Proceedings of

Seminar on Partial Differential Equations in Osaka 2012

— in honor of Professor Hiroki Tanabeǒs 80th birthday —

(Osaka University, August 20 ‐ 24, 2012)

These proceedings are dedicated to Professor Hiroki Tanabe.

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Edited by

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Hiroki Tanabe, born on February 23, 1932, was brought up in Tokyo. He majored in Mathematics at the University of Tokyo, obtained his Bachelor's degree in 1954, his Master's degree in 1956, and successively entered the doctor's course. Leaving the doctor's course, he was appointed to Assistant at Osaka University in 1956. He received the title of Doctor of Science from Osaka University in 1960. After being promoted to Lecturer and Associate Professor, he was promoted to Professor at Osaka University in 1967. After retiring from Osaka University in 1995, he worked as Professor at Otemon Gakuin University until 2002.

Tanabe acted as a member of editorial board of mathematical journals: Osaka Journal of Mathematics, Funkcialaj Ekvacioj, and Advances in Mathematical Science and Applications.

In 1964 he received the Fujihara Award from the Fujihara Foundation of Science for his famous work on the theory of abstract evolution equations and its applications. In 2011 he was awarded the Order of the Sacred Treasure (Zuihou Shou) from the Japanese Government for his distinguished accomplishments in education and researches.

Hiroki Tanabe made a profound contribution to developing the study of functional analytic methods for partial differential equations. We here describe some of them which are classified in five subjects.

He presented definitive results of constructing a fundamental solution for non-autonomous linear abstract parabolic evolution equations. For the case where the domains of linear operators appearing in equations are temporally constant the work has been done independently with Sobolevskiĭ and is now called the Sobolevskiĭ and Tanabe theory. For the case where the domains of linear operators are temporally variable it has been done jointly with Kato. The Kato-Tanabe condition is now widely known as one of the fundamental sufficient conditions for solving various parabolic equations in a unified way.

He introduced a very general formula for estimating the distribution of eigenvalues of elliptic operators which are determined from coercive sesquilinear forms. His method enables us to know the distribution of eigenvalues under minimum assumptions of the regularity of coefficient functions in the elliptic operators.

The L^1 space is a specifically important space in which the parabolic equations should be handled, although one cannot prove directly the generation of analytic semigroups for elliptic operators differently from the L^p spaces for $1 < p < \infty$. Tanabe introduced a new method of utilizing integral kernels to show the L^1 generation of analytic semigroups and together to construct an L^1 fundamental solution to parabolic equations.

He developed these profound results to the advanced study on abstract evolution equations with memory and abstract evolution equations of degenerate type. Partial differential equations containing memory effects form one of the important classes of problems to be studied in mathematical engineering. He first showed methods how to construct a fundamental solution to these equations which are at the same time available to a wide class of problems.

Abstract degenerate equations were first studied by Favini systematically. Tanabe then contributed deeply to develop the theory by newly introducing useful methods for constructing a fundamental solution to the equations.

Hiroki Tanabe is a world-famous researcher, driven by his powerful will to cast light on the difficult problems of mathematics. His accomplishment as a scientist is enormous and his mind is both proactive and profound. However, it is not just his performance as a scientist that we praise on his 80th birthday; his kindness, his courtesy and his good nature make him an invaluable and true leader for all of us.

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Preface

This volume is the proceedings of a five day seminar on partial differential equations held at Osaka University in August, 2012, under the support of Osaka University. Such a seminar is annually held and is organized in rotation by Tanabe's school members. Seminar 2012 was specially organized in honor of his 80th birthday. The volume is divided into two parts. The first consists of the reports of the speakers of Seminar 2012. The second is devoted to the reports of contributors to the annual seminar.

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Atsushi Yagi and Yoshitaka Yamamoto (Editors)

Program of Seminar 2012

20 August (Monday)

13:20-13:30 Opening

13:30-14:20 Tadashi Kawanago (Tokyo Institute of Technology)
Codimension - n bifurcation theorems applicable to the numerical verification

14:30-15:20 Kiyoko Furuya (Ochanomizu University)
On the formally self-adjoint Schrödinger operators

15:40-16:10 Atsushi Kosaka (Osaka Prefecture University)
Structure of solutions to the Emden equation on a geodesic ball in a sphere

16:20-17:10 Takeshi Wada (Kumamoto University)
On well-posedness for nonlinear Schrödinger equations with power nonlinearity in fractional order Sobolev spaces

21 August (Tuesday)

10:00-10:50 Shoji Yotsutani (Ryukoku University)
Existence and stability of stationary solutions to a multidimensional SKT cross-diffusion equation

11:00-11:50 Masato Iida (University of Miyazaki)
On the fast reaction limit of a reaction-diffusion system whose interaction terms are non-relative to each other

13:30-14:20 Tohru Ozawa (Waseda University)
Sharp bilinear estimate on the Klein-Gordon equations

14:30-15:20 Nguyen Van Mau (Hanoi University)
Solvability theory of a class of singular integral equations with rotations

15:40-16:10 Yoich Enatsu (Waseda University)
Lyapunov functional for disease transmission models with delays and its applications

16:10-16:40 Kazuhiro Oeda (Waseda University)
Effect of a protection zone and cross-diffusion on a prey-predator model

16:40-17:10 Yuki Kaneko (Waseda University)
Asymptotic behavior of radially symmetric solutions for a free boundary problem related to an ecological model

22 August (Wednesday)

10:00-10:50 Hiroko Okochi (Tokyo University of Pharmacy and Life Science)

Conditions for transitions of level set patterns in nonlinear differential equations

11:00-11:50 Jin-Mun Jeong (Pukyong National University)

Control problems for semilinear differential equations with local Lipschitz continuity

13:30-14:20 Hiroki Tanabe (Osaka University, professor emeritus)

Identification problem for degenerate parabolic equations

14:30-15:20 Angelo Favini (Bologna University)

Degenerate differential equations of parabolic type and inverse problems

16:00-16:50 Yoshio Tsutsumi (Kyoto University)

Gibbs measure for the isothermal Falk model

18:00~ Banquet

23 August (Thursday)

11:00-11:50 Goro Akagi (Kobe University)

A variational approach to gradient flows

13:30-14:20 Shin-ichi Nakagiri (Kobe University)

Boundary controllability of nonlocal advection-diffusion equations

14:30-15:20 Katsuyuki Ishii (Kobe University)

On the convergence of an area minimizing scheme for anisotropic mean curvature flow

15:30-16:00 Doan Duy Hai (Osaka University)

A temporal discretization method for advection-diffusion-reaction equations

16:00-16:50 Koji Kikuchi (Shizuoka University)

Linear approximation of a system of quasilinear hyperbolic equations having linear growth energy functional

24 August (Friday)

10:00-10:50 Atsushi Yagi (Osaka University)

Hölder type maximal regularity for parabolic evolution equations

11:00-11:50 Davide Guidetti (Bologna University)

On partial reconstruction of source terms in parabolic problems

12:00-12:10 Closing

Contents

Part I: Reports by Speakers

Codimension-n bifurcation theorems applicable to the numerical verification

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Abstract. This article is a digest version of our paper [K5]. We establish codimension-n bifurcation theorems applicable to the numerical verification methods. They are generalization of codimension-1 bifurcation theorems in [K1].

