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CONFORMALLY FLAT HYPERSURFACES WITH BIANCHI-TYPE GUICHARD NET

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Abstract

We obtain a partial classification result for generic 3-dimensional conformally flat hypersurfaces in the conformal 4-sphere: explicit analytic data are obtained for conformally flat hypersurfaces with Bianchi-type canonical Guichard net. This is the first classification result for conformally flat hypersurfaces without additional symmetry. We discuss the curved flat associated family for conformally flat hypersurfaces and show that it descends to an associated family of conformally flat hypersurfaces. The associated family of conformally flat hypersurfaces with Bianchi-type Guichard net is investigated.

1. Introduction

A generic conformally flat hypersurface of the conformal 4-sphere comes with a special coordinate system, its "canonical principal Guichard net". These are curvature line coordinates (x, y, z) so that, in particular, the coordinate system defines a triply orthogonal system in the intrinsic geometry of the hypersurface, i.e., the coordinate surfaces of different families, say $x = \text{const}$ and $y = \text{const}$, intersect orthogonally. Further, these coordinates form a Guichard net, that is, they satisfy a zero trace condition (2.1) for the induced conformal structure. This condition allows to encode the induced conformal structure in terms of a single real-valued function $(x, y, z) \mapsto \varphi(x, y, z)$, which in turn allows to classify conformally flat hypersurfaces in terms of these functions, satisfying a system of partial differential equations, see [6, Lemma 1].

As a conformally flat hypersurface also admits conformal coordinates, these can be used to map the system of coordinate surfaces to Euclidean 3-space to obtain a Guichard net in $\mathbb{R}^3$. The conformally flat hypersurface can then (locally) be reconstructed in an essentially unique way from this Guichard net in Euclidean space, that is, from the intrinsic geometry of the hypersurface equipped with this special coordinate system, see [5, §2.4.6]. Thus the classification of generic 3-dimensional conformally flat hypersurfaces in S^4 is equivalent to a classification of such Guichard nets in Euclidean 3-space or, equivalently, the conformal 3-sphere. In this context it is an interesting

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question to understand the relation between the geometries of the hypersurface and its canonical principal Guichard net.

Note that this correspondence between conformally flat hypersurfaces and their canonical principal Guichard nets is established for *parametrized* Guichard nets. The Guichard condition allows for rescaling of the coordinate functions by a real constant λ: this is closely related to the associated family of a conformally flat hypersurface, as we shall discuss in the last section of this paper. Thus the aforementioned relation between conformally flat hypersurfaces and Guichard nets in Euclidean 3-space requires fixing the scaling of the coordinate functions to obtain uniqueness up to Möbius transformation in the correspondence, cf. [9, Corollaries 3.1.1 and 3.2.1].

Some examples and simple classification results were obtained in [4] and [5, Section 2.4]. Based on [9], we obtained a classification of the fairly large class of conformally flat hypersurfaces with cyclic Guichard net, that is, where one family of coordinate lines are circular arcs in the intrinsic geometry of the hypersurface, in [6]. This class is characterized by separation of variables in the system of differential equations for the real valued function φ describing the induced conformal class.

The aim of this paper is to provide a class of examples, where the characterizing function φ cannot be written as the sum of a function of one variable and one of two variables. Or, otherwise said, where the canonical principal Guichard net is not cyclic.

After clarifying the technology to be used in Section 2, we discuss the intrinsic geometry of the conformally flat hypersurfaces obtained from a relatively simple class of solutions φ that do not give rise to cyclic Guichard nets in general. We find that there is a constant sectional curvature representative of the induced conformal structure so that all (coordinate) surfaces of the canonical Guichard net have constant Gauss curvature, that is, the triply orthogonal system is a Bianchi system in three different ways, see [8, §22]:

We say that a triply-orthogonal system of surfaces in a constant sectional curvature space is of Bianchi-type if all surfaces have constant Gauss curvature.

Imposing this geometric condition for a non-cyclic Guichard net, the corresponding conformally flat hypersurface turns out to have an induced conformal structure given by a function φ of that type as we shall see in Section 4. Hence we obtain a classification result:

Main Theorem. *Let $\varphi(x, y, z) = g(ax + by + cz)$, where $g'^2 = C - A\cos 2g$ and $a, b, c, A, C \in \mathbb{R}$ satisfy $abc \neq 0$ and $A \neq 0, \pm C$. Then φ defines a conformally flat hypersurface f in the conformal 4-sphere so that its canonical principal Guichard net is non-cyclic and of (triply) Bianchi type for a suitable choice of constant sectional curvature representative of the induced conformal structure. Conversely, any conformally flat hypersurface with this intrinsic geometry comes from such a function φ.*

In the process of proving the latter part of the theorem in Section 4, we obtain another class of conformally flat hypersurfaces, satisfying milder conditions on its canonical Guichard net.

The classifications of 3-dimensional generic conformally flat hypersurfaces, Guichard nets and the functions φ satisfying a certain set of partial differential equations, respectively, are all equivalent. However, it is a highly non-trivial task to establish relations between the geometric properties of a hypersurface and its canonical Guichard net and the analytic properties of the induced conformal structure. In Section 2 we also discuss some alternative analytic data that may facilitate the understanding of the interplay between geometry and analysis. However, as this paper shows, the analysis of even seemingly simple classes of conformally flat hypersurfaces is getting increasingly laborious as symmetry is lost. Hence another, more geometric characterization of conformally flat hypersurfaces seems desirable for obtaining a complete classification.

2. The setup

We will work in a Möbius geometric realm, within the same framework as in our previous paper [6] (for more background details on the formalism the reader is referred to Blaschke's classic [1], to Cartan's original paper [3], or to [5]). In particular, our hypersurfaces will "live" in the projective light cone

$$S^4 \cong L^5/\mathbb{R}, \quad \text{where} \quad L^5 := \{y \in \mathbb{R}^6_1 \mid |y|^2 = 0\}$$

denotes the light cone in Minkowski 6-space $\mathbb{R}^6_1$: by $|\cdot|^2$ we denote the quadratic form of the Minkowski inner product on $\mathbb{R}^6_1$, so that $|y|^2 > 0$, $|y|^2 = 0$ or $|y|^2 < 0$ according to whether y is spacelike, lightlike or timelike, respectively; note that rescaling of a light cone lift of a hypersurface corresponds to a conformal change of the induced metric,

$$f \to f' = e^{\psi} f \Rightarrow \mathrm{I} = \langle df, df \rangle \to \mathrm{I}' = \langle d(e^{\psi} f), d(e^{\psi} f) \rangle = e^{2\psi}\mathrm{I}.$$

Hyperspheres of the conformal 4-sphere are, in this framework, encoded by spacelike lines in $\mathbb{R}^6_1$ or either of two unit representatives

$$s \in S^5_1 := \{y \in \mathbb{R}^6_1 \mid |y|^2 = 1\}.$$

Note that the quadrics

$$\mathcal{Q}^4 := \{y \in L^5 \mid \langle y, Q \rangle = -1\}$$

are spaces of constant curvature $\kappa = -|Q|^2$ (see [5, Section 1.4]), and that the "stereographic projections" between any two such quadrics are conformal. In particular, a conformally flat hypersurface in the conformal 4-sphere can be placed in any of these quadrics by means of the scaling $-f/\langle f, Q\rangle$ (provided that $\langle f, Q\rangle \neq 0$, that is, f does not hit the infinity boundary of $\mathcal{Q}^4$ in case $|Q|^2 \geq 0$). Thus, if $(\omega_1, \omega_2, \omega_3)$ denotes a principal orthonormal co-frame for a hypersurface $\mathfrak{f}\colon M^3 \to \mathcal{Q}^4$ with principal curvatures κ_i (in the space form $\mathcal{Q}^4$) then its (real) conformal fundamental forms are

$$\begin{aligned}\gamma_1 &= \sqrt{|\kappa_3-\kappa_1|\,|\kappa_1-\kappa_2|}\,\omega_1,\\ \gamma_2 &= \sqrt{|\kappa_1-\kappa_2|\,|\kappa_2-\kappa_3|}\,\omega_2,\\ \gamma_3 &= \sqrt{|\kappa_2-\kappa_3|\,|\kappa_3-\kappa_1|}\,\omega_3.\end{aligned}$$

We shall see later (Section 2.2 below) that the γ_i are indeed conformal invariants—up to permutation and choice of sign: in the case of a *generic* hypersurface, that is, a hypersurface with pairwise distinct principal curvatures, an ordering of the principal curvatures can be used to remove the permutation ambiguity. The sign ambiguity will play no geometric role other than sign choices for a tangential frame and we will therefore not account for it.

The conformal fundamental forms are closed, $d\gamma_i = 0$, if and only if the hypersurface is conformally flat, see [5, §2.3.3]. Consequently, by (locally) integrating the conformal fundamental forms, we obtain a canonical coordinate system (x, y, z) for any conformally flat hypersurface as soon as we assume the hypersurface to be generic: its *canonical principal Guichard net*,

$$dx = \gamma_1, \quad dy = \gamma_2, \quad dz = \gamma_3.$$

It is obvious that the obtained coordinate system is principal; in order to see that it defines a Guichard net (cf. [5, §2.4.4]) assume, without loss of generality, that κ_3 is the middle principal curvature, that is,

$$(\kappa_2-\kappa_3)(\kappa_3-\kappa_1) > 0 :$$

then the induced metric of $\mathfrak{f}$ becomes

$$\mathrm{I} = l_1^2\,dx^2 + l_2^2\,dy^2 + l_3^2\,dz^2 \quad \text{with} \quad l_1^2 + l_2^2 = l_3^2. \tag{2.1}$$

Hence we can write

$$\mathrm{I} = e^{2\psi}\{\cos^2\varphi\,dx^2 + \sin^2\varphi\,dy^2 + dz^2\} \tag{2.2}$$

on the coordinate domain $U \subset \mathbb{R}^3$ with suitable functions φ and ψ.

Here the function ψ reflects the choice of lift $\mathfrak{f}$, taking values in a particular space form $\mathcal{Q}^4 \subset L^5$, whereas the function φ encodes the conformal geometry of the hypersurface.

By [11], see also [5, §2.3.5], the conformal fundamental forms and Wang's Möbius curvature

$$W := \frac{\kappa_2 - \kappa_3}{\kappa_2 - \kappa_1}$$

form a complete set of conformal invariants for generic hypersurfaces in the conformal 4-sphere: two hypersurfaces are conformally equivalent if and only if they have the same conformal fundamental forms and Möbius curvature.

