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RANDOM WALKS AND KURAMOCHI BOUNDARIES OF INFINITE NETWORKS

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Abstract

In this paper, we study a connected non-parabolic, or transient, network compactified with the Kuramochi boundary, and show that the random walk converges almost surely to a random variable valued in the harmonic boundary, and a function of finite Dirichlet energy converges along the random walk to a random variable almost surely and in L^2. We also give integral representations of solutions of Poisson equations on the Kuramochi compactification.

1. Introduction

Ancona, Lyons and Peres [1] showed that a function of finite Dirichlet energy on a transient network converges along the random walk almost surely and in L^2. In this paper, we concern the Kuramochi boundary of the network and proves that the random walk converges almost surely to a random variable valued in the harmonic boundary, and a function of finite Dirichlet energy converges along the random walk to a random variable almost surely and in L^2.

Let $G = (V, E)$ be a graph with the set of vertices V and the set of edges E that consists of pairs of vertices. In this paper, a graph admits no loops and multiple edges, and the set of vertices is finite or countably infinite. We say that a vertex x is adjacent to another y if $\{x, y\}$ belongs to E and write $x \sim y$ to indicate it. We also write $|xy|$ for $\{x, y\}$. By a path in G, we mean a sequence of vertices $c = (x_0, x_1, \dots, x_n)$ such that $x_i \sim x_{i+1}$ $(i = 0, 1, \dots, n-1)$, and we say that c connects x_0 to x_n. G is called a connected graph if for any pair of vertices x and y, there exist paths connecting them.

We are now given an admissible weight r on the set of edges E, that is a positive function on E with the property that

$$c(x) = \sum_{y \sim x} \frac{1}{r(|xy|)} < +\infty, \quad \forall x \in V.$$

An admissible weight r gives rise to a distance d_r on V, called the geodesic distance of Γ, by taking $r(e)$ as the length of an edge e and by assigning to each pair

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of vertices x and y the infimum of the length of paths connecting them. In this paper, we call such a couple of a graph and an admissible weight a network.

Given a connected network $\Gamma = (V, E, r)$, a nonnegative quadratic form $(\mathcal{E}_\Gamma, D[\mathcal{E}_\Gamma])$ on the space $l(V)$ of functions on V can be defined as follows:

$$D[\mathcal{E}_\Gamma] = \left\{ u \in l(V) \;\middle|\; \sum_{x \sim y} \frac{|u(x) - u(y)|^2}{r(|xy|)} < +\infty \right\};$$

$$\mathcal{E}_\Gamma(u, v) = \frac{1}{2} \sum_{x \sim y} \frac{(u(x) - u(y))(v(x) - v(y))}{r(|xy|)}, \quad u, v \in D[\mathcal{E}_\Gamma].$$

The domain $D[\mathcal{E}_\Gamma]$ endowed with an inner product $\mathcal{E}_\Gamma(u, v) + u(o)v(o)$, where o is a fixed point of V, becomes a Hilbert space.

Let $D_0[\mathcal{E}_\Gamma]$ be the closure of the set of finitely supported functions on V in $D[\mathcal{E}_\Gamma]$. We say that Γ is non-parabolic if

$$\sup \left\{ \frac{|u(x)|^2}{\mathcal{E}_\Gamma(u, u)} \;\middle|\; u \in D_0[\mathcal{E}_\Gamma],\ \mathcal{E}_\Gamma(u, u) > 0 \right\} < +\infty$$

for some $x \in V$. We recall here the fact that the following conditions are mutually equivalent:

(i) Γ is non-parabolic,

(ii) $D_0[\mathcal{E}_\Gamma]$ contains no constant functions,

(iii) $D_0[\mathcal{E}_\Gamma] \neq D[\mathcal{E}_\Gamma]$ (see [14]).

If these are the cases, $D[\mathcal{E}_\Gamma]$ is decomposed into the direct sum of $D_0[\mathcal{E}_\Gamma]$ and the space $H_{\mathcal{E}_\Gamma}$ of harmonic functions of finite Dirichlet sums on V that is the orthogonal complement of $D_0[\mathcal{E}_\Gamma]$ relative to the form; a function h on V belongs to $H_{\mathcal{E}_\Gamma}$ if and only if $h \in D[\mathcal{E}_\Gamma]$ and $L^c h(x) := \sum_{y \sim x} (h(x) - h(y))/r(|xy|) = 0$ for all $x \in V$.

Let $\{p(x, y) \mid x, y \in V\}$ be transition probabilities on V defined by

$$p(x, y) = \frac{c(|xy|)}{c(x)}, \quad x, y \in V,$$

where $c(|xy|) = r(|xy|)^{-1}$ and $c(x) = \sum_{y \sim x} c(|xy|)$. It is well known that Γ is non-parabolic if and only if the (reversible) Markov chain is transient.

Ancona, Lyons and Peres [1] proved the following

Theorem 1. *Let $\Gamma = (V, E, r)$ be a connected non-parabolic network and $\{X_n\}$ the Markov chain. Then for any $u \in D[\mathcal{E}_\Gamma]$, the sequence $\{u(X_n)\}$ converges almost surely and in L^2. If $u = h + g$, where $h \in H_{\mathcal{E}_\Gamma}$ and $g \in D_0[\mathcal{E}_\Gamma]$, is the Royden decomposition of u, then $\lim_{n\to\infty} u(X_n) = \lim_{n\to\infty} h(X_n)$ almost surely.*

To state our main results, we introduce the Kuramochi compactification of a connected infinite network $\Gamma = (V, E, r)$.

A compactification of any (discrete) set X is a compact Hausdorff space which contains X as a dense subset and which induces the discrete topology on X. It is known that given a family Φ of bounded functions on X, there exists an (up to canonical homeomorphisms) unique compactification $\mathcal{C}(X, \Phi)$ of X with the following properties (see e.g. [2]):

(i) every function of Φ extends to a continuous function on $\mathcal{C}(X, \Phi)$, and

(ii) the extended functions separate the points of the boundary $\partial\mathcal{C}(X,\Phi) = \mathcal{C}(X,\Phi) \setminus V$.

We remark that if Ψ is a subfamily of Φ, then the identity map extends to a continuous map from $\mathcal{C}(X, \Phi)$ onto $\mathcal{C}(X, \Psi)$, and if Φ_0 is a subfamily of Φ and each function of Φ is a finite linear combination of functions in Φ_0, then $\mathcal{C}(X, \Phi)$ and $\mathcal{C}(X, \Phi_0)$ are canonically homeomorphic; in particular, if in addition, X and Φ_0 is countable, then $\mathcal{C}(X, \Phi)$ is metrizable.

The compactification relative to the space of bounded functions in $D[\mathcal{E}_\Gamma]$, $BD[\mathcal{E}_\Gamma]$, is called the Royden compactification of the network Γ and denoted by $\mathcal{R}(\mathcal{E}_\Gamma)$. The boundary $\partial\mathcal{R}(\mathcal{E}_\Gamma)$ is called the Royden boundary of Γ. There is an important part of the Royden boundary refered to as the harmonic boundary of Γ which is defined by $\Delta(\mathcal{E}_\Gamma) = \{x \in \partial\mathcal{R}(\mathcal{E}_\Gamma) \mid g(x) = 0 \text{ for all } g \in BD_0[\mathcal{E}_\Gamma]\}$. It is known (see [15], [6], [11, Chapter VI]) that Γ is non-parabolic if and only if the harmonic boundary is not empty, and also that if $\partial\mathcal{R}(\mathcal{E}_\Gamma) \setminus \Delta(\mathcal{E}_\Gamma)$ is not empty, then any set of a single point there is not a G_δ set and for a nonempty closed subset F in $\partial\mathcal{R}(\mathcal{E}_\Gamma) \setminus \Delta(\mathcal{E}_\Gamma)$, there exists a function $g \in D_0[\mathcal{E}_\Gamma]$ such that $g(x)$ tends to infinity as $x \in V \to F$.

We recall a basic fact concerning Dirichlet problems on the Royden boundary $\partial\mathcal{R}(\mathcal{E}_\Gamma)$ (see [11, Chapter VI]): for any continuous function f on $\partial\mathcal{R}(\mathcal{E}_\Gamma)$, there exists a unique harmonic function H_f on Γ such that for any $\xi \in \Delta(\mathcal{E}_\Gamma)$, $\lim_{x\in V\to\xi} H_f(x) = f(\xi)$, and $\sup_V |H_f| \le \max_{\Delta(\mathcal{E}_\Gamma)} |f|$. Given a point $a \in V$, letting $\bar{\nu}_a(f) = H_f(a)$ for $f \in C(\partial\mathcal{R}(\mathcal{E}_\Gamma))$, we have a Radon measure $\bar{\nu}_a$ on $\partial\mathcal{R}(\mathcal{E}_\Gamma)$, called the harmonic measure with respect to the point a. In view of Harnack's inequality, $\bar{\nu}_a$ and $\bar{\nu}_b$ are mutually absolutely continuous for any pair of points $a, b \in V$, and the harmonic measures are supported on the harmonic boundary.

