Title	ANTIPODAL SETS OF SYMMETRIC R-SPACES
Author(s)	Tanaka, Makiko Sumi; Tasaki, Hiroyuki
Citation	Osaka Journal of Mathematics. 50(1) P.161-P.169
Issue Date	2013-03
Text Version	publisher
URL	https://doi.org/10.18910/24476
DOI	10.18910/24476
rights	

Tanaka, M.S. and Tasaki, H.
Osaka J. Math.
50 (2013), 161–169

ANTIPODAL SETS OF SYMMETRIC *R*-SPACES

MAKIKO SUMI TANAKA and HIROYUKI TASAKI

(Received January 31, 2011, revised June 13, 2011)

Abstract

We show that antipodal sets of symmetric R-spaces have the following properties. Any antipodal set is included in a great antipodal set and any two great antipodal sets are congruent.

1. Introduction

We assume that M is a Riemannian symmetric space and denote by s_x the geodesic symmetry at $x \in M$. A subset S of M is called an *antipodal set*, if $s_x(y) = y$ for every points x and y in S. The 2-*number* $\#_2 M$ of M is the supremum of the cardinalities of antipodal sets of M. We call an antipodal set in M *great* if its cardinality attains $\#_2 M$. These were introduced by Chen and Nagano [1].

If the orbit of the linear isotropy action of a Riemannian symmetric space is a Riemannian symmetric space, the orbit is called a *symmetric R-space*. We show some fundamental properties of antipodal sets of Riemannian symmetric spaces, in particular symmetric R-spaces.

Chen–Nagano [1] gave no explicit proof of the finiteness of 2-number of a symmetric space M in [1], so we give a proof of the finiteness of 2-number in Section 2.

Hermitian symmetric spaces of compact type have realizations as orbits of the adjoint representations of compact semisimple Lie groups. We show that the antipodal sets of Hermitian symmetric spaces of compact type are clearly described in orbits of the adjoint representations. We review this realization in Section 3 and using this we prove the following properties of antipodal sets of Hermitian symmetric spaces of compact type.

(A) Any antipodal set is included in a great antipodal set.

(B) Any two great antipodal sets are congruent.

In general we say that two subsets S_1 and S_2 of a Riemannian manifold M are *congruent* if there exists an element g of the identity component of the group of all isometries of M which satisfies $S_2 = gS_1$.

2000 Mathematics Subject Classification. Primary 53C35; Secondary 53C40.

The second author was partly supported by the Grant-in-Aid for Science Research (C) 2009 (No. 21540063), Japan Society for the Promotion of Science.

A submanifold L is called a *real form* of a Kähler manifold M, if there exists an involutive anti-holomorphic isometry σ of M satisfying

$$L = \{x \in M \mid \sigma(x) = x\}.$$

Any real form is a totally geodesic Lagrangian submanifold. We note that any real form of Hermitian symmetric spaces of compact type is a symmetric R-space and that any symmetric R-space is a real form of a Hermitian symmetric space of compact type, which are shown in [4]. Using these results on Hermitian symmetric spaces of compact type, we prove that antipodal sets of symmetric R-spaces satisfy the properties (A) and (B) in Section 4.

Antipodal sets of the adjoint group of $SU(4)$ do not satisfy the property (A), which is shown in Section 5.

The authors would like to thank the referee, whose comments improved the manuscript.

2. Finiteness of 2-numbers

Let M be a compact connected Riemannian symmetric space and let S be an antipodal set of M. Then S is a subset of the fixed point set $F(s_x, M) = \{p \in M \mid s_x(p) = p\}$ for every x in S. So x is an isolated point in S and S is a discrete set. Hence S is finite because of the compactness of M. Since we cannot find the proof of the finiteness of $\#_2 M$ in [1], we give it here.

Proposition 2.1. *The 2-number $\#_2 M$ is finite.*

Proof. We prove the proposition by induction on $\dim M$. If $\dim M = 0$, the 2-number $\#_2 M = 1$. We assume that the 2-number of any compact connected Riemannian symmetric space whose dimension is less than $\dim M$ is finite and prove that $\#_2 M$ is finite. If we assume that $\#_2 M = \infty$, then there exits a sequence of antipodal sets $A_1, A_2, \dots$ which satisfies $\lim_{i\to\infty} \#A_i = \infty$. Since the isometry group of M acts transitively on M, we may assume that there is a point x which is contained in every A_i. Then we have $A_i \subset F(s_x, M)$ for every i. We denote each connected component of $F(s_x, M)$ by M_k^+ ($k = 1, \dots, r$), which is called a polar ([1]). Then we have

$$A_i = \bigcup_{k=1}^{r} (A_i \cap M_k^+). \tag{2.1}$$

Since $\lim_{i\to\infty} \#A_i = \infty$, we have $\lim_{i\to\infty} \#(A_i \cap M_k^+) = \infty$ for some k. This means that $\#_2 M_k^+ = \infty$, which contradicts the assumption of the induction. Hence $\#_2 M$ is finite. $\square$

