Generalization of a theorem of Peter J. Cameron

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GENERALIZATION OF A THEOREM OF PETER J. CAMERON

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Peter J. Cameron [3] has shown that a primitive permutation group \( G \) has rank at most 4 if the stabilizer \( G_\alpha \) of a point \( \alpha \) is doubly transitive on all its nontrivial suborbits except one.

The purpose of this paper is to prove the following two theorems, one of which extends the Cameron's result.

**Theorem 1.** Let \( G \) be a primitive permutation group on a finite set \( \Omega \), and all nontrivial \( G \)-orbits in Cartesian product \( \Omega \times \Omega \) be \( \Gamma_1, \ldots, \Gamma_s, \Delta_1, \ldots, \Delta_t \), where \( G_\alpha \) is doubly transitive on \( \Gamma_i(\alpha) = \{ \beta | (\alpha, \beta) \in \Gamma_i \} \), \( 1 \leq i \leq s \) and not doubly transitive on \( \Delta_i(\alpha) \), \( 1 \leq i \leq t \). Suppose that \( G \) has no subdegree smaller than 4 and that \( t > 1 \). Then, we have

\[
 s \leq 2t - r
\]

where \( r = \# \Delta_i \setminus \Delta_i = \sum_{j=1}^{s} \Gamma_j \cap \Gamma_j, \ 1 \leq j \leq s \). Moreover if \( r = 1 \), then we have

\[
 s \leq 2t - 2
\]

(For the notation \( \Gamma_j^n \), see the section 1)

**Theorem 2.** Under the hypothesis of Theorem 1, if \( r = t \), then \( s = t = 2 \), and \( G \) is isomorphic to the small Janko simple group and \( G_\alpha \) is isomorphic to \( \text{PSL}(2, 11) \).
For the case of \( t \geq 3 \), I don't know the example satisfying the equality \( s = 2t - r \), and when \( r = 1 \), the example satisfying the equality \( s = 2t - 2 \). I know only three examples with \( t = 2 \) and \( s = 2 \).

The small Janko simple group \( J_1 \) of order 175560 has a primitive rank 5 representation of degree 266 in which the stabilizer of a point is isomorphic to \( \text{PSL}(2, 11) \) and acts doubly transitively on suborbits of lengths 11 and 12; the other suborbit lengths are 110 and 132 (See Livingstone [7]). The Mathieu group \( M_{12} \) has a primitive rank 5 representation of degree 144 in which the stabilizer of a point is isomorphic to \( \text{PSL}(2, 11) \) and acts doubly transitively on two suborbits of length 11; the other suborbit lengths are 55 and 66 (See Cameron [4]).

The group \( \left[ \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \right] S_4 \) has a primitive rank 5 representation of degree 27 in which the stabilizer of a point is \( S_4 \) and acts doubly transitively on two suborbits of length 4; the other suborbit lengths are 6 and 12. I conjecture that it may even be true that \( s \) is at most \( t \).

1. Preliminaries

Let \( G \) be a transitive permutation group on a finite set \( \Omega \), and \( \Delta \) be a subset of the Cartesian product \( \Omega \times \Omega \) which is fixed by \( G \) (acting in the natural way on \( \Omega \times \Omega \)), then \( \Delta(\alpha) = \{ (\beta, \alpha) | (\alpha, \beta) \in \Delta \} \) is a subset of \( \Omega \) fixed by \( G_\alpha \). This procedure sets up a one-to-one correspondence between \( G \)-orbits in \( \Omega \times \Omega \) and \( G_\alpha \)-orbits in \( \Omega \).

The number of such orbits is called the rank of \( G \).

\[ \Delta^* = \{ (\beta, \alpha) | (\alpha, \beta) \in \Delta \} \] is the subset of \( \Omega \times \Omega \) fixed by \( G \) paired with \( \Delta \); \( \Delta \) is self-paired if \( \Delta = \Delta^* \). Note that

\[ |\Delta(\alpha)| = |\Delta^*(\alpha)| = |\Delta|/|\Omega| \]. If \( \Gamma \) and \( \Delta \) are fixed sets of \( G \)
in $\Omega \times \Omega$, let $\Pi \Delta$ denote the set $\{(\alpha', \beta') | \text{there exists } \gamma \in \Omega \text{ with } (\alpha, \gamma) \in \Pi', (\gamma, \beta) \in \Delta; \alpha \neq \beta\}$; this is also a fixed set of $G$.

The diagonal $\{(\alpha, \alpha) | \alpha \in \Omega\}$ is a trivial $G$-orbit. If $\Pi'$ is a nontrivial $G$-orbits in $\Omega \times \Omega$, the $\Pi'$-graph is the regular directed graph whose point set is $\Omega$ and whose edges are precisely the ordered pairs in $\Pi'$. A connected component of any such graph is a block of imprimitivity for $G$. $G$ is primitive if and only if each such graph is connected.

For a $G$-orbit $\Pi'$ in $\Omega \times \Omega$, the basis matrix $C = C(\Pi')$ is the matrix whose rows and columns are indexed by $\Omega$, with $(\alpha', \beta)$ entry 1 if $(\alpha', \beta) \in \Pi'$, 0 otherwise. All of the basis matrices form a basis of the centralizer algebra of the permutation matrices in $G$.

Let $G$ be a group which acts as a permutation group on $\Omega$, and $\pi$ the permutation character of $G$ i.e. the integer-valued function on $G$ defined by $\pi(g) =$ number of fixed points of $g$.

The formula

$$(\pi, 1)_G = \frac{1}{|G|} \sum_{g \in G} \pi(g) = \text{number of orbits of } G,$$

is well-known. If $G$ acts as a permutation group on $\Omega_1$ and $\Omega_2$, with permutation characters $\pi_1$ and $\pi_2$, the number $m$ of $G$-orbits in $\Omega_1 \times \Omega_2$ is

$$m = (\pi_1 \pi_2, 1)_G = (\pi_1, \pi_2)_G^G.$$

In particular, if $G$ is a transitive permutation group on $\Omega$ with permutation character $\pi$, the rank $r$ of $G$ is given by
If $G$ acts doubly transitively on $\Omega_1$ and $\Omega_2$,

\[(r_1, r_2)^G = 2 \text{ or } 1 \quad \text{according as } r_1 = r_2 \text{ or } r_1 \neq r_2.\]

Lastly, we note that if $G$ is a primitive permutation group on $\Omega$, then for $\alpha, \beta (\neq) \subset \Omega$, either $G_\alpha \neq G_\beta$ or $G$ is a regular group of prime degree ([8], Prop. 8.6); primitive groups with a subdegree 2 are Frobenius groups of prime degree ([8], Theorem 18.7); primitive groups with a subdegree 3 are classified by W. J. Wong [9].

2. Lemmata

Throughout this section, we suppose that $G$ is a primitive but not doubly transitive group on a finite set $\Omega$, and $\Gamma_1, \Gamma_2, \ldots$ are $G$-orbits in $\Omega \times \Omega$ such that $G_\alpha$ is doubly transitive on $\Gamma_1(\alpha)$, $i = 1, 2, \ldots$; $\pi_i$ and $\pi_i^*$ are the permutation characters of $G_\alpha$ on $\Gamma_1(\alpha)$ and $\Gamma_1^*(\alpha)$, respectively, and let $c_i = C(\pi_i)$, $c_i^* = C(\pi_i^*)$.

**Lemma 1.** (P. J. Cameron [2]. Proposition 1.2) $G_\alpha$ is doubly transitive on $\Gamma_i^*(\alpha)$.

**Lemma 2.** (P. J. Cameron [3]. Lemma 1) $\Gamma_i^* \cap \Gamma_i$ is a $G$-orbit in $\Omega \times \Omega$, and if $|\Gamma_i(\alpha)| > 2$, then $G_\alpha$ is not doubly transitive on $\Gamma_i^* \cap \Gamma_i(\alpha)$. 

Lemma 3. (P. J. Cameron [2]. Theorem 2.2)

For \((\alpha, \beta) \in P^*_1 P^*_2\), we put
\[v_i = |P^*_1(\alpha)|\] and
\[k_i = |P^*_1(\alpha) \cap P^*_2(\beta)|.\]

Then \(k_i < v_i\) and
\[|P^*_1 P^*_2(\alpha)| = \frac{v_i(v_i - 1)}{k_i}.\]

If \(v_i > 2\), then \(k_i \leq \frac{v_i - 1}{2}\); when particularly \(k_i = \frac{v_i - 1}{2}\), then \(v_i = 3\) or 5.

In the following, we set
\[|P^*_1(\alpha)| = v_i, \quad |P^*_1 P^*_2(\alpha)| = \frac{v_i(v_i - 1)}{k_i}.\]

Lemma 4. (P. J. Cameron [2]. Lemma 2.1)

\[|P^*_1 P^*_2(\alpha)| = |P^*_1 P^*_1(\alpha)|.\]

Lemma 5.
\[P^*_1 P^*_2(\alpha) \neq P^*_2 P^*_2(\alpha)\] if and only if \(\frac{|P^*_1 P^*_2(\alpha)|}{|P^*_1 P^*_2(\alpha)|} > 1\)

Proof. If \(\frac{|P^*_1 P^*_2(\alpha)|}{|P^*_1(\alpha)| \cdot |P^*_2(\alpha)|} \geq 1\), we have \(|P^*_1(\alpha) \cap P^*_2(\beta)| > 1\)

for some \((\alpha, \beta) \in P^*_1 P^*_2\). For \((\gamma_1, \gamma_2) \neq (\gamma_1, \gamma_2) \in P^*_1(\alpha) \cap P^*_2(\beta),\)
\((\gamma_1, \gamma_2) \in P^*_1 P^*_1\) and \((\gamma_1, \gamma_2) \in P^*_2 P^*_2\). So \(P^*_1 P^*_1 = P^*_2 P^*_2\).

Conversely, if \(\gamma_1, \gamma_2 \in P^*_1 P^*_2\) for \((\gamma_1, \gamma_2) \in P^*_1(\alpha) \cap P^*_2(\beta)\), we can choose \(\alpha'\) and \(\beta'\) such that \(\alpha \in P^*_1(\gamma_1) \cap P^*_2(\gamma_2), \beta \in P^*_2(\gamma_1) \cap P^*_2(\gamma_2)\).

Since \(\gamma_1, \gamma_2 \in P^*_1 P^*_2(\alpha), \gamma_1, \gamma_2 \in P^*_2(\beta),\) \(|P^*_1 P^*_2(\alpha)| > 1\).

Therefore \(\frac{|P^*_1 P^*_2(\alpha)|}{|P^*_1(\alpha)| \cdot |P^*_2(\alpha)|} > 1\).
Lemma 6. \( \Gamma_1^* \cap \Gamma_2^* \) is the union of at most two \( G \)-orbits in \( \Omega \times \Omega \), and \( \mathcal{K}_1 = \mathcal{K}_2 \) if and only if \( \Gamma_1^* \cap \Gamma_2^* \) is the union of two \( G \)-orbits in \( \Omega \times \Omega \).

Proof. Since \((\mathcal{K}_1 \mathcal{K}_2, 1)_G = (\mathcal{K}_1, \mathcal{K}_2)_G \leq 2\), and \( \mathcal{K}_1 \mathcal{K}_2 \) is the permutation character of \( G \) on \( \Gamma_1^*(\chi) \times \Gamma_2^*(\chi) \), \( G \) has at most two orbits in \( \{(\alpha, \gamma, \delta) | (\alpha, \gamma) \in \Gamma_1^*, (\alpha, \delta) \in \Gamma_2^* \} \), and hence, \( \Gamma_1^* \cap \Gamma_2^* \) is the union of at most two \( G \)-orbits. If \( \mathcal{K}_1 \neq \mathcal{K}_2 \), then \( G \) is transitive on \( \{(\alpha, \gamma, \delta) | (\alpha, \gamma) \in \Gamma_1^*, (\alpha, \delta) \in \Gamma_2^* \} \), and hence, \( \Gamma_1^* \cap \Gamma_2^* \) is a \( G \)-orbit in \( \Omega \times \Omega \). Now, we shall assume that \( \mathcal{K}_1 = \mathcal{K}_2 \) and \( \Gamma_1^* \cap \Gamma_2^* \) is a \( G \)-orbit in \( \Omega \times \Omega \). We put \( v = v_1 = v_2 \), and \( m = |\Gamma_1^*(\alpha) \cap \Gamma_2^*(\delta)| \) for \((\alpha, \delta) \in \Gamma_1^* \cap \Gamma_2^* \).

If \( m = 1 \), then since \( \Gamma_1^* \cap \Gamma_2^* \) is a \( G \)-orbit, \( G \) is transitive on \( \{(\alpha, \gamma, \delta) | (\gamma, \alpha) \in \Gamma_1^*, (\gamma, \delta) \in \Gamma_2^* \} \). Therefore \( (\mathcal{K}_1, \mathcal{K}_2)_G = 1 \), and hence, \( \mathcal{K}_1 \neq \mathcal{K}_2 \). This is contrary to the assumption.

If \( m > 1 \), then there exist quadrilaterals \((\alpha, \gamma_1, \delta, \gamma_2)\) whose edges are successively \( \Gamma_1^*, \Gamma_2^*, \Gamma_2^* \) and \( \Gamma_1^* \); and whose vertices are all distinct. Counting all of them in two ways, we have

\[
|\Omega| \frac{v}{m} m(m - 1) = |\Omega| \frac{v(v - 1)}{k_1 k_2},
\]

so

\[
v(m - 1) = (v - 1)k_2.
\]

Hence, \( v = k_2 \). This is impossible by Lemma 3.
Lemma 7. If \( \Gamma_i \neq \Gamma_i^* \), then \( \Gamma_i \bigcirc \Gamma_i^* \neq \Gamma_i \bigcirc \Gamma_i^* \).

Proof. Now assume \( \Gamma_i \bigcirc \Gamma_i^* \neq \Gamma_i \bigcirc \Gamma_i^* \) or \( \Gamma_i \neq \Gamma_i^* \), then we have the following figure,

![Diagram showing the relationship between \( \Gamma_i \), \( \Gamma_i^* \), and \( \Gamma_i \bigcirc \Gamma_i^* \).]

and hence, \( \Gamma_i \bigcirc \Gamma_i^* \bigcirc \Gamma_i \). Since \( \Gamma_i \bigcirc \Gamma_i^* \) is the union of at most two G-orbits in \( \Omega \times \Omega \), we have \( \Gamma_i \bigcirc \Gamma_i^* = \Gamma_i \bigcirc \Gamma_i^* \bigcirc \Gamma_i \). By the assumption of this lemma, \( |\Gamma_i \bigcirc \Gamma_i^*(x)| = |\Gamma_i \bigcirc \Gamma_i^*(x)| \).