2.1. Canonical lift. The meaning of φ for the conformal geometry of f becomes clear from the following consideration, cf. [5, §2.4.6]: let $f := e^{-\psi}\mathfrak{f}$ with ψ from (2.2) denote a rescaling of the immersion $\mathfrak{f}$ into $\mathcal{Q}^4$ and note that such a rescaling does not change the conformally flat hypersurface as an immersion into the conformal 4-sphere $S^4 \cong L^5/\mathbb{R}$. Then we choose an adapted Möbius geometric frame

$$(s_1, s_2, s_3, s, f, \hat{f}) \simeq F : U \to O_1(6),$$

where $s_1 = f_x/|f_x|$, etc., s is an enveloped sphere congruence, that is, it defines a unit normal field for $f : U \to L^5$, and $\hat{f} \perp s_i$, s is the unique lightlike vector field so that $\langle \hat{f}, f \rangle = 1$, cf. [6, Section 2] or [5, Section 1.7]. Additionally, we may use the freedom of choice of an enveloped sphere congruence to fix s to be the curvature sphere congruence for the z-direction, that is, we choose s to satisfy $s_z \perp f_z$ or, equivalently, $\eta_3 = 0$ in (2.3): if $\mathfrak{t} \perp Q$ denotes the unit normal field of $\mathfrak{f}$ in $\mathcal{Q}^4$ then we take $s = \mathfrak{t} + \kappa_3\mathfrak{f}$. Then, using the compatibility conditions, the structure equations $dF = F\Phi$ for F can be written entirely in terms of φ:

$$\Phi = \begin{pmatrix} 0 & \omega_{12} & -\omega_{31} & -\eta_1 & \omega_1 & \chi_1 \\ -\omega_{12} & 0 & \omega_{23} & -\eta_2 & \omega_2 & \chi_2 \\ \omega_{31} & -\omega_{23} & 0 & -\eta_3 & \omega_3 & \chi_3 \\ \eta_1 & \eta_2 & \eta_3 & 0 & 0 & \chi \\ -\chi_1 & -\chi_2 & -\chi_3 & -\chi & 0 & 0 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} &\omega_{12} = -(\varphi_y\,dx + \varphi_x\,dy), \quad \omega_1 = \cos\varphi\,dx, \quad \eta_1 = \sin\varphi\,dx, \quad \chi_1 = \sigma_1 + \frac{1}{2}\omega_1, \\ &\omega_{23} = \varphi_z\cos\varphi\,dy, \quad \omega_2 = \sin\varphi\,dy, \quad \eta_2 = -\cos\varphi\,dy, \quad \chi_2 = \sigma_2 + \frac{1}{2}\omega_2, \\ &\omega_{31} = \varphi_z\sin\varphi\,dx, \quad \omega_3 = dz, \quad \eta_3 = 0, \quad \chi_3 = \sigma_3 - \frac{1}{2}\omega_3, \\ &\chi = \varphi_z\,dz \end{aligned} \tag{2.3}$$

and σ_i are the Schouten forms of the induced metric $\mathrm{I} = \cos^2\varphi\, dx^2 + \sin^2\varphi\, dy^2 + dz^2$, see [6, (3.10)]:

$$\begin{aligned}
\sigma_1 &= -\left\{\frac{\varphi_{xx} - \varphi_{yy} - \varphi_{zz}}{\sin 2\varphi} - \frac{\varphi_z^2}{2}\right\}\omega_1 - \frac{\varphi_{xz}}{\sin\varphi}\omega_3, \\
\sigma_2 &= -\left\{\frac{\varphi_{xx} - \varphi_{yy} + \varphi_{zz}}{\sin 2\varphi} - \frac{\varphi_z^2}{2}\right\}\omega_2 + \frac{\varphi_{yz}}{\cos\varphi}\omega_3, \\
\sigma_3 &= -\frac{\varphi_{xz}}{\sin\varphi}\omega_1 + \frac{\varphi_{yz}}{\cos\varphi}\omega_2 + \left\{\frac{\varphi_{xx} - \varphi_{yy} - \varphi_{zz}\cos 2\varphi}{\sin 2\varphi} + \frac{\varphi_z^2}{2}\right\}\omega_3.
\end{aligned} \tag{2.4}$$

The remaining compatibility conditions then yield a system of partial differential equations for φ as a necessary and (locally) sufficient condition for the existence of a conformally flat hypersurface with φ defining its induced conformal structure via (2.2), cf. [6, Lemma 1]:

Lemma 1. *Parametrizing a conformally flat hypersurface $f : \mathbb{R}^3 \supset U \to S^4$ in the conformal 4-sphere by its canonical principal Guichard net, its induced conformal structure is given by*

$$\mathrm{I} = \cos^2\varphi\, dx^2 + \sin^2\varphi\, dy^2 + dz^2, \tag{2.5}$$

where φ satisfies

$$\begin{aligned}
0 &= (\varphi_{yz}\tan\varphi)_x + (\varphi_{xz}\cot\varphi)_y, \\
0 &= \left(\frac{\varphi_{xx} - \varphi_{yy} + \varphi_{zz}}{\sin 2\varphi}\right)_x - \frac{\varphi_z^2}{\sin^2\varphi}\left(\frac{\varphi_x}{\varphi_z}\right)_z, \\
0 &= \left(\frac{-\varphi_{xx} + \varphi_{yy} + \varphi_{zz}}{\sin 2\varphi}\right)_y + \frac{\varphi_z^2}{\cos^2\varphi}\left(\frac{\varphi_y}{\varphi_z}\right)_z, \\
0 &= \left(\frac{\varphi_{xx} + \varphi_{yy} + \varphi_{zz}}{\sin 2\varphi}\right)_z - \frac{\varphi_z^2}{\cos^2\varphi}\left(\frac{\varphi_y}{\varphi_z}\right)_y + \frac{\varphi_z^2}{\sin^2\varphi}\left(\frac{\varphi_x}{\varphi_z}\right)_x.
\end{aligned} \tag{2.6}$$

Conversely, if φ satisfies (2.6) *then it gives* (*locally*) *rise to a unique* (*up to Möbius transformation*) *generic conformally flat hypersurface with* (2.5) *as its induced conformal class in terms of its canonical principal Guichard net.*

Note that $\sin 2\varphi \neq 0$ as we assume (2.5) to define a non-degenerate metric; the formal requirement $\varphi_z \neq 0$ in the equations (2.6) is clearly analytically unproblematic. However, note that the case $\varphi_z \equiv 0$ leads to a special case of our previous classification in [6]: the normal bundle $\mathrm{span}\{s, f, \hat{f}\}$ is flat,

$$0 = d\chi = \varphi_{yz}\, dy \wedge dz - \varphi_{xz}\, dz \wedge dx,$$

if the z-lines are circular arcs, cf. [6, Lemma 2]; in particular, in this case the canonical Guichard net (x, y, z) is a cyclic system. We will come back to this case later, in Section 4.1.

Equivalent formulations of the equations (2.6) may facilitate finding further solutions: for example, the first three of these conditions can be formulated as the integrability $d\alpha = 0$ of the 1-form

$$\alpha := -\varphi_{xz} \cot\varphi \, dx + \varphi_{yz} \tan\varphi \, dy + \frac{\varphi_{xx} - \varphi_{yy} - \varphi_{zz}\cos 2\varphi}{\sin 2\varphi} \, dz$$

see [6, Lemma 1], whereas the last can be formulated as the integrability of the 2-form

$$\beta := \varphi_{xz} \cot\varphi \, dy \wedge dz + \varphi_{yz} \tan\varphi \, dz \wedge dx - \frac{(\varphi_{xx} - \varphi_{yy})\cos 2\varphi - \varphi_{zz}}{\sin 2\varphi} \, dx \wedge dy;$$

on the other hand, the last equation can also be formulated as a condition on the divergence of α:

$$(-\varphi_{xz}\cot\varphi)_x + (\varphi_{yz}\tan\varphi)_y + \left(\frac{\varphi_{xx} - \varphi_{yy} - \varphi_{zz}\cos 2\varphi}{\sin 2\varphi}\right)_z = (\varphi_x^2 + \varphi_y^2 + \varphi_z^2)_z.$$

Here we compute divergence and gradient with respect to the flat metric $dx^2 + dy^2 + dz^2$ of the coordinate domain of the canonical principal Guichard net.

If we let $\star$ denote the Hodge-$\star$ operator with respect to this metric and set $\eta := -\omega_{12}$, another interesting formulation of (2.6) can be given in terms of the forms

$$\theta^\pm := \alpha \pm \star\beta = \begin{cases} \tan\varphi(\star\, d\eta + d\varphi_z), \\ \cot\varphi(\star\, d\eta - d\varphi_z). \end{cases}$$

Clearly, the compatibility conditions (2.6) now read

$$d\theta^+ = -d\theta^- \quad \text{and} \quad d(\star\,\theta^+) = d(\star\,\theta^-).$$

As

$$\begin{aligned} d\theta^+ &= \frac{2d\varphi}{\sin 2\varphi} \wedge \theta^+ + \tan\varphi \, d\star d\eta, \quad & d\star\theta^+ &= \frac{2d\varphi}{\sin 2\varphi} \wedge \star\,\theta^+ + \tan\varphi \, d\star d\varphi_z, \\ d\theta^- &= \frac{-2d\varphi}{\sin 2\varphi} \wedge \theta^- + \cot\varphi \, d\star d\eta, \quad & d\star\theta^- &= \frac{-2d\varphi}{\sin 2\varphi} \wedge \star\,\theta^- - \cot\varphi \, d\star d\varphi_z, \end{aligned}$$

these become

$$0 = d\star d\eta + d\varphi \wedge (\theta^+ - \theta^-) \quad \text{and} \quad 0 = d\star d\varphi_z + d\varphi \wedge \star(\theta^+ + \theta^-).$$

A key problem, however, in determining conformally flat hypersurfaces from either the forms α and β or from $\theta^\pm$ will be to extract φ or the connection form (2.3) from these forms.

2.2. Change of lift. In our analysis below, we will repeatedly need to change our light cone lift f of the conformally flat hypersurface and, consequently, its adapted frame F. Thus we change the lift of the hypersurface,

$$f \to f' := e^{\psi} f, \tag{2.7}$$

where ψ denotes some function, while keeping the curvature sphere congruence s as part of our frame:

$$F = (s_1, s_2, s_3, s, f, \hat{f}) \to F' = (s_1', s_2', s_3', s, f', \hat{f}'),$$

where

$$s_i \to s_i' := s_i + \partial_i \psi f \quad \text{and} \quad \hat{f} \to \hat{f}' := e^{-\psi}\left\{\hat{f} - \sum_j \partial_j \psi s_j - \frac{1}{2}\sum_j (\partial_j \psi)^2 f\right\}$$

since $\hat{f}' \perp s_i', s, \hat{f}'$ is lightlike and satisfies $\langle \hat{f}', f' \rangle = 1$; here $(\partial_1, \partial_2, \partial_3)$ denotes the dual frame field of the orthonormal co-frame $(\omega_1, \omega_2, \omega_3)$ of the induced metric $\mathrm{I} = \langle df, df \rangle$.

