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Doctoral Thesis

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Abstract

In this paper we present a relation among the multiple zeta values which generalizes simultaneously the "Sum formula" and the "duality" theorem. As an application, we give a formula for the special values at positive integral points of a certain zeta function of Arakawa-Kaneko in terms of multiple harmonic series.

Introduction

The multiple zeta values (or Euler-Zagier sums) seem to be related to many kind of mathematical subjects. Recently A. Graville, D. Zagier and others proved a conjecture known as the "Sum formula" (or "Sum conjecture") (Theorem 1.3) which gives a remarkable relation between the multiple zeta values and special values of Riemann zeta function. In this note we prove a generalization of the "Sum formula" which is at the same time a generalization of another remarkable identity referred to as the "duality" theorem (Theorem 1.2).

The multiple zeta values are defined for integers $k_1, \ldots, k_{n-1} \geq 1$ and $k_n \geq 2$ by

$$\zeta(k_1, k_2, \ldots, k_n) = \sum_{0<m_1<m_2<\cdots<m_n} \frac{1}{{m_1}^{k_1}{m_2}^{k_2}\cdots{m_n}^{k_n}}.$$

In this paper we shall prove the following theorem.

For any index set $(k_1, k_2, \ldots, k_n)$ satisfying the condition above and any integer $l \geq 0$, we define

$$Z(k_1, k_2, \ldots, k_n; l) = \sum_{\substack{c_1+c_2+\cdots+c_n=l \\ \forall c_j \geq 0}} \zeta(k_1 + c_1, k_2 + c_2, \ldots, k_n + c_n).$$

For any integer $s \geq 1$ and $a_1, b_1, a_2, b_2, \ldots, a_s, b_s \geq 1$, we define two index sets which are "dual" to each other by

$$\mathsf{k} = (\underbrace{1, \ldots, 1}_{a_1 - 1}, b_1 + 1, \underbrace{1, \ldots, 1}_{a_2 - 1}, b_2 + 1, \ldots, \underbrace{1, \ldots, 1}_{a_s - 1}, b_s + 1)$$

and

$$\mathsf{k}' = (\underbrace{1, \ldots, 1}_{b_s = 1}, a_s + 1, \underbrace{1, \ldots, 1}_{b_{s-1} = 1}, a_{s-1} + 1, \ldots, \underbrace{1, \ldots, 1}_{b_1 = 1}, a_1 + 1).$$

Our main theorem is then the following.

Theorem 2.1

$$Z(\mathsf{k}'; l) = Z(\mathsf{k}; l).$$

Note that, if we put $s = a_1 = 1$ in above, then $\zeta(\mathsf{k})$ is a Riemann zeta value, and the identity above is nothing but the "Sum formula ". We also note that, if we put $l = 0$, then the above theorem gives $\zeta(\mathsf{k}') = \zeta(\mathsf{k})$, the duality theorem (Theorem 1.2).

On the other hand, poly-Bernoulli numbers were defined by M. Kaneko[9] using the polylogarithms. Recently T. Arakawa and M. Kaneko[1] defined a new function $\xi_k(s)$ which has poly-Bernoulli numbers as the special values of non-positive integral points. They gave some expressions of the function by using the multiple zeta values. (We shall explain some of their results in section 3.)

As an application of the main theorem, we present another theorem that the special values at positive integral points of the zeta function $\xi_k(s)$ are also the special values of a certain multiple harmonic series. Namely, we can state the theorem as follows.

Theorem 3.3

For integers $k \geq 1$ and $n \geq 1$, we have

$$\xi_k(n) = \sum_{0<m_1\leq m_2\leq\cdots\leq m_n} \frac{1}{m_1 m_2 \cdots m_{n-1} {m_n}^{k+1}}.$$

In section 1, we shall review the definition and some properties of the multiple harmonic series (including the multiple zeta values). We shall prove our main theorem in section 2. In section 3, we shall review the zeta function related to poly-Bernoulli numbers defined by T. Arakawa and M. Kaneko[1], and show the relation to multiple harmonic series.

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1 Multiple Harmonic Series

In this section, we shall review the definition and some properties of the multiple harmonic series (including the multiple zeta values).

For integers $n \geq 1$, $k_1, k_2, \ldots, k_{n-1} \geq 1$ and $k_n \geq 2$, we define $\zeta(k_1, k_2, \ldots, k_n)$ and $\zeta^*(k_1, k_2, \ldots, k_n)$ as follows.

$$\begin{aligned} \zeta(k_1, k_2, \ldots, k_n) &= \sum_{0<m_1<m_2<\cdots<m_n} \frac{1}{{m_1}^{k_1} {m_2}^{k_2} \cdots {m_n}^{k_n}}, \\ \zeta^*(k_1, k_2, \ldots, k_n) &= \sum_{0<m_1\leq m_2\leq\cdots\leq m_n} \frac{1}{{m_1}^{k_1} {m_2}^{k_2} \cdots {m_n}^{k_n}}. \end{aligned}$$

Note that

$\zeta(k_1, k_2, \ldots, k_n) = A(k_n, k_{n-1}, \ldots, k_1)$ and $\zeta^*(k_1, k_2, \ldots, k_n) = S(k_n, k_{n-1}, \ldots, k_1)$

in Hoffman's notation[6].