3. Hermitian symmetric spaces of compact type

It is well-known that Hermitian symmetric spaces of compact type are realized as adjoint orbits. Here we summarize it in order to prepare for the remaining part of this article.

Let $\mathfrak{g}$ be a compact semisimple Lie algebra and let $G = \mathrm{Int}(\mathfrak{g})$. We take a G-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$. Let $J \in \mathfrak{g}$ be an nonzero element which satisfies $(\mathrm{ad}\, J)^3 = -\mathrm{ad}\, J$. Then the G-orbit $M = G\cdot J$ is a Hermitian symmetric space of compact type with respect to the induced metric from $\langle\ ,\ \rangle$. Let K be the isotropy subgroup at J. Then the Lie algebra $\mathfrak{k}$ of K is

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid [J, X] = 0\}.$$

Let

$$\mathfrak{m} = \{[J, X] \mid X \in \mathfrak{g}\},$$

then we have an orthogonal direct sum decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$. $\mathfrak{k}$ is the $(+1)$-eigenspace and $\mathfrak{m}$ is the (-1)-eigenspace of the involutive automorphism $e^{\pi\,\mathrm{ad}\, J}$ of $\mathfrak{g}$ respectively. $\mathrm{ad}\, J$ is a complex structure of $\mathfrak{m}$ which can be identified with the tangent space of M at J.

Conversely, every Hermitian symmetric space of compact type is obtained like this.

We prove the following theorem using the notation and results mentioned above.

Theorem 3.1. *Let M be a Hermitian symmetric space of compact type and take $X, Y \in M$. $s_X(Y) = Y$ if and only if $[X, Y] = 0$. Moreover the following conditions* (A) *and* (B) *hold.*
(A) *Any antipodal set is included in a great antipodal set.*
(B) *Any two great antipodal sets are congruent.*
A great antipodal set of M is represented as $M \cap \mathfrak{t}$ for a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. In particular, a great antipodal set of M is an orbit of the Weyl group of $\mathfrak{g}$.

REMARK 3.2. After submitting the first draft of this paper we found Sánchez [3]. Lemma 5 in [3] implies (A) of Theorem 3.1 and Corollary 6 in [3] implies the description of a great antipodal set in Theorem 3.1.

Proof. The geodesic symmetry s_J of M at J is represented by

$$s_J(gJ) = e^{\pi\,\mathrm{ad}\, J} g e^{\pi\,\mathrm{ad}\, J} J = e^{\pi\,\mathrm{ad}\, J} gJ \quad (g \in G),$$

so we have

$$s_J(X) = e^{\pi\,\mathrm{ad}\, J} X \quad (X \in M).$$

The automorphism $e^{\pi \operatorname{ad} J}$ of $\mathfrak{g}$ is also the involutive automorphism which determines the symmetric pair associated with M. Hence

$$F(s_J, M) = M \cap \mathfrak{k}.$$

For any $X \in M$ we denote by $\mathfrak{k}_X$ the centralizer of X and set $\mathfrak{m}_X = [X, \mathfrak{g}]$. Then

$$\mathfrak{g} = \mathfrak{k}_X + \mathfrak{m}_X$$

is the canonical direct sum decomposition of $\mathfrak{g}$ associated with the involutive automorphism $e^{\pi \operatorname{ad} X}$. Similarly to the case of J we obtain

$$s_X(Y) = e^{\pi \operatorname{ad} X} Y \quad (Y \in M).$$

Now we prove that $s_X(Y) = Y$ if and only if $[X, Y] = 0$. If $[X, Y] = 0$, then $s_X(Y) = Y$. Conversely we suppose that $s_X(Y) = Y$. This implies $e^{\pi \operatorname{ad} X} Y = Y$ and $Y \in \mathfrak{k}_X$. Hence we obtain $[X, Y] = 0$.