Now we consider a subspace $Q(\mathcal{E}_\Gamma)$ of $BD[\mathcal{E}_\Gamma]$ which consists of functions u such that $\mathcal{E}_\Gamma(u, v) = 0$ for all $v \in D[\mathcal{E}_\Gamma]$ vanishing on a finite subset of V. The compactification relative to $Q(\mathcal{E}_\Gamma)$ is called the Kuramochi compactification of the network Γ and denoted by $\mathcal{K}(\mathcal{E}_\Gamma)$ (see [9]). The identity map of V extends to a continuous map from $\mathcal{R}(\mathcal{E}_\Gamma)$ onto $\mathcal{K}(\mathcal{E}_\Gamma)$. We denote by ρ_Γ the induced map from the Royden boundary $\partial\mathcal{R}(\mathcal{E}_\Gamma)$ onto the Kuramochi boundary $\partial\mathcal{K}(\mathcal{E}_\Gamma)$. Let $\Delta^K(\mathcal{E}_\Gamma) = \rho_\Gamma(\Delta(\mathcal{E}_\Gamma))$ and $\nu_a = \rho_{\Gamma *}\bar{\nu}_a$ $(a \in V)$. Here and after, we fix a point $o \in V$ and write ν for ν_o.

We will prove that the Kuramochi compactification $\mathcal{K}(\mathcal{E}_\Gamma)$ of a connected, non-parabolic network Γ admits a compatible metric $d^{\mathcal{E}_\Gamma}$ such that for each Lipschitz function $f : (\mathcal{K}(\mathcal{E}_\Gamma), d^{\mathcal{E}_\Gamma}) \to \mathbf{R}$, the sequence $\{f(X_n)\}$ is almost surely convergent. This shows that the Markov chain $\{X_n\}$ converges to a random variable X_∞ in $\mathcal{K}(\mathcal{E}_\Gamma)$. In fact, a result by Ancona, Lyons and Peres [1] states that *if $\mathcal{M}$ is a complete separable metric space and $\{Y_n\}$ is a process such that for each bounded Lipschitz function*

$f : \mathcal{M} \to \boldsymbol{R}$, the sequence $\{f(Y_n)\}$ is almost surely convergent, then the process $\{Y_n\}$ in $\mathcal{M}$ is already almost surely convergent.

We will now take an appropriate measure on V. Given two vertices x and y of Γ, we define a nonnegative number $R_{\mathcal{E}_\Gamma}(x, y)$, called the effective resistance between x and y, by

$$R_{\mathcal{E}_\Gamma}(x, y) = \sup\left\{ \frac{|u(x) - u(y)|^2}{\mathcal{E}_\Gamma(u, u)} \;\middle|\; u \in D[\mathcal{E}_\Gamma],\ \mathcal{E}_\Gamma(u, u) > 0 \right\}.$$

It is known that $R_{\mathcal{E}_\Gamma}(x, y) \le d_r(x, y)$ for all $x, y \in V$ and $R_{\mathcal{E}_\Gamma}$ induces a distance on V (see e.g., [5]). Choose a measure μ on V in such a way that $\mu(V) = \sum_{x\in V} \mu(x) = 1$ and

$$\int_V R_{\mathcal{E}_\Gamma}(o, x)^2 \, d\mu(x) \left(= \sum_{x\in V} R_\Gamma(o, x)^2 \mu(x) \right) < +\infty.$$

Under the condition, it is proved in [5] that $D[\mathcal{E}_\Gamma] \subset L^2(V, \mu)$, the embedding is compact, $(\mathcal{E}_\Gamma, D[\mathcal{E}_\Gamma])$ is a regular Dirichlet form in $L^2(\mathcal{K}(\mathcal{E}_\Gamma), \mu)$, and the Royden decomposition is stated in such a way that a function $u \in D[\mathcal{E}_\Gamma]$ is expressed as

$$u(x) = \int_{\partial\mathcal{K}(\mathcal{E}_\Gamma)} \tau(u) \, d\nu_x + g(x), \quad x \in V,\ g \in D_0[\mathcal{E}_\Gamma],$$

where $\tau(u)$ is a function in $L^2(\partial\mathcal{K}(\mathcal{E}_\Gamma), \nu)$ $(= L^2(\Delta^K(\mathcal{E}_\Gamma), \nu))$. We define a Radon measure $\bar{\mu}$ on the Kuramochi compactification $\mathcal{K}(\mathcal{E}_\Gamma)$ by

$$\bar{\mu}(f) = \int_V f \, d\mu + \int_{\partial\mathcal{K}(\mathcal{E}_\Gamma)} f \, d\nu$$

for $f \in C(\mathcal{K}(\mathcal{E}_\Gamma))$. Then any function u of $D[\mathcal{E}_\Gamma]$ coupled with $\tau(u)$ can be considered as a function in $L^2(\mathcal{K}(\mathcal{E}_\Gamma), \bar{\mu})$.

Our main results are stated in the following

Theorem 2. *Let $\Gamma = (V, E, r)$ be a connected non-parabolic network. Then the following assertions hold*:

(i) *$(\mathcal{E}_\Gamma, D[\mathcal{E}_\Gamma])$ is a regular Dirichlet form on $L^2(\mathcal{K}(\mathcal{E}_\Gamma), \bar{\mu})$.*

(ii) *There exists a $\Delta^K(\mathcal{E}_\Gamma)$-valued random variable X_∞ such that in the $\mathcal{K}(\mathcal{E}_\Gamma)$-topology, the Markov chain X_n almost surely converges to X_∞ as $n \to \infty$, the measure ν_{X_n} converges weakly to the delta measure δ_{X_∞} almost surely as $n \to \infty$, and for any $u \in D[\mathcal{E}_\Gamma]$, $u(X_n)$ converges to $\tau(u)(X_\infty)$ almost surely and in L^2 as $n \to \infty$.*

(iii) *Let $(\bar{L}^{\mathcal{E}_\Gamma}, D[\bar{L}^{\mathcal{E}_\Gamma}])$ be the self-adjoint operator associated with the regular Dirichlet form $(\mathcal{E}_\Gamma, D[\mathcal{E}_\Gamma])$. For a function $f \in L^2(\mathcal{K}(\mathcal{E}_\Gamma), \bar{\mu})$, there exists a solution u, unique up to additive constants, of equation: $\bar{L}^{\mathcal{E}_\Gamma} u = f$ if and only if $\bar{\mu}(f) = 0$; in particular, the solution is harmonic on V if f vanishes there.*

We briefly explain the contents of the paper. In section 1, we introduce a resistance form of a connected non-parabolic network and its Kuramochi compactification, and prove Theorem 2 (i) for a resistance form. In Section 2, Theorem 2 (ii) for a resistance form is discussed. The last section is devoted to investigating Poisson equations on the Kuramochi compactification of a resistance form.

2. Resistance forms

In this section, we introduce the Kuramochi compactification of a resistance form of a connected non-parabolic network and prove Theorem 2 (i) for a resistance form.

Let $\Gamma = (V, E, r)$ be a connected non-parabolic network. A nonnegative quadratic form $\mathcal{E}$ on a subspace $D[\mathcal{E}]$ of $D[\mathcal{E}_\Gamma]$ is called a resistance form of the network Γ if it satisfies the following properties:
(i) $D_0[\mathcal{E}_\Gamma] + \mathbf{R} \subset D[\mathcal{E}] \subset D[\mathcal{E}_\Gamma]$,
(ii) $\mathcal{E}(1, 1) = 0$,
(iii) $\mathcal{E}_\Gamma(u, u) \leq \mathcal{E}(u, u)$ for all $u \in D[\mathcal{E}]$ and $\mathcal{E}(u, v) = \mathcal{E}_\Gamma(u, v)$ for all $u \in D[\mathcal{E}]$ and $v \in D_0[\mathcal{E}_\Gamma]$,
(iv) for $u \in D[\mathcal{E}]$, $\bar{u} = \max\{0, \min\{1, u\}\}$ belongs to $D[\mathcal{E}]$ and $\mathcal{E}(\bar{u}, \bar{u}) \leq \mathcal{E}(u, u)$,
(v) $D[\mathcal{E}]$ becomes a Hilbert space with inner product $(u,v) = \mathcal{E}(u,v)+u(o)v(o)$, where o is a fixed vertex of V. When we restrict $\mathcal{E}_\Gamma$ to $D_0[\mathcal{E}_\Gamma] + \mathbf{R}$, we have the minimal resistance form denoted by $(\mathcal{E}_\Gamma^0, D_0[\mathcal{E}_\Gamma] + \mathbf{R})$.

For any pair of vertexes x, y, we have a nonnegative number $R_\mathcal{E}(x, y)$, called the effective resistance relative to $\mathcal{E}$ between x and y, defined by

$$R_\mathcal{E}(x, y) = \sup\left\{ \frac{|u(x) - u(y)|^2}{\mathcal{E}(u, u)} \;\middle|\; u \in D[\mathcal{E}],\ \mathcal{E}(u, u) > 0 \right\}.$$

Then it follows from the definitions above that

$$R_{\mathcal{E}_\Gamma^0}(x, y) \leq R_\mathcal{E}(x, y) \leq R_{\mathcal{E}_\Gamma}(x, y), \quad x, y \in V.$$

We remark that $R_{\mathcal{E}_\Gamma}(x, y) \leq d_r(x, y)$ for $x, y \in V$, and $R_\mathcal{E}$ induces a distance on V (see e.g., [5, Theorem 1.12, Proposition 2.6]). We write $H_\mathcal{E}$ for the space of functions u in $D[\mathcal{E}]$ which are harmonic on V, i.e.,

$$L^c u(x) := \sum_{y \sim x} \frac{u(x) - u(y)}{r(|xy|)} = 0, \quad \forall x \in V.$$

