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# QUASITORIC MANIFOLDS HOMEOMORPHIC TO HOMOGENEOUS SPACES

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## Abstract

We present some classification results for quasitoric manifolds  $M$  with  $p_1(M) = -\sum a_i^2$  for some  $a_i \in H^2(M)$  which admit an action of a compact connected Lie-group  $G$  such that  $\dim M/G \leq 1$ . In contrast to Kuroki's work [7, 6] we do not require that the action of  $G$  extends the torus action on  $M$ .

## 1. Introduction

Quasitoric manifolds are certain  $2n$ -dimensional manifolds on which an  $n$ -dimensional torus acts such that the orbit space of this action may be identified with a simple convex polytope. They were first introduced by Davis and Januszkiewicz [2] in 1991.

In [7, 6] Kuroki studied quasitoric manifolds  $M$  which admit an extension of the torus action to an action of some compact connected Lie-group  $G$  such that  $\dim M/G \leq 1$ . Here we drop the condition that the  $G$ -action extends the torus action in the case where the first Pontrjagin-class of  $M$  is equal to the negative of a sum of squares of elements of  $H^2(M)$ . In this note all cohomology groups are taken with coefficients in  $\mathbb{Q}$ . We have the following two results.

**Theorem 1.1.** *Let  $M$  be a quasitoric manifold with  $p_1(M) = -\sum a_i^2$  for some  $a_i \in H^2(M)$  which is homeomorphic (or diffeomorphic) to a homogeneous space  $G/H$  with  $G$  a compact connected Lie-group. Then  $M$  is homeomorphic (diffeomorphic) to  $\prod S^2$ . In particular, all Pontrjagin-classes of  $M$  vanish.*

**Theorem 1.2.** *Let  $M$  be a quasitoric manifold with  $p_1(M) = -\sum a_i^2$  for some  $a_i \in H^2(M)$ . Assume that the compact connected Lie-group  $G$  acts smoothly and almost effectively on  $M$  such that  $\dim M/G = 1$ . Then  $G$  has a finite covering group of the form  $\prod SU(2)$  or  $\prod SU(2) \times S^1$ . Furthermore  $M$  is diffeomorphic to a  $S^2$ -bundle over a product of two-spheres.*

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The proofs of these theorems are based on Hauschild's study [4] of spaces of  $q$ -type. A space of  $q$ -type is defined to be a topological space  $X$  satisfying the following cohomological properties:

- The cohomology ring  $H^*(X)$  is generated as a  $\mathbb{Q}$ -algebra by elements of degree two, i.e.  $H^*(X) = \mathbb{Q}[x_1, \dots, x_n]/I_0$  and  $\deg x_i = 2$ .
- The defining ideal  $I_0$  contains a definite quadratic form  $Q$ .

The note is organised as follows. In Section 2 we show that a quasitoric manifold  $M$  with  $p_1(M) = -\sum a_i^2$  for some  $a_i \in H^2(M)$  is of  $q$ -type. In Section 3 we prove Theorem 1.1. In Section 4 we recall some properties of cohomogeneity one manifolds. In Section 5 we prove Theorem 1.2.

The results presented in this note form part of the outcome of my Ph.D. thesis [10] written under the supervision of Prof. Anand Dessai at the University of Fribourg. I would like to thank Anand Dessai for helpful discussions.

## 2. Quasitoric manifolds with $p_1(M) = -\sum a_i^2$

In this section we study quasitoric manifolds  $M$  with  $p_1(M) = -\sum a_i^2$  for some  $a_i \in H^2(M)$ . To do so we first introduce some notations from [4] and [5, Chapter VII]. For a topological space  $X$  we define the topological degree of symmetry of  $X$  as

$$N_t(X) = \max\{\dim G; G \text{ compact Lie-group, } G \text{ acts effectively on } X\}.$$

Similarly one defines the semi-simple degree of symmetry of  $X$  as

$$N_t^{ss}(X) = \max\{\dim G; G \text{ compact semi-simple Lie-group, } G \text{ acts effectively on } X\}$$

and the torus-degree of symmetry as

$$T_t(X) = \max\{\dim T; T \text{ torus, } T \text{ acts effectively on } X\}.$$

In the above definitions we assume that all groups act continuously.