1. Introduction and our main result

By the recent growth of the computer power, we can observe numerically bifurcation phenomena of solutions for a lot of differential equations and systems. It is often difficult, however, to analyze such phenomena by the use of pure analytical methods. On the other hand, it is becoming more and more possible to analyze the bifurcation phenomena rigorously by numerical verification methods based on some appropriate bifurcation theorems. Actually, by using a symmetry-breaking bifurcation theorem [K1, Theorem 3.1] and the numerical verification methods, we proved the existence of a $\mathbb{Z}_2$-symmetry breaking bifurcation point for a nonlinear forced vibration system described by a wave equation in [K2], and Nakao, Watanabe, Yamamoto, Nishida and Kim verified some symmetry-breaking bifurcation points for two-dimensional Rayleigh-Bénard heat convection system in [WN] and [NWYNK].

In this article, we describe codimension-n bifurcation theorems applicable to the numerical verification methods, which were recently established in the author's paper [K5]. They are generalization of codimension-1 bifurcation theorems in [K1].

Here, we present our main theorem. Let X and Y be real Banach spaces. Let X_1 and X_2 be closed subspaces of X, and Y_1 and Y_2 be closed subspaces of Y. We assume that $X = X_1 \oplus X_2$ and $Y = Y_1 \oplus Y_2$. Here, $\oplus$ means the direct sum. Let $n \in \mathbb{N}$ and $g \in C^2(\mathbb{R}^n \times X, Y)$ have the following properties:

$$g(\Lambda, v) \in Y_1 \quad \text{for any} \quad \Lambda \in \mathbb{R}^n \quad \text{and} \quad v \in X_1, \tag{1.1}$$

$$g_u(\Lambda, v)X_j \subset Y_j \quad \text{for any} \quad \Lambda \in \mathbb{R}^n \quad \text{and} \quad v \in X_1 \quad (j = 1, 2), \tag{1.2}$$

$$g_{uu}(\Lambda, v_1)v_2 w \in Y_2 \quad \text{for any} \quad \Lambda \in \mathbb{R}^n,\ v_1, v_2 \in X_1 \quad \text{and} \quad w \in X_2. \tag{1.3}$$

We denote by $\boldsymbol{e}_1 := (1, 0, \cdots, 0) \in \mathbb{R}^n$ the first row vector of the identity matrix of order n. We define $J : \mathbb{R}^n \oplus X \to \mathbb{R}^n \oplus Y$ by

$$(1.4) \qquad J \begin{pmatrix} \Lambda \\ v \\ w \end{pmatrix} := \begin{pmatrix} lw - \boldsymbol{e}_1 \\ g(\Lambda, v) \\ g_u(\Lambda, v)w \end{pmatrix} \in \mathbb{R}^n \times Y_1 \times Y_2 \quad \text{for} \quad (\Lambda, v, w) \in \mathbb{R}^n \times X_1 \times X_2.$$

Here, $l \in \mathcal{L}(X, \mathbb{R}^n)$ and we assume that $lv = 0$ for any $v \in X_1$. We define projections p and P by

$$(1.5) \qquad p : (r_1, r_2, \cdots, r_n) \in \mathbb{R}^n \longmapsto r_1 \in \mathbb{R},$$

$$(1.6) \qquad P : (r_1, r_2, \cdots, r_n) \in \mathbb{R}^n \longmapsto (r_2, \cdots, r_n) \in \mathbb{R}^{n-1} \quad \text{for the case} \quad n \geq 2.$$

In what follows, we always set formally $\mathbb{R}^{n-1} \times Y := Y$ for the case: $n = 1$. We define $G : \mathbb{R}^n \times X \to \mathbb{R}^{n-1} \times Y$ by

$$(1.7) \qquad G := g \quad (n = 1) \quad \text{and} \quad G \begin{pmatrix} \Lambda \\ u \end{pmatrix} := \begin{pmatrix} Plu \\ g(\Lambda, u) \end{pmatrix} \quad (n \geq 2).$$

We set $Z := (pl)^{-1}(0) = \{u \in X \,;\, plu = 0\}$ and $\mathbb{R}_+ := (0, \infty)$. Our main theorem is the following:

Theorem 1.1. *In addition to the assumptions above we assume that* $(\Lambda_0, v_0, w_0) \in \mathbb{R}^n \times X_1 \times X_2$ *satisfies the following* (H1) *and* (H2)*:*

(H1) *A point* $(\Lambda, v, w) = (\Lambda_0, v_0, w_0)$ *is an isolated solution of the extended system* $J(\Lambda, v, w) = 0$,

(H2) *The linear operator* $g_u(\Lambda_0, v_0)|_{X_1} : X_1 \to Y_1$ *is bijective.*

Then, the point (Λ_0, v_0) *is a bifurcation point of the equation* $G(\Lambda, v) = 0$*. Exactly, there exist an open neighborhood* W *of* (Λ_0, v_0) *in* $\mathbb{R}^n \times X$, $a \in \mathbb{R}_+$, $\zeta \in C^1((-a, a), \mathbb{R}^n)$, $\eta \in C^1((-a, a), Z)$, *an open neighborhood* V *of 0 in* $\mathbb{R}^n$ *and* $q \in C^2(\Lambda_0 + V, X_1)$ *such that* $\zeta(0) = \Lambda_0$, $\eta(0) = 0$, $q(\Lambda_0) = v_0$ *and*

$$(1.8)\; G^{-1}(0) \cap W = \{(\Lambda, q(\Lambda)) \,;\, (\Lambda, q(\Lambda)) \in W\} \cup \{(\zeta(\alpha),\, \alpha w_0 + \alpha\eta(\alpha) + q(\zeta(\alpha))) \,;\, |\alpha| < a\}.$$

Roughly speaking, the well-known pitchfork bifurcation theorem [CR, Theorem 1.7] by Crandall and Rabinowitz is equivalent to our Theorem 1.1 with $n = 1$, $X_1 = \{0\}$ and $Y_1 = \{0\}$ (see Section 2). We immediately obtain a $\mathbb{Z}_2$-symmetry breaking bifurcation theorem [K1, Theorem 3.1] by setting $n = 1$ in our Theorem 1.1 and by choosing the symmetric subspace of X as X_1 and the anti-symmetric subspace as X_2. We can apply our Theorem 1.1 to Hopf bifurcation by setting $n = 2$ and choosing an appropriate space of periodic functions as X, the subspace of steady functions as X_1 and the complementary subspace of X_1 as X_2 (see [K5, Section 4] for the details).

We note that most known bifurcation theorems are not applicable directly to the numerical verification methods. We explain this point by Hopf bifurcation as an example. We consider the next autonomous ordinary differential equation:

(O) $\quad \dot{y} = f(\lambda, y), \quad y, f(\lambda, y) \in \mathbb{R}^d.$

Hopf bifurcation theorem. *The point $(\lambda, y) = (\lambda_0, y_0)$ is a Hopf bifurcation point of the equation (O), i.e. a branch of periodic solutions of (O) bifurcates at the point $(\lambda, y) = (\lambda_0, y_0)$ from a branch of steady solutions of (O) with the initial period $2\pi\sigma_0$, provided the following conditions (C1)-(C4) are satisfied:*

(C1) $\quad f(\lambda_0, y_0) = 0,$

(C2) $\pm i$ *are the simple eigenvalues of* $\sigma_0 f_y(\lambda_0, y_0)$ *(So, by the implicit function theorem, the matrix* $\sigma_0 f_y(\lambda, y(\lambda))$ *has a pair of complex conjugate of eigenvalues* $\mu(\lambda)$, $\overline{\mu(\lambda)}$ *with* $\mu(\lambda_0) = i$*),*

(C3) *(Transversality condition of eigenvalues)* $\quad Re\, \mu'(\lambda_0) \neq 0,$

(C4) $\quad ik$ *is not an eigenvalue of* $\sigma_0 f_y(\lambda_0, y_0)$ *for* $k \in \mathbb{Z} - \{-1, 1\}$.

