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*Doctoral Thesis*

# A Generalization of the Duality and the Sum formula on the Multiple Zeta Values

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Department of Mathematics, Graduate School of Science  
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## Abstract

In this paper we present a relation among the multiple zeta values which generalizes simultaneously the “Sum formula” and the “duality” theorem. As an application, we give a formula for the special values at positive integral points of a certain zeta function of Arakawa-Kaneko in terms of multiple harmonic series.

## Introduction

The multiple zeta values (or Euler-Zagier sums ) seem to be related to many kind of mathematical subjects. Recently A. Graville, D. Zagier and others proved a conjecture known as the “Sum formula” (or “Sum conjecture”) (Theorem 1.3 ) which gives a remarkable relation between the multiple zeta values and special values of Riemann zeta function. In this note we prove a generalization of the “Sum formula” which is at the same time a generalization of another remarkable identity referred to as the “duality” theorem (Theorem 1.2 ).

The multiple zeta values are defined for integers  $k_1, \dots, k_{n-1} \geq 1$  and  $k_n \geq 2$  by

$$\zeta(k_1, k_2, \dots, k_n) = \sum_{0 < m_1 < m_2 < \dots < m_n} \frac{1}{m_1^{k_1} m_2^{k_2} \dots m_n^{k_n}}.$$

In this paper we shall prove the following theorem.

For any index set  $(k_1, k_2, \dots, k_n)$  satisfying the condition above and any integer  $l \geq 0$ , we define

$$Z(k_1, k_2, \dots, k_n; l) = \sum_{\substack{c_1 + c_2 + \dots + c_n = l \\ \forall c_j \geq 0}} \zeta(k_1 + c_1, k_2 + c_2, \dots, k_n + c_n).$$

For any integer  $s \geq 1$  and  $a_1, b_1, a_2, b_2, \dots, a_s, b_s \geq 1$ , we define two index sets which are “dual” to each other by

$$k = (\underbrace{1, \dots, 1}_{a_1 - 1}, b_1 + 1, \underbrace{1, \dots, 1}_{a_2 - 1}, b_2 + 1, \dots, \underbrace{1, \dots, 1}_{a_s - 1}, b_s + 1)$$

and

$$k' = (\underbrace{1, \dots, 1}_{b_s = 1}, a_s + 1, \underbrace{1, \dots, 1}_{b_{s-1} = 1}, a_{s-1} + 1, \dots, \underbrace{1, \dots, 1}_{b_1 = 1}, a_1 + 1).$$

Our main theorem is then the following.

**Theorem 2.1**

$$Z(k'; l) = Z(k; l).$$

Note that, if we put  $s = a_1 = 1$  in above, then  $\zeta(k)$  is a Riemann zeta value, and the identity above is nothing but the “Sum formula”. We also note that, if we put  $l = 0$ , then the above theorem gives  $\zeta(k') = \zeta(k)$ , the duality theorem (Theorem 1.2).

On the other hand, poly-Bernoulli numbers were defined by M. Kaneko[9] using the polylogarithms. Recently T. Arakawa and M. Kaneko[1] defined a new function  $\xi_k(s)$  which has poly-Bernoulli numbers as the special values of non-positive integral points. They gave some expressions of the function by using the multiple zeta values. (We shall explain some of their results in section 3.)

As an application of the main theorem, we present another theorem that the special values at positive integral points of the zeta function  $\xi_k(s)$  are also the special values of a certain multiple harmonic series. Namely, we can state the theorem as follows.

### Theorem 3.3

For integers  $k \geq 1$  and  $n \geq 1$ , we have

$$\xi_k(n) = \sum_{0 < m_1 \leq m_2 \leq \dots \leq m_n} \frac{1}{m_1 m_2 \dots m_{n-1} m_n^{k+1}}.$$

In section 1, we shall review the definition and some properties of the multiple harmonic series (including the multiple zeta values). We shall prove our main theorem in section 2. In section 3, we shall review the zeta function related to poly-Bernoulli numbers defined by T. Arakawa and M. Kaneko[1], and show the relation to multiple harmonic series.

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## 1 Multiple Harmonic Series

In this section, we shall review the definition and some properties of the multiple harmonic series (including the multiple zeta values).