So

\[
v_i^2 = |\Gamma_i \bigcirc \Gamma_i^*(x)| = |\Gamma_i \bigcirc \Gamma_i^*(x)| + |\Gamma_i \bigcirc \Gamma_i^*(x)| = \frac{2v_i(v_i - 1)}{k_i},
\]

\[v_i k_i = 2(v_i - 1)\]

Therefore, \( v_i = 2 \). All of the suborbits of the primitive group with a subdegree 2 are self-paired. This is contrary to the assumption of this Lemma.

Lemma 8. Let \( \Sigma_1 \bigcirc \Sigma_2 \) be the union of two G-orbits \( \Sigma_1 \) and \( \Sigma_2 \). We set \( v = v_1 = v_2, s_i = c(\Sigma_i), s_i = |\Sigma_i(x)|, i = 1, 2, \) and \( c_1 c_2 = a_1 s_1 + a_2 s_2 \). Then we have
Proof. i) Assume \( s_1 \leq v \). Then \( (\Pi_1^*, \Pi(\Sigma_1)) = 1 \) or 2 according as \( \Pi_1^* \neq \Pi(\Sigma_1) \) or \( \Pi_1^* = \Pi(\Sigma_1) \) where \( \Pi(\Sigma_1) \) is the permutation character of \( G_\chi \) on \( \Sigma_1(\alpha) \). If \( \Pi_1^* \neq \Pi(\Sigma_1) \), for \( \delta \in \Sigma_1(\alpha) \), \( G_\chi \delta \) is transitive on \( \Gamma_1^*(\alpha) \). Thus \( \Gamma_1^*(\alpha) = \Gamma_2^*(\delta) \).

Therefore \( G_\chi = G_1 \Gamma_1^*(\alpha) \) \( = G_2 \Gamma_2^*(\delta) \) \( = G_\delta \). This is impossible.

So we have \( \Pi_1^* = \Pi(\Sigma_1) \), and hence, \( s_1 = v \) and \( G_\chi \) is doubly transitive on \( \Sigma_1(\alpha) \).

ii) For the matrix \( F \) such that any entry is 1, we have

\[
F(C_{1,2}^*) = v^2 F \text{ and } F(a_1 s_1 + a_2 s_2) = (a_1 s_1 + a_2 s_2) F,
\]

so

\[
v^2 = a_1 s_1 + a_2 s_2.
\]

iii) The existence of the following figure is equivalent to

\[
\Gamma_1^* \Gamma_1 = \Gamma_2^* \Gamma_2.
\]
It holds also that the figure exists if and only if $a_i \geq 2$
for $i = 1$ or $2$.

iv) By ii), $v^2 = a_1v(v - 1) + a_2s_2$. Since $s_2 \geq v$,
a_1 = a_2 = 1 and $s_2 = v$. Therefore we conclude that
$\Gamma_1^* \Gamma_2$ contains some $\Gamma_i$ by i), and
$\Gamma_1^* \Gamma_1^* \neq \Gamma_2^* \Gamma_2^*$ by iii).

Lemma 9. If $\Gamma_1 \neq \Gamma_2$, $G_\alpha$ is not doubly transitive on $\Gamma_1^* \Gamma_2(\alpha)$.

Proof. Assume that $G_\alpha$ is doubly transitive on $\Gamma_1^* \Gamma_2(\alpha)$.  
If $|\Gamma_1^* \Gamma_2(\alpha)| \neq |\Gamma_1(\alpha)|$, then $G_\alpha$ has different permutation
characters on $\Gamma_1^* (\alpha)$ and $\Gamma_1^* \Gamma_2(\alpha)$.
Hence, for $(\alpha, \gamma) \in \Gamma_1^*$, $G_{\alpha, \gamma}$ is
transitive on $\Gamma_1^* \Gamma_2(\alpha)$, so, $\Gamma_2(\gamma) = \Gamma_2^* \Gamma_2(\alpha)$. Therefore
$G_\gamma = G_\alpha \Gamma_2(\gamma) = G_\alpha \Gamma_2^* \Gamma_2(\alpha) = G_\alpha$. This is impossible. Thus,
we obtain $|\Gamma_2^* \Gamma_1(\alpha)| = |\Gamma_1^* \Gamma_2(\alpha)| = |\Gamma_1(\alpha)|$. On the other hand,
for $(\delta, \gamma) \in \Gamma_2^*$, $\Gamma_1(\gamma) \subset \Gamma_2^* \Gamma_1(\delta)$. So, $\Gamma_2^* \Gamma_1(\delta) = \Gamma_1(\gamma)$.
This is also impossible.

Lemma 10. Assume $\Gamma_1^* = \Gamma_2^* = \Gamma_1$ and $\Gamma_1^* \Gamma_2$ be the union of
two $G$-orbits $\Sigma_1$ and $\Sigma_2$; put $|\Gamma_1^*(\alpha)| = |\Gamma_2^*(\alpha)| = v$,
$|\Gamma_1^* \Gamma_1^*(\alpha)| = v(v - 1)$, $|\Sigma_i(\alpha)| = s_i$, $i = 1, 2$; and $|\Gamma_2^*(\gamma)| \cap \Sigma_2(\alpha)|$ = $t$ for $\gamma \in \Gamma_1^*(\alpha)$. Then, we have the following quadratic
equation for $t$.

$$
\frac{v(v - t)^2}{s_1} + \frac{vt}{s_2} - v - k(v - 1) = 0.
$$
Particularly, i) when \( s_1 \geq \frac{v(v-1)}{k} \), the quadratic equation has at most one root for \( 0 < t < v \); ii) when \( t = 1 \), then \( s_2 = v \), \( s_1 = \frac{v(v-1)}{k+1} \) and \( G_\alpha \) is doubly transitive on \( \Sigma_2(\alpha) \).

Proof. For \( \gamma_1, \gamma_2 \neq \emptyset \in [1] \), counting arguments show that

\[
\left| \Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2) \cap \Sigma_1(\alpha) \right| = \frac{(v-t)\{v(v-t)-s_1\}}{(v-1)s_1},
\]

so

\[
k = \left| \Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2) \cap \Sigma_2(\alpha) \right| = \frac{(v-t)\{v(v-t)-s_1\}}{(v-1)s_1} + \frac{t(vt-s_2)}{(v-1)s_2},
\]

\[
(v-1)k = \frac{v(v+t)^2}{s_1} - (v - t) + \frac{vt^2}{s_2} - t = \frac{v(v-t)^2}{s_1} + \frac{vt^2}{s_2} - v,
\]

\[
0 = \frac{v(v-t)^2}{s_1} + \frac{vt^2}{s_2} - v - k(v-1).
\]

We shall prove the latter assertions. We put

\[
f(t) = \frac{v(v-t)^2}{s_1} + \frac{vt^2}{s_2} - v - k(v-1).
\]

When \( s_1 \geq \frac{v(v-1)}{k} \), then \( f(0) < 0 \). Since the coefficient of \( t^2 \) in \( f(t) \) is positive, \( f(t) \) has at most one root for \( 0 < t < v \).

When \( t = 1 \), then \( s_2 \leq v \). By Lemma 8,i) \( s_2 \geq v \). So \( s_2 = v \), and hence, \( G_\alpha \) is doubly transitive on \( \Sigma_2(\alpha) \), and \( s_1 = \frac{v(v-1)}{k+1} \).

Lemma 11. Let \( \Gamma_1 \cap \Gamma_2 \) be the union of two \( G \)-orbits \( \Sigma_1 \) and \( \Sigma_2 \), and \( G_\alpha \) doubly transitive on \( \Sigma_1(\alpha) \) and \( \Sigma_2(\alpha) \), then

\[
\left| \Gamma_1(\alpha) \right| = \left| \Gamma_2(\alpha) \right| \leq 3.
\]
Proof. This lemma due to P. J. Cameron. ([3], Lemma 4.) We put $|\Gamma_1^*(\alpha)| = |\Gamma_2^*(\alpha)| = \nu$, and assume $|\Sigma_1(\alpha)| \neq \nu$. Then, $G_\alpha$ has the different permutation characters on $\Gamma_1^*(\alpha)$ and $\Sigma_1(\alpha)$, so, for $(\alpha', \delta') \in \Sigma_1$, $G_{\alpha', \delta'}$ is transitive on $\Gamma_1^*(\alpha)$. Hence, $\Gamma_1^*(\alpha) = \Gamma_2^*(\delta)$. Therefore, $G_\alpha = G_{\Gamma_1^*(\alpha)} = G_{\Gamma_2^*(\delta)} = G_{\delta}$. This is impossible. Thus we conclude that $|\Sigma_1(\alpha)| = \nu$. In the same way, we have $|\Sigma_2(\alpha)| = \nu$.

Now, if $\Gamma_1^* \neq \Gamma_2^*$, then by Lemma 5 $|\Gamma_1^* \Gamma_2^*(\alpha)| = |\Gamma_1^*(\alpha)| = \nu^2$. Therefore, $\nu^2 = |\Gamma_1^* \Gamma_2^*(\alpha)| = |\Sigma_1(\alpha)| + |\Sigma_2(\alpha)| = 2\nu$, so $\nu = 2$. Thus, when $\nu > 2$, we obtain that $\Gamma_1^* \Gamma_1^* = \Gamma_2^* \Gamma_2^*$. For $y \in \Gamma_1^*(\alpha)$, we put $t = |\Gamma_2(y) \cap \Sigma_1(\alpha)|$. Then, for $(Y_1, Y_2) \subseteq \Gamma_1^* \Gamma_1^*$, by Lemma 10 we have the following equation.

$$k_2 = \frac{|\Gamma_2(Y_1) \cap \Gamma_2(Y_2)|}{v^2} = \frac{1}{v-1} \left\{ (v-t)^2 + t^2 - \nu \right\}$$

If $t = \frac{v}{2}$, $|\Gamma_2(Y_1) \cap \Gamma_2(Y_2)| = \nu + \frac{v^2}{2(v-1)}$ is not integer, so $t \leq \frac{v-1}{2}$ or $t \geq \frac{v+1}{2}$. Hence $k_2 = v - \frac{2t(v-t)}{v-1} \geq v - \frac{1}{2}(v+1) = \frac{1}{2}(v-1)$. But $k_2 \leq \frac{1}{2}(v-1)$ by Lemma 3, so equality holds, and thus $v = 3$ or 5 by Lemma 3, and $t = \frac{1}{2}(v+1)$ or $\frac{1}{2}(v-1)$. Counting arguments show that $|\Gamma_2(Y_1) \cap \Gamma_2(Y_2) \cap \Sigma_1(\alpha)| = \frac{t(t-1)}{v-1}$ for $Y_1, Y_2(\neq) \subseteq \Gamma_1^*(\alpha)$. Therefore $v - 1$ divides $t(t-1)$; this excludes $v = 5$, and so $v = 3$.

**Lemma 12.** For $\Gamma_1^*, \Gamma_2^*, \Gamma_3^*$, if $\Sigma$ is a $G$-orbit contained in $\Gamma_1^* \Gamma_2^* \cap \Gamma_1^* \Gamma_3^*$, and $|\Gamma_1(\alpha)| \geq 3$, then $G_\alpha$ is not doubly transitive on $\Sigma(\alpha)$. 

Proof. \( \Sigma^* \Gamma^*_1 \supset \Gamma^*_2 \cup \Gamma^*_3 \). If \( G^\alpha \) is doubly transitive on \( \Sigma(\alpha) \), \( \Sigma^* \Gamma_1^* \) is the union of at most two \( G \)-orbits by Lemma 6, so \( \Sigma^* \Gamma_1^* = \Gamma^*_2 \cup \Gamma^*_3 \). This is contrary to Lemma 11.

Lemma 13. If \( \Gamma^*_1 \Gamma^*_1 = \Gamma^*_2 \Gamma^*_2 \) and \( \Gamma^*_1 \neq \Gamma^*_2 \), then \( |v_1 - v_2| \geq 2 \), and
\[
|\Gamma^*_1 \Gamma^*_2(\alpha)| > |\Gamma^*_1 \Gamma^*_2(\alpha)|.
\]

Proof: For \((\alpha, \delta) \in \Gamma^*_1 \Gamma^*_2\), we put
\[
m = |\Gamma^*_1(\alpha) \cap \Gamma^*_2(\delta)|.
\]
Count in two ways quadrilaterals \((\alpha, \gamma_1, \delta, \gamma_2)\) with \( \gamma_1 \neq \gamma_2 \) whose edges are successively \( \Gamma^*_1, \Gamma^*_2, \Gamma^*_2, \) and \( \Gamma^*_1 \); then we have
\[
|\Omega| \frac{v_2(v_2 - 1)}{k_2} k_1 k_2 = |\Omega| \frac{v_1 v_2}{m} m(m - 1),
\]
so
\[
(v_2 - 1)k_1 = v_1(m - 1) \quad (1)
\]
If \( v_1 = v_2 \), then \( k_1 = v_1 \). This is impossible. If \( v_1 = v_2 + 1 \), then \( k_1 \geq \frac{v_1}{2} \), and hence, by Lemma 3 \( v_1 = 2, v_2 = 1 \). This is also impossible. Thus we can conclude that \( |v_1 - v_2| \geq 2 \).

Assume \( |\Gamma^*_1 \Gamma^*_2(\alpha)| = \frac{v_1(v_1 - 1)}{k_1} = |\Gamma^*_1 \Gamma^*_2(\alpha)| = \frac{v_1 v_2}{m} \). Then
\[
k_1 v_2 \geq m(v_1 - 1) \quad (2)
\]
From \( \Gamma^*_1 \Gamma^*_1 = \Gamma^*_2 \Gamma^*_2 \), we have also
\[
k_2 v_1 \geq m(v_2 - 1) \quad (3)
\]
Therefore, (1) and (2) yield

\[ v_1 \leq k_1 + m. \] (4)

By Lemma 3 and (3), we have

\[ 2v_2 \leq \frac{v_2(v_2-1)}{k_2} \leq \frac{v_1v_2}{m}, \]

so

\[ 2 \leq m \leq \frac{v_1}{2}. \] (5)

Thus (4) and (5) yield

\[ k_1 \geq \frac{1}{2}v_1. \]

This is contrary to Lemma 3.

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**Lemma 14.** (P. J. Cameron [3]) If \( \Gamma_1^* \Gamma_1^* = \Gamma_2^* \Gamma_2^* \), then \( \Gamma_1^* \Gamma_1^* \neq \Gamma_2^* \Gamma_2^* \).

**Proof.** We shall prove this lemma in a different way from P. J. Cameron's. Assume \( \Gamma_1^* \Gamma_1^* = \Gamma_1^* \Gamma_2^* = \Gamma_2^* \Gamma_2^* \). We put

\[ |\Gamma_1^* \Gamma_2^* (\alpha)| = \frac{v_1(v_1-1)}{k_1} = |\Gamma_2^* \Gamma_2^* (\alpha)| = \frac{v_2(v_2-1)}{k_2} = |\Gamma_1^* \Gamma_2^* (\alpha)| \]

\[ = \frac{v_1v_2}{m}, \]

where \( m = |\Gamma_1^* (\alpha) \cap \Gamma_2^* (\delta)| \) for \((\alpha, \delta) \in \Gamma_1^* \Gamma_2^* \). Then it is trivial that \( m > 1 \) from the above formula, and hence, \( \Gamma_1^* \Gamma_1^* = \Gamma_2^* \Gamma_2^* \).

