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GLOBAL MONODROMY MODULO 5 OF THE QUINTIC-MIRROR FAMILY

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Abstract

The quintic-mirror family is a well-known one-parameter family of Calabi–Yau threefolds. A complete description of the global monodromy group of this family is not yet known. In this paper, we give a presentation of the global monodromy group in the general linear group of degree 4 over the ring of integers modulo 5.

1. Introduction

The quintic-mirror family \((W_\lambda)_{\lambda \in U} \to \mathbb{P}^1\) is a family, whose restriction \(f \colon (W_\lambda)_{\lambda \in U} \to U\) on \(U := \mathbb{P}^1 - \{0, 1, \infty\}\) is a smooth projective family of Calabi–Yau manifolds. Fix \(b \in U\) and let \(\langle \cdot, \cdot \rangle\) be the anti-symmetric bilinear form on \(H^3(W_b, \mathbb{Z})\) defined by the cup product. The global monodromy group \(\Gamma\) is the image of the representation \(\pi_1(U, b) \to \text{Aut}(H^3(W_b, \mathbb{Z}), \langle \cdot, \cdot \rangle)\) corresponding to the local system \(R^3\pi_*\mathcal{F}\) with the fiber \(H^3(W_b, \mathbb{Z})\) over \(b\). When we take a symplectic basis, we can identify \(\text{Aut}(H^3(W_b, \mathbb{Z}), \langle \cdot, \cdot \rangle)\) with \(\text{Sp}(4, \mathbb{Z})\).

In this paper, we are concerned with a description of \(\Gamma\). Matrix presentations of the generators of \(\Gamma\) are well studied and it is also known that \(\Gamma\) is Zariski dense in \(\text{Sp}(4, \mathbb{Z})\) (e.g. [1], [3]). However, it is not known whether the index of \(\Gamma\) in \(\text{Sp}(4, \mathbb{Z})\) is finite or not (e.g. [2]). A direct approach for this problem is to describe \(\Gamma\) explicitly. In the main theorem of this paper, we give a presentation of \(\Gamma\) in \(\text{GL}(4, \mathbb{Z}/5\mathbb{Z})\), which is a small attempt toward a description of \(\Gamma\).

On the other hand, Chen, Yang and Yui find a congruence subgroup \(\Gamma(5, 5)\) of \(\text{Sp}(4, \mathbb{Z})\) of finite index, which contains \(\Gamma\) in [2]. Combining their result and our main theorem, we can construct a smaller congruence subgroup \(\hat{\Gamma}(5, 5)\) of \(\text{Sp}(4, \mathbb{Z})\) of finite index, which contains \(\Gamma\). However this result is merely the fact that \(\hat{\Gamma}(5, 5)\) contains \(\Gamma\). After all, the index of \(\Gamma\) in \(\text{Sp}(4, \mathbb{Z})\) is still unknown.

2. The quintic-mirror family

The quintic-mirror family was constructed by Greene and Plesser. We review the construction of the quintic-mirror family after [4].

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Let $\psi \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, and let

$$Q_\psi = \{x \in \mathbb{P}^4 \mid x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5\psi x_1 x_2 x_3 x_4 x_5 = 0\}.$$ 

A finite group $G$, which is abstractly isomorphic to $(\mathbb{Z}/5\mathbb{Z})^3$, acts on $Q_\psi$ as follows.

- $\mu_5$: the multiplicative group of the 5-th root of 1.
- $G = ((\alpha_1, \ldots, \alpha_5) \in (\mathbb{Z}/5\mathbb{Z})^5 \mid \alpha_1 = \cdots = \alpha_5)$.
- $G \times Q_\psi \to Q_\psi$, \((\alpha_1, \ldots, \alpha_5), (x_1, \ldots, x_5)) \mapsto (\alpha_1 x_1, \ldots, \alpha_5 x_5).$$

When we take the quotient of the hypersurface $Q_\psi$ by $G$, canonical singularities appear. For $\psi \in \mathbb{C} \subset \mathbb{P}^1$, it is known that there is a simultaneous minimal desingularization of these singularities, and we have the one-parameter family $(W_\psi)_{\psi \in \mathbb{P}^1}$ whose fibres are listed as follows:

- When $\psi$ belongs to $\mu_5 \subset \mathbb{C} \subset \mathbb{P}^1$, $W_\psi$ has one ordinary double point.
- $W_\infty$ is a normal crossing divisor in the total space.
- The other fibres of $(W_\psi)_{\psi \in \mathbb{P}^1}$ are smooth with Hodge numbers $h^{p,q} = 1$ for $p + q = 3$, $p, q \geq 0$.

