OUKA
The University of Osaka
Institutional Knowledge Archive

Title	RANDOM WALKS AND KURAMOCHI BOUNDARIES OF INFINITE NETWORKS
Author(s)	Kasue, Atsushi
Citation	Osaka Journal of Mathematics. 2013, 50(1), p. 31-51
Version Type	VoR
URL	https://doi.org/10.18910/24478
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

https://ir.library.osaka-u.ac.jp/

The University of Osaka

Kasue, A.
Osaka J. Math.
50 (2013), 31–51

RANDOM WALKS AND KURAMOCHI BOUNDARIES OF INFINITE NETWORKS

ATSUSHI KASUE

(Received April 25, 2011)

Abstract

In this paper, we study a connected non-parabolic, or transient, network compactified with the Kuramochi boundary, and show that the random walk converges almost surely to a random variable valued in the harmonic boundary, and a function of finite Dirichlet energy converges along the random walk to a random variable almost surely and in L^2. We also give integral representations of solutions of Poisson equations on the Kuramochi compactification.

1. Introduction

Ancona, Lyons and Peres [1] showed that a function of finite Dirichlet energy on a transient network converges along the random walk almost surely and in L^2. In this paper, we concern the Kuramochi boundary of the network and proves that the random walk converges almost surely to a random variable valued in the harmonic boundary, and a function of finite Dirichlet energy converges along the random walk to a random variable almost surely and in L^2.

Let $G = (V, E)$ be a graph with the set of vertices V and the set of edges E that consists of pairs of vertices. In this paper, a graph admits no loops and multiple edges, and the set of vertices is finite or countably infinite. We say that a vertex x is adjacent to another y if $\{x, y\}$ belongs to E and write $x \sim y$ to indicate it. We also write $|xy|$ for $\{x, y\}$. By a path in G, we mean a sequence of vertices $c = (x_0, x_1, \ldots, x_n)$ such that $x_i \sim x_{i+1}$ $(i = 0, 1, \ldots, n-1)$, and we say that c connects x_0 to x_n. G is called a connected graph if for any pair of vertices x and y, there exist paths connecting them.

We are now given an admissible weight r on the set of edges E, that is a positive function on E with the property that

$$c(x) = \sum_{y \sim x} \frac{1}{r(|xy|)} < +\infty, \quad \forall x \in V.$$

An admissible weight r gives rise to a distance d_r on V, called the geodesic distance of Γ, by taking $r(e)$ as the length of an edge e and by assigning to each pair

2010 Mathematics Subject Classification. Primary 53C21; Secondary 58D17, 58J50.

of vertices x and y the infimum of the length of paths connecting them. In this paper, we call such a couple of a graph and an admissible weight a network.

Given a connected network $\Gamma = (V, E, r)$, a nonnegative quadratic form $(\mathcal{E}_\Gamma, D[\mathcal{E}_\Gamma])$ on the space $l(V)$ of functions on V can be defined as follows:

$$D[\mathcal{E}_\Gamma] = \left\{ u \in l(V) \;\middle|\; \sum_{x\sim y} \frac{|u(x)-u(y)|^2}{r(|xy|)} < +\infty \right\};$$

$$\mathcal{E}_\Gamma(u, v) = \frac{1}{2} \sum_{x\sim y} \frac{(u(x)-u(y))(v(x)-v(y))}{r(|xy|)}, \quad u, v \in D[\mathcal{E}_\Gamma].$$

The domain $D[\mathcal{E}_\Gamma]$ endowed with an inner product $\mathcal{E}_\Gamma(u, v) + u(o)v(o)$, where o is a fixed point of V, becomes a Hilbert space.

Let $D_0[\mathcal{E}_\Gamma]$ be the closure of the set of finitely supported functions on V in $D[\mathcal{E}_\Gamma]$. We say that Γ is non-parabolic if

$$\sup \left\{ \frac{|u(x)|^2}{\mathcal{E}_\Gamma(u, u)} \;\middle|\; u \in D_0[\mathcal{E}_\Gamma],\ \mathcal{E}_\Gamma(u, u) > 0 \right\} < +\infty$$

for some $x \in V$. We recall here the fact that the following conditions are mutually equivalent:

(i) Γ is non-parabolic,

(ii) $D_0[\mathcal{E}_\Gamma]$ contains no constant functions,

(iii) $D_0[\mathcal{E}_\Gamma] \neq D[\mathcal{E}_\Gamma]$ (see [14]).

If these are the cases, $D[\mathcal{E}_\Gamma]$ is decomposed into the direct sum of $D_0[\mathcal{E}_\Gamma]$ and the space $H_{\mathcal{E}_\Gamma}$ of harmonic functions of finite Dirichlet sums on V that is the orthogonal complement of $D_0[\mathcal{E}_\Gamma]$ relative to the form; a function h on V belongs to $H_{\mathcal{E}_\Gamma}$ if and only if $h \in D[\mathcal{E}_\Gamma]$ and $L^c h(x) := \sum_{y\sim x}(h(x) - h(y))/r(|xy|) = 0$ for all $x \in V$.

Let $\{p(x, y) \mid x, y \in V\}$ be transition probabilities on V defined by

$$p(x, y) = \frac{c(|xy|)}{c(x)}, \quad x, y \in V,$$

where $c(|xy|) = r(|xy|)^{-1}$ and $c(x) = \sum_{y\sim x} c(|xy|)$. It is well known that Γ is non-parabolic if and only if the (reversible) Markov chain is transient.

Ancona, Lyons and Peres [1] proved the following

Theorem 1. *Let $\Gamma = (V, E, r)$ be a connected non-parabolic network and $\{X_n\}$ the Markov chain. Then for any $u \in D[\mathcal{E}_\Gamma]$, the sequence $\{u(X_n)\}$ converges almost surely and in L^2. If $u = h + g$, where $h \in H_{\mathcal{E}_\Gamma}$ and $g \in D_0[\mathcal{E}_\Gamma]$, is the Royden decomposition of u, then $\lim_{n\to\infty} u(X_n) = \lim_{n\to\infty} h(X_n)$ almost surely.*

To state our main results, we introduce the Kuramochi compactification of a connected infinite network $\Gamma = (V, E, r)$.

A compactification of any (discrete) set X is a compact Hausdorff space which contains X as a dense subset and which induces the discrete topology on X. It is known that given a family Φ of bounded functions on X, there exists an (up to canonical homeomorphisms) unique compactification $\mathcal{C}(X, \Phi)$ of X with the following properties (see e.g. [2]):

(i) every function of Φ extends to a continuous function on $\mathcal{C}(X, \Phi)$, and

(ii) the extended functions separate the points of the boundary $\partial\mathcal{C}(X,\Phi) = \mathcal{C}(X,\Phi)\setminus V$.

We remark that if Ψ is a subfamily of Φ, then the identity map extends to a continuous map from $\mathcal{C}(X, \Phi)$ onto $\mathcal{C}(X, \Psi)$, and if Φ_0 is a subfamily of Φ and each function of Φ is a finite linear combination of functions in Φ_0, then $\mathcal{C}(X, \Phi)$ and $\mathcal{C}(X, \Phi_0)$ are canonically homeomorphic; in particular, if in addition, X and Φ_0 is countable, then $\mathcal{C}(X, \Phi)$ is metrizable.

The compactification relative to the space of bounded functions in $D[\mathcal{E}_\Gamma]$, $BD[\mathcal{E}_\Gamma]$, is called the Royden compactification of the network Γ and denoted by $\mathcal{R}(\mathcal{E}_\Gamma)$. The boundary $\partial\mathcal{R}(\mathcal{E}_\Gamma)$ is called the Royden boundary of Γ. There is an important part of the Royden boundary refered to as the harmonic boundary of Γ which is defined by $\Delta(\mathcal{E}_\Gamma) = \{x \in \partial\mathcal{R}(\mathcal{E}_\Gamma) \mid g(x) = 0 \text{ for all } g \in BD_0[\mathcal{E}_\Gamma]\}$. It is known (see [15], [6], [11, Chapter VI]) that Γ is non-parabolic if and only if the harmonic boundary is not empty, and also that if $\partial\mathcal{R}(\mathcal{E}_\Gamma) \setminus \Delta(\mathcal{E}_\Gamma)$ is not empty, then any set of a single point there is not a G_δ set and for a nonempty closed subset F in $\partial\mathcal{R}(\mathcal{E}_\Gamma) \setminus \Delta(\mathcal{E}_\Gamma)$, there exists a function $g \in D_0[\mathcal{E}_\Gamma]$ such that $g(x)$ tends to infinity as $x \in V \to F$.

We recall a basic fact concerning Dirichlet problems on the Royden boundary $\partial\mathcal{R}(\mathcal{E}_\Gamma)$ (see [11, Chapter VI]): for any continuous function f on $\partial\mathcal{R}(\mathcal{E}_\Gamma)$, there exists a unique harmonic function H_f on Γ such that for any $\xi \in \Delta(\mathcal{E}_\Gamma)$, $\lim_{x\in V\to\xi} H_f(x) = f(\xi)$, and $\sup_V |H_f| \le \max_{\Delta(\mathcal{E}_\Gamma)} |f|$. Given a point $a \in V$, letting $\bar{\nu}_a(f) = H_f(a)$ for $f \in C(\partial\mathcal{R}(\mathcal{E}_\Gamma))$, we have a Radon measure $\bar{\nu}_a$ on $\partial\mathcal{R}(\mathcal{E}_\Gamma)$, called the harmonic measure with respect to the point a. In view of Harnack's inequality, $\bar{\nu}_a$ and $\bar{\nu}_b$ are mutually absolutely continuous for any pair of points $a, b \in V$, and the harmonic measures are supported on the harmonic boundary.