Thus, by Lemma 13, \( |\Gamma_1^* \Gamma_2^* (\alpha)| < |\Gamma_1^* \Gamma_1^* (\alpha)| = |\Gamma_1^* \Gamma_1^* (\alpha)| \). This is contrary to assumption.
Now we shall investigate from Lemma 15 to Lemma 22 the necessary condition that the intersection of \( \Gamma_1^* \Gamma_2 \) and \( \Gamma_2^* \Gamma_3 \) for \( \Gamma_1, \Gamma_2, \Gamma_3 \) (\( \neq \)) is not empty.

**Lemma 15.** If \( \Pi_1 = \Pi_2 \neq \Pi_3 \) and \( \Pi_2^* = \Pi_3^* \), or \( \Pi_1 = \Pi_2 = \Pi_3 \) and \( \Pi_2^* \neq \Pi_3^* \), then \( \Gamma_1^* \Gamma_2 \cap \Gamma_1^* \Gamma_3 = \emptyset \).

**Proof.** Assume \( \Pi_1 = \Pi_2 \neq \Pi_3 \) and \( \Pi_2^* = \Pi_3^* \). Then we have 
\[ v_1 = v_2 = v_3. \]
We put \( v = v_1 = v_2 = v_3 \). By Lemma 13, \( \Gamma_1^* \Gamma_2 \neq \Gamma_3^* \Gamma_3 \), and hence, 
\[ |\Gamma_1^* \Gamma_3| = |\Gamma_3^* \Gamma_3| = v^2 \]
by Lemma 5.

If \( \Gamma_1^* \Gamma_2 \cap \Gamma_1^* \Gamma_3 \neq \emptyset \), then since \( \Gamma_1^* \Gamma_3 \) is a G-orbit and \( \Gamma_1^* \Gamma_2 \) is a union of two G-orbits, we have \( \Gamma_1^* \Gamma_2 \not\supset \Gamma_1^* \Gamma_3 \). Therefore 
\[ |\Gamma_1^* \Gamma_2| > |\Gamma_1^* \Gamma_3| = v^2. \]
This is impossible.

Similarly, we can prove the lemma for the case of \( \Pi_1 = \Pi_2 = \Pi_3 \) and \( \Pi_2^* \neq \Pi_3^* \).

**Lemma 16.** If \( \Pi_1^* \neq \Pi_2^* \), \( \Pi_2^* \neq \Pi_3^* \) and \( \Pi_2 \neq \Pi_3 \), then 
\[ \Gamma_1^* \Gamma_2 \cap \Gamma_1^* \Gamma_3 = \emptyset. \]

**Proof.** By the assumption, \( \Gamma_1^* \Gamma_2 \), \( \Gamma_1^* \Gamma_3 \) and \( \Gamma_2^* \Gamma_3 \) are G-orbits. Assume \( \Gamma_1^* \Gamma_2 = \Gamma_1^* \Gamma_3 \). For \( (\alpha, \delta) \in \Gamma_1^* \Gamma_2 \), we put 
\[ |\Gamma_1^* (\alpha) \cap \Gamma_2 (\delta)| = m_2 \quad \text{and} \quad |\Gamma_1^* (\alpha) \cap \Gamma_3 (\delta)| = m_3. \]

For \( \gamma_1, \gamma_2 \neq \emptyset \in \Gamma_1^* (\alpha) \), we put 
\[ |\Gamma_2^* (\gamma_1) \cap \Gamma_3^* (\gamma_2)| = x. \]

Then, since \( \Gamma_1^* \Gamma_1 = \Gamma_2^* \Gamma_3 \), we have 
\[ \frac{v_1(v_1 - 1)}{k_1} = |\Gamma_1^* \Gamma_1| = |\Gamma_2^* \Gamma_3| = \frac{v_2 v_3}{x}. \]
so
\[ v_1 (v_1 - 1)x = v_2 v_3^k_1. \]  \hspace{1cm} (1)

Count in two ways quadrilaterals \((\alpha, \gamma, \delta, \rho)\) whose edges are successively \(\Gamma_1^1, \Gamma_2^1, \Gamma_3 \) and \(\Gamma_1^\rho\), then we have
\[ |\Omega| \frac{v_1 (v_1 - 1)}{k_1} k_1 x = |\Omega| \frac{v_1 v_3}{m_3} m_2 m_3, \]
so
\[ (v_1 - 1)x = v_3 m_2. \]  \hspace{1cm} (2)

(1) and (2) yield
\[ v_1 m_2 = k_1 v_2. \]  \hspace{1cm} (3)

If \(m_2 > 1\), there exist quadrilaterals \((\alpha', \beta_1, \delta, \beta_2)\) whose edges are successively \(\Gamma_1^1, \Gamma_2^1, \Gamma_2 \) and \(\Gamma_1^\rho\), whose vertices are all distinct; count all of them in two ways, we have
\[ |\Omega| \frac{v_1 (v_1 - 1)}{k_1} k_1 k_2 = |\Omega| \frac{v_1 v_2}{m_2} m_2 (m_2 - 1), \]
so
\[ (v_1 - 1)k_2 = v_2 (m_2 - 1). \]

On the other hand, from \(\Gamma_1^\rho \Gamma_1 = \Gamma_2^\rho \Gamma_2\),
\[ v_2 (v_2 - 1)k_1 = v_1 (v_1 - 1)k_2 = v_1 v_2 (m_2 - 1), \]
so
\[ v_1 (m_2 - 1) = (v_2 - 1)k_1. \]  \hspace{1cm} (4)
(3) and (4) yield

\[ v_1 = k_1. \]

This is contrary to Lemma 3.

Thus, we have \( m_2 = m_3 = 1 \) and \( v_1 = k_1 v_2 \). For \( (\alpha, \gamma) \in \Gamma_1 \), \( G_{\alpha, \gamma} \) is transitive on \( \Gamma_1^* (\alpha) \setminus \{ \gamma \} \) and since \( \mathcal{P}_1 \neq \mathcal{P}_2 \), it is also transitive on \( \Gamma_2^* (\gamma) \). Count in two ways \( (\gamma', \delta) \) such that \( \gamma' \in \Gamma_1 (\alpha) \setminus \{ \gamma \} \), \( \delta \in \Gamma_2^* (\gamma) \) and \( (\gamma', \delta) \in \Gamma_3^* \), then we have

\[(v_1 - 1)x = v_2 = \frac{v_1}{k_1}. \]

This is impossible.

\[
\text{Lemma 17.} \quad \text{If} \quad \mathcal{P}_1 \neq \mathcal{P}_2, \quad \mathcal{P}_2 \neq \mathcal{P}_3 \quad \text{and} \quad \Gamma_1 \circ \Gamma_2 = \Gamma_2 \circ \Gamma_1, \quad \text{then} \quad \Gamma_1^* \cap \Gamma_2^* \cap \Gamma_3^* = \emptyset.
\]

Proof. Assume \( \Gamma_1 \circ \Gamma_2 = \Gamma_2 \circ \Gamma_1 \). By Lemma 16, \( \mathcal{P}_2^* = \mathcal{P}_3^* \). We put \( v = v_1, \ w = v_2 = v_3, \ m = |\Gamma_1^* (\alpha) \cap \Gamma_2^* (\delta)| = |\Gamma_1^* (\alpha) \cap \Gamma_3^* (\delta)| > 1 \) for \( (\alpha, \delta) \in \Gamma_1^* \circ \Gamma_2^* \), and \( x = |\Gamma_2^* (\gamma_1) \cap \Gamma_3^* (\gamma_2)| \) for \( \gamma_1, \gamma_2 \neq (\alpha) \).

Count in two ways quadrilaterals \( (\alpha, \gamma_1, \delta, \gamma_2) \) whose edges are successively \( \Gamma_1^*, \Gamma_2^*, \Gamma_3^* \) and \( \Gamma_1 \), then we have

\[ |\Omega| \frac{v(v-1)}{k_1} k_1 x = |\Omega| \frac{vw}{m} \text{ mm}, \]

so

\[(v - 1)x = wm. \] (1)
Next, count in two ways quadrilaterals \((\alpha, \gamma_1, \delta, \gamma_2)\) whose edges are successively \(\delta_1, \gamma_1, \gamma_2, \delta_1\) and whose vertices are all distinct; then

\[
|\Omega|^{(v-1)k_1k_2} = |\Omega|^{vw(m-1)},
\]

\[
(v-1)k_2 = w(m-1).
\]  \(2\)

(1) and (2) yield

\[(v-1)(x-k_2) = w, \text{ that is, } x > k_2 \geq 1. \]  \(3\)

Since \(x \geq 2\), there exist quadrilaterals \((\gamma, \delta_1, \gamma', \delta_2)\) whose edges are successively \(\gamma_1, \gamma_2, \gamma_3, \gamma_4\), whose vertices are all distinct, and \((\gamma, \gamma') \leq \gamma_1 \gamma_1^* = \gamma_2 \gamma_2^*\); count all of them in two ways, then

\[
|\Omega|^{w(w-1)\lambda} = |\Omega|^{w(w-1)x(x-1)},
\]

\[
(\lambda = |\gamma_1^* \cap \gamma_2^* \cap \gamma_3^* | \text{ for } \delta_1, \delta_2 (\neq) \in \gamma_1^* \gamma_1^* )
\]

so

\[
\lambda = \frac{x(x-1)}{k_2}.
\]

By the definition of \(\lambda\), \(\lambda \leq k_2\). On the other hand, since \(x > k_2\),

\[
\lambda = \frac{x(x-1)}{k_2} > k_2.
\]

This is a contradiction.

**Lemma 18.** If \(\gamma_1^* \neq \gamma_2^*\), \(\gamma_1^* \neq \gamma_3^*\) and \(\gamma_1 \gamma_2^* = \gamma_1 \gamma_3^*\), then

\[
c_1^* c_2^* = c_1^* c_3^*.
\]
Proof. By Lemma 6, \( \Sigma = \bigvee_{1}^{\gamma} \Sigma_{2}^{\gamma} = \bigvee_{1}^{\gamma} \Sigma_{3}^{\gamma} \) is a \( G \)-orbit. Let \( S = \Sigma(\Sigma) \), \( C_{1}C_{2} = m_{2}S \), \( C_{1}C_{3} = m_{3}S \) and \( |\Sigma(\Sigma)| = s \).

For the matrix \( F \) such that the value of any entry is 1, we have

\[ v_{1}v_{2}F = F(C_{1}C_{2}^{*}) = F(m_{2}S) = m_{2}sF, \]

so

\[ v_{1}v_{2} = m_{2}s. \]

Similarly

\[ v_{1}v_{3} = m_{3}s. \]

On the other hand, by Lemma 16, \( \Pi_{2} = \Pi_{3} \), and hence, \( v_{2} = v_{3} \).

So, \( m_{2} = m_{3} \). Thus we can conclude that \( C_{1}C_{2}^{*} = C_{1}C_{3}^{*} \).

\textbf{Lemma 19.} If \( C_{1}C_{2}^{*} = C_{1}C_{3}^{*} \) and \( |\bigvee_{1}^{\gamma}(\Sigma)| = v_{1} > 3 \), then we have

i) \( \Pi_{2} = \Pi_{3} \), \( \Pi_{1}^{*} \neq \Pi_{2}^{*} \), \( \Pi_{3}^{*} \).

ii) \( \bigvee_{1}^{\gamma} \Sigma_{1} \neq \bigvee_{2}^{\gamma} \Sigma_{2} \), \( \bigvee_{1}^{\gamma} \Sigma_{1} \neq \bigvee_{3}^{\gamma} \Sigma_{3} \).

iii) \( v_{1} = v_{2} + 1 = v_{3} + 1 \), \( |\bigvee_{2}^{\gamma}(\Sigma_{1}) \cap \bigvee_{3}^{\gamma}(\Sigma_{2})| = 1 \) for

\( (\gamma_{1}, \gamma_{2}) \leq \bigvee_{1}^{\gamma} \Sigma_{1} \).

iv) \( |\bigvee_{1}^{\gamma} \Pi_{1}^{\gamma}(\Sigma)| = \frac{v_{1}(v_{1} - 1)}{2}. \)

Proof. By the assumption \( \bigvee_{1}^{\gamma} \Sigma_{2}^{\gamma} = \bigvee_{1}^{\gamma} \Sigma_{3}^{\gamma} \). For the matrix \( F \) such that the value of any entry is 1, we have

\[ F(C_{1}C_{2}^{*}) = (FC_{1})C_{2}^{*} = (v_{1}F)C_{2}^{*} = v_{1}(FC_{2}^{*}) = v_{1}v_{2}F. \]

Similarly

\[ F(C_{1}C_{3}^{*}) = v_{1}v_{3}F. \]
So

\[ v_2 = v_3. \]

We shall show that \( v_1 \neq v_2 = v_3. \) Assume \( v = v_1 = v_2 = v_3 \) and put \( D = C(I^*_1 \Gamma_1). \) If \( \Gamma_1^* \Gamma_2 = \Gamma_2^* \Gamma_2, \) then \( \left| \Gamma_1^* \Gamma_3 (\alpha) \right| = \left| \Gamma_1^* \Gamma_2 (\alpha) \right| \neq \left| \Gamma_1^* (\alpha) \cdot \Gamma_2 (\alpha) \right| = \left| \Gamma_1^* \Gamma_2 (\alpha) \right| , \) therefore \( \Gamma_1^* \Gamma_1 = \Gamma_3^* \Gamma_3 \) by Lemma 5. We put \( k = k_1 = k_2 = k_3. \)

\[
C_1^* (C_1^* C_2^*) = (C_1^* C_1^*) C_2^* = (vE + kD) C_2^* = vC_2^* + k(v - 1)C_2^* + \\
\text{terms no involving } C_2^*.
\]

Similarly

\[
C_1^* (C_1^* C_3^*) = vC_3^* + k(v - 1)C_3^* + \text{terms not involving } C_3^*.
\]

So

\[
(vE + kD) C_2^* = \{v + k(v - 1)\} C_3^* + \text{terms not involving } C_3^*.
\]

Since the coefficients of the basis matrices in \( DC_2^* \) are at most \( v, \) the above formula is impossible.

