Generalization of a theorem of Peter J. Cameron

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GENERALIZATION OF A THEOREM OF PETER J. CAMERON

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Peter J. Cameron [3] has shown that a primitive permutation group G has rank at most 4 if the stabilizer G_α of a point α is doubly transitive on all its nontrivial suborbits except one.

The purpose of this paper is to prove the following two theorems, one of which extends the Cameron's result.

Thorem 1. Let G be a primitive permutation group on a finite set Ω, and all nontrivial G-orbits in Cartesian product $\Omega\times\Omega$ be $\Gamma_1, \ldots, \Gamma_s, \Delta_1, \ldots, \Delta_t$, where G_α is doubly transitive on $\Gamma_i(\alpha) = \{\beta \mid (\alpha, \beta) \in \Gamma_i\}$, $1 \leqq i \leqq s$ and not doubly transitive on $\Delta_i(\alpha)$, $1 \leqq i \leqq t$. Suppose that G has no subdegree smaller than 4 and that $t > 1$. Then, we have

$$s \leqq 2t - r,$$

where $r = \#\{\Delta_i \mid \Delta_i = \Gamma_j^* \circ \Gamma_j,\ 1 \leqq j \leqq s\}$. Moreover if $r = 1$, then we have

$$s \leqq 2t - 2.$$

(For the notation $\Gamma_j^* \circ \Gamma_j$, see the section 1)

Theorem 2. Under the hypothesis of Theorem 1, if $r = t$, then $s = t = 2$, and G is isomorphic to the small Janko simple group and G_α is isomorphic to PSL(2, 11).

For the case of $t \geq 3$, I don't know the example satisfying the equality $s = 2t - r$, and when $r = 1$, the example satisfying the equality $s = 2t - 2$. I know only three exmaples with $t = 2$ and $s = 2$.

The small Janko simple group J_1 of order 175560 has a primitive rank 5 represestation of degree 266 in which the stabilizer of a point is isomorphic to PSL(2, 11) and acts doubly transitively on suborbits of lengths 11 and 12; the other suborbit lengths are 110 and 132 (See Livingstone [7]). The Mathieu group M_{12} has a primitive rank 5 representation of degree 144 in which the stabilizer of a point is isomorphic to PSL(2, 11) and acts doubly transitively on two suborbits of length 11; the other suborbit lengths are 55 and 66 (See Cameron [4]).

The group $[Z_3 \times Z_3 \times Z_3]S_4$ has a primitive rank 5 representation of degree 27 in which the stabilizer of a point is S_4 and acts doubly transitively on two suborbits of length 4; the other suborbit lengths are 6 and 12. I conjecture that it may even be true that s is at most t.

1. Preliminaries

Let G be a transitive permutation group on a finite set Ω, and Δ be a subset of the Cartesian product $\Omega \times \Omega$ which is fixed by G (acting in the natural way on $\Omega\times\Omega$), then $\Delta(\alpha) = \{\beta \in \Omega \mid (\alpha, \beta) \in \Delta\}$ is a subset of Ω fixed by G_α. This procedure sets up a one-to-one correspondence between G-orbits in $\Omega\times\Omega$ and G_α-orbits in Ω. The number of such orbits is called the rank of G.
$\Delta^* = \{(\beta, \alpha) \mid (\alpha, \beta) \in \Delta\}$ is the subset of $\Omega\times\Omega$ fixed by G paired with Δ; Δ is self-paired if $\Delta = \Delta^*$. Note that $|\Delta(\alpha)| = |\Delta^*(\alpha)| = |\Delta|/|\Omega|$. If Γ and Δ are fixed sets of G

in $\Omega\times\Omega$, let $\Gamma\Delta$ denote the set $\{(\alpha,\beta)|$ there exists $\gamma\in\Omega$ with $(\alpha,\gamma)\in\Gamma$, $(\gamma,\beta)\in\Delta$; $\alpha\neq\beta\}$; this is also a fixed set of G. The diagonal $\{(\alpha,\alpha)|\alpha\in\Omega\}$ is a trivial G-orbit. If Γ is a nontrivial G-orbits in $\Omega\times\Omega$, the Γ-graph is the reqular directed graph whose point set is Ω and whose edges are precisely the ordered pairs in Γ. A connected component of any such graph is a block of imprimitivity for G. G is primitive if and only if each such graph is connected.

For a G-orbit Γ in $\Omega\times\Omega$, the basis matrix $C = C(\Gamma)$ is the matrix whose rows and columns are indexed by Ω, with (α,β) entry 1 if $(\alpha,\beta)\in\Gamma$, 0 otherwise. All of the basis matrices form a basis of the centralizer algebra of the permutation matrices in G.

Let G be a group which acts as a permutation group on Ω, and π the permutation character of G i.e. the integer-valued function on G defined by $\pi(g)$ = number of fixed points of g. The formula

$$(\pi, 1)_G = \frac{1}{|G|}\sum_{g\in G}\pi(g) = \text{number of orbits of } G,$$

is well-known. If G acts as a permutation group on Ω_1 and Ω_2, with permutation characters π_1 and π_2, the number m of G-orbits in $\Omega_1\times\Omega_2$ is

$$m = (\pi_1\pi_2, 1)_G = (\pi_1, \pi_2)_G.$$

In particular, if G is a transitive permutation group on Ω with permutation character π, the rank r of G is given by

$$r = (\pi, \pi)_G = \text{sum of squares of multiplicities of irreducible consitituents of } \pi$$

If G acts doubly transitively on Ω_1 and Ω_2,

$$(\pi_1, \pi_2)_G = 2 \text{ or } 1 \text{ according as } \pi_1 = \pi_2 \text{ or } \pi_1 \neq \pi_2.$$

Lastly, we note that if G is a primitive permutation group on Ω, then for $\alpha, \beta\ (\neq) \in \Omega$, either $G_\alpha \neq G_\beta$ or G is a regular group of prime degree ([8], Prop. 8.6); primitive groups with a subdegree 2 are Frobenius groups of prime degree ([8], Theorem 18.7); primitive groups with a subdegree 3 are classified by W. J. Wong [9].

2. Lemmata

Throughout this section, we suppose that G is a primitive but not doubly transitive group on a finite set Ω, and $\Gamma_1, \Gamma_2, \ldots$ are G-orbits in $\Omega\times\Omega$ such that G_α is doubly transitive on $\Gamma_i(\alpha)$, $i = 1, 2, \ldots$; π_i and π_i^* are the permutation characters of G_α on $\Gamma_i(\alpha)$ and $\Gamma_i^*(\alpha)$, respectively, and let $c_i = c(\Gamma_i)$, $c_i^* = c(\Gamma_i^*)$.

Lemma 1. (P. J. Cameron [2]. Proposition 1.2)
G_α is doubly transitive on $\Gamma_i^*(\alpha)$.

Lemma 2. (P. J. Cameron [3]. Lemma 1)
$\Gamma_i^*\circ\Gamma_i$ is a G-orbit in $\Omega\times\Omega$, and if $|\Gamma_i(\alpha)| > 2$, then G_α is not doubly transitive on $\Gamma_i^*\circ\Gamma_i(\alpha)$.

Lemma 3. (P. J. Cameron [2]. Theorem 2.2)
For $(\alpha, \beta) \in \Gamma_i \circ \Gamma_i^*$, we put $v_i = |\Gamma_i(\alpha)|$ and $k_i = |\Gamma_i(\alpha) \cap \Gamma_i(\beta)|$.
Then $k_i < v_i$ and $|\Gamma_i \circ \Gamma_i^*(\alpha)| = \frac{v_i(v_i-1)}{k_i}$.
If $v_i > 2$, then $k_i \leqq \frac{v_i-1}{2}$; when particulary $k_i = \frac{v_i-1}{2}$, then $v_i = 3$ or 5.

In the following, we set

$$|\Gamma_i(\alpha)| = v_i, \quad |\Gamma_i \circ \Gamma_i^*(\alpha)| = \frac{v_i(v_i-1)}{k_i}.$$

Lemma 4. (P. J. Cameron [2]. Lemma 2.1)

$$|\Gamma_i^* \circ \Gamma_i(\alpha)| = |\Gamma_i \circ \Gamma_i^*(\alpha)|.$$

Lemma 5. $\Gamma_i^* \circ \Gamma_i \neq \Gamma_2^* \circ \Gamma_2$ if and only if $|\Gamma_1 \circ \Gamma_2^*(\alpha)| = |\Gamma_1(\alpha)| \cdot |\Gamma_2(\alpha)|$.

Proof. If $|\Gamma_1 \circ \Gamma_2^*(\alpha)| < |\Gamma_1(\alpha)| \cdot |\Gamma_2(\alpha)|$, we have $|\Gamma_1(\alpha) \cap \Gamma_2(\beta)| > 1$ for some $(\alpha, \beta) \in \Gamma_1 \circ \Gamma_2^*$. For $\gamma_1, \gamma_2 (\neq) \in \Gamma_1(\alpha) \cap \Gamma_2(\beta)$, $(\gamma_1, \gamma_2) \in \Gamma_1^* \circ \Gamma_1$ and $(\gamma_1, \gamma_2) \in \Gamma_2^* \circ \Gamma_2$. So $\Gamma_1^* \circ \Gamma_1 = \Gamma_2^* \circ \Gamma_2$.
Conversely, if $\Gamma_1^* \circ \Gamma_1 = \Gamma_2^* \circ \Gamma_2$, for $(\gamma_1, \gamma_2) \in \Gamma_1^* \circ \Gamma_1 = \Gamma_2^* \circ \Gamma_2$ we can choose α and β such that $\alpha \in \Gamma_1^*(\gamma_1) \cap \Gamma_1^*(\gamma_2)$, $\beta \in \Gamma_2^*(\gamma_1) \cap \Gamma_2^*(\gamma_2)$.
Since $\Gamma_1(\alpha) \cap \Gamma_2(\beta) \ni \gamma_1, \gamma_2$, $|\Gamma_1(\alpha) \cap \Gamma_2(\beta)| > 1$.
Therefore $|\Gamma_1 \circ \Gamma_2^*(\alpha)| < |\Gamma_1(\alpha)| \cdot |\Gamma_2(\alpha)|$.

Lemma 6. $\Gamma_1^* \circ \Gamma_2$ is the union of at most two G-orbits in $\Omega \times \Omega$, and $\pi_1 = \pi_2$ if and only if $\Gamma_1^* \circ \Gamma_2$ is the union of two G-orbits in $\Omega \times \Omega$.

Proof. Since $(\pi_1\pi_2, 1)_G = (\pi_1, \pi_2)_G \leq 2$, and $\pi_1\pi_2$ is the permutation character of G_α on $\Gamma_1(\alpha) \times \Gamma_2(\alpha)$, G has at most two orbits in $\{(\alpha, \gamma, \delta) \mid (\alpha, \gamma) \in \Gamma_1,\ (\alpha, \delta) \in \Gamma_2\}$, and hence, $\Gamma_1^* \circ \Gamma_2$ is the union of at most two G-orbits. If $\pi_1 \neq \pi_2$, then G is transitive on $\{(\alpha, \gamma, \delta) \mid (\alpha, \gamma) \in \Gamma_1,\ (\alpha, \delta) \in \Gamma_2\}$, and hence, $\Gamma_1^* \circ \Gamma_2$ is a G-orbit in $\Omega \times \Omega$. Now, we shall assume that $\pi_1 = \pi_2$ and $\Gamma_1^* \circ \Gamma_2$ is a G-orbit in $\Omega \times \Omega$. We put $v = v_1 = v_2$, and $m = |\Gamma_1^*(\alpha) \cap \Gamma_2^*(\delta)|$ for $(\alpha, \delta) \in \Gamma_1^* \circ \Gamma_2$.
If $m = 1$, then since $\Gamma_1^* \circ \Gamma_2$ is a G-orbit, G is transitive on $\{(\alpha, \gamma, \delta) \mid (\gamma, \alpha) \in \Gamma_1,\ (\gamma, \delta) \in \Gamma_2\}$. Therfore $(\pi_1, \pi_2)_G = 1$, and hence, $\pi_1 \neq \pi_2$, This is contrary to the assumption
If $m > 1$, then there exist quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$ whose edges are successively Γ_1^*, Γ_2, Γ_2^* and Γ_1; and whose vertices are all distinct. Counting all of them in two ways, we have

$$|\Omega| \frac{vv}{m} m(m - 1) = |\Omega| \frac{v(v - 1)}{k_1} k_1 k_2,$$

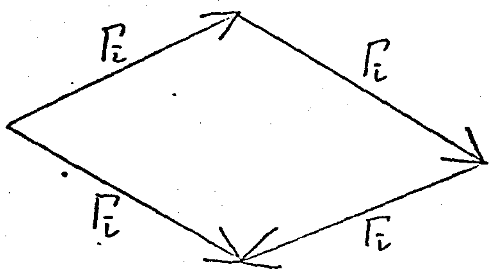
so

$$v(m - 1) = (v - 1)k_2.$$

Hence, $v = k_2$. This is impossible by Lemma 3.

Lemma 7. If $\Gamma_i\circ\Gamma_i^* \neq \Gamma_i^*\circ\Gamma_i$, then $\Gamma_i\circ\Gamma_i \not\supset \Gamma_i\circ\Gamma_i^*,\ \Gamma_i^*\circ\Gamma_i$.

Proof. Now assume $\Gamma_i\circ\Gamma_i \supset \Gamma_i\circ\Gamma_i^*$ or $\Gamma_i^*\circ\Gamma_i$, then we have the following figure,



and hence, $\Gamma_i\circ\Gamma_i \supset \Gamma_i\circ\Gamma_i^* \cup \Gamma_i^*\circ\Gamma_i$. Since $\Gamma_i\circ\Gamma_i$ is the union of at most two G-orbits in $\Omega\times\Omega$, we have $\Gamma_i\circ\Gamma_i = \Gamma_i\circ\Gamma_i^* \cup \Gamma_i^*\circ\Gamma_i$. By the assumption of this lemma, $|\Gamma_i\circ\Gamma_i(\alpha)| =$ $|\Gamma_i(\alpha)|\cdot|\Gamma_i(\alpha)| = v_i^2$.

So

$$v_i^2 = |\Gamma_i\circ\Gamma_i(\alpha)| = |\Gamma_i\circ\Gamma_i^*(\alpha)| + |\Gamma_i^*\circ\Gamma_i(\alpha)| = \frac{2v_i(v_i - 1)}{k_i},$$

$$v_i k_i = 2(v_i - 1).$$

Therfore, $v_i = 2$. All of the suborbits of the primitive group with a subdegree 2 are self-paired. This is contrary to the assumption of this Lemma.