It is known that we have relations as

$$\begin{aligned}\zeta^*(k_1,k_2) &= \zeta(k_1,k_2)+\zeta(k_1+k_2),\\ \zeta^*(k_1,k_2,k_3) &= \zeta(k_1,k_2,k_3)+\zeta(k_1+k_2,k_3)+\zeta(k_1,k_2+k_3)+\zeta(k_1+k_2+k_3),\end{aligned}$$

and

$$\zeta^*(\underbrace{1,1,\ldots,1}_{m-1},k+1)$$

$$=\sum_{n=1}^{m}\sum_{\substack{b_1+b_2+\cdots+b_n=m-n\\ \forall b_j\geq 0}}\zeta(b_1+1,b_2+1,\ldots,b_{n-1}+1,b_n+k+1),$$

(cf.[6]).

Hoffman[6] studied these series. Following theorem is one of his results.

Theorem 1. 1 (Hoffman[6]) *For any integers $k\geq 1$, $i_1,i_2,\ldots,i_{k-1}\geq 1$ and $i_k\geq 2$, we have*

$$\sum_{\substack{a_1+a_2+\cdots+a_k=1\\ \forall a_j\geq 0}}\zeta(a_1+i_1,a_2+i_2,\ldots,a_k+i_k)$$

$$=\sum_{\substack{1\leq l\leq k\\ i_l\geq 2}}\sum_{j=0}^{i_l-2}\zeta(i_1,\ldots,i_{l-1},j+1,i_l-j,i_{l+1},\ldots,i_k).$$

Next, we review two interesting properties of the multiple zeta values. One is called "duality" and another is called "Sum formula" of the multiple zeta values.

First, we review the definition of "Drinfel'd integral " following Zagier[14]. For $\varepsilon_1=1,\varepsilon_k=0$ and $\varepsilon_2,\ldots,\varepsilon_{k-1}\in\{0,1\}$, we define

$$I(\varepsilon_1,\ldots,\varepsilon_k)=\int\!\cdots\!\int_{0<t_1<\cdots<t_k<1}\frac{dt_1}{A_{\varepsilon_1}(t_1)}\cdots\frac{dt_k}{A_{\varepsilon_k}(t_k)},$$

where we denote $A_0(t) = t$ and $A_1(t) = 1-t$. It is known that there is an identity between the multiple zeta values and "Drinfel'd integral ", namely we have (cf.[14])

$$\zeta(k_1, \ldots, k_n) = I(1, \underbrace{0, \ldots, 0}_{k_1-1}, 1, \underbrace{0, \ldots, 0}_{k_2-1}, \ldots, 1, \underbrace{0, \ldots, 0}_{k_n-1}).$$

We also know

$$I(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) = I(1-\varepsilon_k, 1-\varepsilon_{k-1}, \ldots, 1-\varepsilon_1),$$

(cf.[14]).

For any integer $s \geq 1$ and $a_1, b_1, a_2, b_2, \ldots, a_s, b_s \geq 1$, we define two index sets which are "dual" to each other by

$$\mathsf{k} = (\underbrace{1, \ldots, 1}_{a_1-1}, b_1+1, \underbrace{1, \ldots, 1}_{a_2-1}, b_2+1, \ldots, \underbrace{1, \ldots, 1}_{a_s-1}, b_s+1)$$

and

$$\mathsf{k}' = (\underbrace{1, \ldots, 1}_{b_s-1}, a_s+1, \underbrace{1, \ldots, 1}_{b_{s-1}-1}, a_{s-1}+1, \ldots, \underbrace{1, \ldots, 1}_{b_1-1}, a_1+1).$$

Then the following theorem is known.

Theorem 1. 2 (**"Duality" cf.** [2][14]) *For any index set* k *and its dual index set* k' *we have*

$$\zeta(\mathsf{k}') = \zeta(\mathsf{k}).$$

Next, we state the theorem called "Sum formula" conjectured by C. Moen and M. Hoffman, and proved by A. Granville, D. Zagier and others.

Theorem 1. 3 (**"Sum formula" cf.** [2][6]) *For* $0 < n < k$ *we have*

$$\sum_{\substack{k_1, k_2, \ldots, k_{n-1} \geq 1, k_n \geq 2, \\ k_1+k_2+\cdots+k_n = k}} \zeta(k_1, k_2, \ldots, k_n) = \zeta(k).$$

2 Main Theorem

In this section we shall give our main theorem.

For any index set $\mathsf{k} = (k_1, k_2, \ldots, k_n)$ and for any integer $l \geq 0$, we define Z as

$$Z(\mathsf{k}; l) = \sum_{\substack{c_1+c_2+\cdots+c_n=l \\ \forall c_j \geq 0}} \zeta(k_1 + c_1, k_2 + c_2, \ldots, k_n + c_n).$$

Now, we state our main theorem.