For integers  $n \geq 1$ ,  $k_1, k_2, \dots, k_{n-1} \geq 1$  and  $k_n \geq 2$ , we define  $\zeta(k_1, k_2, \dots, k_n)$  and  $\zeta^*(k_1, k_2, \dots, k_n)$  as follows.

$$\begin{aligned} \zeta(k_1, k_2, \dots, k_n) &= \sum_{0 < m_1 < m_2 < \dots < m_n} \frac{1}{m_1^{k_1} m_2^{k_2} \dots m_n^{k_n}}, \\ \zeta^*(k_1, k_2, \dots, k_n) &= \sum_{0 < m_1 \leq m_2 \leq \dots \leq m_n} \frac{1}{m_1^{k_1} m_2^{k_2} \dots m_n^{k_n}}. \end{aligned}$$

Note that

$$\zeta(k_1, k_2, \dots, k_n) = A(k_n, k_{n-1}, \dots, k_1) \text{ and } \zeta^*(k_1, k_2, \dots, k_n) = S(k_n, k_{n-1}, \dots, k_1)$$

in Hoffman's notation[6].

It is known that we have relations as

$$\begin{aligned}\zeta^*(k_1, k_2) &= \zeta(k_1, k_2) + \zeta(k_1 + k_2), \\ \zeta^*(k_1, k_2, k_3) &= \zeta(k_1, k_2, k_3) + \zeta(k_1 + k_2, k_3) + \zeta(k_1, k_2 + k_3) + \zeta(k_1 + k_2 + k_3),\end{aligned}$$

and

$$\begin{aligned}\zeta^*(\underbrace{1, 1, \dots, 1}_{m-1}, k+1) \\ = \sum_{n=1}^m \sum_{\substack{b_1+b_2+\dots+b_n=m-n \\ \forall b_j \geq 0}} \zeta(b_1+1, b_2+1, \dots, b_{n-1}+1, b_n+k+1),\end{aligned}$$

(cf.[6]).

Hoffman[6] studied these series. Following theorem is one of his results.

**Theorem 1.1 (Hoffman[6])** *For any integers  $k \geq 1$ ,  $i_1, i_2, \dots, i_{k-1} \geq 1$  and  $i_k \geq 2$ , we have*

$$\begin{aligned}\sum_{\substack{a_1+a_2+\dots+a_k=1 \\ \forall a_j \geq 0}} \zeta(a_1+i_1, a_2+i_2, \dots, a_k+i_k) \\ = \sum_{\substack{1 \leq l \leq k \\ i_l \geq 2}} \sum_{j=0}^{i_l-2} \zeta(i_1, \dots, i_{l-1}, j+1, i_l-j, i_{l+1}, \dots, i_k).\end{aligned}$$

Next, we review two interesting properties of the multiple zeta values. One is called “duality” and another is called “Sum formula” of the multiple zeta values.

First, we review the definition of “Drinfel'd integral ” following Zagier[14]. For  $\varepsilon_1 = 1, \varepsilon_k = 0$  and  $\varepsilon_2, \dots, \varepsilon_{k-1} \in \{0, 1\}$ , we define

$$I(\varepsilon_1, \dots, \varepsilon_k) = \int \cdots \int_{0 < t_1 < \cdots < t_k < 1} \frac{dt_1}{A_{\varepsilon_1}(t_1)} \cdots \frac{dt_k}{A_{\varepsilon_k}(t_k)},$$

where we denote  $A_0(t) = t$  and  $A_1(t) = 1 - t$ . It is known that there is an identity between the multiple zeta values and “Drinfel’d integral”, namely we have (cf.[14])

$$\zeta(k_1, \dots, k_n) = I(1, \underbrace{0, \dots, 0}_{k_1 - 1}, 1, \underbrace{0, \dots, 0}_{k_2 - 1}, \dots, 1, \underbrace{0, \dots, 0}_{k_n - 1}).$$

We also know

$$I(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k) = I(1 - \varepsilon_k, 1 - \varepsilon_{k-1}, \dots, 1 - \varepsilon_1),$$

(cf.[14]).

For any integer  $s \geq 1$  and  $a_1, b_1, a_2, b_2, \dots, a_s, b_s \geq 1$ , we define two index sets which are “dual” to each other by

$$k = (\underbrace{1, \dots, 1}_{a_1 - 1}, b_1 + 1, \underbrace{1, \dots, 1}_{a_2 - 1}, b_2 + 1, \dots, \underbrace{1, \dots, 1}_{a_s - 1}, b_s + 1)$$

and

$$k' = (\underbrace{1, \dots, 1}_{b_s - 1}, a_s + 1, \underbrace{1, \dots, 1}_{b_{s-1} - 1}, a_{s-1} + 1, \dots, \underbrace{1, \dots, 1}_{b_1 - 1}, a_1 + 1).$$

Then the following theorem is known.

**Theorem 1.2** ( “Duality” cf. [2][14]) *For any index set  $k$  and its dual index set  $k'$  we have*

$$\zeta(k') = \zeta(k).$$

Next, we state the theorem called “Sum formula” conjectured by C. Moen and M. Hoffman, and proved by A. Granville, D. Zagier and others.