It is then straightforward to compute the effect on the structure equations, $\Phi \to \Phi'$. Clearly,

$$\omega_i \to \omega_i' = e^{\psi} \omega_i \quad \text{and} \quad \eta_i \to \eta_i' = -\langle ds, s_i' \rangle = \eta_i \tag{2.8}$$

since f envelops s so that $ds \perp f$. Then

$$\begin{aligned} \omega_{ij} &\to \omega_{ij}' = \langle s_i', ds_j' \rangle = \omega_{ij} + \partial_j \psi \omega_i - \partial_i \psi \omega_j, \\ \chi &\to \chi' = -\langle ds, \hat{f}' \rangle = e^{-\psi}\left\{\chi - \sum_j \partial_j \psi \eta_j\right\}, \\ \chi_i &\to \chi_i' = -\langle ds_i', \hat{f}' \rangle = e^{-\psi}\{\chi_i - \tau_i\} \end{aligned} \tag{2.9}$$

with

$$\tau_i := \sum_j \left(\partial_j \partial_i \psi \omega_j + \partial_j \psi \omega_{ij} - \partial_i \psi \partial_j \psi \omega_j + \frac{1}{2}(\partial_j \psi)^2 \omega_i\right). \tag{2.10}$$

From the Gauss equations

$$\begin{aligned} \omega_i' \wedge \sigma_j' + \sigma_i' \wedge \omega_j' &= \eta_i' \wedge \eta_j' + \omega_i' \wedge \chi_j' + \chi_i' \wedge \omega_j' \\ &= \eta_i \wedge \eta_j + \omega_i \wedge (\chi_j - \tau_j) + (\chi_i - \tau_i) \wedge \omega_j \\ &= \omega_i \wedge (\sigma_j - \tau_j) + (\sigma_i - \tau_i) \wedge \omega_j \end{aligned}$$

we infer that the τ_i carry, up to scaling, the change of the Schouten tensor under conformal change of metric:

$$\sigma_i \to \sigma_i' = e^{-\psi}(\sigma_i - \tau_i). \tag{2.11}$$

Observe that these computations also show that the conformal fundamental forms and Wang's Möbius curvature are indeed conformal invariants and, in particular, do not depend on a choice of ambient space form of a hypersurface. Namely, as long as we work with a principal frame so that $\eta_i = a_i\omega_i$ with the "principal curvatures" a_i of f with respect to s as a unit normal field—note that if $f = \mathfrak{f}$ takes values in a space form $\mathcal{Q}^4$ and $s = \mathfrak{t}$, $\mathfrak{t} \perp Q$, is its tangent plane congruence, then $a_i = \kappa_i$ are the principal curvatures of $\mathfrak{f}$ in $\mathcal{Q}^4$—we learn from (2.8) that a renormalization $f \to f' = e^{\psi} f$ does neither affect the conformal fundamental forms

$$\begin{aligned} \gamma_1 &= \sqrt{|a_3 - a_1|\,|a_1 - a_2|}\,\omega_1, \\ \gamma_2 &= \sqrt{|a_1 - a_2|\,|a_2 - a_3|}\,\omega_2, \\ \gamma_3 &= \sqrt{|a_2 - a_3|\,|a_3 - a_1|}\,\omega_3 \end{aligned} \tag{2.12}$$

nor Wang's Möbius curvature

$$W = \frac{a_2 - a_3}{a_2 - a_1} \tag{2.13}$$

as $\omega_i' = e^{\psi}\omega_i$ and $a_i' = e^{-\psi}a_i$. Moreover, as the γ_i and W only depend on the differences of the a_i, a change $s \to s + af$ of the enveloped sphere congruence does not change these invariants either as $a_i \to a_i - a$.

Starting from the structure equations (2.3), we hence obtain

(2.14)
$$\begin{aligned} &\omega_1' = e^{\psi}\cos\varphi\,dx, \quad \eta_1' = \sin\varphi\,dx, \quad \omega_{12}' = (\psi_y\cot\varphi - \varphi_y)\,dx - (\psi_x\tan\varphi + \varphi_x)\,dy, \\ &\omega_2' = e^{\psi}\sin\varphi\,dy, \quad \eta_2' = -\cos\varphi\,dy, \quad \omega_{23}' = (\psi_z\sin\varphi + \varphi_z\cos\varphi)\,dy - \frac{\psi_y}{\sin\varphi}\,dz, \\ &\omega_3' = e^{\psi}\,dz, \quad \eta_3' = 0, \quad \omega_{31}' = \frac{\psi_x}{\cos\varphi}\,dz - (\psi_z\cos\varphi - \varphi_z\sin\varphi)\,dx \end{aligned}$$

and

$$\chi' = e^{-\psi}\{-\psi_x\tan\varphi\,dx + \psi_y\cot\varphi\,dy + \varphi_z\,dz\}. \tag{2.15}$$

Now the principal curvatures k_{ij}' of the coordinate surfaces $\omega_i' = 0$ in direction ∂_j' are given by

$$k_{ij}' = -\mathrm{I}'\big(\nabla'_{\partial_j'}\partial_i', \partial_j'\big) = -\langle\partial_j' s_i', s_j'\rangle = \omega_{ij}'(\partial_j'), \tag{2.16}$$

that is, $\omega'_{ij} = k'_{ij}\omega'_j - k'_{ji}\omega'_i$ as expected from Dupin's theorem. Hence, in particular (cf. [6, p. 314]):

surface	x-direction	y-direction	z-direction
$x = \text{const}$		$k'_{12} = -e^{-\psi}\left(\frac{\psi_x}{\cos\varphi} + \frac{\varphi_x}{\sin\varphi}\right)$	$k'_{13} = -e^{-\psi}\,\frac{\psi_x}{\cos\varphi}$
$y = \text{const}$	$k'_{21} = -e^{-\psi}\left(\frac{\psi_y}{\sin\varphi} - \frac{\varphi_y}{\cos\varphi}\right)$		$k'_{23} = -e^{-\psi}\,\frac{\psi_y}{\sin\varphi}$
$z = \text{const}$	$k'_{31} = -e^{-\psi}(\psi_z - \varphi_z \tan\varphi)$	$k'_{32} = -e^{-\psi}(\psi_z + \varphi_z \cot\varphi)$	

Finally observe that, using the relation between χ_i and the Schouten forms σ_i from (2.3), the Ricci equations read

$$0 = \sum_j \omega'_j \wedge \chi'_j = \sum_j \omega'_j \wedge \sigma'_j \quad \text{and} \quad 0 = d\chi' + \sum_j \eta'_j \wedge \chi'_j = d\chi' + \sum_j \eta'_j \wedge \sigma'_j,$$

showing that the normal bundle of f' as an immersion into $\mathbb{R}^6_1$ becomes flat, $d\chi' = 0$, if and only if the Schouten tensor becomes diagonal: writing $\eta'_i = a'_i\omega'_i$ as before and $\sigma'_i = \sum_j s'_{ij}\omega'_j$, these two equations yield

$$s'_{ij} = s'_{ji} \quad \text{and} \quad d\chi' = 0 \Leftrightarrow 0 = (a'_i - a'_j)s'_{ij}$$

for $i, j \in \{1, 2, 3\}$. Hence the claim follows since the hypersurface was assumed to be generic.

3. A special solution

Let $a, b, c \in \mathbb{R}$ and g a real function. With the ansatz

$$\varphi(x, y, z) := g(ax + by + cz)$$

the integrability conditions (2.6) reduce to either $c = 0$ and $a^2 = b^2$ or to

$$0 = \left(\frac{g''}{\sin 2g}\right)'.$$

The first case leads to a particular class of conformally flat hypersurfaces with cyclic Guichard net: as $c = 0$ the coordinate surfaces $z \equiv \text{const}$ become totally umbilic in the intrinsic geometry of the hypersurface which is therefore a “conformal product hypersurface”, see [5, §2.4.12].

The second case is the one of interest to us: here

$$g'' = A \sin 2g \Leftrightarrow g'^2 = C - A \cos 2g$$

with suitable constants $A, C \in \mathbb{R}$, that is, g satisfies a pendulum equation and hence is an elliptic function:

$$g(t) = \phi_p(\sqrt{(C-A)}t), \quad \text{where} \quad p^2 = \frac{2A}{A-C}$$

and ϕ_p denotes the Jacobi Amplitude function; in particular,

$$\sin g(t) = \mathrm{sn}_p(\sqrt{C-A}t) \quad \text{and} \quad \cos g(t) = \mathrm{cn}_p(\sqrt{C-A}t).$$

Our aim is to investigate the intrinsic geometry of the lift

$$f' := \frac{1}{g'} f$$

of the generic conformally flat hypersurface obtained from φ via Lemma 1: it will turn out that the intrinsic structure induced by this lift leads to the Bianchi-type Guichard nets with constant ambient curvature that we seek.

First note that we can expect the intrinsic geometry to be independent of our initial distinction of the z-direction as the induced metric of this lift is independent of that choice in the following sense: let $\partial\varphi$ denote any directional derivative of φ and rewrite

$$\begin{aligned} \mathrm{I}' &= \frac{1}{(\partial\varphi)^2}\{\cos^2\varphi\, dx^2 + \sin^2\varphi\, dy^2 + dz^2\} \\ &= \frac{\cos^2\varphi}{(\partial\varphi)^2}\{dx^2 + \sinh^2\tilde{\varphi}\, dy^2 + \cosh^2\tilde{\varphi}\, dz^2\} \end{aligned}$$

by choosing $\tilde{\varphi}$ so that $\cosh\tilde{\varphi} = 1/\cos\varphi$ and $\sinh\tilde{\varphi} = \tan\varphi$; then $\partial\tilde{\varphi} = \partial\varphi/\cos\varphi$ so that

$$\mathrm{I}' = \frac{1}{(\partial\tilde{\varphi})^2}\{dx^2 + \sinh^2\tilde{\varphi}\, dy^2 + \cosh^2\tilde{\varphi}\, dz^2\},$$

leading to an alternative representation of the induced metric of f', but now distinguishing the x-direction, cf. [6, (2.3)]. Clearly, with $\cosh\tilde{\varphi} = 1/\sin\varphi$ and $\sinh\tilde{\varphi} = \cot\varphi$ a similar argument holds for a distinction of the y-direction.

Thus, in particular, we can expect the surfaces $\omega_i = 0$ of the Guichard net and their orthogonal lines to share similar geometric properties.

3.1. Sectional curvature and normal connection. Thus we set $\psi(x, y, z) = -\ln g'(ax+by+cz)$ and employ the transformation formulas of the last section: firstly, we obtain from (2.14)

$$\begin{aligned}
&\omega_1' = \frac{1}{g'}\cos g\,dx, \quad \eta_1' = \sin g\,dx, \quad \omega_{12}' = -\frac{1}{g'}\{b(A+C)\,dx - a(A-C)\,dy\},\\
&\omega_2' = \frac{1}{g'}\sin g\,dy, \quad \eta_2' = -\cos g\,dy, \quad \omega_{23}' = -\frac{\cos g}{g'}\{c(A-C)\,dy - 2bA\,dz\},\\
&\omega_3' = \frac{1}{g'}\,dz, \quad \eta_3' = 0, \quad \omega_{31}' = -\frac{\sin g}{g'}\{2aA\,dz - c(A+C)\,dx\}.
\end{aligned} \tag{3.1}$$

It is now much simpler to directly determine the Schouten forms σ_i' of the induced metric I' from the curvature forms,

$$\omega_i' \wedge \sigma_j' + \sigma_i' \wedge \omega_j' = \varrho_{ij}' := d\omega_{ij}' + \omega_{ik}' \wedge \omega_{kj}'$$

than to employ the transformation formulas (2.11).