It is difficult to check rigorously by numerical methods the *simplicity* condition (C2) and the *dynamic* condition (C3). In Theorem 1.1 these conditions correspond to *static* conditions, i.e. regularity conditions for linear operators. It is not difficult to verify them by some numerical method. See [K5, Section 4.4] for the details.

We omit the proofs of our results described in this article. See [K5] for them. For a numerical example see [K5, Section 4], where we applied Theorem 1.1 with $n = 2$ and our numerical verification method to a parabolic system called Brusslator model to verify that it has a Hopf bifurcation point.

The author asks the readers interested in this article to contact him by e-mail. He is willing to send [K5] (a PDF document) to them.

2. Basic bifurcation theorems

Theorem 2.1 in this section is a generalized version of [K1, Theorem 2.1] and can be regarded as as a refined version of Theorem 1.1 with $X_1 = \{0\}$ and $Y_1 = \{0\}$.

Let X and Y be Banach spaces and U be an open neighborhood of 0 in X. Let $n \in \mathbb{N}$ and V be an open neighborhood of 0 in $\mathbb{R}^n$. Let $f \in C^1(V \times U, Y)$ be a map such that

(2.1) $\quad f(\Lambda, 0) = 0 \quad \text{for} \quad \text{any} \quad \Lambda = (\Lambda_1, \cdots, \Lambda_n) \in V$

and the partial Fréchet derivative $f_{\Lambda_k u}$ exists and is continuous for $k = 1, \cdots, n$. We denote $f_{\Lambda u} = (f_{\Lambda_1 u}, \cdots, f_{\Lambda_n u})$ for simplicity.
We define $H : V \times X \to \mathbb{R}^n \times Y$ by

$$(2.2) \qquad H \begin{pmatrix} \Lambda \\ u \end{pmatrix} := \begin{pmatrix} lu - \boldsymbol{e}_1 \\ f_u(\Lambda, 0)u \end{pmatrix}.$$

Here, $l \in \mathcal{L}(X, \mathbb{R}^n)$. In what follows, we often use the same notations as in Section 1. We define $F : V \times U \to \mathbb{R}^{n-1} \times Y$ by

$$F := f \quad (n = 1) \quad \text{and} \quad F \begin{pmatrix} \Lambda \\ u \end{pmatrix} := \begin{pmatrix} Plu \\ f(\Lambda, u) \end{pmatrix} \quad (n \geq 2).$$

We set $Z := (pl)^{-1}(0) = \{u \in X \,;\, plu = 0\}$.

Theorem 2.1. *In addition to the assumptions above we assume*

(H) *There exists $u_0 \in U$ such that $(\Lambda, u) = (0, u_0)$ is an isolated solution of the extended system $H(\Lambda, u) = 0$.*

Then there exist an open neighborhood W of (0,0) in $\mathbb{R}^n \times X$, $a \in \mathbb{R}_+$ and continuous functions $\zeta : (-a, a) \to \mathbb{R}^n$, $\eta : (-a, a) \to Z$ such that $\zeta(0) = 0$, $\eta(0) = 0$ such that

$$(2.3) \qquad F^{-1}(0) \cap W = \{(\Lambda, 0) \,;\, (\Lambda, 0) \in W\} \cup \{(\zeta(\alpha),\, \alpha u_0 + \alpha\eta(\alpha)) \,;\, |\alpha| < a\}.$$

Moreover, if $f \in C^{k+1}$ ($k \in \mathbb{N}$) then $\zeta, \eta \in C^k$.

For simplicity, we write $DH^0 := DH(0, u_0)$, $f_u^0 := f_u(0,0)$, $F_u^0 := F_u(0,0)$, $F_{\Lambda u}^0 := F_{\Lambda u}(0,0)$ and so on. We have

$$(2.4) \qquad DH^0 \begin{pmatrix} \Lambda \\ u \end{pmatrix} = \begin{pmatrix} lu \\ \Lambda \cdot f_{\Lambda u}^0 u_0 + f_u^0 u \end{pmatrix} = \begin{pmatrix} plu \\ \Lambda \cdot F_{\Lambda u}^0 u_0 + F_u^0 u \end{pmatrix}.$$

In view of the next result, we may consider Theorem 2.1 as a generalized version of [CR, Theorem 1.7].

Proposition 2.1. *The condition (H) in Theorem 2.1 is equivalent to the following (i) and (ii):*

(i) $\dim \mathcal{N}(F_u^0) = 1$ *and* $\mathit{codim}\, \mathcal{R}(F_u^0) = n$.

(ii) $\exists u_0 \in U$ *such that*

$$(2.5) \qquad plu_0 = 1, \quad \mathcal{N}(F_u^0) = \mathit{span}\,\{u_0\},$$

$$(2.6) \qquad \mathit{span}(F_{\Lambda u}^0 u_0) \oplus \mathcal{R}(F_u^0) = \mathbb{R}^{n-1} \times Y.$$

Here, we denote $\mathit{span}(F_{\Lambda u}^0 u_0) := \{\Lambda \cdot F_{\Lambda u}^0 u_0 \,;\, \Lambda \in \mathbb{R}^n\}$.

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ON THE FORMALLY SELF-ADJOINT SCHERÖDINGER OPERATORS

KIYOKO FURUYA

Dedicated to Professor Hiroki Tanabe on his 80th birthday

ABSTRACT. A Schrödinger operator, (not self-adjoint but) formally self-adjoint, generates a (not unitary but) contraction semigroup. Our class of potentials U in Schrödinger equation is wide enough : the real measurable potential U should be locally essentially bounded except a closed set of measure zero.

1. INTRODUCTION

We shall construct a family of unique solutions to the Schrödinger equation in $\mathbb{R}^N$

$$h\frac{\partial}{\partial t}u(t,x) = \frac{ih^2}{2m}\triangle u(t,x) - iU(x)u(t,x), \quad u(0,x) = \varphi(x), \tag{1}$$

for $U \in L^{\infty}_{loc}(\mathbb{R}^N \backslash \mathcal{N}, \mathbb{R})$ where $\mathcal{N}$ is a closed set of measure 0. For further information, see (2), (4) . Here h and m are positive constants.

Remark 1. *We define a sequence of functions $\{U_n\}_{n\in\mathbb{N}}$ as follows:*

$$U_n(x) = \begin{cases} n & \text{if} \quad n < U(x), \\ U(x) & \text{if} \quad -n \le U(x) \le n \\ -n & \text{if} \quad U(x) < -n \end{cases} \qquad for \quad n \in \mathbb{N} \tag{2}$$

It is easily checked that $U_n(x) = \min\Big\{n, \max\{-n, U(x)\}\Big\}$. Then we shall approximate the potential U by U_n. The unique solution obtained by this approximation seems to correspond to the case no particle comes from infinity in the example above. This solution seems natural for the theory of path integrals. The physical meaning of the solution by Nelson[11] *is unclear to the author.*

Definition 1. *Let B be a densely defined operator on Hilbert space $\mathcal{H}$. Then*
(i) *B is essentially self-adjoint if and only if it has a unique self-adjoint extension, necessarily its closure $\bar{B}$,*
(ii) *B is formally self-adjoint if $\langle B\varphi, \psi\rangle = \langle \varphi, B\psi\rangle$ for all φ and ψ in $\mathcal{H}$.*

We consider a closed extension of (not necessarily essentially self-adjoint but) formally self-adjoint operator $iA \equiv -(h^2/2m)\triangle + U$ on $\mathcal{C}_0^{\infty}(\mathbb{R}^N \backslash \mathcal{N})$. Here $\mathcal{C}_0^{\infty}(E)$ denote the set of all infinitely differentiable functions with compact support in E. The semigroup of our solution family, which is obtained by the approximation (2), is not necessarily a group of unitary operators but a semigroup of contractions. Our result improves one of the Nelson [11]'s, which says the contraction semigroup of his solution family exists (*not for all but*) for a. e. $m > 0$ and for any $U \in \mathcal{C}(\mathbb{R}^N \backslash \mathcal{N}_0; \mathbb{R})$, where $\mathcal{N}_0$ is a closed subset of capacity 0. Our results is closely related to the theory of path integrals (see Furuya [4]).