Now we consider a subspace $Q(\mathcal{E}_\Gamma)$ of $BD[\mathcal{E}_\Gamma]$ which consists of functions u such that $\mathcal{E}_\Gamma(u, v) = 0$ for all $v \in D[\mathcal{E}_\Gamma]$ vanishing on a finite subset of V. The compactification relative to $Q(\mathcal{E}_\Gamma)$ is called the Kuramochi compactification of the network Γ and denoted by $\mathcal{K}(\mathcal{E}_\Gamma)$ (see [9]). The identity map of V extends to a continuous map from $\mathcal{R}(\mathcal{E}_\Gamma)$ onto $\mathcal{K}(\mathcal{E}_\Gamma)$. We denote by ρ_Γ the induced map from the Royden boundary $\partial\mathcal{R}(\mathcal{E}_\Gamma)$ onto the Kuramochi boundary $\partial\mathcal{K}(\mathcal{E}_\Gamma)$. Let $\Delta^K(\mathcal{E}_\Gamma) = \rho_\Gamma(\Delta(\mathcal{E}_\Gamma))$ and $\nu_a = \rho_{\Gamma *}\bar{\nu}_a$ $(a \in V)$. Here and after, we fix a point $o \in V$ and write ν for ν_o.

We will prove that the Kuramochi compactification $\mathcal{K}(\mathcal{E}_\Gamma)$ of a connected, non-parabolic network Γ admits a compatible metric $d^{\mathcal{E}_\Gamma}$ such that for each Lipschitz function $f : (\mathcal{K}(\mathcal{E}_\Gamma), d^{\mathcal{E}_\Gamma}) \to \mathbf{R}$, the sequence $\{f(X_n)\}$ is almost surely convergent. This shows that the Markov chain $\{X_n\}$ converges to a random variable X_∞ in $\mathcal{K}(\mathcal{E}_\Gamma)$. In fact, a result by Ancona, Lyons and Peres [1] states that *if $\mathcal{M}$ is a complete separable metric space and $\{Y_n\}$ is a process such that for each bounded Lipschitz function*

$f : \mathcal{M} \to \boldsymbol{R}$, the sequence $\{f(Y_n)\}$ is almost surely convergent, then the process $\{Y_n\}$ in $\mathcal{M}$ is already almost surely convergent.

We will now take an appropriate measure on V. Given two vertices x and y of Γ, we define a nonnegative number $R_{\mathcal{E}_\Gamma}(x, y)$, called the effective resistance between x and y, by

$$R_{\mathcal{E}_\Gamma}(x, y) = \sup\left\{ \frac{|u(x) - u(y)|^2}{\mathcal{E}_\Gamma(u, u)} \;\middle|\; u \in D[\mathcal{E}_\Gamma],\ \mathcal{E}_\Gamma(u, u) > 0 \right\}.$$

It is known that $R_{\mathcal{E}_\Gamma}(x, y) \le d_r(x, y)$ for all $x, y \in V$ and $R_{\mathcal{E}_\Gamma}$ induces a distance on V (see e.g., [5]). Choose a measure μ on V in such a way that $\mu(V) = \sum_{x \in V} \mu(x) = 1$ and

$$\int_V R_{\mathcal{E}_\Gamma}(o, x)^2 \, d\mu(x) \left(= \sum_{x \in V} R_\Gamma(o, x)^2 \mu(x) \right) < +\infty.$$

Under the condition, it is proved in [5] that $D[\mathcal{E}_\Gamma] \subset L^2(V, \mu)$, the embedding is compact, $(\mathcal{E}_\Gamma, D[\mathcal{E}_\Gamma])$ is a regular Dirichlet form in $L^2(\mathcal{K}(\mathcal{E}_\Gamma), \mu)$, and the Royden decomposition is stated in such a way that a function $u \in D[\mathcal{E}_\Gamma]$ is expressed as

$$u(x) = \int_{\partial \mathcal{K}(\mathcal{E}_\Gamma)} \tau(u) \, d\nu_x + g(x), \quad x \in V,\ g \in D_0[\mathcal{E}_\Gamma],$$

where $\tau(u)$ is a function in $L^2(\partial \mathcal{K}(\mathcal{E}_\Gamma), \nu)$ $(= L^2(\Delta^K(\mathcal{E}_\Gamma), \nu))$. We define a Radon measure $\bar{\mu}$ on the Kuramochi compactification $\mathcal{K}(\mathcal{E}_\Gamma)$ by

$$\bar{\mu}(f) = \int_V f \, d\mu + \int_{\partial \mathcal{K}(\mathcal{E}_\Gamma)} f \, d\nu$$

for $f \in C(\mathcal{K}(\mathcal{E}_\Gamma))$. Then any function u of $D[\mathcal{E}_\Gamma]$ coupled with $\tau(u)$ can be considered as a function in $L^2(\mathcal{K}(\mathcal{E}_\Gamma), \bar{\mu})$.

Our main results are stated in the following

Theorem 2. *Let $\Gamma = (V, E, r)$ be a connected non-parabolic network. Then the following assertions hold*:

(i) *$(\mathcal{E}_\Gamma, D[\mathcal{E}_\Gamma])$ is a regular Dirichlet form on $L^2(\mathcal{K}(\mathcal{E}_\Gamma), \bar{\mu})$.*

(ii) *There exists a $\Delta^K(\mathcal{E}_\Gamma)$-valued random variable X_∞ such that in the $\mathcal{K}(\mathcal{E}_\Gamma)$-topology, the Markov chain X_n almost surely converges to X_∞ as $n \to \infty$, the measure ν_{X_n} converges weakly to the delta measure δ_{X_∞} almost surely as $n \to \infty$, and for any $u \in D[\mathcal{E}_\Gamma]$, $u(X_n)$ converges to $\tau(u)(X_\infty)$ almost surely and in L^2 as $n \to \infty$.*

(iii) *Let $(\bar{L}^{\mathcal{E}_\Gamma}, D[\bar{L}^{\mathcal{E}_\Gamma}])$ be the self-adjoint operator associated with the regular Dirichlet form $(\mathcal{E}_\Gamma, D[\mathcal{E}_\Gamma])$. For a function $f \in L^2(\mathcal{K}(\mathcal{E}_\Gamma), \bar{\mu})$, there exists a solution u, unique up to additive constants, of equation: $\bar{L}^{\mathcal{E}_\Gamma} u = f$ if and only if $\bar{\mu}(f) = 0$; in particular, the solution is harmonic on V if f vanishes there.*

We briefly explain the contents of the paper. In section 1, we introduce a resistance form of a connected non-parabolic network and its Kuramochi compactification, and prove Theorem 2 (i) for a resistance form. In Section 2, Theorem 2 (ii) for a resistance form is discussed. The last section is devoted to investigating Poisson equations on the Kuramochi compactification of a resistance form.

2. Resistance forms

In this section, we introduce the Kuramochi compactification of a resistance form of a connected non-parabolic network and prove Theorem 2 (i) for a resistance form.

Let $\Gamma = (V, E, r)$ be a connected non-parabolic network. A nonnegative quadratic form $\mathcal{E}$ on a subspace $D[\mathcal{E}]$ of $D[\mathcal{E}_\Gamma]$ is called a resistance form of the network Γ if it satisfies the following properties:

(i) $D_0[\mathcal{E}_\Gamma] + \mathbf{R} \subset D[\mathcal{E}] \subset D[\mathcal{E}_\Gamma]$,

(ii) $\mathcal{E}(1, 1) = 0$,

(iii) $\mathcal{E}_\Gamma(u, u) \leq \mathcal{E}(u, u)$ for all $u \in D[\mathcal{E}]$ and $\mathcal{E}(u, v) = \mathcal{E}_\Gamma(u, v)$ for all $u \in D[\mathcal{E}]$ and $v \in D_0[\mathcal{E}_\Gamma]$,

(iv) for $u \in D[\mathcal{E}]$, $\bar{u} = \max\{0, \min\{1, u\}\}$ belongs to $D[\mathcal{E}]$ and $\mathcal{E}(\bar{u}, \bar{u}) \leq \mathcal{E}(u, u)$,

(v) $D[\mathcal{E}]$ becomes a Hilbert space with inner product $(u, v) = \mathcal{E}(u, v) + u(o)v(o)$, where o is a fixed vertex of V. When we restrict $\mathcal{E}_\Gamma$ to $D_0[\mathcal{E}_\Gamma] + \mathbf{R}$, we have the minimal resistance form denoted by $(\mathcal{E}_\Gamma^0, D_0[\mathcal{E}_\Gamma] + \mathbf{R})$.

For any pair of vertexes x, y, we have a nonnegative number $R_\mathcal{E}(x, y)$, called the effective resistance relative to $\mathcal{E}$ between x and y, defined by

$$R_\mathcal{E}(x, y) = \sup\left\{ \frac{|u(x) - u(y)|^2}{\mathcal{E}(u, u)} \;\middle|\; u \in D[\mathcal{E}],\ \mathcal{E}(u, u) > 0 \right\}.$$

Then it follows from the definitions above that

$$R_{\mathcal{E}_\Gamma^0}(x, y) \leq R_\mathcal{E}(x, y) \leq R_{\mathcal{E}_\Gamma}(x, y), \quad x, y \in V.$$

We remark that $R_{\mathcal{E}_\Gamma}(x, y) \leq d_r(x, y)$ for $x, y \in V$, and $R_\mathcal{E}$ induces a distance on V (see e.g., [5, Theorem 1.12, Proposition 2.6]). We write $H_\mathcal{E}$ for the space of functions u in $D[\mathcal{E}]$ which are harmonic on V, i.e.,

$$L^c u(x) := \sum_{y \sim x} \frac{u(x) - u(y)}{r(|xy|)} = 0, \quad \forall x \in V.$$

Given $x, z \in V$, there exist functions $g_{x,z} \in D[\mathcal{E}]$ and $h_{x,z} \in H_\mathcal{E}$ respectively satisfying $\mathcal{E}(g_{x,z}, u) = u(x) - u(z)$ for all $u \in D[\mathcal{E}]$ and $\mathcal{E}(h_{x,z}, h) = h(x) - h(z)$ for all $h \in H_\mathcal{E}$. We write $g_z^\mathcal{E}(x, y)$ and $h_z^\mathcal{E}(x, y)$ respectively for $g_{x,z}(y)$ and $h_{x,z}(y)$. It is easy to see that $g_z^\mathcal{E}(x, y) = g_z^\mathcal{E}(y, x)$ and $h_z^\mathcal{E}(x, y) = h_z^\mathcal{E}(y, x)$. We notice that $R_\mathcal{E}(x, y) = g_x^\mathcal{E}(y, y) = g_x^\mathcal{E}(y, z) + g_y^\mathcal{E}(x, z)$ and $g_z^\mathcal{E}(x, y) = (1/2)\{R_\mathcal{E}(x, z) + R_\mathcal{E}(z, y) - R_\mathcal{E}(x, y)\}$ for all

