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THE MODULI SPACE OF CATANESE–CILIBERTO–ISHIDA SURFACES

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Abstract

We determine the moduli space of the surfaces of general type studied by Catanese, Ciliberto and Hirotaka Ishida by using the family of Hesse cubic curves.

Introduction

A minimal surface S of general type over $\mathbf{C}$ is called *Catanese–Ciliberto surface* if it satisfies $p_g(S) = q(S) = 1$ and $K_S^2 = 3$. Then the Albanese map $a\colon S \to E$ is a surjection to an elliptic curve $E = E(S)$. The general fiber of the morphism a is a smooth irreducible curve, and the genus g is known to be two or three [1].

In [3], Hirotaka Ishida studied the case $g = 3$. In this case, $V = a_* K_S \otimes_{\mathcal{O}_E} K_E^{-1}$ is a locally free sheaf of rank three, and the natural rational map $\phi\colon S \to \mathbf{P}_E(V)$ is a morphism [1, Theorem 3.1]. In [3], the surface is defined to be *of Type* I if ϕ is an embedding and a has only one singular fiber. Ishida studied precisely the Catanese–Ciliberto surfaces of Type I. In this paper, we call this type of surface a CCI surface by taking the initials of Catanese, Ciliberto and Hirotaka Ishida. He got the following theorem [3, Theorem 0.2].

Theorem 0.1. *Let E be an elliptic curve defined over* $\mathbf{C}$. *If E has an automorphism of complex multiplication type, then there exist exactly two isomorphism classes of CCI surfaces S with $E = E(S)$. Otherwise, there exist exactly four isomorphism classes of such CCI surfaces.*

If we take an isogeny $q\colon E' \to E$ of degree three, there exists a natural coordinate system of the $\mathbf{P}^2$-bundle $\mathbf{P}(q^*V)$. In [3], he fixed one of such coverings and showed that there exist exactly four equations which define the pullbacks of CCI surfaces. Of course, the CCI surfaces over E are recovered by descending the surfaces defined by these equations. Only one of these equations is of Fermat type whereas the others are not. However, by the elementary theory of abelian varieties, we know that there exist

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exactly four isogenies of degree three for a given E, and each covering has an equation of Fermat type defining the pullback of a CCI surface. We show that the four CCI surfaces obtained by the Fermat type equations of different coverings are distinct from each other. Namely, the Fermat type equations are enough for getting all CCI surfaces over an elliptic curve E, if we consider all isogenies of degree three.

In this paper, we construct a CCI surface $S(\rho)$ for each $\rho \in \mathbf{C} \setminus \{1\}$ by using this Fermat type equation, and we show that $\mathbf{C} \setminus \{1\}$ is the coarse moduli space of the CCI surfaces. For each $\rho \in \mathbf{C} \setminus \{1\}$, the j-invariant of the elliptic curve $E(S(\rho))$ is given by

$$j(\rho) = \frac{27\rho(\rho+8)^3}{(\rho-1)^3}$$

(cf. [2, p. 456] and [6]). Furthermore, we construct a global family of CCI surfaces with the parameter μ in the threefold covering $T = \mathbf{C} \setminus \{1, \omega, \omega^2\}$ of $\mathbf{C} \setminus \{1\}$ defined by $\rho = \mu^3$ by using the family of Hesse cubic curves on T.

1. The coordinate transformation on an elliptic curve

We use the following lemma which follows from the Riemann–Roch theorem.

Lemma 1.1. *Let E be an elliptic curve defined over $\mathbf{C}$ and P a point on it. Then we have* $\mathrm{H}^0(E, \mathcal{O}_E(P)) = \mathbf{C}$.

Let E be an elliptic curve defined over $\mathbf{C}$ with a fixed additive group structure. Let $3_E \colon \tilde{E} \to E$ be the morphism defined by $\tilde{E} = E$ and $3_E(x) = [3x]$, where [ ] indicates the calculation by the group law. Since $\operatorname{Ker} 3_E \simeq (\mathbf{Z}/3\mathbf{Z})^2$, there exist exactly four unramified coverings of degree three of E which correspond to the four subgroups of index three of $(\mathbf{Z}/3\mathbf{Z})^2$.

Take a set of generators $\{P_{10}, P_{01}\}$ of $\operatorname{Ker} 3_E$, and define $P_{ij} = [iP_{10} + jP_{01}]$ in $\tilde{E}$ for $0 \le i, j \le 2$. We denote by $\begin{bmatrix} a_{02} & a_{12} & a_{22} \\ a_{01} & a_{11} & a_{21} \\ a_{00} & a_{10} & a_{20} \end{bmatrix}$ the divisor $\sum_{i=0}^{2} \sum_{j=0}^{2} a_{ij} P_{ij}$ with the support contained in $\operatorname{Ker} 3_E$. Since $[P_{10} + P_{20}] = [P_{01} + P_{02}]$ in $\tilde{E}$, there exists a nonzero rational function f_{00} on $\tilde{E}$ whose divisor (f_{00}) is equal to $P_{10} + P_{20} - P_{01} - P_{02}$ by Abel's theorem.

Let σ and τ be the translations of $\tilde{E}$ defined by $\sigma \colon x \mapsto [x + P_{10}]$ and $\tau \colon x \mapsto [x + P_{01}]$, respectively. Set $f_{ij} = (\sigma^{-i}\tau^{-j})^* f_{00}$ for $0 \le i, j \le 2$. Then

$$(f_{00}) = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, (f_{10}) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, (f_{20}) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix},$$

$$(f_{01}) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}, (f_{11}) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, (f_{21}) = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$(f_{02}) = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, (f_{12}) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, (f_{22}) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Lemma 1.2. *The rational function* $f_{00} f_{11} f_{10}^{-1} f_{01}^{-1}$ *is equal to the constant* ω *or* $\omega^{-1} = \omega^2$, *where* $\omega = (-1 + \sqrt{3}i)/2$.

Proof. Since

$$(f_{00}) - (f_{10}) = (f_{01}) - (f_{11}) = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix},$$

we have $(f_{00} f_{11} f_{10}^{-1} f_{01}^{-1}) = 0$. Hence, this function is a nonzero constant α. Since

$$\alpha^3 = \frac{f_{00} f_{11}}{f_{10} f_{01}} \cdot (\tau^{-1})^* \left(\frac{f_{00} f_{11}}{f_{10} f_{01}} \right) \cdot (\tau^{-2})^* \left(\frac{f_{00} f_{11}}{f_{10} f_{01}} \right) = \frac{f_{00} f_{11}}{f_{10} f_{01}} \cdot \frac{f_{01} f_{12}}{f_{11} f_{02}} \cdot \frac{f_{02} f_{10}}{f_{12} f_{00}} = 1,$$

α is 1, ω or ω^2. If $\alpha = 1$, then

$$\frac{f_{00}}{f_{10}} = \frac{f_{01}}{f_{11}} = (\tau^{-1})^* \left(\frac{f_{00}}{f_{10}} \right).$$

Hence f_{00}/f_{10} is τ-invariant and descends to a rational function on $\tilde{E}/(\tau)$ with a single pole of order one at the image of P_{00}. This contradicts Lemma 1.1 since $\tilde{E}/(\tau)$ is an elliptic curve. Hence α is ω or ω^2. □

If $f_{00} f_{11} f_{10}^{-1} f_{01}^{-1} = \omega$, then we can make it ω^{-1} by exchanging P_{10} and P_{01} and by redefining f_{ij}'s for the new (P_{10}, P_{01}). Actually, this value is equal to the inverse of the Weil pairing $\mathbf{e}_3(P_{10}, P_{01})$ (cf. [8], [7, III, §8]). From now on, we assume $f_{00} f_{11} f_{10}^{-1} f_{01}^{-1} = \omega^{-1}$.

It is easy to check that

$$(f_{10} f_{20} f_{01} f_{02}) = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = (f_{00}).$$

Hence, there exists a nonzero constant $\epsilon \in \mathbf{C}^\times$ with $f_{00} = \epsilon f_{10} f_{20} f_{01} f_{02}$.

Lemma 1.3. $\delta = f_{00} + f_{01} + f_{02}$ *is a nonzero constant function on* $\tilde{E}$.

Proof. Since f_{00}, f_{01} and f_{02} are in $\mathrm{H}^0(\tilde{E}, \mathcal{O}_E(P_{00} + P_{01} + P_{02}))$, δ is also in this vector space. Since δ is τ-invariant, it descends to a rational function $\bar{\delta}$ on the quotient

$\tilde{E}/(\tau)$. This function may have at most a single pole of order one at the image of P_{00}. Since $\tilde{E}/(\tau)$ is an elliptic curve, this implies that $\bar{\delta}$ is a constant by Lemma 1.1. Hence δ is also a constant. We have to prove that it is nonzero.