**Theorem 1.3** ( “Sum formula” cf. [2][6]) *For  $0 < n < k$  we have*

$$\sum_{\substack{k_1, k_2, \dots, k_{n-1} \geq 1, k_n \geq 2, \\ k_1 + k_2 + \dots + k_n = k}} \zeta(k_1, k_2, \dots, k_n) = \zeta(k).$$



## 2 Main Theorem

In this section we shall give our main theorem.

For any index set  $k = (k_1, k_2, \dots, k_n)$  and for any integer  $l \geq 0$ , we define  $Z$  as

$$Z(k; l) = \sum_{\substack{c_1 + c_2 + \dots + c_n = l \\ \forall c_j \geq 0}} \zeta(k_1 + c_1, k_2 + c_2, \dots, k_n + c_n).$$

Now, we state our main theorem.

**Theorem 2.1** *For any index set  $k$  and its dual index set  $k'$  and for any integer  $l \geq 0$ , we have*

$$Z(k'; l) = Z(k; l).$$

**Remark** Note that, if we put  $s = a_1 = 1$ , then  $\zeta(k)$  is a Riemann zeta value, and the identity above is nothing but the “Sum formula” ( Theorem 1.3 ). We also note that, if we put  $l = 0$ , then the above theorem gives  $\zeta(k') = \zeta(k)$ , the duality theorem ( Theorem 1.2 ). Theorem 2.1 also contains Theorem 1.1 ( Theorem 5.1 in M. Hoffman[6] ) as a special case when  $l = 1$ .

*Proof* We fix an index set

$$k = (\underbrace{1, \dots, 1}_{a_1 - 1}, b_1 + 1, \underbrace{1, \dots, 1}_{a_2 - 1}, b_2 + 1, \dots, \underbrace{1, \dots, 1}_{a_s - 1}, b_s + 1).$$

Using “Drinfel’d integral” (we reviewed in section 1), for integers  $l_i \geq 0$  satisfying  $l_1 + \dots + l_s = l$ , and for integers  $d_i$  satisfying  $1 \leq d_i \leq a_i + l_i$  for  $i = 1, \dots, s$ , we put  $S_k$  as follows.

$$S_k(d_1, \dots, d_s; l_1, \dots, l_s) = \sum_{\substack{\varepsilon_{i,2} + \dots + \varepsilon_{i,a_i+l_i} = d_i - 1 \\ \varepsilon_{i,2}, \dots, \varepsilon_{i,a_i+l_i} \in \{0,1\} \text{ for } \forall i}} I(1, \varepsilon_{1,2}, \dots, \varepsilon_{1,a_1+l_1}, \underbrace{0, \dots, 0}_{b_1}, \\ 1, \varepsilon_{2,2}, \dots, \varepsilon_{2,a_2+l_2}, \underbrace{0, \dots, 0}_{b_2}, \dots, 1, \varepsilon_{s,2}, \dots, \varepsilon_{s,a_s+l_s}, \underbrace{0, \dots, 0}_{b_s}).$$

Then we have

$$Z(k; l) = \sum_{\substack{l_1 + l_2 + \dots + l_s = l \\ l_i \geq 0 \text{ for } \forall i}} S_k(a_1, \dots, a_s; l_1, \dots, l_s).$$

We put  $m = l + \sum_{i=1}^s a_i + b_i$ . If we fix the values  $l_i \geq 0$  for  $i = 1, \dots, s$  such that  $l_1 + \dots + l_s = l$ , and make a generating function of  $S_k$ , we have

$$\begin{aligned}
& \sum_{1 \leq d_i \leq a_i + l_i \text{ for } \forall i} \left( S_k(d_1, \dots, d_s; l_1, \dots, l_s) \prod_{j=1}^s X_j^{d_j-1} \right) \\
&= \sum_{\substack{\varepsilon_{i,2}, \dots, \varepsilon_{i,a_i+l_i} \in \{0,1\} \text{ for } \forall i}} \left( I(1, \varepsilon_{1,2}, \dots, \varepsilon_{1,a_1+l_1}, \underbrace{0, \dots, 0}_{b_1}, \right. \\
&\quad \left. \dots, 1, \varepsilon_{s,2}, \dots, \varepsilon_{s,a_s+l_s}, \underbrace{0, \dots, 0}_{b_s} \right) \prod_{j=1}^s X_j^{\varepsilon_{j,2} + \dots + \varepsilon_{j,a_j+l_j}} \\
&= \int \dots \int_{0 < t_1 < \dots < t_m < 1} \left( \frac{1}{1-t_1} \left( \frac{1}{t_2} + \frac{X_1}{1-t_2} \right) \dots \left( \frac{1}{t_{a_1+l_1}} + \frac{X_1}{1-t_{a_1+l_1}} \right) \right. \\
&\quad \left. \times \frac{1}{t_{a_1+l_1+1}} \frac{1}{t_{a_1+l_1+2}} \dots \frac{1}{t_{a_1+l_1+b_1}} \right) \dots \dots \\
&\times \left( \frac{1}{1-t_{m-a_s-l_s-b_s+1}} \left( \frac{1}{t_{m-a_s-l_s-b_s+2}} + \frac{X_s}{1-t_{m-a_s-l_s-b_s+2}} \right) \dots \left( \frac{1}{t_{m-b_s}} + \frac{X_s}{1-t_{m-b_s}} \right) \right. \\
&\quad \left. \times \frac{1}{t_{m-b_s+1}} \frac{1}{t_{m-b_s+2}} \dots \frac{1}{t_m} \right) dt_1 \dots dt_m.
\end{aligned}$$