$x, y, z \in V$, and also that $h_z^{\mathcal{E}}(x, x) = \sup\{|h(x) - h(z)|^2/\mathcal{E}(h, h) \mid h \in H_{\mathcal{E}},\ \mathcal{E}(h, h) > 0\}$ (see [5, 7.2]). Since Γ is assumed to be non-parabolic, given a vertex $x \in V$, there exists uniquely a function $g_x \in D_0[\mathcal{E}_\Gamma]$ such that $\mathcal{E}_\Gamma(g_x, v) = v(x)$ for all $v \in D_0[\mathcal{E}_\Gamma]$. We write $g_\Gamma^0(x, y)$ for $g_x(y)$. It holds also that $g_\Gamma^0(x, y) = g_\Gamma^0(y, x)$. These functions are related as follows:

$$(1)\quad g_z^{\mathcal{E}}(x, y) = h_z^{\mathcal{E}}(x, y) + (g_\Gamma^0(x, y) - g_\Gamma^0(x, z) - g_\Gamma^0(y, z) + g_\Gamma^0(z, z)), \quad x, y, z \in V$$

(see [5, 7.2]).

Now as in the case of the form $\mathcal{E}_\Gamma$, we consider a subspace $Q(\mathcal{E})$ of $D[\mathcal{E}]$ which consists of functions u such that $\mathcal{E}(u, v) = 0$ for all $v \in D[\mathcal{E}]$ vanishing on a finite subset of V. The compactification relative to $Q(\mathcal{E})$ is called the Kuramochi compactification of the network Γ relative to the resistance form $\mathcal{E}$, and denoted by $\mathcal{K}(\mathcal{E})$. The identity map of V extends to a continuous map from the Royden compactification $\mathcal{R}(\mathcal{E}_\Gamma)$ of Γ onto $\mathcal{K}(\mathcal{E})$. We denote by $\rho_{\mathcal{E}}$ the induced map from the Royden boundary $\partial\mathcal{R}(\mathcal{E}_\Gamma)$ onto the Kuramochi boundary $\partial\mathcal{K}(\mathcal{E})$. Let $\Delta^K(\mathcal{E}) = \rho_{\mathcal{E}}(\Delta(\mathcal{E}_\Gamma))$ and $\nu_a = \rho_{\mathcal{E}*}\bar{\nu}_a$ ($a \in V$). Here and after, we fix a point $o \in V$ and write ν for ν_o.

We take a positive function μ on V and consider it as a measure on V, $\mu = \sum_{x\in V} \mu(x)\delta_x$. In what follows, μ is chosen in such a way that $\mu(V) = 1$,

$$(2)\qquad \int_V R_{\mathcal{E}}(o, x)^2\, d\mu(x) < +\infty.$$

The measure μ extends to a Radon measure, denoted by the same letter, on the Kuramochi compactification. Here we recall some results in [5, 7.3]:

(i) $D[\mathcal{E}] \subset L^2(\mathcal{K}(\mathcal{E}), \mu)$.

(ii) Any function $u \in D[\mathcal{E}]$ can be written in the Royden decomposition as

$$u(x) = \int_{\partial\mathcal{K}(\mathcal{E})} \tau(u)\, d\nu_x + g(x), \quad x \in V,\ g \in D_0[\mathcal{E}],$$

where $\tau(u)$ is a function in $L^2(\partial\mathcal{K}(\mathcal{E}), \nu)$.

(iii) $(\mathcal{E}, D[\mathcal{E}])$ is a regular Dirichlet form on $L^2(\mathcal{K}(\mathcal{E}), \mu)$.

(iv) The domain $D[L^{\mathcal{E}}]$ of the self-adjoint operator $L^{\mathcal{E}}$ associated to the Dirichlet form $\mathcal{E}$ is embedded in the space of continuous functions on $\mathcal{K}(\mathcal{E})$, and $D[L^{\mathcal{E}}]$ is dense both in the Banach space $C(\mathcal{K}(\mathcal{E}))$ of continuous functions on $\mathcal{K}(\mathcal{E})$ and the Hilbert space $(D[\mathcal{E}], \mathcal{E} + \delta_o^2)$.

(v) The domain $D[\mathcal{E}]$ is compactly embedded into $L^2(\mathcal{K}(\mathcal{E}), \mu)$.

Now we define a Radon measure $\bar{\mu}$ on the Kuramochi compactification $\mathcal{K}(\mathcal{E})$ by

$$\bar{\mu}(f) = \int_V f\, d\mu + \int_{\partial\mathcal{K}(\mathcal{E})} f\, d\nu, \quad f \in C(\mathcal{K}(\mathcal{E})).$$

Then any function u of $D[\mathcal{E}]$ coupled with $\tau(u)$ can be considered as a function in $L^2(\mathcal{K}(\mathcal{E}), \bar{\mu})$.

Given $u \in D[\mathcal{E}]$, if we write h for the harmonic part $\int_{\partial\mathcal{K}(\mathcal{E})} \tau(u) d\nu_x$ of u, then we have the following basic identity:

$$(3)\quad \int_{\partial\mathcal{K}(\mathcal{E})} \tau(u)^2\, d\nu_x - \left(\int_{\partial\mathcal{K}(\mathcal{E})} \tau(u)\, d\nu_x\right)^2 = \sum_{y\in V} g_\Gamma^0(x,y) \sum_{z\sim y} \frac{(h(y)-h(z))^2}{r(|yz|)}, \quad x \in V,$$

from which we can deduce that

$$(4)\qquad \int_{\partial\mathcal{K}(\mathcal{E})} \tau(u)^2\, d\nu_x \le 2g_\Gamma^0(x,x)\mathcal{E}(u,u) + 2u(x)^2, \quad x \in V$$

(see [5, Lemma 7.8]). Using this inequality, we get

$$\int_{\partial\mathcal{K}(\mathcal{E})} \tau(u)^2\, d\nu_x \le 2\left(g_\Gamma^0(x,x) + \frac{1}{\mu(x)}\right)\left(\mathcal{E}(u,u) + \int_V u^2\, d\mu\right),$$

since $\mu(x)u(x)^2 \le \int_V u^2\, d\mu$. This shows in particular that the norm $\mathcal{E}(u,u)^{1/2} + \left(\int_V u^2\, d\mu\right)^{1/2} + \left(\int_{\partial\mathcal{K}(\mathcal{E})} \tau(u)^2\, d\nu\right)^{1/2}$ is equivalent to the norm $\mathcal{E}(u,u)^{1/2} + \left(\int_V u^2\, d\mu\right)^{1/2}$. Since $(\mathcal{E}, D[\mathcal{E}])$ is a regular Dirichlet form on $L^2(\mathcal{K}(\mathcal{E}), \mu)$, we can thus deduce the following

Theorem 3. *Let $\Gamma = (V, E, r)$ be a connected non-parabolic network and $\mathcal{E}$ a resistance form of Γ. Then the Dirichlet form $(\mathcal{E}, D[\mathcal{E}])$ on $L^2(\mathcal{K}(\mathcal{E}), \bar{\mu})$ is regular.*

Let $(\bar{L}^{\mathcal{E}}, D[\bar{L}^{\mathcal{E}}])$ be the self-adjoint operator associated with the regular Dirichlet form $\mathcal{E}$ in $L^2(\mathcal{K}(\mathcal{E}), \bar{\mu})$. For $u \in D[\bar{L}^{\mathcal{E}}]$, we note that

$$\bar{L}^{\mathcal{E}}u(x) = \frac{1}{\mu(x)} L^c u(x) = \frac{1}{\mu(x)} \sum_{y\sim x} \frac{u(x)-u(y)}{r(|xy|)}, \quad x \in V.$$

The restriction of $\bar{L}^{\mathcal{E}}u$ to the Kuramochi boundary is denoted by $N^{\mathcal{E}}u$. Then we have

$$\mathcal{E}(u,v) = \int_V v L^c u\, d\mu^c + \int_{\partial\mathcal{K}(\mathcal{E})} \tau(v) N^{\mathcal{E}} u\, d\nu, \quad v \in D[\mathcal{E}].$$

It is a consequence from the definitions of $L^{\mathcal{E}}$ and $\bar{L}^{\mathcal{E}}$ that

$$D[L^{\mathcal{E}}] = \{u \in D[\bar{L}^{\mathcal{E}}] \mid N^{\mathcal{E}}u = 0 \text{ in } L^2(\partial\mathcal{K}(\mathcal{E}), \nu)\} \quad (\subset C(\mathcal{K}(\mathcal{E}))).$$

We remark that $Q(\mathcal{E})$ is a subspace of $D[L^{\mathcal{E}}]$. In fact, let u be a function in $Q(\mathcal{E})$. Then there exists a finite subset A of V such that $\mathcal{E}(u,v) = 0$ for all $v \in D[\mathcal{E}]$ which

vanishes on A. Let χ_A be the characteristic function of A. Then for any $v \in D[\mathcal{E}]$, we have

$$\mathcal{E}(u, v) = \mathcal{E}(u, \chi_A v) = \sum_{x \in A} v(x) L^c u(x)$$

and hence we get

$$|\mathcal{E}(u, v)| \leq \left(\int_A (L^{\mathcal{E}} u)^2 \, d\mu \right)^{1/2} \left(\int_V v^2 \, d\mu \right)^{1/2}.$$

This shows that $u \in D[L^{\mathcal{E}}]$.