The computation can be further facilitated by the following observation: from (2.15) we learn that the normal connection

$$\begin{aligned}
\chi' &= 2aA\sin^2 g\,dx - 2bA\cos^2 g\,dy + c(C - A\cos 2g)\,dz\\
&= a(A-C)\,dx - b(A+C)\,dy + g'^2(a\,dx + b\,dy + c\,dz).
\end{aligned}$$

Hence the normal bundle $\mathrm{span}\{s, f', \hat{f}'\}$ is flat as, clearly, $d\chi' = 0$. As a consequence, the Schouten tensor is diagonal,

$$\sigma_i' \wedge \omega_i' = 0$$

so that only the diagonal terms need to be computed: we find that the above curvature forms

$$\varrho_{ij}' = \kappa\omega_i' \wedge \omega_j' \quad \text{with} \quad \kappa = -\{2a^2A(A-C) + 2b^2A(A+C) + c^2(A^2 - C^2)\}$$

so that

$$\sigma_i' = \frac{\kappa}{2}\omega_i'$$

and the induced metric I' has constant sectional curvature κ.

3.2. Geometry of the Guichard net. From (3.1) we read off the principal curvatures of the coordinate surfaces (as codimension 1 submanifolds in the conformally flat hypersurface):

surface	x-direction	y-direction	z-direction
$x = \text{const}$		$k'_{12} = \dfrac{a(A-C)}{\sin g}$	$k'_{13} = 2aA \sin g$
$y = \text{const}$	$k'_{21} = \dfrac{b(A+C)}{\cos g}$		$k'_{23} = 2bA \cos g$
$z = \text{const}$	$k'_{31} = c(A+C) \tan g$	$k'_{32} = c(A-C) \cot g$	

Clearly, the coordinate surfaces $x = \text{const}$, $y = \text{const}$ and $z = \text{const}$ have constant (extrinsic) Gauss curvatures $2a^2 A(A-C)$, $2b^2 A(A+C)$ and $c^2(A^2 - C^2)$, respectively. Note that the Gauss curvature does not change within each Lamé family, $x = \text{const}$, $y = \text{const}$ or $z = \text{const}$, respectively.

Thus, the triply orthogonal system is a special type of a triply Bianchi system, see [8, §22].

Since, by Dupin's theorem, the surfaces of a triply orthogonal system intersect in curvature lines the dual frame field $(\partial'_1, \partial'_2, \partial'_3)$ of the orthonormal co-frame $(\omega'_1, \omega'_2, \omega'_3)$ yields a parallel frame along each coordinate curve, that is, the frames are torsion free and their respective curvatures are

line	normal ∂'_1	normal ∂'_2	normal ∂'_3
x-lines		$k'_{21} = \dfrac{b(A+C)}{\cos g}$	$k'_{31} = c(A+C) \tan g$
y-lines	$k'_{12} = \dfrac{a(A-C)}{\sin g}$		$k'_{32} = c(A-C) \cot g$
z-lines	$k'_{13} = 2aA \sin g$	$k'_{23} = 2bA \cos g$	

Thus the (Frenet) curvatures κ_i and torsions τ_i of the coordinate curves are given by

$$\kappa_1^2 = k'^2_{31} + k'^2_{21} \quad \text{and} \quad \tau_1 = \frac{k'_{21}\partial'_1 k'_{31} - k'_{31}\partial'_1 k'_{21}}{k'^2_{21} + k'^2_{31}},$$

and cyclic permutations of the indices, so that

$$\begin{aligned}
\kappa_1^2 &= \frac{(A+C)^2(b^2 + c^2 \sin^2 g)}{\cos^2 g}, \quad \tau_1 = \frac{abcg'^2}{b^2 + c^2 \sin^2 g}; \\
\kappa_2^2 &= \frac{(A-C)^2(a^2 + c^2 \cos^2 g)}{\sin^2 g}, \quad \tau_2 = \frac{abcg'^2}{a^2 + c^2 \cos^2 g}; \\
\kappa_3^2 &= 4A^2(a^2 \sin^2 g + b^2 \cos^2 g), \quad \tau_3 = \frac{-abcg'^2}{a^2 \sin^2 g + b^2 \cos^2 g}.
\end{aligned}$$

Consequently our coordinate lines satisfy the condition relating curvature and torsion of a Kirchhoff rod, see [7, (3.10)]:

$$\begin{aligned}
\kappa_1^2\Big(\tau_1 - \frac{abc(A+C)}{b^2+c^2}\Big) &= -abc(A+C)^2\Big\{A + \frac{b^2A - c^2C}{b^2+c^2}\Big\},\\
\kappa_2^2\Big(\tau_2 + \frac{abc(A-C)}{a^2+c^2}\Big) &= abc(A-C)^2\Big\{A + \frac{a^2A + c^2C}{a^2+c^2}\Big\},\\
\kappa_3^2\Big(\tau_3 + \frac{2abcA}{a^2-b^2}\Big) &= -4abcA^2\Big\{C - A\frac{a^2+b^2}{a^2-b^2}\Big\};
\end{aligned}$$

note that for the third equation to make sense we need to exclude the case $a^2 = b^2$, where the curves have constant curvature κ_3 but non-constant torsion τ_3.

The elliptic differential equation [7, (3.11)] for the squared curvature of a Kirchhoff rod is, however, not satisfied in our case—though the squared curvatures κ_i^2 clearly are elliptic functions, as algebraic expressions in sn_p and cn_p.

We summarize the properties relevant to our main classification result:

Proposition 2. *Let $\varphi(x, y, z) = g(ax + by + cz)$, where $a, b, c \in \mathbb{R}$ and the real function g satisfies*

$$g'^2 = C - A\cos 2g$$

for some $A, C \in \mathbb{R}$. Then φ defines a conformally flat hypersurface f so that its canonical principal Guichard net consists of constant Gauss curvature surfaces for a suitable choice of constant sectional curvature metric in the conformal structure induced by f as their ambient geometry.

This proposition provides the first part of our Main Theorem.

4. Conformally flat hypersurfaces with Bianchi-type Guichard net

In this section we aim to convince ourselves of the converse: if f is a conformally flat hypersurface for which we can choose a constant sectional curvature representative of the induced conformal structure so that all (coordinate) surfaces of its canonical Guichard net have constant Gauss curvature, then the hypersurface comes from the construction discussed in the previous section. In fact, we shall see that it is sufficient to assume that we can choose a light cone lift f' with flat normal bundle for the hypersurface, that is, so that the Schouten tensor of the induced metric becomes diagonal.

Thus we start with an undetermined lift (2.7) of a conformally flat hypersurface and assume flatness of its normal bundle,

$$0 = d\chi' \Leftrightarrow \begin{cases} 0 = \psi_{xy} - \psi_x\psi_y + \psi_x\varphi_y \tan\varphi - \psi_y\varphi_x \cot\varphi,\\ 0 = \psi_{yz} - \psi_y\psi_z - \psi_y\varphi_z \cot\varphi - \varphi_{yz}\tan\varphi,\\ 0 = \psi_{xz} - \psi_x\psi_z + \psi_x\varphi_z \tan\varphi + \varphi_{xz}\cot\varphi \end{cases} \tag{4.1}$$

from (2.15); then the derivatives of the principal curvatures (2.16) simplify to

$$\begin{aligned}
(k'_{12})_y &= -\frac{e^{-\psi}}{\sin\varphi}(\varphi_{xy} - \varphi_x\varphi_y \cot\varphi), \quad (k'_{13})_y = -\frac{e^{-\psi}}{\sin\varphi}\varphi_x\psi_y;\\
(k'_{12})_z &= \frac{e^{-\psi}}{\sin\varphi}\varphi_x(\psi_z + \varphi_z \cot\varphi), \quad (k'_{13})_z = \frac{e^{-\psi}}{\sin\varphi}\varphi_{xz};\\
(k'_{21})_z &= -\frac{e^{-\psi}}{\cos\varphi}\varphi_y(\psi_z - \varphi_z \tan\varphi), \quad (k'_{23})_z = -\frac{e^{-\psi}}{\cos\varphi}\varphi_{yz};\\
(k'_{21})_x &= \frac{e^{-\psi}}{\cos\varphi}(\varphi_{xy} + \varphi_x\varphi_y \tan\varphi), \quad (k'_{23})_x = \frac{e^{-\psi}}{\cos\varphi}\psi_x\varphi_y;\\
(k'_{31})_x &= \frac{2e^{-\psi}}{\sin 2\varphi}(\varphi_{xz} + \varphi_x\varphi_z \tan\varphi), \quad (k'_{32})_x = \frac{2e^{-\psi}}{\sin 2\varphi}\varphi_z(\psi_x + \varphi_x \cot\varphi);\\
(k'_{31})_y &= -\frac{2e^{-\psi}}{\sin 2\varphi}\varphi_z(\psi_y - \varphi_y \tan\varphi), \quad (k'_{32})_y = -\frac{2e^{-\psi}}{\sin 2\varphi}(\varphi_{yz} - \varphi_y\varphi_z \cot\varphi).
\end{aligned}$$

Hence the coordinate surfaces have constant Gauss curvatures (not necessarily the same in each Lamé family) if and only if

$$\begin{aligned}
0 &= \psi_x\left\{\varphi_{xy} - \frac{2\varphi_x\varphi_y\cos 2\varphi}{\sin 2\varphi}\right\} + \varphi_x\{\psi_x\psi_y - \psi_x\varphi_y\tan\varphi + \varphi_x\psi_y\cot\varphi\}\\
&= (\psi_x + \varphi_x\cot\varphi)\left\{\varphi_{xz} + \frac{2\varphi_x\varphi_z}{\sin 2\varphi}\right\} + \varphi_x\left\{\psi_x\psi_z - \psi_x\varphi_z\tan\varphi - \frac{\varphi_x\varphi_z}{\sin^2\varphi}\right\},\\
0 &= \psi_y\left\{\varphi_{xy} - \frac{2\varphi_x\varphi_y\cos 2\varphi}{\sin 2\varphi}\right\} + \varphi_y\{\psi_x\psi_y - \psi_x\varphi_y\tan\varphi + \varphi_x\psi_y\cot\varphi\}\\
&= (\psi_y - \varphi_y\tan\varphi)\left\{\varphi_{yz} - \frac{2\varphi_y\varphi_z}{\sin 2\varphi}\right\} + \varphi_y\left\{\psi_y\psi_z + \psi_y\varphi_z\cot\varphi - \frac{\varphi_y\varphi_z}{\cos^2\varphi}\right\},\\
0 &= (\psi_z + \varphi_z\cot\varphi)\left\{\varphi_{xz} + \frac{2\varphi_x\varphi_z}{\sin 2\varphi}\right\} + \varphi_z\left\{\psi_x\psi_z - \psi_x\varphi_z\tan\varphi - \frac{\varphi_x\varphi_z}{\sin^2\varphi}\right\}\\
&= (\psi_z - \varphi_z\tan\varphi)\left\{\varphi_{yz} - \frac{2\varphi_y\varphi_z}{\sin 2\varphi}\right\} + \varphi_z\left\{\psi_y\psi_z + \psi_y\varphi_z\cot\varphi - \frac{\varphi_y\varphi_z}{\cos^2\varphi}\right\}.
\end{aligned} \tag{4.2}$$

Eliminating the quadratic expressions of the derivatives of ψ we then arrive at three equations

$$\begin{aligned}
0 &= (\psi_x\varphi_y - \varphi_x\psi_y)(\varphi_{xy}\cos\varphi\sin\varphi - \varphi_x\varphi_y(\cos^2\varphi - \sin^2\varphi)),\\
0 &= (\psi_y\varphi_z - \varphi_y\psi_z)(\varphi_{yz}\cos\varphi\sin\varphi - \varphi_y\varphi_z),\\
0 &= (\psi_z\varphi_x - \varphi_z\psi_x)(\varphi_{xz}\cos\varphi\sin\varphi + \varphi_z\varphi_x)
\end{aligned} \tag{4.3}$$

that will govern our analysis.

