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Author(s)	Wiemeler, Michael
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QUASITORIC MANIFOLDS HOMEOMORPHIC TO HOMOGENEOUS SPACES

MICHAEL WIEMELER

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Abstract

We present some classification results for quasitoric manifolds M with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$ which admit an action of a compact connected Lie-group G such that $\dim M/G \leq 1$. In contrast to Kuroki's work [7, 6] we do not require that the action of G extends the torus action on M .

1. Introduction

Quasitoric manifolds are certain $2n$ -dimensional manifolds on which an n -dimensional torus acts such that the orbit space of this action may be identified with a simple convex polytope. They were first introduced by Davis and Januszkiewicz [2] in 1991.

In [7, 6] Kuroki studied quasitoric manifolds M which admit an extension of the torus action to an action of some compact connected Lie-group G such that $\dim M/G \leq 1$. Here we drop the condition that the G -action extends the torus action in the case where the first Pontrjagin-class of M is equal to the negative of a sum of squares of elements of $H^2(M)$. In this note all cohomology groups are taken with coefficients in \mathbb{Q} . We have the following two results.

Theorem 1.1. *Let M be a quasitoric manifold with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$ which is homeomorphic (or diffeomorphic) to a homogeneous space G/H with G a compact connected Lie-group. Then M is homeomorphic (diffeomorphic) to $\prod S^2$. In particular, all Pontrjagin-classes of M vanish.*

Theorem 1.2. *Let M be a quasitoric manifold with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$. Assume that the compact connected Lie-group G acts smoothly and almost effectively on M such that $\dim M/G = 1$. Then G has a finite covering group of the form $\prod SU(2)$ or $\prod SU(2) \times S^1$. Furthermore M is diffeomorphic to a S^2 -bundle over a product of two-spheres.*

The proofs of these theorems are based on Hauschild's study [4] of spaces of q -type. A space of q -type is defined to be a topological space X satisfying the following cohomological properties:

- The cohomology ring $H^*(X)$ is generated as a \mathbb{Q} -algebra by elements of degree two, i.e. $H^*(X) = \mathbb{Q}[x_1, \dots, x_n]/I_0$ and $\deg x_i = 2$.
- The defining ideal I_0 contains a definite quadratic form Q .

The note is organised as follows. In Section 2 we show that a quasitoric manifold M with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$ is of q -type. In Section 3 we prove Theorem 1.1. In Section 4 we recall some properties of cohomogeneity one manifolds. In Section 5 we prove Theorem 1.2.

The results presented in this note form part of the outcome of my Ph.D. thesis [10] written under the supervision of Prof. Anand Dessai at the University of Fribourg. I would like to thank Anand Dessai for helpful discussions.

2. Quasitoric manifolds with $p_1(M) = -\sum a_i^2$

In this section we study quasitoric manifolds M with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$. To do so we first introduce some notations from [4] and [5, Chapter VII]. For a topological space X we define the topological degree of symmetry of X as

$$N_t(X) = \max\{\dim G; G \text{ compact Lie-group, } G \text{ acts effectively on } X\}.$$

Similarly one defines the semi-simple degree of symmetry of X as

$$N_t^{ss}(X) = \max\{\dim G; G \text{ compact semi-simple Lie-group, } G \text{ acts effectively on } X\}$$

and the torus-degree of symmetry as

$$T_t(X) = \max\{\dim T; T \text{ torus, } T \text{ acts effectively on } X\}.$$

In the above definitions we assume that all groups act continuously.

Another important invariant of a topological space X used in [4] is the so called embedding dimension of its rational cohomology ring. For a local \mathbb{Q} -algebra A , we denote by $\text{edim } A$ the embedding dimension of A . By definition, we have $\text{edim } A = \dim_{\mathbb{Q}} \mathfrak{m}_A / \mathfrak{m}_A^2$, where \mathfrak{m}_A is the maximal ideal of A . In case that $A = \bigoplus_{i \geq 0} A^i$ is a positively graded local \mathbb{Q} -algebra, \mathfrak{m}_A is the augmentation ideal $A_+ = \bigoplus_{i > 0} A^i$. If furthermore A is generated by its degree two part, then $\mathfrak{m}_A^2 = \bigoplus_{i > 2} A^i$. Therefore for a quasitoric manifold M over the polytope P we have $\text{edim } H^*(M) = \dim_{\mathbb{Q}} H^2(M) = m - n$ where m is the number of facets of P and n is its dimension.