Lemma 8. Let $\Gamma_1^*\circ\Gamma_2$ be the union of two G-orbits Σ_1 and Σ_2. We set $v = v_1 = v_2$, $S_i = C(\Sigma_i)$, $s_i = |\Sigma_i(\alpha)|$, $i = 1, 2$, and $C_1^*C_2 = a_1S_1 + a_2S_2$. Then we have

i) $s_1, s_2 \geqq v$. If $s_1 = v$, G_α is double transitive on $\sum_1(\alpha)$.

ii) $v^2 = a_1 s_1 + a_2 s_2$

iii) $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*$ if and only if $a_1 = a_2 = 1$

iv) if $s_1 = v(v-1)$, then $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*$ and $\Gamma_1^* \circ \Gamma_2$ containes some Γ_i

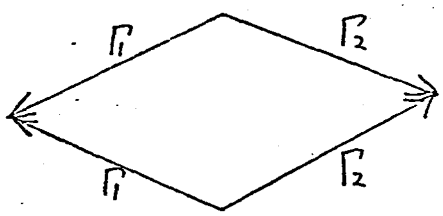
Proof. i) Assume $s_1 \leqq v$. Then $(\pi_1^*, \pi(\Sigma_1)) = 1$ or 2 according as $\pi_1^* \neq \pi(\Sigma_1)$ or $\pi_1^* = \pi(\Sigma_1)$ where $\pi(\Sigma_1)$ is the permutation character of G_α on $\Sigma_1(\alpha)$. If $\pi_1^* \neq \pi(\Sigma_1)$, for $\delta \in \Sigma_1(\alpha)$, $G_{\alpha,\delta}$ is transitive on $\Gamma_1^*(\alpha)$. Thus $\Gamma_1^*(\alpha) = \Gamma_2^*(\delta)$. Therefore $G_\alpha = G_{\{\Gamma_1^*(\alpha)\}} = G_{\{\Gamma_2^*(\delta)\}} = G_\delta$. This is impossible. So we have $\pi_1^* = \pi(\Sigma_1)$, and hence, $s_1 = v$ and G_α is doubly transitive on $\Sigma_1(\alpha)$.

ii) For the matrix F such that any entry is 1, we have

$$F(C_1^* C_2) = v^2 F \text{ and } F(a_1 S_1 + a_2 S_2) = (a_1 s_1 + a_2 s_2)F,$$

so
$$v^2 = a_1 s_1 + a_2 s_2.$$

iii) The existence of the following figure is equvalent to $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^*$.



It holds also that the figure exists if and only if $a_i \geqq 2$ for $i = 1$ or 2.

iv) By ii), $v^2 = a_1 v(v-1) + a_2 s_2$. Since $s_2 \geqq v$, $a_1 = a_2 = 1$ and $s_2 = v$. Therefore we conclude that $\Gamma_1^* \circ \Gamma_2$ containes some Γ_i by i), and $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*$ by iii).

Lemma 9. If $\pi_1 \neq \pi_2$, G_α is not doubly transitive on $\Gamma_1^* \circ \Gamma_2(\alpha)$.

Proof. Assume that G_α is doubly transitive on $\Gamma_1^* \circ \Gamma_2(\alpha)$. If $|\Gamma_1^* \circ \Gamma_2(\alpha)| \neq |\Gamma_1(\alpha)|$, then G_α has different permutation characters on $\Gamma_1^*(\alpha)$ and $\Gamma_1^* \circ \Gamma_2(\alpha)$. Hence, for $(\alpha, \gamma) \in \Gamma_1^*$, $G_{\alpha,\gamma}$ is transitive on $\Gamma_1^* \circ \Gamma_2(\alpha)$, so, $\Gamma_2(\gamma) = \Gamma_1^* \circ \Gamma_2(\alpha)$. Therefore $G_\gamma = G_{\{\Gamma_2(\gamma)\}} = G_{\{\Gamma_1^* \circ \Gamma_2(\alpha)\}} = G_\alpha$. This is impossible. Thus, we obtain $|\Gamma_2^* \circ \Gamma_1(\alpha)| = |\Gamma_1^* \circ \Gamma_2(\alpha)| = |\Gamma_1(\alpha)|$. On the other hand, for $(\delta, \gamma) \in \Gamma_2^*$, $\Gamma_1(\gamma) \subset \Gamma_2^* \circ \Gamma_1(\delta)$. So, $\Gamma_2^* \circ \Gamma_1(\delta) = \Gamma_1(\gamma)$. This is also impossible.

Lemma 10. Assume $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^*$ and $\Gamma_1^* \circ \Gamma_2$ be the union of two G-orbits Σ_1 and Σ_2; put $|\Gamma_1(\alpha)| = |\Gamma_2(\alpha)| = v$, $|\Gamma_i \circ \Gamma_i^*(\alpha)| = \frac{v(v-1)}{k}$, $|\Sigma_i(\alpha)| = s_i$, $i = 1, 2$; and $|\Gamma_2(\gamma) \cap \Sigma_2(\alpha)| = t$ for $\gamma \in \Gamma_1^*(\alpha)$. Then, we have the following quadratic equation for t

$$\frac{v(v-t)^2}{s_1} + \frac{vt^2}{s_2} - v - k(v-1) = 0.$$

Particulary, i) when $s_1 \geqq \frac{v(v-1)}{k}$, the quadratic equation has at most one root for $0 < t < v$; ii) when $t = 1$, then $s_2 = v$, $s_1 = \frac{v(v-1)}{k+1}$ and G_α is doubly transitive on $\Sigma_2(\alpha)$.

Proof. For $\gamma_1, \gamma_2\ (\neq) \in \Gamma_1^*(\alpha)$, counting arguments show that

$$|\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2) \cap \Sigma_1(\alpha)| = \frac{(v-t)\{v(v-t)-s_1\}}{(v-1)s_1},$$

$$|\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2) \cap \Sigma_2(\alpha)| = \frac{t(vt-s_2)}{(v-1)s_2},$$

so

$$k = |\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2)| = \frac{(v-t)\{v(v-t)-s_1\}}{(v-1)s_1} + \frac{t(vt-s_2)}{(v-1)s_2},$$

$$(v-1)k = \frac{v(v+t)^2}{s_1} - (v-t) + \frac{vt^2}{s_2} - t = \frac{v(v-t)^2}{s_1} + \frac{vt^2}{s_2} - v,$$

$$0 = \frac{v(v-t)^2}{s_1} + \frac{vt^2}{s_2} - v - k(v-1).$$

We shall prove the latter assertions. We put

$$f(t) = \frac{v(v-t)^2}{s_1} + \frac{vt^2}{s_2} - v - k(v-1).$$

When $s_1 \geqq \frac{v(v-1)}{k}$, then $f(0) < 0$. Since the coefficient of t^2 in $f(t)$ is positive, $f(t)$ has at most one root for $0 < t < v$. When $t = 1$, then $s_2 \leqq v$. By Lemma 8, i) $s_2 \geqq v$. So $s_2 = v$, and hence, G_α is doubly transitive on $\Sigma_2(\alpha)$, and $s_1 = \frac{v(v-1)}{k+1}$.

Lemma 11. Let $\Gamma_1^* \circ \Gamma_2$ be the union of two G-orbits Σ_1 and Σ_2, and G_α doubly transitive on $\Sigma_1(\alpha)$ and $\Sigma_2(\alpha)$, then $|\Gamma_1(\alpha)| = |\Gamma_2(\alpha)| \leqq 3$.

Proof. This lemma due to P. J. Cameron. ([3], Lemma 4.) We put $|\Gamma_1(\alpha)| = |\Gamma_2(\alpha)| = v$, and assume $|\Sigma_1(\alpha)| \neq v$. Then, G_α has the different permutation characters on $\Gamma_1^*(\alpha)$ and $\Sigma_1(\alpha)$, so, for $(\alpha, \delta) \in \Sigma_1$, $G_{\alpha,\delta}$ is transitive on $\Gamma_1^*(\alpha)$. Hence, $\Gamma_1^*(\alpha) = \Gamma_2^*(\delta)$. Therefore, $G_\alpha = G_{\{\Gamma_1^*(\alpha)\}} = G_{\{\Gamma_2^*(\delta)\}} = G_\delta$. This is impossible. Thus we conclude that $|\Sigma_1(\alpha)| = v$. In the same way, we have $|\Sigma_2(\alpha)| = v$.

Now, if $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*$, then by Lemma 5 $|\Gamma_1^* \circ \Gamma_2(\alpha)| = |\Gamma_1^*(\alpha)|\,|\Gamma_2(\alpha)| = v^2$. Therefore, $v^2 = |\Gamma_1^* \circ \Gamma_2(\alpha)| = |\Sigma_1(\alpha)| + |\Sigma_2(\alpha)| = 2v$, so $v = 2$. Thus, when $v > 2$, we obtain that $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^*$. For $\gamma \in \Gamma_1^*(\alpha)$, we put $t = |\Gamma_2(\gamma) \cap \Sigma_1(\alpha)|$. Then, for $(\gamma_1, \gamma_2) \in \Gamma_1 \circ \Gamma_1^*$, by Lemma 10 we have the following equation.

$$\begin{aligned} k_2 &= |\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2)| = \frac{1}{v-1}\{(v-t)^2 + t^2 - v\} \\ &= v - \frac{2t(v-t)}{v-1}. \end{aligned}$$

If $t = \frac{v}{2}$, $|\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2)| = v + \frac{v^2}{2(v-1)}$ is not integer, so $t \leqq \frac{v-1}{2}$ or $t \geqq \frac{v+1}{2}$. Hence $k_2 = v - \frac{2t(v-t)}{v-1} \geqq v - \frac{1}{2}(v+1) = \frac{1}{2}(v-1)$. But $k_2 \leqq \frac{1}{2}(v-1)$ by Lemma 3, so equality holds, and thus $v = 3$ or 5 by Lemma 3, and $t = \frac{1}{2}(v+1)$ or $\frac{1}{2}(v-1)$. Counting arguments show that $|\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2) \cap \Sigma_1(\alpha)| = \frac{t(t-1)}{v-1}$ for $\gamma_1, \gamma_2 (\neq) \in \Gamma_1^*(\alpha)$. Therefore $v - 1$ divides $t(t-1)$; this excludes $v = 5$, and so $v = 3$.

Lemma 12. For $\Gamma_1, \Gamma_2, \Gamma_3$, if Σ is a G-orbit contained in $\Gamma_1^* \circ \Gamma_2 \cap \Gamma_1^* \circ \Gamma_3$, and $|\Gamma_1(\alpha)| > 3$; then G_α is not doubly transitive on $\Sigma(\alpha)$.

Proof. $\Sigma^* \circ \Gamma_1^* \supset \Gamma_2^* \cup \Gamma_3^*$. If G_α is doubly transitive on $\Sigma(\alpha)$, $\Sigma^* \circ \Gamma_1^*$ is the union of at most two G-orbits by Lemma 6, so $\Sigma^* \circ \Gamma_1^* = \Gamma_2^* \dot{\cup} \Gamma_3^*$. This is contrary to Lemma 11.

Lemma 13. If $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^*$ and $\pi_1 \neq \pi_2$, then $|v_1 - v_2| \geqq 2$, and $|\Gamma_1 \circ \Gamma_1^*(\alpha)| > |\Gamma_1^* \circ \Gamma_2(\alpha)|$.

Proof. For $(\alpha, \delta) \in \Gamma_1^* \circ \Gamma_2$, we put

$$m = |\Gamma_1^*(\alpha) \cap \Gamma_2^*(\delta)|.$$

Count in two ways quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$ with $\gamma_1 \neq \gamma_2$ whose edges are successively Γ_1^*, Γ_2, Γ_2^*, and Γ_1; then we have

$$|\Omega| \frac{v_2(v_2-1)}{k_2} k_2 k_1 = |\Omega| \frac{v_1 v_2}{m} m(m-1),$$

so

$$(v_2 - 1)k_1 = v_1(m - 1). \tag{1}$$

If $v_1 = v_2$, then $k_1 = v_1$. This is impossible. If $v_1 = v_2 + 1$, then $k_1 \geqq \frac{v_1}{2}$, and hence, by Lemma 3 $v_1 = 2$, $v_2 = 1$. This is also impossible. Thus we can conclude that $|v_1 - v_2| \geqq 2$.

Assume $|\Gamma_1 \circ \Gamma_1^*(\alpha)| = \frac{v_1(v_1-1)}{k_1} = |\Gamma_1^* \circ \Gamma_2(\alpha)| = \frac{v_1 v_2}{m}$. Then

$$k_1 v_2 \geqq m(v_1 - 1). \tag{2}$$

From $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^*$, we have also

$$k_2 v_1 \geqq m(v_2 - 1). \tag{3}$$

Therefore, (1) and (2) yield

$$v_1 \leqq k_1 + m. \tag{4}$$

By Lemma 3 and (3), we have

$$2v_2 \leqq \frac{v_2(v_2-1)}{k_2} \leqq \frac{v_1 v_2}{m},$$

so

$$2 \leqq m \leqq \frac{v_1}{2}. \tag{5}$$

Thus (4) and (5) yield

$$k_1 \geqq \frac{1}{2} v_1.$$

This is contrary to Lemma 3.

Lemma 14. (P. J. Cameron [3]) If $\Gamma_1 \circ \Gamma_1^* = \Gamma_1 \circ \Gamma_2^*$, then $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*$.

Proof. We shall prove this lemma in a different way from P. J. Cameron's. Assume $\Gamma_1 \circ \Gamma_1^* = \Gamma_1 \circ \Gamma_2^* = \Gamma_2 \circ \Gamma_2^*$. We put

$$\left|\Gamma_1 \circ \Gamma_1^*(\alpha)\right| = \frac{v_1(v_1-1)}{k_1} = \left|\Gamma_2 \circ \Gamma_2^*(\alpha)\right| = \frac{v_2(v_2-1)}{k_2} = \left|\Gamma_1 \circ \Gamma_2^*(\alpha)\right| = \frac{v_1 v_2}{m},$$

where $m = \left|\Gamma_1(\alpha) \cap \Gamma_2(\delta)\right|$ for $(\alpha, \delta) \in \Gamma_1 \circ \Gamma_2^*$. Then it is trivial that $m > 1$ from the above formula, and hence, $\Gamma_1^* \circ \Gamma_1 = \Gamma_2^* \circ \Gamma_2$. Thus, by Lemma 13, $\left|\Gamma_1 \circ \Gamma_2^*(\alpha)\right| < \left|\Gamma_1^* \circ \Gamma_1(\alpha)\right| = \left|\Gamma_1 \circ \Gamma_1^*(\alpha)\right|$. This is contrary to assumption.

Now we shall investigate from Lemma 15 to Lemma 22 the necessary condition that the intersection of $\Gamma_1^* \circ \Gamma_2$ and $\Gamma_1^* \circ \Gamma_3$ for Γ_1, Γ_2, Γ_3 $(\neq)$ is not empty.

Lemma 15. If $\pi_1 = \pi_2 \neq \pi_3$ and $\pi_2^* = \pi_3^*$, or $\pi_1 = \pi_2 = \pi_3$ and $\pi_2^* \neq \pi_3^*$, then $\Gamma_1^* \circ \Gamma_2 \cap \Gamma_1^* \circ \Gamma_3 = \emptyset$.

Proof. Assume $\pi_1 = \pi_2 \neq \pi_3$ and $\pi_2^* = \pi_3^*$. Then we have $v_1 = v_2 = v_3$. We put $v = v_1 = v_2 = v_3$. By Lemma 13, $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_3 \circ \Gamma_3^*$, and hence, $|\Gamma_1^* \circ \Gamma_3(\alpha)| = |\Gamma_1^*(\alpha)| \cdot |\Gamma_3(\alpha)| = v^2$ by Lemma 5. If $\Gamma_1^* \circ \Gamma_2 \cap \Gamma_1^* \circ \Gamma_3 \neq \emptyset$, then since $\Gamma_1^* \circ \Gamma_3$ is a G-orbit and $\Gamma_1^* \circ \Gamma_2$ is a union of two G-orbits, we have $\Gamma_1^* \circ \Gamma_2 \supsetneqq \Gamma_1^* \circ \Gamma_3$. Therefore $|\Gamma_1^* \circ \Gamma_2(\alpha)| > |\Gamma_1^* \circ \Gamma_3(\alpha)| = v^2$. This is impossible. Similarly, we can prove the lemma for the case of $\pi_1 = \pi_2 = \pi_3$ and $\pi_2^* \neq \pi_3^*$.