Theorem 2. 1 *For any index set* k *and its dual index set* k' *and for any integer* $l \geq 0$, *we have*

$$Z(\mathsf{k}'; l) = Z(\mathsf{k}; l).$$

Remark Note that, if we put $s = a_1 = 1$, then $\zeta(\mathsf{k})$ is a Riemann zeta value, and the identity above is nothing but the "Sum formula " (Theorem 1.3). We also note that, if we put $l = 0$, then the above theorem gives $\zeta(\mathsf{k}') = \zeta(\mathsf{k})$, the duality theorem (Theorem 1.2). Theorem 2.1 also contains Theorem 1.1 (Theorem 5.1 in M. Hoffman[6]) as a special case when $l = 1$.

Proof We fix an index set

$$\mathsf{k} = (\underbrace{1, \ldots, 1}_{a_1 - 1}, b_1 + 1, \underbrace{1, \ldots, 1}_{a_2 - 1}, b_2 + 1, \ldots, \underbrace{1, \ldots, 1}_{a_s - 1}, b_s + 1).$$

Using "Drinfel'd integral" (we reviewed in section 1), for integers $l_i \geq 0$ satisfying $l_1 + \cdots + l_s = l$, and for integers d_i satisfying $1 \leq d_i \leq a_i + l_i$ for $i = 1, \ldots, s$, we put S_{k} as follows.

$$S_{\mathsf{k}}(d_1, \ldots, d_s; l_1, \ldots, l_s) = \sum_{\substack{\varepsilon_{i,2}+\cdots+\varepsilon_{i,a_i+l_i}=d_i-1 \\ \varepsilon_{i,2},\ldots,\varepsilon_{i,a_i+l_i}\in\{0,1\} \text{ for } \forall i}} I(1, \varepsilon_{1,2}, \ldots, \varepsilon_{1,a_1+l_1}, \underbrace{0, \ldots, 0}_{b_1},$$
$$1, \varepsilon_{2,2}, \ldots, \varepsilon_{2,a_2+l_2}, \underbrace{0, \ldots, 0}_{b_2}, \ldots\ldots, 1, \varepsilon_{s,2}, \ldots, \varepsilon_{s,a_s+l_s}, \underbrace{0, \ldots, 0}_{b_s}).$$

Then we have

$$Z(\mathsf{k}; l) = \sum_{\substack{l_1+l_2+\cdots+l_s=l \\ l_i \geq 0 \text{ for } \forall i}} S_{\mathsf{k}}(a_1, \ldots, a_s; l_1, \ldots, l_s).$$

We put $m = l + \sum_{i=1}^{s} a_i + b_i$. If we fix the values $l_i \geq 0$ for $i = 1,\dots,s$ such that $l_1 + \cdots + l_s = l$, and make a generating function of S_{k}, we have

$$\sum_{1\leq d_i\leq a_i+l_i \text{ for } \forall i} \left(S_{\mathsf{k}}(d_1,\dots,d_s;l_1,\dots,l_s)\prod_{j=1}^{s} X_j^{d_j-1}\right)$$

$$= \sum_{\varepsilon_{i,2},\dots,\varepsilon_{i,a_i+l_i}\in\{0,1\} \text{ for } \forall i} \Big(I(1,\varepsilon_{1,2},\dots,\varepsilon_{1,a_1+l_1},\underbrace{0,\dots,0}_{b_1},$$

$$\dots,1,\varepsilon_{s,2},\dots,\varepsilon_{s,a_s+l_s},\underbrace{0,\dots,0}_{b_s}) \prod_{j=1}^{s} X_j^{\varepsilon_{j,2}+\cdots+\varepsilon_{j,a_j+l_j}}\Big)$$

$$= \int\!\!\cdots\!\!\int_{0<t_1<\cdots<t_m<1} \Bigg(\frac{1}{1-t_1}\left(\frac{1}{t_2}+\frac{X_1}{1-t_2}\right)\cdots\left(\frac{1}{t_{a_1+l_1}}+\frac{X_1}{1-t_{a_1+l_1}}\right)$$

$$\times\frac{1}{t_{a_1+l_1+1}}\frac{1}{t_{a_1+l_1+2}}\cdots\frac{1}{t_{a_1+l_1+b_1}}\Bigg)\cdots\cdots$$

$$\times\Bigg(\frac{1}{1-t_{m-a_s-l_s-b_s+1}}\left(\frac{1}{t_{m-a_s-l_s-b_s+2}}+\frac{X_s}{1-t_{m-a_s-l_s-b_s+2}}\right)\cdots\left(\frac{1}{t_{m-b_s}}+\frac{X_s}{1-t_{m-b_s}}\right)$$

$$\times\frac{1}{t_{m-b_s+1}}\frac{1}{t_{m-b_s+2}}\cdots\frac{1}{t_m}\Bigg)dt_1\cdots dt_m.$$