By the action of

$$\alpha \in \mu_5, \quad (x_1, \ldots, x_5) \mapsto (x_1, \ldots, x_4, \alpha^{-1} x_5),$$

we have the isomorphism from the fibre over $\psi$ to the fibre over $\alpha \psi$. Let $\lambda = \psi^5$ and let

$$(W_\lambda)_{\lambda \in \mathbb{P}^1} \cong ((W_\psi)_{\psi \in \mathbb{P}^1})/\mu_5$$

$$\downarrow$$

$$\lambda\text{-plane} \cong (\psi\text{-plane})/\mu_5.$$

This family $(W_\lambda)_{\lambda \in \mathbb{P}^1}$ is the so-called quintic-mirror family. (For more details of the above construction, see e.g. [4], [5].)

### 3. Monodromy

Let $b \in \mathbb{P}^1 - \{0, 1, \infty\}$ on the $\lambda$-plane. In [1], Candelas, de la Ossa, Green and Parks constructed a symplectic basis $\{A^1, A^2, B_1, B_2\}$ of $H_3(W_b, \mathbb{Z})$ and calculated the monodromies around $\lambda = 0, 1, \infty$ on the period integrals of a holomorphic 3-form on this basis. By the relation in [5, Appendix C] between the symplectic basis $\{\beta^1, \beta^2, \alpha_1, \alpha_2\}$ of $H^3(W_b, \mathbb{Z})$, which is defined to be the dual basis of $\{B_1, B_2, A^1, A^2\}$, and the period
integrals, we have the matrix representations of the local monodromies for the basis \( \{ \beta^1, \beta^2, \alpha_1, \alpha_2 \} \). We recall their results.

Matrix representations \( A, T, T_\infty \) of local monodromies around \( 0, 1, \infty \) for the basis \( \{ \beta^1, \beta^2, \alpha_1, \alpha_2 \} \) are as follows:

\[
A = \begin{pmatrix}
11 & 8 & -5 & 0 \\
-4 & -3 & 1 & 0 \\
20 & 15 & -9 & 0 \\
-5 & -3 & 1 & 0
\end{pmatrix}, \quad
T = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad
T_\infty = \begin{pmatrix}
-9 & -3 & 5 & 0 \\
0 & 1 & 0 & 0 \\
-20 & -5 & 11 & 0 \\
-15 & 5 & 8 & 1
\end{pmatrix}.
\]

In particular, the above \( A \) and \( T \) are the inverse matrices of the matrices \( A \) and \( T \) in the lists of [1], respectively.

Let \( \langle , \rangle \) be the anti-symmetric bilinear form on \( H^3(W_b, \mathbb{Z}) \) defined by the cup product. The global monodromy \( \Gamma \) is \( \text{Im}(\pi_1(\mathbb{P}^1 - \{ 0, 1, \infty \}) \to \text{Aut}(H^3(W_b, \mathbb{Z}), \langle , \rangle) \). When we take \( \{ \beta^1, \beta^2, \alpha_1, \alpha_2 \} \) as the basis of \( H^3(W_b, \mathbb{Z}) \), \( \text{Aut}(H^3(W_b, \mathbb{Z}), \langle , \rangle) \) is identified with \( \text{Sp}(4, \mathbb{Z}) \), and \( \Gamma \) is the subgroup of \( \text{Sp}(4, \mathbb{Z}) \) which is generated by \( A \) and \( T \).

We can partially normalize \( A \) and \( T \) simultaneously as follows.

**Lemma.** There exists \( P \in \text{GL}(4, \mathbb{Q}) \) such that

\[
p^{-1}A^{-1}p = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
5 & 5 & 5 & -4
\end{pmatrix}, \quad
p^{-1}T^{-1}p = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

**Proof.** We take \( P = \begin{pmatrix}
5 & -3 & 0 & 0 \\
0 & 0 & 1 & 0 \\
10 & -5 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \). The assertion follows. \( \square \)

4. Main result

Let \( \Gamma' = \{ P^{-1}XP \in \text{GL}(4, \mathbb{Z}) \mid X \in \Gamma \} \), and let \( \rho : \text{GL}(4, \mathbb{Z}) \to \text{GL}(4, \mathbb{Z}/5\mathbb{Z}) \) be the natural projection. Define \( \tilde{\Gamma} = \rho(\Gamma') \). We will study \( \tilde{\Gamma} \).

