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Osaka University
RANDOM WALKS AND KURAMOCHI BOUNDARIES
OF INFINITE NETWORKS

ATSUSHI KASUE

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Abstract

In this paper, we study a connected non-parabolic, or transient, network compactified with the Kuramochi boundary, and show that the random walk converges almost surely to a random variable valued in the harmonic boundary, and a function of finite Dirichlet energy converges along the random walk to a random variable almost surely and in $L^2$. We also give integral representations of solutions of Poisson equations on the Kuramochi compactification.

1. Introduction

Ancona, Lyons and Peres [1] showed that a function of finite Dirichlet energy on a transient network converges along the random walk almost surely and in $L^2$. In this paper, we concern the Kuramochi boundary of the network and proves that the random walk converges almost surely to a random variable valued in the harmonic boundary, and a function of finite Dirichlet energy converges along the random walk to a random variable almost surely and in $L^2$.

Let $G = (V, E)$ be a graph with the set of vertices $V$ and the set of edges $E$ that consists of pairs of vertices. In this paper, a graph admits no loops and multiple edges, and the set of vertices is finite or countably infinite. We say that a vertex $x$ is adjacent to another $y$ if $\{x, y\}$ belongs to $E$ and write $x \sim y$ to indicate it. We also write $[xy]$ for $\{x, y\}$. By a path in $G$, we mean a sequence of vertices $c = (x_0, x_1, \ldots, x_n)$ such that $x_i \sim x_{i+1}$ ($i = 0, 1, \ldots, n - 1$), and we say that $c$ connects $x_0$ to $x_n$. $G$ is called a connected graph if for any pair of vertices $x$ and $y$, there exist paths connecting them.

We are now given an admissible weight $r$ on the set of edges $E$, that is a positive function on $E$ with the property that

$$c(x) = \sum_{y \sim x} \frac{1}{r([xy])} < +\infty, \forall x \in V.$$  

An admissible weight $r$ gives rise to a distance $d_r$ on $V$, called the geodesic distance of $\Gamma$, by taking $r(e)$ as the length of an edge $e$ and by assigning to each pair

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of vertices \( x \) and \( y \) the infimum of the length of paths connecting them. In this paper, we call such a couple of a graph and an admissible weight a network.

Given a connected network \( \Gamma = (V, E, r) \), a nonnegative quadratic form \( (\mathcal{E}_\Gamma, D[\mathcal{E}_\Gamma]) \) on the space \( l(V) \) of functions on \( V \) can be defined as follows:

\[
D[\mathcal{E}_\Gamma] = \left\{ u \in l(V) \left| \sum_{x \sim y} \frac{|u(x) - u(y)|^2}{r(|xy|)} < +\infty \right. \right\};
\]

\[
\mathcal{E}_\Gamma(u, v) = \frac{1}{2} \sum_{x \sim y} (u(x) - u(y))(v(x) - v(y))/r(|xy|), \quad u, v \in D[\mathcal{E}_\Gamma].
\]

The domain \( D[\mathcal{E}_\Gamma] \) endowed with an inner product \( \mathcal{E}_\Gamma(u, v) + u(o)v(o) \), where \( o \) is a fixed point of \( V \), becomes a Hilbert space.

Let \( D_0[\mathcal{E}_\Gamma] \) be the closure of the set of finitely supported functions on \( V \) in \( D[\mathcal{E}_\Gamma] \). We say that \( \Gamma \) is non-parabolic if

\[
\sup \left\{ \frac{|u(x)|^2}{\mathcal{E}_\Gamma(u, u)} \left| u \in D_0[\mathcal{E}_\Gamma], \mathcal{E}_\Gamma(u, u) > 0 \right. \right\} < +\infty
\]

for some \( x \in V \). We recall here the fact that the following conditions are mutually equivalent:

(i) \( \Gamma \) is non-parabolic,
(ii) \( D_0[\mathcal{E}_\Gamma] \) contains no constant functions,
(iii) \( D_0[\mathcal{E}_\Gamma] \neq D[\mathcal{E}_\Gamma] \) (see [14]).

If these are the cases, \( D[\mathcal{E}_\Gamma] \) is decomposed into the direct sum of \( D_0[\mathcal{E}_\Gamma] \) and the space \( H_{\mathcal{E}_\Gamma} \) of harmonic functions of finite Dirichlet sums on \( V \) that is the orthogonal complement of \( D_0[\mathcal{E}_\Gamma] \) relative to the form; a function \( h \) on \( V \) belongs to \( H_{\mathcal{E}_\Gamma} \) if and only if \( h \in D[\mathcal{E}_\Gamma] \) and \( L^r h(x) := \sum_{y \sim x} (h(x) - h(y))/r(|xy|) = 0 \) for all \( x \in V \).

Let \( \{p(x, y) \mid x, y \in V\} \) be transition probabilities on \( V \) defined by

\[
p(x, y) = \frac{c(|xy|)}{c(x)}, \quad x, y \in V,
\]

where \( c(|xy|) = r(|xy|)^{-1} \) and \( c(x) = \sum_{y \sim x} c(|xy|) \). It is well known that \( \Gamma \) is non-parabolic if and only if the (reversible) Markov chain is transient.

Ancona, Lyons and Peres [1] proved the following

**Theorem 1.** Let \( \Gamma = (V, E, r) \) be a connected non-parabolic network and \( \{X_n\} \) the Markov chain. Then for any \( u \in D[\mathcal{E}_\Gamma] \), the sequence \( \{u(X_n)\} \) converges almost surely and in \( L^2 \). If \( u = h + g \), where \( h \in H_{\mathcal{E}_\Gamma} \) and \( g \in D_0[\mathcal{E}_\Gamma] \), is the Royden decomposition of \( u \), then \( \lim_{n \to \infty} u(X_n) = \lim_{n \to \infty} h(X_n) \) almost surely.

To state our main results, we introduce the Kuramochi compactification of a connected infinite network \( \Gamma = (V, E, r) \).
A compactification of any (discrete) set $X$ is a compact Hausdorff space which contains $X$ as a dense subset and which induces the discrete topology on $X$. It is known that given a family $\Phi$ of bounded functions on $X$, there exists an (up to canonical homeomorphisms) unique compactification $C(X, \Phi)$ of $X$ with the following properties (see e.g. [2]):

(i) every function of $\Phi$ extends to a continuous function on $C(X, \Phi)$, and

(ii) the extended functions separate the points of the boundary $\partial C(X, \Phi) = C(X, \Phi) \setminus V$.

We remark that if $\Psi$ is a subfamily of $\Phi$, then the identity map extends to a continuous map from $C(X, \Phi)$ onto $C(X, \Psi)$, and if $\Phi_0$ is a subfamily of $\Phi$ and each function of $\Phi$ is a finite linear combination of functions in $\Phi_0$, then $C(X, \Phi)$ and $C(X, \Phi_0)$ are canonically homeomorphic; in particular, if in addition, $X$ and $\Phi_0$ is countable, then $C(X, \Phi)$ is metrizable.

The compactification relative to the space of bounded functions in $D[\mathcal{E}_\Gamma]$, $BD[\mathcal{E}_\Gamma]$, is called the Royden compactification of the network $\Gamma$ and denoted by $\mathcal{R}(\mathcal{E}_\Gamma)$. The boundary $\partial \mathcal{R}(\mathcal{E}_\Gamma)$ is called the Royden boundary of $\Gamma$. There is an important part of the Royden boundary referred to as the harmonic boundary of $\Gamma$ which is defined by $\Delta(\mathcal{E}_\Gamma) = \{ x \in \partial \mathcal{R}(\mathcal{E}_\Gamma) \mid g(x) = 0 \text{ for all } g \in BD_0[\mathcal{E}_\Gamma] \}$. It is known (see [15], [6], [11, Chapter VI]) that $\Gamma$ is non-parabolic if and only if the harmonic boundary is not empty, and also that if $\partial \mathcal{R}(\mathcal{E}_\Gamma) \setminus \Delta(\mathcal{E}_\Gamma)$ is not empty, then any set of a single point there is not a $G_\delta$ set and for a nonempty closed subset $F$ in $\partial \mathcal{R}(\mathcal{E}_\Gamma) \setminus \Delta(\mathcal{E}_\Gamma)$, there exists a function $g \in D_0[\mathcal{E}_\Gamma]$ such that $g(x)$ tends to infinity as $x \in V \to F$.

We recall a basic fact concerning Dirichlet problems on the Royden boundary $\partial \mathcal{R}(\mathcal{E}_\Gamma)$ (see [11, Chapter VI]): for any continuous function $f$ on $\partial \mathcal{R}(\mathcal{E}_\Gamma)$, there exists a unique harmonic function $H_f$ on $\Gamma$ such that for any $\xi \in \Delta(\mathcal{E}_\Gamma)$, $\lim_{x \to \xi} H_f(x) = f(\xi)$, and $\sup_{V \to \xi} |H_f| \leq \max_{\Delta(\mathcal{E}_\Gamma)} |f|$. Given a point $a \in V$, letting $\widetilde{v}_a(f) = H_f(a)$ for $f \in C(\partial \mathcal{R}(\mathcal{E}_\Gamma))$, we have a Radon measure $\widetilde{v}_a$ on $\partial \mathcal{R}(\mathcal{E}_\Gamma)$, called the harmonic measure with respect to the point $a$. In view of Harnack’s inequality, $\widetilde{v}_a$ and $\widetilde{v}_b$ are mutually absolutely continuous for any pair of points $a, b \in V$, and the harmonic measures are supported on the harmonic boundary.