Lemma 2.1. *Let M be a quasitoric manifold with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$. Then M is a manifold of q -type.*

Proof. The discussion at the beginning of Section 3 of [8] together with Corollary 6.8 of [2, p. 448] shows that there are a basis u_{n+1}, \dots, u_m of $H^2(M)$ and $\lambda_{i,j} \in \mathbb{Z}$ such that

$$p_1(M) = \sum_{i=n+1}^m u_i^2 + \sum_{j=1}^n \left(\sum_{i=n+1}^m \lambda_{i,j} u_i \right)^2.$$

Therefore

$$\begin{aligned} 0 &= \sum_{i=n+1}^m u_i^2 + \sum_{j=1}^n \left(\sum_{i=n+1}^m \lambda_{i,j} u_i \right)^2 + \sum_i a_i^2 \\ &= \sum_{i=n+1}^m u_i^2 + \sum_{j=1}^n \left(\sum_{i=n+1}^m \lambda_{i,j} u_i \right)^2 + \sum_j \left(\sum_{i=n+1}^m \mu_{i,j} u_i \right)^2 \end{aligned}$$

with some $\mu_{i,j} \in \mathbb{Q}$ follows.

Because

$$\sum_{i=n+1}^m X_i^2 + \sum_{j=1}^n \left(\sum_{i=n+1}^m \lambda_{i,j} X_i \right)^2 + \sum_j \left(\sum_{i=n+1}^m \mu_{i,j} X_i \right)^2$$

is a positive definite bilinear form the statement follows. \square

Proposition 2.2. *Let M be a quasitoric manifold of q -type over the n -dimensional polytope P . Then we have for the number m of facets of P :*

$$m \geq 2n$$

Proof. By Theorem 3.2 of [4, p. 563], we have

$$n \leq T_t(M) \leq \text{edim } H^*(M) = m - n.$$

Therefore we have $2n \leq m$. \square

REMARK 2.3. The inequality in the above proposition is sharp, because for $M = S^2 \times \dots \times S^2$ we have $m = 2n$ and $p_1(M) = 0$.

By Theorem 5.13 of [4, p. 573], we have for a manifold M of q -type that $N_t^{ss} \leq \dim M + \text{edim } M$. Hence, for a quasitoric manifold M , we get:

Proposition 2.4. *Let M as in Proposition 2.2. Then we have*

$$N_t^{ss}(M) \leq 2n + m - n = n + m.$$

REMARK 2.5. The inequality in the above proposition is sharp because for $M = S^2 \times \cdots \times S^2$ we have $m = 2n$ and $SU(2) \times \cdots \times SU(2)$ acts on M and has dimension $3n$.

3. Quasitoric manifolds which are also homogeneous spaces

In this section we prove Theorem 1.1. Recall from Lemma 2.1 that a quasitoric manifold M with first Pontrjagin-class equal to the negative of the sum of squares of elements of $H^2(M)$ is a manifold of q -type.

Let M be a quasitoric manifold over the polytope P which is also a homogeneous space and is of q -type.

Let G be a compact connected Lie-group and $H \subset G$ a closed subgroup such that M is homeomorphic or diffeomorphic to G/H . Because $\chi(M) > 0$ and M is simply connected, we have $\text{rank } G = \text{rank } H$ and H is connected. Therefore we may assume that G is semi-simple and simply connected.

Let T be a maximal torus of G . Then $(G/H)^T$ is non-empty. By Theorem 5.9 of [4, p.572], the isotropy group G_x of a point $x \in (G/H)^T$ is a maximal torus of G . Hence, H is a maximal torus of G .

Now it follows from Theorem 3.3 of [4, p.563] that

$$T_i(G/H) = \text{rank } G.$$

Because M is quasitoric, we have $n \leq T_i(G/H)$. Combining these inequations, we get

$$\dim G - \dim H = \dim M = 2n \leq 2 \text{rank } G.$$

This equation implies that $\dim G \leq 3 \text{rank } G$.