Lemma 16. If $\pi_1^* \neq \pi_2^*$, $\pi_1^* \neq \pi_3^*$ and $\pi_2 \neq \pi_3$, then $\Gamma_1 \circ \Gamma_2^* \cap \Gamma_1 \circ \Gamma_3^* = \emptyset$.

Proof. By the assumption, $\Gamma_1 \circ \Gamma_2^*$, $\Gamma_1 \circ \Gamma_3^*$ and $\Gamma_2^* \circ \Gamma_3$ are G-orbits. Assume $\Gamma_1 \circ \Gamma_2^* = \Gamma_1 \circ \Gamma_3^*$. For $(\alpha, \delta) \in \Gamma_1 \circ \Gamma_2^*$, we put

$$|\Gamma_1(\alpha) \cap \Gamma_2(\delta)| = m_2 \quad \text{and} \quad |\Gamma_1(\alpha) \cap \Gamma_3(\delta)| = m_3.$$

For γ_1, γ_2 $(\neq) \in \Gamma_1(\alpha)$, we put

$$|\Gamma_2^*(\gamma_1) \cap \Gamma_3^*(\gamma_2)| = x.$$

Then, since $\Gamma_1^* \circ \Gamma_1 = \Gamma_2^* \circ \Gamma_3$, we have

$$\frac{v_1(v_1-1)}{k_1} = |\Gamma_1^* \circ \Gamma_1(\alpha)| = |\Gamma_2^* \Gamma_3(\alpha)| = \frac{v_2 v_3}{x},$$

so

$$v_1(v_1 - 1)x = v_2v_3k_1. \tag{1}$$

Count in two ways quadrilaterals $(\alpha, \gamma', \delta, \Upsilon)$ whose edges are successively Γ_1, Γ_2^*, Γ_3 and Γ_1^*, then we have

$$|\Omega|\frac{v_1(v_1-1)}{k_1}k_1x = |\Omega|\frac{v_1v_3}{m_3}m_2m_3,$$

so

$$(v_1 - 1)x = v_3m_2. \tag{2}$$

(1) and (2) yield

$$v_1m_2 = k_1v_2. \tag{3}$$

If $m_2 > 1$, there exist quadrilaterals $(\alpha, \beta_1, \delta, \beta_2)$ whose edges are successively Γ_1, Γ_2^*, Γ_2 and Γ_1^*, whose vertices are all distinct; count all of them in two ways, we have

$$|\Omega|\frac{v_1(v_1-1)}{k_1}k_1k_2 = |\Omega|\frac{v_1v_2}{m_2}m_2(m_2-1),$$

so

$$(v_1 - 1)k_2 = v_2(m_2 - 1).$$

On the other hand, from $\Gamma_1^* \circ \Gamma_1 = \Gamma_2^* \circ \Gamma_2$,

$$v_2(v_2 - 1)k_1 = v_1(v_1 - 1)k_2 = v_1v_2(m_2 - 1),$$

so

$$v_1(m_2 - 1) = (v_2 - 1)k_1. \tag{4}$$

(3) and (4) yield

$$v_1 = k_1.$$

This is contrary to Lemma 3.

Thus, we have $m_2 = m_3 = 1$ and $v_1 = k_1v_2$. For $(\alpha, \gamma) \in \Gamma_1$, $G_{\alpha,\gamma}$ is transitive on $\Gamma_1(\alpha)\setminus\{\gamma\}$ and since $\pi_1^* \neq \pi_2^*$, it is also transitive on $\Gamma_2^*(\gamma)$. Count in two ways (γ', δ) such that $\gamma' \in \Gamma_1(\alpha)\setminus\{\gamma\}$, $\delta \in \Gamma_2^*(\gamma)$ and $(\gamma', \delta) \in \Gamma_3^*$, then we have

$$(v_1 - 1)x = v_2 = \frac{v_1}{k_1}.$$

This is impossible.

Lemma 17. If $\pi_1 \neq \pi_2$, $\pi_1 \neq \pi_3$ and $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^*$, then $\Gamma_1^* \circ \Gamma_2 \cap \Gamma_1^* \circ \Gamma_3 = \emptyset$.

Proof. Assume $\Gamma_1^* \circ \Gamma_2 = \Gamma_1^* \circ \Gamma_3$. By Lemma 16, $\pi_2^* = \pi_3^*$. We put $v = v_1$, $w = v_2 = v_3$, $m = |\Gamma_1^*(\alpha) \cap \Gamma_2^*(\delta)| = |\Gamma_1^*(\alpha) \cap \Gamma_3^*(\delta)| > 1$ for $(\alpha, \delta) \in \Gamma_1^* \circ \Gamma_2$, and $x = |\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2)|$ for $\gamma_1, \gamma_2\ (\neq) \in \Gamma_1^*(\alpha)$.

Count in two ways quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$ whose edges are successively Γ_1^*, Γ_2, Γ_3^* and Γ_1; then we have

$$|\Omega|\frac{v(v-1)}{k_1}k_1x = |\Omega|\frac{vw}{m}mm,$$

so

$$(v - 1)x = wm. \qquad (1)$$

Next, count in two ways quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$ whose edges are successively $\Gamma_1^*, \Gamma_2, \Gamma_2^*, \Gamma_1$ and whose vertices are all distinct; then

$$|\Omega|\frac{v(v-1)}{k_1}k_1k_2 = |\Omega|\frac{vw}{m}m(m-1),$$

$$(v-1)k_2 = w(m-1). \tag{2}$$

(1) and (2) yield

$$(v-1)(x-k_2) = w, \text{ that is, } x > k_2 \geqq 1. \tag{3}$$

Since $x \geqq 2$, there exist quadrilaterals $(\gamma, \delta_1, \gamma', \delta_2)$ whose edges are successively $\Gamma_3, \Gamma_2^*, \Gamma_2$ and Γ_3^*, whose vertices are all distinct, and $(\gamma, \gamma') \in \Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^*$; count all of them in two ways, then

$$|\Omega| w(w-1)\lambda = |\Omega|\frac{w(w-1)}{k_2}x(x-1),$$

$$(\lambda = |\Gamma_2^*(\delta_1) \cap \Gamma_2^*(\Gamma_2) \cap \Gamma_1 \circ \Gamma_1^*(\gamma)| \quad \text{for } \delta_1, \delta_2 \ (\neq) \in \Gamma_3(\gamma)\ )$$

so

$$\lambda = \frac{x(x-1)}{k_2}.$$

By the definition of λ, $\lambda \leqq k_2$. On the other hand, since $x > k_2$, $\lambda = \frac{x(x-1)}{k_2} > k_2$. This is a contradiction.

Lemma 18. If $\pi_1^* \neq \pi_2^*$, $\pi_1^* \neq \pi_3^*$ and $\Gamma_1 \circ \Gamma_2^* = \Gamma_1 \circ \Gamma_3^*$, then $\zeta_1\zeta_2^* = \zeta_1\zeta_3^*$.

Proof. By Lemma 6 $\Sigma = \Gamma_1 \circ \Gamma_2^* = \Gamma_1 \circ \Gamma_3^*$ is a G-orbit. Let $S = \mathfrak{S}(\Sigma)$, $C_1C_2^* = m_2S$, $C_1C_3^* = m_3S$ and $|\Sigma(\alpha)| = s$.

For the matrix F such that the value of any entry is 1, we have

$$v_1v_2F = F(C_1C_2^*) = F\ (m_2S) = m_2sF,$$

so

$$v_1v_2 = m_2s.$$

Similarly

$$v_1v_3 = m_3s.$$

On the other hand, by Lemma 16, $\pi_2 = \pi_3$, and hence, $v_2 = v_3$. So, $m_2 = m_3$. Thus we can conclude that $C_1C_2^* = C_1C_3^*$.

Lemma 19. If $C_1C_2^* = C_1C_3^*$ and $|\Gamma_1(\alpha)| = v_1 > 3$, then we have

i) $\pi_2 = \pi_3$, $\pi_1^* \neq \pi_2^*, \pi_3^*$.

ii) $\Gamma_1^* \circ \Gamma_1 \neq \Gamma_2^* \circ \Gamma_2$, $\Gamma_1^* \circ \Gamma_1 \neq \Gamma_3^* \circ \Gamma_3$.

iii) $v_1 = v_2 + 1 = v_3 + 1$, $|\Gamma_2^*(\gamma_1) \cap \Gamma_3^*(\gamma_2)| = 1$ for $(\gamma_1, \gamma_2) \in \Gamma_1^* \circ \Gamma_1$.

iv) $|\Gamma_1^* \circ \Gamma_1(\alpha)| = \dfrac{v_1(v_1-1)}{2}$.

Proof. By the assumption $\Gamma_1 \circ \Gamma_2^* = \Gamma_1 \circ \Gamma_3^*$. For the matrix F such that the value of any entry is 1, we have

$$F(C_1C_2^*) = (FC_1)C_2^* = (v_1F)C_2^* = v_1(FC_2^*) = v_1v_2F.$$

Similarly

$$F(C_1C_3^*) = v_1v_3F.$$

So

$$v_2 = v_3.$$

We shall show that $v_1 \neq v_2 = v_3$. Assume $v = v_1 = v_2 = v_3$ and put $D = C(\Gamma_1^* \circ \Gamma_1)$. If $\Gamma_1^* \circ \Gamma_1 = \Gamma_2^* \circ \Gamma_2$, then $|\Gamma_1 \circ \Gamma_3^*(\alpha)| = |\Gamma_1 \circ \Gamma_2^*(\alpha)| \neq |\Gamma_1(\alpha)| \cdot |\Gamma_2(\alpha)| = |\Gamma_1(\alpha)| \cdot |\Gamma_3(\alpha)|$, therefore $\Gamma_1^* \circ \Gamma_1 = \Gamma_3^* \circ \Gamma_3$ by Lemma 5. We put $k = k_1 = k_2 = k_3$.

$$C_1^*(C_1 C_2^*) = (C_1^* C_1) C_2^* = (vE + kD) C_2^* = vC_2^* + k(v-1)C_2^* +$$
terms no involving C_2^*.

Similarly

$$C_1^*(C_1 C_3^*) = vC_3^* + k(v-1)C_3^* + \text{terms not involving } C_3^*.$$

So

$$(vE + kD)C_2^* = \{v + k(v-1)\} C_3^* + \text{terms not involving } C_3^*.$$

Since the coefficients of the basis matrices in DC_2^* are at most v, the above formula is impossible.

Next, if $\Gamma_1^* \circ \Gamma_1 \neq \Gamma_2^* \circ \Gamma_2$, then $\Gamma_1^* \circ \Gamma_1 \neq \Gamma_3^* \circ \Gamma_3$, and DC_3^* does not involve C_3^*. Now

$$C_1^*(C_1 C_2^*) = (C_1^* C_1) C_2^* = (vE + k_1 D) C_2^*,$$
$$C_1^*(C_1 C_3^*) = (C_1^* C_1) C_3^* = (vE + k_1 D) C_3^* = vC_3^* +$$
terms not involving C_3^*,

and hence, $k_1 DC_2^* = vC_3^* + \text{terms not involving } C_3^*$.

For $(\gamma_1, \gamma_2) \in \Gamma_1^* \circ \Gamma_1$ and $(\gamma_1, \delta) \in \Gamma_3^*$, we put

$$x = |\Gamma_3^*(\gamma_1) \cap \Gamma_2^*(\gamma_2)| \text{ and } t = |\Gamma_1^* \circ \Gamma_1(\gamma_1) \cap \Gamma_2(\delta)| .$$

Then from the above formula we have

$$t = \frac{v}{k_1}. \tag{1}$$

Counting in two ways triplilaterals $(\gamma_1, \delta, \gamma_2)$ whose edges are successively Γ_3^*, Γ_2 and $\Gamma_1^* \circ \Gamma_1$, we have

$$\frac{v(v-1)}{k_1} x = vt . \tag{2}$$

(1) and (2) yield

$$(v - 1)x = v,$$

which is a contradiction. Thus we can conclude that $v_1 \neq v_2 = v_3$, and hence, $\pi_2^* \neq \pi_1^* \neq \pi_3^*$. Therefore, we obtain $\pi_2 = \pi_3$ by Lemma 16, $\Gamma_2^* \circ \Gamma_2 \neq \Gamma_1^* \circ \Gamma_1 \neq \Gamma_3^* \circ \Gamma_3$ by Lemma 17, and hence we have i) and ii) of Lemma.

For $(\alpha, \gamma) \in \Gamma_1$, count in two ways the ordered pairs (γ', δ) such that $\gamma' \in \Gamma_1(\alpha) \setminus \{\gamma\}$, $\delta \in \Gamma_2^*(\gamma)$ and $(\gamma', \delta) \in \Gamma_3^*$; then since $\Gamma_1^* \circ \Gamma_1 \neq \Gamma_3^* \circ \Gamma_3$ we have

$$(v_1 - 1)x = v_2. \tag{3}$$

Now, we shall show that $x = 1$. Assume $x > 1$, then there exist quadrilaterals $(\gamma, \delta_1, \gamma', \delta_2)$ whose edges are successively Γ_2^*, Γ_3, Γ_3^* and Γ_2 whose edges are all distinct, and $(\gamma, \gamma') \in \Gamma_1^* \circ \Gamma_1$; count all of them in two ways, then we have

$$|\Omega| v_2(v_2 - 1)\lambda = |\Omega| \frac{v_1(v_1-1)}{k_1} x(x - 1),$$

$$(\lambda = |\Gamma_1^* \circ \Gamma_1(\gamma) \cap \Gamma_3(\delta_1) \cap \Gamma_3(\delta_2)| \text{ for } (\gamma, \delta_1), (\gamma, \delta_2)\ (\neq)$$
$$\in \Gamma_2^*,\ (\delta_1, \delta_2) \in \Gamma_2 \circ \Gamma_2^*),$$

so

$$(v_2 - 1)\lambda k_1 = v_1(x - 1) = (v_1 - 1)x + x - v_1 = v_2 + x - v_1.$$

Therefore, $x \geqq v_1 - 1$. If $x = v_1$ then $(v_2 - 1)\lambda k_1 = v_2$, which is a contradiction. If $x > v_1$, then $v_2 = (v_1 - 1)x > \frac{v_1(v_1-1)}{k_1}$. So $(\pi_2^*, \pi(\Gamma_1^* \circ \Gamma_1(\gamma)))_{G_\gamma} = 1$, where $\pi(\Gamma_1^* \circ \Gamma_1(\gamma))$ is the permutation character of G_γ on $\Gamma_1^* \circ \Gamma_1(\gamma)$. Hence, for $(\gamma, \gamma') \in \Gamma_1^* \circ \Gamma_1$, $G_{\gamma,\gamma'}$ is transitive on $\Gamma_2^*(\gamma)$. So $\Gamma_2^*(\gamma) = \Gamma_3^*(\gamma')$. This is impossible.

Thus we have $x = v_1 - 1$, $k_1 = \lambda = 1$, $v_2 = (v_1 - 1)^2$ and $|\Gamma_1^* \circ \Gamma_1(\gamma) \cap \Gamma_3(\delta)| = v_1$ for $(\gamma, \delta) \in \Gamma_2^*$.