Here, referring to [5, Proposition 4.1, Theorem 7.11], we mention the following propositions:

(I) The following conditions are mutually equivalent:
  (i) $\sup_{x \in V} g_\Gamma^0(x, x)$ is finite.
  (ii) Any $g \in D_0[\mathcal{E}_\Gamma]$ is bounded.
  (iii) $\partial R(\mathcal{E}_\Gamma) = \Delta(\mathcal{E}_\Gamma)$, that is, for any bounded $g \in D_0[\mathcal{E}_\Gamma]$, $g(x)$ tends to zero as $x \in V \to \infty$.
  (iv) For any $g \in D_0[\mathcal{E}_\Gamma]$, $g(x)$ tends to zero as $x \in V \to \infty$.

(II) $\sup_{x,y \in V} R_{\mathcal{E}}(x, y)$ is bounded if and only if every $f \in D[\mathcal{E}]$ is bounded.

(III) The following conditions are mutually equivalent:
  (i) $\sup_{x,y \in V} h_x^{\mathcal{E}}(y, y)$ is finite.
  (ii) Any $h \in H_{\mathcal{E}}$ is bounded.
  (iii) For any $u \in D[\mathcal{E}]$, $\tau(u)$ is continuous on $\Delta^K(\mathcal{E})$.
  (iv) A nonnegative subharmonic function u in $D[\mathcal{E}]$ is bounded.

Now we prove the following

Theorem 4. *Let $\Gamma = (V, E, r)$ be a connected non-parabolic network and $\mathcal{E}$ a resistance form of Γ. Then $D[\mathcal{E}]$ is compactly embedded in $L^2(\mathcal{K}(\mathcal{E}), \bar{\mu})$ if $\sup_{x \in V} g_\Gamma^0(x,x)$ is finite.*

Proof. Let $\{u_n\}$ be a sequence in $D[\mathcal{E}]$ such that $\mathcal{E}(u_n, u_n) + u_n(o)^2$ is bounded as $n \to \infty$. Let h_n be the harmonic part of u_n. Then we have

$$\begin{aligned} u_n(o) - h_n(o) &= \mathcal{E}_\Gamma(g_\Gamma^0(o, *), u_n - h_n) \\ &= \mathcal{E}_\Gamma(g_\Gamma^0(o, *), u_n) \end{aligned}$$

and hence

$$h_n(o)^2 \leq 2u_n(o)^2 + 2g_\Gamma^0(o, o)\mathcal{E}(u_n, u_n).$$

Thus we see that $\mathcal{E}(h_n, h_n) + h_n(o)^2$ are bounded as $n \to \infty$. Since $D[\mathcal{E}]$ is compactly embedded in $L^2(\mathcal{K}(\mathcal{E}), \mu)$, passing to a subsequence, we may assume that u_n and h_n

respectively converge to functions u and h in $L^2(\mathcal{K}(\mathcal{E}), \mu)$, where h is the harmonic part of u. Let $v_n = u_n - u$ and $k_n = h_n - h$. Then in view of (3), we have

$$\int_{\partial\mathcal{K}(\mathcal{E})} \tau(v_n)^2 \, d\nu_o = k_n(o)^2 + \sum_{x\in V} g_\Gamma^0(o, x) \sum_{y\sim x} \frac{(k_n(x) - k_n(y))^2}{r(|xy|)}.$$

Given $\varepsilon > 0$, let $V_\varepsilon = \{x \in V \mid g_\Gamma^0(o, x) \geq \varepsilon\}$. Since $g_\Gamma^0(o, x)$ tends to 0 as $x \in V$ goes to infinity by the assumption: $\sup_{x\in V} g_\Gamma^0(x, x) < +\infty$, V_ε is a finite subset of V. Therefore for sufficiently large n,

$$k_n(o)^2 + \sum_{x\in V_\varepsilon} g_\Gamma^0(o, x) \sum_{y\sim x} \frac{(k_n(x) - k_n(y))^2}{r(|xy|)} < \varepsilon.$$

Since

$$\sum_{x\in V\setminus V_\varepsilon} g_\Gamma^0(o, x) \sum_{y\sim x} \frac{(k_n(x) - k_n(y))^2}{r(|xy|)} < \varepsilon \mathcal{E}_\Gamma(v_n, v_n),$$

we get

$$\int_{\partial\mathcal{K}(\mathcal{E})} \tau(v_n)^2 \, d\nu < \varepsilon\left(1 + \sup_n \mathcal{E}_\Gamma(v_n, v_n)\right)$$

for n large enough. This shows that $\int_{\partial\mathcal{K}(\mathcal{E})} \tau(v_n)^2 \, d\nu$ tends to 0 as $n \to \infty$. Thus we can deduce that $D[\mathcal{E}]$ is compactly embedded in $L^2(\mathcal{K}(\mathcal{E}), \bar{\mu})$. □

REMARK. Let $\Gamma = (V, E, r)$ be a connected infinite network and $\mathcal{E}$ a resistance form of Γ.

(i) Let $D[\mathcal{E}^*] = \{\tau(u) \mid u \in D[\mathcal{E}]\}$ $(\subset L^2(\partial\mathcal{K}(\mathcal{E}), \nu))$ and $\mathcal{E}^*(\tau(u), \tau(v)) = \mathcal{E}(h_u, h_v)$ for $u, v \in D[\mathcal{E}]$, where h_u denotes the harmonic part of u in the Royden decomposition. Then $(\mathcal{E}^*, D[\mathcal{E}^*])$ is a regular Dirichlet form on $L^2(\partial\mathcal{K}(\mathcal{E}), \nu)$.

(ii) Let $(\mathcal{F}, D[\mathcal{F}])$ be a Dirichlet form on a closed subspace of $L^2(\partial\mathcal{K}(\mathcal{E}), \nu)$ with $\mathcal{F}(1, 1) = 0$, and define a form $(\mathcal{E}_\mathcal{F}, D[\mathcal{E}_\mathcal{F}])$ by

$$\mathcal{E}_\mathcal{F}(u, v) = \mathcal{E}(u, v) + \mathcal{F}(\tau(u), \tau(v)); \quad D[\mathcal{E}_\mathcal{F}] = \{u \in D[\mathcal{E}] \mid \tau(u) \in D[\mathcal{F}]\}.$$

Then $\mathcal{E}_\mathcal{F}$ is a resistance form of Γ. Moreover for a positive number t, we set $\mathcal{E}_{\mathcal{F};t}(u, v) = \mathcal{E}(u, v) + t\mathcal{F}(\tau(u), \tau(v))$. Then the limit of the forms as $t \to +\infty$ also gives a resistance form of Γ.

(iii) Given a finite subset K of V, we can define a Dirichlet form on the space $l(K)$ of functions on K by letting $\mathcal{E}_K^*(u, u) = \inf\{\mathcal{E}(\tilde{u}, \tilde{u}) \mid \tilde{u} \in D[\mathcal{E}], \tilde{u} = u \text{ on } K\}$ for $u \in l(K)$. Then we get a finite connected network $\Gamma_K^* = (K, E_K, r_K)$ such that the effective resistance of Γ_K^* between two points of K is equal to the effective resistance

relative to $\mathcal{E}$ (cf. [7, Theorem 2.1.12, Corollary 2.1.13], [5, Theorem 1.13]). Thus if we take an increasing sequence $\{V_n\}$ of finite subsets of V such that $V = \bigcup_n V_n$, then Γ endowed with the resistance form $\mathcal{E}$ can be considered as a limit of finite networks $\{\Gamma^*_{V_n}\}$ (see [5]). Conversely if we have a sequence $\{\Gamma'_n\}$ of finite networks such that the set of vertices of Γ'_n includes the vertex boundary of V_n, namely the set of vertexes of V_n which are adjacent to those outside of V_n, we get a sequence $\{\Gamma''_n\}$ of finite networks obtained by joining the subnetwork Γ_n of Γ generated by V_n with Γ'_n through the vertex boundary of V_n. Since the effective resistance of Γ''_n between two points of V_n is bounded by the effective resistance of $\mathcal{E}_\Gamma$ between them, by taking subsequence if necessarily, we have a resistance form $\mathcal{E}$ of Γ such that for any pair of points of V, the effective resistance of Γ''_n between them (for large n) converges to the effective resistance relative to $\mathcal{E}$ as $n \to \infty$ (see [5, 7.4]).

3. Random walks

We consider a connected non-parabolic network $\Gamma = (V, E, r)$ endowed with a measure $\mu\colon V \to (0, +\infty)$ satisfying (2) and the random walk $\{X_n\}$ of Γ.

Let $(\mathcal{M}, d_\mathcal{M})$ be a complete separable metric space. Define a set $D[\mathcal{E}_{\Gamma,\mathcal{M}}]$ of maps of V to $\mathcal{M}$ and a functional $\mathcal{E}_{\Gamma,\mathcal{M}}$ on $D[\mathcal{E}_{\Gamma,\mathcal{M}}]$ by

$$D[\mathcal{E}_{\Gamma,\mathcal{M}}] = \left\{ \phi\colon V \to \mathcal{M} \;\middle|\; \sum_{x\sim y} \frac{d_\mathcal{M}(\phi(x), \phi(y))^2}{r(|xy|)} < +\infty \right\};$$

$$\mathcal{E}_{\Gamma,\mathcal{M}}(\phi) = \frac{1}{2} \sum_{x\sim y} \frac{d_\mathcal{M}(\phi(x), \phi(y))^2}{r(|xy|)}, \quad \phi \in D[\mathcal{E}_{\Gamma,\mathcal{M}}].$$

A map $\phi\colon V \to \mathcal{M}$ in $D[\mathcal{E}_{\Gamma,\mathcal{M}}]$ is called a Dirichlet finite map. The composition $f \circ \phi$ of a Lipschitz function f on $\mathcal{M}$ and a Dirichlet finite map $\phi\colon V \to \mathcal{M}$ belongs to $D[\mathcal{E}_\Gamma]$. Thus applying the result of [1] mentioned in the introduction, we see that the sequence $\{f(\phi(X_n))\}$ is almost surely convergent, and the process $\phi(X_n)$ is already almost surely convergent in $\mathcal{M}$.