Let $f_{00}(P_{11}) = \alpha$ and $f_{00}(P_{12}) = \beta$. For the involution ι of $\tilde{E}$ fixing the unit P_{00}, we get the equality $\iota^* f_{00} = f_{00}$ since $(\iota^* f_{00}) = (f_{00})$ and they have same nonzero value at P_{00}. Hence we have $f_{00}(P_{22}) = \alpha$ and $f_{00}(P_{21}) = \beta$. Since the value of δ at P_{10} is $f_{00}(P_{10}) + f_{00}(P_{12}) + f_{00}(P_{11}) = 0 + \beta + \alpha = \alpha + \beta$, it suffices to show that $\alpha + \beta$ is not zero. Since the value of $f_{00} f_{11} f_{10}^{-1} f_{01}^{-1}$ at P_{22} is equal to

$$f_{00}(P_{22}) f_{00}(P_{11}) f_{00}(P_{12})^{-1} f_{00}(P_{21})^{-1} = \alpha^2 \beta^{-2},$$

we have $\omega^{-1} = \alpha^2\beta^{-2}$ by Lemma 1.2 and our assumption. Hence $\alpha^2 \neq \beta^2$, and in particular $\alpha + \beta \neq 0$. $\square$

Let $F_0 = \tilde{E}/(\tau)$ and $F_1 = \tilde{E}/(\sigma)$. We denote by p_i the natural surjection $\tilde{E} \to F_i$ for $i = 0, 1$. Then there exists an unramified morphism $q_i \colon F_i \to E$ of degree three with $3_E = q_i \cdot p_i$ for each $i = 0, 1$.

$$\begin{tikzcd} \tilde{E} \arrow[r, "p_1"] \arrow[d, "p_0"'] & F_1 \arrow[d, "q_1"] \\ F_0 \arrow[r, "q_0"] & E \end{tikzcd}$$

There exist points P_0, P_1, P_2 on F_0 such that $p_0^{-1}(\{P_i\}) = \{P_{i0}, P_{i1}, P_{i2}\}$ for $i = 0, 1, 2$. σ induces an automorphism $\bar{\sigma}$ of order three of F_0, and we have $\bar{\sigma}(P_0) = P_1$, $\bar{\sigma}(P_1) = P_2$ and $\bar{\sigma}(P_2) = P_0$. Similarly, there exist Q_0, Q_1, Q_2 on F_1 such that $p_1^{-1}(\{Q_i\}) = \{P_{0i}, P_{1i}, P_{2i}\}$ for $i = 0, 1, 2$. τ induces an automorphism $\bar{\tau}$ of order three of F_1 with $\bar{\tau}(Q_0) = Q_1$, $\bar{\tau}(Q_1) = Q_2$ and $\bar{\tau}(Q_2) = Q_0$.

DEFINITION 1.4. Let X be an algebraic variety and $\mathcal{F}$ a coherent sheaf on it. We say a group G *acts on the pair* $(X, \mathcal{F})$ if G acts on X, an isomorphism $\phi_g \colon g^*\mathcal{F} \to \mathcal{F}$ is given for each $g \in G$ and the diagram

$$\begin{tikzcd} (gh)^*\mathcal{F} \arrow[rr, "\phi_{gh}"] \arrow[dr, "h^*\phi_g"'] & & \mathcal{F} \\ & h^*\mathcal{F} \arrow[ur, "\phi_h"'] & \end{tikzcd}$$

commutes for $g, h \in G$ (cf. [4, Definition 1.6]). For a section $s \in \mathcal{F}(U)$ on an open set $U \subset X$ and for an element $g \in G$, we denote simply by $g^*(s)$ the element $\phi_g(g^*(s))$ of $\mathcal{F}(g^{-1}(U))$.

Let V be an indecomposable vector bundle of rank three on E with $\det V \simeq \mathcal{O}_E(0_E)$ where 0_E is the unit of E. Then there exists an isomorphism

$$q_0^* V \simeq \mathcal{O}_{F_0}(P_0) \oplus \mathcal{O}_{F_0}(P_1) \oplus \mathcal{O}_{F_0}(P_2)$$

of vector bundles on F_0 (cf. [3, 1.1]). We take a section Z_0 of $\mathrm{H}^0(F_0, q_0^* V)$ which corresponds to a nonzero section of the component $\mathrm{H}^0(F_0, \mathcal{O}_{F_0}(P_0))$. Since E is a quotient of F_0 by $(\bar{\sigma})$, this group acts on the pair $(F_0, q_0^* V)$. Let $Z_2 = \bar{\sigma}^*(Z_0)$ and $Z_1 = \bar{\sigma}^*(Z_2)$. Then Z_i corresponds to a nonzero section of $\mathrm{H}^0(F_0, \mathcal{O}_{F_0}(P_i))$ for each $i = 0, 1, 2$. Namely, we can write

$$q_0^* V = \mathcal{O}_{F_0}(P_0) Z_0 \oplus \mathcal{O}_{F_0}(P_1) Z_1 \oplus \mathcal{O}_{F_0}(P_2) Z_2. \tag{1.1}$$

Similarly, we write

$$q_1^* V = \mathcal{O}_{F_1}(Q_0) W_0 \oplus \mathcal{O}_{F_1}(Q_1) W_1 \oplus \mathcal{O}_{F_1}(Q_2) W_2 \tag{1.2}$$

by sections $W_0, W_1, W_2 \in \mathrm{H}^0(F_1, q_1^* V)$ satisfying $W_0 = \bar{\tau}^*(W_1)$ and $W_1 = \bar{\tau}^*(W_2)$.

Note that Z_0 is determined up to multiplications of nonzero constants. Actually, $\mathrm{H}^0(F_0, q_0^* V)$ is three dimensional and elements outside $\mathbf{C} Z_0$ are not zero at P_0. If we replace Z_0 by aZ_0 for a nonzero constant a, then Z_1 and Z_2 are replaced by aZ_1 and aZ_2, respectively. These relations are similar for W_0, W_1, W_2.

The pullback of the sheaf $q_i^* V$ to $\tilde{E}$ is equal to $3_E^* V$ for $i = 0, 1$. We denote the pullbacks of Z_i's and W_i's to $3_E^* V$ on $\tilde{E}$ by the same symbols.

Lemma 1.5. *Assume that we fixed a choice of* $\{Z_0, Z_1, Z_2\}$. *The sections* Z_i*'s and* W_i*'s of* $3_E^* V$ *satisfy the relations*

$$\begin{aligned} W_0 &= f_{00} Z_0 + f_{10} Z_1 + f_{20} Z_2, \\ W_1 &= f_{01} Z_0 + f_{11} Z_1 + f_{21} Z_2, \\ W_2 &= f_{02} Z_0 + f_{12} Z_1 + f_{22} Z_2 \end{aligned}$$

for a suitable choice of $\{W_0, W_1, W_2\}$.

Proof. By (1.1) and (1.2), we have equalities

$$\begin{aligned} 3_E^* V &= \mathcal{O}_{\tilde{E}}(P_{00} + P_{01} + P_{02}) Z_0 \oplus \mathcal{O}_{\tilde{E}}(P_{10} + P_{11} + P_{12}) Z_1 \oplus \mathcal{O}_{\tilde{E}}(P_{20} + P_{21} + P_{22}) Z_2 \\ &= \mathcal{O}_{\tilde{E}}(P_{00} + P_{10} + P_{20}) W_0 \oplus \mathcal{O}_{\tilde{E}}(P_{01} + P_{11} + P_{21}) W_1 \oplus \mathcal{O}_{\tilde{E}}(P_{02} + P_{12} + P_{22}) W_2. \end{aligned}$$

Hence we can express $W_j = \sum_{i=0}^{2} g_{ij} Z_i$ by rational functions

$$g_{ij} \in \mathrm{H}^0(\tilde{E}, \mathcal{O}_{\tilde{E}}(P_{i0} + P_{i1} + P_{i2})).$$

Since W_0 has zeros at P_{00}, P_{10} and P_{20} in the vector bundle $3_E^* V$, each g_{i0} must has zeros of order at least one at these points as a section of $\mathcal{O}_{\tilde{E}}(P_{i0} + P_{i1} + P_{i2})$. Namely, we have inequalities of divisors

$$
\begin{aligned}
(g_{00}) + P_{00} + P_{01} + P_{02} &\geq P_{00} + P_{10} + P_{20},\\
(g_{10}) + P_{10} + P_{11} + P_{12} &\geq P_{00} + P_{10} + P_{20},\\
(g_{20}) + P_{20} + P_{21} + P_{22} &\geq P_{00} + P_{10} + P_{20}.
\end{aligned}
$$

Since the first inequality implies

$$(g_{00}) \geq P_{10} + P_{20} - P_{01} - P_{02} = (f_{00}),$$

g_{00} is zero or a nonzero constant multiple of f_{00}, i.e., $g_{00} = a_{00} f_{00}$ for an element $a_{00} \in \mathbf{C}$. Similarly, we have $g_{10} = a_{10} f_{10}$ and $g_{20} = a_{20} f_{20}$ for some elements $a_{10}, a_{20} \in \mathbf{C}$. Furthermore, by considering W_1 and W_2 similarly, we have an expression $g_{ij} = a_{ij} f_{ij}$ with an element $a_{ij} \in \mathbf{C}$ for every pair (i, j).