Now, count in two ways quadrilaterals $(\alpha, \gamma_1, \gamma_2, \gamma_3)$ such that $(\alpha, \gamma_1) \in \Gamma_2$, (α, γ_2), $(\alpha, \gamma_3) \in \Gamma_3$, and (γ_1, γ_2), $(\gamma_1, \gamma_3) \in \Gamma_1^* \circ \Gamma_1$, $\gamma_2 \neq \gamma_3$; then we have

$$|\Omega| v_3(v_3 - 1)\lambda' = |\Omega| v_2 v_1(v_1 - 1),$$

$$(\lambda' = |\Gamma_1^* \circ \Gamma_1(\gamma_2) \cap \Gamma_1^* \circ \Gamma_1(\gamma_3) \cap \Gamma_2(\alpha)| \text{ for } \gamma_2, \gamma_3\ (\neq)$$
$$\in \Gamma_3(\alpha))$$

so

$$\lambda' = \frac{v_1(v_1-1)}{v_3-1} = \frac{v_1(v_1-1)}{(v_1-1)^2-1} = \frac{v_1-1}{v_1-2}.$$

Therefore, $v_1 = 3$. This is contrary to the hypossesis of Lemma. Thus we can conclude that $x = 1$, and hence, by (3) we have $v_1 = v_2 + 1 = v_3 + 1$. This proves Lemma iii).

Lastly, we shall show that $k_1 = 2$. If $k_1 = 1$, then
$\left|\Gamma_1^* \circ \Gamma_1(\alpha)\right| = v_1(v_1 - 1) \leqq \left|\Gamma_2^* \circ \Gamma_3(\alpha)\right| \leqq v_2 v_3 = (v_1 - 1)^2$
This is impossible. Now, we have

$$u = \left|\Gamma_1^* \circ \Gamma_1(\gamma) \cap \Gamma_3(\delta)\right| = \frac{v_1}{k_1} \text{ for } (\gamma, \delta) \in \Gamma_2^*,$$
$$\text{and } 2 \leqq k_1 < \frac{v_1}{2}.$$

Count again in two ways quadrilaterals $(\alpha, \gamma_1, \gamma_2, \gamma_3)$ such that $(\alpha, \gamma_1) \in \Gamma_2$, (α, γ_2), $(\alpha, \gamma_3) \in \Gamma_3$ and (γ_1, γ_2), $(\gamma_1, \gamma_3) \in \Gamma_1^* \circ \Gamma_1$, $\gamma_2 \neq \gamma_3$; then

$$|\Omega|(v_1 - 1)(v_1 - 2)\lambda'' = |\Omega|(v_1 - 1)\left(\frac{v_1}{k_1} - 1\right)\frac{v_1}{k_1},$$
$$\left(\lambda'' = \left|\Gamma_1^* \circ \Gamma_1(\gamma_2) \cap \Gamma_1^* \circ \Gamma_1(\gamma_3) \cap \Gamma_2(\alpha)\right| \text{ for } \gamma_2, \gamma_3\ (\neq) \in \Gamma_3(\alpha)\right)$$

so

$$\lambda'' = \frac{v_1(v_1 - k_1)}{(v_1 - 2)k_1^2} = \frac{u(u-1)k_1^2}{(k_1 u - 2)k_1^2} = \frac{u(u-1)}{k_1 u - 2}.$$

If u is odd, then $k_1 u - 2$ divedes $u - 1$. This is impossible. We put $u = 2u_0$, then

$$\lambda'' = \frac{2u_0(2u_0 - 1)}{2k_1 u_0 - 2} = \frac{u_0(2u_0 - 1)}{k_1 u_0 - 1}.$$

Therefore, we conclude that $k_1 = 2$.

Lemma 20. If $\pi_1 = \pi_2 \neq \pi_3$ and $\Gamma_1^* \circ \Gamma_2 \cap \Gamma_1^* \circ \Gamma_3 \neq \emptyset$, then $v_1 = v_2 = v_3 + 1$, $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*$ and $\Gamma_1^* \circ \Gamma_2 = \Gamma_1^* \circ \Gamma_3 \cup \Gamma_i$ for some Γ_i.

Proof. By assumption, $\Sigma = \Gamma_1^* \circ \Gamma_3$ is a G-orbit contained in $\Gamma_1^* \circ \Gamma_2$. We put $v = v_1 = v_2$, $w = v_3$, $|\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2)| = x$ for $(\gamma_1, \gamma_2) \in \Gamma_1 \circ \Gamma_1^*$, $|\Gamma_1^*(\alpha) \cap \Gamma_2^*(\delta)| = y$ and $|\Gamma_1^*(\alpha) \cap \Gamma_3^*(\delta)| = m$ for $(\alpha, \delta) \in \Sigma$, $|\Gamma_2(\gamma) \cap \Sigma(\alpha)| = t$ for $(\alpha, \gamma) \in \Gamma_1^*$.
By Lemma 15, $\pi_2^* \neq \pi_3^*$, and hence, $\Gamma_2 \circ \Gamma_3^*$ is a G-orbit. We have

$$\frac{v(v-1)}{k_1} = |\Gamma_1 \circ \Gamma_1^*(\gamma_1)| = |\Gamma_2 \circ \Gamma_3^*(\gamma_1)| = \frac{vw}{x},$$

so

$$(v - 1)x = wk_1. \tag{1}$$

We have also $|\Sigma(\alpha)| = \frac{vw}{m} = \frac{vt}{y}$,
and so

$$wy = tm. \tag{2}$$

Count in two ways quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$ whose edges are successively Γ_1^*, Γ_2, Γ_3^* and Γ_1, then we have

$$|\Omega|\frac{v(v-1)}{k_1}k_1 x = |\Omega|\frac{vw}{m}my,$$

so

$$(v - 1)x = wy. \tag{3}$$

(1) and (3) yield

$$y = k_1. \tag{4}$$

From (2) and (3),

$$(v - 1)x = tm. \tag{5}$$

We shall show that $m = 1$. If $m > 1$, then there exist quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$ whose edges are successively Γ_1^*, Γ_3, Γ_3^* and Γ_1, whose vertices are all distinct; count all of them in two ways, then we have

$$|\Omega|\frac{w(w-1)}{k_3}k_3k_1 = |\Omega|\frac{vw}{m}m(m-1),$$

so

$$(w - 1)k_1 = v(m - 1).$$

On the other hand, from (3) and (4)

$$(w - 1)k_1 = wk_1 - k_1 = (v - 1)x - k_1,$$

therefore

$$v(m - 1) = (v - 1)x - k_1,$$

so

$$0 \lneq v(x - m + 1) = x + k_1 < 2v. \tag{6}$$

(6) yields

$$x = m, \; v = m + k_1. \tag{7}$$

From (5) and (7),

$$t = v - 1. \tag{8}$$

Thus $|\Sigma(\alpha)| = \frac{vt}{y} = \frac{v(v-1)}{k_1}$.

If $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^*$, then by Lemma 10, $|\Sigma(\alpha)| = \frac{v(v-1)}{k_1+1}$. This is a contradiction. So we have $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*$, and hence,

$$1 = y = k_1. \tag{9}$$

Therefore we have $m = v - 1$ from (7) and (9), and $w = (v-1)^2$ from (2) and (8). So

$$|\Gamma_1 \circ \Gamma_1^*(\alpha)| = |\Gamma_3 \circ \Gamma_3^*(\alpha)| = \frac{w(w-1)}{k_3}$$

$$\geqq 2w = 2(v-1)^2 > v(v-1).$$

This is impossible. Thus, we can conclude that $m = 1$, and then by (5) $t = v - 1$, $x = 1$ and $|\Sigma(\alpha)| = \frac{v(v-1)}{k_1}$. By Lemma 10, $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*$, and hence, $1 = y = k_1$. Therefore, by (2) $w = v - 1$, $|\Sigma(\alpha)| = v(v-1)$. By Lemma 8 iv), $\Gamma_1^* \circ \Gamma_2 = \Sigma \cup \Gamma_i$ for some Γ_i.

Lemma 21. <u>If $\Gamma_1^* \circ \Gamma_2 \cap \Gamma_1^* \circ \Gamma_3 \neq \emptyset$, and $v_1, v_2, v_3 > 3$, then the following hold;</u>

<u>i) if $\pi_1 = \pi_2 = \pi_3$, then $\pi_2^* = \pi_3^*$</u>

<u>ii) if $\pi_1 = \pi_2 \neq \pi_3$, then $\pi_2^* \neq \pi_3^*$ and $v_1 = v_2 = v_3 + 1$.</u>

<u>iii) if $\pi_1 \neq \pi_2, \pi_3$, then $\pi_2^* = \pi_3^*$, $c_1^* c_2 = c_1^* c_3$ and $v_1 = v_2 + 1 = v_3 + 1$.</u>

Proof. We have this assertion by arranging from Lemma 15 to Lemma 20.

Lemma 22. Suppose that $\Gamma_1^* \circ \Gamma_2$ and $\Gamma_1^* \circ \Gamma_3$ contain a G-orbit Σ in $\Omega \times \Omega$, and $\pi_1 = \pi_2 = \pi_3$, $|\Gamma_1(\alpha)| > 3$. For γ_1, γ_2 $(\neq)$ $\Gamma_1^*(\alpha)$ and $\delta \in \Sigma(\alpha)$, the following hold;

i) if $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* = \Gamma_3 \circ \Gamma_3^*$, then $|\Gamma_1^*(\alpha) \cap \Gamma_2^*(\delta)| > 1$, $|\Gamma_1^*(\alpha) \cap \Gamma_3^*(\delta)| > 1$ and $|\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2) \cap \Sigma(\alpha)| > 1$.

ii) if $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* \neq \Gamma_3 \circ \Gamma_3^*$, then $|\Gamma_1^*(\alpha) \cap \Gamma_2^*(\delta)| >$ $|\Gamma_1^*(\alpha) \cap \Gamma_3^*(\delta)| = |\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2)| = 1$, $|\Sigma(\alpha)| = \frac{v(v-1)}{k_1+1}$, and $\Gamma_1^* \circ \Gamma_2$ contains some Γ_k.

iii) if $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*, \Gamma_3 \circ \Gamma_3^*$, then $|\Gamma_1^*(\alpha) \cap \Gamma_2^*(\delta)| = |\Gamma_1^*(\alpha) \cap \Gamma_3^*(\delta)|$ $= |\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2)| = 1$, $|\Sigma(\alpha)| = v(v-1)$, and $\Gamma_1^* \circ \Gamma_2$ contains some Γ_i and $\Gamma_1^* \circ \Gamma_3$ contains another Γ_j.

Proof. Put $|\Sigma(\alpha) \cap \Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2)| = \lambda$ for γ_1, γ_2 $(\neq) \in \Gamma_1^*(\alpha)$. $|\Gamma_1^*(\alpha) \cap \Gamma_2^*(\delta)| = x_2$, $|\Gamma_1^*(\alpha) \cap \Gamma_3^*(\delta)| = x_3$ for $(\alpha, \delta) \in \Sigma$. Count in two ways quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$ whose edges are successively $\Gamma_1^*, \Gamma_2, \Gamma_3^*$ and Γ_1, and $(\alpha, \delta) \in \Sigma$; then we have

$$|\Omega| \frac{v(v-1)}{k_1} k_1 \lambda = |\Omega|\, |\Sigma(\alpha)|\, x_2 x_3,$$

so

$$v(v-1)\lambda = |\Sigma(\alpha)|\, x_2 x_3. \tag{1}$$

Assume $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*, \Gamma_3 \circ \Gamma_3^*$. Then we have $|\Gamma_1^*(\alpha) \cap \Gamma_2^*(\delta)|$ $= |\Gamma_1^*(\alpha) \cap \Gamma_3^*(\delta)| = 1$. By (1)

$$v(v-1)\lambda = |\Sigma(\alpha)|.$$

Since $|\Sigma(\alpha)| \leqq v(v-1)$, we have $\lambda = 1$ and $|\Sigma(\alpha)| = v(v-1)$. By Lemma 8 iv), $\Gamma_1^*\circ\Gamma_2 = \Sigma \cup \Gamma_i$ and $\Gamma_1^*\circ\Gamma_3 = \Sigma \cup \Gamma_j$ for some Γ_i, Γ_j. By Lemma 8, iii), we have $C_1^*C_2 = S + C_i$, $C_1^*C_3 = S + C_j$. ($S = C(\Sigma)$) If $C_i = C_j$, then $C_1^*C_2 = C_1^*C_3$, and hence, by Lemma 19 $\pi_1 \neq \pi_2, \pi_3$. This is contrary to the hypothesis of this lemma. Thus $C_i \neq C_j$, that is, $\Gamma_i \neq \Gamma_j$. So $\Sigma(\alpha)\cap\Gamma_2(\gamma_1)\cap\Gamma_3(\gamma_2) = \Gamma_2(\gamma_1)\cap\Gamma_3(\gamma_2)$. Therefore $|\Gamma_2(\gamma_1)\cap\Gamma_3(\gamma_2)| = |\Sigma(\alpha)\cap\Gamma_2(\gamma_1)\cap\Gamma_3(\gamma_2)| = \lambda = 1$. Thus we have iii) of Lemma.

Next assume $\Gamma_1\circ\Gamma_1^* = \Gamma_2\circ\Gamma_2^* \neq \Gamma_3\circ\Gamma_3^*$. Then we have $|\Gamma_1^*(\alpha)\cap\Gamma_3^*(\delta)| = 1$. By (1)

$$v(v-1)\lambda = |\Sigma(\alpha)|\, x_2. \tag{2}$$

Count in two ways triplilaterals (α, δ, γ) whose edges are successibly Σ, Γ_2^* and Γ_1, then we have

$$|\Sigma(\alpha)|\, x_2 \leqq v(v-1). \tag{3}$$

If $x_2 = 1$, then $|\Sigma(\alpha)| = v(v-1)$ by (2) and (3). By Lemma 8. iv), $\Gamma_1\circ\Gamma_1^* \neq \Gamma_2\circ\Gamma_2^*$. This is contrary to the assumption. Therefore we have $x_2 > 1$, $\lambda = 1$ and $|\Sigma(\alpha)|\, x_2 = v(v-1)$. Since $|\Sigma(\alpha)|\, x_2 = v(v-1)$, $|\Sigma(\alpha)\cap\Gamma_2(\gamma)| = v-1$ for $(\alpha, \gamma) \in \Gamma_1^*$. By Lemma 10. ii), $|\Sigma(\alpha)| = \dfrac{v(v-1)}{k_1 + 1}$ and $\Gamma_1^*\circ\Gamma_2$ contains some Γ_i.

Now we shall show that $\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2) = \Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2) \cap \Sigma(\alpha)$, for $\gamma_1, \gamma_2 \in \Gamma_1^*(\alpha)$. If $\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2) \supsetneq \Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2) \cap \Sigma(\alpha)$, then $\Gamma_1^* \circ \Gamma_2 = \Gamma_1^* \circ \Gamma_3$. But $|\Gamma_1^* \circ \Gamma_2(\alpha)| = |\Sigma(\alpha)| + |\Gamma_i(\alpha)| = \frac{v(v-1)}{k_1 + 1} + v < v^2$ and $|\Gamma_1^* \circ \Gamma_3(\alpha)| = v^2$. This is impossible. Therefore, $|\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2)| = |\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2) \cap \Sigma(\alpha)| = \lambda = 1$. Thus we have ii) of Lemma.