For  $0 < t_1, t_2, \dots, t_k < 1$ , we consider the following integral.

$$\begin{aligned}
& \int_{t_1}^{t_k} \left( \dots \left( \int_{t_1}^{t_5} \left( \int_{t_1}^{t_4} \left( \int_{t_1}^{t_3} \left( \frac{1}{t_2} + \frac{X}{1-t_2} \right) \left( \frac{1}{t_3} + \frac{X}{1-t_3} \right) \left( \frac{1}{t_4} + \frac{X}{1-t_4} \right) \right. \right. \right. \right. \right. \\
&\quad \left. \left. \left. \left. \dots \left( \frac{1}{t_{k-1}} + \frac{X}{1-t_{k-1}} \right) dt_2 \right) dt_3 \right) dt_4 \right) \dots \right) dt_{k-1} \\
&= \int_{t_1}^{t_k} \left( \dots \left( \int_{t_1}^{t_6} \left( \int_{t_1}^{t_5} \left( \int_{t_1}^{t_4} \left( \log \frac{t_3}{t_1} + X \log \frac{1-t_1}{1-t_3} \right) \left( \frac{1}{t_3} + \frac{X}{1-t_3} \right) \left( \frac{1}{t_4} + \frac{X}{1-t_4} \right) \right. \right. \right. \right. \right. \\
&\quad \left. \left. \left. \left. \dots \left( \frac{1}{t_{k-1}} + \frac{X}{1-t_{k-1}} \right) dt_3 \right) dt_4 \right) dt_5 \right) \dots \right) dt_{k-1} \\
&= \frac{1}{2} \int_{t_1}^{t_k} \left( \dots \left( \int_{t_1}^{t_7} \left( \int_{t_1}^{t_6} \left( \int_{t_1}^{t_5} \left( \log \frac{t_4}{t_1} + X \log \frac{1-t_1}{1-t_4} \right)^2 \left( \frac{1}{t_4} + \frac{X}{1-t_4} \right) \left( \frac{1}{t_5} + \frac{X}{1-t_5} \right) \right. \right. \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \cdots \left( \frac{1}{t_{k-1}} + \frac{X}{1-t_{k-1}} \right) dt_4 \Big) dt_5 \Big) dt_6 \Big) \cdots \Big) dt_{k-1} \\
&= \frac{1}{6} \int_{t_1}^{t_k} \left( \cdots \left( \int_{t_1}^{t_8} \left( \int_{t_1}^{t_7} \left( \int_{t_1}^{t_6} \left( \log \frac{t_5}{t_1} + X \log \frac{1-t_1}{1-t_5} \right)^3 \left( \frac{1}{t_5} + \frac{X}{1-t_5} \right) \left( \frac{1}{t_6} + \frac{X}{1-t_6} \right) \right. \right. \right. \right. \right. \\
&\quad \left. \left. \left. \left. \cdots \left( \frac{1}{t_{k-1}} + \frac{X}{1-t_{k-1}} \right) dt_5 \right) dt_6 \right) dt_7 \right) \cdots \right) dt_{k-1} \\
&= \cdots = \frac{1}{(k-2)!} \left( \log \frac{t_k}{t_1} + X \log \frac{1-t_1}{1-t_k} \right)^{k-2}.
\end{aligned}$$

By similar argument as above, for  $0 < t_1, t_2, \dots, t_k \leq 1$  we have

$$\begin{aligned}
& \int_{t_1}^{t_k} \left( \cdots \left( \int_{t_1}^{t_5} \left( \int_{t_1}^{t_4} \left( \int_{t_1}^{t_3} \frac{1}{t_2} \frac{1}{t_3} \frac{1}{t_4} \cdots \frac{1}{t_{k-1}} dt_2 \right) dt_3 \right) dt_4 \right) \cdots \right) dt_{k-1} \\
&= \frac{1}{(k-2)!} \left( \log \frac{t_k}{t_1} \right)^{k-2}.
\end{aligned}$$