Since E is the quotient of $\tilde{E}$ by the action of the group $\mathrm{Ker}\, 3_E = (\sigma, \tau)$, this group acts also on the pair $(\tilde{E}, 3_E^* V)$. We have $\tau^* W_j = W_{j-1}$ for $j = 1, 2$ and $\tau^* Z_i = Z_i$ for $i = 0, 1, 2$. Hence we have

$$\sum_{i=0}^{2} a_{i,j-1} f_{i,j-1} Z_i = W_{j-1} = \tau^* W_j = \tau^* \left(\sum_{i=0}^{2} a_{ij} f_{ij} Z_i \right) = \sum_{i=0}^{2} a_{ij} f_{i,j-1} Z_i$$

for $j = 1, 2$. This implies $a_{i2} = a_{i1} = a_{i0}$ for $i = 0, 1, 2$. On the other hand, since $\sigma^* W_0 = W_0$, $\sigma^* Z_i = Z_{i-1}$ for $i = 1, 2$ and $\sigma^* Z_0 = Z_2$, we have

$$a_{00} f_{00} Z_0 + a_{10} f_{10} Z_1 + a_{20} f_{20} Z_2 = W_0 = \sigma^* W_0 = a_{00} f_{20} Z_2 + a_{10} f_{00} Z_0 + a_{20} f_{10} Z_1.$$

Hence $a_{00} = a_{10} = a_{20}$. Thus we know that all a_{ij}'s are equal to a nonzero constant a. If we choose W_0 such that $a = 1$, then we have the equalities in the lemma. □

Recall that ϵ is the nonzero constant with the equality $f_{00} = \epsilon f_{10} f_{20} f_{01} f_{02}$. Set $f_0 = \epsilon f_{00} f_{01} f_{02}$. Clearly, f_0 is invariant by τ^*. Hence we have equalities

$$f_0 = \epsilon f_{00} f_{01} f_{02} = \frac{f_{00}^2}{f_{10} f_{20}} = \frac{f_{01}^2}{f_{11} f_{21}} = \frac{f_{02}^2}{f_{12} f_{22}}. \tag{1.3}$$

Set $f_1 = (\sigma^{-1})^* f_0$ and $f_2 = (\sigma^{-2})^* f_0$. By applying $(\sigma^{-1})^*$ and $(\sigma^{-2})^*$ to (1.3), we get the equalities

$$f_1 = \epsilon f_{10} f_{11} f_{12} = \frac{f_{10}^2}{f_{20} f_{00}} = \frac{f_{11}^2}{f_{21} f_{01}} = \frac{f_{12}^2}{f_{22} f_{02}} \tag{1.4}$$

and

$$f_2 = \epsilon f_{20} f_{21} f_{22} = \frac{f_{20}^2}{f_{00} f_{10}} = \frac{f_{21}^2}{f_{01} f_{11}} = \frac{f_{22}^2}{f_{02} f_{12}}. \tag{1.5}$$

By these equalities, we have

$$f_0 f_1 f_2 = \frac{f_{00}^2}{f_{10} f_{20}} \cdot \frac{f_{10}^2}{f_{20} f_{00}} \cdot \frac{f_{20}^2}{f_{00} f_{10}} = 1. \tag{1.6}$$

Similarly, set $f_0' = (\epsilon f_{00} f_{10} f_{20})^{-1}$, $f_1' = (\tau^{-1})^* f_0'$ and $f_2' = (\tau^{-2})^* f_0'$. These are invariant by σ^*, and we have equalities

$$f_0' = (\epsilon f_{00} f_{10} f_{20})^{-1} = \frac{f_{01} f_{02}}{f_{00}^2} = \frac{f_{11} f_{12}}{f_{10}^2} = \frac{f_{21} f_{22}}{f_{20}^2}, \tag{1.7}$$

$$f_1' = (\epsilon f_{01} f_{11} f_{21})^{-1} = \frac{f_{02} f_{00}}{f_{01}^2} = \frac{f_{12} f_{10}}{f_{11}^2} = \frac{f_{22} f_{20}}{f_{21}^2}, \tag{1.8}$$

$$f_2' = (\epsilon f_{02} f_{12} f_{22})^{-1} = \frac{f_{00} f_{01}}{f_{02}^2} = \frac{f_{10} f_{11}}{f_{12}^2} = \frac{f_{20} f_{21}}{f_{22}^2}. \tag{1.9}$$

The equality $f_0' f_1' f_2' = 1$ follows from these equalities.

By applying $(\sigma^{-1})^*$ and $(\sigma^{-2})^*$ to the equality of Lemma 1.3, we get the equalities

$$\delta = f_{00} + f_{01} + f_{02} = f_{10} + f_{11} + f_{12} = f_{20} + f_{21} + f_{22}. \tag{1.10}$$

Proposition 1.6. *The equality*

$$\begin{aligned}
&\epsilon \delta^{-1} (f_0' W_0^4 + f_1' W_1^4 + f_2' W_2^4) \\
&= (f_0 Z_0^4 + f_1 Z_1^4 + f_2 Z_2^4) \\
&\quad + 4(f_0 Z_0^3 (Z_1 + Z_2) + f_1 Z_1^3 (Z_0 + Z_2) + f_2 Z_2^3 (Z_0 + Z_1)) \\
&\quad + 6(f_0^{-1} Z_1^2 Z_2^2 + f_1^{-1} Z_0^2 Z_2^2 + f_2^{-1} Z_0^2 Z_1^2) \\
&\quad + 12 Z_0 Z_1 Z_2 (Z_0 + Z_1 + Z_2)
\end{aligned}$$

holds in $\mathrm{Sym}^4_{\mathcal{O}_{\tilde{E}}} (3^*_E V)$.

Proof. By the equalities (1.7), (1.8) and (1.9), we have

$$\epsilon (f_0' W_0^4 + f_1' W_1^4 + f_2' W_2^4) = \frac{W_0^4}{f_{00} f_{10} f_{20}} + \frac{W_1^4}{f_{01} f_{11} f_{21}} + \frac{W_2^4}{f_{02} f_{12} f_{22}}. \tag{1.11}$$

By substituting $f_{0j} Z_0 + f_{1j} Z_1 + f_{2j} Z_2$ for W_j of the right-hand side of (1.11) for $j = 0, 1, 2$, we get a polynomial in Z_0, Z_1 and Z_2. It suffices to check that this polynomial is δ times the right-hand side of the equality of the proposition. We will check the coefficient of each monomial.

The coefficient of Z_0^4 is

$$\frac{f_{00}^4}{f_{00}\,f_{10}\,f_{20}}+\frac{f_{01}^4}{f_{01}\,f_{11}\,f_{21}}+\frac{f_{02}^4}{f_{02}\,f_{12}\,f_{22}}= f_{00}\,f_0+ f_{01}\,f_0+ f_{02}\,f_0 = f_0\delta$$

by (1.3) and (1.10). Similarly, the coefficients of Z_1^4 and Z_2^4 are $f_1\delta$ and $f_2\delta$, respectively.

Since $4!/(3!\ 1!\ 1!)=4$, the coefficient of $Z_0^3Z_1$ is

$$\frac{4\,f_{00}^3\,f_{10}}{f_{00}\,f_{10}\,f_{20}}+\frac{4\,f_{01}^3\,f_{11}}{f_{01}\,f_{11}\,f_{21}}+\frac{4\,f_{02}^3\,f_{12}}{f_{02}\,f_{12}\,f_{22}}=4\,f_{10}\,f_0+4\,f_{11}\,f_0+4\,f_{12}\,f_0=4\,f_0\delta$$

by (1.3) and (1.10). That of $Z_0^3Z_2$ is also $4\,f_0\delta$. Similarly, the coefficients of $Z_1^3Z_0$ and $Z_1^3Z_2$ are both $4\,f_1\delta$ and those of $Z_2^3Z_0$ and $Z_2^3Z_1$ are both $4\,f_2\delta$.

Since $4!/(2!\ 2!)=6$, the coefficient of $Z_1^2Z_2^2$ is

$$\frac{6\,f_{10}^2\,f_{20}^2}{f_{00}\,f_{10}\,f_{20}}+\frac{6\,f_{11}^2\,f_{21}^2}{f_{01}\,f_{11}\,f_{21}}+\frac{6\,f_{12}^2\,f_{22}^2}{f_{02}\,f_{12}\,f_{22}}=6\,f_{00}\,f_0^{-1}+6\,f_{01}\,f_0^{-1}+6\,f_{02}\,f_0^{-1}=6\,f_0^{-1}\delta$$

by (1.3) and (1.10). Similarly, the coefficients of $Z_0^2Z_2^2$ and $Z_0^2Z_1^2$ are $6\,f_1^{-1}\delta$ and $6\,f_2^{-1}\delta$, respectively.