Last assume $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* = \Gamma_3 \circ \Gamma_3^*$. We shall show that $x_2 = |\Gamma_1^*(\alpha) \cap \Gamma_2^*(\delta)| > 1$ and $x_3 = |\Gamma_1^*(\alpha) \cap \Gamma_3^*(\delta)| > 1$. We note that $k_1 = k_2 = k_3$, therefore we put $k = k_1 = k_2 = k_3$. If $x_2 = x_3 = 1$, by (1) we have $|\Sigma(\alpha)| = v(v-1)$. By Lemma 8. iv) $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*$, $\Gamma_3 \circ \Gamma_3^*$. This is contrary to the assumption.

If $x_2 > x_3 = 1$, we have $|\Sigma(\alpha)| = \frac{v(v-1)}{k+1}$ as before, and $x_2 = k + 1$. We put $\Gamma_1^* \circ \Gamma_3 = \Sigma \cup \Sigma'$,

$$x = |\Gamma_1^*(\alpha) \cap \Gamma_3^*(\delta')| \quad \text{for } (\alpha, \delta') \in \Sigma', \text{ and}$$

$$t = |\Gamma_3(\gamma_1) \cap \Sigma(\alpha)| = \frac{v-1}{k+1} \quad \text{for } (\alpha, \gamma_1) \in \Gamma_1^*.$$

Since $\Gamma_1 \circ \Gamma_1^* = \Gamma_3 \circ \Gamma_3^*$ and $x_3 = 1$, there exist quadrilaterals $(\alpha, \gamma_1, \delta', \gamma_2)$, with $\gamma_1 \neq \gamma_2$ and $(\alpha, \delta') \in \Sigma'$, whose edges are successively Γ_1^*, Γ_3, Γ_3^* and Γ_1. Count all of them in two ways then we have

$$|\Omega| \frac{v(v-1)}{k} kk = |\Omega| \frac{v(v-\frac{v-1}{k+1})}{x} x(x-1),$$

so

$$x-1 = \frac{(v-1)k}{v-\frac{v-1}{k+1}} = \frac{t(k+1)k}{t(k+1)+1-t} = \frac{tk(k+1)}{tk+1}.$$

Therefore $t = 1$, and hence, $v = k + 2$. This is impossible by Lemma 3. Thus we have $x_2 > 1$ and $x_3 > 1$.

Now we shall show that $\lambda > 1$. If $\lambda = 1$, by (1) we have

$$v(v-1) = |\Sigma(\alpha)|\, x_2 x_3$$

Since $x_2 > 1$, there exist quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$, with $\gamma_1 \neq \gamma_2$ and $(\alpha, \delta) \in \Sigma$, whose edges are successively Γ_1^*, Γ_2, Γ_2^* and Γ_1. Count all of them in two ways, then we have

$$|\Omega| \frac{v(v-1)}{k} k\lambda_2 = |\Omega|\, |\Sigma(\alpha)|\, x_2(x_2-1),$$

$$(\lambda_2 = |\Gamma_2(\gamma_1) \cap \Gamma_2(\gamma_2) \cap \Sigma(\alpha)| \text{ for } \gamma_1, \gamma_2\ (\neq) \in \Gamma_1^*(\alpha))$$

so

$$\lambda_2 = \frac{|\Sigma(\alpha)|\, x_2(x_2-1)}{v(v-1)},$$

and by (1),

$$\lambda_2 = \frac{x_2-1}{x_3}.$$

Thus $\frac{x_2-1}{x_3}$ is a positive integer. Since $x_3 > 1$, in the same way, we have that $\frac{x_3-1}{x_2}$ is a positive integer. This is impossible. Thus we have i) of Lemma.

Lemma 23. If $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^*$ and $\pi_1 \neq \pi_2$, then for any $\Gamma_i, \Gamma_j\ (\neq)$, $\Gamma_i \circ \Gamma_j^* \not\supset \Gamma_1 \circ \Gamma_1^*$.

Proof. Assume $\Gamma_i \circ \Gamma_j^* \supset \Gamma_1 \circ \Gamma_1^*$. Note that $|v_1 - v_2| \geqq 2$ by Lemma 13, and hence, $\pi_1^* \neq \pi_2^*$. If $\{\Gamma_i, \Gamma_j\} = \{\Gamma_1, \Gamma_2\}$, then since $\Gamma_i \circ \Gamma_j^*$ is a G-orbit, $\Gamma_i \circ \Gamma_j^* = \Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^*$. This is a contrary to Lemma 14. Therefore we can assume that $\Gamma_i \neq \Gamma_1, \Gamma_2$.
If $\Gamma_j = \Gamma_1$, then $\Gamma_2^* \circ \Gamma_1 \cap \Gamma_2^* \circ \Gamma_i \neq \emptyset$. By Lemma 21 we have $v_2 = v_1 - 1$.

This is a contradiction. Thus we have $\{\Gamma_i, \Gamma_j\} \cap \{\Gamma_1, \Gamma_2\} = \emptyset$.

From $v_1 \neq v_2$, we may assume $v_i \neq v_1$. Since $\Gamma_1^* \circ \Gamma_i \cap \Gamma_1^* \circ \Gamma_j \neq \emptyset$, $v_i = v_1 - 1$ by Lemma 21. On the other hand, from $|v_1 - v_2| \geqq 2$, $v_i \neq v_2$. Since $\Gamma_2^* \circ \Gamma_i \cap \Gamma_2^* \circ \Gamma_j \neq \emptyset$, in the same way, we have $v_i = v_2 - 1$. This is a contradiction.

Lemma 24. If $\Gamma_1\circ\Gamma_1^* = \Gamma_2\circ\Gamma_2^* = \Gamma_3\circ\Gamma_3^* = \Delta$, $\pi_1 = \pi_2 = \pi_3$ and $|\Gamma_1(\alpha)| > 3$, then $\Gamma_1\circ\Gamma_2^* \not\supset \Delta$ or $\Gamma_1\circ\Gamma_3^* \not\supset \Delta$.

Proof. Assume $\Gamma_1\circ\Gamma_2^* \supset \Delta$ and $\Gamma_1\circ\Gamma_3^* \supset \Delta$. We put $v = v_1 = v_2 = v_3$ and $k = k_1 = k_2 = k_3$. Since $\pi_1 = \pi_2 = \pi_3$, we have $\pi_1^* = \pi_2^* = \pi_3^*$ by Lemma 21. We shall show that $\Gamma_1^*\circ\Gamma_1 = \Gamma_2^*\circ\Gamma_2 = \Gamma_3^*\circ\Gamma_3$.
If $\Gamma_1^*\circ\Gamma_1 \neq \Gamma_2^*\circ\Gamma_2, \Gamma_3^*\circ\Gamma_3$, $|\Delta(\alpha)| = v(v-1)$ by Lemma 22. iii).
Since $|\Gamma_1^*\circ\Gamma_1(\alpha)| = |\Gamma_1\circ\Gamma_1^*(\alpha)| = |\Delta(\alpha)| = v(v-1)$, we have $\Gamma_2\circ\Gamma_2^* \neq \Gamma_3\circ\Gamma_3^*$ by Lemma 8. iv). If $\Gamma_1^*\circ\Gamma_1 = \Gamma_2^*\circ\Gamma_2 \neq \Gamma_3^*\circ\Gamma_3$, $|\Delta(\alpha)| = \frac{v(v-1)}{k+1}$ by Lemma 22. ii). This is impossible. Thus we can conclude that $\Gamma_1^*\circ\Gamma_1 = \Gamma_2^*\circ\Gamma_2 = \Gamma_3^*\circ\Gamma_3$. If $\Gamma_1\circ\Gamma_2^* = \Gamma_1\circ\Gamma_3^*$, then $C_1C_2^* = C_1C_3^*$ by Lemma 10, i); and hence, $v_1 = v_2 + 1$ by Lemma 19, iii). This is contrary to the hypothesis of this lemma. We shall show that $k > 1$. If $k = 1$, $|\Gamma_1^*\circ\Gamma_1(\alpha)| = \frac{v(v-1)}{k} = v(v-1)$. Since $\Gamma_2^*\circ\Gamma_3 \supset \Gamma_1^*\circ\Gamma_1$, $\Gamma_2\circ\Gamma_2^* \neq \Gamma_3\circ\Gamma_3^*$ by Lemma 8, iv). This is contrary to the assumption. Count in two ways quadrilaterals $(\alpha, \gamma_1, \delta, \gamma_2)$ whose edges are successively Γ_1, Γ_2^*, Γ_3 and Γ_1^*; then we have

$$|\Omega|\frac{v(v-1)}{k}kx = |\Omega|\frac{v(v-1)}{k}x_2x_3,$$

so

$$kx = x_2x_3. \tag{1}$$

Here we put $x_2 = |\Gamma_1(\alpha)\cap\Gamma_2(\delta)|$, $x_3 = |\Gamma_1(\alpha)\cap\Gamma_3(\delta)|$ for

$(\alpha, \delta) \in \Delta$ and $x = |\Gamma_2^*(\gamma_1) \cap \Gamma_3^*(\gamma_2) \cap \Delta(\alpha)|$ for $\gamma_1, \gamma_2\ (\neq) \in \Gamma_1(\alpha)$.

We shall show that x, x_2 and x_3 are smaller than k. If $x_2 \geqq k$, then for $(\alpha, \gamma) \in \Gamma_1$, $|\Delta(\alpha) \cap \Gamma_2^*(\gamma)| \geqq v - 1$. Of course, $|\Delta(\alpha) \cap \Gamma_2^*(\gamma)| \leqq v - 1$, and hence, $|\Delta(\alpha) \cap \Gamma_2^*(\gamma)| = v - 1$. By Lemma 10, ii), we have $|\Delta(\alpha)| = \frac{v(v-1)}{k+1}$, which is a contradiction. We can prove in the same way that $x_3 < k$. Then, (1) yields

$$x < x_2, \quad x_3 < k. \tag{2}$$

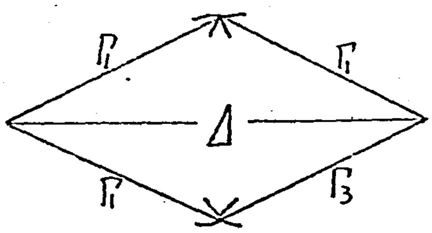
Now

$$C_1(C_2^*C_3) = C_1(xD' + yS'),$$

$$(C_1C_2^*)C_3 = (x_2D + y_2S)C_3 = x_2(v-1)C_3 + \text{terms not involving } C_3.$$

$$(\Delta' = \Gamma_1^* \circ \Gamma_1,\ \Gamma_1 \circ \Gamma_2^* = \Delta \cup \Sigma,\ \Gamma_2^* \circ \Gamma_3 = \Delta' \cup \Sigma',$$
$$D = C(\Delta),\ D' = C(\Delta'),\ S = C(\Sigma) \text{ and } S' = C(\Sigma'))$$

Since $x_2 > x$ and the coefficient of C_3 contained in C_1D' is at most $v - 1$, C_3 is contained in C_1S', that is, $\Gamma_1^* \circ \Gamma_3 \supset \Sigma'$. On the other hand, since $\Gamma_1 \circ \Gamma_3^* \supset \Delta$, there exists the following figure.



Therefore $\Gamma_1^* \circ \Gamma_3 \supset \Delta'$. Thus $\Gamma_1^* \circ \Gamma_3 = \Delta' \cap \Sigma' = \Gamma_2^* \circ \Gamma_3$.

By Lemma 10, i) we have $C_1^*C_3 = C_2^*C_3$. So, $\pi_1 \neq \pi_3$ by Lemma 19, i). This is contrary to the hypothesis of this lemma.

Lemma 25. If v_1, v_2, v_3 and $v_4 > 3$, then the following figures don't exist.

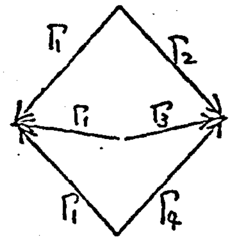


Fig. 1

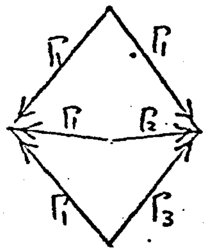


Fig. 2

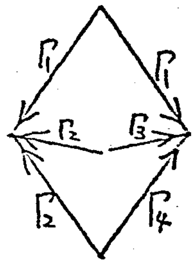


Fig. 3

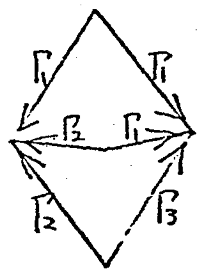


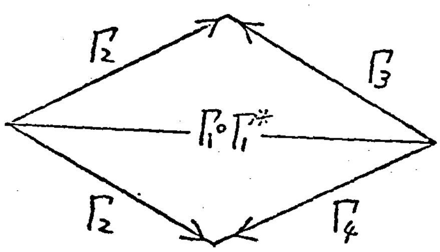
Fig. 4

Proof. For each figure above, we assume its existence and show that it implies a contradiction.

Non-existence of Fig 1.

Case I. $\pi_1 \neq \pi_2, \pi_3, \pi_4$.

By Lemma 18 and Lemma 19, $v_1 = v_2 + 1 = v_3 + 1 = v_4 + 1$, $|\Gamma_1 \circ \Gamma_1^*(\alpha)| = \frac{v_1(v_1-1)}{2}$, $|\Gamma_2(\alpha) \cap \Gamma_3(\delta)| = |\Gamma_2(\alpha) \cap \Gamma_4(\delta)| = 1$ for $(\alpha, \delta) \in \Gamma_1 \circ \Gamma_1^*$ and $\pi_2^* = \pi_3^* = \pi_4^*$. Now let us consider the following figure.



Then by Lemma 22, i) and iii), we have

$$\left|\Gamma_1 \circ \Gamma_1^*(\alpha)\right| = v_2(v_2-1) = (v_1-1)(v_1-2).$$

Thus,

$$\left|\Gamma_1 \circ \Gamma_1^*(\alpha)\right| = \frac{v_1(v_1-1)}{2} = (v_1-1)(v_1-2),$$

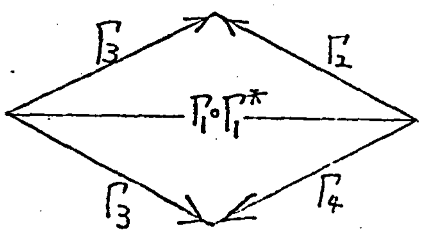
so

$$v_1 = 4,\ v_2 = v_3 = v_4 = 3.$$

This is contrary to the hypothesis of this lemma.

Case II. $\pi_1 = \pi_2 \neq \pi_3, \pi_4$.

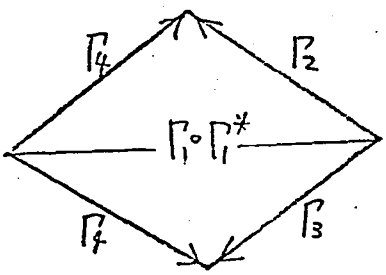
By Lemma 21, $v_1 = v_2 = v_3 + 1 = v_4 + 1$ and $\pi_3^* = \pi_4^* \neq \pi_2^*$. But considering the following figure,



we have $v_3 = v_2 + 1$ by Lemma 20. This is impossible.

Case III. $\pi_1 = \pi_2 = \pi_3 \neq \pi_4$.