Let S be an antipodal set of M. It follows from the result above that $[X, Y] = 0$ for any $X, Y \in S$. We denote by $S_{\mathbb{R}}$ the subspace spanned by S. The subspace $S_{\mathbb{R}}$ is an abelian subalgebra of $\mathfrak{g}$. So we can take a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ including $S_{\mathbb{R}}$. We get $S \subset M \cap \mathfrak{t}$. It follows that any great antipodal set of M is $M \cap \mathfrak{t}$ for a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and that the conditions (A) and (B) hold. □

4. Symmetric *R*-spaces

In this section we show that antipodal sets of real forms of Hermitian symmetric spaces of compact type satisfy the conditions (A) and (B). This implies that antipodal sets of symmetric R-spaces also satisfy the conditions (A) and (B).

We first prepare the following lemma.

Lemma 4.1. *Any real form of a compact Kähler manifold of positive holomorphic sectional curvature is connected.*

REMARK 4.2. We need the above lemma in the case of Hermitian symmetric spaces of compact type and the statement in this case is stated in Proposition 3.2 of [2]. The above lemma is a generalization of this.

Proof. Let X be a compact Kähler manifold of positive holomorphic sectional curvature and $\tau \colon X \to X$ be an involutive anti-holomorphic isometry determining a real form L. Each connected component of $L = F(\tau, X)$ is a totally geodesic Lagrangian submanifold. If there exist more than one connected component of L, they intersect by Lemma 3.1 of [7], which is a contradiction. Therefore L is connected. □

Theorem 4.3. *Let M be a Hermitian symmetric space of compact type and $\tau\colon M \to M$ be an involutive anti-holomorphic isometry determining a real form $L = F(\tau, M)$. We define an automorphism I_τ of G by*

$$I_\tau : G \to G;\ g \mapsto \tau g \tau^{-1}.$$

We assume that L contains J. Let $\mathfrak{g} = \mathfrak{l} + \mathfrak{p}$ be the canonical direct sum decomposition determined by I_τ. We have $L = M \cap \mathfrak{p}$. Moreover the following conditions (A) *and* (B) *hold.*
(A) *Any antipodal set is included in a great antipodal set.*
(B) *Any two great antipodal sets are congruent.*
A great antipodal set of L is represented as $M \cap \mathfrak{a}$ for a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. In particular, a great antipodal set of L is an orbit of the Weyl group of the symmetric pair determined by I_τ.

Proof. Since $M = G \cdot J \subset \mathfrak{g}$, any point x in M is represented by $x = g \cdot J$ for $g \in G$.

$$\tau(x) = (\tau g) \cdot J = (\tau g \tau^{-1} \tau) \cdot J = (I_\tau(g)) \cdot J,$$

thus we have

$$F(I_\tau, G) \cdot J \subset L.$$

Since L contains J, s_J and τ are commutative, thus I_{s_J} and I_τ are commutative and dI_{s_J} and dI_τ are simultaneously diagonalizable.

$$\mathfrak{k} = \mathfrak{k} \cap \mathfrak{l} + \mathfrak{k} \cap \mathfrak{p}, \quad \mathfrak{m} = \mathfrak{m} \cap \mathfrak{l} + \mathfrak{m} \cap \mathfrak{p}$$

are direct sum decompositions. For $X \in T_J M \cong \mathfrak{m}$ we have

$$\tau(\mathrm{Exp}_J X) = \tau(\exp X \cdot J) = I_\tau(\exp X) \cdot J = \exp(dI_\tau(X)) \cdot J,$$

so we obtain

$$T_J L = \{X \in \mathfrak{m} \mid dI_\tau(X) = X\} = \mathfrak{m} \cap \mathfrak{l}.$$

The Lie algebra of $F(I_\tau, G)$ is $\mathfrak{l}$ and L is connected by Lemma 4.1. Thus

$$L = \mathrm{Exp}_J(\mathfrak{m} \cap \mathfrak{l}) \subset \exp(\mathfrak{l}) \cdot J = F(I_\tau, G)_0 \cdot J \subset F(I_\tau, G) \cdot J \subset L$$

are all equal, where $F(I_\tau, G)_0$ is the identity component of $F(I_\tau, G)$. In particular, we get $L = F(I_\tau, G)_0 \cdot J$, which means that L is a symmetric R-space.