For $0 < t_1, t_2, \dots, t_k < 1$, we consider the following integral.

$$\int_{t_1}^{t_k}\Bigg(\cdots\Bigg(\int_{t_1}^{t_5}\Bigg(\int_{t_1}^{t_4}\Bigg(\int_{t_1}^{t_3}\left(\frac{1}{t_2}+\frac{X}{1-t_2}\right)\left(\frac{1}{t_3}+\frac{X}{1-t_3}\right)\left(\frac{1}{t_4}+\frac{X}{1-t_4}\right)$$

$$\cdots\left(\frac{1}{t_{k-1}}+\frac{X}{1-t_{k-1}}\right)dt_2\Bigg)dt_3\Bigg)dt_4\Bigg)\cdots\Bigg)dt_{k-1}$$

$$=\int_{t_1}^{t_k}\Bigg(\cdots\Bigg(\int_{t_1}^{t_6}\Bigg(\int_{t_1}^{t_5}\Bigg(\int_{t_1}^{t_4}\left(\log\frac{t_3}{t_1}+X\log\frac{1-t_1}{1-t_3}\right)\left(\frac{1}{t_3}+\frac{X}{1-t_3}\right)\left(\frac{1}{t_4}+\frac{X}{1-t_4}\right)$$

$$\cdots\left(\frac{1}{t_{k-1}}+\frac{X}{1-t_{k-1}}\right)dt_3\Bigg)dt_4\Bigg)dt_5\Bigg)\cdots\Bigg)dt_{k-1}$$

$$=\frac{1}{2}\int_{t_1}^{t_k}\Bigg(\cdots\Bigg(\int_{t_1}^{t_7}\Bigg(\int_{t_1}^{t_6}\Bigg(\int_{t_1}^{t_5}\left(\log\frac{t_4}{t_1}+X\log\frac{1-t_1}{1-t_4}\right)^2\left(\frac{1}{t_4}+\frac{X}{1-t_4}\right)\left(\frac{1}{t_5}+\frac{X}{1-t_5}\right)$$

$$\cdots\left(\frac{1}{t_{k-1}}+\frac{X}{1-t_{k-1}}\right)dt_4\Big)dt_5\Big)dt_6\Big)\cdots\Big)dt_{k-1}$$

$$=\frac{1}{6}\int_{t_1}^{t_k}\Big(\cdots\Big(\int_{t_1}^{t_8}\Big(\int_{t_1}^{t_7}\Big(\int_{t_1}^{t_6}\left(\log\frac{t_5}{t_1}+X\log\frac{1-t_1}{1-t_5}\right)^3\left(\frac{1}{t_5}+\frac{X}{1-t_5}\right)\left(\frac{1}{t_6}+\frac{X}{1-t_6}\right)$$

$$\cdots\left(\frac{1}{t_{k-1}}+\frac{X}{1-t_{k-1}}\right)dt_5\Big)dt_6\Big)dt_7\Big)\cdots\Big)dt_{k-1}$$

$$=\ \cdots\ =\frac{1}{(k-2)!}\left(\log\frac{t_k}{t_1}+X\log\frac{1-t_1}{1-t_k}\right)^{k-2}.$$

By similar argument as above, for $0<t_1,t_2,\ldots,t_k\leq 1$ we have

$$\int_{t_1}^{t_k}\left(\cdots\left(\int_{t_1}^{t_5}\left(\int_{t_1}^{t_4}\left(\int_{t_1}^{t_3}\frac{1}{t_2}\frac{1}{t_3}\frac{1}{t_4}\cdots\frac{1}{t_{k-1}}dt_2\right)dt_3\right)dt_4\right)\cdots\right)dt_{k-1}$$

$$=\frac{1}{(k-2)!}\left(\log\frac{t_k}{t_1}\right)^{k-2}.$$

We use these arguments and Fubini's theorem, arrange t_i and put $t_{2s+1}=1$, then we can write the above generating function as follows.

$$\prod_{i=1}^{s}\frac{1}{(b_i-1)!(a_i+l_i-1)!}\int\cdots\int_{0<t_1<t_2<\cdots<t_{2s}<1}\Big(\frac{1}{1-t_1}\left(\log\left(\frac{t_2}{t_1}+X_1\frac{1-t_1}{1-t_2}\right)\right)^{a_1+l_1-1}$$

$$\times\frac{1}{t_2}\left(\log\frac{t_3}{t_2}\right)^{b_1-1}\Big)\cdots\cdots$$

$$\times\left(\frac{1}{1-t_{2s-1}}\left(\log\left(\frac{t_{2s}}{t_{2s-1}}+X_s\frac{1-t_{2s-1}}{1-t_{2s}}\right)\right)^{a_s+l_s-1}\frac{1}{t_{2s}}\left(\log\frac{1}{t_{2s}}\right)^{b_s-1}\right)dt_1\cdots dt_{2s}$$

$$=\left(\prod_{i=1}^{s}((b_i-1)!(a_i+l_i-1)!)\right)^{-1}\int\cdots\int_{0<t_1<t_2<\cdots<t_{2s}<1}$$

$$\left(\prod_{i=1}^{s}\left(\log\frac{t_{2i}}{t_{2i-1}}+X_i\log\frac{1-t_{2i-1}}{1-t_{2i}}\right)^{a_i+l_i-1}\left(\log\frac{t_{2i+1}}{t_{2i}}\right)^{b_i-1}\right)\frac{dt_1\,dt_2\,dt_3\cdots dt_{2s}}{(1-t_1)t_2(1-t_3)\cdots t_{2s}}.$$