We use these arguments and Fubini's theorem, arrange  $t_i$  and put  $t_{2s+1} = 1$ , then we can write the above generating function as follows.

$$\begin{aligned}
& \prod_{i=1}^s \frac{1}{(b_i-1)!(a_i+l_i-1)!} \int \cdots \int_{0 < t_1 < t_2 < \cdots < t_{2s} < 1} \left( \frac{1}{1-t_1} \left( \log \left( \frac{t_2}{t_1} + X_1 \frac{1-t_1}{1-t_2} \right) \right)^{a_1+l_1-1} \right. \\
&\quad \left. \times \frac{1}{t_2} \left( \log \frac{t_3}{t_2} \right)^{b_1-1} \right) \cdots \cdots \\
&\quad \times \left( \frac{1}{1-t_{2s-1}} \left( \log \left( \frac{t_{2s}}{t_{2s-1}} + X_s \frac{1-t_{2s-1}}{1-t_{2s}} \right) \right)^{a_s+l_s-1} \frac{1}{t_{2s}} \left( \log \frac{1}{t_{2s}} \right)^{b_s-1} \right) dt_1 \cdots dt_{2s} \\
&= \left( \prod_{i=1}^s ((b_i-1)!(a_i+l_i-1)!) \right)^{-1} \int \cdots \int_{0 < t_1 < t_2 < \cdots < t_{2s} < 1} \\
&\quad \left( \prod_{i=1}^s \left( \log \frac{t_{2i}}{t_{2i-1}} + X_i \log \frac{1-t_{2i-1}}{1-t_{2i}} \right)^{a_i+l_i-1} \left( \log \frac{t_{2i+1}}{t_{2i}} \right)^{b_i-1} \right) \frac{dt_1 dt_2 dt_3 \cdots dt_{2s}}{(1-t_1)t_2(1-t_3) \cdots t_{2s}}.
\end{aligned}$$

Now we pick up the coefficient of  $\prod_{i=1}^s X_i^{a_i-1}$ , then we have

$$S_k(a_1, \dots, a_s; l_1, \dots, l_s)$$

$$\begin{aligned}
&= \left( \prod_{i=1}^s (l_i! (a_i - 1)! (b_i - 1)!) \right)^{-1} \int \cdots \int_{0 < t_1 < t_2 < \cdots < t_{2s} < 1} \left( \prod_{i=1}^s \left( \log \frac{t_{2i}}{t_{2i-1}} \right)^{l_i} \right) \\
&\quad \times \prod_{i=1}^s \left( \left( \log \frac{1 - t_{2i-1}}{1 - t_{2i}} \right)^{a_i-1} \left( \log \frac{t_{2i+1}}{t_{2i}} \right)^{b_i-1} \right) \frac{dt_1 dt_2 dt_3 \cdots dt_{2s}}{(1 - t_1) t_2 (1 - t_3) \cdots t_{2s}}.
\end{aligned}$$

Since we put

$$Z(k; l) = \sum_{l_1 + l_2 + \cdots + l_s = l} S_k(a_1, \dots, a_s; l_1, \dots, l_s),$$

we can write  $Z$  as follows

$$\begin{aligned}
Z(k; l) &= \sum_{l_1 + l_2 + \cdots + l_s = l} \left( \left( \prod_{i=1}^s (l_i! (a_i - 1)! (b_i - 1)!) \right)^{-1} \int \cdots \int_{0 < t_1 < t_2 < \cdots < t_{2s} < 1} \left( \prod_{i=1}^s \left( \log \frac{t_{2i}}{t_{2i-1}} \right)^{l_i} \right) \right. \\
&\quad \times \prod_{i=1}^s \left( \left( \log \frac{1 - t_{2i-1}}{1 - t_{2i}} \right)^{a_i-1} \left( \log \frac{t_{2i+1}}{t_{2i}} \right)^{b_i-1} \right) \frac{dt_1 dt_2 dt_3 \cdots dt_{2s}}{(1 - t_1) t_2 (1 - t_3) \cdots t_{2s}} \Big) \\
&= \left( l! \prod_{i=1}^s ((a_i - 1)! (b_i - 1)!) \right)^{-1} \int \cdots \int_{0 < t_1 < t_2 < \cdots < t_{2s} < 1} \left( \log \left( \prod_{i=1}^s \frac{t_{2i}}{t_{2i-1}} \right) \right)^l \\
&\quad \times \prod_{i=1}^s \left( \left( \log \frac{1 - t_{2i-1}}{1 - t_{2i}} \right)^{a_i-1} \left( \log \frac{t_{2i+1}}{t_{2i}} \right)^{b_i-1} \right) \frac{dt_1 dt_2 dt_3 \cdots dt_{2s}}{(1 - t_1) t_2 (1 - t_3) \cdots t_{2s}}.
\end{aligned}$$