Now we consider a resistance form $\mathcal{E}$ of Γ. For any $x, y \in V$, let

$$d^\mathcal{E}(x, y) = \left(\int_V (g^\mathcal{E}_\mu(x, z) - g^\mathcal{E}_\mu(y, z))^2 \, d\mu(z) \right)^{1/2},$$

where we set $g^\mathcal{E}_\mu(x, y) = \int_V g^\mathcal{E}_z(x, y)\, d\mu(z)$. Then it is proved in [5, Theorem 3.10] that $d^\mathcal{E}$ gives a compatible metric on $\mathcal{K}(\mathcal{E})$. In what follows, $\mathcal{K}(\mathcal{E})$ is equipped with the distance $d^\mathcal{E}$.

Now we prove the following

Lemma 5. *The inclusion map I of V into the metric space $(\mathcal{K}(\mathcal{E}), d^\mathcal{E})$ is a Dirichlet finite map and $\mathcal{E}_{\Gamma,\mathcal{K}(\mathcal{E})}(I) = \iint_{V\times V} R_\mathcal{E}(z, w)\, d\mu(z)\, d\mu(w)$.*

Proof. We have

$$\begin{aligned}
\mathcal{E}_{\Gamma,\mathcal{K}(\mathcal{E})}(I) &= \sum_{x\sim y} \frac{d^{\mathcal{E}}(x,y)^2}{r(|xy|)} \\
&= \sum_{x\sim y} \int_V \frac{(g_\mu^{\mathcal{E}}(x,z) - g_\mu^{\mathcal{E}}(y,z))^2}{r(|xy|)}\, d\mu(z) \\
&= \int_V \sum_{x\sim y} \frac{(g_\mu^{\mathcal{E}}(x,z) - g_\mu^{\mathcal{E}}(y,z))^2}{r(|xy|)}\, d\mu(z) \\
&= \int_V \mathcal{E}(g_\mu^{\mathcal{E}}(z,*), g_\mu^{\mathcal{E}}(z,*))\, d\mu(z) \\
&= \int_V g_\mu^{\mathcal{E}}(z,z)\, d\mu(z) \\
&= \iint_{V\times V} R_{\mathcal{E}}(z,w)\, d\mu(z)\, d\mu(w) \\
&< \int_V R_{\mathcal{E}}(z,o)\, d\mu(z) + \int_V R_{\mathcal{E}}(o,w)\, d\mu(w) = 2\int_V R_{\mathcal{E}}(z,o)\, d\mu(z).
\end{aligned}$$

This completes the proof of the lemma. □

Theorem 6. *Let $\Gamma = (V, E, r)$ be a connected non-parabolic network and $\mathcal{E}$ a resistance form of Γ. Then there exists a $\Delta^K(\mathcal{E})$-valued random variable $X_\infty^{\mathcal{E}}$ such that the process X_n almost surely converges to $X_\infty^{\mathcal{E}}$ in $\mathcal{K}(\mathcal{E})$, the measure ν_{X_n} converges weakly to the delta measure $\delta_{X_\infty^{\mathcal{E}}}$ almost surely, and for any $u \in D[\mathcal{E}]$, $u(X_n)$ converges to $\tau(u)(X_\infty^{\mathcal{E}})$ almost surely and in L^2 as $n \to \infty$.*

Proof. Lemma 5 and the result in [1] stated above imply that the process $\{X_n\}$ is Cauchy in $\mathcal{K}(\mathcal{E})$ almost surely. Let $X_\infty^{\mathcal{E}} = \lim_{n\to\infty} X_n$. We recall here that $D[L^{\mathcal{E}}]$ is densely embedded in both $C(\mathcal{K}(\mathcal{E}))$ and $D[\mathcal{E}]$. Then together with Theorem 1, we see that for $u \in D[L^{\mathcal{E}}]$,

$$\lim_{n\to\infty} u(X_n) = \lim_{n\to\infty} \int_{\partial\mathcal{K}(\mathcal{E})} \tau(u)\, d\nu_{X_n} = \tau(u)(X_\infty^{\mathcal{E}}).$$

Moreover it follows that ν_{X_n} weakly converges to $\delta_{X_\infty^{\mathcal{E}}}$ almost surely, and since the support of the measure ν_x coincides with $\Delta^K(\mathcal{E})$, it follows that $X_\infty^{\mathcal{E}}$ is a $\Delta^K(\mathcal{E})$-valued random variable, and further it is easy to see that the image is dense in $\Delta^K(\mathcal{E})$.

Now we want to show that for $u \in D[\mathcal{E}]$, $u(X_n)$ converges to $\tau(u)(X_\infty^{\mathcal{E}})$ in L^2. We fix a point $a \in V$. For any positive number ε, we take a function $u_\varepsilon \in D[L^{\mathcal{E}}]$ such that

$\mathcal{E}(u-u_\varepsilon)+(u-u_\varepsilon)(a)^2<\varepsilon$. Let $h_\varepsilon(x)=\int_{\partial K(\mathcal{E})}\tau(u-u_\varepsilon)\,d\nu_x$ and $g_\varepsilon=u-u_\varepsilon-h_\varepsilon\in D_0[\mathcal{E}_0]$. Then we have

$$\begin{aligned}\mathbf{E}_a[(u-u_\varepsilon)^2(X_n)]&\le 2\mathbf{E}_a[h_\varepsilon^2(X_n)]+2\mathbf{E}_a[g_\varepsilon^2(X_n)]\\&\le 2\int_{\partial K(\mathcal{E})}\tau(h_\varepsilon)^2\,d\nu_a+2\mathbf{E}_a[g_\varepsilon^2(X_n)],\end{aligned}$$

where we have used the fact that h_ε^2 is subharmonic, so that

$$\mathbf{E}_a[h_\varepsilon^2(X_n)]\le\int_{\partial K(\mathcal{E})}\tau(h_\varepsilon)^2\,d\nu_a.$$

In view of (4), we observe that

$$\begin{aligned}\int_{\partial K(\mathcal{E})}\tau(h_\varepsilon)^2\,d\nu_a&\le 2g_\Gamma^0(a,a)\mathcal{E}(u-u_\varepsilon,u-u_\varepsilon)+2(u-u_\varepsilon)(a)^2\\&\le 2(g_\Gamma^0(a,a)+1)\varepsilon.\end{aligned}$$

Thus we obtain

$$\mathbf{E}_a[(u-u_\varepsilon)^2(X_n)]\le 4(g_\Gamma^0(a,a)+1)\varepsilon+2\mathbf{E}_a[g_\varepsilon^2(X_n)].$$

Using this, we have

$$\begin{aligned}&\mathbf{E}_a[(u(X_n)-\tau(u)(X_\infty^{\mathcal{E}}))^2]\\&\le 4\mathbf{E}_a[h_\varepsilon^2(X_n)]+4\mathbf{E}_a[\tau(u-u_\varepsilon)^2(X_\infty^{\mathcal{E}})]+2\mathbf{E}_a[(u_\varepsilon(X_n)-\tau(u_\varepsilon)(X_\infty^{\mathcal{E}}))^2]\\&\le 16(g_\Gamma^0(a,a)+1)\varepsilon+8\mathbf{E}_a[g_\varepsilon^2(X_n)]+4\int_{\partial K(\mathcal{E})}\tau(h_\varepsilon)^2\,d\nu_a\\&\quad+2\mathbf{E}_a[(u_\varepsilon(X_n)-\tau(u_\varepsilon)(X_\infty^{\mathcal{E}}))^2]\\&\le 24(g_\Gamma^0(a,a)+1)\varepsilon+8\mathbf{E}_a[g_\varepsilon^2(X_n)]+2\mathbf{E}_a[(u_\varepsilon(X_n)-\tau(u_\varepsilon)(X_\infty)^{\mathcal{E}})^2].\end{aligned}$$

Thus we get

$$\limsup_{n\to\infty}\mathbf{E}_a[(u(X_n)-\tau(u)(X_\infty^{\mathcal{E}}))^2]\le 24(g_\Gamma^0(a,a)+1)\varepsilon.$$

Letting ε go to zero, we see that $\lim_{n\to\infty}\mathbf{E}_a[(u(X_n)-\tau(u)(X_\infty^{\mathcal{E}}))^2]=0$. This completes the proof of the theorem. □

Now we consider a map ϕ from the network Γ to a simply connected, complete separable geodesic space $(\mathcal{M},d_{\mathcal{M}})$ of nonpositive curvature (cf. [4], [13]). For any $x\in V$, there exists uniquely a point of $\mathcal{M}$, denoted by $P\phi(x)$, such that

$$\sum_{y\sim x}\frac{d_{\mathcal{M}}(P\phi(x),\phi(y))^2}{r(|xy|)}=\inf_{q\in\mathcal{M}}\sum_{y\sim x}\frac{d_{\mathcal{M}}(q,\phi(y))^2}{r(|xy|)};$$

$P\phi(x)$ is the center of mass of the measure $\sum_{y\sim x} r(|xy|)^{-1}\delta_{\phi(y)}$ on $\mathcal{M}$. A map $\phi\colon V \to \mathcal{M}$ is said to be harmonic if $P\phi(x) = \phi(x)$ at any $x \in V$. A harmonic map $\phi\colon V \to \mathcal{M}$ pulls convex functions η on an open subset $A \subset \mathcal{M}$ back to subharmonic functions $\eta\circ\phi$ on $\phi^{-1}(A)$ (see [4, Proposition 12.3 (Jensen's inequality)]).

Now we prove the following

Theorem 7. *Let ϕ be a map from a connected non-parabolic network $\Gamma = (V, E, r)$ to a simply connected, complete, separable geodesic space $(\mathcal{M}, d_{\mathcal{M}})$ of non-positive curvature. Let $\phi\colon V \to \mathcal{M}$ be a Dirichlet finite harmonic map. Then the image $\phi(V)$ is contained in the convex hull $\mathcal{C}(L)$ of the set L of points to which $\phi(X_n)$ converges almost surely.*

Moreover $\phi(V)$ is bounded if any $h \in H_{\mathcal{E}_\Gamma}$ is bounded. In particular, ϕ must be constant if $H_{\mathcal{E}_\Gamma} = \boldsymbol{R}$, that is, Γ admits no non-constant Dirichlet finite harmonic functions.