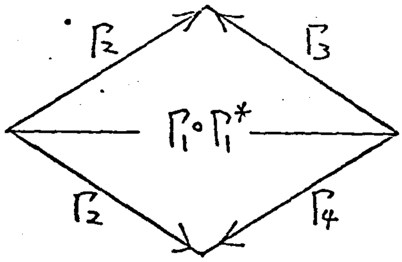
By Lemma 20, $v_1 = v_2 = v_3 = v_4 + 1$. But since there exists the following figure,



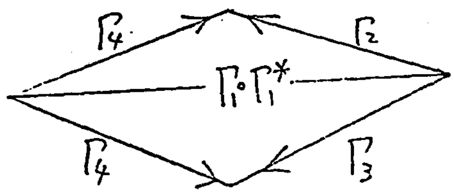
we have $v_4 = v_3 + 1 = v_2 + 1$ by Lemma 21 , which is a contradiction.

Case IV. $\pi_1 = \pi_2 = \pi_3 = \pi_4$, $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* = \Gamma_3 \circ \Gamma_3^* = \Gamma_4 \circ \Gamma_4^*$.

Existence of the following figure is contrary to Lemma 24.



Case V. $\pi_1 = \pi_2 = \pi_3 = \pi_4$, $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* = \Gamma_3 \circ \Gamma_3^* \neq \Gamma_4 \circ \Gamma_4^*$. Since $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* = \Gamma_3 \circ \Gamma_3^*$, we have by Lemma 22, i) $|\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2)| > 1$ for $(\gamma_1, \gamma_2) \in \Gamma_1 \circ \Gamma_1^*$, and hence, $\Gamma_2^* \circ \Gamma_2 = \Gamma_3^* \circ \Gamma_3$. So, we have $|\Gamma_1 \circ \Gamma_1^*(\alpha)| < v_1(v_1-1)$ by Lemma 8, iv). On the other hand, since $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* = \Gamma_3 \circ \Gamma_3^* \neq \Gamma_4 \circ \Gamma_4^*$, we have by Lemma 22, ii) $|\Gamma_4(\gamma_1) \cap \Gamma_2(\gamma_2)| = |\Gamma_4(\gamma_1) \cap \Gamma_3(\gamma_2)| = 1$ for $(\gamma_1, \gamma_2) \in \Gamma_1 \circ \Gamma_1^*$. Then from the existence of the following figure,



we have $|\Gamma_1 \circ \Gamma_1^*(\alpha)| = v_1(v_1-1)$ by Lemma 22, which is a contradiction.

Case VI. $\pi_1 = \pi_2 = \pi_3 = \pi_4$, $\Gamma_1 \circ \Gamma_1^* = \Gamma_2 \circ \Gamma_2^* \neq \Gamma_3 \circ \Gamma_3^*$, $\Gamma_4 \circ \Gamma_4^*$. There exist the following figures, where Σ is a G-orbit.

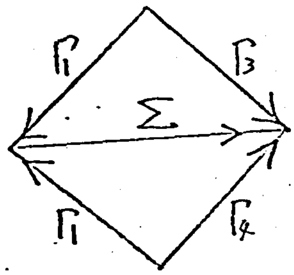


Fig. a

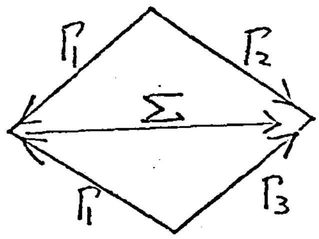


Fig. b

From Fig. a, we have $|\Sigma(\alpha)| = v_1(v_1-1)$ by Lemma 22, iii).

On the other hand, from Fig. b, we have $|\Sigma(\alpha)| = \dfrac{v_1(v_1-1)}{k_1+1}$ by Lemma 22, ii), which is a contradiction.

Case VII. $\pi_1 = \pi_2 = \pi_3 = \pi_4$, $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*$, $\Gamma_3 \circ \Gamma_3^*$, $\Gamma_4 \circ \Gamma_4^*$.

From $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*$, $\Gamma_3 \circ \Gamma_3^*$, we have $|\Gamma_2(\gamma_1) \cap \Gamma_3(\gamma_2)| = 1$ for $\gamma_1, \gamma_2\ (\neq) \in \Gamma_1^*(\alpha)$, by Lemma 22, iii).

Similarly from $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*$, $\Gamma_4 \circ \Gamma_4^*$, we have $|\Gamma_2(\gamma_1) \cap \Gamma_4(\gamma_2)| = 1$ for $\gamma_1, \gamma_2\ (\neq) \in \Gamma_1^*(\alpha)$. From $\Gamma_2 \circ \Gamma_3^* \cap \Gamma_2 \circ \Gamma_4^* \supset \Gamma_1 \circ \Gamma_1^*$,

we have by Lemma 22

$$|\Gamma_1 \circ \Gamma_1^*(\alpha)| = v_1(v_1 - 1). \tag{1}$$

By Lemma 21, $\pi_2^* = \pi_3^* = \pi_4^*$. Therefore we have by Lemma 8, iv)

$$\Gamma_2^* \circ \Gamma_2 \neq \Gamma_3^* \circ \Gamma_3, \quad \Gamma_3^* \circ \Gamma_3 \neq \Gamma_4^* \circ \Gamma_4 \text{ and } \Gamma_4^* \circ \Gamma_4 \neq \Gamma_2^* \circ \Gamma_2$$

$$\text{and } \Gamma_i \circ \Gamma_j^* \ (2 \leqq i, j(\neq) \leqq 4) \text{ contains some } \Gamma_k. \tag{2}$$

We put

$$v = v_1 = v_2 = v_3 = v_4,\ \Gamma_1 \circ \Gamma_1^* = \Delta_1,\ \Gamma_2^* \circ \Gamma_2 = \Delta_2,$$

$$\Gamma_2 \circ \Gamma_3^* = \Delta_1 \cup \Gamma_i,\ \Gamma_3^* \circ \Gamma_4 = \Delta_2 \cup \Sigma',\ \text{and } D_1 = C(\Delta_1),$$

$$D_2 = C(\Delta_2),\ S' = C(\Sigma') \text{ and } s' = |\Sigma'(\alpha)|.$$

Now,

$$(C_2C_3^*)C_4 = (D_1+C_i)C_4 = (v-1)C_3 + \cdots .$$

The coefficient of C_3 of the above equation is $v - 1$ or v by (2).

Next,

$$C_2(C_3^*C_4) = C_2(D_2+xS'),$$

so

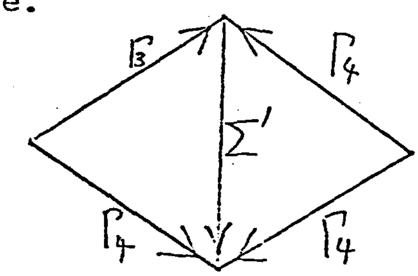
$$v^2 = \frac{v(v-1)}{k_2} + xs'.$$

By Lemma 8, i), $s' \geqq v$,

so

$$x \leqq v - \frac{v-1}{k_2} \leqq v - 2. \qquad (3)$$

We shall show that $\Gamma_4^* \circ \Gamma_4 \neq \Sigma'$. If $\Gamma_4^* \circ \Gamma_4 = \Sigma'$, there exists the following figure.



Since $\Gamma_3\circ\Gamma_4^* = \Delta_1 \cup \Gamma_j$, we have $\Gamma_4\circ\Gamma_4^* = \Delta_1 = \Gamma_1\circ\Gamma_1^*$. This is contrary to the assumption of this case. From $\Gamma_2\circ\Gamma_4^* \cap \Gamma_3\circ\Gamma_4^* \supset \Delta_1$ and (2), for $\gamma_1, \gamma_2\ (\neq) \in \Gamma_4(\alpha)$ we have by Lemma 22, iii)

$$\Gamma_2^*(\gamma_1) \cap \Gamma_3^*(\gamma_2) = 1 . \tag{4}$$

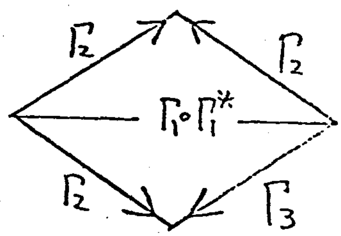
If $\Gamma_2\circ\Sigma'$ contains Γ_3, then we have $\Gamma_2^*\circ\Gamma_3 = \Gamma_4^*\circ\Gamma_4 \cup \Sigma'$, and by (4)

$$C_2S' = (v-\frac{v-1}{k_4})C_3 + \text{terms not involving } C_3.$$

When $k_4 = 1$, $v - \frac{v-1}{k_4} = 1$. So $\Gamma_2\circ\Delta_2$ contains Γ_3, by ( ).

When $k_4 > 1$, $v - 1 > v - \frac{v-1}{k_4} > \frac{v}{2}$. So, $x = 1$, and hence $\Gamma_2\circ\Delta_2$ contains Γ_3.

In all cases, we can conclude that $\Gamma_2\circ\Delta_2$ contains Γ_3, and hence, $\Gamma_2^*\circ\Gamma_3 \supset \Delta_2$. Thus, we have the following figure.



So, $\Gamma_1\circ\Gamma_1^* = \Gamma_2\circ\Gamma_2^*$. This is contrary to the assumption.

Non-existence of Fig. 2.

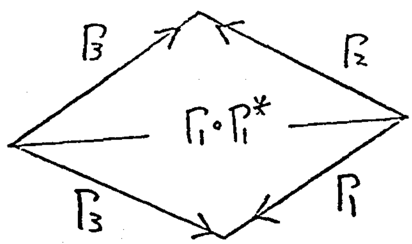
Case I. $\pi_1 \neq \pi_2, \pi_3$.

From $\Gamma_1^*\circ\Gamma_2 \cap \Gamma_1^*\circ\Gamma_3 \neq \emptyset$ and $\pi_1 \neq \pi_2, \pi_3$, we have $|\Gamma_1\circ\Gamma_1^*(\alpha)| = \frac{v_1(v_1-1)}{2}$, $v_1 = v_2 + 1$ and $\Gamma_1\circ\Gamma_1^* \neq \Gamma_2\circ\Gamma_2^*$ by Lemma 21 and

Lemma 19. On the other hand, $|\Gamma_1\circ\Gamma_1^*(\alpha)| = |\Gamma_1^*\circ\Gamma_1(\alpha)| = |\Gamma_1^*\circ\Gamma_2(\alpha)|$ $= |\Gamma_1^*(\alpha)|\cdot|\Gamma_2(\alpha)| = v_1(v_1-1)$. This is impossible.

Case II. $\pi_1 = \pi_2 \neq \pi_3$.

By Lemma 20, $v_1 = v_2 = v_3 + 1$. On the other hand, from the existence of following figure,



We have $v_3 = v_2 + 1 = v_1 + 1$ by Lemma 21, iii). This is impossible.

Case III. $\pi_1 = \pi_2 = \pi_3$, $\Gamma_1\circ\Gamma_1^* = \Gamma_2\circ\Gamma_2^* = \Gamma_3\circ\Gamma_3^*$.

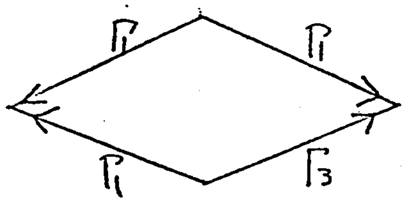
By Lemma 22, for $(\alpha, \delta) \in \Gamma_1^*\circ\Gamma_1$, $1 < |\Gamma_1^*(\alpha)\cap\Gamma_2^*(\delta)|$ and $1 < |\Gamma_1^*(\alpha)\cap\Gamma_3^*(\delta)|$. The counting auguments show that $|\Gamma_1^*(\alpha)\cap\Gamma_2^*(\delta)| = |\Gamma_1(\gamma_1)\cap\Gamma_2(\gamma_2)|$ and $|\Gamma_1^*(\alpha)\cap\Gamma_3^*(\delta)|$ $= |\Gamma_1(\gamma_1)\cap\Gamma_3(\gamma_2)|$ for $(\gamma_1, \gamma_2) \in \Gamma_1\circ\Gamma_1^*$. Therefore, $\Gamma_1^*\circ\Gamma_1 = \Gamma_2^*\circ\Gamma_2$ $= \Gamma_3^*\circ\Gamma_3$. Now $\Gamma_1^*\cdot\Gamma_2 \supset \Gamma_1^*\circ\Gamma_1$ and $\Gamma_1^*\circ\Gamma_3 \supset \Gamma_1^*\circ\Gamma_1$. Since we can show that $\pi_1^* = \pi_2^* = \pi_3^*$ by Lemma 21, we have a contradiction by Lemma 24.

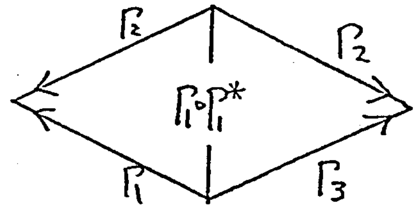
Case IV. $\pi_1 = \pi_2 = \pi_3$, $\Gamma_1\cdot\Gamma_1^* = \Gamma_2\cdot\Gamma_2^* \neq \Gamma_3\circ\Gamma_3^*$.

From $\Gamma_1^*\circ\Gamma_2 \cap \Gamma_1^*\circ\Gamma_3 \supset \Gamma_1^*\circ\Gamma_1$, we have $|\Gamma_1^*\circ\Gamma_1(\alpha)| = \frac{v(v-1)}{k_1+1}$ by Lemma 22. This is impossible.

Case V. $\pi_1 = \pi_2 = \pi_3$, $\Gamma_1\circ\Gamma_1^* \neq \Gamma_2\circ\Gamma_2^*$, $\Gamma_3\cdot\Gamma_3^*$.

By Lemma 21, we have $\pi_1^* = \pi_2^* = \pi_3^*$. By Lemma 22, iii), $|\Gamma_1 \circ \Gamma_1^*(\alpha)| = v(v-1)$, and by Lemma 8, iv), $\Gamma_1^* \circ \Gamma_1 \neq \Gamma_2^* \circ \Gamma_2$.



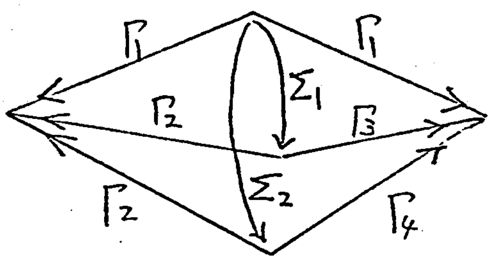


From the existence of the above figures, we have $\Gamma_1^* \circ \Gamma_3 = \Gamma_1^* \circ \Gamma_1 \cup \Gamma_2^* \circ \Gamma_2$. Therefore,

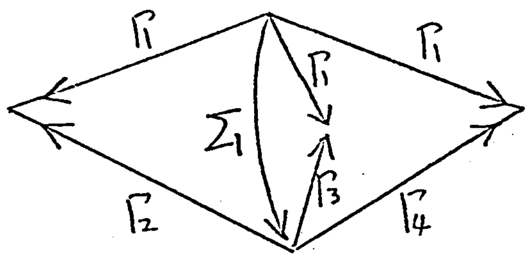
$$
\begin{aligned}
v^2 &= |\Gamma_1(\alpha)| \cdot |\Gamma_3(\alpha)| = |\Gamma_1^* \circ \Gamma_3(\alpha)| \\
&= |\Gamma_1^* \circ \Gamma_1(\alpha)| + |\Gamma_2^* \circ \Gamma_2(\alpha)| = v(v-1) + \frac{v(v-1)}{k_2}.
\end{aligned}
$$

This is impossible.