4.1. The case study. Clearly, (4.3) yields eight cases to consider.

First note that, if φ only depends on two variables, then the corresponding coordinate surfaces become totally umbilic and our hypersurface is a conformally flat "conformal product hypersurface", see [5, §2.4.12]. Hence, disregarding this case in our analysis, we shall assume that everywhere

$$\varphi_x\varphi_y\varphi_z \neq 0.$$

Secondly, we disregard the case of cyclic Guichard nets, which was fully analysed in [6]: as already mentioned above (see Section 2.1),

$$0 = \varphi_{xz} = \varphi_{yz} \tag{4.4}$$

leads to a cyclic system formed by the (then circular) z-lines; the x- or y-lines form a cyclic system if $\tilde{\varphi}_{xy} = \tilde{\varphi}_{xz} = 0$, where $\tanh\tilde{\varphi} = \sin\varphi$ so that $dx^2 + \sinh^2\tilde{\varphi}\,dy^2 + \cosh^2\tilde{\varphi}\,dz^2$ gives the induced conformal structure, that is,

$$0 = \varphi_{xy} + \varphi_x\varphi_y\tan\varphi = \varphi_{xz} + \varphi_x\varphi_z\tan\varphi; \tag{4.5}$$

or $\tilde{\varphi}_{xy} = \tilde{\varphi}_{yz} = 0$, where $\tanh\tilde{\varphi} = \cos\varphi$ so that $\sinh^2\tilde{\varphi}\,dx^2 + dy^2 + \cosh^2\tilde{\varphi}\,dz^2$ yields the induced conformal structure, that is,

$$0 = \varphi_{xy} - \varphi_x\varphi_y\cot\varphi = \varphi_{yz} - \varphi_y\varphi_z\cot\varphi, \tag{4.6}$$

respectively. Thus the equations (4.4)–(4.6) characterize conformally flat hypersurfaces with cyclic Guichard nets.

Now, from (4.3), we have the following cases to consider:

(i) At least two of the quantities $\varphi_{xy} - 2\varphi_x\varphi_y\cos 2\varphi/\sin 2\varphi$, $\varphi_{yz} - 2\varphi_y\varphi_z/\sin 2\varphi$ and $\varphi_{xz} + 2\varphi_z\varphi_x/\sin 2\varphi$ do not vanish. In this case, since $\varphi_x\varphi_y\varphi_z \neq 0$,

$$\frac{\psi_x}{\varphi_x} = \frac{\psi_y}{\varphi_y} = \frac{\psi_z}{\varphi_z}. \tag{4.7}$$

We will see in Section 4.3 that this case leads to the class of conformally flat hypersurfaces discussed in the previous section.

(ii) At least two of the quantities $\varphi_{xy} - 2\varphi_x\varphi_y\cos 2\varphi/\sin 2\varphi$, $\varphi_{yz} - 2\varphi_y\varphi_z/\sin 2\varphi$ and $\varphi_{xz} + 2\varphi_z\varphi_x/\sin 2\varphi$ vanish identically. We shall see below, by algebra, that this implies that the third vanishes as well,

$$\varphi_{xy} - \frac{2\varphi_x\varphi_y\cos 2\varphi}{\sin 2\varphi} = \varphi_{yz} - \frac{2\varphi_y\varphi_z}{\sin 2\varphi} = \varphi_{xz} + \frac{2\varphi_z\varphi_x}{\sin 2\varphi} \equiv 0. \tag{4.8}$$

This case will not lead to conformally flat hypersurfaces with the sought intrinsic structure but it will provide a new class of conformally flat hypersurfaces, see Section 4.2.

To convince ourselves of the first claim in (ii) note that, with the ansatz

$$\psi_x = -\xi\varphi_x \cot\varphi, \quad \psi_y = \zeta\varphi_y \tan\varphi, \quad \psi_z = \frac{2\mu\varphi_z}{\sin 2\varphi},$$

our original six equations (4.2) yield

$$\begin{aligned}
\varphi_{xy} - \frac{2\varphi_x\varphi_y \cos 2\varphi}{\sin 2\varphi} &= 0 \Rightarrow 0 = -\xi\zeta + \xi + \zeta;\\
\varphi_{yz} - \frac{2\varphi_y\varphi_z}{\sin 2\varphi} &= 0 \Rightarrow 0 = \zeta(\mu + \cos^2\varphi) - 1;\\
\varphi_{xz} + \frac{2\varphi_z\varphi_x}{\sin 2\varphi} &= 0 \Rightarrow 0 = \xi(\mu - \sin^2\varphi) + 1.
\end{aligned}$$

The three right hand side equations are not independent: if any two of them are satisfied then so is the third; hence one of the functions, say μ, remains free while the other two are determined:

$$\xi = -\frac{1}{\mu - \sin^2\varphi} \quad \text{and} \quad \zeta = \frac{1}{\mu + \cos^2\varphi}.$$

Now consider case (ii): two of $\varphi_{xy} - 2\varphi_x\varphi_y \cos 2\varphi/\sin 2\varphi$, $\varphi_{yz} - 2\varphi_y\varphi_z/\sin 2\varphi$ and $\varphi_{xz} + 2\varphi_z\varphi_x/\sin 2\varphi$ vanish identically, say

$$\varphi_{yz} - \frac{2\varphi_y\varphi_z}{\sin 2\varphi} = \varphi_{xz} + \frac{2\varphi_z\varphi_x}{\sin 2\varphi} = 0.$$

Then, from (4.2), either

$$\varphi_{xy} - \frac{2\varphi_x\varphi_y \cos 2\varphi}{\sin 2\varphi} = 0 \quad \text{or} \quad \xi = \zeta = 0;$$

however, we have $1 = -\xi(\mu - \sin^2\varphi) = \zeta(\mu + \cos^2\varphi)$ so that the case $\xi = \zeta = 0$ cannot occur. Similar arguments show that the vanishing of any other two of the considered quantities implies the vanishing of all three.

Thus we are left to consider the consequences of (4.7) and (4.8) above.

4.2. A new class of conformally flat hypersurfaces. We start by considering case (ii): the equations (4.8) can be reformulated as

$$(\ln\tan\varphi)_{xy} = (\ln\sin\varphi)_{yz} = (\ln\cos\varphi)_{zx} \equiv 0, \tag{4.9}$$

showing that

$$\sin\varphi(x, y, z) = \frac{\beta(x, z)}{\gamma(x, y)} \quad \text{and} \quad \cos\varphi(x, y, z) = \frac{\alpha(y, z)}{\gamma(x, y)}$$

with suitable functions α, β and γ. The Pythagorean law then yields

$$\gamma^2(x, y) = \beta^2(x, z) + \alpha^2(y, z)$$

so that there are functions X, Y and Z of one variable with

$$\gamma^2(x, y) = X(x) - Y(y), \quad \beta^2(x, z) = X(x) - Z(z), \quad \alpha^2(y, z) = Z(z) - Y(y)$$

and, consequently, $X > Z > Y$ and

$$\sin\varphi = \sqrt{\frac{X-Z}{X-Y}}, \quad \cos\varphi = \sqrt{\frac{Y-Z}{Y-X}}, \quad \tan\varphi = \sqrt{\frac{X-Z}{Z-Y}}.$$

It is now a laborious but straightforward computation to see that the first of the equations (2.6) for φ to define a conformally flat hypersurface is identically satisfied whereas the remaining three become

$$\begin{aligned}
\frac{X'''}{X-Y} - \frac{2X'X''}{(X-Y)^2} + \frac{X'(X'^2-Y'^2)}{(X-Y)^3} &= \frac{X'''}{X-Z} - \frac{2X'X''}{(X-Z)^2} + \frac{X'(X'^2+Z'^2)}{(X-Z)^3},\\
\frac{Y'''}{Y-X} - \frac{2Y'Y''}{(Y-X)^2} + \frac{Y'(Y'^2-X'^2)}{(Y-X)^3} &= \frac{Y'''}{Y-Z} - \frac{2Y'Y''}{(Y-Z)^2} + \frac{Y'(Y'^2+Z'^2)}{(Y-Z)^3},\\
\frac{Z'''}{Y-Z} + \frac{2Z'Z''}{(Y-Z)^2} + \frac{Z'(Z'^2+Y'^2)}{(Y-Z)^3} &= \frac{Z'''}{X-Z} + \frac{2Z'Z''}{(X-Z)^2} + \frac{Z'(Z'^2+X'^2)}{(X-Z)^3}.
\end{aligned}$$

Again, we obtain separation of variables: writing both sides in each equation as $-2AX'$, $-2BY'$ and $-2CZ'$ with functions $A = A(x)$, $B = B(y)$ and $C = C(z)$, respectively, these equations become

$$\begin{aligned}
0 &= Y'^2 + 2A(Y-X)^3 - \frac{X'''}{X'}(Y-X)^2 - 2X''(Y-X) - X'^2,\\
0 &= Z'^2 - 2A(Z-X)^3 + \frac{X'''}{X'}(Z-X)^2 + 2X''(Z-X) + X'^2;\\
0 &= X'^2 + 2B(X-Y)^3 - \frac{Y'''}{Y'}(X-Y)^2 - 2Y''(X-Y) - Y'^2,\\
0 &= Z'^2 - 2B(Z-Y)^3 + \frac{Y'''}{Y'}(Z-Y)^2 + 2Y''(Z-Y) + Y'^2;\\
0 &= Y'^2 + 2C(Y-Z)^3 + \frac{Z'''}{Z'}(Y-Z)^2 + 2Z''(Y-Z) + Z'^2,\\
0 &= X'^2 + 2C(X-Z)^3 + \frac{Z'''}{Z'}(X-Z)^2 + 2Z''(X-Z) + Z'^2.
\end{aligned} \tag{4.10}$$

Taking second derivatives with respect to y and x, respectively, of the last two equations we find

$$0 = \left\{\frac{X'''}{X'} + 6CX\right\} + \left\{\frac{Z'''}{Z'} - 6CZ\right\} = \left\{\frac{Y'''}{Y'} + 6CY\right\} + \left\{\frac{Z'''}{Z'} - 6CZ\right\},$$

which shows that $0 = C'(X - Y)$ so that C is constant and hence

$$0 = X''' + (6CX + L)X' = Y''' + (6CY + L)Y' = Z''' - (6CZ + L)Z'$$

with some $L \in \mathbb{R}$. Thus integrating the equation for Z we learn that Z satisfies an elliptic equation

$$Z'^2 = p(Z) := 2CZ^3 + LZ^2 + 2MZ + N,$$

where $M, N \in \mathbb{R}$ are suitable coefficients; re-inserting this result into the last two equations yields similar elliptic equations for X and Y:

$$0 = X'^2 + p(X) = Y'^2 + p(Y).$$

The first and third of the equations (4.10) then read

$$0 = Y'^2 + p(Y) - 2(A - C)(X - Y)^3 = X'^2 + p(X) + 2(B - C)(X - Y)^3,$$

implying that $A = B = C$; the remaining two equations then just recover the elliptic equation for Z. Thus the equations (4.10) are equivalent to a set of three similar elliptic equations for X, Y and Z, respectively:

Lemma 3. *Given three functions X, Y, Z of one variable with $X > Z > Y$ and non-vanishing derivatives,*

$$\mathrm{I} = \frac{Y - Z}{Y - X}\,dx^2 + \frac{X - Z}{X - Y}\,dy^2 + dz^2$$

represents the induced conformal structure of a conformally flat hypersurface in terms of its canonical principal Guichard net (x, y, z) if and only if

$$0 = X'^2 + p(X) = Y'^2 + p(Y) = Z'^2 - p(Z),$$

where $p(t)$ is a cubic polynomial that changes sign at least twice.