Next, if \( \Gamma_1^* \Gamma_1 \neq \Gamma_2^* \Gamma_2, \) then \( \Gamma_1^* \Gamma_1 \neq \Gamma_3^* \Gamma_3, \) and \( DC_3^* \) does not involve \( C_3^*. \) Now

\[
C_1^* (C_1^* C_2^*) = (C_1^* C_1^*) C_2^* = (vE + k_1 D) C_2^*,
\]

\[
C_1^* (C_1^* C_3^*) = (C_1^* C_1^*) C_3^* = (vE + k_1 D) C_3^* = vC_3^* + \\
\text{terms not involving } C_3^*,
\]

and hence, \( k_1 DC_2^* = vC_3^* + \text{terms not involving } C_3^*. \)
For \((\gamma_1', \gamma_2) \in \Gamma_1^* \Gamma_1^*\) and \((\gamma_1, \delta) \in \Gamma_3^*\), we put
\[
x = |\Gamma_3^*(\gamma_1') \cap \Gamma_2^*(\gamma_2)| \quad \text{and} \quad t = |\Gamma_1^* \Gamma_1^*(\gamma_1) \cap \Gamma_2^*(\delta)|.
\]
Then from the above formula we have
\[
t = \frac{v}{k_1}.
\] (1)

Counting in two ways triplilaterals \((\gamma_1, \delta, \gamma_2)\) whose edges are successively \(\Gamma_3^*, \Gamma_2^*\) and \(\Gamma_1^* \Gamma_1^*\), we have
\[
\frac{v(v-1)}{k_1} x = vt. \quad \text{(2)}
\]
(1) and (2) yield
\[(v - 1)x = v,
\]
which is a contradiction. Thus we can conclude that \(v_1 \neq v_2 = v_3\), and hence, \(\Pi_2^* \neq \Pi_3^* \neq \Pi_1^*\). Therefore, we obtain \(\Pi_2 = \Pi_3\) by Lemma 16, \(\Pi_2^* \Pi_2^* \neq \Pi_1^* \Pi_1^* \neq \Pi_3^* \Pi_3\) by Lemma 17, and hence we have i) and ii) of Lemma.

For \((\alpha, \gamma) \in \Gamma_1^*\), count in two ways the ordered pairs \((\gamma', \delta)\) such that \(\gamma' \in \Gamma_1^*(\alpha) \setminus \{\gamma\}, \quad \delta \in \Gamma_2^*(\gamma)\) and \((\gamma', \delta) \in \Gamma_3^*\), then since \(\Gamma_1^* \Gamma_1^* \neq \Gamma_3^* \Gamma_3\) we have
\[(v_1 - 1)x = v_2. \quad \text{(3)}
\]

Now, we shall show that \(x = 1\). Assume \(x > 1\), then there exist quadrilaterals \((\gamma, \delta_1, \gamma', \delta_2)\) whose edges are successively \(\Pi_2^*, \Pi_3^*, \Pi_1^* \Pi_1^*\) and \(\Gamma_2\) whose edges are all distinct, and \((\gamma, \gamma') \in \Gamma_1^* \Gamma_1^*\); count all of them in two ways, then we have
\[ |\Omega| v_2(v_2 - 1)\lambda = |\Omega| v_1(v_1 - 1) \frac{x(x - 1)}{k_1}, \]

\[(\lambda = |\prod^*_{1\circ} P_1(\gamma) \cap P_3(\delta_1) \cap P_3(\delta_2)| \text{ for } (\gamma, \delta_1), (\gamma, \delta_2) \neq \)

\[\in \prod^*_{2}, (\delta_1, \delta_2) \in P_2 \circ P_2 \), \]

so

\[(v_2 - 1)\lambda k_1 = v_1(x - 1) = (v_1 - 1)x + x - v_1 = v_2 + x - v_1. \]

Therefore, \(x \geq v_1 - 1\). If \(x = v_1\) then \((v_2 - 1)\lambda k_1 = v_2\), which is a contradiction. If \(x > v_1\), then \(v_2 = (v_1 - 1)x > \frac{v_1(v_1 - 1)}{k_1} \).

So \((\prod^*_{2}, \prod(\prod^*_{1\circ} P_1(\gamma)))_{\gamma} = 1\), where \(\prod(\prod^*_{1\circ} P_1(\gamma))\) is the permutation character of \(G_{\gamma}\) on \(\prod^*_{1\circ} P_1(\gamma)\). Hence, for \((\gamma, \gamma') \in \prod^*_{1\circ} P_1\), \(G_{\gamma}, \gamma,\) is transitive on \(\prod^*_{2}(\gamma)\). So \(\prod^*_{2}(\gamma) = \prod^*_{3}(\gamma')\). This is impossible.

Thus we have \(x = v_1 - 1, k_1 = \lambda = 1, v_2 = (v_1 - 1)^2\) and

\[|\prod^*_{1\circ} P_1(\gamma) \cap P_3(\delta)| = v_1 \text{ for } (\gamma, \delta) \in \prod^*_2. \]

Now, count in two ways quadrilaterals \((\omega, \gamma_1, \gamma_2, \gamma_3)\) such that \((\omega, \gamma_1) \in P_2, (\omega, \gamma_2), (\omega, \gamma_3) \in P_3,\) and \((\gamma_1, \gamma_2), (\gamma_1, \gamma_3) \in \prod^*_{1\circ} P_1,\gamma_2 \neq \gamma_3;\) then we have

\[|\Omega| v_3(v_3 - 1)\lambda' = |\Omega| v_2 v_1(v_1 - 1), \]

\[(\lambda' = |\prod^*_{1\circ} P_1(\gamma_2) \cap \prod^*_{1\circ} P_1(\gamma_3) \cap P_3(\omega)| \text{ for } \gamma_2, \gamma_3 \neq \)

\[\in P_3(\delta)) \]

so

\[\lambda' = \frac{v_1(v_1 - 1)}{v_3 - 1} = \frac{v_1(v_1 - 1)}{(v_1 - 1)^2 - 1} = \frac{v_1 - 1}{v_1 - 2}. \]
Therefore, $v_1 = 3$. This is contrary to the hyposesis of Lemma.
Thus we can conclude that $x = 1$, and hence, by (3) we have $v_1 = v_2 + 1 = v_3 + 1$. This proves Lemma iii).

Lastly, we shall show that $k_1 = 2$. If $k_1 = 1$, then
$$\left| \Gamma_1^* \cap \Gamma_1 (\xi) \right| = v_1 (v_1 - 1) \leq \left| \Gamma_2^* \cap \Gamma_3 (\alpha) \right| \leq v_2 v_3 = (v_1 - 1)^2$$
This is impossible. Now, we have
$$u = \left| \Gamma_1^* \cap \Gamma_1 (\gamma) \cap \Gamma_3 (\delta) \right| = \frac{v_1}{k_1} \text{ for } (\gamma, \delta) \in \Gamma_2^*,$$
and $2 \leq k_1 < \frac{v_1}{2}$.

Count again in two ways quadrilaterals $(\alpha', \gamma_1, \gamma_2, \gamma_3)$ such that $(\alpha', \gamma_1) \in \Gamma_2$, $(\alpha', \gamma_2)$, $(\alpha', \gamma_3) \in \Gamma_3$ and $(\gamma_1, \gamma_2)$, $(\gamma_1, \gamma_3) \in \Gamma_1^* \cap \Gamma_1$, $\gamma_2 \neq \gamma_3$; then
$$|\Omega|(v_1 - 1)(v_1 - 2)\lambda'' = |\Omega|(v_1 - 1)\frac{v_1}{k_1} - 1)^2 = \frac{v_1}{k_1}.$$
$$\lambda'' = \left| \Gamma_1^* \cap \Gamma_1 (\gamma_2) \cap \Gamma_1^* \cap \Gamma_3 (\gamma_3) \cap \Gamma_2 (\xi) \right| \text{ for } \gamma_2, \gamma_3 (\neq) \in \Gamma_3 (\xi))$$

so
$$\lambda'' = \frac{v_1(v_1-k_1)}{(v_1-2)k_1} = \frac{u(u-1)k_1^2}{(k_1 u-2)} = \frac{u(u-1)}{k_1 u-2}.$$

If $u$ is odd, then $k_1 u-2$ divides $u-1$. This is impossible.

We put $u = 2u_0$, then
$$\lambda'' = \frac{2u_0(2u_0-1)}{2k_1 u_0-2} = \frac{u_0(2u_0-1)}{k_1 u_0-1}.$$

Therefore, we conclude that $k_1 = 2$. 
Lemma 20. If \( \mathcal{P}_1 = \mathcal{P}_2 \neq \mathcal{P}_3 \) and \( \mathcal{P}_1 \cap \mathcal{P}_2 \cap \mathcal{P}_3 = \emptyset \), then \( v_1 = v_2 = v_3 + 1 \), \( \mathcal{P}_1 \mathcal{P}_1^* \neq \mathcal{P}_2 \mathcal{P}_2^* \) and \( \mathcal{P}_1 \mathcal{P}_2 = \mathcal{P}_1^* \mathcal{P}_3 \cup \mathcal{P}_1^* \mathcal{P}_1^* \) for some \( \mathcal{P}_1^* \).

Proof. By assumption, \( \sum = \mathcal{P}_1 \mathcal{P}_3^* \) is a G-orbit contained in \( \mathcal{P}_1 \mathcal{P}_2 \). We put \( v = v_1 = v_2 \), \( w = v_3 \), \( |\mathcal{P}_2 (\gamma_1) \cap \mathcal{P}_3 (\gamma_2)| = x \) for \( (\gamma_1, \gamma_2) \in \mathcal{P}_1 \mathcal{P}_1^* \), \( |\mathcal{P}_1^* (\alpha) \cap \mathcal{P}_2^* (\delta)| = y \) and \( |\mathcal{P}_1^* (\alpha) \cap \mathcal{P}_3^* (\delta)| = m \) for \( (\alpha, \delta) \in \sum \), \( |\mathcal{P}_2 (\gamma) \cap \sum (\alpha)| = t \) for \( (\alpha, \gamma) \in \mathcal{P}_1 \).

By Lemma 15, \( \mathcal{P}_2 \neq \mathcal{P}_3^* \), and hence, \( \mathcal{P}_1 \mathcal{P}_3^* \) is a G-orbit. We have

\[
\frac{v(v-1)}{k_1} = |\mathcal{P}_1^* \mathcal{P}_1^* (\gamma_1)| = |\mathcal{P}_2^* \mathcal{P}_3^* (\gamma_1)| = \frac{vw}{x},
\]

so

\[
(v - 1)x = wk_1. \tag{1}
\]

We have also \( |\sum (\alpha)| = \frac{vw}{m} = \frac{vt}{y} \), and so

\[
wy = tm. \tag{2}
\]

Count in two ways quadrilaterals \( (\alpha, \gamma_1, \delta, \gamma_2) \) whose edges are successively \( \mathcal{P}_1^*, \mathcal{P}_2, \mathcal{P}_3^* \) and \( \mathcal{P}_1^* \), then we have

\[
|\Omega| \frac{v(v-1)}{k_1} k_1x = |\Omega| \frac{vw}{m} my,
\]

so

\[
(v - 1)x = wy. \tag{3}
\]

(1) and (3) yield

\[
y = k_1. \tag{4}
\]
From (2) and (3),

\[(v - 1)x = tm. \tag{5}\]

We shall show that \(m = 1\). If \(m > 1\), then there exist quadrilaterals \((\alpha, \gamma_1, \delta, \gamma_2)\) whose edges are successively \(\Gamma_1^*, \Gamma_3^*, \Gamma_3^*\) and \(\Gamma_2^*\), whose vertices are all distinct; count all of them in two ways, then we have

\[|\Omega|^{w(w-1)}k_3k_1 = |\Omega|^{vw}_m(m-1),\]

so

\[(w - 1)k_1 = v(m - 1).\]

On the other hand, from (3) and (4)

\[(w - 1)k_1 = wk_1 - k_1 = (v - 1)x - k_1,\]

therefore

\[v(m - 1) = (v - 1)x - k_1,\]

so

\[0 < v(x - m + 1) = x + k_1 < 2v. \tag{6}\]

(6) yields

\[x = m, \ v = m + k_1. \tag{7}\]

From (5) and (7),

\[t = v - 1. \tag{8}\]
Thus \(|\Sigma(\alpha)| = \frac{vt}{y} = \frac{v(v-1)}{k_1}\).

If \(\Gamma_1^* \neq \Gamma_2^* \), then by Lemma 10, \(|\Sigma(\alpha)| = \frac{v(v-1)}{k_1+1}\).

This is a contradiction. So we have \(\Gamma_1^* \neq \Gamma_2^* \), and hence,

\[1 = y = k_1.\] (9)

Therefore we have \(m = v - 1\) from (7) and (9), and \(w = (v - 1)^2\) from (2) and (8). So

\[|\Gamma_1^*| = |\Gamma_2^*| = \frac{w(v-1)}{k_3}\]

\[\geq 2w = 2(v - 1)^2 > v(v - 1).\]

This is impossible. Thus, we can conclude that \(m = 1\), and then

by (5) \(t = v - 1\), \(x = 1\) and \(|\Sigma(\alpha)| = \frac{v(v-1)}{k_1}\). By Lemma 10,

\(\Gamma_1^* \neq \Gamma_2^* \), and hence, \(1 = y = k_1\). Therefore, by (2) \(w = v - 1\), \n
\(|\Sigma(\alpha)| = v(v - 1)\). By Lemma 8 iv), \(\Gamma_1^* \Gamma_2^* = \sum \Gamma_1^* \) for some \(\Gamma_1^*\).

Lemma 21. If \(\Gamma_1^* \cap \Gamma_2^* \cap \Gamma_3^* \neq \emptyset\), and \(v_1, v_2, v_3 > 3\), then the following hold:

i) if \(\Gamma_1^* = \Gamma_2^* = \Gamma_3^*\), then \(\pi_2 = \pi_3\)

ii) if \(\pi_1 = \pi_2 \neq \pi_3\), then \(\pi_2 \neq \pi_3^*\) and \(v_1 = v_2 = v_3 + 1\).

iii) if \(\pi_1 \neq \pi_2, \pi_3\), then \(\pi_2 = \pi_3^*, \quad C_1C_2 = C_1C_3^*\)

\[\text{and } v_1 = v_2 + 1 = v_3 + 1.\]

Proof. We have this assertion by arranging from Lemma 15 to Lemma 20.
Lemma 2.2: Suppose that $P_1^* P_2$ and $P_3^* P_3^*$ contain a $G$-orbit $\Sigma$ in $\Omega \times \Omega$, and $P_1^* = P_2 = P_3^*$, $|\Gamma_1^*(\alpha)| > 3$. For $\gamma_1, \gamma_2 (\neq) \Gamma_1^*(\alpha)$ and $\delta \in \Sigma(\alpha)$, the following hold:

i) if $P_1^* P_1^* = P_2^* P_2^* = P_3^* P_3^*$, then $|\Gamma_1^*(\alpha) \cap P_2^*(\delta)| > 1$, $|\Gamma_1^*(\alpha) \cap P_3^*(\delta)| > 1$ and $|\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2) \cap \Sigma(\alpha)| > 1$.

ii) if $P_1^* P_1^* = P_2^* P_2^* \neq P_3^* P_3^*$, then $|\Gamma_1^*(\alpha) \cap P_2^*(\delta)| > 1$, $|\Gamma_1^*(\alpha) \cap P_3^*(\delta)| = 1$, $|\Sigma(\alpha)| = \frac{v(v-1)}{k_1+1}$, and $P_1^* P_2$ contains some $\Gamma_k$.

iii) if $P_1^* P_1^* \neq P_2^* P_2^*, P_3^* P_3^*$, then $|\Gamma_1^*(\alpha) \cap P_2^*(\delta)| = |\Gamma_1^*(\alpha) \cap P_3^*(\delta)| = 1$, $|\Sigma(\alpha)| = v(v-1)$, and $P_1^* P_2$ contains some $\Gamma_k$ and $P_1^* P_3$ contains another $\Gamma_k$.