For a simple simply connected Lie-group G' we have $\dim G' \geq 3 \text{rank } G'$ and $\dim G' = 3 \text{rank } G'$ if and only if $G' = SU(2)$. Therefore we have $G = \prod SU(2)$ and $M = \prod SU(2)/T^1 = \prod S^2$. This proves Theorem 1.1.

4. Cohomogeneity one manifolds

Here we discuss some facts about closed cohomogeneity one Riemannian G -manifolds M with orbit space a compact interval $[-1, 1]$. We follow [3, p.39-44] in this discussion.

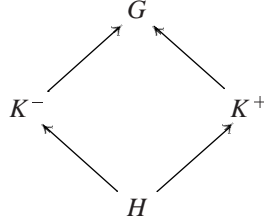
We fix a normal geodesic $c: [-1, 1] \rightarrow M$ perpendicular to all orbits. We denote by H the principal isotropy group $G_{c(0)}$, which is equal to the isotropy group $G_{c(t)}$ for $t \in]-1, 1[$, and by K^\pm the isotropy groups of $c(\pm 1)$.

Then M is the union of tabular neighbourhoods of the non-principal orbits $Gc(\pm 1)$ glued along their boundary, i.e., by the slice theorem we have

$$(4.1) \quad M = G \times_{K^-} D_- \cup G \times_{K^+} D_+,$$

where D_{\pm} are discs. Furthermore $K^{\pm}/H = \partial D_{\pm} = S_{\pm}$ are spheres.

Note that M may be reconstructed from the following diagram of groups.



The construction of such a group diagram from a cohomogeneity one manifold may be reversed. Namely, if such a group diagram with $K^{\pm}/H = S_{\pm}$ spheres is given, then one may construct a cohomogeneity one G -manifold from it. We also write these diagrams as $H \subset K^{-}$, $K^{+} \subset G$.

Now we give a criterion for two group diagrams yielding up to G -equivariant diffeomorphism the same manifold M .

Lemma 4.1 ([3, p.44]). *The group diagrams $H \subset K^{-}$, $K_1^{+} \subset G$ and $H \subset K^{-}$, $K_2^{+} \subset G$ yield the same cohomogeneity one manifold up to equivariant diffeomorphism if there is an $a \in N_G(H)^0$ with $K_1^{+} = aK_2^{+}a^{-1}$.*

5. Quasitoric manifolds with cohomogeneity one actions

In this section we study quasitoric manifolds M which admit a smooth action of a compact connected Lie-group G which has an orbit of codimension one. As before we do not assume that the G -action on M extends the torus action. We have the following lemma:

Lemma 5.1. *Let M be a quasitoric manifold of dimension $2n$ which is of q -type. Assume that the compact connected Lie-group G acts almost effectively and smoothly on M such that $\dim M/G = 1$. Then we have:*

- (1) *The singular orbits are given by G/T where T is a maximal torus of G .*
- (2) *The Euler-characteristic of M is $2\#W(G)$.*
- (3) *The principal orbit type is given by G/S , where $S \subset T$ is a subgroup of codimension one.*
- (4) *The center Z of G has dimension at most one.*
- (5) *$\dim G/T = 2n - 2$.*

Proof. At first note that M/G is an interval $[-1, 1]$ and not a circle because M is simply connected. We start with proving (1). Let T be a maximal torus of G . By passing to a finite covering group of G we may assume $G = G' \times Z'$ with G' a compact

connected semi-simple Lie-group and Z' a torus. Let $x \in M^T$. Then the isotropy group G_x has maximal rank in G . Therefore G_x splits as $G'_x \times Z'$.

By Theorem 5.9 of [4, p.572], G'_x is a maximal torus of G' . Therefore we have $G_x = T$.