Since $4!/(2!\ 1!\ 1!\ 1!)=12$, the coefficient of $Z_0^2Z_1Z_2$ is

$$\frac{12\,f_{00}^2\,f_{10}\,f_{20}}{f_{00}\,f_{10}\,f_{20}}+\frac{12\,f_{01}^2\,f_{11}\,f_{21}}{f_{01}\,f_{11}\,f_{21}}+\frac{12\,f_{02}^2\,f_{12}\,f_{22}}{f_{02}\,f_{12}\,f_{22}}=12\,f_{00}+12\,f_{01}+12\,f_{02}=12\delta$$

by (1.10). Similarly, those of $Z_0Z_1^2Z_2$ and $Z_0Z_1Z_2^2$ are also 12δ. □

2. Defining equations of CCI surfaces

In [3], Ishida considered CCI surfaces S with $E(S)=E$ for an elliptic curve E. In this case, $V=a_*K_S\otimes_{\mathcal{O}_E}K_E^{-1}$ is an indecomposable vector bundle of rank three. Since $\deg(\det V)=1$, we may assume $\det V\simeq\mathcal{O}_E(0_E)$. Since such vector bundle is unique up to isomorphisms, any CCI surface with $E(S)=E$ is embedded in $\mathbf{P}(V)$ for a common V. He takes a triple covering $\varphi\colon E'\to E$ of the elliptic curve E, and describes the defining equations of the surfaces pullbacked by this covering map, where E' is denoted by $\tilde{E}$ in [3, 1.1].

Let E' be $F_0=\tilde{E}/(\tau)$ in Section 1. Then $q_0\colon F_0\to E$ is an isogeny of degree three. We define *the relative canonical coordinates* $(Z_0:Z_1:Z_2)$ of $\mathbf{P}(q_0^*V)$ by (1.1). One of the four equations defining the pullbacks of CCI surfaces is

$$\Psi_1 = f\,Z_0^4+g\,Z_1^4+h\,Z_2^4=0,$$

where f is an explicitly given rational function on F_0 with $(f) = -2P_0 + P_1 + P_2$, and $g = (\bar{\sigma}^{-1})^* f$ and $h = (\bar{\sigma}^{-2})^* f$ (cf. [3, Proposition 2.8]). We denote by S_0 the CCI surface whose pullback to $\mathbf{P}(q_0^* V)$ is defined by this Fermat type equation. Since we consider similar equations on the elliptic curve $F_1 = \tilde{E}/(\sigma)$ and others, we denote by $\Psi_1^{(0)}$ this Ψ_1 in order to avoid the confusion.

Proposition 2.1. *There exists a nonzero constant c_0 with*

$$\Psi_1^{(0)} = c_0(f_0 Z_0^4 + f_1 Z_1^4 + f_2 Z_2^4)$$

for f_0, f_1 and f_2 defined in Section 1.

Proof. Since $f_i = \epsilon f_{i0} f_{i1} f_{i2}$, these rational functions descend to those of $F_0 = \tilde{E}/(\tau)$. Since

$$(f_0) = \begin{bmatrix} -2 & 1 & 1 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{bmatrix},$$

we have $(f_0) = -2P_0 + P_1 + P_2$ as a rational function of F_0. Hence $f = c_0 f_0$ for a nonzero constant c_0. We get the lemma since $f_1 = (\bar{\sigma}^{-1})^* f_0$ and $f_2 = (\bar{\sigma}^{-2})^* f_0$. $\square$

Now, we consider the case $E' = F_1$, we should replace Z_0, Z_1, Z_2 in [3, 1.2] by W_0, W_1, W_2 in Section 1. Then the Fermat type equation of [3, Proposition 2.8] is

$$\Psi_1^{(1)} = f^{(1)} W_0^4 + g^{(1)} W_1^4 + h^{(1)} W_2^4 = 0$$

for the relative canonical coordinates $(W_0 : W_1 : W_2)$ of $\mathbf{P}(q_1^* V)$, where $f^{(1)}$ is a rational function on F_1 with $(f^{(1)}) = -2Q_0 + Q_1 + Q_2$, and $g^{(1)} = (\bar{\tau}^{-1})^* f^{(1)}$ and $h^{(1)} = (\bar{\tau}^{-2})^* f^{(1)}$.

Similarly to Proposition 2.1, we get the following in view of the equality

$$(f_0') = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ -2 & -2 & -2 \end{bmatrix}.$$

Proposition 2.2. *There exists a nonzero constant c_1 with*

$$f^{(1)} W_0^4 + g^{(1)} W_1^4 + h^{(1)} W_2^4 = c_1(f_0' W_0^4 + f_1' W_1^4 + f_2' W_2^4)$$

for f_0', f_1' and f_2' defined in Section 1.

For nonzero rational functions u_0, u_1, u_2 of F_0, we set

$$\begin{aligned}\Psi(u_0, u_1, u_2) = (&u_0 Z_0^4 + u_1 Z_1^4 + u_2 Z_2^4)\\ &+ 4(u_0 Z_0^3(Z_1 + Z_2) + u_1 Z_1^3(Z_0 + Z_2) + u_2 Z_2^3(Z_0 + Z_1))\\ &+ 6(u_0^{-1} Z_1^2 Z_2^2 + u_1^{-1} Z_0^2 Z_2^2 + u_2^{-1} Z_0^2 Z_1^2)\\ &+ 12 Z_0 Z_1 Z_2(Z_0 + Z_1 + Z_2).\end{aligned}$$

Note that $\Psi(f_0, f_1, f_2)$ is the right-hand side of the equality of Proposition 1.6.

Lemma 2.3. *Let $S_1 \subset \mathbf{P}(V)$ be the CCI surface whose pullback to $\mathbf{P}(q_1^* V)$ is defined by the equation $\Psi_1^{(1)} = 0$. Then the pullback of S_1 to $\mathbf{P}(q_0^* V)$ is defined by $\Psi(f_0, f_1, f_2) = 0$.*

Proof. Recall that $3_E = q_0 \cdot p_0 = q_1 \cdot p_1$ for $3_E \colon \tilde{E} \to E$. The pullback of S_1 to $\mathbf{P}(q_1^* V)$ is defined by $f_0' W_0^4 + f_1' W_1^4 + f_2' W_2^4 = 0$ by Proposition 2.2. Hence Propositions 1.6 implies that the pullback of S_1 to $\mathbf{P}(3_E^* V)$ is equal to the pullback of the surface $Y \subset \mathbf{P}(q_0^* V)$ defined by $\Psi(f_0, f_1, f_2) = 0$. Since the morphism $\mathbf{P}(3_E^* V) \to \mathbf{P}(q_0^* V)$ is an etale surjection, Y is equal to the pullback of S_1 to $\mathbf{P}(q_0^* V)$. $\square$

In [3], a global section Ψ_ζ of $\mathrm{Sym}^4_{\mathcal{O}_{F_0}}(q_0^* V)$ is defined by

$$\begin{aligned}\Psi_\zeta = (&f Z_0^4 + g Z_1^4 + h Z_2^4)\\ &+ 4(f Z_0^3(Z_1 + Z_2) + g Z_1^3(Z_0 + Z_2) + h Z_2^3(Z_0 + Z_1))\\ &- 6\zeta^{-2}(gh Z_1^2 Z_2^2 + fh Z_0^2 Z_2^2 + fg Z_0^2 Z_1^2)\\ &- 12\zeta^2 Z_0 Z_1 Z_2(Z_0 + Z_1 + Z_2)\end{aligned}$$

for $\zeta \in \mathbf{C} \setminus \{0\}$. He showed [3, Proposition 2.8] that the equation $\Psi_\zeta = 0$ defines the pullback of a CCI surface if $\zeta^3 = -2\beta$, where β is an explicitly given constant satisfying $fgh = -4\beta^2$ in [3, 1.2].

Proposition 2.4. *The three equations $\Psi_\zeta = 0$ with $\zeta^3 = -2\beta$ in* [3, Proposition 2.8] *are equivalent to $\Psi(f_0, f_1, f_2) = 0$, $\Psi(\omega f_0, \omega f_1, \omega f_2) = 0$ and $\Psi(\omega^2 f_0, \omega^2 f_1, \omega^2 f_2) = 0$.*

Proof. Since f, g, h are rational functions of F_0 with $(f) = -2P_0 + P_1 + P_2$, $g = (\bar{\sigma}^{-1})^* f$ and $h = (\bar{\sigma}^{-2})^* f$, there exists a nonzero constant c with $c = f/f_0 = g/f_1 = h/f_2$. Since $f_0 f_1 f_2 = 1$ by (1.6), we have $c^3 = -4\beta^2$. Then we have $\Psi_\zeta = c\Psi(f_0, f_1, f_2)$ for $\zeta = 2\beta/c$, since $-6\zeta^{-2} gh = 6c/f_0$, $-6\zeta^{-2} fh = 6c/f_1$, $-6\zeta^{-2} fg = 6c/f_2$ and $-12\zeta^2 = 12c^3/c^2 = 12c$. Since $\zeta^3 = 8\beta^3/c^3 = -2\beta$, the equation $\Psi(f_0, f_1, f_2) = 0$ is equivalent to $\Psi_\zeta = 0$ in [3, Proposition 2.8] for this ζ.