Given $x, z \in V$, there exist functions $g_{x,z} \in D[\mathcal{E}]$ and $h_{x,z} \in H_\mathcal{E}$ respectively satisfying $\mathcal{E}(g_{x,z}, u) = u(x) - u(z)$ for all $u \in D[\mathcal{E}]$ and $\mathcal{E}(h_{x,z}, h) = h(x) - h(z)$ for all $h \in H_\mathcal{E}$. We write $g_z^\mathcal{E}(x, y)$ and $h_z^\mathcal{E}(x, y)$ respectively for $g_{x,z}(y)$ and $h_{x,z}(y)$. It is easy to see that $g_z^\mathcal{E}(x, y) = g_z^\mathcal{E}(y, x)$ and $h_z^\mathcal{E}(x, y) = h_z^\mathcal{E}(y, x)$. We notice that $R_\mathcal{E}(x, y) = g_x^\mathcal{E}(y,y) = g_x^\mathcal{E}(y,z) + g_y^\mathcal{E}(x,z)$ and $g_z^\mathcal{E}(x,y) = (1/2)\{R_\mathcal{E}(x,z) + R_\mathcal{E}(z,y) - R_\mathcal{E}(x,y)\}$ for all

$x, y, z \in V$, and also that $h_z^{\mathcal{E}}(x, x) = \sup\{|h(x) - h(z)|^2/\mathcal{E}(h, h) \mid h \in H_{\mathcal{E}},\ \mathcal{E}(h, h) > 0\}$ (see [5, 7.2]). Since Γ is assumed to be non-parabolic, given a vertex $x \in V$, there exists uniquely a function $g_x \in D_0[\mathcal{E}_\Gamma]$ such that $\mathcal{E}_\Gamma(g_x, v) = v(x)$ for all $v \in D_0[\mathcal{E}_\Gamma]$. We write $g_\Gamma^0(x, y)$ for $g_x(y)$. It holds also that $g_\Gamma^0(x, y) = g_\Gamma^0(y, x)$. These functions are related as follows:

$$(1) \quad g_z^{\mathcal{E}}(x, y) = h_z^{\mathcal{E}}(x, y) + (g_\Gamma^0(x, y) - g_\Gamma^0(x, z) - g_\Gamma^0(y, z) + g_\Gamma^0(z, z)), \quad x, y, z \in V$$

(see [5, 7.2]).

Now as in the case of the form $\mathcal{E}_\Gamma$, we consider a subspace $Q(\mathcal{E})$ of $D[\mathcal{E}]$ which consists of functions u such that $\mathcal{E}(u, v) = 0$ for all $v \in D[\mathcal{E}]$ vanishing on a finite subset of V. The compactification relative to $Q(\mathcal{E})$ is called the Kuramochi compactification of the network Γ relative to the resistance form $\mathcal{E}$, and denoted by $\mathcal{K}(\mathcal{E})$. The identity map of V extends to a continuous map from the Royden compactification $\mathcal{R}(\mathcal{E}_\Gamma)$ of Γ onto $\mathcal{K}(\mathcal{E})$. We denote by $\rho_{\mathcal{E}}$ the induced map from the Royden boundary $\partial\mathcal{R}(\mathcal{E}_\Gamma)$ onto the Kuramochi boundary $\partial\mathcal{K}(\mathcal{E})$. Let $\Delta^K(\mathcal{E}) = \rho_{\mathcal{E}}(\Delta(\mathcal{E}_\Gamma))$ and $\nu_a = \rho_{\mathcal{E}*}\bar{\nu}_a$ ($a \in V$). Here and after, we fix a point $o \in V$ and write ν for ν_o.

We take a positive function μ on V and consider it as a measure on V, $\mu = \sum_{x \in V} \mu(x)\delta_x$. In what follows, μ is chosen in such a way that $\mu(V) = 1$,

$$(2) \qquad \int_V R_{\mathcal{E}}(o, x)^2 \, d\mu(x) < +\infty.$$

The measure μ extends to a Radon measure, denoted by the same letter, on the Kuramochi compactification. Here we recall some results in [5, 7.3]:

(i) $D[\mathcal{E}] \subset L^2(\mathcal{K}(\mathcal{E}), \mu)$.

(ii) Any function $u \in D[\mathcal{E}]$ can be written in the Royden decomposition as

$$u(x) = \int_{\partial\mathcal{K}(\mathcal{E})} \tau(u) \, d\nu_x + g(x), \quad x \in V,\ g \in D_0[\mathcal{E}],$$

where $\tau(u)$ is a function in $L^2(\partial\mathcal{K}(\mathcal{E}), \nu)$.

(iii) $(\mathcal{E}, D[\mathcal{E}])$ is a regular Dirichlet form on $L^2(\mathcal{K}(\mathcal{E}), \mu)$.

(iv) The domain $D[L^{\mathcal{E}}]$ of the self-adjoint operator $L^{\mathcal{E}}$ associated to the Dirichlet form $\mathcal{E}$ is embedded in the space of continuous functions on $\mathcal{K}(\mathcal{E})$, and $D[L^{\mathcal{E}}]$ is dense both in the Banach space $C(\mathcal{K}(\mathcal{E}))$ of continuous functions on $\mathcal{K}(\mathcal{E})$ and the Hilbert space $(D[\mathcal{E}], \mathcal{E} + \delta_o^2)$.

(v) The domain $D[\mathcal{E}]$ is compactly embedded into $L^2(\mathcal{K}(\mathcal{E}), \mu)$.

Now we define a Radon measure $\bar{\mu}$ on the Kuramochi compactification $\mathcal{K}(\mathcal{E})$ by

$$\bar{\mu}(f) = \int_V f \, d\mu + \int_{\partial\mathcal{K}(\mathcal{E})} f \, d\nu, \quad f \in C(\mathcal{K}(\mathcal{E})).$$

Then any function u of $D[\mathcal{E}]$ coupled with $\tau(u)$ can be considered as a function in $L^2(\mathcal{K}(\mathcal{E}), \bar{\mu})$.

Given $u \in D[\mathcal{E}]$, if we write h for the harmonic part $\int_{\partial\mathcal{K}(\mathcal{E})} \tau(u) d\nu_x$ of u, then we have the following basic identity:

$$(3) \quad \int_{\partial\mathcal{K}(\mathcal{E})} \tau(u)^2 \, d\nu_x - \left(\int_{\partial\mathcal{K}(\mathcal{E})} \tau(u) \, d\nu_x \right)^2 = \sum_{y \in V} g_\Gamma^0(x, y) \sum_{z \sim y} \frac{(h(y) - h(z))^2}{r(|yz|)}, \quad x \in V,$$

from which we can deduce that

$$(4) \quad \int_{\partial\mathcal{K}(\mathcal{E})} \tau(u)^2 \, d\nu_x \le 2 g_\Gamma^0(x, x) \mathcal{E}(u, u) + 2u(x)^2, \quad x \in V$$

(see [5, Lemma 7.8]). Using this inequality, we get

$$\int_{\partial\mathcal{K}(\mathcal{E})} \tau(u)^2 \, d\nu_x \le 2 \left(g_\Gamma^0(x, x) + \frac{1}{\mu(x)} \right) \left(\mathcal{E}(u, u) + \int_V u^2 \, d\mu \right),$$

since $\mu(x) u(x)^2 \le \int_V u^2 \, d\mu$. This shows in particular that the norm $\mathcal{E}(u, u)^{1/2} + \left(\int_V u^2 \, d\mu\right)^{1/2} + \left(\int_{\partial\mathcal{K}(\mathcal{E})} \tau(u)^2 \, d\nu\right)^{1/2}$ is equivalent to the norm $\mathcal{E}(u, u)^{1/2} + \left(\int_V u^2 \, d\mu\right)^{1/2}$. Since $(\mathcal{E}, D[\mathcal{E}])$ is a regular Dirichlet form on $L^2(\mathcal{K}(\mathcal{E}), \mu)$, we can thus deduce the following

Theorem 3. *Let $\Gamma = (V, E, r)$ be a connected non-parabolic network and $\mathcal{E}$ a resistance form of Γ. Then the Dirichlet form $(\mathcal{E}, D[\mathcal{E}])$ on $L^2(\mathcal{K}(\mathcal{E}), \bar{\mu})$ is regular.*

Let $(\bar{L}^{\mathcal{E}}, D[\bar{L}^{\mathcal{E}}])$ be the self-adjoint operator associated with the regular Dirichlet form $\mathcal{E}$ in $L^2(\mathcal{K}(\mathcal{E}), \bar{\mu})$. For $u \in D[\bar{L}^{\mathcal{E}}]$, we note that

$$\bar{L}^{\mathcal{E}} u(x) = \frac{1}{\mu(x)} L^c u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} \frac{u(x) - u(y)}{r(|xy|)}, \quad x \in V.$$

The restriction of $\bar{L}^{\mathcal{E}} u$ to the Kuramochi boundary is denoted by $N^{\mathcal{E}} u$. Then we have

$$\mathcal{E}(u, v) = \int_V v L^c u \, d\mu^c + \int_{\partial\mathcal{K}(\mathcal{E})} \tau(v) N^{\mathcal{E}} u \, d\nu, \quad v \in D[\mathcal{E}].$$

It is a consequence from the definitions of $L^{\mathcal{E}}$ and $\bar{L}^{\mathcal{E}}$ that

$$D[L^{\mathcal{E}}] = \{u \in D[\bar{L}^{\mathcal{E}}] \mid N^{\mathcal{E}} u = 0 \text{ in } L^2(\partial\mathcal{K}(\mathcal{E}), \nu)\} \quad (\subset C(\mathcal{K}(\mathcal{E}))).$$

We remark that $Q(\mathcal{E})$ is a subspace of $D[L^{\mathcal{E}}]$. In fact, let u be a function in $Q(\mathcal{E})$. Then there exists a finite subset A of V such that $\mathcal{E}(u, v) = 0$ for all $v \in D[\mathcal{E}]$ which vanishes on A. Let χ_A be the characteristic function of A. Then for any $v \in D[\mathcal{E}]$, we have

$$\mathcal{E}(u, v) = \mathcal{E}(u, \chi_A v) = \sum_{x \in A} v(x) L^c u(x)$$

and hence we get

$$|\mathcal{E}(u, v)| \leq \left(\int_A (L^{\mathcal{E}} u)^2 \, d\mu \right)^{1/2} \left(\int_V v^2 \, d\mu \right)^{1/2}.$$

This shows that $u \in D[L^{\mathcal{E}}]$.