Let \( \tilde{A} = \rho(P^{-1}A^{-1}P) \), \( \tilde{T} = \rho(P^{-1}T^{-1}P) \in \text{GL}(4, \mathbb{Z}/5\mathbb{Z}) \). By a simple calculation, we obtain

\[
\tilde{A}^n = \begin{pmatrix}
1 & n & 3n(n + 4) & n(n + 1)(4n + 1) \\
0 & 1 & n & 2n(n + 1) \\
0 & 0 & 1 & 4n \\
0 & 0 & 0 & 1
\end{pmatrix} \in \text{GL}(4, \mathbb{Z}/5\mathbb{Z}).
\]
Let \( \hat{\Gamma} \) be
\[
\begin{pmatrix}
1 & n & 3n^2 + 2n & a \\
0 & 1 & n & b \\
0 & 0 & 1 & c \\
0 & 0 & 0 & 1
\end{pmatrix} \in \text{GL}(4, \mathbb{Z}/5\mathbb{Z}), \quad n, a, b, c \in \mathbb{Z}/5\mathbb{Z}.
\]
\( \hat{\Gamma} \) is a subgroup of \( \text{GL}(4, \mathbb{Z}/5\mathbb{Z}) \) which contains \( \bar{A} \) and \( \bar{T} \). The following Theorem and Corollary are the main results of this paper.

**Theorem.** \( \hat{\Gamma} = \bar{\Gamma} \).

**Proof.** \( \hat{\Gamma} \subseteq \bar{\Gamma} \) follows from what we just mentioned. So we shall prove the converse inclusion.

From the presentations of elements of \( \hat{\Gamma} \), we see that \( \hat{\Gamma} \) is generated by
\[
\bar{A}, \bar{T}, E_1 = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Therefore, it is enough to show \( E_1 \) and \( E_2 \) belong to \( \bar{\Gamma} \). In fact, we have
\[
E_2 = \bar{A} \bar{T} \bar{A}^4 \bar{T}, \quad E_1 = (E_2^2 \bar{A}^2 \bar{T}^4 \bar{A}^3 \bar{T})^4.
\]
Hence \( E_1, E_2 \in \bar{\Gamma} \).

**Corollary.** Let \( X \in \Gamma \). Then the characteristic polynomial of \( X \) is
\[
x^4 + (5m + 1)x^3 + (5n + 1)x^2 + (5m + 1)x + 1,
\]
where \( m, n \) are some integers. In particular, if \( X \) is not the unit matrix and the order of \( X \) is finite, then the order of \( X \) is 5 and the eigenvalues of \( X \) are \( \exp(2\pi i/5) \), \( \exp(4\pi i/5) \), \( \exp(6\pi i/5) \), \( \exp(8\pi i/5) \).

**Proof.** We shall prove the first part. Let \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) be the eigenvalues of \( X \). Then the characteristic polynomial \( p(X) \) of \( X \) is
\[
x^4 - \left( \sum_{1 \leq i \leq j \leq k \leq 4} \lambda_i \lambda_j \lambda_k \right)x^3 + \left( \sum_{1 \leq i \leq j \leq 4} \lambda_i \lambda_j \right)x^2 - \left( \sum_{1 \leq i \leq 4} \lambda_i \right)x + 1.
\]
On the other hand, the characteristic polynomial $p(X^{-1})$ of $X^{-1}$ is

$$x^4 - \left( \sum_{1 \leq i \leq j \leq k \leq 4} \frac{1}{\lambda_i \lambda_j \lambda_k} \right) x^3 + \left( \sum_{1 \leq i \leq j \leq k \leq 4} \frac{1}{\lambda_i \lambda_j} \right) x^2 - \left( \sum_{1 \leq i \leq j \leq k \leq 4} \frac{1}{\lambda_i} \right) x + 1 = x^4 - \left( \sum_{1 \leq i \leq 4} \lambda_i \right) x^3 + \left( \sum_{1 \leq i \leq j \leq 4} \lambda_i \lambda_j \lambda_k \right) x^2 - \left( \sum_{1 \leq i \leq j \leq k \leq 4} \lambda_i \lambda_j \lambda_k \right) x + 1.$$  

Since $X \in \text{Sp}(4, \mathbb{Z})$, $p(X) = p(X^{-1})$. So $p(X)$ is the form $x^4 + ax^3 + bx^2 + ax + 1$, where $a, b \in \mathbb{Z}$. It follows from the theorem that $a \equiv -4, b \equiv 6 \mod 5$. Hence the claim of the first part follows.