Next, we prepare for change of the variables. We denote  $t_{2s+1} = 1$ , and put

$$x_{2i-1} = \log \frac{1 - t_{2i-1}}{1 - t_{2i}} \quad \text{and} \quad x_{2i} = \log \frac{t_{2i+1}}{t_{2i}} \quad (\text{for } i = 1, 2, \dots, s).$$

Note that for  $i = 1, 2, \dots, s$  we have

$$\begin{aligned}
t_{2i-1} &= 1 - e^{x_{2i-1}} (1 - e^{-x_{2i}} (1 - e^{x_{2i+1}} (\cdots (1 - e^{-x_{2s}})) \cdots)) \\
&= 1 + \sum_{j=2i-1}^{2s} \left( (-1)^j \exp \left( \sum_{r=2i-1}^j (-1)^{r-1} x_r \right) \right), \\
t_{2i} &= e^{-x_{2i}} (1 - e^{x_{2i+1}} (1 - e^{-x_{2i+2}} (\cdots (1 - e^{-x_{2s}})) \cdots)) \\
&= \sum_{j=2i}^{2s} \left( (-1)^j \exp \left( \sum_{r=2i}^j (-1)^{r-1} x_r \right) \right)
\end{aligned}$$

and

$$\frac{t_{2i}}{t_{2i+1}} = \exp(-x_{2i}).$$

We also have

$$\frac{\partial t_{2i-1}}{\partial x_{2i-1}} = (t_{2i-1} - 1), \quad \frac{\partial t_{2i}}{\partial x_{2i}} = -t_{2i},$$

and for  $i < j$  we have

$$\frac{\partial t_j}{\partial x_i} = 0.$$

So we have

$$\frac{dt_1 dt_2 dt_3 \cdots dt_{2s}}{(1-t_1)t_2(1-t_3)\cdots t_{2s}} = dx_1 dx_2 dx_3 \cdots dx_{2s}.$$

Next, we can calculate as follows.

$$\begin{aligned} \prod_{i=1}^s \frac{t_{2i}}{t_{2i-1}} &= \frac{1}{t_1} \prod_{i=1}^s \frac{t_{2i}}{t_{2i+1}} = \frac{1}{t_1} \prod_{i=1}^s e^{-x_{2i}} \\ &= \frac{\exp(-\sum_{i=1}^s x_{2i})}{1 + \sum_{j=1}^{2s} \left( (-1)^j \exp\left(\sum_{r=1}^j (-1)^{r-1} x_r\right) \right)} \\ &= \left( \exp\left(\sum_{\substack{i=2 \\ i:\text{even}}}^{2s} x_i\right) + \sum_{j=1}^{2s} \left( (-1)^j \exp\left(\sum_{\substack{r=1 \\ r:\text{odd}}}^j x_r + \sum_{\substack{r=j+1 \\ r:\text{even}}}^{2s} x_r\right) \right) \right)^{-1} \\ &= \left( \sum_{j=0}^{2s} \left( (-1)^j \exp\left(\sum_{\substack{r=1 \\ r:\text{odd}}}^j x_r + \sum_{\substack{r=j+1 \\ r:\text{even}}}^{2s} x_r\right) \right) \right)^{-1}. \end{aligned}$$

Hereafter we denote by  $f(x_1, x_2, \dots, x_{2s})$  the inverse of the right-hand side of above equality, namely we can write

$$\prod_{i=1}^s \frac{t_{2i}}{t_{2i-1}} = f(x_1, x_2, \dots, x_{2s})^{-1}.$$

For  $i = 1, 2, \dots, s$ , we also note that

$$\begin{aligned} t_{2i-1} < t_{2i} &\Leftrightarrow x_{2i-1} = \log \frac{1-t_{2i-1}}{1-t_{2i}} > 0, \\ t_{2i} < t_{2i+1} &\Leftrightarrow x_{2i} = \log \frac{t_{2i+1}}{t_{2i}} > 0, \end{aligned}$$

and  $t_1 > 0$  means

$$f(x_1, x_2, \dots, x_{2s}) = t_1 \prod_{i=1}^s e^{x_{2i}} > 0.$$

Now, we change the variables and rewrite  $Z(k; l)$  as

$$\begin{aligned}
Z(k; l) &= \left( l! \prod_{i=1}^s ((a_i - 1)!(b_i - 1)!) \right)^{-1} \int \cdots \int_{\substack{x_i > 0, \ 1 \leq i \leq 2s, \\ f(x_1, x_2, \dots, x_{2s}) > 0}} \left( \log \left( f(x_1, x_2, \dots, x_{2s})^{-1} \right) \right)^l \\
&\quad \times \prod_{i=1}^s \left( x_{2i-1}^{a_i-1} x_{2i}^{b_i-1} \right) dx_1 dx_2 \cdots dx_{2s}.
\end{aligned}$$