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2. Schrödinger equation

For simplicity we consider the following normalized equation :

$$\frac{\partial}{\partial t}u(t,x) = i\triangle u(t,x) - iU(x)u(t,x), \quad u(0,x) = \varphi(x) \quad \text{for} \quad \varphi \in H^{(2)}(\mathbb{R}^N;\mathbb{C}), \tag{3}$$

where $H^{(2)}(\mathbb{R}^N;\mathbb{C})$ denote the Sobolev space of L^2-functions with first and second distributional derivatives also in L^2 on $\mathbb{R}^N$ to $\mathbb{C}$.

If $\triangle - U$ is essentially self-adjoint, the operator family $\{T(t)\}$ defined by $T(t)\varphi = u(t)$ is uniquely extended to a group of unitary operators from $L^2(\mathbb{R}^N;\mathbb{C}\big)$ to $L^2(\mathbb{R}^N;\mathbb{C})$. Let $\mathcal{N}$ be a fixed closed subset of $\mathbb{R}^N$ of measure 0 and $\mathcal{D} = \{D\}$ be the maximum family such that each element $D \subset \overline{D} \subset \mathbb{R}^N \backslash \mathcal{N}$ is a finite union of connected bounded open sets. The family $\mathcal{D} = \{D\}$ satisfies $\bigcup_{D\in\mathcal{D}} D = \mathbb{R}^N\backslash\mathcal{N}$. We denote the restriction of f to D by $f|_D$. We use the following notation

$$L^\infty_{loc}(\mathbb{R}^N\backslash\mathcal{N};\mathbb{R}) = \Big\{ f \;\Big|\; f(x) \in \mathbb{R},\ x \in \mathbb{R}^N,\ f|_D \in L^\infty(D;\mathbb{R}),\ \forall D \in \mathcal{D} \Big\}. \tag{4}$$

Let $U \in L^\infty_{loc}(\mathbb{R}^N\backslash\mathcal{N},\mathbb{R})$. We assume for any neighbourhood of any point of $\mathcal{N}$, U is not essentially bounded. By this assumption, U uniquely determines $\mathcal{N}$ in the following sense :

$$\mathcal{N} = \bigcap_\nu \{\mathcal{N}_\nu \;\Big|\; U \in L^\infty_{loc}(\mathbb{R}^N\backslash\mathcal{N}_\nu;\mathbb{R})\}. \tag{5}$$

Let

$$B_n = \{\ x \in \mathbb{R}^N \;\Big|\; -n < U(x) < n\ \} \quad for \quad n \in \mathbb{N}. \tag{6}$$

We have $B_m \supset B_n$ for $m > n$ and

$$\text{for any } D \in \mathcal{D}, \text{ there exists } B_n \text{ such that } D \subset \overline{D} \subset B_n. \tag{7}$$

We denote $U_n(x) = \min\Big\{n, \max\{-n, U(x)\}\Big\}$. Thus $U_n \in L^\infty(\mathbb{R}^N;\mathbb{R})$. For $U \in L^\infty_{loc}(\mathbb{R}^N\backslash\mathcal{N};\mathbb{R})$ we consider the approximative equation

$$\frac{d}{dt}u_n(t) = A_n u_n(t) \quad where\ A_n = i(\triangle - U_n). \tag{8}$$

In this case the operator $-iA_n$ is essentially self-adjoint. Hence the semigroup $\{T_n(t)\}$ generated by $-iA_n$ is the family of solutions to (8) and is a group of unitary operators : $\|T_n(t)\varphi\| = \|\varphi\|$ for $t \in \mathbb{R}$ and $\varphi \in L^2(\mathbb{R}^N;\mathbb{C})$.

The main theorem in this paper is the following :

Theorem 1. *For any $U \in L^\infty_{loc}(\mathbb{R}^N\backslash\mathcal{N},\mathbb{R})$, there exists a closed extension of $(i\triangle - iU)|_{C_0^\infty(\mathbb{R}^N\backslash\mathcal{N})}$ in $L^2(\mathbb{R}^N;\mathbb{C})$ to $L^2(\mathbb{R}^N;\mathbb{C})$ which generates a unique contraction C_0-semigroup $\{T(t) \;\big|\; t \geq 0\}$ such that*

$$T(t)\varphi = w\text{-}\lim_{n\to\infty} T_n(t)\varphi \quad for \quad all \quad \varphi \in L^2(\mathbb{R}^N;\mathbb{C}), \tag{9}$$

where $T_n(t)\varphi$ is the solution to (8) *and* w-lim *means the weak convergence.*

3. Preliminaries

In this section we begin by introducing some terminology and notation and present those aspects of the basic theory which are required in subsequent sections.

3.1. Filter.

Definition 2. *Given a set E, a partial ordering $\subset$ can be defined on the powerset $\mathcal{P}(E)$ by subset inclusion. Define a filter $\mathcal{F}$ on E as a subset of $\mathcal{P}(E)$ with the following properties:*
(i) $\emptyset \notin \mathcal{F}$.*(The empty set is not in $\mathcal{F}$.)*
(ii) *If $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \bigcap B \in \mathcal{F}$. ($\mathcal{F}$ is closed under finite meets.)*
(iii) *If $A \in \mathcal{F}$ and $A \subset B$, then $B \in \mathcal{F}$. (Therefore $E \in \mathcal{F}$.)*

Definition 3. *Let $\mathcal{B}$ is a subset of $\mathcal{P}(E)$. $\mathcal{B}$ is called filter base on E if and only if*
(i) *The intersection of any two sets of $\mathcal{B}$ contains a set of $\mathcal{B}$,*
(ii) *$\mathcal{B}$ is non-empty and the empty set is not in $\mathcal{B}$*

Let X be a topological space.

Definition 4. *$\mathcal{U}(x)$ is called the neighourhood filter at point x for X if and only if $\mathcal{U}(x)$ is the set of all topological neighbourhoods of the point x.*

Definition 5. *To say that filter base $\mathcal{B}$ converges to x, denoted $\mathcal{B} \to x$, means that for every neighourhood U of x, there is a $B \in \mathcal{B}$ such that $B \subset U$. In this case, x is called a limit of $\mathcal{B}$ and $\mathcal{B}$ is called a convergent filter base*

Lemma 1. *X is a Hausdorff space if and only if every filter base on X has at most one limit.*

Definition 6. *A filter $\mathcal{F}$ in a topological spacec is called ultra filter if having the property that no other filter exists in the space having among its subsets all the subsets in the given filter.*

For details concerning the filter, we refer to Bourbak[1].

3.2. Compact open topology.

Definition 7. *A linear topological space X is called a locally convex linear topological space, or, in short, a locally convex space, if and only if its open sets $\ni 0$ contains a convex, balanced and absorbing open set.*

Let X and X' be two linear spaces over the complex field $\mathbb{C}$ and a scalar product $\langle x, x' \rangle \in \mathbb{C}$ for $x \in X$ and $x' \in X'$ be defined.