Now we consider a subspace $Q(\mathcal{E}_\Gamma)$ of $BD[\mathcal{E}_\Gamma]$ which consists of functions $u$ such that $\mathcal{E}_\Gamma(u, v) = 0$ for all $v \in D[\mathcal{E}_\Gamma]$ vanishing on a finite subset of $V$. The compactification relative to $Q(\mathcal{E}_\Gamma)$ is called the Kuramochi compactification of the network $\Gamma$ and denoted by $K(\mathcal{E}_\Gamma)$ (see [9]). The identity map of $V$ extends to a continuous map from $\mathcal{R}(\mathcal{E}_\Gamma)$ onto $K(\mathcal{E}_\Gamma)$. We denote by $\rho_\Gamma$ the induced map from the Royden boundary $\partial \mathcal{R}(\mathcal{E}_\Gamma)$ onto the Kuramochi boundary $\partial K(\mathcal{E}_\Gamma)$. Let $\Delta^K(\mathcal{E}_\Gamma) = \rho_\Gamma(\Delta(\mathcal{E}_\Gamma))$ and $v_a = \rho_\Gamma \circ \widetilde{v}_a (a \in V)$. Here and after, we fix a point $o \in V$ and write $v$ for $v_o$.

We will prove that the Kuramochi compactification $K(\mathcal{E}_\Gamma)$ of a connected, non-parabolic network $\Gamma$ admits a compatible metric $d^E$ such that for each Lipschitz function $f : (K(\mathcal{E}_\Gamma), d^E) \to \mathbb{R}$, the sequence $\{ f(X_n) \}$ is almost surely convergent. This shows that the Markov chain $\{ X_n \}$ converges to a random variable $X_\infty$ in $K(\mathcal{E}_\Gamma)$. In fact, a result by Ancona, Lyons and Peres [1] states that if $\mathcal{M}$ is a complete separable metric space and $\{ Y_n \}$ is a process such that for each bounded Lipschitz function
\( f : \mathcal{M} \to \mathbb{R} \), the sequence \( \{ f(Y_n) \} \) is almost surely convergent, then the process \( \{ Y_n \} \)
in \( \mathcal{M} \) is already almost surely convergent.

We will now take an appropriate measure on \( V \). Given two vertices \( x \) and \( y \) of \( \Gamma \), we define a nonnegative number \( R_{\mathcal{E}_r}(x, y) \), called the effective resistance between \( x \) and \( y \), by

\[
R_{\mathcal{E}_r}(x, y) = \sup \left\{ \frac{|u(x) - u(y)|^2}{\mathcal{E}_r(u, u)} \mid u \in D[\mathcal{E}_r], \mathcal{E}_r(u, u) > 0 \right\}.
\]

It is known that \( R_{\mathcal{E}_r}(x, y) \leq d_r(x, y) \) for all \( x, y \in V \) and \( R_{\mathcal{E}_r} \) induces a distance on \( V \) (see e.g., [5]). Choose a measure \( \mu \) on \( V \) in such a way that \( \mu(V) = \sum_{x \in V} \mu(x) = 1 \) and

\[
\int_V R_{\mathcal{E}_r}(o, x)^2 d\mu(x) \left( = \sum_{x \in V} R_{\mathcal{E}_r}(o, x)^2 \mu(x) \right) < +\infty.
\]

Under the condition, it is proved in [5] that \( D[\mathcal{E}_r] \subset L^2(V, \mu) \), the embedding is compact, \( (\mathcal{E}_r, D[\mathcal{E}_r]) \) is a regular Dirichlet form in \( L^2(K(\mathcal{E}_r), \mu) \), and the Royden decomposition is stated in such a way that a function \( u \in D[\mathcal{E}_r] \) is expressed as

\[
u(x) = \int_{\partial K(\mathcal{E}_r)} \tau(u) d\nu_x + g(x), \quad x \in V, \quad g \in D_0[\mathcal{E}_r],
\]

where \( \tau(u) \) is a function in \( L^2(\partial K(\mathcal{E}_r), \nu) = L^2(\Delta^K(\mathcal{E}_r), \nu) \). We define a Radon measure \( \tilde{\mu} \) on the Kuramochi compactification \( K(\mathcal{E}_r) \) by

\[
\tilde{\mu}(f) = \int_V f \, d\mu + \int_{\partial K(\mathcal{E}_r)} f \, d\nu
\]

for \( f \in C(K(\mathcal{E}_r)) \). Then any function \( u \) of \( D[\mathcal{E}_r] \) coupled with \( \tau(u) \) can be considered as a function in \( L^2(K(\mathcal{E}_r), \tilde{\mu}) \).

Our main results are stated in the following

**Theorem 2.** Let \( \Gamma = (V, E, r) \) be a connected non-parabolic network. Then the following assertions hold:

(i) \( (\mathcal{E}_r, D[\mathcal{E}_r]) \) is a regular Dirichlet form on \( L^2(K(\mathcal{E}_r), \tilde{\mu}) \).

(ii) There exists a \( \Delta^K(\mathcal{E}_r) \)-valued random variable \( X_\infty \) such that in the \( K(\mathcal{E}_r) \)-topology, the Markov chain \( X_n \) almost surely converges to \( X_\infty \) as \( n \to \infty \), the measure \( \nu_{X_n} \) converges weakly to the delta measure \( \delta_{X_\infty} \) almost surely as \( n \to \infty \), and for any \( u \in D[\mathcal{E}_r] \), \( u(X_n) \) converges to \( \tau(u)(X_\infty) \) almost surely and in \( L^2 \) as \( n \to \infty \).

(iii) Let \( (L^2_r, D[L^2_r]) \) be the self-adjoint operator associated with the regular Dirichlet form \( (\mathcal{E}_r, D[\mathcal{E}_r]) \). For a function \( f \in L^2(K(\mathcal{E}_r), \tilde{\mu}) \), there exists a solution \( u \), unique up to additive constants, of equation: \( L^2_r u = f \) if and only if \( \tilde{\mu}(f) = 0 \); in particular, the solution is harmonic on \( V \) if \( f \) vanishes there.
We briefly explain the contents of the paper. In section 1, we introduce a resistance form of a connected non-parabolic network and its Kuramochi compactification, and prove Theorem 2 (i) for a resistance form. In Section 2, Theorem 2 (ii) for a resistance form is discussed. The last section is devoted to investigating Poisson equations on the Kuramochi compactification of a resistance form.

2. Resistance forms

In this section, we introduce the Kuramochi compactification of a resistance form of a connected non-parabolic network and prove Theorem 2 (i) for a resistance form.

Let $\Gamma = (V, E, \tau)$ be a connected non-parabolic network. A nonnegative quadratic form $E$ on a subspace $D[E]$ of $D[E^\Gamma]$ is called a resistance form of the network $\Gamma$ if it satisfies the following properties:

(i) $D_0[E^\Gamma] + R \subset D[E] \subset D[E^\Gamma]$,
(ii) $E(1, 1) = 0$,
(iii) $E^\Gamma(u, u) \leq E(u, u)$ for all $u \in D[E]$ and $E(u, v) = E^\Gamma(u, v)$ for all $u \in D[E]$ and $v \in D_0[E^\Gamma]$,
(iv) for $u \in D[E]$, $\bar{u} = \max\{0, \min\{1, u\}\}$ belongs to $D[E]$ and $E(\bar{u}, \bar{u}) \leq E(u, u)$,
(v) $D[E]$ becomes a Hilbert space with inner product $(u, v) = E(u, v) + u(o)v(o)$, where $o$ is a fixed vertex of $V$. When we restrict $E^\Gamma$ to $D_0[E^\Gamma] + R$, we have the minimal resistance form denoted by $(E^\Gamma_0, D_0[E^\Gamma] + R)$.