Now we pick up the coefficient of $\prod_{i=1}^{s}X_i^{a_i-1}$, then we have

$$S_{\mathrm{k}}(a_1,\ldots,a_s;l_1,\ldots,l_s)$$

$$= \left(\prod_{i=1}^{s}(l_i!(a_i-1)!(b_i-1)!)\right)^{-1} \int\cdots\int_{0<t_1<t_2<\cdots<t_{2s}<1} \left(\prod_{i=1}^{s}\left(\log\frac{t_{2i}}{t_{2i-1}}\right)^{l_i}\right)$$

$$\times \prod_{i=1}^{s}\left(\left(\log\frac{1-t_{2i-1}}{1-t_{2i}}\right)^{a_i-1}\left(\log\frac{t_{2i+1}}{t_{2i}}\right)^{b_i-1}\right)\frac{dt_1\,dt_2\,dt_3\cdots dt_{2s}}{(1-t_1)t_2(1-t_3)\cdots t_{2s}}.$$

Since we put

$$Z(\mathsf{k};l) = \sum_{l_1+l_2+\cdots+l_s=l} S_{\mathsf{k}}(a_1,\ldots,a_s;l_1,\ldots,l_s),$$

we can write Z as follows

$$Z(\mathsf{k};l) = \sum_{l_1+l_2+\cdots+l_s=l}\left(\left(\prod_{i=1}^{s}(l_i!(a_i-1)!(b_i-1)!)\right)^{-1} \int\cdots\int_{0<t_1<t_2<\cdots<t_{2s}<1} \left(\prod_{i=1}^{s}\left(\log\frac{t_{2i}}{t_{2i-1}}\right)^{l_i}\right)\right.$$

$$\left.\times \prod_{i=1}^{s}\left(\left(\log\frac{1-t_{2i-1}}{1-t_{2i}}\right)^{a_i-1}\left(\log\frac{t_{2i+1}}{t_{2i}}\right)^{b_i-1}\right)\frac{dt_1\,dt_2\,dt_3\cdots dt_{2s}}{(1-t_1)t_2(1-t_3)\cdots t_{2s}}\right)$$

$$= \left(l!\prod_{i=1}^{s}((a_i-1)!(b_i-1)!)\right)^{-1} \int\cdots\int_{0<t_1<t_2<\cdots<t_{2s}<1} \left(\log\left(\prod_{i=1}^{s}\frac{t_{2i}}{t_{2i-1}}\right)\right)^{l}$$

$$\times \prod_{i=1}^{s}\left(\left(\log\frac{1-t_{2i-1}}{1-t_{2i}}\right)^{a_i-1}\left(\log\frac{t_{2i+1}}{t_{2i}}\right)^{b_i-1}\right)\frac{dt_1\,dt_2\,dt_3\cdots dt_{2s}}{(1-t_1)t_2(1-t_3)\cdots t_{2s}}.$$

Next, we prepare for change of the variables. We denote $t_{2s+1}=1$, and put

$$x_{2i-1} = \log\frac{1-t_{2i-1}}{1-t_{2i}} \quad\text{and}\quad x_{2i} = \log\frac{t_{2i+1}}{t_{2i}} \quad (\text{for } i=1,2,\ldots,s).$$

Note that for $i=1,2,\ldots,s$ we have

$$\begin{aligned}
t_{2i-1} &= 1-e^{x_{2i-1}}(1-e^{-x_{2i}}(1-e^{x_{2i+1}}(\cdots(1-e^{-x_{2s}}))\cdots) \\
&= 1+\sum_{j=2i-1}^{2s}\left((-1)^j\exp\left(\sum_{r=2i-1}^{j}(-1)^{r-1}x_r\right)\right), \\
t_{2i} &= e^{-x_{2i}}(1-e^{x_{2i+1}}(1-e^{-x_{2i+2}}(\cdots(1-e^{-x_{2s}}))\cdots) \\
&= \sum_{j=2i}^{2s}\left((-1)^j\exp\left(\sum_{r=2i}^{j}(-1)^{r-1}x_r\right)\right)
\end{aligned}$$

and

$$\frac{t_{2i}}{t_{2i+1}} = \exp\left(-x_{2i}\right).$$

We also have

$$\frac{\partial t_{2i-1}}{\partial x_{2i-1}} = (t_{2i-1} - 1), \qquad \frac{\partial t_{2i}}{\partial x_{2i}} = -t_{2i},$$

and for $i < j$ we have

$$\frac{\partial t_j}{\partial x_i} = 0.$$

So we have

$$\frac{dt_1\, dt_2\, dt_3 \cdots dt_{2s}}{(1-t_1)t_2(1-t_3)\cdots t_{2s}} = dx_1\, dx_2\, dx_3 \cdots dx_{2s}.$$