Definition 8. *Let X be topological vector space. The weak topology on X, denote by $\sigma(X, X')$, is the weakest topology such that all elements of X' remains continuous.*

Definition 9. *The strong topology β of X' is the topology of uniform convergence on every $\sigma(X, X')$-bounded set in X. X'_β denotes $(X')_\beta$.*

Definition 10. *τ_0 is the locally convex topology on X, defined by the seminorm system $\mathcal{P} = \{p_\gamma \bigm| p_\gamma(f) = \sup_{g \in C_\gamma} |\langle f, g \rangle|, C_\gamma \in \mathcal{C}\}$, where $\mathcal{C} = \{C_\gamma\}$ denotes the family of the compact subsets of X'_β. τ_0 is called the compact open topology.*

In the case of Banach space J. Dieudonné has proved the following theorem.

Theorem 2 (Dieudonné[2])**.** *The bounded weak* topology in a Banach space is identical with the compact open tpoplogy*

We denote by X'^* the space of linear functionals bounded on every bounded set in X'_β.

Proposition 1 (Kōmura and Furuya[9] Proposition 1)**.** *Let* $\overline{X}_{\tau_0}$ *be the completion of the space* X_{τ_0}*. Then we have:*

$$(X'_\beta)' \subset \overline{X}_{\tau_0} \subset X'^*.$$

Lemma 2 (Kōmura and Furuya[9] Lemma 5)**.** *Let* $x'' \in X''$*.* $x'' \in \overline{X}_{\tau_0}$ *if and only if* x'' *is* $\sigma(X', X)$*-continuous on every* τ_0*-equi-continuous set* $\{U_p^o | U_p \in \mathcal{U}_{\tau_0}\}$*. Here* U_p^o *is a polar set of* U_p*.*

Corollary 1. *If* X *is a Banach space, we have* $X'' = \overline{X}_{\tau_0}$*.*

4. Existence of weak limit of unitary groups in abstract case

$(\mathcal{H}, \|\cdot\|)$, or simply $\mathcal{H}$, denotes a Hilbert space with norm $\|\cdot\|$. Instead of the convergence of subsequences we use the convergence of filters. We consider an infinite semi-ordered index set $\mathcal{A} = \{\alpha\}$. We assume that there exists an ultra-filter Φ of infinite subsets of $\mathcal{A}$ satisfying

$$\forall\phi \in \Phi,\ \forall\alpha \in \mathcal{A},\ \exists\alpha' \in \phi : \alpha' \succ \alpha. \tag{10}$$

In the following $\{\Phi\}$ denotes the family of ultra-filters whose element satisfies (10).

Remark 2. *Note that we can use subsequences* $\{\alpha_k\}_{k=1}^\infty$ *instead of ultra-filters, if* $\mathcal{H}$ *is separable.*

Let a family $\{T_\alpha(t) \mid -\infty < t < \infty\}_{\alpha\in\mathcal{A}}$ of groups of unitary operators in $\mathcal{H}$ be given. A_α denotes the generator of $\{T_\alpha(t)\}$:

$$\frac{d}{dt}T_\alpha(t)\varphi = A_\alpha T_\alpha(t)\varphi \quad \text{for} \quad \varphi \in D(A_\alpha).$$

Definition 11. *For an ultra-filter* Φ *satisfying (10), the operators* $(I - A_\Phi)^{-1}$ *and* $T_\Phi(t)$ *are defined as follows* :

$$(I - A_\Phi)^{-1}f = w\text{-}\lim_{\alpha\in\phi\in\Phi}(I - A_\alpha)^{-1}f \quad for \quad \forall f \in \mathcal{H}, \tag{11}$$

$$T_\Phi(t)\varphi = w\text{-}\lim_{\alpha\in\phi\in\Phi} T_\alpha(t)\varphi \quad for \quad \forall\varphi \in \mathcal{H}. \tag{12}$$

In this section we shall show the existence of a semigroup $\{T_\Phi(t)\}$ in (12). As is well known, iA_α is self-adjoint : $\langle A_\alpha\varphi, \psi\rangle = -\langle\varphi, A_\alpha\psi\rangle$ for $\varphi, \psi \in D(A_\alpha)$, and the resolvent $(I - A_\alpha)^{-1}$ is a contraction : $\|(I - A_\alpha)^{-1}\| \le 1$. Since a bounded subset of $\mathcal{H}$ is relatively $\sigma(\mathcal{H}, \mathcal{H})$-compact, $\|(I - A_\alpha)^{-1}\varphi\| \le \|\varphi\|$ and $\|T_\alpha(t)\varphi\| = \|\varphi\|$, there exist $w\text{-}\lim_{\alpha\in\phi\in\Phi}(I - A_\alpha)^{-1}\varphi$ and $w\text{-}\lim_{\alpha\in\phi\in\Phi} T_\alpha(t)\varphi$. Hence $(I - A_\Phi)^{-1}$ and $T_\Phi(t)$ are well defined. Note that A_Φ may be multi-valued. The following condition implies A_Φ is single-valued, which will be verified later (See.Theorem 4).

Condition 1. *There exist a dense subspace* $\mathcal{H}_0$ *of* $\mathcal{H}$ *satisfying* $\mathcal{H}_0 \subset \bigcap_{\alpha\in\phi} D(A_\alpha)$ *and a linear operator* $A_0 : \mathcal{H}_0 \longrightarrow \mathcal{H}$ *such that*

$$\forall\psi \in \mathcal{H}_0,\ \exists\alpha(\psi) \in \mathcal{A}\ :\ A_0\psi = A_\alpha\psi \quad for \quad \forall\alpha \succ \alpha(\psi). \tag{13}$$

Throughout this paper, we assume Condition 1 holds. By definition, (11) means

$$\forall f \in \mathcal{H}, \forall\varepsilon > 0, \forall C_\beta \in \mathcal{C}, \exists\phi \in \Phi : \sup_{\alpha\in\phi,\varphi\in C_\beta} |\langle (I - A_\Phi)^{-1} f - (I - A_\alpha)^{-1} f, \varphi\rangle| < \varepsilon.$$

Lemma 3. *For a fixed* $f \in \mathcal{H}$*, put*

$$\varphi_\alpha = (I - A_\alpha)^{-1} f \quad and \quad \varphi_\Phi = (I - A_\Phi)^{-1} f. \tag{14}$$

Under the condition 1, we have

(a) *Let* $f = (I - A_0)\varphi$ *for* $\varphi \in \mathcal{H}_0$ *then* $\varphi = \lim_{\alpha\in\phi\in\Phi}(I - A_\alpha)^{-1} f.$

(b) *Let* $\varphi_\alpha = (I - A_\alpha)^{-1} f$ *and* $\varphi_\Phi = w\text{-}\lim_{\alpha\in\phi\in\Phi} \varphi_\alpha,$ *then*

$$w\text{ - }\lim_{\alpha\in\phi\in\Phi} A_\alpha\varphi_\alpha = A_\Phi\varphi_\Phi.$$

Moreover if A_Φ *is single-valued, it follows that*

$$\mathcal{H}_0 \subset D(A_\Phi), \quad A_\Phi|_{\mathcal{H}_0} = A_0, \tag{15}$$

$$\langle A_\Phi\varphi, \psi\rangle = -\langle\varphi, A_0\psi\rangle \quad for \quad \forall\varphi \in D(A_\Phi) \quad and \quad \forall\psi \in \mathcal{H}_0. \tag{16}$$

Proposition 2. *The range of* $(I - A_\Phi)^{-1} : \mathcal{R}((I - A_\Phi)^{-1}) = (I - A_\Phi)^{-1}\mathcal{H}$ *is dense in* $\mathcal{H}$*.*

We cite Theorem 9 in [9] as Theorem 3. Let X be a reflexive Banach space and $\{T_\alpha(t)\}_{\alpha\in\mathcal{A}}$ be a family of contraction C_0-semigroups in X.