Here, the last assertion follows from the observation that $Y < Z < X$ while

$$p(X) = -X'^2, p(Y) = -Y'^2 < 0 < Z'^2 = p(Z).$$

Note that, as a consequence, $p(t)$ has degree 2 or 3.

Next we wish to convince ourselves that these conformally flat hypersurfaces do not have the intrinsic structure sought, that is, that their induced conformal structure does not have a constant sectional curvature representative so that all coordinate surfaces of the Guichard net have constant Gauss curvature.

To this end, observe that equations (4.2) now read

$$
\begin{aligned}
0 &= \psi_x\psi_y - \psi_x\varphi_y \tan\varphi + \varphi_x\psi_y \cot\varphi, \\
0 &= \psi_y\psi_z + \psi_y\varphi_z \cot\varphi - \frac{\varphi_y\varphi_z}{\cos^2\varphi}, \\
0 &= \psi_x\psi_z - \psi_x\varphi_z \tan\varphi - \frac{\varphi_x\varphi_z}{\sin^2\varphi};
\end{aligned} \tag{4.11}
$$

using these, flatness $d\chi' = 0$ of the normal bundle (4.1) simplifies to

$$0 = \psi_{xy} = \psi_{yz} + (\ln\cos\varphi)_{yz} = \psi_{zx} + (\ln\sin\varphi)_{zx}.$$

Together with the equations (4.9) these yield

$$
\begin{aligned}
0 &= (\psi + \ln\tan\varphi)_{xy} = (\psi + \ln\tan\varphi)_{xz}, \\
0 &= (\psi - \ln\tan\varphi)_{xy} = (\psi - \ln\tan\varphi)_{yz},
\end{aligned}
$$

implying that

$$\psi = \alpha + \beta + \gamma - \ln\sqrt{(X-Z)(Z-Y)}$$

with three functions $\alpha = \alpha(x)$, $\beta = \beta(y)$ and $\gamma = \gamma(z)$ of one variable. Hence

$$\psi_x = \alpha' - \frac{1}{2}\frac{X'}{X-Z}, \quad \psi_y = \beta' - \frac{1}{2}\frac{Y'}{Y-Z}, \quad \psi_z = \gamma' + \frac{1}{2}\left\{\frac{Z'}{X-Z} + \frac{Z'}{Y-Z}\right\}$$

and equations (4.11) read

$$
\begin{aligned}
\alpha'(Y' - 2\beta' Y) &= \beta'(X' - 2\alpha' X), \\
\beta'(Z' - 2\gamma' Z) &= \gamma'(Y' - 2\beta' Y), \\
\gamma'(X' - 2\alpha' X) &= \alpha'(Z' - 2\gamma' Z).
\end{aligned}
$$

Consequently, $\alpha' = \beta' = \gamma' = 0$ or, with some $A \in \mathbb{R}$,

$$\alpha' = \frac{X'}{2(X-A)}, \quad \beta' = \frac{Y'}{2(Y-A)} \quad \text{and} \quad \gamma' = \frac{Z'}{2(Z-A)}$$

so that (up to an irrelevant constant of integration)

$$\psi = \ln\sqrt{\frac{1}{(X-Z)(Z-Y)}} \quad \text{or} \quad \psi = \ln\sqrt{\frac{(X-A)(Y-A)(Z-A)}{(X-Z)(Z-Y)}}.$$

We consider the two corresponding representatives of the induced conformal structure in turn.

Firstly, suppose that $\alpha' = \beta' = \gamma' = 0$ so that we arrive at

$$\mathrm{I} = \frac{dx^2}{(X-Y)(X-Z)} + \frac{dy^2}{(Y-X)(Y-Z)} + \frac{dz^2}{(Z-Y)(X-Z)}$$

as a representative of the induced conformal structure.

In this case the Gauss curvatures of the coordinate surfaces are

$$k'_{12}k'_{13} = \frac{1}{4}X'^2, \quad k'_{21}k'_{23} = \frac{1}{4}Y'^2, \quad k'_{31}k'_{32} = -\frac{1}{4}Z'^2.$$

Hence, all coordinate surfaces have constant Gauss curvature. However observe that, in contrast to the conformally flat hypersurfaces studied in the previous section, the Gauss curvatures are now not constant in each Lamé family. On the other hand, the sectional curvatures of I are

$$\kappa_{12} = \frac{1}{4}\left\{X'^2+Y'^2+Z'^2+2\left(\frac{X'}{X-Y}\right)_x(X-Y)(X-Z)+2\left(\frac{Y'}{X-Y}\right)_y(X-Y)(Y-Z)\right\},$$
$$\kappa_{23} = \frac{1}{4}\left\{-X'^2+Y'^2-Z'^2-2\left(\frac{Y'}{Y-Z}\right)_y(X-Y)(Y-Z)+2\left(\frac{Z'}{Y-Z}\right)_z(X-Z)(Y-Z)\right\},$$
$$\kappa_{31} = \frac{1}{4}\left\{X'^2-Y'^2-Z'^2+2\left(\frac{X'}{X-Z}\right)_x(X-Y)(X-Z)+2\left(\frac{Z'}{X-Z}\right)_z(X-Z)(Y-Z)\right\}.$$

Assuming that these are constant we find

$$0 = (\kappa_{12})_z \Rightarrow p'(Z) = 2(X-Y)\left\{\left(\frac{X'}{X-Y}\right)_x + \left(\frac{Y'}{X-Y}\right)_y\right\},$$
$$0 = (\kappa_{23})_x \Rightarrow p'(X) = 2(Y-Z)\left\{\left(\frac{Y'}{Y-Z}\right)_y - \left(\frac{Z'}{Y-Z}\right)_z\right\},$$
$$0 = (\kappa_{31})_y \Rightarrow p'(Y) = 2(X-Z)\left\{\left(\frac{X'}{X-Z}\right)_x - \left(\frac{Z'}{X-Z}\right)_z\right\},$$

where we used the elliptic differential equations $X'^2 = -p(X)$, $Y'^2 = -p(Y)$ and $Z'^2 = p(Z)$ that X, Y and Z satisfy. Hence $p'(t)$ must be constant, contradicting the fact that p has to be a polynomial of degree at least 2: for example, taking a further z-derivative of $p'(Z)$ we obtain $p''(Z) = 0$ since $Z' \neq 0$.

Hence we cannot have constant sectional curvatures of the induced metric in this case.

Secondly, suppose that

$$\mathrm{I} = (X-A)(Y-A)(Z-A)\left\{\frac{dx^2}{(X-Y)(X-Z)} + \frac{dy^2}{(Y-X)(Y-Z)} + \frac{dz^2}{(Z-Y)(X-Z)}\right\}$$

is the sought representative of the induced conformal structure.

In this case, the Gauss curvatures of the coordinate surfaces become

$$k'_{12}k'_{13} = \frac{1}{4}\frac{X'^2}{(X-A)^3}, \quad k'_{21}k'_{23} = \frac{1}{4}\frac{Y'^2}{(Y-A)^3}, \quad k'_{31}k'_{32} = -\frac{1}{4}\frac{Z'^2}{(Z-A)^3};$$

while the sectional curvatures of our representative of the induced conformal structure are

$$\begin{aligned}
\kappa_{12} = \frac{1}{4}\frac{1}{Z-A}\Bigg\{ & \frac{X'^2}{(X-A)^2} + \frac{Y'^2}{(Y-A)^2} + \frac{Z'^2}{(Z-A)^2} \\
& +2(X-Y)\left\{\left(\frac{X'}{(X-A)(X-Y)}\right)_x \frac{(X-Z)}{(X-A)} + \left(\frac{Y'}{(Y-A)(X-Y)}\right)_y \frac{(Y-Z)}{(Y-A)}\right\}\Bigg\}, \\
\kappa_{23} = \frac{1}{4}\frac{1}{X-A}\Bigg\{ & -\frac{X'^2}{(X-A)^2} + \frac{Y'^2}{(Y-A)^2} - \frac{Z'^2}{(Z-A)^2} \\
& +2(Y-Z)\left\{-\left(\frac{Y'}{(Y-A)(Y-Z)}\right)_y \frac{(X-Y)}{(Y-A)} + \left(\frac{Z'}{(Z-A)(Y-Z)}\right)_z \frac{(X-Z)}{(Z-A)}\right\}\Bigg\}, \\
\kappa_{31} = \frac{1}{4}\frac{1}{Y-A}\Bigg\{ & \frac{X'^2}{(X-A)^2} - \frac{Y'^2}{(Y-A)^2} - \frac{Z'^2}{(Z-A)^2} \\
& +2(X-Z)\left\{\left(\frac{X'}{(X-A)(X-Z)}\right)_x \frac{(X-Y)}{(X-A)} + \left(\frac{Z'}{(Z-A)(X-Z)}\right)_z \frac{(Y-Z)}{(Z-A)}\right\}\Bigg\}.
\end{aligned}$$

Following the same procedure as in the simpler first case, the assumption of the sectional curvatures to be constant leads to a differential equation for the polynomial $p(t)$ of the elliptic equations for X, Y and Z:

$$\begin{aligned}
4\kappa_{12}Z' = & \left(\frac{p(Z)}{(Z-A)^2}\right)_z \\
& -2(X-Y)\left\{\left(\frac{X'}{(X-A)(X-Y)}\right)_x \frac{1}{X-A} + \left(\frac{Y'}{(Y-A)(X-Y)}\right)_y \frac{1}{Y-A}\right\} Z', \\
4\kappa_{23}X' = & \left(\frac{p(X)}{(X-A)^2}\right)_x \\
& -2(Y-Z)\left\{\left(\frac{Y'}{(Y-A)(Y-Z)}\right)_y \frac{1}{Y-A} - \left(\frac{Z'}{(Z-A)(Y-Z)}\right)_z \frac{1}{Z-A}\right\} X', \\
4\kappa_{31}Y' = & \left(\frac{p(Y)}{(Y-A)^2}\right)_y \\
& -2(X-Z)\left\{\left(\frac{X'}{(X-A)(X-Z)}\right)_x \frac{1}{X-A} - \left(\frac{Z'}{(Z-A)(X-Z)}\right)_z \frac{1}{Z-A}\right\} Y'.
\end{aligned}$$

Hence

$$\left(\frac{p(t)}{(t-A)^2}\right)' \equiv \text{const}$$

so that $p(t)$ has a perfect square factor; on the other hand, $p(t)$ has degree at most 3 so that, in particular, it cannot change sign twice as required for the functions X, Y and Z to define a conformally flat hypersurface.