Proof. Put $|\Sigma(\alpha) \cap P_2(\gamma_1) \cap P_3(\gamma_2)| = \lambda$ for $\gamma_1, \gamma_2 (\neq) \in \Gamma_1^*(\alpha)$. $|\Gamma_1^*(\alpha) \cap P_2^*(\delta)| = x_2$, $|\Gamma_1^*(\alpha) \cap P_3^*(\delta)| = x_3$ for $(\alpha, \delta) \in \Sigma$.

Count in two ways quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$ whose edges are successively $P_1^*, P_2^*, P_3^*$ and $P_1^*$ and $(\alpha, \delta) \in \Sigma$; then we have

$$|\Omega| \frac{v(v-1)}{k_1} \lambda = |\Omega| |\Sigma(\alpha)| x_2 x_3,$$

so

$$v(v-1) \lambda = |\Sigma(\alpha)| x_2 x_3. \quad (1)$$

Assume $P_1^* P_1^* \neq P_2^* P_2^*, P_3^* P_3^*$. Then we have $|\Gamma_1^*(\alpha) \cap P_2^*(\delta)| = |\Gamma_1^*(\alpha) \cap P_3^*(\delta)| = 1$. By (1)

$$v(v-1) \lambda = |\Sigma(\alpha)|.$$
Since $|\Sigma(\mathcal{A})| \leq v(v - 1)$, we have $\lambda = 1$ and $|\Sigma(\mathcal{A})| = v(v - 1)$.

By Lemma 8 iv), $\mathcal{C}_1^* \mathcal{P}_2 = \sum \bigcup \mathcal{P}_i$ and $\mathcal{C}_1^* \mathcal{P}_3 = \sum \bigcup \mathcal{P}_j$ for some $\mathcal{P}_i$, $\mathcal{P}_j$.

By Lemma 8, iii), we have $c_1^*c_2 = S + c_i$, $c_1^*c_3 = S + c_j$. ($S = C(\Sigma)$)

If $c_1 = c_j$, then $c_1^*c_2 = c_1^*c_3$, and hence, by Lemma 19 $\mathcal{P}_1 \neq \mathcal{P}_2$, $\mathcal{P}_3$.

This is contrary to the hypothesis of this lemma. Thus $c_i \neq c_j$, that is, $\mathcal{P}_i \neq \mathcal{P}_j$.

Therefore $|\mathcal{P}_2(\gamma_1) \cap \mathcal{P}_3(\gamma_2)| = |\Sigma(\mathcal{A}) \cap \mathcal{P}_2(\gamma_1) \cap \mathcal{P}_3(\gamma_2)| = \lambda = 1$.

Thus we have iii) of Lemma.

Next assume $\mathcal{P}_1^* \mathcal{P}_2^* \neq \mathcal{P}_1^* \mathcal{P}_2^*$. Then we have $|\mathcal{P}_1(\mathcal{A}) \cap \mathcal{P}_2(\delta)| = 1$. By (1)

$$v(v - 1)\lambda = |\Sigma(\mathcal{A})| x_2.$$  

(2)

Count in two ways triplilaterals $(\mathcal{A}, \delta, \gamma)$ whose edges are successively $\Sigma$, $\mathcal{P}_2^*$ and $\mathcal{P}_1^*$, then we have

$$|\Sigma(\mathcal{A})| x_2 \leq v(v - 1).$$  

(3)

If $x_2 = 1$, then $|\Sigma(\mathcal{A})| = v(v - 1)$ by (2) and (3). By Lemma 8. iv), $\mathcal{P}_1^* \mathcal{P}_2^* \neq \mathcal{P}_2^* \mathcal{P}_2^*$. This is contrary to the assumption. Therefore we have $x_2 > 1$, $\lambda = 1$ and $|\Sigma(\mathcal{A})| x_2 = v(v - 1)$. Since $|\Sigma(\mathcal{A})| x_2 = v(v - 1)$, $|\Sigma(\mathcal{A}) \cap \mathcal{P}_2(\gamma)| = v - 1$ for $(\mathcal{A}, \gamma) \in \mathcal{P}_1^*$.

By Lemma 10. ii), $|\Sigma(\mathcal{A})| = \frac{v(v - 1)}{k_1 + 1}$ and $\mathcal{P}_1^* \mathcal{P}_2$ contains some $\mathcal{P}_1$. 


Now we shall show that \( P_2(\gamma_1) \cap P_3(\gamma_2) = P_2(\gamma_1) \cap P_3(\gamma_2) \cap \Sigma(\alpha) \), for \( \gamma_1, \gamma_2 \in P_1(\alpha) \). If \( P_2(\gamma_1) \cap P_3(\gamma_2) \cap \Sigma(\alpha) \neq \emptyset \), then \( |P_2(\gamma_1)| = |P_3(\gamma_2)| \). But \( |P_2(\gamma_1)| = |P_3(\gamma_2)| = 1 \). This is impossible. Therefore, \( |P_2(\gamma_1) \cap P_3(\gamma_2)| = \lambda = 1 \). Thus we have ii) of Lemma.

Last assume \( P_1^0 P_1^* = P_2^0 P_2^* = P_3^0 P_3^* \). We shall show that \( x_2 = |P_2^*(\alpha) \cap P_2^*(\delta)| > 1 \) and \( x_3 = |P_3^*(\alpha) \cap P_3^*(\delta)| > 1 \). We note that \( k_1 = k_2 = k_3 \), therefore we put \( k = k_1 = k_2 = k_3 \). If \( x_2 = x_3 = 1 \), by (1) we have \( |\Sigma(\alpha)| = v(v - 1) \). By Lemma 8. iv)

\[ |P_1^0 P_1^* \neq P_2^0 P_2^*, P_3^0 P_3^* \]. This is contrary to the assumption.

If \( x_2 > x_3 = 1 \), we have \( |\Sigma(\alpha)| = \frac{v(v - 1)}{k + 1} \) as before, and \( x_2 = k + 1 \).

We put \( P_1^0 P_3 = \Sigma \cup \Sigma' \).

\[ x = |P_1^*(\alpha) \cap P_3^*(\delta')| \quad \text{for } (\alpha, \delta') \in \Sigma', \text{ and} \]

\[ t = |P_3(\gamma_1) \cap \Sigma(\alpha)| = \frac{x(v - 1)}{k + 1} \quad \text{for } (\alpha, \gamma_1) \in P_1^* . \]

Since \( P_1^0 P_1^* = P_3^0 P_3^* \) and \( x_3 = 1 \), there exist quadrilaterals \( (\alpha', \gamma_1', \delta', \tau_1' \), with \( \gamma_1' \neq \gamma_2 \) and \( (\alpha', \delta') \in \Sigma' \), whose edges are successively \( P_1^*, P_3, P_3^* \) and \( P_1^0 \). Count all of them in two ways then we have

\[ |\Omega| \frac{v(v - 1) k}{k + 1} = |\Omega| \frac{v(v - 1)}{x} x(x - 1) , \]

so

\[ x - 1 = \frac{(v - 1) k}{v - 1} = \frac{t(k + 1) k}{t(k + 1) + 1 - t} = \frac{tk(k + 1)}{tk + 1} . \]
Therefore $t = 1$, and hence, $v = k + 2$. This is impossible by Lemma 3.

Thus we have $x_2 > 1$ and $x_3 > 1$.

Now we shall show that $\lambda > 1$. If $\lambda = 1$, by (1) we have

$$v(v - 1) = |\Sigma(\alpha)| x_2 x_3$$

Since $x_2 > 1$, there exist quadrilaterals $(\alpha', \gamma_1, \delta, \gamma_2)$, with $\gamma_1 \neq \gamma_2$ and $(\alpha', \delta) \in \Sigma$. whose edges are successively $\Gamma^*_1$, $\Gamma_2$, $\Gamma_2^*$ and $\Gamma_1^*$. Count all of them in two ways, then we have

$$|\Omega| \frac{v(v-1)k}{k} = |\Omega| |\Sigma(\alpha)| x_2 (x_2 - 1),$$

$$\Omega_2 = |\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2) \cap \Sigma(\alpha)|$$

for $\gamma_1, \gamma_2 (\neq) \in \Gamma_1^*(\alpha))$

so

$$\lambda_2 = \frac{|\Sigma(\alpha)| x_2 (x_2 - 1)}{v(v - 1)}.$$

and by (1),

$$\lambda_2 = \frac{x_2 - 1}{x_3}.$$

Thus $\frac{x_2 - 1}{x_3}$ is a positive integer. Since $x_3 > 1$, in the same way, we have that $\frac{x_3 - 1}{x_2}$ is a positive integer. This is impossible.

Thus we have i) of Lemma.

Lemma 23. If $\Gamma_1 \Gamma_2^* = \Gamma_2 \Gamma_2^*$ and $\pi_1 \neq \pi_2$, then for any $\Gamma_1, \Gamma_j (\neq), \Gamma_1 \Gamma_j^* \ni \Gamma_1^* \Gamma_1$. 

Proof. Assume $\Gamma_1^* \supset \Gamma_1^* \cap \Gamma_1$. Note that $|v_1 - v_2| \geq 2$
by Lemma 13, and hence, $\kappa_1 \neq \kappa_2$. If $\{\Gamma_1, \Gamma_j\} = \{\Gamma_1, \Gamma_2\}$, then
since $\Gamma_1^* \cap \Gamma_j^*$ is a G-orbit, $\Gamma_1^* \cap \Gamma_j^* = \Gamma_1^* \cap \Gamma_2^* = \Gamma_2^*$. This is a contrary
to Lemma 14. Therefore we can assume that $\Gamma_1 \neq \Gamma_1, \Gamma_2$.

If $\Gamma_j = \Gamma_1$, then $\Gamma_2 \cap \Gamma_1 \cap \Gamma_2 \neq \emptyset$. By Lemma 21 we have $v_2 = v_1 - 1$.

This is a contradiction. Thus we have $\Gamma_1 \cap \Gamma_2 \emptyset$.

From $v_1 \neq v_2$, we may assume $v_i \neq v_1$. Since $\Gamma_1^* \cap \Gamma_1^* \neq \emptyset$, $v_i = v_1 - 1$ by Lemma 21.

On the other hand, from $|v_1 - v_2| \geq 2$, $v_i \neq v_2$. Since $\Gamma_2^* \cap \Gamma_2^* \neq \emptyset$, in the
same way, we have $v_i = v_2 - 1$. This is a contradiction.
Lemma 24. If \( \Gamma_1 \ast \Gamma_2 = \Gamma_3 \ast \Gamma_3 = \Delta, \ \Gamma_1 = \Gamma_2 = \Gamma_3 \) and \( |\Gamma_1(\alpha)| > 3 \), then \( \Gamma_1 \ast \Gamma_2 \ast \Delta \) or \( \Gamma_1 \ast \Gamma_3 \ast \Delta \).

Proof. Assume \( \Gamma_1 \ast \Gamma_2 \ast \Delta \) and \( \Gamma_1 \ast \Gamma_3 \ast \Delta \). We put \( v = v_1 = v_2 = v_3 \) and \( k = k_1 = k_2 = k_3 \). Since \( \Gamma_1 = \Gamma_2 = \Gamma_3 \), we have \( \Gamma_1^* = \Gamma_2^* = \Gamma_3^* \) by Lemma 21. We shall show that \( \Gamma_1^* \Gamma_1 = \Gamma_2^* \Gamma_2 = \Gamma_3^* \Gamma_3 \).

If \( \Gamma_1^* \Gamma_1 \neq \Gamma_2^* \Gamma_2, \ \Gamma_3^* \Gamma_3 \), \( |\Delta(\alpha)| = v(v-1) \) by Lemma 22. iii).

Since \( |\Gamma_1^* \Gamma_1(\alpha)| = |\Gamma_1 \Gamma_1(\alpha)| = |\Delta(\alpha)| = v(v-1) \), we have \( \Gamma_2^* \Gamma_2 \neq \Gamma_3^* \Gamma_3 \) by Lemma 8. iv). If \( \Gamma_1^* \Gamma_1 = \Gamma_2^* \Gamma_2 = \Gamma_3^* \Gamma_3 \), then \( c_1 c_2 = c_1 c_3 \) by Lemma 10, i); and hence, \( v_1 = v_2 + 1 \) by Lemma 19, iii).

This is contrary to the hypothesis of this lemma. We shall show that \( k > 1 \). If \( k = 1 \), \( |\Gamma_1 \Gamma_1(\alpha)| = \frac{v(v-1)}{k} \) by Lemma 22. ii). This is impossible. Thus we can conclude that \( \Gamma_1^* \Gamma_1 = \Gamma_2^* \Gamma_2 = \Gamma_3^* \Gamma_3 \).

If \( \Gamma_1^* \Gamma_1 = \Gamma_2^* \Gamma_2 = \Gamma_3^* \Gamma_3 \), then \( c_1 c_2 = c_1 c_3 \) by Lemma 10, i); and hence, \( v_1 = v_2 + 1 \) by Lemma 19, iii).

This is contrary to the assumption. Count in two ways quadrilaterals \((\alpha, \gamma_1, \delta, \gamma_2)\) whose edges are successively \( \Gamma_1, \Gamma_1^*, \Gamma_3 \) and \( \Gamma_1^* \); then we have

\[
|\Omega| \frac{v(v-1)}{k} x_2 x_3 = |\Omega| \frac{v(v-1)}{k} x_2 x_3,
\]

so

\[
k x = x_2 x_3. \tag{1}
\]

Here we put \( x_2 = |\Gamma_1(\alpha) \cap \Gamma_2(\delta)|, \ x_3 = |\Gamma_1(\alpha) \cap \Gamma_3(\delta)| \) for
We shall show that \( x, x_2 \) and \( x_3 \) are smaller than \( k \). If \( x_2 \geq k \), then for \( (\alpha, \gamma) \in \mathcal{P}_1 \), \( |A(\alpha) \cap \mathcal{P}_2^*(\gamma)| \geq v - 1 \).

Of course, \( |A(\alpha) \cap \mathcal{P}_2^*(\gamma)| \leq v - 1 \), and hence, \( |A(\alpha) \cap \mathcal{P}_2^*(\gamma)| = v - 1 \). By Lemma 10, ii), we have \( |A(\alpha)| = \frac{v(v - 1)}{k + 1} \), which is a contradiction. We can prove in the same way that \( x_3 < k \).