Theorem 3 (Kōmura and Furuya [9] Theorem 9)**.** *Suppose for some filter* Φ

$$\forall f \in X,\ \exists\varphi_\Phi = w\text{-}\lim_{\alpha\in\phi\in\Phi}(I - A_\alpha)^{-1} f. \tag{17}$$

Thus the operator $(I - A_\Phi)^{-1}$ *is defined. If the range* $\mathcal{R}((I - A_\Phi)^{-1})$ *is dense in* X*,* A_Φ *is a densely defined closed operator and generates a semigroup* $\{T_\Phi(t)\}$:

$$w\text{-}\lim_{\alpha\in\phi\in\Phi} T_\alpha(t)x = T_\Phi(t)x \quad for \quad \forall x \in X. \tag{18}$$

Moreover, we have $\{T_\Phi(t)\}$ *is a contraction* C_0*-semigroup in* X*.*

Theorem 4. *Under condition 1,* A_Φ *is a closed operator and generates a contraction* C_0*-semigroup* $\{T_\Phi(t)\}$*.*

Proof. Since the range $\mathcal{R}((I - A_\Phi)^{-1})$ is dense in $\mathcal{H}$ by Proposition 2, our Theorm follows from Theorem 3. □

5. Approximation by bounded domains

Let $\mathcal{D} = \{D\}$ be the maximum family such that each element $D \subset \overline{D} \subset \mathbb{R}^N \backslash \mathcal{N}$ is a finite union of connected bounded open sets. For $D \in \mathcal{D}$, $L^2(D;\mathbb{C})$ denotes the L^2-space on D to $\mathbb{C}$ and $H^{(1)}(D;\mathbb{C})$ denote the Sobolev space of L^2-functions with first distributional derivatives also in L^2 on D to $\mathbb{C}$. $H^{(2)}(D;\mathbb{C})$ denote the Sobolev space of L^2-functions with first and second distributional derivatives also in L^2 on D to $\mathbb{C}$ with norm $\|\cdot\|_{(2)}$. $H_0^{(1)}(D;\mathbb{C})$ is defined as the closure in $H^{(1)}(D;\mathbb{C})$ of $\mathcal{C}_0^\infty(D;\mathbb{C})$. For $U \in L^\infty(D;\mathbb{R})$, the functional $\Psi^D(\varphi) \equiv \frac{1}{2}\|(-\triangle)^{-1/2}\varphi\|^2 + \frac{1}{2}\|\sqrt{U+C}\varphi\|^2$ is lower semicontinuous and convex, where $C = \max\{0, -\text{ess inf } U\}$. The domain of Ψ^D is $H^{(1)}(D;\mathbb{C})$.

Definition 12. *We denote by Ψ_0^D if the domain is restricted to the closure of $\mathcal{C}_0^\infty(D;\mathbb{C})$: $D(\Psi_0^D) = H_0^{(1)}(D;\mathbb{C})$.*

Definition 13. *Let $\Psi : \mathcal{H} \to]-\infty, +\infty]$ be a propery convex function. The subdifferential of Ψ is the (possibly multivalued) operator $\partial\Psi : \mathcal{H} \to \mathcal{H}$ defined by*

$$\partial\Psi(x) = \{w \in \mathcal{H}; \Psi(x) - \Psi(v) \leq (w, x-v),\ \forall v \in \mathcal{H}\}.$$

Since

$$-\partial\Psi_0^D = \triangle - U - C \quad \text{and} \quad D(\partial\Psi_0^D) = H_0^{(1)}(D;\mathbb{C}) \bigcap H^{(2)}(D;\mathbb{C}),$$

the equation in $L^2(D;\mathbb{C})$ with Dirichlet condition is written as

$$\frac{d}{dt}u_D(t) = -i\partial\Psi_0^D(u_D(t)) \Big(= i(\triangle - U - C)u_D(t)\Big) \quad \text{and} \quad u_D(0) = \varphi. \tag{19}$$

If the boundary ∂D of D is smooth, the normal derivative ∂_n is defined on ∂D, and we have

$$-\partial\Psi^D = \triangle - U - C \quad \text{where} \quad D(\partial\Psi^D) = \{\varphi \in H^{(2)}(D;\mathbb{C}) \;\Big|\; \partial_n\varphi|_{\partial D} = 0\}.$$

Hence the equation in $L^2(D;\mathbb{C})$ with (generalized) Neumann condition is written as

$$\frac{d}{dt}u_D(t) = -i\partial\Psi^D(u_D(t)) \quad \text{and} \quad u_D(0) = \varphi. \tag{20}$$

The semigroup $\{T_D(t)\}, T_D(t)\varphi = e^{-iCt}u_D(t)$, of solution family to (19) or (20) is a group of unitary operators, respectively. We define an order in $\mathcal{D}$ as follows :

$$D_\alpha, D_\beta \in \mathcal{D} : \quad D_\alpha \prec D_\beta \Longleftrightarrow \overline{D_\alpha} \subset D_\beta. \tag{21}$$

We consider an ultra-filter $\Phi = \{\phi\}$ whose element ϕ consists of infinite subsets of $\mathcal{D}$ satisfying (10) for $\mathcal{A} = \mathcal{D}$ in the next section :

$$\lim_{D \in \phi \in \Phi} D = \bigcup_{D \in \phi \in \Phi} D = \mathbb{R}^N \backslash \mathcal{N}.$$

Proposition 3. *We define an operator $T_\Phi(t)$ for $T_D(t)$ associated with* (19) *or* (20) *by*

$$T_\Phi(t)\varphi = w\text{-}\lim_{D \in \phi \in \Phi} T_D(t)\varphi \Big(= \tau_0\text{-}\lim_{D \in \phi \in \Phi} T_D(t)\varphi\Big) \quad \textit{for} \quad \forall\varphi \in L^2. \tag{22}$$

Then $\{T_\Phi(t)\}$ is a contraction semigroup.

6. Weak limit of unitary groups

6.1. **Existence in L^2-case.**

Theorem 5. *For each approximation $\{A_n\}$ or $\{A_D\}$, the limit $T_\Phi(t) = \lim_\Phi exp(tA_n)$ or $\lim_\Phi exp(tA_D)$ exists and $\{T_\Phi(t)\}$ is a contraction C_0-semigroup. Here $A_D = \partial\Phi_0^D$ in (19) or $\partial\Phi^D$ in (20).*

Lemma 4. *Condition 1 is satisfied for $\mathcal{H}_0 = C_0^\infty(\mathbb{R}^N \backslash \mathcal{N}; \mathbb{C})$.*

We consider an ultra-filter $\mathcal{D}_1 = \{\delta\}$ by the order (21) whose element ϕ consists of infinite subsets of $\mathcal{D}$. $\mathcal{D}_1$ satisfies (10) and

$$\lim_{D\in\delta\in\mathcal{D}_1} D = \bigcup_{D\in\delta\in\mathcal{D}_1} D = \mathbb{R}^N \backslash \mathcal{N}. \tag{23}$$

Lemma 5. *Let $\varphi_\Phi = w\text{-}\lim_{m\in\phi_1\in\Phi}(I - A_m)^{-1}f$ for $f \in L^2(\mathbb{R}^N;\mathbb{C})$. Then on any fixed $D \in \mathcal{D}$, the filter $\{\varphi_m; \varphi_m = (I - A_m)^{-1}f, m \in \phi \in \Phi\}$ strongly converge to φ_Φ on D, that is,*

$$\forall D \in \mathcal{D},\ \forall\varepsilon > 0,\ \exists\phi \in \Phi : \left\|(\varphi_\Phi - \varphi_m)\Big|_D\right\| < \varepsilon \quad for\ all \quad m \in \phi, \tag{24}$$

and the filter $\{\varphi_\Phi|_D; D \in \mathcal{D}_1\}$ strongly converge to φ_Φ, that is,

$$\forall\varepsilon > 0,\ \exists D \in \mathcal{D} \quad such\ that \quad \left\|\varphi_\Phi - \varphi_\Phi\Big|_D\right\| < \varepsilon. \tag{25}$$