Non-existence of Fig. 3.



For the above figure, if $\Sigma_1 = \Sigma_2$ then there exists the following figure.



This is contrary to non-existence of Fig. 1. Thus we have $\Sigma_1 \neq \Sigma_2$, $\pi_1^* = \pi_2^*$, $\Gamma_1 \circ \Gamma_2^* = \Sigma_1 \cup \Sigma_2$ and G_α is not doubly transitive on $\Sigma_1(\alpha)$ and $\Sigma_2(\alpha)$ by Lemma 12. So, by Lemma 20 we have $\pi_1^* = \pi_2^* = \pi_3^* = \pi_4^*$. Also $\Gamma_1^* \circ \Gamma_1 = \Gamma_2^* \circ \Gamma_2 = \Gamma_3^* \circ \Gamma_3 = \Gamma_4^* \circ \Gamma_4$ by Lemma 22. From $\Gamma_2^* \circ \Gamma_3 \cap \Gamma_2^* \circ \Gamma_4 \supset \Gamma_1^* \circ \Gamma_1$, this is contrary to Lemma 24.

Non-existence of Fig. 4.

There exist the following figures.

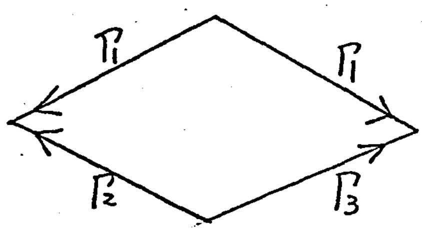


Fig. a

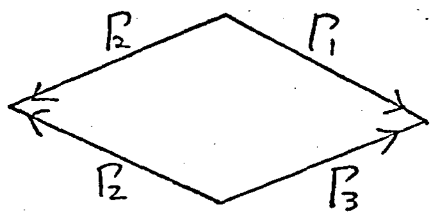


Fig. b

Case I. $\pi_1^* \neq \pi_2^*$.

By Lemma 21, we have $v_1 = v_2 + 1$ from Fig. a, and $v_2 = v_1 + 1$ from Fig. b. This is impossible.

Case II. $\pi_1^* = \pi_2^* \neq \pi_3^*$.

By Lemma 20, we have $v_1 = v_2 = v_3 + 1$ and $\Gamma_2 \circ \Gamma_2^* \neq \Gamma_1 \circ \Gamma_1^*$ from Fig. b. On the other hand, $\Gamma_2 \circ \Gamma_2^* = \Gamma_3 \circ \Gamma_1^* \subset \Gamma_2 \circ \Gamma_1^* \supset \Gamma_1 \circ \Gamma_1^*$, and $\Gamma_2 \circ \Gamma_1^*$ has some Γ_i by Lemma 20, and hence, $\Gamma_2 \circ \Gamma_2^* = \Gamma_1 \circ \Gamma_1^*$. This is impossible.

Case III. $\pi_1^* = \pi_2^* = \pi_3^*$, $\Gamma_1^* \circ \Gamma_1 = \Gamma_2^* \circ \Gamma_2 = \Gamma_3^* \circ \Gamma_3$.

By assumption, $\Gamma_2^* \circ \Gamma_1 \cap \Gamma_2^* \circ \Gamma_3 \supset \Gamma_1^* \circ \Gamma_1 = \Gamma_2^* \circ \Gamma_2 = \Gamma_3^* \circ \Gamma_3$, which contrary to Lemma 24.

Case IV. $\pi_1^* = \pi_2^* = \pi_3^*$, $\Gamma_1^* \circ \Gamma_1 = \Gamma_2^* \circ \Gamma_2 \neq \Gamma_3^* \circ \Gamma_3$.

From Fig. a, $\Gamma_1 \circ \Gamma_2^* = \Gamma_1 \circ \Gamma_1^* \cup \Gamma_i$ for some Γ_i by Lemma 22. So, $\Gamma_1 \circ \Gamma_2^* \cap \Gamma_1 \circ \Gamma_3^* = \Gamma_1 \circ \Gamma_1^*$ and $\Gamma_1 \circ \Gamma_1^*(\alpha) = \frac{v(v-1)}{k_1+1}$. This is impossible.

Case V. $\pi_1^* = \pi_2^* = \pi_3^*$, $\Gamma_1^* \circ \Gamma_1 = \Gamma_3^* \circ \Gamma_3 \neq \Gamma_2^* \circ \Gamma_2$.

We put $\Sigma = \Gamma_1 \circ \Gamma_3^* \cap \Gamma_1 \circ \Gamma_2^*$.
By Lemma 22, $|\Sigma(\alpha)| = \frac{v(v-1)}{k_1+1}$.

From that $\Gamma_1 \circ \Gamma_2^* \supset \Gamma_1 \circ \Gamma_1^*$, we have $\Gamma_1 \circ \Gamma_2^* = \Sigma \cup \Gamma_1 \circ \Gamma_1^*$. So $v^2 = \frac{v(v-1)}{k_1+1} + \frac{v(v-1)}{k_1}$. Therefore $k_1 = 1$ and $v - 1 = k_1 + 1 = 2$. This is contrary to the hypothesis of this lemma.

Case VI. $\pi_1^* = \pi_2^* = \pi_3^*$, $\Gamma_1^* \circ \Gamma_1 \neq \Gamma_2^* \circ \Gamma_2$, $\Gamma_3^* \circ \Gamma_3$.

We put $\Sigma = \Gamma_1 \circ \Gamma_2^* \cap \Gamma_1 \circ \Gamma_3^*$. By Lemma 22, we have $\Gamma_1 \circ \Gamma_2^* = \Sigma \cup \Gamma_i$, $\Gamma_1 \circ \Gamma_3^* = \Sigma \cup \Gamma_j$ from some Γ_i, Γ_j and $|\Sigma(\alpha)| = v(v-1)$. Since $\Gamma_1 \circ \Gamma_2^* \supset \Gamma_1 \circ \Gamma_1^*$ and $\Gamma_1 \circ \Gamma_3^* \supset \Gamma_2 \circ \Gamma_2^*$, we have $\Gamma_1 \circ \Gamma_1^* = \Sigma = \Gamma_2 \circ \Gamma_2^*$. On the other hand, since $\Gamma_1^* \circ \Gamma_2 \supset \Gamma_1^* \circ \Gamma_1$ and $|\Gamma_1^* \circ \Gamma_1(\alpha)| = v(v-1)$, $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^*$ by Lemma 8, iv). This is impossible.

Lemma 26. For Γ_1, Γ_2 and Γ_3, suppose that $\Gamma_1 \circ \Gamma_2^* \cap \Gamma_1 \circ \Gamma_3^*$ contains a G-gobits Σ in $\Omega \times \Omega$, and $v_1, v_2, v_3 > 3$. Then, there does not exist Γ_i such that $\Gamma_i \circ \Gamma_i^* = \Sigma$.

Proof. From non-existences of Fig. 2, Fig. 3, Fig. 4 of Lemma 24, we have this assertion.

Lemma 27. (P. J. Cameron [3], Prop.)
If $\Gamma_i^* \neq \Gamma_i$ and $\Gamma_i \circ \Gamma_i \subseteqq \Gamma_i \cup \Gamma_i^* \cup (\Gamma_i \cup \Gamma_i^*) \cup (\Gamma_i^* \circ \Gamma_i)$, then G has rank 4.

3. Proof of Theorem 1.

We put

$$x_i = \#\{\Gamma_j \mid \Delta_i = \Gamma_j \circ \Gamma_j^*\},$$

$$y_i = \#\{(\Gamma_k, \Gamma_\ell) \mid \quad \Gamma_k \circ \Gamma_\ell^* \supset \Delta_i\}$$

and assume that $x_1 \geqq \cdots \geqq x_r > x_{r+1} = \cdots = x_t = 0$. Counting in two ways triplilaterals $(\Gamma_k, \Gamma_\ell, \Delta_i)$ such that $\Gamma_k \circ \Gamma_\ell^* \supset \Delta_i$, we have by Lemma 9 and 11

$$s^2 \leqq \sum_{i=1}^{t} y_i .$$

The equality means that, for any Γ_i and Γ_j, we cannot have $\Gamma_i \circ \Gamma_j^* = \Delta_k \cup \Delta_\ell$, $\Delta_k \neq \Delta_\ell$.

When $x_i > 0$, by Lemma 26 $y_i \leqq x_i + s$. When $x_i = 0$, by non-existence of Fig. 1 of Lemma 25 $y_i \leqq 2s$. Therefore

$$s^2 \leqq \sum_{i=1}^{t} y_i \leqq \sum_{i=1}^{r} (x_i + s) + 2(t - r)s,$$

so

$$s^2 \leqq (r + 1)s + 2(t - r)s,$$

$$s \leqq 2t - r + 1. \qquad (1)$$

Now, let $\Delta_1 = \Gamma_{i_0} \circ \Gamma^*_{i_0}$ and we put

$$A = \left\{ \{\Gamma_i, \Gamma_j\} : \text{unordered pair} \;\middle|\; \Gamma_i \circ \Gamma^*_j \supset \Delta_1, \Gamma_i \neq \Gamma_j \right\},$$

$$B = \left\{ \Gamma_i \;\middle|\; \{\Gamma_i, \Gamma_j\} \in A \right\}.$$

For $\{\Gamma_i, \Gamma_j\}, \{\Gamma_k, \Gamma_\ell\}$ $(\neq) \in A$, $\{\Gamma_i, \Gamma_j\} \cap \{\Gamma_k, \Gamma_\ell\} = \emptyset$ by Lemma 26. Therefore $|B| \doteq 2|A|$. Furthermore, for $\{\Gamma_i, \Gamma_j\}$, $\{\Gamma_k, \Gamma_\ell\}$ $(\neq) \in A$, and for Γ_m, Γ_n $(\neq) \notin B$, $\Gamma^*_{i_0} \circ \Gamma_i \cap \Gamma^*_{i_0} \circ \Gamma_j$, $\Gamma^*_{i_0} \circ \Gamma_k \cap \Gamma^*_{i_0} \circ \Gamma_\ell$, $\Gamma^*_{i_0} \circ \Gamma_m$, $\Gamma^*_{i_0} \circ \Gamma_n$ are disjoint to each other by non-existence of Fig. 1 of Lemma 25. Thus we have

$$|A| + (s - |B|) = s - |A| \leqq t, \tag{2}$$

and by Lemma 26

$$|A| - 1 \leqq t - r. \tag{3}$$

Assume $s = 2t - r + 1$. Since the equality of (1) hold $y_1 = x_1 + s$, and hence $|A| = \frac{s}{2}$ and $\frac{s}{2} - 1 \leqq t - r$ by (3), and hence, $2t - r + 1 = s \leqq 2t - 2r + 2$. So $r = 1$. Therefore, if $r > 1$, we conclude that $s \leqq 2t - r$.

We shall show that when $r = 1$, $s \leqq 2t - 2$. Assume $r = 1$ and $2t \geqq s \geqq 2t - 1$, and put $\Delta = \Gamma_i \circ \Gamma^*_i$, $1 \leqq i \leqq s$. If $\pi_i \neq \pi_j$ for some Γ_i and Γ_j, then by Lemma 23, $\Delta \not\subset \Gamma_k \circ \Gamma^*_\ell$ for any $\Gamma_k, \Gamma_\ell\,(\neq)$, and hence, $\Gamma_i^* \circ \Gamma_k \cap \Gamma_i^* \circ \Gamma_\ell = \emptyset$. So $s \leqq t$. This is contrary to the assumption that $t \geqq 2$. Thus, it holds that $\pi_1 = \pi_2 = \cdots = \pi_s$.

Now, Suppose $\Gamma_i \circ \Gamma_j = \Delta \cup \Gamma_k^*$ for some Γ_i, Γ_j and Γ_k, and put $D = C(\Delta)$, $\Gamma_j \circ \Gamma_k = \Delta' \cup \Gamma_i^*$, $D' = C(\Delta')$, $t = |\Gamma_i(\alpha) \cap \Gamma_j^*(\beta)|$ for $(\alpha, \beta) \in \Gamma_k^*$, $x = |\Gamma_i(\alpha) \cap \Gamma_j^*(\delta)|$ for $(\alpha, \delta) \in \Delta$, $v = v_1 = v_2 = \cdots$, $k = k_1 = k_2 = \cdots$. Then we have

$$(C_iC_j)C_k = (tC_k^* + xD)C_k = tvI + tkD + xDC_k,$$

$$C_i(C_jC_k) = C_i(t'C_i^* + x'D') = t'vI + t'kD + x'C_iD'.$$

$$(t' = |\Gamma_j(\alpha) \cap \Gamma_k^*(\beta)| \text{ for } (\alpha, \beta) \in \Gamma_i^*, \quad x' = |\Gamma_j(\alpha) \cap \Gamma_k^*(\delta)|$$
$$\text{for } (\alpha, \delta) \in \Delta'.)$$

We have $t = t'$ by counting in two ways triplilaterals (α, β, γ) whose edges are successively Γ_i, Γ_j and Γ_k, and have $|\Delta(\alpha)| = |\Delta'(\alpha)|$ and $x = x'$ by Lemma 10.

So,

$$C_iD' = DC_k = (v - 1)C_k + \cdots .$$

If $C_i \neq C_k$, $|\Delta'(\alpha)| = \frac{v(v-1)}{k+1}$ by Lemma 10. This is impossible. Thus $C_i = C_k$. Similarly, $C_j = C_k$.

When $S = 2t$, then the equality of (1) holds. Therefore, for any Γ_i, there exists Γ_j such that $\Gamma_i \circ \Gamma_j = \Delta \cup \Gamma_k^*$ for some Γ_k^*. So, as is shown above, $\Gamma_i = \Gamma_j = \Gamma_k$. Therefore we have for any Γ_i.

$$\Gamma_i \neq \Gamma_i^*, \; \Gamma_i \circ \Gamma_i = \Delta \cup \Gamma_i^* \text{ and } \Gamma_i \circ \Gamma_m^* \cap \Gamma_i \circ \Gamma_n^* = \emptyset$$

$$\text{for } \Gamma_m \neq \Gamma_n, \Gamma_n^*.$$

When $s = 2t - 1$, then $|A| \leqq t - 1$, and from (2) $s - |A| \leqq t$. So $|A| = t - 1$. Therefore, there is a unique Γ_u such that for any Γ_i ($\neq \Gamma_u$), $\Gamma_i \circ \Gamma_u^* \not\supset \Delta$. We shall show that for any Γ_i, Γ_j ($\neq$), $\Gamma_i \circ \Gamma_j^*$ contains some Γ_k. Assume $\Gamma_i \circ \Gamma_j^* = \Delta_k \cup \Delta_\ell$ for some Γ_i, Γ_j ($\neq$). Count in two ways the paired (Γ_m, Δ_n) such that $\Gamma_i \circ \Gamma_m^*$ contains Δ_n, then by Lemma 25, we have

$$2t = s + 1 \leqq \#\{(\Gamma_m, \Delta_n) \mid \Gamma_i \circ \Gamma_m^* \supset \Delta_n\} \leqq 2t.$$

So, equality holds. Thus for any Δ_k, there exist Γ_p and Γ_q ($\neq$) such that $\Gamma_i \circ \Gamma_p^*$ and $\Gamma_i \circ \Gamma_q^*$ contains Δ_k. Therefore we may choose Γ_a such that $\Gamma_i \circ \Gamma_u^* \cap \Gamma_i \circ \Gamma_a^* \neq \emptyset$ and $\Gamma_a \neq \Gamma_u$. Then $\Gamma_a \circ \Gamma_u^* \supset \Gamma_i \circ \Gamma_i^* = \Delta$. This is impossible. Thus, again as is shown above, we can conclude that for any Γ_i ($\neq \Gamma_u$),

$$\Gamma_i \neq \Gamma_i^*, \ \Gamma_i \circ \Gamma_i = \Delta \cup \Gamma_i^* \text{ and}$$

$$\Gamma_i \circ \Gamma_m^* \cap \Gamma_i \circ \Gamma_n^* = \emptyset \quad \text{for } \Gamma_m \neq \Gamma_n, \Gamma_n^*.$$

Thus if $s \geqq 2t - 1$, there exists Γ_i such that

$$\Gamma_i \neq \Gamma_i^* \text{ and } \Gamma_i \circ \Gamma_i = \Gamma_i \circ \Gamma_i^* \cup \Gamma_i^*.$$

By Lemma 27, this show that G has rank 4. This is impposible for $s \geqq 2t - 1$ and $t \geqq 2$.