Here, referring to [5, Proposition 4.1, Theorem 7.11], we mention the following propositions:

(I) The following conditions are mutually equivalent:
- (i) $\sup_{x \in V} g_\Gamma^0(x, x)$ is finite.
- (ii) Any $g \in D_0[\mathcal{E}_\Gamma]$ is bounded.
- (iii) $\partial R(\mathcal{E}_\Gamma) = \Delta(\mathcal{E}_\Gamma)$, that is, for any bounded $g \in D_0[\mathcal{E}_\Gamma]$, $g(x)$ tends to zero as $x \in V \to \infty$.
- (iv) For any $g \in D_0[\mathcal{E}_\Gamma]$, $g(x)$ tends to zero as $x \in V \to \infty$.

(II) $\sup_{x,y \in V} R_{\mathcal{E}}(x, y)$ is bounded if and only if every $f \in D[\mathcal{E}]$ is bounded.

(III) The following conditions are mutually equivalent:
- (i) $\sup_{x,y \in V} h_x^{\mathcal{E}}(y, y)$ is finite.
- (ii) Any $h \in H_{\mathcal{E}}$ is bounded.
- (iii) For any $u \in D[\mathcal{E}]$, $\tau(u)$ is continuous on $\Delta^K(\mathcal{E})$.
- (iv) A nonnegative subharmonic function u in $D[\mathcal{E}]$ is bounded.

Now we prove the following

Theorem 4. *Let $\Gamma = (V, E, r)$ be a connected non-parabolic network and $\mathcal{E}$ a resistance form of Γ. Then $D[\mathcal{E}]$ is compactly embedded in $L^2(\mathcal{K}(\mathcal{E}), \bar{\mu})$ if $\sup_{x \in V} g_\Gamma^0(x,x)$ is finite.*

Proof. Let $\{u_n\}$ be a sequence in $D[\mathcal{E}]$ such that $\mathcal{E}(u_n, u_n) + u_n(o)^2$ is bounded as $n \to \infty$. Let h_n be the harmonic part of u_n. Then we have

$$\begin{aligned} u_n(o) - h_n(o) &= \mathcal{E}_\Gamma(g_\Gamma^0(o, *), u_n - h_n) \\ &= \mathcal{E}_\Gamma(g_\Gamma^0(o, *), u_n) \end{aligned}$$

and hence

$$h_n(o)^2 \leq 2u_n(o)^2 + 2g_\Gamma^0(o, o)\mathcal{E}(u_n, u_n).$$

Thus we see that $\mathcal{E}(h_n, h_n) + h_n(o)^2$ are bounded as $n \to \infty$. Since $D[\mathcal{E}]$ is compactly embedded in $L^2(\mathcal{K}(\mathcal{E}), \mu)$, passing to a subsequence, we may assume that u_n and h_n

respectively converge to functions u and h in $L^2(\mathcal{K}(\mathcal{E}), \mu)$, where h is the harmonic part of u. Let $v_n = u_n - u$ and $k_n = h_n - h$. Then in view of (3), we have

$$\int_{\partial\mathcal{K}(\mathcal{E})} \tau(v_n)^2\, d\nu_o = k_n(o)^2 + \sum_{x\in V} g_\Gamma^0(o, x) \sum_{y\sim x} \frac{(k_n(x) - k_n(y))^2}{r(|xy|)}.$$

Given $\varepsilon > 0$, let $V_\varepsilon = \{x \in V \mid g_\Gamma^0(o, x) \geq \varepsilon\}$. Since $g_\Gamma^0(o, x)$ tends to 0 as $x \in V$ goes to infinity by the assumption: $\sup_{x\in V} g_\Gamma^0(x, x) < +\infty$, V_ε is a finite subset of V. Therefore for sufficiently large n,

$$k_n(o)^2 + \sum_{x\in V_\varepsilon} g_\Gamma^0(o, x) \sum_{y\sim x} \frac{(k_n(x) - k_n(y))^2}{r(|xy|)} < \varepsilon.$$

Since

$$\sum_{x\in V\setminus V_\varepsilon} g_\Gamma^0(o, x) \sum_{y\sim x} \frac{(k_n(x) - k_n(y))^2}{r(|xy|)} < \varepsilon\mathcal{E}_\Gamma(v_n, v_n),$$

we get

$$\int_{\partial\mathcal{K}(\mathcal{E})} \tau(v_n)^2\, d\nu < \varepsilon\left(1 + \sup_n \mathcal{E}_\Gamma(v_n, v_n)\right)$$

for n large enough. This shows that $\int_{\partial\mathcal{K}(\mathcal{E})} \tau(v_n)^2\, d\nu$ tends to 0 as $n \to \infty$. Thus we can deduce that $D[\mathcal{E}]$ is compactly embedded in $L^2(\mathcal{K}(\mathcal{E}), \bar{\mu})$. □

REMARK. Let $\Gamma = (V, E, r)$ be a connected infinite network and $\mathcal{E}$ a resistance form of Γ.

(i) Let $D[\mathcal{E}^*] = \{\tau(u) \mid u \in D[\mathcal{E}]\}$ $(\subset L^2(\partial\mathcal{K}(\mathcal{E}), \nu))$ and $\mathcal{E}^*(\tau(u), \tau(v)) = \mathcal{E}(h_u, h_v)$ for $u, v \in D[\mathcal{E}]$, where h_u denotes the harmonic part of u in the Royden decomposition. Then $(\mathcal{E}^*, D[\mathcal{E}^*])$ is a regular Dirichlet form on $L^2(\partial\mathcal{K}(\mathcal{E}), \nu)$.

(ii) Let $(\mathcal{F}, D[\mathcal{F}])$ be a Dirichlet form on a closed subspace of $L^2(\partial\mathcal{K}(\mathcal{E}), \nu)$ with $\mathcal{F}(1, 1) = 0$, and define a form $(\mathcal{E}_\mathcal{F}, D[\mathcal{E}_\mathcal{F}])$ by

$$\mathcal{E}_\mathcal{F}(u, v) = \mathcal{E}(u, v) + \mathcal{F}(\tau(u), \tau(v)); \quad D[\mathcal{E}_\mathcal{F}] = \{u \in D[\mathcal{E}] \mid \tau(u) \in D[\mathcal{F}]\}.$$

Then $\mathcal{E}_\mathcal{F}$ is a resistance form of Γ. Moreover for a positive number t, we set $\mathcal{E}_{\mathcal{F};t}(u, v) = \mathcal{E}(u, v) + t\mathcal{F}(\tau(u), \tau(v))$. Then the limit of the forms as $t \to +\infty$ also gives a resistance form of Γ.

(iii) Given a finite subset K of V, we can define a Dirichlet form on the space $l(K)$ of functions on K by letting $\mathcal{E}_K^*(u, u) = \inf\{\mathcal{E}(\tilde{u}, \tilde{u}) \mid \tilde{u} \in D[\mathcal{E}],\ \tilde{u} = u \text{ on } K\}$ for $u \in l(K)$. Then we get a finite connected network $\Gamma_K^* = (K, E_K, r_K)$ such that the effective resistance of Γ_K^* between two points of K is equal to the effective resistance relative to $\mathcal{E}$ (cf. [7, Theorem 2.1.12, Corollary 2.1.13], [5, Theorem 1.13]). Thus if we take an increasing sequence $\{V_n\}$ of finite subsets of V such that $V = \bigcup_n V_n$, then Γ endowed with the resistance form $\mathcal{E}$ can be considered as a limit of finite networks $\{\Gamma^*_{V_n}\}$ (see [5]). Conversely if we have a sequence $\{\Gamma'_n\}$ of finite networks such that the set of vertices of Γ'_n includes the vertex boundary of V_n, namely the set of vertexes of V_n which are adjacent to those outside of V_n, we get a sequence $\{\Gamma''_n\}$ of finite networks obtained by joining the subnetwork Γ_n of Γ generated by V_n with Γ'_n through the vertex boundary of V_n. Since the effective resistance of Γ''_n between two points of V_n is bounded by the effective resistance of $\mathcal{E}_\Gamma$ between them, by taking subsequence if necessarily, we have a resistance form $\mathcal{E}$ of Γ such that for any pair of points of V, the effective resistance of Γ''_n between them (for large n) converges to the effective resistance relative to $\mathcal{E}$ as $n \to \infty$ (see [5, 7.4]).

3. Random walks

We consider a connected non-parabolic network $\Gamma = (V, E, r)$ endowed with a measure $\mu\colon V \to (0, +\infty)$ satisfying (2) and the random walk $\{X_n\}$ of Γ.