For any pair of vertexes $x, y$, we have a nonnegative number $R_{E}(x, y)$, called the effective resistance relative to $E$ between $x$ and $y$, defined by

$$R_{E}(x, y) = \sup \left\{ \frac{u(x) - u(y)}{E(u, u)} \mid u \in D[E], E(u, u) > 0 \right\}.$$ 

Then it follows from the definitions above that

$$R_{E^\Gamma_0}(x, y) \leq R_{E}(x, y) \leq R_{E^\Gamma}(x, y), \quad x, y \in V.$$ 

We remark that $R_{E^\Gamma}(x, y) \leq d_r(x, y)$ for $x, y \in V$, and $R_{E}$ induces a distance on $V$ (see e.g., [5, Theorem 1.12, Proposition 2.6]). We write $H_{E}$ for the space of functions $u \in D[E]$ which are harmonic on $V$, i.e.,

$$L_{E}u(x) := \sum_{y \sim x} \frac{u(x) - u(y)}{r(|xy|)} = 0, \quad \forall x \in V.$$ 

Given $x, z \in V$, there exist functions $g_{x,z} \in D[E]$ and $h_{x,z} \in H_{E}$ respectively satisfying $E(g_{x,z}, u) = u(x) - u(z)$ for all $u \in D[E]$ and $E(h_{x,z}, h) = h(x) - h(z)$ for all $h \in H_{E}$. We write $g_{x,z}^E(x, y)$ and $h_{x,z}^E(x, y)$ respectively for $g_{x,z}(y)$ and $h_{x,z}(y)$. It is easy to see that $g_{x,z}^E(x, y) = g_{x,z}^E(y, x)$ and $h_{x,z}^E(x, y) = h_{x,z}^E(y, x)$. We notice that $R_{E}(x, y) = g_{x}^E(y, y) = g_{x}^E(y, z) + g_{x}^E(x, z)$ and $g_{x}^E(x, y) = (1/2)\{R_{E}(x, z) + R_{E}(z, y) - R_{E}(x, y)\}$ for all

$$R_{E}(x, y) = g_{x}^E(y, y) = g_{x}^E(y, z) + g_{x}^E(x, z)$$

and

$$g_{x}^E(x, y) = (1/2)\{R_{E}(x, z) + R_{E}(z, y) - R_{E}(x, y)\}$$
x, y, z ∈ V, and also that \( h^E_t(x, x) = \sup\{[h(x) - h(z)]^2/\mathcal{E}(h, h) \mid h ∈ H_\mathcal{E}, \mathcal{E}(h, h) > 0\} \) (see [5, 7.2]). Since \( \Gamma \) is assumed to be non-parabolic, given a vertex \( x ∈ V \), there exists uniquely a function \( g_x \in D_0[\mathcal{E}_\Gamma] \) such that \( \mathcal{E}_\Gamma(g_x, v) = \nu(x) \) for all \( v ∈ D_0[\mathcal{E}_\Gamma] \). We write \( g^0(x, y) \) for \( g_x(y) \). It holds also that \( g^0_\Gamma(x, y) = g^0_\Gamma(y, x) \). These functions are related as follows:

\[ 1 \quad g^E_\Gamma(x, y) = h^E_\Gamma(x, y) + (g^0_\Gamma(x, y) - g^0_\Gamma(x, z) - g^0_\Gamma(y, z) + g^0_\Gamma(z, z)), \quad x, y, z ∈ V \]

(see [5, 7.2]).

Now as in the case of the form \( \mathcal{E}_\Gamma \), we consider a subspace \( Q(\mathcal{E}) \) of \( D[\mathcal{E}] \) which consists of functions \( u \) such that \( \mathcal{E}(u, v) = 0 \) for all \( v ∈ D[\mathcal{E}] \) vanishing on a finite subset of \( V \). The compactification relative to \( Q(\mathcal{E}) \) is called the Kuramochi compactification of the network \( \Gamma \) relative to the resistance form \( \mathcal{E} \), and denoted by \( K(\mathcal{E}) \). The identity map of \( V \) extends to a continuous map from the Royden compactification \( R(\mathcal{E}_\Gamma) \) of \( \Gamma \) onto \( K(\mathcal{E}) \). We denote by \( \rho_\mathcal{E} \) the induced map from the Royden boundary \( \partial R(\mathcal{E}_\Gamma) \) onto the Kuramochi boundary \( \partial K(\mathcal{E}) \). Let \( \mathcal{E}_\mathcal{K}(\mathcal{E}) = \rho_\mathcal{E}(\mathcal{E}_\Gamma) \) and \( \nu_a = \rho_\mathcal{E}_a \bar{\nu}_a \) \((a ∈ V)\). Here and after, we fix a point \( o ∈ V \) and write \( \nu \) for \( \nu_o \).

We take a positive function \( \mu \) on \( V \) and consider it as a measure on \( V \), \( \mu = \sum_{s ∈ V} \mu(x)δ_s \). In what follows, \( \mu \) is chosen in such a way that \( \mu(V) = 1 \),

\[ 2 \quad \int_V R_\mathcal{E}(o, x)^2 d\mu(x) < +∞. \]

The measure \( \mu \) extends to a Radon measure, denoted by the same letter, on the Kuramochi compactification \( K(\mathcal{E}) \). Here we recall some results in [5, 7.3]:

(i) \( \mathcal{E}[\mathcal{E}] \subset L^2(\mathcal{K}(\mathcal{E}), \mu) \).

(ii) Any function \( u ∈ D[\mathcal{E}] \) can be written in the Royden decomposition as

\[ u(x) = \int_{\partial \mathcal{K}(\mathcal{E})} \tau(u) \, d\nu \, g(x), \quad x ∈ V, \, g ∈ D_0[\mathcal{E}], \]

where \( \tau(u) \) is a function in \( L^2(\partial \mathcal{K}(\mathcal{E}), v) \).

(iii) \( (\mathcal{E}, D[\mathcal{E}]) \) is a regular Dirichlet form on \( L^2(\mathcal{K}(\mathcal{E}), \mu) \).

(iv) The domain \( D[L^\mathcal{E}] \) of the self-adjoint operator \( L^\mathcal{E} \) associated to the Dirichlet form \( \mathcal{E} \) is embedded in the space of continuous functions on \( \mathcal{K}(\mathcal{E}) \), and \( D[L^\mathcal{E}] \) is dense both in the Banach space \( C(\mathcal{K}(\mathcal{E})) \) of continuous functions on \( \mathcal{K}(\mathcal{E}) \) and the Hilbert space \( (D[\mathcal{E}], \mathcal{E} + \delta^2_o) \).

(v) The domain \( D[\mathcal{E}] \) is compactly embedded into \( L^2(\mathcal{K}(\mathcal{E}), \mu) \).

Now we define a Radon measure \( \bar{\mu} \) on the Kuramochi compactification \( \mathcal{K}(\mathcal{E}) \) by

\[ \bar{\mu}(f) = \int_V f \, d\mu + \int_{\partial \mathcal{K}(\mathcal{E})} f \, d\nu, \quad f ∈ C(\mathcal{K}(\mathcal{E})). \]

Then any function \( u \) of \( D[\mathcal{E}] \) coupled with \( \tau(u) \) can be considered as a function in \( L^2(\mathcal{K}(\mathcal{E}), \bar{\mu}) \).
Given \( u \in D[\mathcal{E}] \), if we write \( h \) for the harmonic part \( \int_{\partial \mathcal{K}(\mathcal{E})} \tau(u) d\nu \) of \( u \), then we have the following basic identity:

\[
(3) \quad \int_{\partial \mathcal{K}(\mathcal{E})} \tau(u)^2 d\nu - \left( \int_{\partial \mathcal{K}(\mathcal{E})} \tau(u) d\nu \right)^2 = \sum_{y \in V} g^0_T(x, y) \sum_{z \sim y} \frac{(h(y) - h(z))^2}{r(|yz|)}, \quad x \in V,
\]

from which we can deduce that

\[
(4) \quad \int_{\partial \mathcal{K}(\mathcal{E})} \tau(u)^2 d\nu \leq 2g^0_T(x, x)\mathcal{E}(u, u) + 2u(x)^2, \quad x \in V
\]

(see [5, Lemma 7.8]). Using this inequality, we get

\[
\int_{\partial \mathcal{K}(\mathcal{E})} \tau(u)^2 d\nu \leq 2\left( g^0_T(x, x) + \frac{1}{\mu(x)} \right) \left( \mathcal{E}(u, u) + \int_{\nu} u^2 d\mu \right),
\]

since \( \mu(x)u(x)^2 \leq \int_{\nu} u^2 d\mu \). This shows in particular that the norm \( \mathcal{E}(u, u)^{1/2} + \left( \int_{\nu} u^2 d\mu \right)^{1/2} \) is equivalent to the norm \( \mathcal{E}(u, u)^{1/2} + \left( \int_{\nu} u^2 d\mu \right)^{1/2} \).

Since \( (\mathcal{E}, D[\mathcal{E}]) \) is a regular Dirichlet form on \( L^2(\mathcal{K}(\mathcal{E}), \mu) \), we can thus deduce the following

**Theorem 3.** Let \( \Gamma = (V, E, r) \) be a connected non-parabolic network and \( \mathcal{E} \) a resistance form of \( \Gamma \). Then the Dirichlet form \( (\mathcal{E}, D[\mathcal{E}]) \) on \( L^2(\mathcal{K}(\mathcal{E}), \mu) \) is regular.