Others are checked similarly. Namely, we have $\Psi_\zeta = \omega c \Psi(\omega f_0, \omega f_1, \omega f_2)$ for $\zeta = 2\omega^2\beta/c$ and $\Psi_\zeta = \omega^2 c \Psi(\omega^2 f_0, \omega^2 f_1, \omega^2 f_2)$ for $\zeta = 2\omega\beta/c$. $\square$

There are two unramified coverings of E of degree three besides F_0 and F_1. Namely, these are $F_2 = \tilde{E}/(\sigma\tau)$ and $F_3 = \tilde{E}/(\sigma\tau^{-1})$. If we take $(P'_{10}, P'_{01}) = (P_{11}, P_{01})$ as the basis of $\operatorname{Ker} 3_E$ at the beginning of Section 1, then F_2 plays the role of F_1 while F_0 and the automorphism $\bar{\sigma}$ of it are not changed.

Let $P'_{ij} = [iP'_{10} + jP'_{01}]$ for $0 \le i, j \le 2$. We take a rational function g_{00} with the divisor

$$(g_{00}) = P'_{10} + P'_{20} - P'_{01} + P'_{02} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and define $g_{ij} = ((\sigma\tau)^{-i}\tau^{-j})^* g_{00}$ for $0 \le i, j \le 2$.

Lemma 2.5. *The assumption* $f_{00} f_{11} f_{10}^{-1} f_{01}^{-1} = \omega^{-1}$ *in* Section 1 *implies the equality* $g_{00} g_{11} g_{10}^{-1} g_{01}^{-1} = \omega^{-1}$.

Proof. Since

$$\left(\frac{g_{00}}{g_{10}}\right) = (g_{00}) - (g_{10}) = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \left(\frac{f_{00}}{f_{10}}\right),$$

there exists a nonzero constant a with $g_{00}/g_{10} = af_{00}/f_{10}$. By applying $(\tau^{-1})^*$ to this equality, we have $g_{01}/g_{11} = af_{01}/f_{11}$. Hence we have

$$\frac{g_{00}g_{11}}{g_{10}g_{01}} = \frac{g_{00}}{g_{10}} \cdot \frac{g_{11}}{g_{01}} = \frac{af_{00}}{f_{10}} \cdot \frac{f_{11}}{af_{01}} = \frac{f_{00} f_{11}}{f_{10} f_{01}} = \omega^{-1}. \qquad \square$$

Let $q_2 \colon F_2 \to E$ be the covering map. For the similar Fermat type equation

$$\Psi_1^{(2)} = f^{(2)} T_0^4 + g^{(2)} T_1^4 + h^{(2)} T_2^4 = 0$$

defined for $E' = F_2$, we can apply Lemma 2.3. Namely, let $S_2 \subset \mathbf{P}^2(V)$ be the CCI surface whose pullback in $\mathbf{P}^2(q_2^* V)$ is defined by $\Psi_1^{(2)} = 0$ for the relative canonical coordinates $(T_0 : T_1 : T_2)$. Then the pullback of S_2 to $\mathbf{P}^2(q_0^* V)$ is defined by $\Psi(g_0, g_1, g_2) = 0$, where $g_0 = g_{00}^2/g_{10}g_{20}$, $g_1 = (\bar{\sigma}^{-1})^* g_0$ and $g_2 = (\bar{\sigma}^{-2})^* g_0$.

Lemma 2.6. *In above notation, we have* $g_0 = \omega f_0$, $g_1 = \omega f_1$ *and* $g_2 = \omega f_2$. *Hence the pullback of* S_2 *to* $\mathbf{P}^2(q_0^* V)$ *is defined by* $\Psi(\omega f_0, \omega f_1, \omega f_2) = 0$.

Proof. It suffices to show the first one since others are its translations on F_0. We have $(\sigma\tau)^*(g_{10}/g_{00}) = g_{00}/g_{20}$, while

$$(2.12) \qquad (\sigma\tau)^* \left(\frac{f_{10}}{f_{00}}\right) = \frac{f_{02}}{f_{22}} = \frac{f_{02} f_{20}}{f_{22} f_{00}} \cdot \frac{f_{00}}{f_{20}} = (\sigma\tau)^* \left(\frac{f_{10} f_{01}}{f_{00} f_{11}}\right) \cdot \frac{f_{00}}{f_{20}} = \frac{\omega f_{00}}{f_{20}}.$$

Since we have the equality of divisors $(f_{00}/f_{10}) = (g_{00}/g_{10})$, we have an equality $g_{00}/g_{10} = cf_{00}/f_{10}$ for a nonzero constant c. By using (2.12), we have

$$g_0 = \frac{g_{00}^2}{g_{10}g_{20}} = \frac{g_{00}}{g_{10}} \cdot (\sigma\tau)^*\left(\frac{g_{10}}{g_{00}}\right) = \frac{cf_{00}}{f_{10}} \cdot c^{-1}(\sigma\tau)^*\left(\frac{f_{10}}{f_{00}}\right) = \frac{\omega f_{00}^2}{f_{10}f_{20}} = \omega f_0.$$

□

The calculation for $F_3 = \tilde{E}/(\sigma\tau^{-1})$ is similar. Let h_{00} be the rational function with

$$(h_{00}) = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

and let $h_{ij} = ((\sigma\tau^{-1})^{-i}\tau^{-j})^* h_{00}$ for $0 \le i, j \le 2$.

Let $q_3\colon F_3 \to E$ be the covering map, and consider the similar Fermat type equation

$$\Psi_1^{(3)} = f^{(3)}U_0^4 + g^{(3)}U_1^4 + h^{(3)}U_2^4 = 0$$

for the relative canonical coordinates $(U_0 : U_1 : U_2)$ of $\mathbf{P}(q_3^*V)$. Let $S_3 \subset \mathbf{P}^2(V)$ be the CCI surface whose pullback to $\mathbf{P}^2(q_3^*V)$ is defined by $\Psi_1^{(3)} = 0$.

Lemma 2.7. *For $h_0 = h_{00}^2/h_{10}h_{20}$, $h_1 = (\bar{\sigma}^{-1})^*h_0$ and $h_2 = (\bar{\sigma}^{-2})^*h_0$, we have $h_0 = \omega^2 f_0$, $h_1 = \omega^2 f_1$ and $h_2 = \omega^2 f_2$. Hence the pullback of S_3 to $\mathbf{P}^2(q_0^*V)$ is defined by $\Psi(\omega^2 f_0, \omega^2 f_1, \omega^2 f_2) = 0$.*

Proof. It suffices to show the first equality. We have $(\sigma\tau^{-1})^*(h_{10}/h_{00}) = h_{00}/h_{20}$, while

$$(2.13)\qquad (\sigma\tau^{-1})^*\left(\frac{f_{10}}{f_{00}}\right) = \frac{f_{01}}{f_{21}} = \frac{f_{20}f_{01}}{f_{00}f_{21}} \cdot \frac{f_{00}}{f_{20}} = \sigma^*\left(\frac{f_{00}f_{11}}{f_{10}f_{01}}\right) \cdot \frac{f_{00}}{f_{20}} = \frac{\omega^2 f_{00}}{f_{20}}.$$

Since $(f_{00}/f_{10}) = (h_{00}/h_{10})$ as divisors, there exists a nonzero constant c with $h_{00}/h_{10} = cf_{00}/f_{10}$. By using (2.13), we have

$$h_0 = \frac{h_{00}^2}{h_{10}h_{20}} = \frac{h_{00}}{h_{10}} \cdot (\sigma\tau^{-1})^*\left(\frac{h_{10}}{h_{00}}\right) = \frac{cf_{00}}{f_{10}} \cdot c^{-1}(\sigma\tau^{-1})^*\left(\frac{f_{10}}{f_{00}}\right) = \frac{\omega^2 f_{00}^2}{f_{10}f_{20}} = \omega^2 f_0.$$

We get the lemma by Lemma 2.3. □

Theorem 2.8. *Let S be a CCI surface with $E(S) = E$. Then S is isomorphic to the surface S_i in $\mathbf{P}(V)$ whose pullback to $\mathbf{P}(p_i^*V)$ is defined by the equation $\Psi_1^{(i)} = 0$ for an i with $0 \le i \le 3$. If $j \ne i$, then the pullback of S_i to $\mathbf{P}(p_j^*V)$ is not defined by a Fermat type equation for the relative canonical coordinates.*

Proof. By [3, Proposition 2.8], for any CCI surface S with $E(S) = E$, there exists an embedding $S \subset \mathbf{P}(V)$ such that the pullback of S to $\mathbf{P}(q_0^* V)$ is defined by $\Psi_1^{(0)} = 0$ or $\Psi_\zeta = 0$ for a ζ with $\zeta^3 = -2\beta$. The last three equations are equivalent to $\Psi(\omega^i f_0, \omega^i f_1, \omega^i f_2) = 0$ for $i = 0, 1, 2$ by Proposition 2.4. Hence we get the first part of the theorem by Lemmas 2.3, 2.6 and 2.7. The last part of the theorem follows from Lemma 2.3 by retaking the basis of $\operatorname{Ker} 3_E$ so that F_j and F_i play the roles of F_0 and F_1, respectively. □

3. The moduli space of CCI surfaces

Every elliptic curve E in this section is assumed to have a fixed null element 0_E. For an element $x \in E$, the translation of E defined by $y \mapsto [x + y]$ is denoted by T_x, while the involution defined by $y \mapsto -y$ is denoted by ι.