Then, (1) yields

\[
x < x_2, \quad x_3 < k. \tag{2}
\]

Now

\[
C_1(C_2^*C_3) = C_1(xD' + yS'),
\]

\[
(C_1C_2^*)C_3 = (x_2D' + y_2S)C_3 = x_2(v - 1)C_3 + \text{terms not involving } C_3.
\]

\[
(\Delta' = \mathcal{P}_1^* \cap \mathcal{P}_1, \quad \mathcal{P}_1 \cap \mathcal{P}_2^* = \Delta' \cup \Sigma, \quad \mathcal{P}_2^* \cap \mathcal{P}_3 = \Delta' \cup \Sigma'),
\]

\[
D = C(\Delta), \quad D' = C(\Delta'), \quad S = C(\Sigma) \text{ and } S' = C(\Sigma').
\]

Since \( x_2 > x \) and the coefficient of \( C_3 \) contained in \( C_1D' \) is at most \( v - 1 \), \( C_3 \) is contained in \( C_1S' \), that is, \( \mathcal{P}_1^* \cap \mathcal{P}_3 \supset \Sigma' \).

On the other hand, since \( \mathcal{P}_1^* \cap \mathcal{P}_3 \supset \Delta \), there exists the following figure.

Therefore \( \mathcal{P}_1^* \cap \mathcal{P}_3 \supset \Delta' \). Thus \( \mathcal{P}_1^* \cap \mathcal{P}_3 = \Delta' \cap \Sigma' = \mathcal{P}_2^* \cap \mathcal{P}_3 \).
By Lemma 10, i) we have $c_1^*c_3 = c_2^*c_3$. So, $\mathcal{P}_1 \neq \mathcal{P}_3$ by Lemma 19, i). This is contrary to the hypothesis of this lemma.

**Lemma 25.** If $v_1, v_2, v_3$ and $v_4 > 3$, then the following figures don’t exist.

![Figures 1-4](image)

**Proof.** For each figure above, we assume its existence and show that it implies a contradiction.

**Non-existence of Fig 1.**

**Case I.** $\mathcal{P}_1 \neq \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$.

By Lemma 18 and Lemma 19, $v_1 = v_2 + 1 = v_3 + 1 = v_4 + 1$,

$$|\Gamma_{\alpha} \cap \Gamma_{\beta}| = \frac{v_1(v_1-1)}{2}, \quad |\Gamma_{\beta} \cap \Gamma_{\gamma}| = |\Gamma_{\beta} \cap \Gamma_{\delta}| = 1 \text{ for } (\alpha, \beta) \in \Gamma^*_{\alpha} \text{ and } \mathcal{P}_2^* = \mathcal{P}_3^* = \mathcal{P}_4^*.$$  

Now let us consider the following figure.

![Figure 5](image)
Then by Lemma 22, i) and iii), we have

$$|\Gamma_1 \circ \Gamma_1^*(\alpha)| = v_2(v_2-1) = (v_1-1)(v_1-2).$$

Thus,

$$|\Gamma_1 \circ \Gamma_1^*(\alpha)| = \frac{v_1(v_1-1)}{2} = (v_1-1)(v_1-2),$$

so

$$v_1 = 4, v_2 = v_3 = v_4 = 3.$$

This is contrary to the hypothesis of this lemma.

Case II. \( \mathcal{G}_1 = \mathcal{G}_2 \neq \mathcal{G}_3, \mathcal{G}_4 \).

By Lemma 21, \( v_1 = v_2 = v_3 + 1 = v_4 + 1 \) and \( \mathcal{G}_3^* = \mathcal{G}_4^* \neq \mathcal{G}_2^* \). But considering the following figure,

we have \( v_3 = v_2 + 1 \) by Lemma 20. This is impossible.

Case III. \( \mathcal{G}_1 = \mathcal{G}_2 = \mathcal{G}_3 \neq \mathcal{G}_4 \).

By Lemma 20, \( v_1 = v_2 = v_3 = v_4 + 1 \). But since there exists the following figure,
we have \( v_4 = v_3 + 1 = v_2 + 1 \) by Lemma 21, which is a contradiction.

**Case IV.** \( \pi_1 = \pi_2 = \pi_3 = \pi_4, \quad \pi_1^* \pi_1 = \pi_2^* \pi_2 = \pi_3^* \pi_3 = \pi_4^* \pi_4 \).

Existence of the following figure is contrary to Lemma 24.

\[
\begin{array}{c}
\bullet \quad \pi_2 \\
\pi_1^* \pi_1 \\
\pi_3 \\
\pi_4 \\
\pi_2 \\
\end{array}
\]

**Case V.** \( \pi_1 = \pi_2 = \pi_3 = \pi_4, \quad \pi_1^* \pi_1 = \pi_2^* \pi_2 = \pi_3^* \pi_3 \neq \pi_4^* \pi_4 \).

Since \( \pi_1^* \pi_1 = \pi_2^* \pi_2 = \pi_3^* \pi_3 \), we have by Lemma 22, i)

\[ |P_2(\gamma_1) \cap P_3(\gamma_2)| > 1 \text{ for } (\gamma_1, \gamma_2) \in P_1^* P_1, \text{ and hence, } \pi_2^* \pi_2 = P_3^* P_3. \]

So, we have \( |P_1^* \pi_1(\alpha)| < v_1(v_1-1) \) by Lemma 8, iv). On the other hand, since \( \pi_1^* \pi_1 = \pi_2^* \pi_2 = \pi_3^* \pi_3 \neq \pi_4^* \pi_4 \), we have by Lemma 22, ii)

\[ |P_4(\gamma_1) \cap P_2(\gamma_2)| = |P_4(\gamma_1) \cap P_3(\gamma_2)| = 1 \text{ for } (\gamma_1, \gamma_2) \in P_1^* P_1. \]

Then from the existence of the following figure,

\[
\begin{array}{c}
\pi_4 \\
\pi_2 \\
\pi_1^* \pi_1 \\
\pi_3 \\
\end{array}
\]

we have \( |P_1 \pi_1(\alpha)| = v_1(v_1-1) \) by Lemma 22, which is a contradiction.

**Case VI.** \( \pi_1 = \pi_2 = \pi_3 = \pi_4, \quad \pi_1^* \pi_1 = \pi_2^* \pi_2 \neq \pi_3^* \pi_3, \quad \pi_4^* \pi_4 \).

There exist the following figures, where \( \Sigma \) is a G-orbit.
From Fig. a, we have $|\Sigma(\alpha)| = v_1(v_1 - 1)$ by Lemma 22, iii).

On the other hand, from Fig. b, we have $|\Sigma(\alpha)| = \frac{v_1(v_1 - 1)}{k_1 + 1}$ by Lemma 22, ii), which is a contradiction.

**Case VII.** $\mathcal{R}_1 = \mathcal{R}_2 = \mathcal{R}_3 = \mathcal{R}_4$, $\Gamma_1^* \Gamma_1^* \neq \Gamma_2^* \Gamma_2^*$; $\Gamma_3^* \Gamma_3^*$, $\Gamma_4^* \Gamma_4^*$.

From $\Gamma_1^* \Gamma_1^* \neq \Gamma_2^* \Gamma_2^*$, $\Gamma_3^* \Gamma_3^*$, we have $|P_2(\gamma_1) \cap P_3(\gamma_2)| = 1$ for $\gamma_1, \gamma_2 (\neq) \in \Gamma_1^*(\alpha)$, by Lemma 22, iii).

Similarly from $\Gamma_1^* \Gamma_1^* \neq \Gamma_2^* \Gamma_2^*$, $\Gamma_4^* \Gamma_4^*$, we have $|P_2(\gamma_1) \cap P_4(\gamma_2)| = 1$ for $\gamma_1, \gamma_2 (\neq) \in \Gamma_1^*(\alpha)$. From $P_2 \Gamma_3 \cap P_2 \Gamma_4 \supset P_1 \Gamma_1^*$,

we have by Lemma 22

$$|\Gamma_1^* \Gamma_1^*(\alpha)| = v_1(v_1 - 1).$$

(1)

By Lemma 21, $\mathcal{R}_2^* = \mathcal{R}_3^* = \mathcal{R}_4^*$. Therefore we have by Lemma 8, iv)

$$\Gamma_2^* \Gamma_2^* \neq \Gamma_3^* \Gamma_3^*; \quad \Gamma_3^* \Gamma_3^* \neq \Gamma_4^* \Gamma_4^* \text{ and } \Gamma_4^* \Gamma_4^* \neq \Gamma_2^* \Gamma_2^*$$

and $\Gamma_1^* \Gamma_j^*$ $(2 \leq i, j(\neq) \leq 4)$ contains some $\Gamma_k^*$. (2)
We put
\[ v = v_1 = v_2 = v_3 = v_4, \quad P_1^* \cdot P_1 = \Delta_1, \quad P_2^* \cdot P_2 = \Delta_2, \]
\[ P_2^* \cdot P_3 = \Delta_1 \cup P_1, \quad P_3^* \cdot P_4 = \Delta_2 \cup \sum', \quad \text{and } D_1 = C(\Delta_1), \]
\[ D_2 = C(\Delta_2), \quad s' = C(\sum') \quad \text{and} \quad s' = |\sum'(\omega)|. \]

Now,
\[ (C_2 C_3^*) C_4 = (D_1 + C_1) C_4 = (v-1) C_3 + \cdots . \]

The coefficient of \( C_3 \) of the above equation is \( v - 1 \) or \( v \) by (2).

Next,
\[ C_2 (C_3^* C_4) = C_2 (D_2 + xS'), \]
so
\[ v^2 = \frac{v(v-1)}{k_2} + xs'. \]

By Lemma 8, i), \( s' \geq v \),
so
\[ x \leq v - \frac{v-1}{k_2} \leq v - 2. \quad (3) \]

We shall show that \( P_4^* \cdot P_4 \neq \sum' \). If \( P_4^* \cdot P_4 = \sum' \), there exists the following figure.

![Diagram](https://via.placeholder.com/150)
Since $P_3^*P_4^* = \Delta_1 \cup \Gamma_3^*$, we have $P_4^*P_4^* = \Delta_1 = P_1^*P_1^*$. This is contrary to the assumption of this case. From $P_2^*P_4^* \cap P_3^*P_4^* \supset \Delta_1$ and (2), for $\gamma_1, \gamma_2 \neq \in P_4^*$ we have by Lemma 22, iii)

$$P_2^*(\gamma_1) \cap P_3^*(\gamma_2) = 1.$$  (4)

If $P_2^* \Sigma'$ contains $P_3$, then we have $P_2^*P_3 = P_4^*P_4 \cup \Sigma'$, and by (4)

$$C_2S' = (v - \frac{v-1}{4})C_3 + \text{terms not involving } C_3.$$  

When $k_4 = 1$, $v - \frac{v-1}{4} = 1$. So $P_2^* \Delta_2$ contains $P_3$, by (1).

When $k_4 \geq 1$, $v - 1 > v - \frac{v-1}{k_4} > \frac{v}{2}$. So, $x = 1$, and hence $P_2^* \Delta_2$ contains $P_3$.

In all cases, we can conclude that $P_2^* \Delta_2$ contains $P_3$, and hence, $P_2^*P_3 \supset \Delta_2$. Thus, we have the following figure.

$$\begin{align*}
\text{\includegraphics[width=0.3\textwidth]{figure.png}}
\end{align*}$$

So, $P_1^*P_1^* = P_2^*P_2^*$. This is contrary to the assumption.

Non-existence of Fig. 2.

Case I. $\pi_1 \neq \pi_2, \pi_3$.

From $P_1^*P_2 \cap P_1^*P_3 \neq \emptyset$ and $\pi_1 \neq \pi_2, \pi_3$, we have $|P_1^*P_1^*(\gamma)|$

$$= \frac{v_1(v_1-1)}{2}, \quad v_1 = v_2 + 1 \text{ and } P_1^*P_1^* \neq P_2^*P_2^* \text{ by Lemma 21 and}$$

$$= \ldots$$
Lemma 19. On the other hand, \( |\Gamma_1 \cdot \Gamma_1^*| = |\Gamma_1^* \cdot \Gamma_1| = |\Gamma_1^* \cdot \Gamma_2| \)

\[ = |\Gamma_1^*| \cdot |\Gamma_2| = v_1(v_1-1). \]

This is impossible.

Case II. \( \Pi_1 = \Pi_2 \neq \Pi_3 \).

By Lemma 20, \( v_1 = v_2 = v_3 + 1 \). On the other hand, from the existence of following figure,

\[ \text{We have } v_3 = v_2 + 1 = v_1 + 1 \text{ by Lemma 21, iii). This is impossible.} \]

Case III. \( \Pi_1 = \Pi_2 = \Pi_3, \Gamma_1 \cdot \Pi_1^* = \Gamma_2 \cdot \Pi_2^* = \Gamma_3 \cdot \Pi_3^* \).

By Lemma 22, for \((\xi, \delta) \in \Gamma_1^* \cdot \Gamma_1, 1 < |\Gamma_1^*(\xi) \cap \Gamma_2^* (\delta)| \) and

\[ 1 < |\Gamma_1^* (\xi) \cap \Gamma_3^* (\delta)|. \]

The counting arguments show that

\[ |\Gamma_1^* (\xi) \cap \Gamma_2^* (\delta)| = |\Gamma_1^*(\xi_1) \cap \Gamma_2^*(\xi_2)| \text{ and } |\Gamma_1^* (\xi) \cap \Gamma_3^* (\delta)| \]

\[ = |\Gamma_1^*(\xi_1) \cap \Gamma_3^*(\xi_2)| \text{ for } (\xi_1, \xi_2) \in \Gamma_1^* \cdot \Gamma_1. \]

Therefore, \( \Gamma_1^* \cdot \Pi_1 = \Gamma_2^* \cdot \Pi_2 = \Gamma_3^* \cdot \Pi_3 \). Now \( \Gamma_1^* \cdot \Pi_2 \supset \Gamma_1^* \cdot \Pi_1 \) and \( \Gamma_1^* \cdot \Pi_3 \supset \Gamma_1^* \cdot \Pi_1 \). Since we can show that \( \Pi_1 = \Pi_2 = \Pi_3 \) by Lemma 21, we have a contradiction by Lemma 24.

Case IV. \( \Pi_1 = \Pi_2 = \Pi_3, \Gamma_1 \cdot \Pi_1^* = \Gamma_2 \cdot \Pi_2^* \neq \Gamma_3 \cdot \Pi_3^* \).

From \( \Gamma_1^* \cdot \Pi_2 \cap \Gamma_1^* \cdot \Pi_3 \supset \Gamma_1^* \cdot \Pi_1 \), we have \( |\Gamma_1^* \cdot \Pi_1| = \frac{v(v-1)}{k_1+1} \) by Lemma 22. This is impossible.