Let \((\tilde{\mathcal{E}}, D[\tilde{\mathcal{E}}])\) be the self-adjoint operator associated with the regular Dirichlet form \( \mathcal{E} \) in \( L^2(\mathcal{K}(\mathcal{E}), \tilde{\mu}) \). For \( u \in D[\tilde{\mathcal{E}}] \), we note that

\[
\tilde{\mathcal{E}} u(x) = \frac{1}{\mu(x)} L^\mathcal{E} u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} \frac{u(x) - u(y)}{r(|xy|)}, \quad x \in V.
\]

The restriction of \( \tilde{\mathcal{E}} u \) to the Kuramochi boundary is denoted by \( N^\mathcal{E} u \). Then we have

\[
\mathcal{E}(u, v) = \int_{\nu} v L^\mathcal{E} u d\mu + \frac{1}{\partial \mathcal{K}(\mathcal{E})} \tau(v) N^\mathcal{E} u d\nu, \quad v \in D[\mathcal{E}].
\]

It is a consequence from the definitions of \( L^\mathcal{E} \) and \( \tilde{\mathcal{E}} \) that

\[
D[L^\mathcal{E}] = \{ u \in D[\tilde{\mathcal{E}}] \mid N^\mathcal{E} u = 0 \text{ in } L^2(\partial \mathcal{K}(\mathcal{E}), \nu) \} \subseteq C(\mathcal{K}(\mathcal{E})).
\]

We remark that \( Q(\mathcal{E}) \) is a subspace of \( D[L^\mathcal{E}] \). In fact, let \( u \) be a function in \( Q(\mathcal{E}) \). Then there exists a finite subset \( A \) of \( V \) such that \( \mathcal{E}(u, v) = 0 \) for all \( v \in D[\mathcal{E}] \) which
vanishes on $A$. Let $\chi_A$ be the characteristic function of $A$. Then for any $v \in D[\mathcal{E}]$, we have
\[
\mathcal{E}(u, v) = \mathcal{E}(u, \chi_A v) = \sum_{x \in A} v(x)L^\mathcal{E} u(x)
\]
and hence we get
\[
|\mathcal{E}(u, v)| \leq \left( \int_A (L^\mathcal{E} u)^2 \, d\mu \right)^{1/2} \left( \int_V v^2 \, d\mu \right)^{1/2}.
\]
This shows that $u \in D[L^\mathcal{E}]$.

Here, referring to [5, Proposition 4.1, Theorem 7.11], we mention the following propositions:

(I) The following conditions are mutually equivalent:
   (i) $\sup_{x \in V} g^0_t(x, x)$ is finite.
   (ii) Any $g \in D_0[\mathcal{E}_t]$ is bounded.
   (iii) $\partial R(\mathcal{E}_t) = \Delta(\mathcal{E}_t)$, that is, for any bounded $g \in D_0[\mathcal{E}_t]$, $g(x)$ tends to zero as $x \in V \to \infty$.
   (iv) For any $g \in D_0[\mathcal{E}_t]$, $g(x)$ tends to zero as $x \in V \to \infty$.

(II) $\sup_{x, y \in V} R_{\mathcal{E}}(x, y)$ is bounded if and only if every $f \in D[\mathcal{E}]$ is bounded.

(III) The following conditions are mutually equivalent:
   (i) $\sup_{x, y \in V} h^\mathcal{E}_t(y, y)$ is finite.
   (ii) Any $h \in H^\mathcal{E}$ is bounded.
   (iii) For any $u \in D[\mathcal{E}]$, $\tau(u)$ is continuous on $\Delta^K(\mathcal{E})$.
   (iv) A nonnegative subharmonic function $u$ in $D[\mathcal{E}]$ is bounded.

Now we prove the following

**Theorem 4.** Let $\Gamma = (V, E, r)$ be a connected non-parabolic network and $\mathcal{E}$ a resistance form of $\Gamma$. Then $D[\mathcal{E}]$ is compactly embedded in $L^2(\mathcal{K}(\mathcal{E}), \mu)$ if $\sup_{x \in V} g^0_t(x, x)$ is finite.

Proof. Let $\{u_n\}$ be a sequence in $D[\mathcal{E}]$ such that $\mathcal{E}(u_n, u_n) + u_n(o)^2$ is bounded as $n \to \infty$. Let $h_n$ be the harmonic part of $u_n$. Then we have
\[
\begin{align*}
u_n(o) - h_n(o) &= \mathcal{E}_r(g^0_t(o, *), u_n - h_n) \\
&= \mathcal{E}_r(g^0_t(o, *), u_n)
\end{align*}
\]
and hence
\[
h_n(o)^2 \leq 2u_n(o)^2 + 2g^0_t(o, o)\mathcal{E}(u_n, u_n).
\]
Thus we see that $\mathcal{E}(h_n, h_n) + h_n(o)^2$ are bounded as $n \to \infty$. Since $D[\mathcal{E}]$ is compactly embedded in $L^2(\mathcal{K}(\mathcal{E}), \mu)$, passing to a subsequence, we may assume that $u_n$ and $h_n$
respectively converge to functions \( u \) and \( h \) in \( L^2(K(E), \mu) \), where \( h \) is the harmonic part of \( u \). Let \( v_n = u_n - u \) and \( k_n = h_n - h \). Then in view of (3), we have

\[
\int_{\mathcal{K}(E)} \tau(v_n) \, dv = k_n(o)^2 + \sum_{x \in V} g^0_\Gamma(o, x) \sum_{y \sim x} \frac{(k_n(x) - k_n(y))^2}{r(|xy|)}.
\]

Given \( \varepsilon > 0 \), let \( V_\varepsilon = \{ x \in V \mid g^0_\Gamma(o, x) \geq \varepsilon \} \). Since \( g^0_\Gamma(o, x) \) tends to 0 as \( x \in V \) goes to infinity by the assumption: \( \sup_{x \in V} g^0_\Gamma(x, x) < +\infty \), \( V_\varepsilon \) is a finite subset of \( V \). Therefore for sufficiently large \( n \),

\[
k_n(o)^2 + \sum_{x \in V_\varepsilon} g^0_\Gamma(o, x) \sum_{y \sim x} \frac{(k_n(x) - k_n(y))^2}{r(|xy|)} < \varepsilon.
\]

Since

\[
\sum_{x \in V \setminus V_\varepsilon} g^0_\Gamma(o, x) \sum_{y \sim x} \frac{(k_n(x) - k_n(y))^2}{r(|xy|)} < \varepsilon \mathcal{E}_\Gamma(v_n, v_n),
\]

we get

\[
\int_{\mathcal{K}(E)} \tau(v_n) \, dv < \varepsilon \left( 1 + \sup_n \mathcal{E}_\Gamma(v_n, v_n) \right)
\]

for \( n \) large enough. This shows that \( \int_{\mathcal{K}(E)} \tau(v_n) \, dv \) tends to 0 as \( n \to \infty \). Thus we can deduce that \( D[E] \) is compactly embedded in \( L^2(K(E), \tilde{\mu}) \).

**Remark.** Let \( \Gamma = (V, E, r) \) be a connected infinite network and \( E \) a resistance form of \( \Gamma \).

(i) Let \( D[E^*] = \{ \tau(u) \mid u \in D[E] \} \subset L^2(\mathcal{K}(E), \nu) \) and \( E^*(\tau(u), \tau(v)) = \mathcal{E}(h_u, h_v) \) for \( u, v \in D[E] \), where \( h_u \) denotes the harmonic part of \( u \) in the Royden decomposition. Then \( (E^*, D[E^*]) \) is a regular Dirichlet form on \( L^2(\mathcal{K}(E), \nu) \).

(ii) Let \( (\mathcal{F}, D[\mathcal{F}]) \) be a Dirichlet form on a closed subspace of \( L^2(\mathcal{K}(E), \nu) \) with \( \mathcal{F}(1, 1) = 0 \), and define a form \( (\mathcal{E}_\mathcal{F}, D[\mathcal{E}_\mathcal{F}]) \) by

\[
\mathcal{E}_\mathcal{F}(u, v) = \mathcal{E}(u, v) + \mathcal{F}(\tau(u), \tau(v)); \quad D[\mathcal{E}_\mathcal{F}] = \{ u \in D[E] \mid \tau(u) \in D[\mathcal{F}] \}.
\]

Then \( \mathcal{E}_\mathcal{F} \) is a resistance form of \( \Gamma \). Moreover for a positive number \( t \), we set \( \mathcal{E}_{\mathcal{F}, t}(u, v) = \mathcal{E}(u, v) + t \mathcal{F}(\tau(u), \tau(v)) \). Then the limit of the forms as \( t \to +\infty \) also gives a resistance form of \( \Gamma \).

(iii) Given a finite subset \( K \) of \( V \), we can define a Dirichlet form on the space \( l(K) \) of functions on \( K \) by letting \( \mathcal{E}_K^*(u, u) = \inf \{ \mathcal{E}(\tilde{u}, \tilde{u}) \mid \tilde{u} \in D[E], \tilde{u} = u \text{ on } K \} \) for \( u \in l(K) \). Then we get a finite connected network \( \Gamma^*_K = (K, E_K, r_K) \) such that the effective resistance of \( \Gamma^*_K \) between two points of \( K \) is equal to the effective resistance
relative to $E$ (cf. [7, Theorem 2.1.12, Corollary 2.1.13], [5, Theorem 1.13]). Thus if we take an increasing sequence $\{V_n\}$ of finite subsets of $V$ such that $V = \bigcup_n V_n$, then $\Gamma$ endowed with the resistance form $E$ can be considered as a limit of finite networks $\{\Gamma_n\}$ (see [5]). Conversely if we have a sequence $\{\Gamma'_n\}$ of finite networks such that the set of vertices of $\Gamma'_n$ includes the vertex boundary of $V_n$, namely the set of vertexes of $V_n$ which are adjacent to those outside of $V_n$, we get a sequence $\{\Gamma''_n\}$ of finite networks obtained by joining the subnetwork $\Gamma_n$ of $\Gamma$ generated by $V_n$ with $\Gamma''_n$ through the vertex boundary of $V_n$. Since the effective resistance of $\Gamma''_n$ between two points of $V_n$ is bounded by the effective resistance of $E_\Gamma$ between them, by taking subsequence if necessarily, we have a resistance form $E$ of $\Gamma$ such that for any pair of points of $V$, the effective resistance of $\Gamma''_n$ between them (for large $n$) converges to the effective resistance relative to $E$ as $n \to \infty$ (see [5, 7.4]).