Let $(\mathcal{M}, d_{\mathcal{M}})$ be a complete separable metric space. Define a set $D[\mathcal{E}_{\Gamma,\mathcal{M}}]$ of maps of V to $\mathcal{M}$ and a functional $\mathcal{E}_{\Gamma,\mathcal{M}}$ on $D[\mathcal{E}_{\Gamma,\mathcal{M}}]$ by

$$D[\mathcal{E}_{\Gamma,\mathcal{M}}] = \left\{\phi\colon V \to \mathcal{M} \;\middle|\; \sum_{x\sim y} \frac{d_{\mathcal{M}}(\phi(x), \phi(y))^2}{r(|xy|)} < +\infty\right\};$$

$$\mathcal{E}_{\Gamma,\mathcal{M}}(\phi) = \frac{1}{2}\sum_{x\sim y} \frac{d_{\mathcal{M}}(\phi(x), \phi(y))^2}{r(|xy|)}, \quad \phi \in D[\mathcal{E}_{\Gamma,\mathcal{M}}].$$

A map $\phi\colon V \to \mathcal{M}$ in $D[\mathcal{E}_{\Gamma,\mathcal{M}}]$ is called a Dirichlet finite map. The composition $f \circ \phi$ of a Lipschitz function f on $\mathcal{M}$ and a Dirichlet finite map $\phi\colon V \to \mathcal{M}$ belongs to $D[\mathcal{E}_\Gamma]$. Thus applying the result of [1] mentioned in the introduction, we see that the sequence $\{f(\phi(X_n))\}$ is almost surely convergent, and the process $\phi(X_n)$ is already almost surely convergent in $\mathcal{M}$.

Now we consider a resistance form $\mathcal{E}$ of Γ. For any $x, y \in V$, let

$$d^{\mathcal{E}}(x, y) = \left(\int_V (g^{\mathcal{E}}_\mu(x, z) - g^{\mathcal{E}}_\mu(y, z))^2 \, d\mu(z)\right)^{1/2},$$

where we set $g^{\mathcal{E}}_\mu(x, y) = \int_V g^{\mathcal{E}}_z(x, y)\, d\mu(z)$. Then it is proved in [5, Theorem 3.10] that $d^{\mathcal{E}}$ gives a compatible metric on $\mathcal{K}(\mathcal{E})$. In what follows, $\mathcal{K}(\mathcal{E})$ is equipped with the distance $d^{\mathcal{E}}$.

Now we prove the following

Lemma 5. *The inclusion map I of V into the metric space $(\mathcal{K}(\mathcal{E}), d^{\mathcal{E}})$ is a Dirichlet finite map and $\mathcal{E}_{\Gamma,\mathcal{K}(\mathcal{E})}(I) = \iint_{V\times V} R_{\mathcal{E}}(z, w)\, d\mu(z)\, d\mu(w)$.*

Proof. We have

$$
\begin{aligned}
\mathcal{E}_{\Gamma,\mathcal{K}(\mathcal{E})}(I) &= \sum_{x\sim y} \frac{d^{\mathcal{E}}(x,y)^2}{r(|xy|)} \\
&= \sum_{x\sim y} \int_V \frac{(g_\mu^{\mathcal{E}}(x,z) - g_\mu^{\mathcal{E}}(y,z))^2}{r(|xy|)}\, d\mu(z) \\
&= \int_V \sum_{x\sim y} \frac{(g_\mu^{\mathcal{E}}(x,z) - g_\mu^{\mathcal{E}}(y,z))^2}{r(|xy|)}\, d\mu(z) \\
&= \int_V \mathcal{E}(g_\mu^{\mathcal{E}}(z,*), g_\mu^{\mathcal{E}}(z,*))\, d\mu(z) \\
&= \int_V g_\mu^{\mathcal{E}}(z,z)\, d\mu(z) \\
&= \iint_{V\times V} R_{\mathcal{E}}(z,w)\, d\mu(z)\, d\mu(w) \\
&< \int_V R_{\mathcal{E}}(z,o)\, d\mu(z) + \int_V R_{\mathcal{E}}(o,w)\, d\mu(w) = 2\int_V R_{\mathcal{E}}(z,o)\, d\mu(z).
\end{aligned}
$$

This completes the proof of the lemma. □

Theorem 6. *Let $\Gamma = (V, E, r)$ be a connected non-parabolic network and $\mathcal{E}$ a resistance form of Γ. Then there exists a $\Delta^K(\mathcal{E})$-valued random variable $X_\infty^{\mathcal{E}}$ such that the process X_n almost surely converges to $X_\infty^{\mathcal{E}}$ in $\mathcal{K}(\mathcal{E})$, the measure ν_{X_n} converges weakly to the delta measure $\delta_{X_\infty^{\mathcal{E}}}$ almost surely, and for any $u \in D[\mathcal{E}]$, $u(X_n)$ converges to $\tau(u)(X_\infty^{\mathcal{E}})$ almost surely and in L^2 as $n \to \infty$.*

Proof. Lemma 5 and the result in [1] stated above imply that the process $\{X_n\}$ is Cauchy in $\mathcal{K}(\mathcal{E})$ almost surely. Let $X_\infty^{\mathcal{E}} = \lim_{n\to\infty} X_n$. We recall here that $D[L^{\mathcal{E}}]$ is densely embedded in both $C(\mathcal{K}(\mathcal{E}))$ and $D[\mathcal{E}]$. Then together with Theorem 1, we see that for $u \in D[L^{\mathcal{E}}]$,

$$\lim_{n\to\infty} u(X_n) = \lim_{n\to\infty} \int_{\partial\mathcal{K}(\mathcal{E})} \tau(u)\, d\nu_{X_n} = \tau(u)(X_\infty^{\mathcal{E}}).$$

Moreover it follows that ν_{X_n} weakly converges to $\delta_{X_\infty^{\mathcal{E}}}$ almost surely, and since the support of the measure ν_x coincides with $\Delta^K(\mathcal{E})$, it follows that $X_\infty^{\mathcal{E}}$ is a $\Delta^K(\mathcal{E})$-valued random variable, and further it is easy to see that the image is dense in $\Delta^K(\mathcal{E})$.

Now we want to show that for $u \in D[\mathcal{E}]$, $u(X_n)$ converges to $\tau(u)(X_\infty^{\mathcal{E}})$ in L^2. We fix a point $a \in V$. For any positive number ε, we take a function $u_\varepsilon \in D[L^{\mathcal{E}}]$ such that

$\mathcal{E}(u-u_\varepsilon)+(u-u_\varepsilon)(a)^2<\varepsilon$. Let $h_\varepsilon(x)=\int_{\partial\mathcal{K}(\mathcal{E})}\tau(u-u_\varepsilon)\,d\nu_x$ and $g_\varepsilon=u-u_\varepsilon-h_\varepsilon\in D_0[\mathcal{E}_0]$. Then we have

$$\begin{aligned}\mathbf{E}_a[(u-u_\varepsilon)^2(X_n)]&\le 2\mathbf{E}_a[h_\varepsilon^2(X_n)]+2\mathbf{E}_a[g_\varepsilon^2(X_n)]\\&\le 2\int_{\partial\mathcal{K}(\mathcal{E})}\tau(h_\varepsilon)^2\,d\nu_a+2\mathbf{E}_a[g_\varepsilon^2(X_n)],\end{aligned}$$

where we have used the fact that h_ε^2 is subharmonic, so that

$$\mathbf{E}_a[h_\varepsilon^2(X_n)]\le\int_{\partial\mathcal{K}(\mathcal{E})}\tau(h_\varepsilon)^2\,d\nu_a.$$

In view of (4), we observe that

$$\begin{aligned}\int_{\partial\mathcal{K}(\mathcal{E})}\tau(h_\varepsilon)^2\,d\nu_a&\le 2g_\Gamma^0(a,a)\mathcal{E}(u-u_\varepsilon,u-u_\varepsilon)+2(u-u_\varepsilon)(a)^2\\&\le 2(g_\Gamma^0(a,a)+1)\varepsilon.\end{aligned}$$

Thus we obtain

$$\mathbf{E}_a[(u-u_\varepsilon)^2(X_n)]\le 4(g_\Gamma^0(a,a)+1)\varepsilon+2\mathbf{E}_a[g_\varepsilon^2(X_n)].$$

Using this, we have

$$\begin{aligned}&\mathbf{E}_a[(u(X_n)-\tau(u)(X_\infty^{\mathcal{E}}))^2]\\&\le 4\mathbf{E}_a[h_\varepsilon^2(X_n)]+4\mathbf{E}_a[\tau(u-u_\varepsilon)^2(X_\infty^{\mathcal{E}})]+2\mathbf{E}_a[(u_\varepsilon(X_n)-\tau(u_\varepsilon)(X_\infty^{\mathcal{E}}))^2]\\&\le 16(g_\Gamma^0(a,a)+1)\varepsilon+8\mathbf{E}_a[g_\varepsilon^2(X_n)]+4\int_{\partial\mathcal{K}(\mathcal{E})}\tau(h_\varepsilon)^2\,d\nu_a\\&\quad+2\mathbf{E}_a[(u_\varepsilon(X_n)-\tau(u_\varepsilon)(X_\infty^{\mathcal{E}}))^2]\\&\le 24(g_\Gamma^0(a,a)+1)\varepsilon+8\mathbf{E}_a[g_\varepsilon^2(X_n)]+2\mathbf{E}_a[(u_\varepsilon(X_n)-\tau(u_\varepsilon)(X_\infty)^{\mathcal{E}})^2].\end{aligned}$$

Thus we get

$$\limsup_{n\to\infty}\mathbf{E}_a[(u(X_n)-\tau(u)(X_\infty^{\mathcal{E}}))^2]\le 24(g_\Gamma^0(a,a)+1)\varepsilon.$$

Letting ε go to zero, we see that $\lim_{n\to\infty}\mathbf{E}_a[(u(X_n)-\tau(u)(X_\infty^{\mathcal{E}}))^2]=0$. This completes the proof of the theorem. □

Now we consider a map ϕ from the network Γ to a simply connected, complete separable geodesic space $(\mathcal{M},d_{\mathcal{M}})$ of nonpositive curvature (cf. [4], [13]). For any $x\in V$, there exists uniquely a point of $\mathcal{M}$, denoted by $P\phi(x)$, such that

$$\sum_{y\sim x}\frac{d_{\mathcal{M}}(P\phi(x),\phi(y))^2}{r(|xy|)}=\inf_{q\in\mathcal{M}}\sum_{y\sim x}\frac{d_{\mathcal{M}}(q,\phi(y))^2}{r(|xy|)};$$

$P\phi(x)$ is the center of mass of the measure $\sum_{y\sim x} r(|xy|)^{-1}\delta_{\phi(y)}$ on $\mathcal{M}$. A map ϕ: $V \to \mathcal{M}$ is said to be harmonic if $P\phi(x) = \phi(x)$ at any $x \in V$. A harmonic map ϕ: $V \to \mathcal{M}$ pulls convex functions η on an open subset $A \subset \mathcal{M}$ back to subharmonic functions $\eta\circ\phi$ on $\phi^{-1}(A)$ (see [4, Proposition 12.3 (Jensen's inequality)]).