Another important invariant of a topological space  $X$  used in [4] is the so called embedding dimension of its rational cohomology ring. For a local  $\mathbb{Q}$ -algebra  $A$ , we denote by  $\text{edim } A$  the embedding dimension of  $A$ . By definition, we have  $\text{edim } A = \dim_{\mathbb{Q}} \mathfrak{m}_A/\mathfrak{m}_A^2$ , where  $\mathfrak{m}_A$  is the maximal ideal of  $A$ . In case that  $A = \bigoplus_{i \geq 0} A^i$  is a positively graded local  $\mathbb{Q}$ -algebra,  $\mathfrak{m}_A$  is the augmentation ideal  $A_+ = \bigoplus_{i > 0} A^i$ . If furthermore  $A$  is generated by its degree two part, then  $\mathfrak{m}_A^2 = \bigoplus_{i > 2} A^i$ . Therefore for a quasitoric manifold  $M$  over the polytope  $P$  we have  $\text{edim } H^*(M) = \dim_{\mathbb{Q}} H^2(M) = m - n$  where  $m$  is the number of facets of  $P$  and  $n$  is its dimension.

**Lemma 2.1.** *Let  $M$  be a quasitoric manifold with  $p_1(M) = -\sum a_i^2$  for some  $a_i \in H^2(M)$ . Then  $M$  is a manifold of  $q$ -type.*

Proof. The discussion at the beginning of Section 3 of [8] together with Corollary 6.8 of [2, p. 448] shows that there are a basis  $u_{n+1}, \dots, u_m$  of  $H^2(M)$  and  $\lambda_{i,j} \in \mathbb{Z}$  such that

$$p_1(M) = \sum_{i=n+1}^m u_i^2 + \sum_{j=1}^n \left( \sum_{i=n+1}^m \lambda_{i,j} u_i \right)^2.$$

Therefore

$$\begin{aligned} 0 &= \sum_{i=n+1}^m u_i^2 + \sum_{j=1}^n \left( \sum_{i=n+1}^m \lambda_{i,j} u_i \right)^2 + \sum_i a_i^2 \\ &= \sum_{i=n+1}^m u_i^2 + \sum_{j=1}^n \left( \sum_{i=n+1}^m \lambda_{i,j} u_i \right)^2 + \sum_j \left( \sum_{i=n+1}^m \mu_{i,j} u_i \right)^2 \end{aligned}$$

with some  $\mu_{i,j} \in \mathbb{Q}$  follows.

Because

$$\sum_{i=n+1}^m X_i^2 + \sum_{j=1}^n \left( \sum_{i=n+1}^m \lambda_{i,j} X_i \right)^2 + \sum_j \left( \sum_{i=n+1}^m \mu_{i,j} X_i \right)^2$$

is a positive definite bilinear form the statement follows.  $\square$

**Proposition 2.2.** *Let  $M$  be a quasitoric manifold of  $q$ -type over the  $n$ -dimensional polytope  $P$ . Then we have for the number  $m$  of facets of  $P$ :*

$$m \geq 2n$$

Proof. By Theorem 3.2 of [4, p. 563], we have

$$n \leq T_t(M) \leq \text{edim } H^*(M) = m - n.$$

Therefore we have  $2n \leq m$ .  $\square$

**REMARK 2.3.** The inequality in the above proposition is sharp, because for  $M = S^2 \times \dots \times S^2$  we have  $m = 2n$  and  $p_1(M) = 0$ .

By Theorem 5.13 of [4, p. 573], we have for a manifold  $M$  of  $q$ -type that  $N_t^{ss} \leq \dim M + \text{edim } M$ . Hence, for a quasitoric manifold  $M$ , we get:

**Proposition 2.4.** *Let  $M$  as in Proposition 2.2. Then we have*

$$N_t^{ss}(M) \leq 2n + m - n = n + m.$$

REMARK 2.5. The inequality in the above proposition is sharp because for  $M = S^2 \times \cdots \times S^2$  we have  $m = 2n$  and  $SU(2) \times \cdots \times SU(2)$  acts on  $M$  and has dimension  $3n$ .

### 3. Quasitoric manifolds which are also homogeneous spaces

In this section we prove Theorem 1.1. Recall from Lemma 2.1 that a quasitoric manifold  $M$  with first Pontrjagin-class equal to the negative of the sum of squares of elements of  $H^2(M)$  is a manifold of  $q$ -type.

Let  $M$  be a quasitoric manifold over the polytope  $P$  which is also a homogeneous space and is of  $q$ -type.

Let  $G$  be a compact connected Lie-group and  $H \subset G$  a closed subgroup such that  $M$  is homeomorphic or diffeomorphic to  $G/H$ . Because  $\chi(M) > 0$  and  $M$  is simply connected, we have  $\text{rank } G = \text{rank } H$  and  $H$  is connected. Therefore we may assume that  $G$  is semi-simple and simply connected.

Let  $T$  be a maximal torus of  $G$ . Then  $(G/H)^T$  is non-empty. By Theorem 5.9 of [4, p.572], the isotropy group  $G_x$  of a point  $x \in (G/H)^T$  is a maximal torus of  $G$ . Hence,  $H$  is a maximal torus of  $G$ .