3. Random walks

We consider a connected non-parabolic network $\Gamma = (V, E, r)$ endowed with a measure $\mu : V \to (0, +\infty)$ satisfying (2) and the random walk $\{X_n\}$ of $\Gamma$.

Let $(\mathcal{M}, d_{\mathcal{M}})$ be a complete separable metric space. Define a set $D[E_{\Gamma, \mathcal{M}}]$ of maps of $V$ to $\mathcal{M}$ and a functional $E_{\Gamma, \mathcal{M}}$ on $D[E_{\Gamma, \mathcal{M}}]$ by

$$D[E_{\Gamma, \mathcal{M}}] = \left\{ \phi : V \to \mathcal{M} \left| \sum_{x \sim y} \frac{d_{\mathcal{M}}(\phi(x), \phi(y))^2}{r(|xy|)} < +\infty \right\} ;
$$

$$E_{\Gamma, \mathcal{M}}(\phi) = \frac{1}{2} \sum_{x \sim y} \frac{d_{\mathcal{M}}(\phi(x), \phi(y))^2}{r(|xy|)}, \quad \phi \in D[E_{\Gamma, \mathcal{M}}].$$

A map $\phi : V \to \mathcal{M}$ in $D[E_{\Gamma, \mathcal{M}}]$ is called a Dirichlet finite map. The composition $f \circ \phi$ of a Lipschitz function $f$ on $\mathcal{M}$ and a Dirichlet finite map $\phi : V \to \mathcal{M}$ belongs to $D[E_\Gamma]$. Thus applying the result of [1] mentioned in the introduction, we see that the sequence $\{f(\phi(X_n))\}$ is almost surely convergent, and the process $\phi(X_n)$ is already almost surely convergent in $\mathcal{M}$.

Now we consider a resistance form $E$ of $\Gamma$. For any $x, y \in V$, let

$$d^E(x, y) = \left( \int_V (g^E_p(x, z) - g^E_p(y, z))^2 \ d\mu(z) \right)^{1/2},$$

where we set $g^E_p(x, y) = \int_V g^E_p(x, y) \ d\mu(z)$. Then it is proved in [5, Theorem 3.10] that $d^E$ gives a compatible metric on $\mathcal{K}(E)$. In what follows, $\mathcal{K}(E)$ is equipped with the distance $d^E$.

Now we prove the following

**Lemma 5.** The inclusion map $I$ of $V$ into the metric space $(\mathcal{K}(E), d^E)$ is a Dirichlet finite map and $E_{\Gamma, \mathcal{K}(E)}(I) = \iint_{V \times V} R_E(z, w) \ d\mu(z) \ d\mu(w).$
Proof. We have
\[
E_{\Gamma, \mathcal{K}(\mathcal{E})}(I) = \sum_{x \sim y} \frac{d^\mathcal{E}(x, y)^2}{r(|xy|)}
\]
\[
= \sum_{x \sim y} \int_{V} \frac{(g^\mathcal{E}_{\mu}(x, z) - g^\mathcal{E}_{\mu}(y, z))^2}{r(|xy|)} d\mu(z)
\]
\[
= \int_{V} \sum_{x \sim y} \frac{(g^\mathcal{E}_{\mu}(x, z) - g^\mathcal{E}_{\mu}(y, z))^2}{r(|xy|)} d\mu(z)
\]
\[
= \int_{V} \mathcal{E}(g^\mathcal{E}_{\mu}(z, *), g^\mathcal{E}_{\mu}(z, *)) d\mu(z)
\]
\[
= \int_{V} g^\mathcal{E}_{\mu}(z, z) d\mu(z)
\]
\[
= \int_{V \times V} R^\mathcal{E}(z, w) d\mu(z) d\mu(w)
\]
\[
< \int_{V} R^\mathcal{E}(z, o) d\mu(z) + \int_{V} R^\mathcal{E}(o, w) d\mu(w) = 2 \int_{V} R^\mathcal{E}(z, o) d\mu(z).
\]
This completes the proof of the lemma.

Theorem 6. Let $\Gamma = (V, E, r)$ be a connected non-parabolic network and $\mathcal{E}$ a resistance form of $\Gamma$. Then there exists a $\Delta^K(\mathcal{E})$-valued random variable $X^\mathcal{E}_\infty$ such that the process $X_n$ almost surely converges to $X^\mathcal{E}_\infty$ in $\mathcal{K}(\mathcal{E})$, the measure $\nu_{X_n}$ converges weakly to the delta measure $\delta_{X^\mathcal{E}_\infty}$ almost surely, and for any $u \in D[\mathcal{E}]$, $u(X_n)$ converges to $\tau(u)(X^\mathcal{E}_\infty)$ almost surely and in $L^2$ as $n \to \infty$.

Proof. Lemma 5 and the result in [1] stated above imply that the process $\{X_n\}$ is Cauchy in $\mathcal{K}(\mathcal{E})$ almost surely. Let $X^\mathcal{E}_\infty = \lim_{n \to \infty} X_n$. We recall here that $D[L^2] = D[L^2]$. Then together with Theorem 1, we see that for $u \in D[L^2]$, $u(X_n)$ converges to $\tau(u)(X^\mathcal{E}_\infty)$ almost surely and in $L^2$ as $n \to \infty$.

Moreover it follows that $\nu_{X_n}$ weakly converges to $\delta_{X^\mathcal{E}_\infty}$ almost surely, and since the support of the measure $\nu_{X}$ coincides with $\Delta^K(\mathcal{E})$, it follows that $X^\mathcal{E}_\infty$ is a $\Delta^K(\mathcal{E})$-valued random variable, and further it is easy to see that the image is dense in $\Delta^K(\mathcal{E})$.
\[ \mathcal{E}(u - u_\varepsilon) + (u - u_\varepsilon)(a)^2 < \varepsilon. \] Let \( h_\varepsilon(x) = \int_{\delta K(\mathcal{E})} \tau(u - u_\varepsilon) \, dv_\varepsilon \) and \( g_\varepsilon = u - u_\varepsilon - h_\varepsilon \in D_0[\mathcal{E}_0]. \) Then we have
\[
E_a[(u - u_\varepsilon)(X_n)] \leq 2E_a[h_\varepsilon^2(X_n)] + 2E_a[g_\varepsilon^2(X_n)] \\
\leq 2 \int_{\delta K(\mathcal{E})} \tau(h_\varepsilon)^2 \, dv_\varepsilon + 2E_a[g_\varepsilon^2(X_n)],
\]
where we have used the fact that \( h_\varepsilon^2 \) is subharmonic, so that
\[
E_a[h_\varepsilon^2(X_n)] \leq \int_{\delta K(\mathcal{E})} \tau(h_\varepsilon)^2 \, dv_\varepsilon.
\]
In view of (4), we observe that
\[
\int_{\delta K(\mathcal{E})} \tau(h_\varepsilon)^2 \, dv_\varepsilon \leq 2g^0_\varepsilon(a, a)\mathcal{E}(u - u_\varepsilon, u - u_\varepsilon) + 2(u - u_\varepsilon)(a)^2 \\
\leq 2(g^0_\varepsilon(a, a) + 1)\varepsilon.
\]
Thus we obtain
\[
E_a[(u - u_\varepsilon)(X_n)] \leq 4(g^0_\varepsilon(a, a) + 1)\varepsilon + 2E_a[g_\varepsilon^2(X_n)].
\]
Using this, we have
\[
E_a[(u(X_n) - \tau(u)(X^\varepsilon_\infty))^2] \\
\leq 4E_a[h_\varepsilon^2(X_n)] + 4E_a[\tau(u - u_\varepsilon)^2(X^\varepsilon_\infty)] + 2E_a[(u_\varepsilon(X_n) - \tau(u_\varepsilon)(X^\varepsilon_\infty))^2] \\
\leq 16(g^0_\varepsilon(a, a) + 1)\varepsilon + 8E_a[g_\varepsilon^2(X_n)] + 4 \int_{\delta K(\mathcal{E})} \tau(h_\varepsilon)^2 \, dv_\varepsilon \\
+ 2E_a[(u_\varepsilon(X_n) - \tau(u_\varepsilon)(X^\varepsilon_\infty))^2] \\
\leq 24(g^0_\varepsilon(a, a) + 1)\varepsilon + 8E_a[g_\varepsilon^2(X_n)] + 2E_a[(u_\varepsilon(X_n) - \tau(u_\varepsilon)(X^\varepsilon_\infty))^2].
\]
Thus we get
\[
\limsup_{n \to \infty} E_a[(u(X_n) - \tau(u)(X^\varepsilon_\infty))^2] \leq 24(g^0_\varepsilon(a, a) + 1)\varepsilon.
\]
Letting \( \varepsilon \) go to zero, we see that \( \lim_{n \to \infty} E_a[(u(X_n) - \tau(u)(X^\varepsilon_\infty))^2] = 0. \) This completes the proof of the theorem. \( \square \)

Now we consider a map \( \phi \) from the network \( \Gamma \) to a simply connected, complete separable geodesic space \( (M, d_M) \) of nonpositive curvature (cf. [4], [13]). For any \( x \in V \), there exists uniquely a point of \( M \), denoted by \( P\phi(x) \), such that
\[
\sum_{y \sim x} \frac{d_M(P\phi(x), \phi(y))^2}{r(|xy|)} = \inf_{q \in M} \sum_{y \sim x} \frac{d_M(q, \phi(y))^2}{r(|xy|)};
\]
\[ P \phi(x) \] is the center of mass of the measure \( \sum_{y \sim x} r(|xy|)^{-1} \delta_{\phi(y)} \) on \( \mathcal{M} \). A map \( \phi: V \to \mathcal{M} \) is said to be harmonic if \( P \phi(x) = \phi(x) \) at any \( x \in V \). A harmonic map \( \phi: V \to \mathcal{M} \) pulls convex functions \( \eta \) on an open subset \( A \subset \mathcal{M} \) back to subharmonic functions \( \eta \circ \phi \) on \( \phi^{-1}(A) \) (see [4, Proposition 12.3 (Jensen’s inequality)]).