Let $\alpha = (F, \Lambda)$ be a pair consisting of an elliptic curve F and a subgroup $\Lambda \subset F$ of order three. Then the CCI surface S_α is constructed as follows.

Let $\Lambda = \{0_F = P_0, P_1, P_2\}$. Since $[P_1 + P_2] = P_0$, there exists a nonzero rational function f_0 with $(f_0) = -2P_0 + P_1 + P_2$ by Abel's theorem. Then $\iota^* f_0 = f_0$ for the involution ι of F since $(\iota^* f_0) = (f_0)$ and these functions have same value at P_0.

Let $f_1 = T_{P_2}^* f_0$ and $f_2 = T_{P_1}^* f_0$. Then $(f_1) = P_0 - 2P_1 + P_2$ and $(f_2) = P_0 + P_1 - 2P_2$. The group $\tilde{\Lambda}$ generated by T_{P_1} and ι is a group isomorphic to the symmetric group of degree three whose action on F induces the permutations of $\{P_0, P_1, P_2\}$ as well as those of $\{f_0, f_1, f_2\}$.

We consider the locally free sheaf $V_\alpha = \mathcal{O}_F(P_0)Z_0 \oplus \mathcal{O}_F(P_1)Z_1 \oplus \mathcal{O}_F(P_2)Z_2$ of rank three on F and the $\mathbf{P}^2$-bundle $\mathbf{P}_F(V_\alpha)$, where Z_0, Z_1 and Z_2 are indeterminates. For a point x in $F \setminus \{P_0, P_1, P_2\}$, the fiber $\mathbf{P}_F(V_\alpha)_x$ is a projective plane with the homogeneous coordinates $(Z_0 : Z_1 : Z_2)$. The action of $\tilde{\Lambda}$ on the pair (F, V_α) is defined so that it induces permutations of $\{Z_0, Z_1, Z_2\}$ (cf. Definition 1.4). Namely, $T_{P_1}^*(Z_0) = Z_2$, $T_{P_1}^*(Z_1) = Z_0$, $T_{P_1}^*(Z_2) = Z_1$ and $\iota^*(Z_0) = Z_0$, $\iota^*(Z_1) = Z_2$, $\iota^*(Z_2) = Z_1$.

This action of $\tilde{\Lambda}$ on (F, V_α) induces that on $\mathbf{P}_F(V_\alpha)$. Namely, we have

$$\begin{aligned} &T_{P_1}((x, (a_0 : a_1 : a_2))) = ([x + P_1], (a_2 : a_0 : a_1)), \\ &T_{P_2}((x, (a_0 : a_1 : a_2))) = ([x + P_2], (a_1 : a_2 : a_0)), \quad \text{and} \\ &\iota((x, (a_0 : a_1 : a_2))) = ([-x], (a_0 : a_2 : a_1)) \end{aligned}$$

for $(x, (a_0 : a_1 : a_2)) \in (F \setminus \{P_0, P_1, P_2\}) \times \mathbf{P}^2$.

Set $E = F/\Lambda$ and define $3_E : \tilde{E} \to E$ as in Section 1. Since $\Lambda \simeq \mathbf{Z}/3\mathbf{Z}$, 3_E factors through the natural map $q : F \to E$. For a suitable choice of the basis $\{P_{10}, P_{01}\}$ of $\operatorname{Ker} 3_E$, F is equal to F_0 in Section 1 and P_0, P_1, P_2 are equal to those in Section 1. Then the functions f_0, f_1 and f_2 in Section 1 play the roles of f_0, f_1 and f_2 above, respectively.

Since the rational functions f_0, f_1, f_2 have no zeros or poles on $F \setminus \{P_0, P_1, P_2\}$, the equation

$$f_0 Z_0^4 + f_1 Z_1^4 + f_2 Z_2^4 = 0 \tag{3.14}$$

defines a family of Fermat quartics in $(F \setminus \{P_0, P_1, P_2\}) \times \mathbf{P}^2$. Let $\tilde{S}_\alpha$ be the closure of this family in $\mathbf{P}_F(V_\alpha)$. Since the equality (3.14) is invariant by the action of $\tilde{\Lambda}$, the subgroup Λ acts freely on the fiber space $\tilde{S}_\alpha \to F$ and ι induces an involution of $\tilde{S}_\alpha$. In particular, the definition of $\tilde{S}_\alpha$ does not depend on the numbering of P_1, P_2.

Let t be a rational function of F which has a simple zero at P_0. For $Y_0 = t^{-1} Z_0$, the $\mathbf{P}^2$-bundle $\mathbf{P}_F(V_\alpha)$ has coordinates $(Y_0 : Z_1 : Z_2)$ in a neighborhood of P_0 since (Y_0, Z_1, Z_2) is a frame of V_α at P_0. The equation (3.14) is

$$t^4 f_0 Y_0^4 + f_1 Z_1^4 + f_2 Z_2^4 = t(t^3 f_0 Y_0^4 + t^{-1} f_1 Z_1^4 + t^{-1} f_2 Z_2^4) = 0$$

for these coordinates. Since f_0 has a pole of order two and f_1, f_2 have simple zeros at P_0, we know the surface $\tilde{S}_\alpha$ is defined by

$$t^3 f_0 Y_0^4 + t^{-1} f_1 Z_1^4 + t^{-1} f_2 Z_2^4 = 0$$

in a neighborhood of P_0. The fiber at P_0 is defined by $a_{01} Z_1^4 + a_{02} Z_2^4 = 0$ in $\mathbf{P}^2$ for the coordinates $(Y_0 : Z_1 : Z_2)$, where a_{01} and a_{02} are the value at P_0 of $t^{-1} f_1$ and $t^{-1} f_2$, respectively. Since a_{01} and a_{02} are nonzero constants, the fiber at P_0 is the union of four distinct lines intersecting at $(1 : 0 : 0)$.

We define S_α to be the quotient surface $\tilde{S}_\alpha / \Lambda$. By [3, Proposition 2.8] and Proposition 2.1, this is a CCI surface with $E(S_\alpha) = E = F/\Lambda$. Let $\bar{P}_0 \in F/\Lambda$ be the image of P_0. Then the surface S_α has a reduced fiber consisting of four lines at $\bar{P}_0$, and other fibers are nonsingular quartics.

Since Λ is a normal subgroup of $\tilde{\Lambda}$, the involution ι of $\tilde{S}_\alpha$ induces that of S_α which we denote also by ι.

Let $\mathcal{X}$ be the set of isomorphism classes of pairs $\alpha = (F, \Lambda)$ of an elliptic curve F and a subgroup Λ of order three.

Theorem 3.1. *The correspondence $\alpha \mapsto S_\alpha$ defines a bijection from $\mathcal{X}$ to to the set of isomorphism classes of CCI surfaces.*

Proof. Let S be a CCI surface. By [3, Proposition 2.8], there exists an embedding $S \subset \mathbf{P}(V)$ for $E = E(S)$ and $V = a_* K_S \otimes K_E^{-1}$. By Theorem 2.8, there exists a unique unramified covering $q(S)\colon F(S) \to E$ of degree three such that the pullback of S to $\mathbf{P}(q(S)^* V)$ is defined by the Fermat type equation. Define $\alpha = (F(S), \operatorname{Ker} q(S))$. Then we have $V_\alpha \simeq q(S)^* V$ and $S_\alpha \simeq S$ by the construction of S_α.

Conversely, Let $\alpha = (F, \Lambda)$ be a pair in $\mathcal{X}$. Then S_α is a CCI surface with $E(S_\alpha) = F/\Lambda$. Since the canonical map $q\colon F \to F/\Lambda$ is unramified of degree three and the surface $\tilde{S}_\alpha$ is defined by the Fermat type equation, we can identify F with $F(S_\alpha)$ in the first part of this proof by Theorem 2.8. Then, clearly $q = q(S_\alpha)$ and $\Lambda = \operatorname{Ker} q(S_\alpha)$. $\square$

For $\alpha = (F, \Lambda)$, we define $\alpha^* = (F/\Lambda, \operatorname{Ker} 3_F/\Lambda)$ where $3_F\colon F \to F$ is the morphism defined by $3_F(x) = [3x]$. Then $\alpha^{**} = \alpha$ for every α and the map $\alpha \mapsto \alpha^*$ is a bijection from $\mathcal{X}$ to itself. Since $F \simeq F/\operatorname{Ker} 3_F$, S_{α^*} is a CCI surface with $E(S_{\alpha^*}) = F$.