Note that,  $f(x_1, x_2, \dots, x_{2s})$  has the following property.

$$\begin{aligned}
f(x_{2s}, x_{2s-1}, \dots, x_1) &= \sum_{j=0}^{2s} \left( (-1)^j \exp \left( \sum_{\substack{r=1 \\ r:\text{odd}}}^j x_{2s-r+1} + \sum_{\substack{r=j+1 \\ r:\text{even}}}^{2s} x_{2s-r+1} \right) \right) \\
&= \sum_{j=0}^{2s} \left( (-1)^j \exp \left( \sum_{\substack{r=2s-j+1 \\ r:\text{even}}}^{2s} x_r + \sum_{\substack{r=1 \\ r:\text{odd}}}^{2s-j} x_r \right) \right) \\
&= \sum_{j=0}^{2s} \left( (-1)^j \exp \left( \sum_{\substack{r=j+1 \\ r:\text{even}}}^{2s} x_r + \sum_{\substack{r=1 \\ r:\text{odd}}}^j x_r \right) \right) \\
&= f(x_1, x_2, \dots, x_{2s}).
\end{aligned}$$

So we complete this proof with the following calculation.

$$\begin{aligned}
Z(k'; l) &= \left( l! \prod_{i=1}^s ((a_i - 1)!(b_i - 1)!) \right)^{-1} \int \cdots \int_{\substack{x_i > 0, \ 1 \leq i \leq 2s, \\ f(x_1, x_2, \dots, x_{2s}) > 0}} \left( \log \left( f(x_1, x_2, \dots, x_{2s})^{-1} \right) \right)^l \\
&\quad \times \prod_{i=1}^s \left( x_{2i-1}^{b_{s-i+1}-1} x_{2i}^{a_{s-i+1}-1} \right) dx_1 dx_2 \cdots dx_{2s} \\
&= \left( l! \prod_{i=1}^s ((a_i - 1)!(b_i - 1)!) \right)^{-1} \int \cdots \int_{\substack{x_i > 0, \ 1 \leq i \leq 2s, \\ f(x_{2s}, x_{2s-1}, \dots, x_1) > 0}} \left( \log \left( f(x_{2s}, x_{2s-1}, \dots, x_1)^{-1} \right) \right)^l \\
&\quad \times \prod_{i=1}^s \left( x_{2s-2i+2}^{b_{s-i+1}-1} x_{2s-2i+1}^{a_{s-i+1}-1} \right) dx_1 dx_2 \cdots dx_{2s} \\
&= \left( l! \prod_{i=1}^s ((a_i - 1)!(b_i - 1)!) \right)^{-1} \int \cdots \int_{\substack{x_i > 0, \ 1 \leq i \leq 2s, \\ f(x_1, x_2, \dots, x_{2s}) > 0}} \left( \log \left( f(x_1, x_2, \dots, x_{2s})^{-1} \right) \right)^l \\
&\quad \times \prod_{i=1}^s \left( x_{2i}^{b_i-1} x_{2i-1}^{a_i-1} \right) dx_1 dx_2 \cdots dx_{2s} \\
&= Z(k; l).
\end{aligned}$$

Q.E.D.

### 3 Application

In this section, we shall give an application of main theorem. First, we review poly-Bernoulli numbers and related zeta functions following the paper of T. Arakawa and M. Kaneko[1] p.9.

Poly-Bernoulli numbers  $B_n^{(k)}$  are a generalization of the classical Bernoulli numbers. They were defined by M. Kaneko as

$$\frac{Li_k(1 - e^{-x})}{e^x - 1} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!},$$

where, for any integer  $k$ ,  $Li_k(z)$  denotes the formal power series (for the  $k$ -th polylogarithm if  $k \geq 1$  and a rational function if  $k \leq 0$ )  $\sum_{m=0}^{\infty} \frac{z^m}{m^k}$ . When  $k = 1$ ,  $B_n^{(1)}$  is the usual Bernoulli number, and when  $k \geq 1$ , the left hand side of the equation of above definition can be written in the form of “iterated integrals” as follows.

$$\underbrace{\frac{1}{e^x - 1} \int_0^x \frac{1}{e^t - 1} \int_0^t \cdots \frac{1}{e^t - 1} \int_0^t \frac{t}{e^t - 1} dt \, dt \, \cdots \, dt}_{(k-1)\text{-times}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}.$$

M. Kaneko studied these values, gave an explicit formula of  $B_n^{(k)}$  and also gave a theorem about its “duality”.

Recently T. Arakawa and M. Kaneko[1] defined the following zeta function  $\xi_k(s)$  for  $k \geq 1$ .