Now we prove the following

Theorem 7. *Let ϕ be a map from a connected non-parabolic network $\Gamma = (V, E, r)$ to a simply connected, complete, separable geodesic space $(\mathcal{M}, d_{\mathcal{M}})$ of non-positive curvature. Let $\phi\colon V \to \mathcal{M}$ be a Dirichlet finite harmonic map. Then the image $\phi(V)$ is contained in the convex hull $\mathcal{C}(L)$ of the set L of points to which $\phi(X_n)$ converges almost surely.*

Moreover $\phi(V)$ is bounded if any $h \in H_{\mathcal{E}_\Gamma}$ is bounded. In particular, ϕ must be constant if $H_{\mathcal{E}_\Gamma} = \boldsymbol{R}$, that is, Γ admits no non-constant Dirichlet finite harmonic functions.

Proof. Let η be a distance function to the convex hull $\mathcal{C}(L)$ of L, that is the smallest closed convex subset containing L in $\mathcal{M}$. Then η^2 is convex and hence $\eta^2\circ\phi$ is subharmonic on V. Thus we have

$$\eta^2 \circ \phi(x) \le \mathbf{E}_x[\eta^2 \circ \phi(X_n)]$$

for any $x \in V$ and all $n = 1, 2, \ldots$. Since $\lim_{n\to\infty} \eta^2(\phi(X_n)) = 0$ almost surely, we get $\eta^2 \circ \phi(x) = 0$, that is, $\phi(x) \in \mathcal{C}(L)$.

Now we suppose that any $h \in H_{\mathcal{E}_\Gamma}$ is bounded. Since this condition is equivalent to the condition that any nonnegative subharmonic function u of $D[\mathcal{E}_\Gamma]$ is bounded, for the distance function η to a point of $\mathcal{M}$, $\eta \circ \phi$ is bounded. Thus $\phi(V)$ must be bounded. Moreover we suppose that Γ admits no non-constant Dirichlet finite harmonic functions. Then $\Delta(\mathcal{E}_\Gamma)$ consists of a single point, and hence so does L. Thus ϕ must be a constant map. This completes the proof of the theorem. □

Let Ω be the set of one-sided infinite paths in a connected non-parabolic network Γ. Given a path $\omega \in \Omega$, the set of limit points of ω in the Royden boundary $\partial\mathcal{R}(\mathcal{E}_\Gamma)$ of Γ is defined as

$$L(\omega) = \overline{\{X_n(\omega)\}} \cap \partial\mathcal{R}(\mathcal{E}_\Gamma).$$

Then we can deduce from Theorem 6 the following

Lemma 8. *For any null family Σ of one-sided infinite paths, one has*

$$\overline{\bigcup\{L(\omega) \mid \omega \in \Omega \setminus \Sigma\}} \supset \Delta(\mathcal{E}_\Gamma).$$

Now we prove the following

Theorem 9. *Let $\phi\colon \Gamma \to (\mathcal{M}, d_{\mathcal{M}})$ be a Dirichlet finite map from a connected non-parabolic network $\Gamma = (V, E, r)$ to a proper metric space $(\mathcal{M}, d_{\mathcal{M}})$, that is a metric space such that any bounded closed subset is compact. Let $\overline{\mathcal{M}} = \mathcal{M} \cup \{\infty_{\mathcal{M}}\}$ be the one-point compactification of $\mathcal{M}$. Then ϕ extends to a continuous map $\bar{\phi}\colon \mathcal{R}(\mathcal{E}_\Gamma) \to \overline{\mathcal{M}}$ from the Royden compactification $\mathcal{R}(\mathcal{E}_\Gamma)$ of Γ to $\overline{\mathcal{M}}$. Moreover there exists a null family Σ in Ω such that $\phi(X_n(\omega))$ converges in $\mathcal{M}$ for all $\omega \in \Omega \setminus \Sigma$ and*

$$\bar{\phi}(\Delta(\mathcal{E}_\Gamma)) = \overline{\left\{\lim_{n\to\infty} \phi(X_n(\omega)) \in \mathcal{M} \;\middle|\; \omega \in \Omega \setminus (\Sigma' \cup \Sigma)\right\}} \cup \{\infty_{\mathcal{M}}\}$$

for any null family Σ' in Ω.

Proof. For a point $x \in \mathcal{M}$, we denote by η_x the distance function to x in $\mathcal{M}$. Let $\Lambda_\phi = \{\xi \in \partial\mathcal{R}(\mathcal{E}_\Gamma) \mid \overline{\eta_x \circ \phi}(\xi) = +\infty\}$, where $\overline{\eta_x \circ \phi}$ stands for the continuous extension of $\eta_x \circ \phi$ to $\mathcal{R}(\mathcal{E}_\Gamma)$ with values in $\mathbf{R} \cup \{\pm\infty\}$. This closed subset is independent of the choice of a reference point x. Now we take a countably infinite dense subset $\{x_i\}$ of $\mathcal{M}$. Let ξ and $\{v_n\}$ be, respectively, a point of $\partial\mathcal{R}(\mathcal{E}_\Gamma) \setminus \Lambda_\phi$ and a sequence in V converging to ξ. Then $\phi(v_n)$ stays in a compact subspace in $\mathcal{M}$. Since $d_X(x_i, \phi(v_n))$ tends to $\overline{\eta_{x_i} \circ \phi}(\xi)$ as $n \to \infty$ for all x_i which are densely distributed in $\mathcal{M}$, we can deduce that as n tends to infinity, $\phi(v_n)$ converges to a point, $\bar{\phi}(\xi)$, in $\mathcal{M}$. By setting $\bar{\phi}(\xi) = \infty_{\mathcal{M}}$ for $\xi \in \Lambda_\phi$, we obtain a continuous map $\bar{\phi}$ from $\mathcal{R}(\mathcal{E}_\Gamma)$ to $\overline{\mathcal{M}}$.

Let Ω_ϕ be the set of one-sided infinite paths along which $\phi(X_n)$ converges in $\mathcal{M}$. For any $j = 1, 2, \ldots$, let $\bar{\phi}(\Delta(\mathcal{E}_\Gamma))_j = \{x \in \mathcal{M} \mid d_{\mathcal{M}}(x, \bar{\phi}(\Delta(\mathcal{E}_\Gamma))) < 1/j\}$ and $A_j = \bar{\phi}(\partial\mathcal{R}(\mathcal{E}_\Gamma)) \setminus \bar{\phi}(\Delta(\mathcal{E}_\Gamma))_j$. Since $\bar{\phi}^{-1}(A_j)$ is disjoint from $\Delta(\mathcal{E}_\Gamma)$, we have by Lemma 5.3 in [15] a function $g_j \in D_0[\mathcal{E}_\Gamma]$ such that $g_j = +\infty$ on $\bar{\phi}^{-1}(A_j) \cap \partial\mathcal{R}(\mathcal{E}_\Gamma)$. On the other hand, it follows from Theorem 1 that $\lim_{n\to\infty} g_j(X_n) = 0$ almost surely. This shows that $\{\omega \in \Omega_\phi \mid \lim_{n\to\infty} \phi(X_n(\omega)) \in A_j\})$ is a null family of paths, and hence, letting $\Sigma = \{\omega \in \Omega_\phi \mid \lim_{n\to\infty} \phi(X_n(\omega)) \in \bigcup_j A_j\}$, we see that $\lim_{n\to\infty} \phi(X_n(\omega)) \in \bar{\phi}(\Delta(\mathcal{E}_\Gamma))$ for all $\omega \in \Omega_\phi \setminus \Sigma$. Moreover by Lemma 8, the assertion holds true. $\square$

REMARK. Relevantly to Theorem 7, we refer to [8] in which a Liouville type theorem for harmonic maps to convex spaces via Markov chains is discussed. For an existence result of Dirichlet finite harmonic maps, see [12]. A connected parabolic network admits no non-constant Dirichlet finite harmonic maps to a simply connected, complete, geodesic space of nonpositive curvature. In fact, Theorem (3.34) in [11] states that a Dirichlet finite subharmonic function on such a network must be constant. We also refer to [3], where it is proved that if on a complete Riemannian manifold M, every harmonic function with finite Dirichlet energy is bounded, then every harmonic map with finite total energy from M into a Cartan–Hadamard manifold must also have bounded image.

4. Poisson equations

Let $\Gamma = (V, E, r)$ be a connected non-parabolic network and $\mathcal{E}$ a resistance form of Γ. In this section, we derive integral representations of solutions of Poisson equations on the Kuramochi compactification of $\mathcal{E}$.

To begin with, we show the following

Lemma 10 (Harnack's inequality). *Let h be a positive harmonic function on V. Then*

$$h(x) \leq \frac{g_\Gamma^0(x, x)}{g_\Gamma^0(y, x)} h(y)$$

for all $x, y \in V$.