We shall show that $dI_\tau(J) = -J$. For $X \in \mathfrak{m} \cap \mathfrak{l} = T_J L$ we have $[J, X] \in \mathfrak{m} \cap \mathfrak{p} = T_J^\perp L$ because L is a Lagrangian submanifold of M. Hence

$$-[J, X] = dI_\tau[J, X] = [dI_\tau J, dI_\tau X] = [dI_\tau J, X].$$

On the other hand for $X \in \mathfrak{m} \cap \mathfrak{p}$ we have $[J, X] \in \mathfrak{m} \cap \mathfrak{l}$. Hence

$$[J, X] = dI_\tau[J, X] = [dI_\tau J, dI_\tau X] = [dI_\tau J, -X].$$

Therefore we obtain

$$[dI_\tau J, X] = -[J, X] \quad (X \in \mathfrak{m}).$$

The action of $\mathrm{ad}(\mathfrak{k})$ on $\mathfrak{m}$ is effective, so we have $dI_\tau J = -J$. Thus $J \in \mathfrak{k} \cap \mathfrak{p}$. We can continue to calculate $\tau(\mathrm{Exp}_J X)$. Since τ is involutive, I_τ and dI_τ are also involutive. For $X, Y \in \mathfrak{g}$ we have

$$\mathrm{ad}(dI_\tau(X))^k \cdot Y = dI_\tau \,\mathrm{ad}(X)^k dI_\tau(Y).$$

If $X \in \mathfrak{m}$, then

$$\begin{aligned} \tau(\mathrm{Exp}_J X) = \exp(dI_\tau(X)) \cdot J &= \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{ad}(dI_\tau(X))^k \cdot J \\ &= dI_\tau \sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{ad}(X)^k dI_\tau J \\ &= dI_\tau \exp X \cdot (-J) = -dI_\tau \mathrm{Exp}_J X. \end{aligned}$$

This implies

$$\tau(x) = -dI_\tau(x) \quad (x \in M)$$

and

$$L = \{x \in M \mid \tau(x) = x\} = M \cap \mathfrak{p}.$$

Let S be an antipodal set of L. It follows from the result above that $[X, Y] = 0$ for any $X, Y \in S$. We denote by $S_{\mathbb{R}}$ the subspace spanned by S. The subspace $S_{\mathbb{R}}$ is an abelian subspace of $\mathfrak{p}$. So we can take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ including $S_{\mathbb{R}}$. We get $S \subset M \cap \mathfrak{a}$. It follows that any great antipodal set of L is $M \cap \mathfrak{a}$ for a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and that the conditions (A) and (B) hold. □

Corollary 4.4. *For a symmetric R-space the following conditions* (A) *and* (B) *hold.*
(A) *Any antipodal set is included in a great antipodal set.*
(B) *Any two great antipodal sets are congruent.*

Proof. Any symmetric R-space is a real form of some Hermitian symmetric space of compact type by a result of [4]. Hence the corollary follows from Theorem 4.3. □

The authors have proved the following theorems in [6].

Theorem 4.5 ([6] Theorem 1.1). *Let M be a Hermitian symmetric space of compact type. If two real forms L_1 and L_2 of M intersect transversally, then $L_1 \cap L_2$ is an antipodal set of L_1 and L_2.*

Theorem 4.6 ([6] Theorem 1.2). *Let M be a Hermitian symmetric space of compact type and let L_1, L_2, L_1', L_2' be real forms of M. We assume that L_1, L_1' are congruent and that L_2, L_2' are congruent. If L_1, L_2 intersect transversally and if L_1', L_2' intersect transversally, then $\#(L_1 \cap L_2) = \#(L_1' \cap L_2')$.*

We can make the conclusion of Theorem 4.6 stronger under a stronger assumption as follows:

Corollary 4.7. *Let M be a Hermitian symmetric space of compact type and let L_1, L_2, L_1', L_2' be real forms of M. We assume that L_1, L_1' are congruent and that L_2,L_2' are congruent. We also assume that L_1,L_2 intersect transversally and that L_1',L_2' intersect transversally. If $\#(L_1 \cap L_2) = \#_2 L_1$, then $L_1 \cap L_2$ and $L_1' \cap L_2'$ are congruent.*

Proof. According to Theorem 4.5 $L_1 \cap L_2$ is an antipodal set of L_1. The assumption $\#(L_1 \cap L_2) = \#_2 L_1$ implies that $L_1 \cap L_2$ is a great antipodal set of L_1. According to Theorem 4.6 $\#(L_1 \cap L_2) = \#(L_1' \cap L_2')$. Since L_1, L_1' are congruent, we have $\#(L_1' \cap L_2') = \#_2 L_1'$. Hence $L_1' \cap L_2'$ is also a great antipodal set of L_1'. We can take $\phi \in I_0(M)$ satisfying $\phi(L_1) = L_1'$, thus $\phi^{-1}(L_1' \cap L_2')$ is a great antipodal set of L_1 and it is congruent to $L_1 \cap L_2$ in L_1 by Theorem 4.3. Therefore $L_1 \cap L_2$ and $L_1' \cap L_2'$ are congruent in M. $\square$

5. The adjoint group of $SU(4)$

In this section we show that antipodal sets of the adjoint group of $SU(4)$ do not satisfy the condition (A).