Next, we can calculate as follows.

$$\begin{aligned}
\prod_{i=1}^{s} \frac{t_{2i}}{t_{2i-1}} &= \frac{1}{t_1} \prod_{i=1}^{s} \frac{t_{2i}}{t_{2i+1}} = \frac{1}{t_1} \prod_{i=1}^{s} e^{-x_{2i}} \\
&= \frac{\exp\left(-\sum_{i=1}^{s} x_{2i}\right)}{1 + \sum_{j=1}^{2s} \left((-1)^j \exp\left(\sum_{r=1}^{j} (-1)^{r-1} x_r\right)\right)} \\
&= \left(\exp\left(\sum_{\substack{i=2 \\ i:\text{even}}}^{2s} x_i \right) + \sum_{j=1}^{2s} \left((-1)^j \exp\left(\sum_{\substack{r=1 \\ r:\text{odd}}}^{j} x_r + \sum_{\substack{r=j+1 \\ r:\text{even}}}^{2s} x_r \right)\right)\right)^{-1} \\
&= \left(\sum_{j=0}^{2s} \left((-1)^j \exp\left(\sum_{\substack{r=1 \\ r:\text{odd}}}^{j} x_r + \sum_{\substack{r=j+1 \\ r:\text{even}}}^{2s} x_r \right)\right)\right)^{-1}.
\end{aligned}$$

Hereafter we denote by $f(x_1, x_2, \ldots, x_{2s})$ the inverse of the right-hand side of above equality, namely we can write

$$\prod_{i=1}^{s} \frac{t_{2i}}{t_{2i-1}} = f(x_1, x_2, \ldots, x_{2s})^{-1}.$$

For $i = 1, 2, \ldots, s$, we also note that

$$\begin{aligned}
t_{2i-1} < t_{2i} \quad &\Leftrightarrow \quad x_{2i-1} = \log \frac{1 - t_{2i-1}}{1 - t_{2i}} > 0, \\
t_{2i} < t_{2i+1} \quad &\Leftrightarrow \quad x_{2i} = \log \frac{t_{2i+1}}{t_{2i}} > 0,
\end{aligned}$$

and $t_1 > 0$ means

$$f(x_1, x_2, \ldots, x_{2s}) = t_1 \prod_{i=1}^{s} e^{x_{2i}} > 0.$$

Now, we change the variables and rewrite $Z(\mathsf{k}; l)$ as

$$Z(\mathsf{k};l)=\left(l!\prod_{i=1}^{s}((a_i-1)!(b_i-1)!)\right)^{-1}\int\cdots\int_{\substack{x_i>0,\ 1\leq i\leq 2s,\\ f(x_1,x_2,\ldots,x_{2s})>0}}\left(\log\left(f(x_1,x_2,\ldots,x_{2s})^{-1}\right)\right)^{l}$$
$$\times\prod_{i=1}^{s}\left(x_{2i-1}^{a_i-1}x_{2i}^{b_i-1}\right)dx_1\,dx_2\cdots dx_{2s}.$$

Note that, $f(x_1,x_2,\ldots,x_{2s})$ has the following property.

$$\begin{aligned}
f(x_{2s},x_{2s-1},\ldots,x_1) &= \sum_{j=0}^{2s}\left((-1)^j\exp\left(\sum_{\substack{r=1\\ r:\text{odd}}}^{j}x_{2s-r+1}+\sum_{\substack{r=j+1\\ r:\text{even}}}^{2s}x_{2s-r+1}\right)\right)\\
&= \sum_{j=0}^{2s}\left((-1)^j\exp\left(\sum_{\substack{r=2s-j+1\\ r:\text{even}}}^{2s}x_r+\sum_{\substack{r=1\\ r:\text{odd}}}^{2s-j}x_r\right)\right)\\
&= \sum_{j=0}^{2s}\left((-1)^j\exp\left(\sum_{\substack{r=j+1\\ r:\text{even}}}^{2s}x_r+\sum_{\substack{r=1\\ r:\text{odd}}}^{j}x_r\right)\right)\\
&= f(x_1,x_2,\ldots,x_{2s}).
\end{aligned}$$

So we complete this proof with the following calculation.