Now we prove the following

**Theorem 7.** Let \( \phi \) be a map from a connected non-parabolic network \( \Gamma = (V, E, r) \) to a simply connected, complete, separable geodesic space \( (\mathcal{M}, d_{\mathcal{M}}) \) of non-positive curvature. Let \( \phi: V \to \mathcal{M} \) be a Dirichlet finite harmonic map. Then the image \( \phi(V) \) is contained in the convex hull \( C(L) \) of the set \( L \) of points to which \( \phi(X_n) \) converges almost surely.

Moreover \( \phi(V) \) is bounded if any \( h \in H_{\mathcal{E}_r} \) is bounded. In particular, \( \phi \) must be constant if \( H_{\mathcal{E}_r} = R \), that is, \( \Gamma \) admits no non-constant Dirichlet finite harmonic functions.

Proof. Let \( \eta \) be a distance function to the convex hull \( C(L) \) of \( L \), that is the smallest closed convex subset containing \( L \) in \( \mathcal{M} \). Then \( \eta^2 \) is convex and hence \( \eta^2 \circ \phi \) is subharmonic on \( V \). Thus we have

\[ \eta^2 \circ \phi(x) \leq E_x[\eta^2 \circ \phi(X_n)] \]

for any \( x \in V \) and all \( n = 1, 2, \ldots \). Since \( \lim_{n \to \infty} \eta^2(\phi(X_n)) = 0 \) almost surely, we get \( \eta^2 \circ \phi(x) = 0 \), that is, \( \phi(x) \in C(L) \).

Now we suppose that any \( h \in H_{\mathcal{E}_r} \) is bounded. Since this condition is equivalent to the condition that any nonnegative subharmonic function \( u \) of \( D[\mathcal{E}_\Gamma] \) is bounded, for the distance function \( \eta \) to a point of \( \mathcal{M} \), \( \eta \circ \phi \) is bounded. Thus \( \phi(V) \) must be bounded. Moreover we suppose that \( \Gamma \) admits no non-constant Dirichlet finite harmonic functions. Then \( \Delta(\mathcal{E}_\Gamma) \) consists of a single point, and hence so does \( L \). Thus \( \phi \) must be a constant map. This completes the proof of the theorem. \( \square \)

Let \( \Omega \) be the set of one-sided infinite paths in a connected non-parabolic network \( \Gamma \). Given a path \( \omega \in \Omega \), the set of limit points of \( \omega \) in the Royden boundary \( \partial \mathcal{R}(\mathcal{E}_\Gamma) \) of \( \Gamma \) is defined as

\[ L(\omega) = [X_\omega] \cap \partial \mathcal{R}(\mathcal{E}_\Gamma). \]

Then we can deduce from Theorem 6 the following

**Lemma 8.** For any null family \( \Sigma \) of one-sided infinite paths, one has

\[ \bigcup \{L(\omega) \mid \omega \in \Omega \setminus \Sigma \} \supset \Delta(\mathcal{E}_\Gamma). \]

Now we prove the following
Theorem 9. Let $\phi: \Gamma \to (\mathcal{M}, d_{\mathcal{M}})$ be a Dirichlet finite map from a connected non-parabolic network $\Gamma = (V, E, r)$ to a proper metric space $(\mathcal{M}, d_{\mathcal{M}})$, that is a metric space such that any bounded closed subset is compact. Let $\overline{\mathcal{M}} = \mathcal{M} \cup \{ \infty_{\mathcal{M}} \}$ be the one-point compactification of $\mathcal{M}$. Then $\phi$ extends to a continuous map $\tilde{\phi}: \mathcal{R}(\mathcal{E}_\Gamma) \to \overline{\mathcal{M}}$ from the Royden compactification $\mathcal{R}(\mathcal{E}_\Gamma)$ of $\Gamma$ to $\overline{\mathcal{M}}$. Moreover there exists a null family $\Sigma$ in $\Omega$ such that $\phi(X_n(\omega))$ converges in $\mathcal{M}$ for all $\omega \in \Omega \setminus \Sigma$ and

$$\tilde{\phi}(\Delta(\mathcal{E}_\Gamma)) = \left\{ \lim_{n \to \infty} \phi(X_n(\omega)) \in \mathcal{M} \left| \omega \in \Omega \setminus (\Sigma' \cup \Sigma) \right\} \cup \{ \infty_{\mathcal{M}} \}$$

for any null family $\Sigma'$ in $\Omega$.

Proof. For a point $x \in \mathcal{M}$, we denote by $\eta_x$ the distance function to $x$ in $\mathcal{M}$. Let $\Lambda_\phi = \{ \xi \in \partial \mathcal{R}(\mathcal{E}_\Gamma) \mid \eta_x \circ \tilde{\phi}(\xi) = +\infty \}$, where $\eta_x \circ \tilde{\phi}$ stands for the continuous extension of $\eta_x \circ \phi$ to $\mathcal{R}(\mathcal{E}_\Gamma)$ with values in $\mathbb{R} \cup \{ \pm \infty \}$. This closed subset is independent of the choice of a reference point $x$. Now we take a countably infinite dense subset $\{ x_i \}$ of $\mathcal{M}$. Let $\xi$ and $\{ v_n \}$ be, respectively, a point of $\partial \mathcal{R}(\mathcal{E}_\Gamma) \setminus \Lambda_\phi$ and a sequence in $V$ converging to $\xi$. Then $\phi(v_n)$ stays in a compact subspace in $\mathcal{M}$. Since $d_{\mathcal{M}}(x_i, \phi(v_n))$ tends to $\eta_{x_i} \circ \tilde{\phi}(\xi)$ as $n \to \infty$ for all $x_i$ which are densely distributed in $\mathcal{M}$, we can deduce that as $n$ tends to infinity, $\phi(v_n)$ converges to a point, $\tilde{\phi}(\xi)$, in $\mathcal{M}$. By setting $\tilde{\phi}(\xi) = \infty_{\mathcal{M}}$ for $\xi \in \Lambda_\phi$, we obtain a continuous map $\tilde{\phi}$ from $\mathcal{R}(\mathcal{E}_\Gamma)$ to $\overline{\mathcal{M}}$.

Let $\Omega_\phi$ be the set of one-sided infinite paths along which $\phi(X_n)$ converges in $\mathcal{M}$. For any $j = 1, 2, \ldots$, let $\tilde{\phi}(\Delta(\mathcal{E}_\Gamma))_j = \{ x \in \mathcal{M} \mid d_{\mathcal{M}}(x, \phi(X_{n}(\omega))) < 1/j \}$ and $A_j = \tilde{\phi}(\mathcal{R}(\mathcal{E}_\Gamma) \setminus \phi(\Delta(\mathcal{E}_\Gamma))_j$. Since $\phi^{-1}(A_j)$ is disjoint from $\Delta(\mathcal{E}_\Gamma)$, we have by Lemma 5.3 in [15] a function $g_j \in D_0[\mathcal{E}_\Gamma]$ such that $g_j = +\infty$ on $\phi^{-1}(A_j) \cap \partial \mathcal{R}(\mathcal{E}_\Gamma)$. On the other hand, it follows from Theorem 1 that $\lim_{n \to \infty} g_j(X_n) = 0$ almost surely. This shows that $\{ \omega \in \Omega_\phi \mid \lim_{n \to \infty} \phi(X_n(\omega)) \in A_j \}$ is a null family of paths, and hence, letting $\Sigma = \{ \omega \in \Omega_\phi \mid \lim_{n \to \infty} \phi(X_n(\omega)) \in \bigcup A_j \}$, we see that $\lim_{n \to \infty} \phi(X_n(\omega)) \in \tilde{\phi}(\Delta(\mathcal{E}_\Gamma))$ for all $\omega \in \Omega_\phi \setminus \Sigma$. Moreover by Lemma 8, the assertion holds true.