Proof. Let η be a distance function to the convex hull $\mathcal{C}(L)$ of L, that is the smallest closed convex subset containing L in $\mathcal{M}$. Then η^2 is convex and hence $\eta^2\circ\phi$ is subharmonic on V. Thus we have

$$\eta^2\circ\phi(x) \le \mathbf{E}_x[\eta^2\circ\phi(X_n)]$$

for any $x \in V$ and all $n = 1, 2, \ldots$. Since $\lim_{n\to\infty} \eta^2(\phi(X_n)) = 0$ almost surely, we get $\eta^2\circ\phi(x) = 0$, that is, $\phi(x) \in \mathcal{C}(L)$.

Now we suppose that any $h \in H_{\mathcal{E}_\Gamma}$ is bounded. Since this condition is equivalent to the condition that any nonnegative subharmonic function u of $D[\mathcal{E}_\Gamma]$ is bounded, for the distance function η to a point of $\mathcal{M}$, $\eta\circ\phi$ is bounded. Thus $\phi(V)$ must be bounded. Moreover we suppose that Γ admits no non-constant Dirichlet finite harmonic functions. Then $\Delta(\mathcal{E}_\Gamma)$ consists of a single point, and hence so does L. Thus ϕ must be a constant map. This completes the proof of the theorem. □

Let Ω be the set of one-sided infinite paths in a connected non-parabolic network Γ. Given a path $\omega \in \Omega$, the set of limit points of ω in the Royden boundary $\partial\mathcal{R}(\mathcal{E}_\Gamma)$ of Γ is defined as

$$L(\omega) = \overline{\{X_n(\omega)\}} \cap \partial\mathcal{R}(\mathcal{E}_\Gamma).$$

Then we can deduce from Theorem 6 the following

Lemma 8. *For any null family Σ of one-sided infinite paths, one has*

$$\overline{\bigcup\{L(\omega) \mid \omega \in \Omega \setminus \Sigma\}} \supset \Delta(\mathcal{E}_\Gamma).$$

Now we prove the following

Theorem 9. *Let $\phi\colon \Gamma \to (\mathcal{M}, d_{\mathcal{M}})$ be a Dirichlet finite map from a connected non-parabolic network $\Gamma = (V, E, r)$ to a proper metric space $(\mathcal{M}, d_{\mathcal{M}})$, that is a metric space such that any bounded closed subset is compact. Let $\overline{\mathcal{M}} = \mathcal{M} \cup \{\infty_{\mathcal{M}}\}$ be the one-point compactification of $\mathcal{M}$. Then ϕ extends to a continuous map $\bar{\phi}\colon \mathcal{R}(\mathcal{E}_\Gamma) \to \overline{\mathcal{M}}$ from the Royden compactification $\mathcal{R}(\mathcal{E}_\Gamma)$ of Γ to $\overline{\mathcal{M}}$. Moreover there exists a null family Σ in Ω such that $\phi(X_n(\omega))$ converges in $\mathcal{M}$ for all $\omega \in \Omega \setminus \Sigma$ and*

$$\bar{\phi}(\Delta(\mathcal{E}_\Gamma)) = \overline{\left\{ \lim_{n\to\infty} \phi(X_n(\omega)) \in \mathcal{M} \;\middle|\; \omega \in \Omega \setminus (\Sigma' \cup \Sigma) \right\}} \cup \{\infty_{\mathcal{M}}\}$$

for any null family Σ' in Ω.

Proof. For a point $x \in \mathcal{M}$, we denote by η_x the distance function to x in $\mathcal{M}$. Let $\Lambda_\phi = \{\xi \in \partial\mathcal{R}(\mathcal{E}_\Gamma) \mid \overline{\eta_x \circ \phi}(\xi) = +\infty\}$, where $\overline{\eta_x \circ \phi}$ stands for the continuous extension of $\eta_x \circ \phi$ to $\mathcal{R}(\mathcal{E}_\Gamma)$ with values in $\mathbf{R} \cup \{\pm\infty\}$. This closed subset is independent of the choice of a reference point x. Now we take a countably infinite dense subset $\{x_i\}$ of $\mathcal{M}$. Let ξ and $\{v_n\}$ be, respectively, a point of $\partial\mathcal{R}(\mathcal{E}_\Gamma) \setminus \Lambda_\phi$ and a sequence in V converging to ξ. Then $\phi(v_n)$ stays in a compact subspace in $\mathcal{M}$. Since $d_X(x_i, \phi(v_n))$ tends to $\overline{\eta_{x_i} \circ \phi}(\xi)$ as $n \to \infty$ for all x_i which are densely distributed in $\mathcal{M}$, we can deduce that as n tends to infinity, $\phi(v_n)$ converges to a point, $\bar{\phi}(\xi)$, in $\mathcal{M}$. By setting $\bar{\phi}(\xi) = \infty_{\mathcal{M}}$ for $\xi \in \Lambda_\phi$, we obtain a continuous map $\bar{\phi}$ from $\mathcal{R}(\mathcal{E}_\Gamma)$ to $\overline{\mathcal{M}}$.

Let Ω_ϕ be the set of one-sided infinite paths along which $\phi(X_n)$ converges in $\mathcal{M}$. For any $j = 1, 2, \ldots$, let $\bar{\phi}(\Delta(\mathcal{E}_\Gamma))_j = \{x \in \mathcal{M} \mid d_{\mathcal{M}}(x, \bar{\phi}(\Delta(\mathcal{E}_\Gamma))) < 1/j\}$ and $A_j = \bar{\phi}(\partial\mathcal{R}(\mathcal{E}_\Gamma)) \setminus \bar{\phi}(\Delta(\mathcal{E}_\Gamma))_j$. Since $\bar{\phi}^{-1}(A_j)$ is disjoint from $\Delta(\mathcal{E}_\Gamma)$, we have by Lemma 5.3 in [15] a function $g_j \in D_0[\mathcal{E}_\Gamma]$ such that $g_j = +\infty$ on $\bar{\phi}^{-1}(A_j) \cap \partial\mathcal{R}(\mathcal{E}_\Gamma)$. On the other hand, it follows from Theorem 1 that $\lim_{n\to\infty} g_j(X_n) = 0$ almost surely. This shows that $\left\{\omega \in \Omega_\phi \;\middle|\; \lim_{n\to\infty} \phi(X_n(\omega)) \in A_j\right\}$ is a null family of paths, and hence, letting $\Sigma = \left\{\omega \in \Omega_\phi \;\middle|\; \lim_{n\to\infty} \phi(X_n(\omega)) \in \bigcup_j A_j\right\}$, we see that $\lim_{n\to\infty} \phi(X_n(\omega)) \in \bar{\phi}(\Delta(\mathcal{E}_\Gamma))$ for all $\omega \in \Omega_\phi \setminus \Sigma$. Moreover by Lemma 8, the assertion holds true. □

REMARK. Relevantly to Theorem 7, we refer to [8] in which a Liouville type theorem for harmonic maps to convex spaces via Markov chains is discussed. For an existence result of Dirichlet finite harmonic maps, see [12]. A connected parabolic network admits no non-constant Dirichlet finite harmonic maps to a simply connected, complete, geodesic space of nonpositive curvature. In fact, Theorem (3.34) in [11] states that a Dirichlet finite subharmonic function on such a network must be constant. We also refer to [3], where it is proved that if on a complete Riemannian manifold M, every harmonic function with finite Dirichlet energy is bounded, then every harmonic map with finite total energy from M into a Cartan–Hadamard manifold must also have bounded image.

4. Poisson equations

Let $\Gamma = (V, E, r)$ be a connected non-parabolic network and $\mathcal{E}$ a resistance form of Γ. In this section, we derive integral representations of solutions of Poisson equations on the Kuramochi compactification of $\mathcal{E}$.

To begin with, we show the following

Lemma 10 (Harnack's inequality). *Let h be a positive harmonic function on V. Then*

$$h(x) \leq \frac{g_\Gamma^0(x, x)}{g_\Gamma^0(y, x)} h(y)$$

for all $x, y \in V$.

Proof. Let $\{V_n\}$ be an increasing sequence of finite subsets of V such that $V = \bigcup_n V_n$. Let D_n be the space of functions on V which vanish outside of V_n. Then for any $x \in V_n$, there exists uniquely a function $g_x \in D_n$ satisfying

$$\mathcal{E}_\Gamma(g_x, u) = u(x)$$

for all $u \in D_n$. We write $g_n(x, y)$ for $g_x(y)$. Fix points $x, y \in V$ and consider V_n for n large enough. Then there exists uniquely a function $p_n \in D_n$ such that $p_n(x) = h(x)$, $p_n(y) = h(y)$, and $L^c p_n(z) = 0$ for any $z \in V_n \setminus \{x, y\}$. The maximum principle ensures that $p_n \leq h$ in V and hence $L^c p_n(x) \geq 0$ and $L^c p_n(y) \geq 0$. Then we have

$$\begin{aligned}
h(x) &= p_n(x) \\
&= \mathcal{E}_\Gamma(g_n(x, *), p_n) \\
&= \sum_{z \in V_n} g_n(x, z) L^c p_n(z) \\
&= g_n(x, x) L^c p_n(x) + g_n(x, y) L^c p_n(y) \\
&= \frac{g_n(x, x)}{g_n(y, x)} g_n(y, x) L^c p_n(x) + \frac{g_n(x, y)}{g_n(y, y)} g_n(y, y) L^c p_n(y) \\
&\leq \frac{g_n(x, x)}{g_n(y, x)} \{g_n(y, x) L^c p_n(x) + g_n(y, y) L^c p_n(y)\} \\
&= \frac{g_n(x, x)}{g_n(y, x)} \mathcal{E}_\Gamma(g_n(y, *), p_n) \\
&= \frac{g_n(x, x)}{g_n(y, x)} p_n(y) \\
&= \frac{g_n(x, x)}{g_n(y, x)} h(y).
\end{aligned}$$