Case V. \( \Pi_1 = \Pi_2 = \Pi_3, \Gamma_1 \cdot \Pi_1^* \neq \Gamma_2 \cdot \Pi_2^* \), \( \Gamma_3 \cdot \Pi_3^* \).
By Lemma 21, we have \( \pi_1^* = \pi_2^* = \pi_3^* \). By Lemma 22, iii),

\[ |\pi_1^* \circ \pi_1^*(x)| = v(v-1), \text{ and by Lemma 8, iv), } \pi_1^* \circ \pi_1 \neq \pi_2^* \circ \pi_2. \]

\[ \begin{array}{c}
\pi_1 \\
\pi_2 \\
\pi_3
\end{array} \quad \begin{array}{c}
\pi_1^* \\
\pi_2^* \\
\pi_3^*
\end{array} \]

From the existence of the above figures, we have \( \pi_1^* \circ \pi_3 = \pi_1^* \circ \pi_1 \cup \pi_2^* \circ \pi_2 \).

Therefore,

\[ v^2 = |\pi_1^*(d)| \cdot |\pi_3^*(d)| = |\pi_1^* \circ \pi_3(d)| = |\pi_1^* \circ \pi_1(d)| + |\pi_2^* \circ \pi_2(d)| = v(v-1) + \frac{v(v-1)}{k_2}. \]

This is impossible.

Non-existence of Fig. 3.

\[ \begin{array}{c}
\Sigma_1 \\
\Sigma_2 \\
\Sigma_3
\end{array} \]

For the above figure, if \( \Sigma_1 = \Sigma_2 \) then there exists the following figure.
This is contrary to non-existence of Fig. 1. Thus we have
\[ \sum_1 \neq \sum_2, \quad \pi_1^* = \pi_2^*, \quad \Gamma_1^* \Gamma_2^* = \sum_1 \cup \sum_2 \] and \( G_\alpha \) is not doubly transitive on \( \sum_1 (\alpha) \) and \( \sum_2 (\alpha) \) by Lemma 12. So, by Lemma 20 we have \( \pi_1^* = \pi_2^* = \pi_3^* = \pi_4^* \). Also \( \Gamma_1^* \Gamma_2 = \Gamma_2^* \Gamma_2 = \Gamma_3^* \Gamma_3 = \Gamma_4^* \Gamma_4 \) by Lemma 22. From \( \Gamma_2^* \Gamma_3 \cap \Gamma_2^* \Gamma_4 \supset \Gamma_1^* \Gamma_1^* \), this is contrary to Lemma 24.

Non-existence of Fig. 4.

There exist the following figures.

\begin{align*}
\text{Fig. a} & \quad \begin{array}{c}
\pi_1 \\
\pi_2 \\
\pi_3 \\
\pi_4
\end{array} \\
\text{Fig. b} & \quad \begin{array}{c}
\pi_1 \\
\pi_2 \\
\pi_3 \\
\pi_4
\end{array}
\end{align*}

Case I. \( \pi_1^* \neq \pi_2^* \).

By Lemma 21, we have \( v_1 = v_2 + 1 \) from Fig. a, and \( v_2 = v_1 + 1 \) from Fig. b. This is impossible.

Case II. \( \pi_1^* = \pi_2^* \neq \pi_3^* \).

By Lemma 20, we have \( v_1 = v_2 = v_3 + 1 \) and \( \Gamma_2^* \Gamma_2 \neq \Gamma_1^* \Gamma_1^* \) from Fig. b. On the other hand, \( \Gamma_2^* \Gamma_2 = \Gamma_3^* \Gamma_1 \subset \Gamma_2^* \Gamma_1 \supset \Gamma_1^* \Gamma_1^* \), and \( \Gamma_2^* \Gamma_1^* \) has some \( \Gamma_1^* \) by Lemma 20, and hence, \( \Gamma_2^* \Gamma_2 = \Gamma_1^* \Gamma_1^* \). This is impossible.

Case III. \( \pi_1^* = \pi_2^* = \pi_3^* \), \( \Gamma_1^* \Gamma_1 = \Gamma_2^* \Gamma_2 = \Gamma_3^* \Gamma_3 \).

By assumption, \( \Gamma_2^* \Gamma_1 \cap \Gamma_2^* \Gamma_3 \supset \Gamma_1^* \Gamma_1 = \Gamma_2^* \Gamma_2 = \Gamma_3^* \Gamma_3 \), which
contrary to Lemma 24.

Case IV. $\pi_1^* = \pi_2^* = \pi_3^*$, \( \pi_1^* \cap \pi_2^* \neq \pi_3^* \). From Fig. a, \( \pi_1^o \pi_2^* = \pi_1^o \pi_1^* \cup \pi_1^* \) for some \( \pi_1 \) by Lemma 22. So, \( \pi_1^o \pi_2^* \cap \pi_1^o \pi_3^* = \pi_1^o \pi_1^* \) and \( \pi_1^o \pi_1^* (\xi) = \frac{v(v-1)}{k_1+1} \). This is impossible.

Case V. $\pi_1^* = \pi_2^* = \pi_3^*$, \( \pi_1^* \cap \pi_2^* \neq \pi_3^* \). We put \( \sum = \pi_1^o \pi_3^* \cap \pi_1^o \pi_2^* \).

By Lemma 22, \( |\pi_1^o \pi_1^* (\xi)| = \frac{v(v-1)}{k_1+1} \).

From that \( \pi_1^o \pi_2^* \supset \pi_1^o \pi_1^* \), we have \( \pi_1^o \pi_2^* = \sum \cup \pi_1^o \pi_1^* \). So \( v^2 = \frac{v(v-1)}{k_1+1} + \frac{v(v-1)}{k_1} \). Therefore \( k_1 = 1 \) and \( v-1 = k_1 + 1 = 2 \). This is contrary to the hypothesis of this lemma.

Case VI. $\pi_1^* = \pi_2^* = \pi_3^*$, \( \pi_1^* \cap \pi_2^* \neq \pi_3^* \). We put \( \sum = \pi_1^o \pi_2^* \cap \pi_1^o \pi_3^* \). By Lemma 22, we have \( \pi_1^o \pi_2^* = \sum \cup \pi_1^* \).

\( \pi_1^o \pi_3^* = \sum \cup \pi_j^* \) from some \( \pi_1^*, \pi_j^* \) and \( |\sum (\xi)| = v(v-1) \). Since \( \pi_1^o \pi_2^* \supset \pi_1^o \pi_1^* \) and \( \pi_1^o \pi_3^* \supset \pi_2^o \pi_2^* \), we have \( \pi_1^o \pi_1^* = \sum = \pi_2^o \pi_2^* \).

On the other hand, since \( \pi_1^o \pi_2^* \supset \pi_1^o \pi_1^* \) and \( |\pi_1^o \pi_1^* (\xi)| = v(v-1) \), \( \pi_1^o \pi_1^* \neq \pi_2^o \pi_2^* \) by Lemma 8, iv). This is impossible.

Lemma 26. For \( \pi_1^*, \pi_2^* \) and \( \pi_3^* \), suppose that \( \pi_1^o \pi_2^* \cap \pi_1^o \pi_3^* \) contains a G-gobits \( \sum \) in \( \Omega \times \Omega \), and \( v_1^*, v_2^*, v_3^* > 3 \). Then, there does not exist \( \pi_1^* \) such that \( \pi_1^o \pi_1^* = \sum \).
Proof. From non-existences of Fig. 2, Fig. 3, Fig. 4 of Lemma 24, we have this assertion.

Lemma 27. (P. J. Cameron [3], Prop.)

If \( \pi_i^* \neq \pi_i \) and \( \pi_i \cap \pi_i = \pi_i \cup \pi_i^* \cup (\pi_i \cup \pi_i^*) \cup (\pi_i^* \cap \pi_i) \), then \( G \) has rank 4.
3. Proof of Theorem 1.

We put

\[ x_i = \# \{ p_j \mid \Delta_i = p_j^* \}, \]
\[ y_i = \# \{ (p_k, p_l) \mid p_k^* p_l^* \supset \Delta_i \} \]

and assume that \( x_1 \geq \cdots \geq x_r > x_{r+1} = \cdots = x_t = 0 \). Counting in two ways triplilaterals \((p_k^*, p_l, \Delta_i)\) such that \( p_k^* p_l^* \supset \Delta_i \), we have by Lemma 9 and 11

\[ s^2 \leq \sum_{i=1}^{t} y_i. \]

The equality means that, for any \( p_i \) and \( p_j \), we cannot have

\[ p_i p_j^* = \Delta_k \cup \Delta_{k'}, \quad \Delta_k \neq \Delta_{k'}. \]

When \( x_i > 0 \), by Lemma 26 \( y_i \leq x_i + s \). When \( x_i = 0 \), by non-existence of Fig. 1 of Lemma 25 \( y_i \leq 2s \). Therefore

\[ s^2 \leq \sum_{i=1}^{t} y_i \leq \sum_{i=1}^{r} (x_i + s) + 2(t - r)s, \]

so

\[ s^2 \leq (r + 1)s + 2(t - r)s, \]

\[ s \leq 2t - r + 1. \] (1)
Now, let $A_1 = \Gamma_{i_0}^* \Gamma_{i_0}$ and we put

$$A = \left\{ \Gamma_i, \Gamma_j \mid \Gamma_i \neq \Gamma_j \right\},$$

$$B = \left\{ \Gamma_i \mid \left\{ \Gamma_i, \Gamma_j \right\} \subseteq A \right\}.$$ 

For $\left\{ \Gamma_i, \Gamma_j \right\}, \left\{ \Gamma_k, \Gamma_l \right\} \in A$, $\left\{ \Gamma_i, \Gamma_j \right\} \cap \left\{ \Gamma_k, \Gamma_l \right\} = \emptyset$ by Lemma 26. Therefore $|B| = 2|A|$. Furthermore, for $\left\{ \Gamma_i, \Gamma_j \right\}, \left\{ \Gamma_k, \Gamma_l \right\} \in A$, and for $\Gamma_m, \Gamma_n \in B$, $\Gamma_i \Gamma_j \cap \Gamma_k \Gamma_l$ are disjoint to each other by non-existence of Fig. 1 of Lemma 25. Thus we have

$$|A| + (s - |B|) = s - |A| \leq t,$$  \hspace{1cm} (2)

and by Lemma 26

$$|A| - 1 \leq t - r.$$  \hspace{1cm} (3)

Assume $s = 2t - r + 1$. Since the equality of (1) hold

$$y_1 = x_1 + s,$$

and hence $|A| = \frac{s}{2}$ and $\frac{s}{2} - 1 \leq t - r$ by (3), and hence, $2t - r + 1 = s \leq 2t - 2r + 2$. So $r = 1$. Therefore, if $r > 1$, we conclude that $s \leq 2t - r$.

We shall show that when $r = 1$, $s \leq 2t - 2$. Assume $r = 1$ and

$$2t \geq s \geq 2t - 1,$$

and put $\Delta = \Gamma_i^* \Gamma_i^*$, $1 \leq i \leq s$. If $\Gamma_i \neq \Gamma_j$, for some $\Gamma_i$ and $\Gamma_j$, then by Lemma 23, $\Delta \in \Gamma_k^* \Gamma_l^*$ for any $\Gamma_k, \Gamma_l \neq \emptyset$, for any $\Gamma_k, \Gamma_l \neq \emptyset$. \hspace{1cm}
and hence, \( \Gamma_i^* \cap \Gamma_j^* \cap \Gamma_k^* \cap \Gamma_l^* = \emptyset \). So \( s \leq t \). This is contrary to the assumption that \( t \geq 2 \). Thus, it holds that \( \pi_1 = \pi_2 = \cdots = \pi_s \).

Now, Suppose \( \Gamma_i^* \cap \Gamma_j^* = \Delta \cup \Gamma_k^* \) for some \( \Gamma_i^* \), \( \Gamma_j^* \) and \( \Gamma_k^* \), and put \( D = C(\Delta) \), \( \Gamma_j^* \cap \Gamma_k^* = \Delta \cup \Gamma_i^* \), \( D' = C(\Delta') \), \( t = |\Gamma_i^* \cap \Gamma_j^*(\beta)| \) for \( (\alpha', \beta) \subseteq \Gamma_i^* \), \( x = |\Gamma_i^* \cap \Gamma_j^*(\delta)| \) for \( (\alpha, \delta) \subseteq \Delta \), \( v = v_1 = v_2 = \cdots \), \( k = k_1 = k_2 = \cdots \). Then we have

\[
(C_i C_j) C_k = (tC_k^* + xD) C_k = tvI + tkD + xDC_k, \\
C_i (C_j C_k) = C_i (t'C_i^* + x'D') = t'vI + t'kD + x'C_i D'.
\]

\[
t' = \left| \Gamma_j^*(\alpha) \cap \Gamma_j^*(\beta) \right| \text{ for } (\alpha', \beta) \subseteq \Gamma_i^*, \quad x' = \left| \Gamma_j^*(\alpha) \cap \Gamma_j^*(\delta) \right| \text{ for } (\alpha, \delta) \subseteq \Delta'.
\]

We have \( t = t' \) by counting in two ways triplilateral \( (\alpha, \beta, \gamma) \) whose edges are successively \( \Gamma_i^* \), \( \Gamma_j^* \) and \( \Gamma_k^* \), and have \( |\Delta(\alpha)| = |\Delta'(\alpha')| \) and \( x = x' \) by Lemma 10.

So,

\[
C_i D' = DC_k = (v - 1) C_k + \cdots
\]

If \( C_i \neq C_k \), \( |\Delta'(\alpha')| = \frac{v(v - 1)}{k + 1} \) by Lemma 10. This is impossible.

Thus \( C_i = C_k \). Similarly, \( C_j = C_k \).

When \( S = 2t \), then the equality of (1) holds. Therefore, for any \( \Gamma_i^* \), there exists \( \Gamma_j^* \) such that \( \Gamma_i^* \cap \Gamma_j^* = \Delta \cup \Gamma_k^* \) for some \( \Gamma_k^* \).

So, as is shown above, \( \Gamma_i^* = \Gamma_j^* = \Gamma_k^* \). Therefore we have for any \( \Gamma_i \).

\[
\Gamma_i \neq \Gamma_i^*, \quad \Gamma_i^* \cap \Gamma_i^* = \Delta \cup \Gamma_i^* \text{ and } \Gamma_i^* \cap \Gamma_i^* \cap \Gamma_i^* = \emptyset
\]

for \( \Gamma_i \neq \Gamma_i^* \).
When \( s = 2t - 1 \), then \(|A| \leq t - 1\), and from (2) \( s - |A| \leq t\).