Thus, again, we cannot have constant sectional curvatures of this representative of the induced conformal structure either.

Consequently, the conformally flat hypersurfaces characterized in the previous lemma do not have the desired intrinsic structure and we can safely ignore them in our classification. Although the geometry of their Guichard nets may be interesting—for example, it is easy to see that all surfaces of the canonical Guichard net are isothermic—a more detailed analysis will be left for another time.

4.3. Conformally flat hypersurfaces with Bianchi-type Guichard net. We now turn to our main case (i): thus we assume now that the equations (4.7) hold, that is, that

$$d\psi = \lambda\, d\varphi$$

with a suitable multiplier λ. Using the linear dependence of the gradients of ψ and φ, the equations (4.2) governing our analysis become

$$\begin{aligned}
0 &= \psi_x(\varphi_{xy} + \varphi_x\psi_y) = \psi_x(\varphi_{xy} + \psi_x\varphi_y),\\
0 &= \psi_y(\varphi_{xy} + \varphi_x\psi_y) = \psi_y(\varphi_{xy} + \psi_x\varphi_y);\\
0 &= (\psi_y - \varphi_y \tan\varphi)(\varphi_{yz} + \varphi_y\psi_z) = (\psi_y - \varphi_y \tan\varphi)(\varphi_{yz} + \psi_y\varphi_z),\\
0 &= (\psi_z - \varphi_z \tan\varphi)(\varphi_{yz} + \varphi_y\psi_z) = (\psi_z - \varphi_z \tan\varphi)(\varphi_{yz} + \psi_y\varphi_z);\\
0 &= (\psi_z + \varphi_z \cot\varphi)(\varphi_{xz} + \varphi_x\psi_z) = (\psi_z + \varphi_z \cot\varphi)(\varphi_{xz} + \psi_x\varphi_z),\\
0 &= (\psi_x + \varphi_x \cot\varphi)(\varphi_{xz} + \varphi_x\psi_z) = (\psi_x + \varphi_x \cot\varphi)(\varphi_{xz} + \psi_x\varphi_z).
\end{aligned} \tag{4.12}$$

Now, if $\lambda = 0$, $\lambda = \tan\varphi$ or $\lambda = -\cot\varphi$, the remaining four equations yield (4.4), (4.5) or (4.6), respectively—that is, the Guichard net is cyclic. Hence, ignoring this case, (4.12) further simplify to

$$\begin{aligned}
0 &= (\varphi_x e^{\psi})_y = (\varphi_x e^{\psi})_z,\\
0 &= (\varphi_y e^{\psi})_z = (\varphi_y e^{\psi})_x,\\
0 &= (\varphi_z e^{\psi})_x = (\varphi_z e^{\psi})_y,
\end{aligned}$$

which imply that

$$d\varphi = e^{-\psi}\, dt, \quad \text{where} \quad t(x, y, z) = \alpha(x) + \beta(y) + \gamma(z)$$

with suitable functions α, β and γ of one variable. In particular, φ and t are functionally dependent so that there is a function g of one variable with $g' \circ t = e^{-\psi}$ and

$$\varphi(x, y, z) = (g \circ t)(x, y, z) = g(\alpha(x) + \beta(y) + \gamma(z)).$$

Now the first of the differential equations (2.6) for φ to define a conformally flat hypersurface reads

$$\begin{aligned} 0 &= \varphi_{xyz} + \varphi_x \varphi_{yz} \tan\varphi - \varphi_y \varphi_{xz} \cot\varphi \\ &= \alpha' \beta' \gamma' \left(\frac{g''}{\sin 2g} \right)'(t) \sin 2\varphi \end{aligned}$$

so that g is an elliptic function,

$$g'' = A \sin 2g \quad \text{and} \quad g'^2 = C - A \cos 2g$$

with suitable constants A and C. Note that

$$\lambda = -\frac{g''}{g'^2} \circ t = \begin{cases} 0 & \text{if} \quad A = 0, \\ -\cot\varphi & \text{if} \quad A = C, \\ \tan\varphi & \text{if} \quad A = -C \end{cases}$$

so that we can assume that $A \neq 0, C, -C$.

The remaining three equations of (2.6) then take the form

$$\begin{aligned} 0 &= \frac{g'}{\sin 2g}(t) \frac{\alpha'''}{\alpha'} + (A + M)\alpha'' + (A - M)\beta'' + (A + N)\gamma'', \\ 0 &= \frac{g'}{\sin 2g}(t) \frac{\beta'''}{\beta'} + (A - M)\alpha'' + (A + M)\beta'' + (A - N)\gamma'', \\ 0 &= \frac{g'}{\sin 2g}(t) \frac{\gamma'''}{\gamma'} + (A + N)\alpha'' + (A - N)\beta'' + (A + M)\gamma'', \end{aligned}$$

where $M := 2(A - C \cos 2g)/\sin^2 2g$ and $N := 2(C - A \cos 2g)/\sin^2 2g$. Hence

$$\begin{aligned} 0 &= \left(\frac{\alpha'''}{\alpha'} + \frac{\beta'''}{\beta'} \right) + \frac{2A \sin 2g}{g'} \{\alpha'' + \beta'' + \gamma''\}, \\ 0 &= \left(\frac{\alpha'''}{\alpha'} - \frac{\gamma'''}{\gamma'} \right) + \frac{2(A - C) \cot g}{g'} \{\alpha'' - \beta'' - \gamma''\}, \\ 0 &= \left(\frac{\beta'''}{\beta'} - \frac{\gamma'''}{\gamma'} \right) + \frac{2(A + C) \tan g}{g'} \{-\alpha'' + \beta'' - \gamma''\} \end{aligned}$$

and taking derivatives with respect to z, y and x, respectively, and using the original equations to eliminate third order derivatives we arrive at

$$0 = \begin{pmatrix} (M+N) & (M-N) & 2M \\ -(M+N) & 2A+(M+N) & 2A \\ 2A+(M-N) & -(M-N) & 2A \end{pmatrix} \begin{pmatrix} \alpha'' \\ \beta'' \\ \gamma'' \end{pmatrix}$$

so that $\alpha'' = \beta'' = -\gamma'' \equiv \text{const}$ since $A + M \not\equiv 0$ on any open set; the original equations then imply that

$$\alpha'' = \beta'' = \gamma'' \equiv 0.$$

Consequently, up to a coordinate translation, φ is of the sought form:

Proposition 4. *Let f be a conformally flat hypersurface with canonical Guichard net* (x,y,z) *and induced conformal structure given by φ satisfying* (2.5). *If the Guichard net is not cyclic and consists of constant Gauss curvature surfaces for a suitable constant sectional curvature representative of the induced conformal structure then there are constants $a, b, c, A, C \in \mathbb{R}$ with $abc \neq 0$ and $A \neq 0, \pm C$ so that, up to a coordinate translation,*

$$\varphi(x, y, z) = g(ax + by + cz), \quad \textit{where} \quad g'^2 = C - A\cos 2g.$$

This proposition yields the second statement of our Main Theorem, thus completing its proof.

5. The associated family

Conformally flat hypersurfaces give rise to cyclic systems with the original conformally flat hypersurface as an orthogonal hypersurface and so that all other orthogonal hypersurfaces of the cyclic system are conformally flat as well, see [5, §2.2.15]: choosing a lift $f' = e^{\psi} f$ of the conformally flat hypersurface, given in terms of its canonical lift

$$f : M^3 \to L^5 \subset \mathbb{R}^6_1$$

of Section 2.1, so that the induced metric becomes flat, its normal bundle as an immersion into the Minkowski space $\mathbb{R}^6_1$ becomes flat as well [5, §2.1.4], hence defining a curved flat in the symmetric space $O(5, 1)/(O(3) \times O(2, 1))$ of circles in the conformal 4-sphere which, geometrically, is an orthogonal cyclic system, see [5, §2.2.3]. Note that this cyclic system depends on the choice of a function ψ, that is, on a choice of a flat lift f' of f.

Writing the structure equations for an adapted Möbius frame of such a flat lift f' as

$$dF' = F'\Phi' \quad \text{with} \quad \Phi' = \Phi'_{\mathfrak{k}} + \Phi'_{\mathfrak{p}},$$

where $\Phi'_{\mathfrak{k}}\colon TM \to \mathfrak{o}(3) \oplus \mathfrak{o}(2,1)$ takes values in the isotropy algebra of the symmetric space of circles and $\Phi'_{\mathfrak{p}}$ encodes the derivative of the Gauss map $p \mapsto d_p f'(T_p M)$ of f' with values in the Grassmannian of spacelike 3-planes, the compatibility conditions decouple:

$$0 = d\Phi'_{\mathfrak{k}} + \frac{1}{2}[\Phi'_{\mathfrak{k}} \wedge \Phi'_{\mathfrak{k}}],$$
$$0 = d\Phi'_{\mathfrak{p}} + [\Phi'_{\mathfrak{k}} \wedge \Phi'_{\mathfrak{p}}],$$
$$0 = [\Phi'_{\mathfrak{p}} \wedge \Phi'_{\mathfrak{p}}],$$

where the last equation encodes the simultaneous flatness of the tangent and normal bundles of f'. As a consequence,

$$\Phi'^{\lambda} := \Phi'_{\mathfrak{k}} + \lambda\Phi'_{\mathfrak{p}}, \quad \lambda \in (0, \infty), \tag{5.1}$$

defines a loop of flat connections: this yields the curved flat associated family of the Gauss map of f', that is, an associated family of the cyclic system associated to a conformally flat hypersurface. Note that we restrict to $\lambda > 0$ here: a change of sign of $\Phi'_{\mathfrak{p}}$ is realized by a simple gauge transformation and does therefore not affect the geometry; using a λ-dependent gauge transformation to blow up the limiting hypersurface in the limiting case $\lambda = 0$, reveals that the hypersurface obtained cannot be generic.

We shall see that this associated family of the cyclic system descends to an associated family for the conformally flat hypersurface, cf. [2].