Remark. Relevantly to Theorem 7, we refer to [8] in which a Liouville type theorem for harmonic maps to convex spaces via Markov chains is discussed. For an existence result of Dirichlet finite harmonic maps, see [12]. A connected parabolic network admits no non-constant Dirichlet finite harmonic maps to a simply connected, complete, geodesic space of nonpositive curvature. In fact, Theorem (3.34) in [11] states that a Dirichlet finite subharmonic function on such a network must be constant. We also refer to [3], where it is proved that if on a complete Riemannian manifold $M$, every harmonic function with finite Dirichlet energy is bounded, then every harmonic map with finite total energy from $M$ into a Cartan–Hadamard manifold must also have bounded image.
4. Poisson equations

Let $\Gamma = (V, E, r)$ be a connected non-parabolic network and $\mathcal{E}$ a resistance form of $\Gamma$. In this section, we derive integral representations of solutions of Poisson equations on the Kuramochi compactification of $\mathcal{E}$.

To begin with, we show the following

**Lemma 10** (Harnack’s inequality). Let $h$ be a positive harmonic function on $V$. Then

$$h(x) \leq \frac{g_0^\Gamma(x, x)}{g_0^\Gamma(y, x)}h(y)$$

for all $x, y \in V$.

Proof. Let $\{V_n\}$ be an increasing sequence of finite subsets of $V$ such that $V = \bigcup_n V_n$. Let $D_n$ be the space of functions on $V$ which vanish outside of $V_n$. Then for any $x \in V_n$, there exists uniquely a function $g_x \in D_n$ satisfying

$$\mathcal{E}_\Gamma(g_x, u) = u(x)$$

for all $u \in D_n$. We write $g_n(x, y)$ for $g_x(y)$. Fix points $x, y \in V$ and consider $V_n$ for $n$ large enough. Then there exists uniquely a function $p_n \in D_n$ such that $p_n(x) = h(x)$, $p_n(y) = h(y)$, and $L^c p_n(z) = 0$ for any $z \in V_n \setminus \{x, y\}$. The maximum principle ensures that $p_n \leq h$ in $V$ and hence $L^c p_n(x) \geq 0$ and $L^c p_n(y) \geq 0$. Then we have

$$h(x) = p_n(x)$$

$$= \mathcal{E}_\Gamma(g_n(x, \ast), p_n)$$

$$= \sum_{z \in V_n} g_n(x, z)L^c p_n(z)$$

$$= g_n(x, x)L^c p_n(x) + g_n(x, y)L^c p_n(y)$$

$$= \frac{g_n(x, x)}{g_n(y, x)}g_n(y, x)L^c p_n(x) + \frac{g_n(x, y)}{g_n(y, y)}g_n(y, y)L^c p_n(y)$$

$$\leq \frac{g_n(x, x)}{g_n(y, x)}\{g_n(y, x)L^c p_n(x) + g_n(y, y)L^c p_n(y)\}$$

$$= \frac{g_n(x, x)}{g_n(y, x)}\mathcal{E}_\Gamma(g_n(y, \ast), p_n)$$

$$= \frac{g_n(x, x)}{g_n(y, x)}p_n(y)$$

$$= \frac{g_n(x, x)}{g_n(y, x)}h(y).$$
Thus we get
\[ h(x) \leq \frac{g_n(x, x)}{g_n(y, x)} h(y) \]
for all large \( n \). As \( n \to \infty \), \( g_n(z, w) \) converges to \( g^0_\Gamma(z, w) \) for any \( (z, w) \in V \times V \), and thus we obtain the required inequality. \( \square \)

In what follows, we take a probability measure \( \mu \) on \( V \) satisfying (2) and
\[ (5) \quad \int_V \frac{g^0_\Gamma(x, x)}{g^0_\Gamma(x, o)} d\mu(x) < +\infty. \]

**Proposition 11.** (i) For any fixed \( x \in V \), \( g^0_\Gamma(x, \ast) \in D[\bar{L}^\varepsilon] \cap D_0[\varepsilon_\Gamma] \) and the harmonic measure \( \nu_x \) with respect to \( x \in V \) is given by
\[ \nu_x = -N^\varepsilon g^0_\Gamma(x, \ast) \nu. \]

(ii) Let
\[ G^0_\mu(x) = \int_V g^0_\Gamma(x, z) d\mu(z), \quad x \in V. \]

Then \( G^0_\mu \) belongs to \( D[\bar{L}^\varepsilon] \) and \( N^\varepsilon G^0_\mu = \int_V N^\varepsilon g^0_\Gamma(x, \ast) d\mu(x) \). Moreover \( N^\varepsilon G^0_\varepsilon \) satisfies
\[ 0 < \mu(0) < -N^\varepsilon G^0_\mu < \int_V \frac{g^0_\Gamma(x, x)}{g^0_\Gamma(x, o)} d\mu(x). \]

**Proof.** For a function \( u \in D[\varepsilon] \), we have
\[
\left| \int_{\partial\varepsilon(\varepsilon)} \tau(u) d\nu_x \right| \leq \int_{\partial\varepsilon(\varepsilon)} |\tau(u)| d\nu_x \\
\leq \frac{g^0_\Gamma(x, x)}{g^0_\Gamma(x, o)} \int_{\partial\varepsilon(\varepsilon)} |\tau(u)| d\nu.
\]

This implies that
\[
|\varepsilon(g^0_\Gamma(x, \ast), u)| = \left| u(x) - \int_{\partial\varepsilon(\varepsilon)} \tau(u) d\nu_x \right|
\]
is bounded by \( \mu(x)^{-1} \int_V |u| d\mu + g^0_\Gamma(x, x)/g^0_\Gamma(x, o) \int_{\partial\varepsilon(\varepsilon)} |\tau(u)| d\nu \). Thus we see that \( g^0_\Gamma(x, \ast) \) belongs to \( D[\bar{L}^\varepsilon] \). Moreover since \( L^\varepsilon g^0_\Gamma(x, \ast) = \delta_x \), we get
\[
u_x - \int_{\partial\varepsilon(\varepsilon)} \tau(u) d\nu_x = \int_V \tau(u)(u(y) L^\varepsilon g^0_\Gamma(x, y) d\mu(y) + \int_{\partial\varepsilon(\varepsilon)} \tau(u) N^\varepsilon g^0_\Gamma(x, \ast) d\nu \\
= u(x) + \int_{\partial\varepsilon(\varepsilon)} \tau(u) N^\varepsilon g^0_\Gamma(x, \ast) d\nu.
\]
In this way, we obtain
\[
\int_{\partial K(E)} \tau(u) \, dv_x = -\int_{\partial K(E)} \tau(u) N^E g^0_\Gamma(x, \ast) \, dv.
\]
This shows the first assertion.

Given \( u \in D[\mathcal{E}] \), let \( g(x) = u(x) - \int_{\partial K(E)} \tau(u) \, dv_x \). Then we have
\[
|\mathcal{E}_\Gamma(G^0_\mu, u)| = |\mathcal{E}_\Gamma(G^0_\mu, g)|
\]
\[
= \left| \int_V g(x) \, d\mu(x) \right|
\]
\[
= \left| \int_V u(x) \, d\mu(x) - \int_V \int_{\partial K(E)} \tau(u) \, dv_x \, d\mu(x) \right|
\]
\[
\leq \int_V |u(x)| \, d\mu(x) + \int_V \int_{\partial K(E)} |\tau(u)| \, dv_x \, d\mu(x)
\]
\[
\leq \int_V |u(x)| \, d\mu(x) + \int_V \frac{g^0_\Gamma(x, x)}{g^0_\Gamma(x, \ast)} \, d\mu(x) \int_{\partial K(E)} |\tau(u)| \, dv.
\]
This shows that \( G^0_\mu \) belongs to \( D[\tilde{L}^E] \). It is easy to see the remaining assertions. This completes the proof of the proposition. \( \square \)

As in Section 3, we now introduce a kernel function \( g^E_\mu \) on \( \mathcal{E} \) by
\[
g^E_\mu(x, y) = \int_V g^E_\mu(x, y) \, d\mu(z), \quad x, y \in V.
\]
Then we have
\[
\mathcal{E}(g^E_\mu(x, \ast), u) = u(x) - \int_V u \, d\mu, \quad u \in D[\mathcal{E}].
\]
In particular, the function \( g^E_\mu(x, \ast) \) for a fixed \( x \in V \) belongs to \( D[L^E] \). Similarly, let
\[
h^E_\mu(x, y) = \int_V h^E_\mu(x, y) \, d\mu(z), \quad x, y \in V.
\]
Then we have
\[
\mathcal{E}(h^E_\mu(x, \ast), h) = h(x) - \int_V h \, d\mu, \quad h \in H_{\mathcal{E}}.
\]
In view of (1), we see that
\[
g^E_\mu(x, y) = h^E_\mu(x, y) + g^0_\Gamma(x, y) - G^0_\mu(x) - G^0_\mu(y) + C_{\Gamma, \mu},
\]
(6)
where we put \( C_{\Gamma, \mu} = \int_V \mathcal{F}_{\Gamma}(z, z) \, d\mu(z) \).