So \(|A| = t - 1\). Therefore, there is a unique \( \mathcal{P}_u \) such that for any \( \mathcal{P}_i (\neq \mathcal{P}_u) \), \( \mathcal{P}_i \mathcal{P}_u^* \not\supseteq \Delta \). We shall show that for any \( \mathcal{P}_i, \mathcal{P}_j (\neq) \), \( \mathcal{P}_i \mathcal{P}_j^* \) contains some \( \mathcal{P}_k \). Assume \( \mathcal{P}_i \mathcal{P}_j^* = \Delta_k \cup \Delta_n \) for some \( \mathcal{P}_i, \mathcal{P}_j (\neq) \).

Count in two ways the paired \( (\mathcal{P}_m, \Delta_n) \) such that \( \mathcal{P}_i \mathcal{P}_m^* \) contains \( \Delta_n \), then by Lemma 25, we have

\[
2t = s + 1 \leq \# \{ (\mathcal{P}_m, \Delta_n) | \mathcal{P}_i \mathcal{P}_m^* \supseteq \Delta_n \} \leq 2t.
\]

So, equality holds. Thus for any \( \Delta_k \), there exist \( \mathcal{P}_p \) and \( \mathcal{P}_q (\neq) \) such that \( \mathcal{P}_i \mathcal{P}_p^* \) and \( \mathcal{P}_i \mathcal{P}_q^* \) contains \( \Delta_k \). Therefore we may choose \( \mathcal{P}_a \) such that \( \mathcal{P}_i \mathcal{P}_u^* \cap \mathcal{P}_i \mathcal{P}_a^* \neq \emptyset \) and \( \mathcal{P}_a \neq \mathcal{P}_u \). Then \( \mathcal{P}_a \mathcal{P}_u^* \supseteq \mathcal{P}_i \mathcal{P}_i^* = \Delta \).

This is impossible. Thus, again as is shown above, we can conclude that for any \( \mathcal{P}_i (\neq \mathcal{P}_u) \),

\[
\mathcal{P}_i \neq \mathcal{P}_i^*, \mathcal{P}_i \mathcal{P}_i^* = \Delta \cup \mathcal{P}_i^* \text{ and } \mathcal{P}_i \mathcal{P}_m^* \cap \mathcal{P}_i \mathcal{P}_n^* = \emptyset \text{ for } \mathcal{P}_m \neq \mathcal{P}_n, \mathcal{P}_m^*.
\]

Thus if \( s \geq 2t - 1 \), there exists \( \mathcal{P}_i \) such that

\[
\mathcal{P}_i \neq \mathcal{P}_i^* \text{ and } \mathcal{P}_i \mathcal{P}_i^* = \mathcal{P}_i \mathcal{P}_i^* \cup \mathcal{P}_i^*.
\]

By Lemma 27, this show that G has rank 4. This is impossible for \( s \geq 2t - 1 \) and \( t \geq 2 \).

### 4. Proof of Theorem 2

When \( r = t \), we have \( s \leq t \) by Theorem 1. On the other hand, from \( s \geq r = t \), we conclude that \( s = t = r \).
We put $\mathcal{P}_i \circ \mathcal{P}_j = \Delta_i$, $A_i = \{ \{ \mathcal{P}_k, \mathcal{P}_l \} : \text{unordered pair} \}$, $\mathcal{P}_k \neq \mathcal{P}_l$. Then $|A_i| = t - r = 0$, so $|A_i| \leq 1$. Count in two ways triplilaterals $(\mathcal{P}_i, \mathcal{P}_j, \Delta_k)$ such that $\mathcal{P}_i \circ \mathcal{P}_j \supseteq \Delta_k$, we have

$$s^2 \leq 3s,$$

so

$$s \leq 3. \quad (1)$$

Case $t = 2$. If $|\mathcal{P}_1(\mathcal{A})| \neq |\mathcal{P}_2(\mathcal{A})|$, by T. Ito [6], $G$ is isomorphic to the small Janko simple group and $G_\mathcal{A}$ is isomorphic to $PSL(2,11)$. We shall prove that the case of $|\mathcal{P}_1(\mathcal{A})| = |\mathcal{P}_2(\mathcal{A})|$ does not occur. We put $|\mathcal{P}_1(\mathcal{A})| = |\mathcal{P}_2(\mathcal{A})| = v$. It is easy to prove that $\mathcal{P}_1 = \mathcal{P}_2$. We shall show that $\mathcal{P}_1$ and $\mathcal{P}_2$ are self paired.

If not, then $\mathcal{P}_1^* = \mathcal{P}_2$. Since $\mathcal{P}_1 \circ \mathcal{P}_1^* \neq \mathcal{P}_2 \circ \mathcal{P}_2^*$, we have that $\mathcal{P}_1 \circ \mathcal{P}_1^* \neq \mathcal{P}_2 \circ \mathcal{P}_2^*$ by Lemma 7. By Lemma 11, there exists a $G$-orbit $\Sigma$ in $\mathcal{P}_1 \circ \mathcal{P}_1$ on which $G_\mathcal{A}$ is not 2-transitive on $\Sigma(\mathcal{A})$, and $\Sigma \neq \Delta_1, \Delta_2$. This is impossible for $t = 2$. Thus, we have

$$\mathcal{P}_1 \circ \mathcal{P}_2 = \Delta_1 \cup \Delta_2.$$ 

So, $v^2 = |\mathcal{P}_1 \circ \mathcal{P}_2(\mathcal{A})| = |\mathcal{P}_1 \circ \mathcal{P}_1(\mathcal{A})| + |\mathcal{P}_2 \circ \mathcal{P}_2(\mathcal{A})| = \frac{v(v-1)}{k_1} + \frac{v(v-1)}{k_2}$. This is impossible.

Case $t = 3$. For this case, the equality of $(1)$ holds. So we have $|A_i| = 1$ for $1 \leq i \leq 3$. We shall show that if $\mathcal{P}_i \circ \mathcal{P}_j^* = \Delta_1$, then $\mathcal{P}_i = \mathcal{P}_1$ or $\mathcal{P}_j = \mathcal{P}_1$. If $\mathcal{P}_i, \mathcal{P}_j \neq \mathcal{P}_1$, then since $\mathcal{P}_i \circ \mathcal{P}_1 \neq \mathcal{P}_1 \circ \mathcal{P}_j$, there exists a $G$-orbit $\Sigma$ in $\mathcal{P}_1 \circ \mathcal{P}_1 \cap \mathcal{P}_1 \circ \mathcal{P}_j$ such that $G$ is not 2-transitive on $\Sigma(\mathcal{A})$ by Lemma 12, and for any $\mathcal{P}_1$, $\mathcal{P}_1 \circ \mathcal{P}_1 \neq \Sigma$ by
Lemma 25. From \( r = t \), this is impossible. Thus we may assume that there exist the following figures.

If \( \mathcal{P}_1 \neq \mathcal{P}_2, \mathcal{P}_3 \), then \( v_1v_2 = |P_1^*P_2(\alpha)| = |P_1^*P_1(\alpha)| = \frac{v_1(v_1-1)}{k_1} \)
from Fig. a, so \( v_1 > v_2 \). Similarly, \( v_3 > v_1 \) from Fig. c. Therefore \( v_3 > v_2 \). On the other hand, \( v_2v_3 = \frac{v_2(v_2-1)}{k_2} \) from Fig. b, so \( v_2 > v_3 \). This is impossible. Thus we have \( \mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P}_3 \). By Lemma 7, \( \mathcal{P}_1, \mathcal{P}_2 \) and \( \mathcal{P}_3 \) are self-paired.

Thus \( \mathcal{P}_1* \mathcal{P}_2 = \mathcal{P}_3 \cup \Delta_1, \mathcal{P}_2* \mathcal{P}_3 = \mathcal{P}_1 \cup \Delta_2, \mathcal{P}_3* \mathcal{P}_1 = \mathcal{P}_2 \cup \Delta_3 \).

Put \( |P_1(\alpha)| = v \), then by Lemma 8, iii) we have

\[ |\Delta_1(\alpha)| = |\Delta_2(\alpha)| = |\Delta_3(\alpha)| = v(v-1). \]

We put

\[ D_i = C(\Delta_i) \text{ and } C_i = C(\mathcal{P}_i), 1 \leq i \leq 3; \]

\[ D_1C_3 = x_1D_1 + x_2D_2 + x_3D_3. \]

Then

\[ x_1 + x_2 + x_3 = v \]
\[ D_2 C_3 = x_2 D_1 + \text{terms not involving } D_1 , \]
\[ D_3 C_3 = x_3 D_1 + \text{terms not involving } D_1 . \]

Now
\[ (C_1 C_2) C_3 = (D_1 + C_3) C_3 = vI + D_3 + D_1 C_3 , \]
\[ C_1 (C_2 C_3) = C_1 (D_2 + C_1) = vI + D_1 + D_2 C_1 . \]

So
\[ D_2 C_1 = D_1 C_3 + D_3 - D_1 = (x_1^{-1}) D_1 + x_2 D_2 + (x_3 + 1) D_3 . \]

Similarly
\[ D_3 C_2 = D_2 C_1 + D_1 - D_2 = x_1 D_1 + (x_2 - 1) D_2 + (x_3 + 1) D_3 . \]

Next
\[ (C_1 C_1) C_3 = (vI + D_1) C_3 = vC_3 + D_1 C_3 , \]
\[ C_1 (C_1 C_3) = C_1 (D_3 + C_2) = C_3 + D_1 + D_3 C_1 . \]

So
\[ D_3 C_1 = D_1 C_3 + (v-1) C_3 - D_1 \]
\[ = (x_1^{-1}) D_1 + x_2 D_2 + x_3 D_3 + (v-1) C_3 . \]

Similarly
\( D_1 C_2 = D_2 C_1 + (v-1)C_1 - D_2 \)

\[ = (x_1 - 1)D_1 + (x_2 - 1)D_2 + (x_3 + 1)D_3 + (v-1)C_1 \]

\( D_2 C_3 = D_3 C_2 + (v-1)C_2 - D_3 \)

\[ = x_1 D_1 + (x_2 - 1)D_2 + x_3 D_3 + (v-1)C_2 \tag{2} \]

Furthermore

\[ (C_1 C_2)C_2 = (vI + D_1)C_2 = vC_2 + D_1 C_2 \]

\[ C_1(C_1 C_2) = C_1(C_3 + D_1) = C_2 + D_3 + D_1 C_1 \]

So

\[ D_1 C_1 = D_1 C_2 + (v-1)C_2 - D_3 \]

\[ = (x_1 - 1)D_1 + (x_2 - 1)D_2 + x_3 D_3 + (v-1)C_1 + (v-1)C_2 \]

Similarly

\[ D_2 C_2 = D_2 C_3 + (v-1)C_3 - D_1 \]

\[ = (x_1 - 1)D_1 + (x_2 - 1)D_2 + x_3 D_3 + (v-1)C_2 + (v-1)C_3 \]

\[ D_3 C_3 = D_3 C_1 + (v-1)C_1 - D_2 \]

\[ = (x_1 - 1)D_1 + (x_2 - 1)D_2 + x_3 D_3 + (v-1)C_3 + (v-1)C_1 \tag{4} \]

Thus (2), (3) and (4) yield

\[ x_1 = x_2, \quad x_1 - 1 = x_3. \]
We put $x_3 = x$, then

$$v = x_1 + x_2 + x_3 = (x+1) + (x+1) + x = 3x + 2. \quad (5)$$

It is easy to show that the graph $(\Omega, \Gamma_1 \cup \Gamma_2 \cup \Gamma_3)$ is a strongly regular graph with parameters $3v, 2, 3$.

From the conditions of the existence of the strongly regular graph, (see [1] p. 97) it holds that

$$(3-2)^2 + 4(3v-3) = 12v - 11 = d^2, \quad (6)$$

(s is a positive integer)

$$m = \frac{3v}{2} \cdot \frac{3}{3} \cdot \frac{1}{d} \left\{ (3v-1+3-2)(d+3-2) - 2\cdot 3 \right\} = \frac{3}{2} v^2 + \frac{3v(v-2)}{2d}. \quad (7)$$

(m is a positive integer)

From (7), $\frac{3v(v-2)}{d}$ is integer, and hence

$$12v - 11 = d^2 \text{ is a divisor of } v^2(v-2)^2.$$  

So

$$12v - 11 \text{ is a divisor of } 11^2 - 13^2.$$
From $v = 3x + 2$, we conclude

$$v = 11.\]$$

Lastly, we shall prove that the primitive group satisfying these conditions does not exist. It is easy to prove that $G_\alpha$ acts faithfully on $\Gamma_1(\alpha)$. We shall show that for $\gamma_1, \gamma_1', (\neq) \in \Gamma_1(\alpha)$, $G_\alpha, \gamma_1, \gamma_1'$ has the fixed points in $\Gamma_1(\alpha) \setminus \{\gamma_1, \gamma_1'\}$.

For $(\alpha, \gamma_1) \in \Gamma_1$, put $\{\gamma_2\} = \Gamma_2(\alpha) \cap \Gamma_3(\gamma_1)$ and $\{\gamma_3\} = \Gamma_3(\alpha) \cap \Gamma_2(\gamma_1)$. Then, $G_\alpha, \gamma_1$ fix $\gamma_2$ and $\gamma_3$. So we must have that $(\gamma_2, \gamma_3) \in \Gamma_1$.

Now, for $\gamma_1, \gamma_1' (\neq) \in \Gamma_1(\alpha)$, put $\{\delta_1\} = \Gamma_1(\gamma_1) \cap \Gamma_2(\gamma_1')$, $\{\delta_2\} = \Gamma_2(\gamma_1') \cap \Gamma_3(\gamma_1'')$. Then $G_\alpha, \gamma_1, \gamma_1'$ fix $\delta_1$ and $\delta_2$.

Since $(\gamma_1, \gamma_1') \in \Gamma_3$, we have $(\delta_1, \delta_2) \notin \Gamma_3$.

Therefore $\Gamma_1(\gamma_1) \cap \Gamma_3(\delta_2) = \{\delta\} \neq \{\delta_1\}$.

So, $G_\alpha, \gamma_1, \gamma_1'$ fix $\delta_1$ and $\delta$. Since $\Gamma_1(\gamma_1) \ni \alpha, \delta_1, \delta (\neq)$, in the same way, we obtain that $G_\alpha, \gamma_1, \gamma_1'$ has the fix points in $\Gamma_1(\alpha) \setminus \{\gamma_1, \gamma_1'\}$. The order of $G_\alpha$ is at most one million.

If $G_\alpha$ is non-solvable, then the minimal normal subgroup of $G_\alpha$ is non-solvable simple. From [5], it is isomorphic to the Mathieu group $M_{11}$ or the transitive extension of the alternating group $A_5$ act on ten points. These groups have not the representation such
that it is doubly-transitive on eleven points and its stabilizer of two points has the additional fixed point. Thus, we can conclude that $G_\infty$ is solvable and the order of $G_\infty$ is 110. So $|G| = |\Omega| \cdot 11 \cdot 10 = 364 \cdot 11 \cdot 10 = 2^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$. $G$ is non-solvable group and $(|G|, 3) = 1$. But there does not exist such group by M. Hall [5].

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References