Proof. Let $\{V_n\}$ be an increasing sequence of finite subsets of V such that $V = \bigcup_n V_n$. Let D_n be the space of functions on V which vanish outside of V_n. Then for any $x \in V_n$, there exists uniquely a function $g_x \in D_n$ satisfying

$$\mathcal{E}_\Gamma(g_x, u) = u(x)$$

for all $u \in D_n$. We write $g_n(x, y)$ for $g_x(y)$. Fix points $x, y \in V$ and consider V_n for n large enough. Then there exists uniquely a function $p_n \in D_n$ such that $p_n(x) = h(x)$, $p_n(y) = h(y)$, and $L^c p_n(z) = 0$ for any $z \in V_n \setminus \{x, y\}$. The maximum principle ensures that $p_n \leq h$ in V and hence $L^c p_n(x) \geq 0$ and $L^c p_n(y) \geq 0$. Then we have

$$\begin{aligned}
h(x) &= p_n(x) \\
&= \mathcal{E}_\Gamma(g_n(x, *), p_n) \\
&= \sum_{z \in V_n} g_n(x, z) L^c p_n(z) \\
&= g_n(x, x) L^c p_n(x) + g_n(x, y) L^c p_n(y) \\
&= \frac{g_n(x, x)}{g_n(y, x)} g_n(y, x) L^c p_n(x) + \frac{g_n(x, y)}{g_n(y, y)} g_n(y, y) L^c p_n(y) \\
&\leq \frac{g_n(x, x)}{g_n(y, x)} \{g_n(y, x) L^c p_n(x) + g_n(y, y) L^c p_n(y)\} \\
&= \frac{g_n(x, x)}{g_n(y, x)} \mathcal{E}_\Gamma(g_n(y, *), p_n) \\
&= \frac{g_n(x, x)}{g_n(y, x)} p_n(y) \\
&= \frac{g_n(x, x)}{g_n(y, x)} h(y).
\end{aligned}$$

Thus we get

$$h(x) \leq \frac{g_n(x,x)}{g_n(y,x)} h(y)$$

for all large n. As $n \to \infty$, $g_n(z,w)$ converges to $g^0_\Gamma(z,w)$ for any $(z,w) \in V \times V$, and thus we obtain the required inequality. $\square$

In what follows, we take a probability measure μ on V satisfying (2) and

$$\int_V \frac{g^0_\Gamma(x,x)}{g^0_\Gamma(x,o)}\, d\mu(x) < +\infty. \tag{5}$$

Proposition 11. (i) *For any fixed* $x \in V$, $g^0_\Gamma(x,*) \in D[\bar{L}^{\mathcal{E}}] \cap D_0[\mathcal{E}_\Gamma]$ *and the harmonic measure* ν_x *with respect to* $x \in V$ *is given by*

$$\nu_x = -N^{\mathcal{E}} g^0_\Gamma(x,*)\nu.$$

(ii) *Let*

$$G^0_\mu(x) = \int_V g^0_\Gamma(x,z)\, d\mu(z), \quad x \in V.$$

Then G^0_μ *belongs to* $D[\bar{L}^{\mathcal{E}}]$ *and* $N^{\mathcal{E}} G^0_\mu = \int_V N^{\mathcal{E}} g^0_\Gamma(x,*)\, d\mu(x)$. *Moreover* $N^{\mathcal{E}} G^0_\Gamma$ *satisfies*

$$0 < \mu(0) < -N^{\mathcal{E}} G^0_\mu < \int_V \frac{g^0_\Gamma(x,x)}{g^0_\Gamma(x,o)}\, d\mu(x).$$

Proof. For a function $u \in D[\mathcal{E}]$, we have

$$\begin{aligned} \left| \int_{\partial\mathcal{K}(\mathcal{E})} \tau(u)\, d\nu_x \right| &\leq \int_{\partial\mathcal{K}(\mathcal{E})} |\tau(u)|\, d\nu_x \\ &\leq \frac{g^0_\Gamma(x,x)}{g^0_\Gamma(x,o)} \int_{\partial\mathcal{K}(\mathcal{E})} |\tau(u)|\, d\nu. \end{aligned}$$

This implies that

$$|\mathcal{E}(g^0_\Gamma(x,*),u)| = \left| u(x) - \int_{\partial\mathcal{K}(\mathcal{E})} \tau(u)\, d\nu_x \right|$$

is bounded by $\mu(x)^{-1} \int_V |u|\, d\mu + g^0_\Gamma(x,x)/g^0_\Gamma(x,o) \int_{\partial\mathcal{K}(\mathcal{E})} |\tau(u)|\, d\nu$. Thus we see that $g^0_\Gamma(x,*)$ belongs to $D[\bar{L}^{\mathcal{E}}]$. Moreover since $L^c g^0_\Gamma(x,*) = \delta_x$, we get

$$\begin{aligned} u(x) - \int_{\partial\mathcal{K}(\mathcal{E})} \tau(u)\, d\nu_x &= \int_V u(y) L^{\mathcal{E}} g^0_\Gamma(x,y)\, d\mu(y) + \int_{\partial\mathcal{K}(\mathcal{E})} \tau(u) N^{\mathcal{E}} g^0_\Gamma(x,*)\, d\nu \\ &= u(x) + \int_{\partial\mathcal{K}(\mathcal{E})} \tau(u) N^{\mathcal{E}} g^0_\Gamma(x,*)\, d\nu. \end{aligned}$$

In this way, we obtain

$$\int_{\partial\mathcal{K}(\mathcal{E})} \tau(u)\, d\nu_x = -\int_{\partial\mathcal{K}(\mathcal{E})} \tau(u) N^{\mathcal{E}} g_\Gamma^0(x,*)\, d\nu.$$

This shows the first assertion.

Given $u \in D[\mathcal{E}]$, let $g(x) = u(x) - \int_{\partial\mathcal{K}(\mathcal{E})} \tau(u)\, d\nu_x$. Then we have

$$\begin{aligned}
|\mathcal{E}_\Gamma(G_\mu^0, u)| &= |\mathcal{E}_\Gamma(G_\mu^0, g)| \\
&= \left| \int_V g(x)\, d\mu(x) \right| \\
&= \left| \int_V u(x)\, d\mu(x) - \int_V \int_{\partial\mathcal{K}(\mathcal{E})} \tau(u)\, d\nu_x\, d\mu(x) \right| \\
&\le \int_V |u(x)|\, d\mu(x) + \int_V \int_{\partial\mathcal{K}(\mathcal{E})} |\tau(u)|\, d\nu_x\, d\mu(x) \\
&\le \int_V |u(x)|\, d\mu(x) + \int_V \frac{g_\Gamma^0(x,x)}{g_\Gamma^0(x,o)}\, d\mu(x) \int_{\partial\mathcal{K}(\mathcal{E})} |\tau(u)|\, d\nu.
\end{aligned}$$

This shows that G_μ^0 belongs to $D[\bar{L}^{\mathcal{E}}]$. It is easy to see the remaining assertions. This completes the proof of the proposition. $\square$

As in Section 3, we now introduce a kernel function $g_\mu^{\mathcal{E}}$ on $\mathcal{E}$ by

$$g_\mu^{\mathcal{E}}(x,y) = \int_V g_z^{\mathcal{E}}(x,y)\, d\mu(z), \quad x, y \in V.$$

Then we have

$$\mathcal{E}(g_\mu^{\mathcal{E}}(x,*), u) = u(x) - \int_V u\, d\mu, \quad u \in D[\mathcal{E}].$$

In particular, the function $g_\mu^{\mathcal{E}}(x,*)$ for a fixed $x \in V$ belongs to $D[L^{\mathcal{E}}]$. Similarly, let

$$h_\mu^{\mathcal{E}}(x,y) = \int_V h_z^{\mathcal{E}}(x,y)\, d\mu(z), \quad x, y \in V.$$

Then we have

$$\mathcal{E}(h_\mu^{\mathcal{E}}(x,*), h) = h(x) - \int_V h\, d\mu, \quad h \in H_{\mathcal{E}}.$$

In view of (1), we see that

$$g_\mu^{\mathcal{E}}(x,y) = h_\mu^{\mathcal{E}}(x,y) + g_\Gamma^0(x,y) - G_\mu^0(x) - G_\mu^0(y) + C_{\Gamma,\mu}, \tag{6}$$

where we put $C_{\Gamma,\mu} = \int_V g^0_\Gamma(z,z)\,d\mu(z)$.