Thus we get

$$h(x) \le \frac{g_n(x,x)}{g_n(y,x)} h(y)$$

for all large n. As $n \to \infty$, $g_n(z,w)$ converges to $g^0_\Gamma(z,w)$ for any $(z,w) \in V \times V$, and thus we obtain the required inequality. □

In what follows, we take a probability measure μ on V satisfying (2) and

$$\int_V \frac{g^0_\Gamma(x,x)}{g^0_\Gamma(x,o)}\, d\mu(x) < +\infty. \tag{5}$$

Proposition 11. (i) *For any fixed $x \in V$, $g^0_\Gamma(x,*) \in D[\bar{L}^{\mathcal{E}}] \cap D_0[\mathcal{E}_\Gamma]$ and the harmonic measure ν_x with respect to $x \in V$ is given by*

$$\nu_x = -N^{\mathcal{E}} g^0_\Gamma(x,*)\nu.$$

(ii) *Let*

$$G^0_\mu(x) = \int_V g^0_\Gamma(x,z)\, d\mu(z), \quad x \in V.$$

Then G^0_μ belongs to $D[\bar{L}^{\mathcal{E}}]$ and $N^{\mathcal{E}} G^0_\mu = \int_V N^{\mathcal{E}} g^0_\Gamma(x,*)\, d\mu(x)$. Moreover $N^{\mathcal{E}} G^0_\Gamma$ satisfies

$$0 < \mu(0) < -N^{\mathcal{E}} G^0_\mu < \int_V \frac{g^0_\Gamma(x,x)}{g^0_\Gamma(x,o)}\, d\mu(x).$$

Proof. For a function $u \in D[\mathcal{E}]$, we have

$$\begin{aligned}\left| \int_{\partial \mathcal{K}(\mathcal{E})} \tau(u)\, d\nu_x \right| &\le \int_{\partial \mathcal{K}(\mathcal{E})} |\tau(u)|\, d\nu_x \\ &\le \frac{g^0_\Gamma(x,x)}{g^0_\Gamma(x,o)} \int_{\partial \mathcal{K}(\mathcal{E})} |\tau(u)|\, d\nu.\end{aligned}$$

This implies that

$$|\mathcal{E}(g^0_\Gamma(x,*), u)| = \left| u(x) - \int_{\partial \mathcal{K}(\mathcal{E})} \tau(u)\, d\nu_x \right|$$

is bounded by $\mu(x)^{-1} \int_V |u|\, d\mu + g^0_\Gamma(x,x)/g^0_\Gamma(x,o) \int_{\partial \mathcal{K}(\mathcal{E})} |\tau(u)|\, d\nu$. Thus we see that $g^0_\Gamma(x,*)$ belongs to $D[\bar{L}^{\mathcal{E}}]$. Moreover since $L^c g^0_\Gamma(x,*) = \delta_x$, we get

$$\begin{aligned}u(x) - \int_{\partial \mathcal{K}(\mathcal{E})} \tau(u)\, d\nu_x &= \int_V u(y) L^{\mathcal{E}} g^0_\Gamma(x,y)\, d\mu(y) + \int_{\partial \mathcal{K}(\mathcal{E})} \tau(u) N^{\mathcal{E}} g^0_\Gamma(x,*)\, d\nu \\ &= u(x) + \int_{\partial \mathcal{K}(\mathcal{E})} \tau(u) N^{\mathcal{E}} g^0_\Gamma(x,*)\, d\nu.\end{aligned}$$

In this way, we obtain

$$\int_{\partial\mathcal{K}(\mathcal{E})} \tau(u)\, d\nu_x = -\int_{\partial\mathcal{K}(\mathcal{E})} \tau(u) N^{\mathcal{E}} g_\Gamma^0(x, *)\, d\nu.$$

This shows the first assertion.

Given $u \in D[\mathcal{E}]$, let $g(x) = u(x) - \int_{\partial\mathcal{K}(\mathcal{E})} \tau(u)\, d\nu_x$. Then we have

$$\begin{aligned}
|\mathcal{E}_\Gamma(G_\mu^0, u)| &= |\mathcal{E}_\Gamma(G_\mu^0, g)| \\
&= \left| \int_V g(x)\, d\mu(x) \right| \\
&= \left| \int_V u(x)\, d\mu(x) - \int_V \int_{\partial\mathcal{K}(\mathcal{E})} \tau(u)\, d\nu_x\, d\mu(x) \right| \\
&\le \int_V |u(x)|\, d\mu(x) + \int_V \int_{\partial\mathcal{K}(\mathcal{E})} |\tau(u)|\, d\nu_x\, d\mu(x) \\
&\le \int_V |u(x)|\, d\mu(x) + \int_V \frac{g_\Gamma^0(x, x)}{g_\Gamma^0(x, o)}\, d\mu(x) \int_{\partial\mathcal{K}(\mathcal{E})} |\tau(u)|\, d\nu.
\end{aligned}$$

This shows that G_μ^0 belongs to $D[\bar{L}^{\mathcal{E}}]$. It is easy to see the remaining assertions. This completes the proof of the proposition. □

As in Section 3, we now introduce a kernel function $g_\mu^{\mathcal{E}}$ on $\mathcal{E}$ by

$$g_\mu^{\mathcal{E}}(x, y) = \int_V g_z^{\mathcal{E}}(x, y)\, d\mu(z), \quad x, y \in V.$$

Then we have

$$\mathcal{E}(g_\mu^{\mathcal{E}}(x, *), u) = u(x) - \int_V u\, d\mu, \quad u \in D[\mathcal{E}].$$

In particular, the function $g_\mu^{\mathcal{E}}(x, *)$ for a fixed $x \in V$ belongs to $D[L^{\mathcal{E}}]$. Similarly, let

$$h_\mu^{\mathcal{E}}(x, y) = \int_V h_z^{\mathcal{E}}(x, y)\, d\mu(z), \quad x, y \in V.$$

Then we have

$$\mathcal{E}(h_\mu^{\mathcal{E}}(x, *), h) = h(x) - \int_V h\, d\mu, \quad h \in H_{\mathcal{E}}.$$

In view of (1), we see that

$$g_\mu^{\mathcal{E}}(x, y) = h_\mu^{\mathcal{E}}(x, y) + g_\Gamma^0(x, y) - G_\mu^0(x) - G_\mu^0(y) + C_{\Gamma,\mu}, \tag{6}$$

where we put $C_{\Gamma,\mu} = \int_V g_\Gamma^0(z, z)\, d\mu(z)$.

Given a function $u \in D[\bar{L}^{\mathcal{E}}]$, we have

$$\begin{aligned} u(x) &= \int_V u\, d\mu + \mathcal{E}(g_\mu^{\mathcal{E}}(x, *), u) \\ &= \int_V u\, d\mu + \int_V g_\mu^{\mathcal{E}}(x, y) L^{\mathcal{E}} u(y)\, d\mu(y) + \int_{\partial \mathcal{K}(\mathcal{E})} g_\mu^{\mathcal{E}}(x, \xi) N^{\mathcal{E}} u(\xi)\, d\nu(\xi) \\ &= \int_V u\, d\mu + \int_V g_\mu^{\mathcal{E}}(x, y) L^{\mathcal{E}} u(y)\, d\mu(y) + \int_{\partial \mathcal{K}(\mathcal{E})} h_\mu^{\mathcal{E}}(x, \xi) N^{\mathcal{E}} u(\xi)\, d\nu(\xi) \\ &\quad - (G_\mu^0(x) - C_{\Gamma,\mu}) \int_{\partial \mathcal{K}(\mathcal{E})} N^{\mathcal{E}} u(\xi)\, d\nu(\xi). \end{aligned}$$

Since

$$\int_V L^{\mathcal{E}} u\, d\mu + \int_{\partial \mathcal{K}(\mathcal{E})} N^{\mathcal{E}} u\, d\nu = \mathcal{E}(u, 1) = 0,$$

by letting

$$\bar{g}_\mu^{\mathcal{E}}(x, y) = g_\mu^{\mathcal{E}}(x, y) + G_\mu^0(x) - C_{\Gamma,\mu},$$

we obtain an integral representation of a function u of $D[\bar{L}^{\mathcal{E}}]$ as follows:

$$u(x) = \int_V u\, d\mu + \int_V \bar{g}_\mu^{\mathcal{E}}(x, y) L^{\mathcal{E}} u(y)\, d\mu(y) + \int_{\partial \mathcal{K}(\mathcal{E})} h_\mu^{\mathcal{E}}(x, \xi) N^{\mathcal{E}} u(\xi)\, d\nu(\xi).$$

Let f be a function in $L^2(\mathcal{K}(\mathcal{E}), \bar{\mu})$. Suppose that $\bar{\mu}(f) = \int_V f\, d\mu + \int_{\partial \mathcal{K}(\mathcal{E})} f\, d\nu = 0$. Then for any $h \in H_{\mathcal{E}}$, we have

$$\begin{aligned} &\left| \int_V hf\, d\mu + \int_{\partial \mathcal{K}(\mathcal{E})} \tau(h) f\, d\nu \right|^2 \\ &= \left| \int_V (h - h(o)) f\, d\mu + \int_{\partial \mathcal{K}(\mathcal{E})} \tau(h - h(0)) f\, d\nu \right|^2 \\ &\le \left(\int_V (h - h(o))^2 d\mu + \int_{\partial \mathcal{K}(\mathcal{E})} \tau(h - h(o))^2\, d\nu \right) \left(\int_V f^2\, d\mu + \int_{\partial \mathcal{K}(\mathcal{E})} f^2\, d\nu \right) \\ &\le \left(\int_V R_{\mathcal{E}}(o, x)\, d\mu(x) + 2g_\Gamma^0(o, o) \right) \left(\int_V f^2\, d\mu + \int_{\partial \mathcal{K}(\mathcal{E})} f^2\, d\nu \right) \mathcal{E}(h, h), \end{aligned}$$

where we have used

$$\int_V (h(x) - h(o))^2\, d\mu(x) \le \int_V h_o^{\mathcal{E}}(x, x)\, d\mu(x) \mathcal{E}(h, h) \le \int_V R_{\mathcal{E}}(o, x)\, d\mu(x) \mathcal{E}(h, h)$$

and

$$\int_{\partial \mathcal{K}(\mathcal{E})} \tau(h - h(o))^2\, d\nu \le 2g_\Gamma^0(o, o) \mathcal{E}(h, h)$$

by (4). For $g \in D_0[\mathcal{E}_\Gamma]$, we have

$$\begin{aligned}\left|\int_V gf\,d\mu\right|^2 &\le \int_V g^2\,d\mu \int_V f^2\,d\mu \\ &\le \int_V \mathcal{E}_\Gamma(g^0_\Gamma(x,*),g)^2\,d\mu(x)\int_V f^2\,d\mu \\ &= \int_V g^0_\Gamma(x,x)\,d\mu(x)\int_V f^2\,d\mu\;\mathcal{E}_\Gamma(g,g) \\ &\le \left(\int_V R_{\mathcal{E}}(o,x)\,d\mu(x)+2g^0_\Gamma(o,o)\right)\int_V f^2\,d\mu\;\mathcal{E}_\Gamma(g,g).\end{aligned}$$