$$\begin{aligned}
Z(\mathsf{k}';l) &= \left(l!\prod_{i=1}^{s}((a_i-1)!(b_i-1)!)\right)^{-1}\int\cdots\int_{\substack{x_i>0,\ 1\leq i\leq 2s,\\ f(x_1,x_2,\ldots,x_{2s})>0}}\left(\log\left(f(x_1,x_2,\ldots,x_{2s})^{-1}\right)\right)^{l}\\
&\qquad\times\prod_{i=1}^{s}\left(x_{2i-1}^{b_{s-i+1}-1}x_{2i}^{a_{s-i+1}-1}\right)dx_1\,dx_2\cdots dx_{2s}\\
&= \left(l!\prod_{i=1}^{s}((a_i-1)!(b_i-1)!)\right)^{-1}\int\cdots\int_{\substack{x_i>0,\ 1\leq i\leq 2s,\\ f(x_{2s},x_{2s-1},\ldots,x_1)>0}}\left(\log\left(f(x_{2s},x_{2s-1},\ldots,x_1)^{-1}\right)\right)^{l}\\
&\qquad\times\prod_{i=1}^{s}\left(x_{2s-2i+2}^{b_{s-i+1}-1}x_{2s-2i+1}^{a_{s-i+1}-1}\right)dx_1\,dx_2\cdots dx_{2s}\\
&= \left(l!\prod_{i=1}^{s}((a_i-1)!(b_i-1)!)\right)^{-1}\int\cdots\int_{\substack{x_i>0,\ 1\leq i\leq 2s,\\ f(x_1,x_2,\ldots,x_{2s})>0}}\left(\log\left(f(x_1,x_2,\ldots,x_{2s})^{-1}\right)\right)^{l}\\
&\qquad\times\prod_{i=1}^{s}\left(x_{2i}^{b_i-1}x_{2i-1}^{a_i-1}\right)dx_1\,dx_2\cdots dx_{2s}\\
&= Z(\mathsf{k};l).
\end{aligned}$$

Q.E.D.

3 Application

In this section, we shall give an application of main theorem. First, we review poly-Bernoulli numbers and related zeta functions following the paper of T. Arakawa and M. Kaneko[1] p.9.

Poly-Bernoulli numbers $B_n^{(k)}$ are a generalization of the classical Bernoulli numbers. They were defined by M. Kaneko as

$$\frac{Li_k(1-e^{-x})}{e^x-1}=\sum_{n=0}^{\infty}B_n^{(k)}\frac{x^n}{n!},$$

where, for any integer k, $Li_k(z)$ denotes the formal power series (for the k-th polylogarithm if $k\geq 1$ and a rational function if $k\leq 0$) $\sum_{m=0}^{\infty}\frac{z^m}{m^k}$. When $k=1$, $B_n^{(1)}$ is the usual Bernoulli number, and when $k\geq 1$, the left hand side of the equation of above definition can be written in the form of "iterated integrals" as follows.

$$\underbrace{\frac{1}{e^x-1}\int_0^x\frac{1}{e^t-1}\int_0^t\cdots\frac{1}{e^t-1}\int_0^t}_{(k-1)\text{-times}}\frac{t}{e^t-1}dt\;dt\;\cdots dt=\sum_{n=0}^{\infty}B_n^{(k)}\frac{x^n}{n!}.$$

M. Kaneko studied these values, gave an explicit formula of $B_n^{(k)}$ and also gave a theorem about its "duality".

Recently T. Arakawa and M. Kaneko[1] defined the following zeta function $\xi_k(s)$ for $k\geq 1$.

$$\xi_k(s)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}}{e^t-1}Li_k(1-e^{-t})dt.$$

They proved the integral converges for $Re(s)>0$ and when $k=1$, $\xi_1(s)$ is equal to $s\zeta(s+1)$. They also gave the following theorems.

Theorem 3. 1 (T. Arakawa and M. Kaneko[1]) (i) *The function $\xi_k(s)$ continues to an entire function of s, and the special values at non-positive integers are given by*

$$\xi_k(-m)=(-1)^m B_m^{(k)}\qquad(m=0,1,2,\ldots).$$

(ii) *The function* $\xi_k(s)$ *can be written in terms of the zeta functions* $\zeta(k_1, k_2, \dots, k_{n-1}; s)$ *as*

$$\xi_k(s) = (-1)^{k-1}\{\zeta(\underbrace{2, 1, \dots, 1}_{k-1}; s) + \zeta(\underbrace{1, 2, 1, \dots, 1}_{k-1}; s) + \cdots + \zeta(\underbrace{1, \dots, 1, 2}_{k-1}; s)$$

$$+ s \cdot \zeta(\underbrace{1, 1, \dots, 1}_{k-1}; s+1)\} + \sum_{j=0}^{k-2}(-1)^j \zeta(k-j) \cdot \zeta(\underbrace{1, 1, \dots, 1}_{j}; s),$$

where we define the single variable function by

$$\zeta(k_1, k_2, \dots, k_{n-1}; s) = \sum_{0<m_1<m_2<\cdots<m_n} \frac{1}{{m_1}^{k_1}{m_2}^{k_2}\cdots {m_{n-1}}^{k_{n-1}}{m_n}^{s}}.$$

For the special values of the zeta function at positive integral points, they got the following theorem.