Because $\dim G - \dim T$ is even, x is contained in a singular orbit. In particular we have

$$(5.1) \quad \chi(M) = \chi(M^T) = \chi(G/K^+) + \chi(G/K^-),$$

where G/K^\pm are the singular orbits. Furthermore we may assume that G/K^+ contains a T -fixed point. This implies

$$(5.2) \quad \chi(G/K^+) = \chi(G/T) = \#W(G) = \#W(G').$$

Now assume that all T -fixed points are contained in the singular orbit G/K^+ . Then we have $(G/K^-)^T = \emptyset$. This implies

$$\chi(M) = \chi(G/K^+) = \#W(G').$$

Now Theorem 5.11 of [4, p.573] implies that M is the homogeneous space $G'/G' \cap T = G/T$. This contradicts our assumption that $\dim M/G = 1$.

Therefore both singular orbits contain T -fixed points. This implies that they are of type G/T . This proves (1). (2) follows from (5.1) and (5.2).

Now we prove (3) and (5). Let $S \subset T$ be a minimal isotropy group. Then T/S is a sphere of dimension $\text{codim}(G/T, M) - 1$. Therefore S is a subgroup of codimension one in T and $\text{codim}(G/T, M) = 2$.

If the center of G has dimension greater than one, then $\dim Z' \cap S \geq 1$. That means that the action is not almost effective. Therefore (4) holds. \square

By Lemma 5.1, we have with the notation of the previous section that K^\pm are maximal tori of G containing $H = S$. In the following we will write $G = G' \times Z'$ with G' a compact connected semi-simple Lie-group and Z' a torus.

Because K^\pm are maximal tori of the identity component $Z_G(S)^0$ of the centraliser of S , there is some $a \in Z_G(S)^0$ such that $K^- = aK^+a^{-1}$. By Lemma 4.1, we may assume that $K^+ = K^- = T$. Now from Theorem 4.1 of [9, p.198] it follows that M is a fiber bundle over G/T with fiber the cohomogeneity one manifold with group diagram $S \subset T$, $T \subset T$. Therefore it is a S^2 -bundle over G/T .

Lemma 5.2. *Let M and G as in the previous lemma. Then we have*

$$T_i(M) \leq \text{rank } G' + 1.$$

Proof. At first we recall the rational cohomology of G/T . By [1, p.67], we have

$$H^*(G/T) \cong H^*(BT)/I$$

where I is the ideal generated by the elements of positive degree which are invariant under the action of the Weyl-group of G . Therefore it follows that

$$\dim_{\mathbb{Q}} H^{\text{odd}}(G/T) = 0 \quad \text{and} \quad \dim_{\mathbb{Q}} H^2(G/T) = \text{rank } G'.$$

Therefore the Serre spectral sequence for the fibration $S^2 \rightarrow M \rightarrow G/T$ degenerates. Hence, we have

$$H^*(M) = H^*(G/T) \otimes H^*(S^2)$$

as $H^*(G/T)$ -modules. In particular, we have

$$\dim_{\mathbb{Q}} H^2(M) = \dim_{\mathbb{Q}} H^2(G/T) + \dim_{\mathbb{Q}} H^2(S^2) = \text{rank } G' + 1.$$

Therefore

$$T_t(M) \leq \text{edim } H^*(M) = \dim_{\mathbb{Q}} H^2(M) = \text{rank } G' + 1$$

follows. □

Theorem 5.3. *Let M and G as in the previous lemmas. Then G has a finite covering group of the form $\prod SU(2)$ or $\prod SU(2) \times S^1$. Furthermore M is diffeomorphic to a S^2 -bundle over a product of two-spheres.*

Proof. Because M is quasitoric we have $n \leq T_t(M)$. By Lemma 5.1 we have

$$\dim G' - \text{rank } G' = \dim G/T = 2n - 2.$$

Now Lemma 5.2 implies

$$\dim G' = 2n - 2 + \text{rank } G' \leq 3 \text{rank } G'.$$

Therefore $\prod SU(2)$ is a finite covering group of G' . This implies the statement about the finite covering group of G .

It follows that $G/T = \prod S^2$. Therefore M is a S^2 -bundle over $\prod S^2$. □

Now Theorem 1.2 follows from Theorem 5.3 and Lemma 2.1.

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The University of Manchester
Oxford Road, Manchester M13 9PL
U.K.

Current address:
Institut für Algebra und Geometrie
Karlsruher Institut für Technologie
Kaiserstrasse 89–93
D-76133 Karlsruhe
Germany
e-mail: michael.wiemeler@kit.edu