Hence for $m > l$, we have

$$\int_D |(I - A_l)\varphi_m(x)|^2 dx = \int_D |(I - A_m)\varphi_m(x)|^2 dx \le \int_{\mathbb{R}^N} |(I - A_m)\varphi_m(x)|^2 dx = \|f\|^2.$$

That is, $\{\varphi_m|_D, m \in \phi\}$ is contained in a bounded subset of $H^{(2)}(\mathbb{R}^N;\mathbb{C})|_D \equiv \{\varphi|_D \bigm| \varphi \in H^{(2)}(\mathbb{R}^N;\mathbb{C})\}$, since two norms $\|\cdot\|_{(2)}$ and $\|\cdot\|_l = \|(I - A_l)^{-1}\cdot\|$ are equivalent on $H^{(2)}(\mathbb{R}^N;\mathbb{C})|_D$. A closed bounded subset of $H^{(2)}(\mathbb{R}^N;\mathbb{C})|_D$ is a compact subset of $L^2(D;\mathbb{C})$, since D is bounded in $\mathbb{R}^N$. Since the filter $\{\varphi_m|_D, m \in \phi \in \Phi\}$ is weakly convergent in $L^2(D;\mathbb{C})$, it is strongly convergent in $L^2(D;\mathbb{C})$.

Proposition 4. *Let $A = A_\Phi$ with domain $D(A) = H_{loc}^{(2)}(\mathbb{R}^N \backslash \mathcal{N};\mathbb{C})$. Then A is a closed operator from $L_{loc}^2(\mathbb{R}^N \backslash \mathcal{N};\mathbb{C})$ to $L_{loc}^2(\mathbb{R}^N \backslash \mathcal{N};\mathbb{C})$.*

Proof. Proof follows from Lemma 5. □

Proof of Theorem 5. From Lemma 4, Proposition 4 and Theorem 4 we obtain Thorem 5.

6.2. **Uniqueness.**

In this subsection we shall show the uniqueness of $T_\Phi(t)$ in (12) for the approximative equation (8).

Let $\Phi = \Big\{\phi = \{n_k\}\ ; n_k \in \mathbb{N}\Big\}$ be an ultra-filter of subsequences of natural numbers.

Theorem 6. *$T_\Phi(t)$ does not depend on Φ.*

We assume the following Assumption:
Assumption 1. $A_{\Phi_1} \neq A_{\Phi_2}$ for two ultra-filters Φ_1 and Φ_2 with $\Phi_1 \neq \Phi_2$.

In the following we shall show that Assumption 1 implies a contradiction. We shall begin with several Lemmas.

Lemma 6. *Suppose $T_{\Phi_1}(t) \neq T_{\Phi_2}(t)$. There exists $\varphi_0 \in C_0^\infty(\mathbb{R}^N \backslash \mathcal{N}; \mathbb{C})$ satisfies*

$$\exists t_1 > 0, \exists c_0 > 0 \quad \textit{such that} \quad \frac{d}{dt}\|T_{\Phi_1}(t)\varphi_0 - T_{\Phi_2}(t)\varphi_0\|\Big|_{t=t_1} \geq c_0. \tag{26}$$

Put $\varphi_1 = T_{\Phi_1}(t_1)\varphi_0$ and $\varphi_2 = T_{\Phi_2}(t_1)\varphi_0$. (26) means

$$\frac{d}{dt}\|T_{\Phi_1}(t)\varphi_1 - T_{\Phi_2}(t)\varphi_2\|\Big|_{t=0} \geq c_0 > 0. \tag{27}$$

Note that $\varphi_1 \in D(A_{\Phi_1})$ and $\varphi_2 \in D(A_{\Phi_2})$
Case 1. In the case that $\varphi_2 \in D(A_{\Phi_1})$.
We have

$$\frac{d^+}{dt}\|T_{\Phi_1}(t)\varphi_1 - T_{\Phi_1}(t)\varphi_2\|\Big|_{t=0} \leq 0, \tag{28}$$

In fact,

$$\|T_{\Phi_1}(h)\left(T_{\Phi_1}(t)\varphi_1 - T_{\Phi_1}(t)\varphi_2\right)\| \leq \|T_{\Phi_1}(t)\varphi_1 - T_{\Phi_1}(t)\varphi_2\| \quad \text{for} \quad \forall h > 0,$$

implies

$$\frac{1}{h}\Big(\|T_{\Phi_1}(t+h)\varphi_1 - T_{\Phi_1}(t+h)\varphi_2\| - \|T_{\Phi_1}(t)\varphi_1 - T_{\Phi_1}(t)\varphi_2\|\Big) \leq 0 \quad \text{for} \quad \forall h > 0.$$

From which the relation (28) follows. Note that the left hand of (28) exists since $\frac{d^+}{dt}T_{\Phi_1}(t)\varphi_1\Big|_{t=0}$ and $\frac{d^+}{dt}T_{\Phi_1}(t)\varphi_2\Big|_{t=0}$ exist for $\varphi_1, \varphi_2 \in D(A_{\Phi_1})$. We have

$$\begin{aligned}\|T_{\Phi_1}(h)\varphi_2 - T_{\Phi_2}(h)\varphi_2\| + \|T_{\Phi_1}(h)\varphi_1 - T_{\Phi_1}(h)\varphi_2\| - \|\varphi_1 - \varphi_2\| \\ \geq \|T_{\Phi_1}(h)\varphi_1 - T_{\Phi_2}(h)\varphi_2\| - \|\varphi_1 - \varphi_2\|.\end{aligned}$$

Hence

$$\frac{d^+}{dt}\|T_{\Phi_1}(t)\varphi_2 - T_{\Phi_2}(t)\varphi_2\|\Big|_{t=0} + \frac{d^+}{dt}\|T_{\Phi_1}(t)\varphi_1 - T_{\Phi_1}(t)\varphi_2\|\Big|_{t=0} \geq \frac{d^+}{dt}\|T_{\Phi_1}(t)\varphi_1 - T_{\Phi_2}(t)\varphi_2\|\Big|_{t=0}.$$

We have by (27) and (28)

$$\begin{aligned}\frac{d^+}{dt}\|T_{\Phi_1}(t)\varphi_2 - T_{\Phi_2}(t)\varphi_2\|\Big|_{t=0} &\geq \frac{d^+}{dt}\|T_{\Phi_1}(t)\varphi_1 - T_{\Phi_2}(t)\varphi_2\|\Big|_{t=0} - \frac{d^+}{dt}\|T_{\Phi_1}(t)\varphi_1 - T_{\Phi_1}(t)\varphi_2\|\Big|_{t=0} \\ &> 0.\end{aligned}$$

Hence

$$\begin{aligned}Re\langle (A_{\Phi_2} - A_{\Phi_1})\varphi_2, \varphi_2\rangle &= Re\frac{d^+}{dt}\langle (T_{\Phi_2}(t) - T_{\Phi_1}(t))\varphi_2, \varphi_2\rangle\Big|_{t=0} \\ &= \frac{1}{2}\frac{d^+}{dt}\|(T_{\Phi_2}(t) - T_{\Phi_1}(t))\varphi_2\|^2\Big|_{t=0} > 0,\end{aligned} \tag{29}$$

Thus we have $A_{\Phi_2}\varphi_2 \neq A_{\Phi_1}\varphi_2$. Since $C_0^\infty(\mathbb{R}^N \backslash \mathcal{N}; \mathbb{C})$ is dense in $L^2(\mathbb{R}^N; \mathbb{C})$, there exists $\psi \in C_0^\infty(\mathbb{R}^N \backslash \mathcal{N}; \mathbb{C})$ such that

$$\langle (A_{\Phi_2} - A_{\Phi_1})\varphi_2, \psi\rangle \neq 0.$$

Nevertheless, from (30) of lemma 7 we obtain that

$$\langle (A_{\Phi_2} - A_{\Phi_1})\varphi_2, \psi\rangle = \langle \varphi_2, ({}^tA_{\Phi_2} - {}^tA_{\Phi_1})\psi\rangle = -\langle \varphi_2, (A_0 - A_0)\psi\rangle = 0.$$

This is a contradiction.