In this way, we see that for any $u \in D[\mathcal{E}]$,

$$\begin{aligned}&\left|\int_V uf\,d\mu+\int_{\partial\mathcal{K}(\mathcal{E})}\tau(u)f\,d\nu\right|^2 \\ &\le \left(\int_V R_{\mathcal{E}}(o,x)\,d\mu(x)+2g^0_\Gamma(o,o)\right)\left(\int_V f^2\,d\mu+\int_{\partial\mathcal{K}(\mathcal{E})}f^2\,d\nu\right)\mathcal{E}(u,u).\end{aligned}$$

This shows that there exists a function ϕ in $D[\mathcal{E}]$, unique up to additive constants, such that

$$\mathcal{E}(u,\phi)=\int_V uf\,d\mu+\int_{\partial\mathcal{K}(\mathcal{E})}\tau(u)f\,d\nu,\quad u\in D[\mathcal{E}],$$

so that ϕ belongs to $D[\bar{L}^{\mathcal{E}}]$, $\bar{L}^{\mathcal{E}}\phi=f$ in $K^2(\mathcal{K}(\mathcal{E}),\bar{\mu})$, and ϕ is expressed in the following way:

$$\phi(x)=\int_V \phi\,d\mu+\int_V \bar{g}^{\mathcal{E}}_\mu(x,y)f(y)\,d\mu(y)+\int_{\partial\mathcal{K}(\mathcal{E})}h^{\mathcal{E}}_\mu(x,\xi)f(\xi)\,d\nu(\xi). \tag{7}$$

In the case where $\bar{\mu}(f)\neq 0$, the function ϕ defined in (7) satisfies $L^{\mathcal{E}}\phi=f$ on V and $N^{\mathcal{E}}\phi=f+\bar{\mu}(f)N^{\mathcal{E}}G^0_\mu$ in $L^2(\partial\mathcal{K}(\mathcal{E}),\nu)$.

In fact, we have

$$\begin{aligned}&\phi(x)-\int_V\phi\,d\mu \\ &=\int_V \bar{g}^{\mathcal{E}}_\mu(x,y)f(y)\,d\mu(y) \\ &\quad+\int_{\partial\mathcal{K}(\mathcal{E})}h^{\mathcal{E}}_\mu(x,\xi)(f(\xi)-\bar{\mu}(f))\,d\nu(\xi)+\bar{\mu}(f)\int_{\partial\mathcal{K}(\mathcal{E})}h^{\mathcal{E}}_\mu(x,\xi)\,d\nu(\xi) \\ &=\int_V \bar{g}^{\mathcal{E}}_\mu(x,y)f(y)\,d\mu(y)+\int_{\partial\mathcal{K}(\mathcal{E})}h^{\mathcal{E}}_\mu(x,\xi)(f(\xi)-\bar{\mu}(f))\,d\nu(\xi)+\bar{\mu}(f)h^{\mathcal{E}}_\mu(x,o)\end{aligned}$$

and

$$N^{\mathcal{E}}h^{\mathcal{E}}_\mu(\xi,o)=1+N^{\mathcal{E}}G^0_\mu(\xi).$$

Thus we have the following

Theorem 12. *Let $\Gamma = (V, E, r)$ be a connected non-parabolic network and $\mathcal{E}$ a resistance form of Γ. A probability measure μ on V satisfying* (2) *and* (5) *is given.*
(i) *For $u \in D[\bar{L}^{\mathcal{E}}]$, one has*

$$u(x) = \int_V u\, d\mu + \int_V \bar{g}_\mu^{\mathcal{E}}(x, y) L^{\mathcal{E}} u(y)\, d\mu(y) + \int_{\partial \mathcal{K}(\mathcal{E})} h_\mu^{\mathcal{E}}(x, \xi) N^{\mathcal{E}} u(\xi)\, d\nu(\xi), \quad x \in V.$$

(ii) *For $f \in L^2(\mathcal{K}(\mathcal{E}), \bar{\mu})$ and a constant c, the function*

$$u(x) = c + \int_V \bar{g}_\mu^{\mathcal{E}}(x, y) f(y)\, d\mu(y) + \int_{\partial \mathcal{K}(\mathcal{E})} h_\mu^{\mathcal{E}}(x, \xi) f(\xi)\, d\nu(\xi), \quad x \in V.$$

belongs to $D[\bar{L}^{\mathcal{E}}]$ and satisfies $\bar{L}^{\mathcal{E}} u = f$ on V and $\bar{L}^{\mathcal{E}} u = f + \bar{\mu}(f) N^{\mathcal{E}} G_\mu^0$ in $L^2(\partial \mathcal{K}(\mathcal{E}), \nu)$. In particular if $\bar{\mu}(f) = 0$, then $\bar{L}^{\mathcal{E}} u = f$ in $L^2(\mathcal{K}(\mathcal{E}), \bar{\mu})$.

Let $D[\mathcal{E}^*] = \{\tau(u) \mid u \in D[\mathcal{E}]\}$ $(\subset L^2(\partial \mathcal{K}(\mathcal{E}), \nu))$ and $\mathcal{E}^*(\tau(u), \tau(v)) = \mathcal{E}(h_u, h_v)$ for $u, v \in D[\mathcal{E}]$, where h_u denotes the harmonic part of u in the Royden decomposition. Let $(L^*, D[L^*])$ be the self-adjoint operator associated to the regular Dirichlet form $(\mathcal{E}^*, D[\mathcal{E}^*])$ on $L^2(\partial \mathcal{K}(\mathcal{E}), \nu)$. The restriction of τ to $H_{\mathcal{E}}$ gives rise to a bijection between $H_{\mathcal{E}}$ and $D[\mathcal{E}^*]$ such that $\tau(H_{\mathcal{E}} \cap D[\bar{L}^{\mathcal{E}}]) = D[L^*]$ and $N^{\mathcal{E}} h = L^* \tau(h)$ for $h \in H_{\mathcal{E}} \cap D[\bar{L}^{\mathcal{E}}]$.

References

[1] A. Ancona, R. Lyons and Y. Peres: *Crossing estimates and convergence of Dirichlet functions along random walk and diffusion paths*, Ann. Probab. **27** (1999), 970–989.
[2] D.I. Cartwright, P.M. Soardi and W. Woess: *Martin and end compactifications for non-locally finite graphs*, Trans. Amer. Math. Soc. **338** (1993), 679–693.
[3] S.Y. Cheng, L.-F. Tam and T.Y.-H. Wan: *Harmonic maps with finite total energy*, Proc. Amer. Math. Soc. **124** (1996), 275–284.
[4] J. Eells and B. Fuglede: Harmonic Maps Between Riemannian Polyhedra, Cambridge Univ. Press, Cambridge, 2001.
[5] A. Kasue: *Convergence of metric graphs and energy forms*, Rev. Mat. Iberoam. **26** (2010), 367–448.
[6] T. Kayano and M. Yamasaki: *Some properties of Royden boundary of an infinite network*, Mem. Fac. Sci. Shimane Univ. **22** (1988), 11–19.
[7] J. Kigami: Analysis on Fractals, Cambridge Tracts in Mathematics **143**, Cambridge Univ. Press, Cambridge, 2001.
[8] K. Kuwae and K.-T. Sturm: *On a Liouville type theorem for harmonic maps to convex spaces via Markov chains*; in Proceedings of RIMS Workshop on Stochastic Analysis and Applications, RIMS Kôkyûroku Bessatsu, B6, Res. Inst. Math. Sci., Kyoto, 177–191, 2008.
[9] A. Murakami and M. Yamasaki: *An introduction of Kuramochi boundary of an infinite network*, Mem. Fac. Sci. Eng. Shimane Univ. Ser. B Math. Sci. **30** (1997), 57–89.

[10] T. Nakamura and M. Yamasaki: *Generalized extremal length of an infinite network*, Hiroshima Math. J. **6** (1976), 95–111.
[11] P.M. Soardi: Potential Theory on Infinite Networks, Lecture Notes in Mathematics **1590**, Springer, Berlin, 1994.
[12] K.-T. Sturm: *Nonlinear Markov operators associated with symmetric Markov kernels and energy minimizing maps between singular spaces*, Calc. Var. Partial Differential Equations **12** (2001), 317–357.
[13] K.-T. Sturm: *Probability measures on metric spaces of nonpositive curvature*; in Heat Kernels and Analysis on Manifolds, Graphs, and Metric Spaces (Paris, 2002), Contemp. Math. **338**, Amer. Math. Soc., Providence, RI., 357–390, 2003.
[14] M. Yamasaki: *Parabolic and hyperbolic infinite networks*, Hiroshima Math. J. **7** (1977), 135–146.
[15] M. Yamasaki: *Discrete Dirichlet potentials on an infinite network*; Potential theory and its related fields (Kyoto, 1986), RIMS Kôkyûroku **610**, Res. Inst. Math. Sci., Kyoto, 51–66, 1987.

Department of Mathematics
Kanazawa University
Kanazawa 920-1192
Japan
e-mail: kasue@se.kanazawa-u.ac.jp