Next we shall prove the latter part. Let $\lambda$ be an eigenvalue of $X$. It follows from $p(X) = p(\tilde{X})$ and $p(X) = p(X^{-1})$ that $\tilde{X}, 1/\lambda, 1/\tilde{X}$ are also eigenvalues of $X$. Since the determinant of $X$ is 1, if 1 or $-1$ is an eigenvalue of $X$, its multiplicity is even. Since the order of $X$ is finite, we can express eigenvalues of $X$ by $\exp(i \theta_1), \exp(-i \theta_1), \exp(i \theta_2), \exp(-i \theta_2)$ ($0 \leq \theta_1, \theta_2 \leq \pi$). Then the characteristic polynomial of $X$ is

$$x^4 - 2(\cos \theta_1 + \cos \theta_2) x^3 + 2(\cos(\theta_1 + \theta_2) + \cos(\theta_1 - \theta_2) + 1)x^2 - 2(\cos \theta_1 + \cos \theta_2)x + 1.$$  

By the claim of the first part of this Corollary, we have

$$-2(\cos \theta_1 + \cos \theta_2) = 5m + 1, \quad 2(\cos(\theta_1 + \theta_2) + \cos(\theta_1 - \theta_2) + 1) = 5n + 1, \quad m, n \in \mathbb{Z}.$$  

By the addition theorem for cosines, we have

$$2(\cos \theta_1 + \cos \theta_2) = -5m - 1, \quad 4 \cos \theta_1 \cos \theta_2 = 5n - 1.$$  

It follows from $-4 \leq 2(\cos \theta_1 + \cos \theta_2) \leq 4$ that $m = 0$ or $-1$. If $m = -1$, then $\cos \theta_1$, $\cos \theta_2 = 1$ and all eigenvalues of $X$ are 1. Since the order of $X$ is finite, $X$ is the unit matrix. This contradicts the assumption that $X$ is not the unit matrix. Hence $m = 0$ and

$$\cos \theta_1 + \cos \theta_2 = -\frac{1}{2}.$$  

It follows from $-4 \leq 4 \cos \theta_1 \cos \theta_2 \leq 4$ that $n = 0$ or 1. If $n = 1$, then $\cos \theta_1 = \pm 1$, $\cos \theta_2 = \pm 1$. This contradicts the fact that $\cos \theta_1 + \cos \theta_2 = -1/2$. Hence $n = 0$ and

$$\cos \theta_1 \cos \theta_2 = -\frac{1}{4}.$$  

Combining these two equations, we have

$$\cos^2 \theta_1 + \frac{1}{2} \cos \theta_1 - \frac{1}{4} = 0.$$
When we solve this equation for $\cos \theta_1$,

$$\cos \theta_1 = \frac{-1 \pm \sqrt{5}}{4}, \quad \sin \theta_1 = \frac{\sqrt{10 \pm 2\sqrt{5}}}{4},$$

$$\cos \theta_2 = \frac{-1 \mp \sqrt{5}}{4}, \quad \sin \theta_2 = \frac{\sqrt{10 \mp 2\sqrt{5}}}{4}.$$

Then we can verify easily that $(\exp(i\theta_1))^5$ and $(\exp(i\theta_2))^5 = 1$. Hence $(\theta_1, \theta_2) = (2\pi/5, 4\pi/5)$ or $(4\pi/5, 2\pi/5)$.

5. A relation to the other result

In this section, we shall compare the main result of this paper with the result of Chen, Yang and Yui. In [2], they find the congruence subgroup $\Gamma(5, 5)$ which contains the global monodromy $\Gamma$. Combining their result and our theorem, we can find a smaller group which contains $\Gamma$.

The congruence subgroup $\Gamma(5, 5)$ is defined by

$$\Gamma(5, 5) = \left\{ X \in \text{Sp}(4, \mathbb{Z}) \left| \gamma \equiv \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \pmod{5} \right. \right\}.$$

$\Gamma(5, 5)$ contains the principal congruence group $\Gamma(5) = \text{Ker}(\text{Sp}(4, \mathbb{Z}) \to \text{Sp}(4, \mathbb{Z}/5\mathbb{Z}))$ as a normal subgroup of finite index.