We first review some results on antipodal sets of compact connected Lie groups obtained in [1]. Let G be a compact connected Lie group with a biinvariant Riemannian metric. The geodesic symmetry s_e at the identity element e of G is given by

$$s_e(y) = y^{-1} \quad (y \in G).$$

For any $x, y \in G$ we have $s_x(y) = xy^{-1}x$. Without loss of generality it is sufficient to consider an antipodal set A containing e. We take $x, y, z \in A$. In this case we have $x^2 = e$ and $y = s_x(y) = xyx$. Hence $xy = yx$. Moreover $s_z(xy) = xy$. Thus $A \cup \{xy\}$ is also an antipodal set. Therefore A is a subgroup of G, if A is a maximal antipodal set containing e. According to the fundamental theorem of finite abelian groups A is isomorphic to a product of some $\mathbb{Z}_2$'s. Hence the 2-number of G is equal to a power of 2.

Now we consider the adjoint group of $SU(4)$. The center of $SU(4)$ is $Z = \{\pm 1_4, \pm i 1_4\}$, which is equal to the kernel of $\mathrm{Ad}\colon SU(4) \to \mathrm{Ad}(SU(4))$. We denote by e the identity element of $\mathrm{Ad}(SU(4))$. Any antipodal set of $\mathrm{Ad}(SU(4))$ containing e is included in $F(s_e, \mathrm{Ad}(SU(4)))$, so we investigate $F(s_e, \mathrm{Ad}(SU(4)))$. We have

$$F(s_e, \mathrm{Ad}(SU(4))) = \mathrm{Ad}(\{x \in SU(4) \mid x^2 \in Z\}).$$

Each connected component of $\{x \in SU(4) \mid x^2 \in Z\}$ is a conjugate class of an element of a maximal torus $S(U(1)^4)$ of $SU(4)$ and we can obtain that all connected components of $F(s_e, \mathrm{Ad}(SU(4)))$ are

$$\begin{aligned} M_0^+ &= \{e\}, \\ M_1^+ &= \mathrm{Ad}(\{gIg^{-1} \mid g \in SU(4)\}), \\ M_2^+ &= \mathrm{Ad}(\{gJg^{-1} \mid g \in SU(4)\}), \end{aligned}$$

where $I = \mathrm{diag}(1, 1, -1, -1)$ and $J = \mathrm{diag}(e^{i\pi/4}, e^{i\pi/4}, e^{i\pi/4}, e^{-i3\pi/4})$. We denote by $G_k(\mathbb{R}^n)$ the real Grassmann manifold consisting of all real subspaces of dimension k in $\mathbb{R}^n$ and by $G_k(\mathbb{C}^n)$ the complex Grassmann manifold consisting of all complex subspaces of dimension k in $\mathbb{C}^n$. We can see

$$\begin{aligned} M_1^+ &\cong G_2(\mathbb{C}^4)/\mathbb{Z}_2 \cong G_2(\mathbb{R}^6), \\ M_2^+ &\cong SU(4)/S(U(3) \times U(1)) \cong G_1(\mathbb{C}^4). \end{aligned}$$

Chen and Nagano [1] showed that $\#_2 G_k(\mathbb{K}^n) = \binom{n}{k}$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ in Proposition 6.1. Hence we have $\#_2 M_1^+ = \binom{6}{2} = 15$ and $\#_2 M_2^+ = 4$. If A is a great antipodal set of M_1^+, then $\{e\} \cup A$ is an antipodal set of $\mathrm{Ad}(SU(4))$. So we have

$$\#_2 \mathrm{Ad}(SU(4)) \geq 1 + \#A = 16.$$

On the other hand

$$\#_2 \mathrm{Ad}(SU(4)) \leq \#_2 M_0^+ + \#_2 M_1^+ + \#_2 M_2^+ = 20$$

by Proposition 1.9 in Chen–Nagano [1]. The 2-number $\#_2 \mathrm{Ad}(SU(4))$ is a power of 2, thus $\#_2 \mathrm{Ad}(SU(4)) = 16$ and $\{e\} \cup A$ is a great antipodal set of $\mathrm{Ad}(SU(4))$. This is a counter example of

$$\#_2 M = \sum_j \#_2 M_j^+,$$

which holds for a symmetric R-space by Theorem 2 in Takeuchi [5].