For each $\mu \in \mathbf{C} \setminus \{1, \omega, \omega^2\}$, the Hesse cubic curve $E(\mu)$ is defined by

$$X_0^3 + X_1^3 + X_2^3 - 3\mu X_0 X_1 X_2 = 0$$

in $\mathbf{P}^2$. It is known that the j-invariant of this elliptic curve is given by

$$j = \frac{27\mu^3(\mu^3+8)^3}{(\mu^3-1)^3} \tag{3.15}$$

(cf. [2, p. 456] and [6, 7.6]).

The point $(1 : -1 : 0) \in E(\mu)$ is defined to be the unit. Then the set of points of order three and the unit is

$$\left\{\begin{matrix} (1:-1:0), & (1:-\omega:0), & (1:-\omega^2:0), \\ (-1:0:1), & (-\omega:0:1), & (-\omega^2:0:1), \\ (0:1:-1), & (0:1:-\omega), & (0:1:-\omega^2) \end{matrix}\right\},$$

which is equal to the set of inflection points of $E(\mu)$. If we set $P_{10} = (1 : -\omega : 0)$ and $P_{01} = (-1 : 0 : 1)$, then we have $P_{i0} = (1 : -\omega^i : 0)$, $P_{i1} = (-\omega^i : 0 : 1)$ and $P_{i2} = (0 : 1 : -\omega^i)$ for $i = 0, 1, 2$, and the condition $f_{00} f_{11} f_{10}^{-1} f_{01}^{-1} = \omega^2$, i.e., $\mathbf{e}_3(P_{10}, P_{01}) = \omega$ is satisfied.

Let $T = \mathbf{C} \setminus \{1, \omega, \omega^2\}$. It is known that T is the fine moduli space of the elliptic curves with level three structure (e_1, e_2) with the Weil pairing $\mathbf{e}_3(e_1, e_2) = \omega$ and $\{E(\mu);\ \mu \in T\}$ is the global family, where the level three structure is defined by $(e_1, e_2) = (P_{10}, P_{01})$ for all μ (cf. [4, Definition 7.1]).

For each $\rho \in \mathbf{C} \setminus \{1\}$, let $F(\rho)$ be the elliptic curve defined by the equation $\rho X_0^3 + \rho X_1^3 + X_2^3 - 3\rho X_0 X_1 X_2 = 0$ if $\rho \neq 0$ and $X_0^3 + X_1^3 + X_2^3 = 0$ if $\rho = 0$, and let $o = (1 : -1 : 0)$, $o' = (1 : -\omega : 0)$, $o'' = (1 : -\omega^2 : 0)$. Then $\{o, o', o''\}$ is a subgroup of $F(\rho)$. We denote by $S(\rho)$ the CCI surface S_{α^*} for the pair $\alpha = (F(\rho), \{o, o', o''\})$. For a Hesse cubic curve $E(\mu)$, the isomorphism $v_\mu\colon E(\mu) \to F(\mu^3)$ is defined by $v_\mu((x_0 : x_1 : x_2)) = (x_0 : x_1 : \mu x_2)$ if $\mu \neq 0$ while it is defined to be the identity map if $\mu = 0$. The points o, o', o'' are fixed by v_μ in $\mathbf{P}^2_{\mathbf{C}}$, and hence $v_\mu(o) = P_{00}$, $v_\mu(o') = P_{10}$ and $v_\mu(o'') = P_{20}$.

Lemma 3.2. *For any pair (F, Λ) of an elliptic curve F and a subgroup Λ of order three, there exists a unique $\rho \in \mathbf{C} \setminus \{1\}$ with an isomorphism $(F, \Lambda) \simeq (F(\rho), \{o, o', o''\})$.*

Proof. Let e_1 be an element of $\Lambda \setminus \{0_F\}$. Since $\Lambda \subset \operatorname{Ker} 3_F \simeq (\mathbf{Z}/3\mathbf{Z})^2$, we can choose $e_2 \in \operatorname{Ker} 3_F \setminus \Lambda$ so that (e_1, e_2) is a level three structure of F. Since the Hesse family is the fine moduli, there exists a $\mu \in T$ and an isomorphism $u\colon F \to E(\mu)$ with $u(e_1) = P_{10}$ and $u(e_2) = P_{01}$. Hence the composite $v_\mu \cdot u$ is an isomorphism satisfying the condition for $\rho = \mu^3$.

Let us prove the uniqueness of ρ. Assume that there exists another isomorphism $w\colon (F, \Lambda) \simeq (F(\eta), \{o, o', o''\})$ for an element $\eta \in \mathbf{C} \setminus \{1\}$. By replacing w with the composite with the involution of F, if necessary, we may assume $w(e_1) = o'$. Let ν be a cubic root of η. If $v_\nu^{-1} \cdot w(e_2) = P_{01}$, then we have an isomorphism $v_\nu^{-1} \cdot w\colon (F, (e_1, e_2)) \simeq (E(\nu), (P_{10}, P_{01}))$, and we get $\nu = \mu$ and $\eta = \rho$ since T is the moduli. Since $v_\nu^{-1} \cdot w(e_1) = P_{10}$, there are two other possibilities of $v_\nu^{-1} \cdot w(e_2)$. Namely, these are P_{11} and P_{21}. We define an automorphism π of $\mathbf{P}^2_{\mathbf{C}}$ by $\pi((x_0 : x_1 : x_2)) = (x_0 : x_1 : \omega x_2)$. Then we have $\pi(E(\nu)) = E(\omega^2 \nu)$, $\pi(P_{10}) = P_{10}$ and $\pi(P_{11}) = P_{01}$. Hence $(E(\nu), (P_{10}, P_{11})) \simeq (E(\omega^2\nu), (P_{10}, P_{01}))$. Similarly, we have $(E(\nu), (P_{10}, P_{21})) \simeq (E(\omega\nu), (P_{10}, P_{01}))$ by π^2. Hence μ is equal to $\omega\nu$ or $\omega^2\nu$. In all cases, we have $\eta = \nu^3 = \mu^3 = \rho$. □

Theorem 3.3. *The correspondences $\rho \mapsto S(\rho)$ define a bijection from $\mathbf{C} \setminus \{1\}$ to the set of isomorphism classes of CCI surfaces.*

Proof. Since $S(\rho) = S_{\alpha^*}$ for $\alpha = (F(\rho), \{o, o', o''\})$, this is a consequence of Theorem 3.1 and Lemma 3.2. □

By this theorem, we can say that $\rho \in \mathbf{C} \setminus \{1\}$ is the moduli parameter of CCI surfaces.

Theorem 3.4. *The j-invariant of the base elliptic curve $E(S)$ of the CCI surface $S = S(\rho)$ is given by the rational function*

$$j(\rho) = \frac{27\rho(\rho+8)^3}{(\rho-1)^3}.$$

This function has ramifications of degree three at $\rho = -8$ and $\rho = 1$. We have $j(\rho) = 0$ for $\rho = -8$ and $\rho = 0$. It has ramifications of degree two at $\rho = 10 - 6\sqrt{3}$ and $\rho = 10 + 6\sqrt{3}$, and $j(\rho) = 1728$ for these ρ. This function defines a finite map from $\mathbf{C} \setminus \{1\}$ to $\mathbf{C}$.

Proof. The first part follows from (3.15) and the relation $\rho = \mu^3$. Set $f(\rho) = j(\rho)$. Then

$$f'(\rho) = \frac{27(\rho+8)^2(\rho^2 - 20\rho - 8)}{(\rho-1)^4}.$$

Hence $f'(\rho) = 0$ for $\rho = -8$ and $\rho = 10 \pm 6\sqrt{3} = (1 \pm \sqrt{3})^3$. Since $f(\rho) = \infty$ only for $\rho = 1$ and $\rho = \infty$, we get the last assertion. □

Lemma 3.5. *Let ρ be an element of $\mathbf{C}\setminus\{1\}$. Then a group automorphism $\varphi\colon F(\rho)\to F(\rho)$ with $\varphi(\{o', o''\}) = \{o', o''\}$ is the identity map or the involution if $\rho \neq 0$. If $\rho = 0$, then there exists a φ of order three with $\varphi(o') = o'$, and the automorphism group is of order six.*

Proof. $F(0)$ is the Fermat cubic curve and the automorphism π of $\mathbf{P}^2_{\mathbf{C}}$ defined by $\pi((x_0 : x_1 : x_2)) = (x_0 : x_1 : \omega x_2)$ induces $\varphi = \varphi_0$ of order three with $\varphi_0(o') = o'$. The group of automorphisms of $F(0)$ is generated by this φ_0 and the involution since the order of the automorphism group of an elliptic curve is at most six. Clearly, all members satisfy $\varphi(\{o', o''\}) = \{o', o''\}$. Suppose $\rho \neq 0$ and φ satisfies $\varphi(o') = o'$. Take a μ with $\mu^3 = \rho$. Then $P = (v_\mu^{-1}\cdot\varphi\cdot v_\mu)(P_{01})$ is P_{01}, P_{11} or P_{21} since $(v_\mu^{-1}\cdot\varphi\cdot v_\mu)(P_{10}) = P_{10}$ and (P_{10}, P) is a level three structure of $E(\mu)$. If it is P_{01}, then φ is the identity since the elliptic curve with level three structure $(E(\mu), (P_{10}, P_{01}))$ has no automorphism other than the identity. As we saw in the proof of Lemma 3.2, $(E(\mu), (P_{10}, P_{11})) \simeq (E(\omega^2\mu), (P_{10}, P_{01}))$ and $(E(\mu), (P_{10}, P_{21})) \simeq (E(\omega\mu), (P_{10}, P_{01}))$. Hence, the other two cases do not occur since T is the moduli. □

Proposition 3.6. *The order of the automorphism group of $S(\rho)$ is six for $\rho = 0$ and two for $\rho \neq 0$.*

Proof. We will show that an automorphism of $S(\rho)$ induces that of $(F(\rho), \Lambda)$ for $\Lambda = \{o, o', o''\}$ and that this correspondence is bijective. Then the proposition follows from Lemma 3.5.