Let $X \in \Gamma(5, 5)$ and express $X$ by

$$\begin{pmatrix} 5x_{11} + 1 & x_{12} & x_{13} & x_{14} \\ 5x_{21} & 5x_{22} + 1 & x_{23} & x_{24} \\ 5x_{31} & 5x_{32} & 5x_{33} + 1 & 5x_{34} \\ 5x_{41} & 5x_{42} & x_{43} & 5x_{44} + 1 \end{pmatrix}, \quad x_{ij} \in \mathbb{Z} \quad (1 \leq i, j \leq 4).$$

Then we have

$$\text{GL}(4, \mathbb{Z}) \ni P^{-1}XP = \begin{pmatrix} 1 & -9x_{31} & -x_{12} + 3x_{32} & -x_{14} + 3x_{34} \\ 0 & 1 & -2x_{12} & -2x_{14} \\ 0 & 0 & 1 & x_{24} \\ 0 & 0 & 0 & 1 \end{pmatrix} \pmod{5}.$$ 

By the main theorem, if $X \in \Gamma$, then $\rho(P^{-1}XP) \in \bar{\Gamma}$ and

$$-9x_{31} \equiv n, \quad -2x_{12} \equiv n, \quad -x_{12} + 3x_{32} \equiv 3n^2 + 2n \pmod{5}.$$
where $n$ is some integer. From a simple calculation, the above equation is equivalent to

\[ x_{31} \equiv 3x_{12}, \quad x_{32} \equiv 4x_{12}^2 + 4x_{12} \quad (\text{mod } 5). \]

So we define

\[
\bar{\Gamma}(5, 5) = \left\{ \begin{pmatrix} 5x_{11} + 1 & x_{12} & x_{13} & x_{14} \\
5x_{21} & 5x_{22} + 1 & x_{23} & x_{24} \\
5x_{31} & 5x_{32} & 5x_{33} + 1 & 5x_{34} \\
5x_{41} & 5x_{42} & x_{43} & 5x_{44} + 1 \end{pmatrix} \in \text{Sp}(4, \mathbb{Z}) \left| \begin{array}{c}
 x_{31} \equiv 3x_{12}, \\
x_{32} \equiv 4x_{12}^2 + 4x_{12} \\
(\text{mod } 5) \end{array} \right. \}
\]

Then we have the following Corollary.

**Corollary.** (i) $\bar{\Gamma}(5, 5)$ is a subgroup of $\Gamma(5, 5)$.
(ii) $\Gamma \subset \bar{\Gamma}(5, 5) \subsetneq \Gamma(5, 5)$.
(iii) $\bar{\Gamma}(5, 5)$ is a congruence subgroup of $\text{Sp}(4, \mathbb{Z})$ of finite index.

**Proof.** Let $\rho' \colon \Gamma(5, 5) \rightarrow \text{GL}(4, \mathbb{Z}), X \mapsto P^{-1}XP$ and let $\pi = \rho \circ \rho' \colon \Gamma(5, 5) \rightarrow \text{GL}(4, \mathbb{Z}/5\mathbb{Z})$. $\bar{\Gamma}(5, 5) = \pi^{-1}(\bar{\Gamma})$ follows from what we just mentioned. Since $\pi$ is a group homomorphism, $\pi^{-1}(\bar{\Gamma})$ is a subgroup of $\Gamma(5, 5)$. Hence the claim of (i) follows.

We can verify easily that $A$ and $T$ belong to $\bar{\Gamma}(5, 5)$. Therefore $\bar{\Gamma}(5, 5)$ contains $\Gamma$.

We shall show that $\bar{\Gamma}(5, 5)$ is a proper subgroup of $\Gamma(5, 5)$. We take $X = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
5 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix}.$

Then $X \in \bar{\Gamma}(5, 5) \subset \Gamma(5, 5)$ and $X \notin \bar{\Gamma}(5, 5)$. Hence the claim of (ii) follows.

Finally, we shall show the claim of (iii). $\bar{\Gamma}(5, 5)$ contains the principal congruence subgroup $\Gamma(25) = \text{Ker}(\text{Sp}(4, \mathbb{Z}) \rightarrow \text{Sp}(4, \mathbb{Z}/25\mathbb{Z}))$ as a normal subgroup. Hence we obtain $|\bar{\Gamma}(5, 5) : \text{Sp}(4, \mathbb{Z})| < |\Gamma(25) : \text{Sp}(4, \mathbb{Z})| = |\text{Sp}(4, \mathbb{Z}/25\mathbb{Z})| < \infty$. \hfill $\square$

**QUESTION.** There are other 13 mirror families of Calabi–Yau threefolds with $h^{2,1} = 1$ as discussed in [2]. Is it possible to find smaller subgroups in those 13 cases as well?

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**References**