Given a function \( u \in D[\bar{L}^E] \), we have

\[
u(x) = \int_V u \, d\mu + \mathcal{E}(g^E_{\mu}(x, \ast), u)
= \int_V u \, d\mu + \int_V g^E_{\mu}(x, y)L^E u(y) \, d\mu(y) + \int_{\partial K(E)} g^E_{\mu}(x, \xi)N^E u(\xi) \, d\nu(\xi)
= \int_V u \, d\mu + \int_V g^E_{\mu}(x, y)L^E u(y) \, d\mu(y) + \int_{\partial K(E)} h^E_{\mu}(x, \xi)N^E u(\xi) \, d\nu(\xi)
= (G^0_{\mu}(x) - C_{\Gamma, \mu}) \int_{\partial K(E)} N^E u(\xi) \, d\nu(\xi).
\]

Since

\[
\int_V L^E u \, d\mu + \int_{\partial K(E)} N^E u \, d\nu = \mathcal{E}(u, 1) = 0,
\]

by letting

\[
\bar{g}^E_{\mu}(x, y) = g^E_{\mu}(x, y) + G^0_{\mu}(x) - C_{\Gamma, \mu},
\]

we obtain an integral representation of a function \( u \) of \( D[\bar{L}^E] \) as follows:

\[
u(x) = \int_V u \, d\mu + \int_V \bar{g}^E_{\mu}(x, y)L^E u(y) \, d\mu(y) + \int_{\partial K(E)} h^E_{\mu}(x, \xi)N^E u(\xi) \, d\nu(\xi).
\]

Let \( f \) be a function in \( L^2(K(E), \mu) \). Suppose that \( \bar{\mu}(f) = \int_V f \, d\mu + \int_{\partial K(E)} f \, d\nu = 0 \). Then for any \( h \in H_E \), we have

\[
\left| \int_V hf \, d\mu + \int_{\partial K(E)} \tau(h) f \, d\nu \right|^2
\]

\[
= \left| \int_V (h - h(o)) f \, d\mu + \int_{\partial K(E)} \tau(h - h(0)) f \, d\nu \right|^2
\]

\[
\leq \left( \int_V (h - h(o))^2 \, d\mu + \int_{\partial K(E)} \tau(h - h(o))^2 \, d\nu \right) \left( \int_V f^2 \, d\mu + \int_{\partial K(E)} f^2 \, d\nu \right)
\]

\[
\leq \left( \int_V \mathcal{E}(o, x) \, d\mu(x) + 2g^0_{\Gamma}(o, o) \right) \left( \int_V f^2 \, d\mu + \int_{\partial K(E)} f^2 \, d\nu \right) \mathcal{E}(h, h),
\]

where we have used

\[
\int_V (h(x) - h(o))^2 \, d\mu(x) \leq \int_V h^E_{\mu}(x, x) \, d\mu(x) \mathcal{E}(h, h) \leq \int_V \mathcal{E}(o, x) \, d\mu(x) \mathcal{E}(h, h)
\]

and

\[
\int_{\partial K(E)} \tau(h - h(o))^2 \, d\nu \leq 2g^0_{\Gamma}(o, o) \mathcal{E}(h, h)
\]
by (4). For \( g \in D_0[\mathcal{E}_\Gamma] \), we have
\[
\left| \int_V g f \, d\mu \right|^2 \leq \int_V g^2 \, d\mu \int_V f^2 \, d\mu 
\leq \int_V \mathcal{E}_\Gamma(g^0_\Gamma(x, \ast), g)^2 \, d\mu(x) \int_V f^2 \, d\mu 
= \int_V g^0_\Gamma(x, x) \, d\mu(x) \int_V f^2 \, d\mu \, \mathcal{E}_\Gamma(g, g) 
\leq \left( \int_V R_\mathcal{E}(o, x) \, d\mu(x) + 2g^0_\Gamma(o, o) \right) \int_V f^2 \, d\mu \, \mathcal{E}_\Gamma(g, g).
\]

In this way, we see that for any \( u \in D[\mathcal{E}] \),
\[
\left| \int_V uf \, d\mu + \int_{\partial K(\mathcal{E})} \tau(u) f \, dv \right|^2 
\leq \left( \int_V R_\mathcal{E}(o, x) \, d\mu(x) + 2g^0_\Gamma(o, o) \right) \left( \int_V f^2 \, d\mu + \int_{\partial K(\mathcal{E})} f^2 \, dv \right) \mathcal{E}(u, u).
\]
This shows that there exists a function \( \phi \) in \( D[\mathcal{E}] \), unique up to additive constants, such that
\[
\mathcal{E}(u, \phi) = \int_V uf \, d\mu + \int_{\partial K(\mathcal{E})} \tau(u) f \, dv, \quad u \in D[\mathcal{E}],
\]
so that \( \phi \) belongs to \( D[\mathcal{L}_\mathcal{E}] \), \( \mathcal{L}_\mathcal{E} \phi = f \) in \( K^2(\mathcal{K}(\mathcal{E}), \tilde{\mu}) \), and \( \phi \) is expressed in the following way:
\[
\phi(x) = \int_V \phi \, d\mu + \int_V \tilde{\mathcal{g}}^\mathcal{E}_\mu(x, y) f(y) \, d\mu(y) + \int_{\partial K(\mathcal{E})} h^\mathcal{E}_\mu(x, \xi) f(\xi) \, dv(\xi).
\]

In the case where \( \tilde{\mu}(f) \neq 0 \), the function \( \phi \) defined in (7) satisfies \( \mathcal{L}_\mathcal{E} \phi = f \) on \( V \) and \( N^\mathcal{E} \phi = f + \tilde{\mu}(f) N^\mathcal{E} G^0_\mu \) in \( L^2(\partial K(\mathcal{E}), v) \).

In fact, we have
\[
\phi(x) - \int_V \phi \, d\mu 
= \int_V \tilde{\mathcal{g}}^\mathcal{E}_\mu(x, y) f(y) \, d\mu(y) 
\quad + \int_{\partial K(\mathcal{E})} h^\mathcal{E}_\mu(x, \xi)(f(\xi) - \tilde{\mu}(f)) \, dv(\xi) + \tilde{\mu}(f) \int_{\partial K(\mathcal{E})} h^\mathcal{E}_\mu(x, \xi) \, dv(\xi) 
= \int_V \tilde{\mathcal{g}}^\mathcal{E}_\mu(x, y) f(y) \, d\mu(y) + \int_{\partial K(\mathcal{E})} h^\mathcal{E}_\mu(x, \xi)(f(\xi) - \tilde{\mu}(f)) \, dv(\xi) + \tilde{\mu}(f) h^\mathcal{E}_\mu(x, o)
\]
and
\[
N^\mathcal{E} h^\mathcal{E}_\mu(\xi, o) = 1 + N^\mathcal{E} G^0_\mu(\xi).
\]
Thus we have the following

**Theorem 12.** Let $\Gamma = (V, E, r)$ be a connected non-parabolic network and $\mathcal{E}$ a resistance form of $\Gamma$. A probability measure $\mu$ on $V$ satisfying (2) and (5) is given.

(i) For $u \in D[\bar{\mathcal{L}}^\mathcal{E}]$, one has

$$u(x) = \int_V u \, d\mu + \int_V \tilde{g}^\mathcal{E}_\mu(x, y) L^\mathcal{E} u(y) \, d\mu(y) + \int_{\partial\mathcal{K}(\mathcal{E})} h^\mathcal{E}_\mu(x, \xi) N^\mathcal{E} u(\xi) \, d\nu(\xi), \quad x \in V.$$ 

(ii) For $f \in L^2(\mathcal{K}(\mathcal{E}), \bar{\mu})$ and a constant $c$, the function

$$u(x) = c + \int_V \tilde{g}^\mathcal{E}_\mu(x, y) f(y) \, d\mu(y) + \int_{\partial\mathcal{K}(\mathcal{E})} h^\mathcal{E}_\mu(x, \xi) f(\xi) \, d\nu(\xi), \quad x \in V.$$ 

belongs to $D[\bar{\mathcal{L}}^\mathcal{E}]$ and satisfies $\bar{\mathcal{L}}^\mathcal{E} u = f$ on $V$ and $\bar{\mathcal{L}}^\mathcal{E} u = f + \tilde{\mu}(f) N^\mathcal{E} G^0_{\mu}$ in $L^2(\partial\mathcal{K}(\mathcal{E}), \nu)$. In particular if $\tilde{\mu}(f) = 0$, then $\bar{\mathcal{L}}^\mathcal{E} u = f$ in $L^2(\mathcal{K}(\mathcal{E}), \bar{\mu})$.

Let $D[\mathcal{E}^*] = \{\tau(u) \mid u \in D[\mathcal{E}]\} \subset L^2(\partial\mathcal{K}(\mathcal{E}), \nu)$ and $\mathcal{E}^*(\tau(u), \tau(v)) = \mathcal{E}(h_u, h_v)$ for $u, v \in D[\mathcal{E}]$, where $h_u$ denotes the harmonic part of $u$ in the Royden decomposition. Let $(L^*, D[L^*])$ be the self-adjoint operator associated to the regular Dirichlet form $(\mathcal{E}^*, D[\mathcal{E}^*])$ on $L^2(\partial\mathcal{K}(\mathcal{E}), \nu)$. The restriction of $\tau$ to $H_{\mathcal{E}}$ gives rise to a bijection between $H_{\mathcal{E}}$ and $D[\mathcal{E}^*]$ such that $\tau(H_{\mathcal{E}} \cap D[\bar{\mathcal{L}}^\mathcal{E}]) = D[L^*]$ and $N^\mathcal{E} h = L^* \tau(h)$ for $h \in H_{\mathcal{E}} \cap D[\bar{\mathcal{L}}^\mathcal{E}]$.

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**References**


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