Let $\Psi\colon S(\rho)\to S(\rho)$ be an automorphism. By the universality of the Albanese map $a\colon S(\rho)\to F(\rho)$, there exists an automorphism ψ of $F(\rho)$ with $\psi\cdot a = a\cdot\Psi$. Here, ψ is a group automorphism since $S(\rho)$ has the unique singular fiber over the unit of $F(\rho)$. Set $F(\rho)^* = F(\rho)/\Lambda$. We define $q\colon F(\rho)^* \to F(\rho)$ by $q(x \bmod \Lambda) = [3x]$. Then, $S(\rho) = S_{(F(\rho)^*, \mathrm{Ker}\, q)}$. Theorem 2.8 implies that q is the unique unramified covering of degree three such that the fiber product $S(\rho)\times_{F(\rho)} F(\rho)^*$ is defined by a Fermat type equation. Since $S(\rho)\times_{F(\rho)}(F(\rho), \psi) \simeq S(\rho)$, we have

$$S(\rho)\times_{F(\rho)}(F(\rho)^*, \psi\cdot q) \simeq (S(\rho)\times_{F(\rho)}(F(\rho), \psi))\times_{F(\rho)} F(\rho)^* \simeq S(\rho)\times_{F(\rho)} F(\rho)^*,$$

i.e., the left-hand side is also defined by a Fermat type equation. Hence q and the composite $\psi\cdot q$ are equivalent coverings of $F(\rho)$. Namely, there exists an automorphism $\bar\psi$ of $F(\rho)^*$ with $q\cdot\bar\psi = \psi\cdot q$. Since $\Lambda = q(\mathrm{Ker}\, 3_{F(\rho)^*})$, we have

$$\psi(\Lambda) = \psi(q(\mathrm{Ker}\, 3_{F(\rho)^*})) = q(\bar\psi(\mathrm{Ker}\, 3_{F(\rho)^*})) = \Lambda.$$

Let ψ be an automorphism of $(F(\rho), \Lambda)$. Since we know that the involution of $F(\rho)$ has a lifting to $S(\rho)$, we may assume that $\psi(o') = o'$ and $\psi(o'') = o''$ in order to find a lifting of ψ to $S(\rho)$ and to show the uniqueness. Since $\psi(\Lambda) = \Lambda$, ψ induces an isomorphism $\bar{\psi}\colon F(\rho)^* \to F(\rho)^*$. Since the determinant of the restriction of ψ to $\operatorname{Ker} 3_{F(\rho)} \simeq (\mathbf{Z}/3\mathbf{Z})^2$ is $1 \in \mathbf{Z}/3\mathbf{Z}$ (cf. [5, IV, Theorem 4]) and since ψ fixes each element of Λ, $\bar{\psi}$ fixes each element of $\Lambda^* = \operatorname{Ker} 3_{F(\rho)}/\Lambda$. Let $\Lambda^* = \{0_{F(\rho)^*} = P_0, P_1, P_2\}$ and $\alpha^* = (F(\rho)^*, \Lambda^*)$. For the locally free sheaf

$$\tilde{V} = \mathcal{O}_{F(\rho)^*}(P_0)Z_0 \oplus \mathcal{O}_{F(\rho)^*}(P_1)Z_1 \oplus \mathcal{O}_{F(\rho)^*}(P_2)Z_2,$$

we define an isomorphism $\Psi\colon \tilde{V} \to \bar{\psi}^*\tilde{V}$ by

$$\Psi(aZ_0 + bZ_1 + cZ_2) = a\bar{\psi}^*(Z_0) + b\bar{\psi}^*(Z_1) + c\bar{\psi}^*(Z_2).$$

Since $(\bar{\psi}^* f_0) = (f_0) = -2P_0 + P_1 + P_2$, there exists a nonzero constant c with $\bar{\psi}^* f_0 = cf_0$. Then we have $(\operatorname{Sym}^4 \Psi)(f_0Z_0^4 + f_1Z_1^4 + f_2Z_2^4) = c(f_0Z_0^4 + f_1Z_1^4 + f_2Z_2^4)$. Hence the automorphism of $\mathbf{P}(\tilde{V})$ induced by Ψ maps $\tilde{S}_{\alpha^*}$ to itself. By taking the quotient of $\tilde{S}_{\alpha^*}$ by Λ^*, we get an automorphism of $S(\rho) = S_{\alpha^*}$ which is a lifting of ψ. The lifting of ψ is unique since Z_0, Z_1, Z_2 are determined up to multiplication by a common nonzero constant. □

Theorem 3.7. *There exists a proper smooth family $\varphi\colon \mathcal{S} \to T$ of algebraic surfaces with the following properties.*
(1) For each $\mu \in T$, the fiber $\mathcal{S}_\mu = \varphi^{-1}(\mu)$ is the CCI surface which corresponds to the pair $(E(\mu), \{P_{00}, P_{10}, P_{20}\})$.
(2) For any CCI surface S, there exists $\mu \in T$ with $\mathcal{S}_\mu \simeq S$.
(3) $\mathcal{S}_\mu \simeq \mathcal{S}_\nu$ if and only if $\mu^3 = \nu^3$.

Proof. Let $\mathcal{E} \subset \mathbf{P}^2_{\mathbf{C}} \times T$ be the the family of cubic curves defined by the equation

$$X_0^3 + X_1^3 + X_2^3 - 3tX_0X_1X_2 = 0,$$

and let $D_0 = \{P_{00}\} \times T$, $D_1 = \{P_{10}\} \times T$ and $D_2 = \{P_{20}\} \times T$.

Let $p\colon \mathcal{E} \to T$ be the projection. Since $\mathcal{E}$ is a smooth family of elliptic curves and the natural isomorphism $\epsilon\colon T \to D_0$ is a section, $\mathcal{E}$ has a structure of an abelian scheme with the identity ϵ (cf. [4, Theorem 6.14]). Since the invertible sheaf $\mathcal{O}_{\mathcal{E}}(-2D_0 + D_1 + D_2)$ is trivial on each fiber, $p_*\mathcal{O}_{\mathcal{E}}(-2D_0 + D_1 + D_2)$ is an invertible sheaf on $T = \operatorname{Spec} \mathbf{C}[X][(X^3 - 1)^{-1}]$. Since $\mathbf{C}[X][(X^3 - 1)^{-1}]$ is a PID, this sheaf is generated by a section u_0 on T. Let $f_0 = p^*u_0$ and f_1 and f_2 the pullbacks of f_0 by translations by D_1 and D_2, respectively, on the abelian scheme $\mathcal{E}$.

Let $\mathcal{V} = \mathcal{O}_{\mathcal{E}}(D_0)Z_0 \oplus \mathcal{O}_{\mathcal{E}}(D_1)Z_1 \oplus \mathcal{O}_{\mathcal{E}}(D_2)Z_2$. Then the $\mathbf{P}^2_{\mathbf{C}}$-bundle $\mathbf{P}_{\mathcal{E}}(\mathcal{V})$ contains $U = (\mathcal{E} \setminus (D_0 \cup D_1 \cup D_2)) \times \mathbf{P}^2_{\mathbf{C}}$ as an open subscheme. We define $\tilde{\mathcal{S}}$ as the closure of the subvariety of U defined by

$$f_0Z_0^4 + f_1Z_1^4 + f_2Z_2^4 = 0.$$

Similarly to the construction of S_α, $\tilde{\mathcal{S}}$ has an action of a cyclic group G of order three induced by the translations by D_2 and D_1. The quotient $\mathcal{S} = \tilde{\mathcal{S}}/G$ is a family of CCI surfaces over T such that the fiber of ρ is equal to S_α for $\alpha = (E(\mu), \{P_{00}, P_{10}, P_{20}\})$. $\square$

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