Given a function $u \in D[\bar{L}^{\mathcal{E}}]$, we have

$$\begin{aligned} u(x) &= \int_V u\,d\mu + \mathcal{E}(g^{\mathcal{E}}_\mu(x,*),u) \\ &= \int_V u\,d\mu + \int_V g^{\mathcal{E}}_\mu(x,y)L^{\mathcal{E}}u(y)\,d\mu(y) + \int_{\partial\mathcal{K}(\mathcal{E})} g^{\mathcal{E}}_\mu(x,\xi)N^{\mathcal{E}}u(\xi)\,d\nu(\xi) \\ &= \int_V u\,d\mu + \int_V g^{\mathcal{E}}_\mu(x,y)L^{\mathcal{E}}u(y)\,d\mu(y) + \int_{\partial\mathcal{K}(\mathcal{E})} h^{\mathcal{E}}_\mu(x,\xi)N^{\mathcal{E}}u(\xi)\,d\nu(\xi) \\ &\qquad - (G^0_\mu(x) - C_{\Gamma,\mu})\int_{\partial\mathcal{K}(\mathcal{E})} N^{\mathcal{E}}u(\xi)\,d\nu(\xi). \end{aligned}$$

Since

$$\int_V L^{\mathcal{E}}u\,d\mu + \int_{\partial\mathcal{K}(\mathcal{E})} N^{\mathcal{E}}u\,d\nu = \mathcal{E}(u,1) = 0,$$

by letting

$$\bar{g}^{\mathcal{E}}_\mu(x,y) = g^{\mathcal{E}}_\mu(x,y) + G^0_\mu(x) - C_{\Gamma,\mu},$$

we obtain an integral representation of a function u of $D[\bar{L}^{\mathcal{E}}]$ as follows:

$$u(x) = \int_V u\,d\mu + \int_V \bar{g}^{\mathcal{E}}_\mu(x,y)L^{\mathcal{E}}u(y)\,d\mu(y) + \int_{\partial\mathcal{K}(\mathcal{E})} h^{\mathcal{E}}_\mu(x,\xi)N^{\mathcal{E}}u(\xi)\,d\nu(\xi).$$

Let f be a function in $L^2(\mathcal{K}(\mathcal{E}),\bar{\mu})$. Suppose that $\bar{\mu}(f) = \int_V f\,d\mu + \int_{\partial\mathcal{K}(\mathcal{E})} f\,d\nu = 0$. Then for any $h \in H_{\mathcal{E}}$, we have

$$\begin{aligned} &\left|\int_V hf\,d\mu + \int_{\partial\mathcal{K}(\mathcal{E})} \tau(h)f\,d\nu\right|^2 \\ &= \left|\int_V (h-h(o))f\,d\mu + \int_{\partial\mathcal{K}(\mathcal{E})} \tau(h-h(0))f\,d\nu\right|^2 \\ &\le \left(\int_V (h-h(o))^2 d\,\mu + \int_{\partial\mathcal{K}(\mathcal{E})} \tau(h-h(o))^2\,d\nu\right)\left(\int_V f^2\,d\mu + \int_{\partial\mathcal{K}(\mathcal{E})} f^2\,d\nu\right) \\ &\le \left(\int_V R_{\mathcal{E}}(o,x)\,d\mu(x) + 2g^0_\Gamma(o,o)\right)\left(\int_V f^2\,d\mu + \int_{\partial\mathcal{K}(\mathcal{E})} f^2\,d\nu\right)\mathcal{E}(h,h), \end{aligned}$$

where we have used

$$\int_V (h(x)-h(o))^2\,d\mu(x) \le \int_V h^{\mathcal{E}}_o(x,x)\,d\mu(x)\mathcal{E}(h,h) \le \int_V R_{\mathcal{E}}(o,x)\,d\mu(x)\mathcal{E}(h,h)$$

and

$$\int_{\partial\mathcal{K}(\mathcal{E})} \tau(h-h(o))^2\,d\nu \le 2g^0_\Gamma(o,o)\mathcal{E}(h,h)$$

by (4). For $g \in D_0[\mathcal{E}_\Gamma]$, we have

$$\begin{aligned}\left|\int_V gf\,d\mu\right|^2 &\le \int_V g^2\,d\mu \int_V f^2\,d\mu \\ &\le \int_V \mathcal{E}_\Gamma(g_\Gamma^0(x,*),g)^2\,d\mu(x)\int_V f^2\,d\mu \\ &= \int_V g_\Gamma^0(x,x)\,d\mu(x)\int_V f^2\,d\mu\,\mathcal{E}_\Gamma(g,g) \\ &\le \left(\int_V R_{\mathcal{E}}(o,x)\,d\mu(x) + 2g_\Gamma^0(o,o)\right)\int_V f^2\,d\mu\ \ \mathcal{E}_\Gamma(g,g).\end{aligned}$$

In this way, we see that for any $u \in D[\mathcal{E}]$,

$$\begin{aligned}&\left|\int_V uf\,d\mu + \int_{\partial\mathcal{K}(\mathcal{E})}\tau(u)f\,d\nu\right|^2 \\ &\le \left(\int_V R_{\mathcal{E}}(o,x)\,d\mu(x) + 2g_\Gamma^0(o,o)\right)\left(\int_V f^2\,d\mu + \int_{\partial\mathcal{K}(\mathcal{E})} f^2\,d\nu\right)\mathcal{E}(u,u).\end{aligned}$$

This shows that there exists a function ϕ in $D[\mathcal{E}]$, unique up to additive constants, such that

$$\mathcal{E}(u,\phi) = \int_V uf\,d\mu + \int_{\partial\mathcal{K}(\mathcal{E})}\tau(u)f\,d\nu, \quad u \in D[\mathcal{E}],$$

so that ϕ belongs to $D[\bar{L}^{\mathcal{E}}]$, $\bar{L}^{\mathcal{E}}\phi = f$ in $K^2(\mathcal{K}(\mathcal{E}),\bar{\mu})$, and ϕ is expressed in the following way:

$$\phi(x) = \int_V \phi\,d\mu + \int_V \bar{g}_\mu^{\mathcal{E}}(x,y)f(y)\,d\mu(y) + \int_{\partial\mathcal{K}(\mathcal{E})} h_\mu^{\mathcal{E}}(x,\xi)f(\xi)\,d\nu(\xi). \tag{7}$$

In the case where $\bar{\mu}(f) \neq 0$, the function ϕ defined in (7) satisfies $L^{\mathcal{E}}\phi = f$ on V and $N^{\mathcal{E}}\phi = f + \bar{\mu}(f)N^{\mathcal{E}}G_\mu^0$ in $L^2(\partial\mathcal{K}(\mathcal{E}),\nu)$.

In fact, we have

$$\begin{aligned}&\phi(x) - \int_V \phi\,d\mu \\ &= \int_V \bar{g}_\mu^{\mathcal{E}}(x,y)f(y)\,d\mu(y) \\ &\quad + \int_{\partial\mathcal{K}(\mathcal{E})} h_\mu^{\mathcal{E}}(x,\xi)(f(\xi)-\bar{\mu}(f))\,d\nu(\xi) + \bar{\mu}(f)\int_{\partial\mathcal{K}(\mathcal{E})} h_\mu^{\mathcal{E}}(x,\xi)\,d\nu(\xi) \\ &= \int_V \bar{g}_\mu^{\mathcal{E}}(x,y)f(y)\,d\mu(y) + \int_{\partial\mathcal{K}(\mathcal{E})} h_\mu^{\mathcal{E}}(x,\xi)(f(\xi)-\bar{\mu}(f))\,d\nu(\xi) + \bar{\mu}(f)h_\mu^{\mathcal{E}}(x,o)\end{aligned}$$

and

$$N^{\mathcal{E}}h_\mu^{\mathcal{E}}(\xi,o) = 1 + N^{\mathcal{E}}G_\mu^0(\xi).$$

Thus we have the following

Theorem 12. *Let $\Gamma = (V, E, r)$ be a connected non-parabolic network and $\mathcal{E}$ a resistance form of Γ. A probability measure μ on V satisfying* (2) *and* (5) *is given.*
(i) *For $u \in D[\bar{L}^{\mathcal{E}}]$, one has*

$$u(x) = \int_V u\, d\mu + \int_V \bar{g}^{\mathcal{E}}_{\mu}(x, y) L^{\mathcal{E}} u(y)\, d\mu(y) + \int_{\partial\mathcal{K}(\mathcal{E})} h^{\mathcal{E}}_{\mu}(x, \xi) N^{\mathcal{E}} u(\xi)\, d\nu(\xi), \quad x \in V.$$

(ii) *For $f \in L^2(\mathcal{K}(\mathcal{E}), \bar{\mu})$ and a constant c, the function*

$$u(x) = c + \int_V \bar{g}^{\mathcal{E}}_{\mu}(x, y) f(y)\, d\mu(y) + \int_{\partial\mathcal{K}(\mathcal{E})} h^{\mathcal{E}}_{\mu}(x, \xi) f(\xi)\, d\nu(\xi), \quad x \in V.$$

belongs to $D[\bar{L}^{\mathcal{E}}]$ and satisfies $\bar{L}^{\mathcal{E}} u = f$ on V and $\bar{L}^{\mathcal{E}} u = f + \bar{\mu}(f) N^{\mathcal{E}} G^0_{\mu}$ in $L^2(\partial\mathcal{K}(\mathcal{E}), \nu)$. In particular if $\bar{\mu}(f) = 0$, then $\bar{L}^{\mathcal{E}} u = f$ in $L^2(\mathcal{K}(\mathcal{E}), \bar{\mu})$.

Let $D[\mathcal{E}^*] = \{\tau(u) \mid u \in D[\mathcal{E}]\}$ $(\subset L^2(\partial\mathcal{K}(\mathcal{E}), \nu))$ and $\mathcal{E}^*(\tau(u), \tau(v)) = \mathcal{E}(h_u, h_v)$ for $u, v \in D[\mathcal{E}]$, where h_u denotes the harmonic part of u in the Royden decomposition. Let $(L^*, D[L^*])$ be the self-adjoint operator associated to the regular Dirichlet form $(\mathcal{E}^*, D[\mathcal{E}^*])$ on $L^2(\partial\mathcal{K}(\mathcal{E}), \nu)$. The restriction of τ to $H_{\mathcal{E}}$ gives rise to a bijection between $H_{\mathcal{E}}$ and $D[\mathcal{E}^*]$ such that $\tau(H_{\mathcal{E}} \cap D[\bar{L}^{\mathcal{E}}]) = D[L^*]$ and $N^{\mathcal{E}} h = L^* \tau(h)$ for $h \in H_{\mathcal{E}} \cap D[\bar{L}^{\mathcal{E}}]$.

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