4. Proof of Theorem 2

When $r = t$, we have $s \leqq t$ by Theorem 1. On the other hand, from $s \geqq r = t$, we conclude that $s = t = r$.

We put $\Gamma_i \circ \Gamma_i^* = \Delta_i$, $A_i = \{\{\Gamma_k, \Gamma_\ell\};$ unordered pair$| \Gamma_k \circ \Gamma_\ell^* \supset \Delta_i$, $\Gamma_k \neq \Gamma_\ell\}$. Then $|A_i| - 1 \leqq t - r = 0$, so $|A_i| \leqq 1$.

Count in two ways triplilaterals $(\Gamma_i, \Gamma_j, \Delta_k)$ such that $\Gamma_i \circ \Gamma_j^* \supset \Delta_k$, we have

$$s^2 \leqq 3s \ ,$$

so

$$s \leqq 3. \qquad (1)$$

Case $t = 2$. If $|\Gamma_1(\alpha)| \neq |\Gamma_2(\alpha)|$, by T. Ito [6], G is isomorphic to the small Janko simple group and G_α is isomorphic to PSL(2.11). We shall prove that the case of $|\Gamma_1(\alpha)| = |\Gamma_2(\alpha)|$ does not occure. We put $|\Gamma_1(\alpha)| = |\Gamma_2(\alpha)| = v$. It is easy to prove that $\pi_1 = \pi_2$. We shall show that Γ_1 and Γ_2 are self paired. If not, then $\Gamma_1^* = \Gamma_2$. Since $\Gamma_1 \circ \Gamma_1^* \neq \Gamma_2 \circ \Gamma_2^* = \Gamma_1^* \circ \Gamma_1$, we have that $\Gamma_1 \circ \Gamma_1 \not\supset \Gamma_1 \circ \Gamma_1^*$, $\Gamma_1^* \circ \Gamma_1 (= \Gamma_2 \circ \Gamma_2^*)$ by Lemma 7. By Lemma 11, there exists a G-orbit Σ in $\Gamma_1 \circ \Gamma_1$ ~~on which~~ such that G_α is not 2-transitive on $\Sigma(\alpha)$, and $\Sigma \neq \Delta_1, \Delta_2$. This is impossible for $t = 2$. Thus, we have $\Gamma_1 \circ \Gamma_2 = \Delta_1 \cup \Delta_2$. So, $v^2 = |\Gamma_1 \circ \Gamma_2(\alpha)| = |\Gamma_1 \circ \Gamma_1(\alpha)| + |\Gamma_2 \circ \Gamma_2(\alpha)|$ $= \dfrac{v(v-1)}{k_1} + \dfrac{v(v-1)}{k_2}$. This is impossible.

Case $t = 3$. For this case, the equality of (1) holds. So we have $|A_i| = 1$ for $1 \leqq i \leqq 3$. We shall show that if $\Gamma_i \circ \Gamma_j^* = \Delta_1$ then $\Gamma_i = \Gamma_1$ or $\Gamma_j = \Gamma_1$. If $\Gamma_i, \Gamma_j \neq \Gamma_1$, then since $\Gamma_1^* \circ \Gamma_i \cap \Gamma_1^* \circ \Gamma_j \neq \emptyset$, there exists a G-orbit Σ in $\Gamma_1^* \circ \Gamma_i \cap \Gamma_1^* \circ \Gamma_j$ such that G is not 2-transitive on $\Sigma(\alpha)$ by Lemma 12, and for any Γ_i, $\Gamma_i \circ \Gamma_i^* \neq \Sigma$ by

Lemma 25. From r = t, this is impossible. Thus we may assume that there exist the following figures.

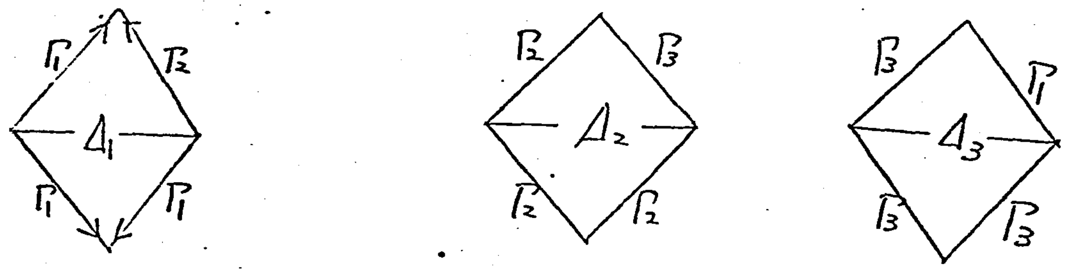


Fig. a Fig. b Fig. c

If $\pi_1 \neq \pi_2, \pi_3$, then $v_1 v_2 = |\Gamma_1^* \circ \Gamma_2(\alpha)| = |\Gamma_1^* \circ \Gamma_1(\alpha)| = \dfrac{v_1(v_1-1)}{k_1}$ from Fig. a, so $v_1 > v_2$. Similarly, $v_3 > v_1$ from Fig. c. Therefore $v_3 > v_2$. On the other hand, $v_2 v_3 = \dfrac{v_2(v_2-1)}{k_2}$ from Fig. b, so $v_2 > v_3$. This is impossible. Thus we have $\pi_1 = \pi_2 = \pi_3$. By Lemma 7, Γ_1, Γ_2 and Γ_3 are self-paired.

Thus $\Gamma_1 \circ \Gamma_2 = \Gamma_3 \cup \Delta_1$, $\Gamma_2 \circ \Gamma_3 = \Gamma_1 \cup \Delta_2$, $\Gamma_3 \circ \Gamma_1 = \Gamma_2 \cup \Delta_3$.
Put $|\Gamma_1(\alpha)| = v$, then by Lemma 8, iii) we have

$$|\Delta_1(\alpha)| = |\Delta_2(\alpha)| = |\Delta_3(\alpha)| = v(v-1).$$

We put

$$D_i = C(\Delta_i) \text{ and } C_i = C(\Gamma_i),\ 1 \leqq i \leqq 3;$$

$$D_1 C_3 = x_1 D_1 + x_2 D_2 + x_3 D_3.$$

Then

$$x_1 + x_2 + x_3 = v$$

$$D_2C_3 = x_2D_1 + \text{terms not involving } D_1\,,$$

(2)

$$D_3C_3 = x_3D_1 + \text{terms not involving } D_1.$$

Now

$$(C_1C_2)C_3 = (D_1 + C_3)C_3 = vI + D_3 + D_1C_3,$$

$$C_1(C_2C_3) = C_1(D_2 + C_1) = vI + D_1 + D_2C_1.$$

So

$$D_2C_1 = D_1C_3 + D_3 - D_1 = (x_1-1)D_1 + x_2D_2 + (x_3+1)D_3.$$

Similarly

$$D_3C_2 = D_2C_1 + D_1 - D_2 = x_1D_1 + (x_2-1)D_2 + (x_3+1)D_3.$$

Next

$$(C_1C_1)C_3 = (vI + D_1)C_3 = vC_3 + D_1C_3\,,$$

$$C_1(C_1C_3) = C_1(D_3 + C_2) = C_3 + D_1 + D_3C_1.$$

So

$$D_3C_1 = D_1C_3 + (v-1)C_3 - D_1$$

$$= (x_1-1)D_1 + x_2D_2 + x_3D_3 + (v-1)C_3.$$

Similarly

$$D_1C_2 = D_2C_1 + (v-1)C_1 - D_2$$

$$= (x_1-1)D_1 + (x_2-1)D_2 + (x_3+1)D_3 + (v-1)C_1,$$

$$D_2C_3 = D_3C_2 + (v-1)C_2 - D_3$$

$$= x_1D_1 + (x_2-1)D_2 + x_3D_3 + (v-1)C_2. \qquad (3)$$

Furthermore

$$(C_1C_1)C_2 = (vI + D_1)C_2 = vC_2 + D_1C_2 \ ,$$

$$C_1(C_1C_2) = C_1(C_3 + D_1) = C_2 + D_3 + D_1C_1$$

So

$$D_1C_1 = D_1C_2 + (v-1)C_2 - D_3$$

$$= (x_1-1)D_1 + (x_2-1)D_2 + x_3D_3 + (v-1)C_1 + (v-1)C_2.$$

Similarly

$$D_2C_2 = D_2C_3 + (v-1)C_3 - D_1$$

$$= (x_1-1)D_1 + (x_2-1)D_2 + x_3D_3 + (v-1)C_2 + (v-1)C_3 \ ,$$

$$D_3C_3 = D_3C_1 + (v-1)C_1 - D_2$$
$$= (x_1-1)D_1 + (x_2-1)D_2 + x_3D_3 + (v-1)C_3 + (v-1)C_1. \qquad (4)$$

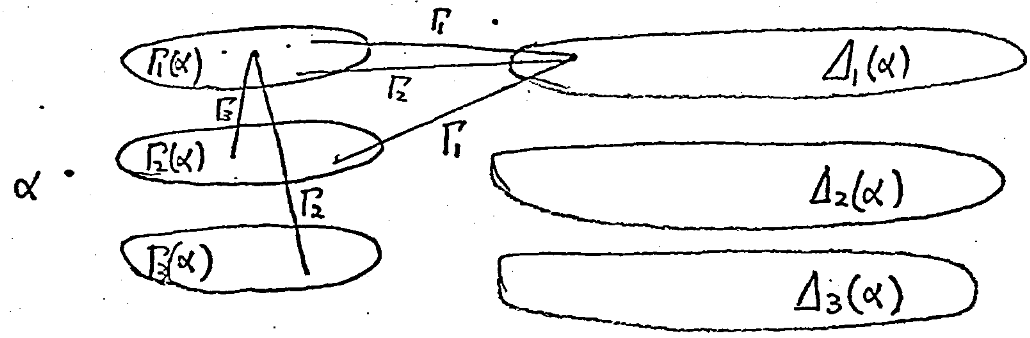
Thus (2), (3) and (4) yield

$$x_1 = x_2, \ x_1 - 1 = x_3.$$

We put $x_3 = x$, then

$$v = x_1 + x_2 + x_3 = (x+1) + (x+1) + x = 3x + 2. \tag{5}$$

It is easy to show that the graph $(\Omega, \Gamma_1 \cup \Gamma_2 \cup \Gamma_3)$ is a strongly regular graph with parameters 3v, 2, 3.



From the conditions of the existence of the strongly regular graph, (see [1] p. 97) it holds that

$$(3-2)^2 + 4(3v-3) = 12v - 11 = d^2, \tag{6}$$

(s is a positive integer)

$$m = \frac{3v}{2\cdot 3\cdot d}\left\{(3v-1+3-2)(d+3-2) - 2\cdot 3\right\} = \frac{3}{2}v^2 + \frac{3v(v-2)}{2d}. \tag{7}$$

(m is a positive integer)

From (7), $\frac{3v(v-2)}{d}$ is integer, and hence

$12v - 11 = d^2$ is a divisor of $v^2(v-2)^2$.

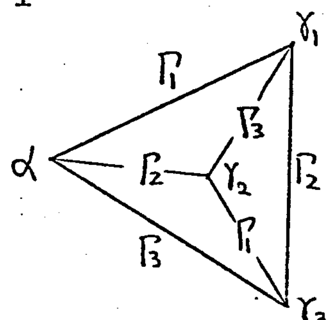
So

$12v - 11$ is a divisor of $11^2 \cdot 13^2$.

From $v = 3x + 2$, we conclude

$$v = 11.$$

Lastly, we shall prove that the primitive group satisfying these conditions does not exist. It is easy to prove that G_α acts faithfully on $\Gamma_1(\alpha)$. We shall show that for γ_1, γ_1' $(\neq)$ $\in \Gamma_1(\alpha)$, $G_{\alpha,\gamma_1,\gamma_1'}$ has the fixed points in $\Gamma_1(\alpha) \setminus \{\gamma_1, \gamma_1'\}$.



For $(\alpha, \gamma_1) \in \Gamma_1$, put $\{\gamma_2\} = \Gamma_2(\alpha) \cap \Gamma_3(\gamma_1)$ and $\{\gamma_3\} = \Gamma_3(\alpha) \cap \Gamma_2(\gamma_1)$. Then, G_{α,γ_1} fix γ_2 and γ_3. So we must have that $(\gamma_2, \gamma_3) \in \Gamma_1$.
Now, for γ_1, γ_1' $(\neq) \in \Gamma_1(\alpha)$, put $\{\delta_1\} = \Gamma_1(\gamma_1) \cap \Gamma_2(\gamma_1')$, $\{\delta_2\} = \Gamma_2(\gamma_1) \cap \Gamma_1(\gamma_1')$. Then $G_{\alpha,\gamma_1,\gamma_1'}$ fix δ_1 and δ_2.
Since $(\gamma_1, \gamma_1') \notin \Gamma_3$, we have $(\delta_1, \delta_2) \notin \Gamma_3$.
Therefore $\Gamma_1(\gamma_1) \cap \Gamma_3(\delta_2) = \{\delta\} \neq \{\delta_1\}$.

So, $G_{\alpha,\gamma_1,\gamma_1'}$ fix δ_1 and δ. Since $\Gamma_1(\gamma_1) \ni \alpha, \delta_1, \delta$ $(\neq)$, in the same way, we obtain that $G_{\alpha,\gamma_1,\gamma_1'}$ has the fix points in $\Gamma_1(\alpha) \setminus \{\gamma_1 \cup \gamma_1'\}$. The order of G_α is at most one million. If G_α is non-solvable, then the minimal normal subfroup of G_α is non-solvable simple. From [5], it is isomorphic to the Mathieu group M_{11} or the transitive extension of the alternating group A_5 act on ten points. These groups have not the representation such that it is doubly-transitive on eleven points and it's stabilizer of two points has the additional fixed point. Thus, we can conclude that G_α is solvable and the order of G_α is 110. So $|G| = |\Omega| \cdot 11 \cdot 10 = 364 \cdot 11 \cdot 10 = 2^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$. G is non-solvable group and $(|G|, 3) = 1$. But there does not exist such group by M. Hall [5].

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