Theorem 3. 2 (T. Arakawa and M. Kaneko[1]) (i) *For* $k \geq 1$ *and* $m \geq 0$,

$$\xi_k(m+1) = \sum_{\substack{a_1+a_2+\cdots+a_k=m \\ \forall a_j \geq 0}} (a_k+1)\zeta(a_1+1, a_2+1, \dots, a_{k-1}+1, a_k+2).$$

(ii) *If* k *is even and* $k \geq 2$, *then*

$$\xi_k(2) = \frac{1}{2}\sum_{i=0}^{k-2}(-1)^i \zeta(i+2)\zeta(k-i).$$

Applying our main theorem (Theorem 2.1) to (i) of Theorem 3.2, we get a relation between the special values of $\xi_k(s)$ and of $\zeta^*(k_1, \dots, k_n)$ as follows.

Theorem 3. 3 *For integers* $k \geq 1$ *and* $m \geq 1$, *we have*

$$\xi_k(m) = \zeta^*(\underbrace{1, 1, \dots, 1}_{m-1}, k+1).$$

Proof By using Theorem 3.2, for positive integers k and m we have

$$\xi_k(m) = \sum_{\substack{a_1+a_2+\cdots+a_k=m-1 \\ \forall a_j \geq 0}} (a_k+1)\zeta(a_1+1, a_2+1, \dots, a_{k-1}+1, a_k+2).$$

We can write

$$\xi_k(m) = \sum_{n=1}^{m} \sum_{\substack{a_1+a_2+\cdots+a_k=m-n \\ \forall a_j \geq 0}} \zeta(a_1+1, a_2+1, \ldots, a_{k-1}+1, a_k+n+1).$$

Now we use Theorem 2.1, then we have

$$\xi_k(m) = \sum_{n=1}^{m} \sum_{\substack{b_1+b_2+\cdots+b_n=m-n \\ \forall b_j \geq 0}} \zeta(b_1+1, b_2+1, \ldots, b_{n-1}+1, b_n+k+1),$$

so we get

$$\xi_k(m) = \zeta^*(\underbrace{1, 1, \ldots, 1}_{m-1}, k+1).$$

Q.E.D.

If we use the known result

$$\zeta(1, k-1) = \frac{k-1}{2}\zeta(k) - \frac{1}{2}\sum_{r=2}^{k-2}\zeta(r)\zeta(k-r)$$

(cf. [7][14]), we can get

$$\begin{aligned} \xi_k(2) &= \zeta^*(1, k+1) = \zeta(1, k+1) + \zeta(k+2) \\ &= \frac{k+3}{2}\zeta(k+2) - \frac{1}{2}\sum_{r=2}^{k}\zeta(r)\zeta(k-r+2) \end{aligned}$$

by Theorem 3.3. In case of k:even, we can check that it matchs Theorem 3.2 (ii).

References

[1] T. Arakawa and M. Kaneko, Multiple zeta values, poly-Bernoulli numbers, and related zeta functions, *preprint (KYUSHU-MPS-1996-20).*

[2] T. Arakawa and M. Kaneko, Multiple zeta values, poly-Bernoulli numbers, and related zeta functions (in Japanese), *Proceedings of the Symposium in Tsudajyuku Univ.*, **2** (1996), 133-144.

[3] V. I. Arnold, The Vassiliev theory of discriminantsand knots, in ECM volume, *Progress in Math.*, **119** (1994), 3-29.

[4] J. M. Borwein and D. M. Bradley, Evaluations of k-fold Euler/Zagier sums: a compendium of results for arbitrary k, *preprint.*

[5] D. Borwein, J. M. Borwein and R. Girgensohn, Explicit evaluation of Euler sums, *Proc. Edin. Math. Soc.*, **38** (1995), 277-294.

[6] M. Hoffman, Multiple harmonic series, *Pacific J. Math.*, **152** (1992), 275-290.

[7] J. G. Huard, K. S. Williams and Zhang Nan-Yue, On Tornheim's double series, *Acta Arithmetica*, **75-2** (1996), 105-117.

[8] C. Jordan, *Calculus of Finite Differences,* Chelsea Publ. Co., New York, 1950.

[9] M. Kaneko, Poly-Bernoulli numbers, *J. de Théorie des Nombres de Bordeaux*, **9** (1997), 221-228.

[10] T. Q. T. Le and J. Murakami, Kontsevich's integral for the Homfly polynomial and relations between values of multiple zeta functions, *Topology and its Applications*, **62** (1995), 193-206.

[11] L. Lewin, *Polylogarithms and Associated Functions,* Tata Institute, Bombey, 1980.

[12] L. Tornheim, Harmonic double series, *Amer. J. Math.*, **72** (1950), 303-314.

[13] C. L. Siegel, *Advanced Analytic Number Theory,* Chelsea Publ. Co., New York, 1950.

[14] D. Zagier, Values of zeta functions and their applications, in ECM volume, *Progress in Math.*, **120** (1994), 497-512.

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