(x)$, $x \in V_T^*$, is given from the product of L-fuzzy relations of (5.9).

$$\mu_{C\sqcup\sqcap G}(x)$$

$$= \sqcup \left[\mu(r_1) \sqcap \mu(r_2/r_1) \sqcap \cdots \sqcap \mu(r_m/r_{m-1}) \right]. \quad (5.44)$$

[C] <u>CONDITIONAL $\sqcap\sqcup$ GRAMMAR</u> (= $C\sqcap\sqcup G$)

(c-1): It is the same as (b-1).

(c-2): $\mu_{C\sqcap\sqcup G}(x)$ is given from (5.10) as follows:

$\mu_{C\sqcap\sqcup G}(x)$

$$= \sqcap\left[\mu(r_1) \sqcup \mu(r_2/r_1) \sqcup \;\ldots.\; \sqcup \;\mu(r_m/r_{m-1})\right]. \tag{5.45}$$

[D] <u>CONDITIONAL COMPOSITE B-FUZZY GRAMMAR</u> (= CCBFG)

(d-1): The weighting space M is the complete Boolean lattice B.

(d-2): $\mu_{CCBFG}(x)$ is as follows:

$$\mu_{CCBFG}(x) = (\alpha \sqcap \mu_{C\sqcup\sqcap BG}(x)) \sqcup (\bar{\alpha} \sqcap \mu_{C\sqcap\sqcup BG}(x)), \tag{5.46}$$

where $\alpha \in B$ and $\bar{\alpha}$ ($\in B$) is the complement of α, and $\mu_{C\sqcup\sqcap BG}(x)$ and $\mu_{C\sqcap\sqcup BG}(x)$ are the grades of the generation of x by $C\sqcup\sqcap BG$ and $C\sqcap\sqcup BG$, which are grammars $C\sqcup\sqcap G$ of [B] and $C\sqcap\sqcup G$ of [C] on complete Boolean lattice B, respectively.

[E] <u>CONDITIONAL (PESSIMISTIC) FUZZY GRAMMAR</u> (= CPFG), or <u>CONDITIONAL MAXIMIN GRAMMAR</u> [39]

(e-1): $L' = [0, 1]$ in (b-1).

(e-2): $\mu_{CPFG}(x)$ is given from (5.12) as follows:

$\mu_{CPFG}(x)$

$$= \max \min \left[\mu(r_1),\ \mu(r_2/r_1),\ \ldots\ ,\ \mu(r_m/r_{m-1}) \right]. \qquad (5.47)$$

[F] CONDITIONAL OPTIMISTIC FUZZY GRAMMAR (= COFG), or CONDITIONAL MINIMAX GRAMMAR

(f-1): It is the same as (e-1),

(f-2): $\mu_{COFG}(x)$ is given from (5.13) as follows:

$\mu_{COFG}(x)$

$$= \min \max \left[\mu(r_1),\ \mu(r_2/r_1),\ \ldots\ ,\ \mu(r_m/r_{m-1}) \right]. \qquad (5.48)$$

[G] CONDITIONAL MIXED FUZZY GRAMMAR (= CMFG)

(g-1): It is the same as (e-1).

(g-2): $\mu_{CMFG}(x)$ is as follows:

$\mu_{CMFG}(x)$

$$= a\ \mu_{CPFG}(x) + b\ \mu_{COFG}(x), \qquad (5.49)$$

where a and b are real numbers such that $a + b = 1$.

[H] <u>CONDITIONAL PHRASE STRUCTURE GRAMMAR</u> (= CG)

(h-1): $L' = \{0, 1\}$ in (b-1) or (e-1).

(h-2): $\mu_{CG}(x)$ is obtained in the same manner as $\mu_{CPFG}(x)$ in (e-2).

<u>Note</u>: CG can be regarded as Programmed Grammars with success fields only defined by Rosenkranz [46].

[I] <u>CONDITIONAL WEIGHTED GRAMMAR</u> (= CWG) [52]

(i-1): M is a set of nonnegative real numbers.

(i-2): $\mu_{CWG}(x)$ is given as

$$\mu_{CWG}(x) = \Sigma\, \mu(r_1)\cdot \mu(r_2/r_1)\cdot \,\cdots\cdots\, \cdot \mu(r_m/r_{m-1}). \qquad (5.50)$$

[J] <u>CONDITIONAL PROBABILISTIC GRAMMAR</u> (= CPG) [52]

(j-1): $M = [0, 1]$ and, in addition, $\mu(r)$ and $\mu(r'/r)$ satisfy the following constraints, respectively.

$$\sum_{r \in J_S} \mu(r) = 1 ,$$

$$\sum_{r' \in J} \mu(r'/r) = 1 ,$$

where J_S is the set of all labels whose rules are initial rules.

(j-2): $\mu_{CPG}(x)$ is given in the same manner as $\mu_{CWG}(x)$ in (i-2).

[K] <u>CONDITIONAL MAX-WEIGHTED GRAMMAR</u> (= CMWG)[52]

(k-1): It is the same as (i-1).

(k-2): We take the maximum in stead of taking Σ in (i-2), i.e.,

$$\mu_{CMWG}(x)$$
$$= \max \left[\mu(r_1) \cdot \mu(r_2/r_1) \cdot \ \cdots\cdots \cdot \mu(r_m/r_{m-1}) \right]. \qquad (5.51)$$

[L] CONDITIONAL MAX-PROBABILISTIC GRAMMAR (= CMPG) [52]

(l-1): It is the same as (j-1).

(l-2): $\mu_{CMPG}(x)$ is defined in the same manner as $\mu_{CMWG}(x)$ in (j-2).

5.5 Conclusions and Remarks

We have derived various kinds of grammars and conditional grammars from a pseudo grammar and a pseudo conditional grammar. As an extension of the pseudo conditional grammar, we can consider the pseudo grammar whose weight of the application of the rule to be used next is conditioned by all the rules used in a derivation. In this case, say, in the case of $\sqcup * G$, the grade of the generation of x is given as

$$\sqcup \left[\mu(r_1) * \mu(r_2/r_1) * \mu(r_3/r_1, r_2) * \cdots\cdots * \mu(r_m/r_1, r_2, \ldots r_{m-1})\right].$$

In the Weighted Grammar of [IX] in section 5.3, we adopted the set of nonnegative real numbers as the weighting space M, and the product and the sum as its operations. In this case, M forms a semiring. Therefore, we hope that more interesting grammars will be formulated by adopting the appropriate algebras such as semiring, ring, and field.

CHAPTER 6

CONCLUSIONS

Although the theory of fuzzy automata and, especially, fuzzy languages is young itself, it offers what appears to be a fertile field for further study. The theory of fuzzy languages might be of relevance in the construction of better models for natural languages and may find some practical applications as information retrieval and machine translation systems. It may also be of use in dealing with problems relating to fuzzy systems and fuzzy algorithms in decision making, pattern recognition and learning process of languages, and other processes involving the manipulation of fuzzy data.

Computers would become more powerful if we could learn how to design computers that can understand natural languages themselves or something close to them and manipulate fuzzy concepts and respond to fuzzy instructions in much the same way as human beings are capable of doing.

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