To this end, we start with our original structure equations (2.3) and investigate the effect of changing the light cone lift as in (2.7) so that the induced metric $|df'|^2$ becomes flat. In particular, the Schouten forms (2.11) of the new metric

$$0 = \sigma'_i = e^{-\psi}(\sigma_i - \tau_i)$$

so that $\tau_i = \sigma_i$; as a consequence the transformation formulas (2.8) and (2.9) read

$$\omega_i \to \omega'_i = e^{\psi}\omega_i,$$
$$\eta_i \to \eta'_i = \eta_i,$$
$$\omega_{ij} \to \omega'_{ij} = \omega_{ij} + \partial_j\psi\,\omega_i - \partial_i\psi\,\omega_j,$$
$$\chi \to \chi' = e^{-\psi}\left\{\chi - \sum_j \partial_j\psi\,\eta_j\right\},$$
$$\chi_i \to \chi'_i = e^{-\psi}\{\chi_i - \sigma_i\}.$$

Now the curved flat associated family is obtained by integrating the structure equations with

$$\omega'_{ij},\ \chi' \quad \text{and} \quad \lambda\omega'_i,\ \lambda\eta'_i,\ \lambda\chi'_i,$$

where $\lambda > 0$, to obtain a 1-parameter family of frames F'^{λ} for the corresponding associated family of cyclic systems. In particular,

$$F'^{\lambda} \simeq (s_1'^{\lambda}, s_2'^{\lambda}, s_3'^{\lambda}, s^{\lambda}, f'^{\lambda}, \hat{f}'^{\lambda})$$

providing a 1-parameter family of conformally flat hypersurfaces f'^{λ}. A priori, this family of conformally flat hypersurfaces depends on our earlier choice of flat lift f'; we shall see that it does not by providing a lift-independent method to define the hypersurfaces.

To this end, we undo the earlier change of lift by letting

$$f'^{\lambda} \to f^{\lambda} := e^{-\psi} f'^{\lambda}$$

so that, in particular, $f' = f'^1 \to f^1 = f$. Employing the transformation formulas (2.8) and (2.9) again, with $\partial_i'^{\lambda} = (e^{-\psi}/\lambda)\partial_i$, we find

$$\begin{aligned}
&\omega_i'^{\lambda} \to \omega_i^{\lambda} = \lambda\omega_i,\\
&\eta_i'^{\lambda} \to \eta_i^{\lambda} = \lambda\eta_i,\\
&\omega_{ij}'^{\lambda} \to \omega_{ij}^{\lambda} = \omega_{ij},\\
&\chi'^{\lambda} \to \chi^{\lambda} = \chi,\\
&\chi_i'^{\lambda} \to \chi_i^{\lambda} = \lambda\chi_i - \left(\lambda - \frac{1}{\lambda}\right)\sigma_i.
\end{aligned}$$

Thus, the 1-parameter family $\lambda \mapsto f^{\lambda}$ of conformally flat hypersurfaces obtained from the curved flat associated family of an associated orthogonal cyclic system can be defined using any light cone lift of the hypersurface (see also [2] for a more general statement and different proof):

Theorem 5. *Let $f: U \to L^5$ (U simply connected) be a light cone lift of a conformally flat hypersurface and let*

$$(s_1, s_2, s_3, s, f, \hat{f}) \simeq F: U \to O_1(6)$$

denote an adapted Möbius geometric frame for f with structure equations $dF = F\Phi$, where

$$\Phi = \begin{pmatrix} 0 & \omega_{12} & -\omega_{31} & -\eta_1 & \omega_1 & \chi_1 \\ -\omega_{12} & 0 & \omega_{23} & -\eta_2 & \omega_2 & \chi_2 \\ \omega_{31} & -\omega_{23} & 0 & -\eta_3 & \omega_3 & \chi_3 \\ \eta_1 & \eta_2 & \eta_3 & 0 & 0 & \chi \\ -\chi_1 & -\chi_2 & -\chi_3 & -\chi & 0 & 0 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 & 0 & 0 \end{pmatrix}.$$

Then the 1*-parameter family of structure equations* $dF^\lambda = F^\lambda \Phi^\lambda$ *obtained by changing*

$$\omega_i \to \lambda\omega_i, \quad \eta_i \to \lambda\eta_i \quad \textit{and} \quad \chi_i \to \lambda\chi_i - \left(\lambda - \frac{1}{\lambda}\right)\sigma_i,$$

where σ_i *denote the Schouten forms of the induced metric, while leaving* ω_{ij} *and* χ *unchanged, is integrable.*

Moreover, the F^λ *are adapted Möbius geometric frames for the conformally flat hypersurfaces* f^λ *obtained from the curved flat associated family given by any flat light cone lift of* f.

In particular, the curved flat associated family for the Gauss map of a flat light cone lift of a conformally flat hypersurface descends to an associated family for the conformally flat hypersurface, as sought.

From (2.12) and (2.13) we learn that the effect of the associated family on the conformal fundamental forms is a rescaling

$$\gamma_i \to \gamma_i^\lambda = \lambda\gamma_i$$

while Wang's Möbius curvature W does not change, as $\omega_i^\lambda = \lambda\omega_i$ and $\eta_i^\lambda = \lambda\eta_i$. As these form a complete set of invariants for a hypersurface in the conformal 4-sphere the hypersurfaces of the family are not Möbius equivalent, see [11] and [5, §2.3.5], see also [9, Corollaries 3.1.1 and 3.2.1].

The coordinate functions of the canonical Guichard net of a generic conformally flat hypersurface are given by integrating its conformal fundamental forms. Hence these become

$$(x^\lambda, y^\lambda, z^\lambda) = (\lambda x, \lambda y, \lambda z).$$

Assuming that the original lift f of the conformally flat hypersurface was the canonical lift of Section 2.1 the induced metric of f^λ is

$$\mathrm{I}^\lambda = |df^\lambda|^2 = \cos^2\varphi(dx^\lambda)^2 + \sin^2\varphi(dy^\lambda)^2 + (dz^\lambda)^2$$

so that all f^λ are canonical lifts and

$$\varphi^\lambda(x^\lambda, y^\lambda, z^\lambda) = \varphi(x, y, z) = \varphi\left(\frac{x^\lambda}{\lambda}, \frac{y^\lambda}{\lambda}, \frac{z^\lambda}{\lambda}\right).$$

Note that the structure equations (2.3) now hold for all λ:

$$\omega_{12} = -(\varphi^\lambda_{y^\lambda}\, dx^\lambda + \varphi^\lambda_{x^\lambda}\, dy^\lambda), \quad \omega_1^\lambda = \cos\varphi^\lambda\, dx^\lambda, \quad \eta_1^\lambda = \sin\varphi^\lambda\, dx^\lambda, \quad \chi_1^\lambda = \sigma_1^\lambda + \frac{1}{2}\omega_1^\lambda,$$
$$\omega_{23} = \varphi^\lambda_{z^\lambda}\cos\varphi^\lambda\, dy^\lambda, \quad \omega_2^\lambda = \sin\varphi^\lambda\, dy^\lambda, \quad \eta_2^\lambda = -\cos\varphi^\lambda\, dy^\lambda, \quad \chi_2^\lambda = \sigma_2^\lambda + \frac{1}{2}\omega_2^\lambda,$$
$$\omega_{31} = \varphi^\lambda_{z^\lambda}\sin\varphi^\lambda\, dx^\lambda, \quad \omega_3^\lambda = dz^\lambda, \quad \eta_3^\lambda = 0, \quad \chi_3^\lambda = \sigma_3^\lambda - \frac{1}{2}\omega_3^\lambda,$$
$$\chi = \varphi^\lambda_{z^\lambda}\, dz^\lambda$$

since $\chi_i^\lambda = \lambda(\chi_i - \sigma_i) + (1/\lambda)\sigma_i$ and $\sigma_i^\lambda = (1/\lambda)\sigma_i$.

In particular, we learn that for a conformally flat hypersurface f with Bianchi-type canonical Guichard net all conformally flat hypersurfaces f^λ of the family have Bianchi-type Guichard net:

$$\varphi^\lambda(x^\lambda, y^\lambda, z^\lambda) = g\left(\frac{a}{\lambda}x^\lambda + \frac{b}{\lambda}y^\lambda + \frac{c}{\lambda}z^\lambda\right).$$

Hence the induced metric of the constant curvature lifts $(1/g')f^\lambda$ become

$$\left|d\left(\frac{1}{g'}f^\lambda\right)\right|^2 = \frac{1}{C - A\cos 2g}\{\cos^2 g(dx^\lambda)^2 + \sin^2 g(dy^\lambda)^2 + (dz^\lambda)^2\} = \lambda^2\left|d\left(\frac{1}{g'}f\right)\right|^2$$

and the (constant) Gauss curvatures as well as their ambient sectional curvature are scaled by $1/\lambda^2$.

However, the canonical Guichard nets are, in general, not Möbius equivalent as *parametrized* triply orthogonal systems (cf. [5, §2.4.6]): for example, the surfaces $x =$ const and $x^\lambda = \lambda x =$ const are not Möbius equivalent even if they have the same constant Gauss curvature after rescaling. In fact, the lifts $(1/\lambda g')f^\lambda$ induce the same metric for all λ, but the principal curvatures of the surface $x = \alpha$ are

$$k'_{12}(y,z) = \frac{a(A-C)}{\sin g(a\alpha + by + cz)} \quad \text{and} \quad k'_{13}(y,z) = 2aA\sin g(a\alpha + by + cz)$$

while the principal curvatures of the surface $x^\lambda = \alpha$ are

$$k'^\lambda_{12}(y,z) = \frac{a(A-C)}{\sin g(a\alpha/\lambda + by + cz)} \quad \text{and} \quad k'^\lambda_{13}(y,z) = 2aA\sin g\left(\frac{a\alpha}{\lambda} + by + cz\right)$$

showing that these surfaces are not congruent in general.

Thus we conclude with the following

Theorem 6. *The associated family of a conformally flat hypersurfaces with Bianchi-type Gui-chard net consists of conformally flat hypersurfaces with Bianchi-type Guichard net.*

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References

[1] W. Blaschke: Vorlesungen über Differentialgeometrie, III, Springer Grundlehren XXIX, Berlin, 1929.

[2] F. Burstall and D. Calderbank: *Conformal submanifold geometry*, IV–V, manuscript (2009)

[3] E. Cartan: *La déformation des hypersurfaces dans l'espace conforme réel à* $n \geq 5$ *dimensions*, Bull. Soc. Math. France **45** (1917), 57–121.

[4] U. Hertrich-Jeromin: *On conformally flat hypersurfaces and Guichard's nets*, Beiträge Algebra Geom. **35** (1994), 315–331.

[5] U. Hertrich-Jeromin: Introduction to Möbius differential geometry, London Mathematical Society Lecture Note Series **300**, Cambridge Univ. Press, Cambridge, 2003.

[6] U. Hertrich-Jeromin and Y. Suyama: *Conformally flat hypersurfaces with cyclic Guichard net,* Internat. J. Math. **18** (2007), 301–329.

[7] S. Kawakubo: *Kirchhoff elastic rods in the three-sphere*, Tohoku Math. J. (2) **56** (2004), 205–235.

[8] E. Salkowski: Dreifach Orthogonal Flächensysteme, Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen D9, Teubner, Leipzig, 1902.

[9] Y. Suyama: *Conformally flat hypersurfaces in Euclidean* 4*-space* II, Osaka J. Math. **42** (2005), 573–598.

[10] Y. Suyama: *A classification and a non-existence theorem for conformally flat hypersurfaces in Euclidean* 4*-space*, Internat. J. Math. **16** (2005), 53–85.

[11] C.P. Wang: *Möbius geometry for hypersurfaces in* S^4, Nagoya Math. J. **139** (1995), 1–20.

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