Now it follows from Theorem 3.3 of [4, p.563] that

$$T_t(G/H) = \text{rank } G.$$

Because  $M$  is quasitoric, we have  $n \leq T_t(G/H)$ . Combining these inequations, we get

$$\dim G - \dim H = \dim M = 2n \leq 2 \text{rank } G.$$

This equation implies that  $\dim G \leq 3 \text{rank } G$ .

For a simple simply connected Lie-group  $G'$  we have  $\dim G' \geq 3 \text{rank } G'$  and  $\dim G' = 3 \text{rank } G'$  if and only if  $G' = SU(2)$ . Therefore we have  $G = \prod SU(2)$  and  $M = \prod SU(2)/T^1 = \prod S^2$ . This proves Theorem 1.1.

### 4. Cohomogeneity one manifolds

Here we discuss some facts about closed cohomogeneity one Riemannian  $G$ -manifolds  $M$  with orbit space a compact interval  $[-1, 1]$ . We follow [3, p.39-44] in this discussion.

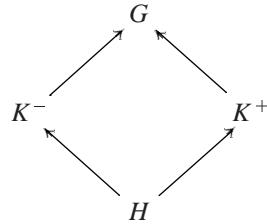
We fix a normal geodesic  $c: [-1, 1] \rightarrow M$  perpendicular to all orbits. We denote by  $H$  the principal isotropy group  $G_{c(0)}$ , which is equal to the isotropy group  $G_{c(t)}$  for  $t \in ]-1, 1[$ , and by  $K^\pm$  the isotropy groups of  $c(\pm 1)$ .

Then  $M$  is the union of tubular neighbourhoods of the non-principal orbits  $Gc(\pm 1)$  glued along their boundary, i.e., by the slice theorem we have

$$(4.1) \quad M = G \times_{K^-} D_- \cup G \times_{K^+} D_+,$$

where  $D_{\pm}$  are discs. Furthermore  $K^{\pm}/H = \partial D_{\pm} = S_{\pm}$  are spheres.

Note that  $M$  may be reconstructed from the following diagram of groups.



The construction of such a group diagram from a cohomogeneity one manifold may be reversed. Namely, if such a group diagram with  $K^{\pm}/H = S_{\pm}$  spheres is given, then one may construct a cohomogeneity one  $G$ -manifold from it. We also write these diagrams as  $H \subset K^-$ ,  $K^+ \subset G$ .

Now we give a criterion for two group diagrams yielding up to  $G$ -equivariant diffeomorphism the same manifold  $M$ .

**Lemma 4.1** ([3, p.44]). *The group diagrams  $H \subset K^-$ ,  $K_1^+ \subset G$  and  $H \subset K^-$ ,  $K_2^+ \subset G$  yield the same cohomogeneity one manifold up to equivariant diffeomorphism if there is an  $a \in N_G(H)^0$  with  $K_1^+ = aK_2^+a^{-1}$ .*

## 5. Quasitoric manifolds with cohomogeneity one actions

In this section we study quasitoric manifolds  $M$  which admit a smooth action of a compact connected Lie-group  $G$  which has an orbit of codimension one. As before we do not assume that the  $G$ -action on  $M$  extends the torus action. We have the following lemma:

**Lemma 5.1.** *Let  $M$  be a quasitoric manifold of dimension  $2n$  which is of  $q$ -type. Assume that the compact connected Lie-group  $G$  acts almost effectively and smoothly on  $M$  such that  $\dim M/G = 1$ . Then we have:*

- (1) *The singular orbits are given by  $G/T$  where  $T$  is a maximal torus of  $G$ .*
- (2) *The Euler-characteristic of  $M$  is  $2 \#W(G)$ .*
- (3) *The principal orbit type is given by  $G/S$ , where  $S \subset T$  is a subgroup of codimension one.*
- (4) *The center  $Z$  of  $G$  has dimension at most one.*
- (5)  $\dim G/T = 2n - 2$ .

**Proof.** At first note that  $M/G$  is an interval  $[-1, 1]$  and not a circle because  $M$  is simply connected. We start with proving (1). Let  $T$  be a maximal torus of  $G$ . By passing to a finite covering group of  $G$  we may assume  $G = G' \times Z'$  with  $G'$  a compact

connected semi-simple Lie-group and  $Z'$  a torus. Let  $x \in M^T$ . Then the isotropy group  $G_x$  has maximal rank in  $G$ . Therefore  $G_x$  splits as  $G'_x \times Z'$ .

By Theorem 5.9 of [4, p.572],  $G'_x$  is a maximal torus of  $G'$ . Therefore we have  $G_x = T$ .