We can see that

$$\begin{aligned} A_1 = \{&\mathrm{Ad}(\mathrm{diag}(e^{i\pi/4}, e^{i\pi/4}, e^{i\pi/4}, e^{-3i\pi/4})), \mathrm{Ad}(\mathrm{diag}(e^{-3i\pi/4}, e^{i\pi/4}, e^{i\pi/4}, e^{i\pi/4})), \\ &\mathrm{Ad}(\mathrm{diag}(e^{i\pi/4}, e^{-3i\pi/4}, e^{i\pi/4}, e^{i\pi/4})), \mathrm{Ad}(\mathrm{diag}(e^{i\pi/4}, e^{i\pi/4}, e^{-3i\pi/4}, e^{i\pi/4}))\} \end{aligned}$$

is a great antipodal set of M_2^+ and

$$A_2 = \{e\} \cup A_1 \\ \cup \{\mathrm{Ad}(\mathrm{diag}(i, i, -i, -i)), \mathrm{Ad}(\mathrm{diag}(i, -i, i, -i)), \mathrm{Ad}(\mathrm{diag}(i, -i, -i, i))\}$$

is an antipodal subgroup of $\mathrm{Ad}(SU(4))$. The centralizer of $\mathrm{Ad}(J)$ is equal to $\mathrm{Ad}(S(U(3)\times U(1)))$. We can get the centralizer of each element of A_1 similarly and the centralizer of A_1 is equal to $\mathrm{Ad}(S(U(1)^4))$. If $\tilde{A}_2$ is an antipodal subset of $\mathrm{Ad}(SU(4))$ including A_2, then $\tilde{A}_2$ is included in the centralizer of A_1 $(\subset A_2)$ and $\tilde{A}_2 \subset \mathrm{Ad}(S(U(1)^4)) \cong U(1)^3$.

$$8 = \#A_2 \leq \#\tilde{A}_2 \leq \#_2\,\mathrm{Ad}(S(U(1)^4)) = 8,$$

hence these are all equal and A_2 is a maximal antipodal subgroup of $\mathrm{Ad}(SU(4))$. As we have showed, $\#_2\mathrm{Ad}(SU(4)) = 16$ and $\#A_2 = 8$. Thus A_2 is not included in any great antipodal subgroup of $\mathrm{Ad}(SU(4))$. Hence $\mathrm{Ad}(SU(4))$ does not satisfy the condition (A).

References

[1] B.-Y. Chen and T. Nagano: *A Riemannian geometric invariant and its applications to a problem of Borel and Serre*, Trans. Amer. Math. Soc. **308** (1988), 273–297.

[2] T. Nagano and M.S. Tanaka: *The involutions of compact symmetric spaces*, IV, Tokyo J. Math. **22** (1999), 193–211.

[3] C.U. Sánchez: *The index number of an R-space*: *an extension of a result of M. Takeuchi's*, Proc. Amer. Math. Soc. **125** (1997), 893–900.

[4] M. Takeuchi: *Stability of certain minimal submanifolds of compact Hermitian symmetric spaces*, Tohoku Math. J. (2) **36** (1984), 293–314.

[5] M. Takeuchi: *Two-number of symmetric R-spaces*, Nagoya Math. J. **115** (1989), 43–46.

[6] M.S. Tanaka and H. Tasaki: *The intersection of two real forms in Hermitian symmetric spaces of compact type*, J. Math. Soc. Japan **64** (2012), 1297–1332.

[7] H. Tasaki: *The intersection of two real forms in the complex hyperquadric*, Tohoku Math. J. (2) **62** (2010), 375–382.

Makiko Sumi Tanaka
Faculty of Science and Technology
Tokyo University of Science
Noda, Chiba, 278-8510
Japan
e-mail: tanaka_makiko@ma.noda.tus.ac.jp

Hiroyuki Tasaki
Division of Mathematics
Faculty of Pure and Applied Sciences
University of Tsukuba
Tsukuba, Ibaraki, 305-8571
Japan
e-mail: tasaki@math.tsukuba.ac.jp