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} Li_k(1 - e^{-t}) dt.$$

They proved the integral converges for  $Re(s) > 0$  and when  $k = 1$ ,  $\xi_1(s)$  is equal to  $s\zeta(s+1)$ . They also gave the following theorems.

**Theorem 3.1** (T. Arakawa and M. Kaneko[1]) (i) *The function  $\xi_k(s)$  continues to an entire function of  $s$ , and the special values at non-positive integers are given by*

$$\xi_k(-m) = (-1)^m B_m^{(k)} \quad (m = 0, 1, 2, \dots).$$

(ii) The function  $\xi_k(s)$  can be written in terms of the zeta functions  $\zeta(k_1, k_2, \dots, k_{n-1}; s)$  as

$$\begin{aligned} \xi_k(s) = & (-1)^{k-1} \{ \underbrace{\zeta(2, 1, \dots, 1; s)}_{k-1} + \underbrace{\zeta(1, 2, 1, \dots, 1; s)}_{k-1} + \dots + \underbrace{\zeta(1, \dots, 1, 2; s)}_{k-1} \\ & + s \cdot \underbrace{\zeta(1, 1, \dots, 1; s+1)}_{k-1} \} + \sum_{j=0}^{k-2} (-1)^j \zeta(k-j) \cdot \underbrace{\zeta(1, 1, \dots, 1; s)}_j, \end{aligned}$$

where we define the single variable function by

$$\zeta(k_1, k_2, \dots, k_{n-1}; s) = \sum_{0 < m_1 < m_2 < \dots < m_n} \frac{1}{m_1^{k_1} m_2^{k_2} \dots m_{n-1}^{k_{n-1}} m_n^s}.$$

For the special values of the zeta function at positive integral points, they got the following theorem.

**Theorem 3.2** (T. Arakawa and M. Kaneko[1]) (i) For  $k \geq 1$  and  $m \geq 0$ ,

$$\xi_k(m+1) = \sum_{\substack{a_1+a_2+\dots+a_k=m \\ \forall a_j \geq 0}} (a_k+1) \zeta(a_1+1, a_2+1, \dots, a_{k-1}+1, a_k+2).$$

(ii) If  $k$  is even and  $k \geq 2$ , then

$$\xi_k(2) = \frac{1}{2} \sum_{i=0}^{k-2} (-1)^i \zeta(i+2) \zeta(k-i).$$

Applying our main theorem ( Theorem 2.1 ) to (i) of Theorem 3.2, we get a relation between the special values of  $\xi_k(s)$  and of  $\zeta^*(k_1, \dots, k_n)$  as follows.

**Theorem 3.3** For integers  $k \geq 1$  and  $m \geq 1$ , we have

$$\xi_k(m) = \zeta^* \left( \underbrace{1, 1, \dots, 1}_{m-1}, k+1 \right).$$

*Proof* By using Theorem 3.2, for positive integers  $k$  and  $m$  we have

$$\xi_k(m) = \sum_{\substack{a_1+a_2+\dots+a_k=m-1 \\ \forall a_j \geq 0}} (a_k+1) \zeta(a_1+1, a_2+1, \dots, a_{k-1}+1, a_k+2).$$



We can write

$$\xi_k(m) = \sum_{n=1}^m \sum_{\substack{a_1+a_2+\dots+a_k=m-n \\ \forall a_j \geq 0}} \zeta(a_1+1, a_2+1, \dots, a_{k-1}+1, a_k+n+1).$$

Now we use Theorem 2.1, then we have

$$\xi_k(m) = \sum_{n=1}^m \sum_{\substack{b_1+b_2+\dots+b_n=m-n \\ \forall b_j \geq 0}} \zeta(b_1+1, b_2+1, \dots, b_{n-1}+1, b_n+k+1),$$

so we get

$$\xi_k(m) = \zeta^*(\underbrace{1, 1, \dots, 1}_{m-1}, k+1).$$

Q.E.D.

If we use the known result

$$\zeta(1, k-1) = \frac{k-1}{2} \zeta(k) - \frac{1}{2} \sum_{r=2}^{k-2} \zeta(r) \zeta(k-r)$$

(cf. [7][14]), we can get

$$\begin{aligned} \xi_k(2) &= \zeta^*(1, k+1) = \zeta(1, k+1) + \zeta(k+2) \\ &= \frac{k+3}{2} \zeta(k+2) - \frac{1}{2} \sum_{r=2}^k \zeta(r) \zeta(k-r+2) \end{aligned}$$

by Theorem 3.3. In case of  $k$  even, we can check that it matches Theorem 3.2 (ii).

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