Because  $\dim G - \dim T$  is even,  $x$  is contained in a singular orbit. In particular we have

$$(5.1) \quad \chi(M) = \chi(M^T) = \chi(G/K^+) + \chi(G/K^-),$$

where  $G/K^\pm$  are the singular orbits. Furthermore we may assume that  $G/K^+$  contains a  $T$ -fixed point. This implies

$$(5.2) \quad \chi(G/K^+) = \chi(G/T) = \#W(G) = \#W(G').$$

Now assume that all  $T$ -fixed points are contained in the singular orbit  $G/K^+$ . Then we have  $(G/K^-)^T = \emptyset$ . This implies

$$\chi(M) = \chi(G/K^+) = \#W(G').$$

Now Theorem 5.11 of [4, p.573] implies that  $M$  is the homogeneous space  $G'/G' \cap T = G/T$ . This contradicts our assumption that  $\dim M/G = 1$ .

Therefore both singular orbits contain  $T$ -fixed points. This implies that they are of type  $G/T$ . This proves (1). (2) follows from (5.1) and (5.2).

Now we prove (3) and (5). Let  $S \subset T$  be a minimal isotropy group. Then  $T/S$  is a sphere of dimension  $\text{codim}(G/T, M) - 1$ . Therefore  $S$  is a subgroup of codimension one in  $T$  and  $\text{codim}(G/T, M) = 2$ .

If the center of  $G$  has dimension greater than one, then  $\dim Z' \cap S \geq 1$ . That means that the action is not almost effective. Therefore (4) holds.  $\square$

By Lemma 5.1, we have with the notation of the previous section that  $K^\pm$  are maximal tori of  $G$  containing  $H = S$ . In the following we will write  $G = G' \times Z'$  with  $G'$  a compact connected semi-simple Lie-group and  $Z'$  a torus.

Because  $K^\pm$  are maximal tori of the identity component  $Z_G(S)^0$  of the centraliser of  $S$ , there is some  $a \in Z_G(S)^0$  such that  $K^- = aK^+a^{-1}$ . By Lemma 4.1, we may assume that  $K^+ = K^- = T$ . Now from Theorem 4.1 of [9, p.198] it follows that  $M$  is a fiber bundle over  $G/T$  with fiber the cohomogeneity one manifold with group diagram  $S \subset T$ ,  $T \subset T$ . Therefore it is a  $S^2$ -bundle over  $G/T$ .

**Lemma 5.2.** *Let  $M$  and  $G$  as in the previous lemma. Then we have*

$$T_t(M) \leq \text{rank } G' + 1.$$

Proof. At first we recall the rational cohomology of  $G/T$ . By [1, p. 67], we have

$$H^*(G/T) \cong H^*(BT)/I$$

where  $I$  is the ideal generated by the elements of positive degree which are invariant under the action of the Weyl-group of  $G$ . Therefore it follows that

$$\dim_{\mathbb{Q}} H^{\text{odd}}(G/T) = 0 \quad \text{and} \quad \dim_{\mathbb{Q}} H^2(G/T) = \text{rank } G'.$$

Therefore the Serre spectral sequence for the fibration  $S^2 \rightarrow M \rightarrow G/T$  degenerates. Hence, we have

$$H^*(M) = H^*(G/T) \otimes H^*(S^2)$$

as  $H^*(G/T)$ -modules. In particular, we have

$$\dim_{\mathbb{Q}} H^2(M) = \dim_{\mathbb{Q}} H^2(G/T) + \dim_{\mathbb{Q}} H^2(S^2) = \text{rank } G' + 1.$$

Therefore

$$T_t(M) \leq \text{edim } H^*(M) = \dim_{\mathbb{Q}} H^2(M) = \text{rank } G' + 1$$

follows. □

**Theorem 5.3.** *Let  $M$  and  $G$  as in the previous lemmas. Then  $G$  has a finite covering group of the form  $\prod SU(2)$  or  $\prod SU(2) \times S^1$ . Furthermore  $M$  is diffeomorphic to a  $S^2$ -bundle over a product of two-spheres.*

Proof. Because  $M$  is quasitoric we have  $n \leq T_t(M)$ . By Lemma 5.1 we have

$$\dim G' - \text{rank } G' = \dim G/T = 2n - 2.$$

Now Lemma 5.2 implies

$$\dim G' = 2n - 2 + \text{rank } G' \leq 3 \text{rank } G'.$$

Therefore  $\prod SU(2)$  is a finite covering group of  $G'$ . This implies the statement about the finite covering group of  $G$ .

It follows that  $G/T = \prod S^2$ . Therefore  $M$  is a  $S^2$ -bundle over  $\prod S^2$ . □

Now Theorem 1.2 follows from Theorem 5.3 and Lemma 2.1.

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