Lemma 7. *Let* $A_0 = A|_{C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})}$ *(see Condition* 1*). We have*

$$A_0 = -{}^tA_{\Phi_1}|_{C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})} = -{}^tA_{\Phi_2}|_{C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})}. \tag{30}$$

Proof. Proof follows from (16) by Lemma 4. □

Corollary 2. *Let* $f \in L^2(\mathbb{R}^N;\mathbb{C})$ *and* $\psi \in C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$*. Then* $\langle T_{\Phi_2}(t)f,\psi\rangle$ *is differentiable in* $t \geq 0$:

$$\frac{d}{dt}\langle T_{\Phi_2}(t)f,\psi\rangle = \langle T_{\Phi_2}(t)A_{\Phi_2}f,\psi\rangle = \langle f, {}^t(T_{\Phi_2}(t){}^tA_{\Phi_2})\psi\rangle = -\langle f, {}^tT_{\Phi_2}(t)A_0\psi\rangle.$$

Case 2. In the case that $\varphi_2 \notin D(A_{\Phi_1})$.

This means

$$\|A_{\Phi_1}\varphi_2\|^2 = \lim_{D\in\delta\in\mathcal{D}_1}\int_D |A_{\Phi_1}\varphi_2|^2 dx = \infty, \tag{31}$$

since $\varphi_2 \in D(A_{\Phi_2}) \subset H_{loc}^{(2)}(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$ and $A_{\Phi_1}\varphi_2 = A_0\varphi_2 \in L_{loc}^2(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$ by Proposition 4.

Lemma 8. *There exists* $\delta > 0$ *such that*

$$0 < Re\frac{d}{dt}\langle T_{\Phi_2}(t)\varphi_2 - T_{\Phi_1}(t)\varphi_1,\psi\rangle\Big|_{t=0}, \tag{32}$$

if $\|\varphi_2-\varphi_1-\psi\| < \delta$ *and* $\psi \in C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$.

Proof. Let δ satisfy

$$0 < \delta < \frac{c_0\|\varphi_2-\varphi_1\|}{2\|A_{\Phi_2}\varphi_2 - A_{\Phi_1}\varphi_1\|}.$$

□

Lemma 9. *Let* δ *be in Lemma 8. Then there exists* $\psi_1 \in D(A_{\Phi_1})$ *with* $\|\varphi_2-\varphi_1-\psi_1\| < \delta$*, such that*

$$Re\frac{d}{dt}\langle T_{\Phi_1}(t)\varphi_2 - T_{\Phi_1}(t)\varphi_1,\psi_1\rangle\Big|_{t=0} = Re\langle A_{\Phi_1}\varphi_2 - A_{\Phi_1}\varphi_1,\psi_1\rangle \leq 0, \tag{33}$$

where

$$\frac{d}{dt}\langle T_{\Phi_1}(t)\varphi_2 - T_{\Phi_1}(t)\varphi_1,\psi_1\rangle = \frac{d}{dt}\langle\varphi_2-\varphi_1, {}^tT_{\Phi_1}(t)\psi_1\rangle\left(= \langle\varphi_2-\varphi_1, \frac{d}{dt}{}^tT_{\Phi_1}(t)\psi_1\rangle\right).$$

Proof. We recall $\|A_{\Phi_1}\varphi_2\| = \infty$ (see (31)). That is, for any $L > 0$ and δ in Lemma 8, there exists $\psi_\varepsilon \in C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$ such that $|\langle A_{\Phi_1}\varphi_2 - A_{\Phi_1}\varphi_1,\psi_\varepsilon\rangle| > L$ and $\|\psi_\varepsilon\| < \delta/2$. Therfore $\langle A_{\Phi_1}\varphi_2 - A_{\Phi_1}\varphi_1, e^{i\theta}\psi_\varepsilon\rangle < -L$ for some real θ.

For $\psi_0 \in C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$ satisfying $\|\varphi_2-\varphi_1-\psi_0\| < \delta/2$, put

$$L := |\langle A_{\Phi_1}(\varphi_2-\varphi_1),\psi_0\rangle|.$$

For $\psi_1 = \psi_0 + e^{i\theta}\psi_\varepsilon$ we have $\|\varphi_2 - \varphi_1 - \psi_1\| < \delta$. Hence by Lemma 8

$$\begin{aligned} Re\langle A_{\Phi_1}\varphi_2 - A_{\Phi_1}\varphi_1, \psi_1\rangle &= Re\langle A_{\Phi_1}\varphi_2 - A_{\Phi_1}\varphi_1, \psi_0 + e^{i\theta}\psi_\varepsilon\rangle \\ &\le |\langle A_{\Phi_1}\varphi_2 - A_{\Phi_1}\varphi_1, \psi_0\rangle| + \langle A_{\Phi_1}\varphi_2 - A_{\Phi_1}\varphi_1, e^{i\theta}\psi_\varepsilon\rangle \\ &\le L - L = 0. \end{aligned}$$

□

Lemma 10. *For $\psi_1 \in C_0^\infty(\mathbb{R}^N\backslash\mathcal{N};\mathbb{C})$ in Lemma 9 we have*

$$Re\frac{d}{dt}\langle (T_{\Phi_2}(t) - T_{\Phi_1}(t))\varphi_2, \psi_1\rangle\Big|_{t=0} > 0, \tag{34}$$

where $\frac{d}{dt}\langle (T_{\Phi_2}(t) - T_{\Phi_1}(t))\,\varphi_2, \psi_1\rangle = \frac{d}{dt}\langle \varphi_2, ({}^tT_{\Phi_2}(t) - {}^tT_{\Phi_1}(t))\,\psi_1\rangle$.

Proof. By using (32) and (33) we get

$$\begin{aligned} 0 &< Re\frac{d}{dt}\langle T_{\Phi_2}(t)\varphi_2 - T_{\Phi_1}(t)\varphi_1, \psi_1\rangle\Big|_{t=0} + Re\frac{d}{dt}\langle T_{\Phi_1}(t)\varphi_1 - T_{\Phi_1}(t)\varphi_2, \psi_1\rangle\Big|_{t=0} \\ &= Re\frac{d}{dt}\langle T_{\Phi_2}(t)\varphi_2 - T_{\Phi_1}(t)\varphi_2, \psi_1\rangle\Big|_{t=0}. \end{aligned}$$

□

On the other hand, from Lemma 7 and Corollary 2 it follows that

$$Re\frac{d}{dt}\langle T_{\Phi_2}(t)\varphi_2 - T_{\Phi_1}(t)\varphi_2, \psi_1\rangle\Big|_{t=0} = Re\langle \varphi_2, ({}^tA_{\Phi_2} - {}^tA_{\Phi_1})\psi_1\rangle = Re\langle \varphi_2, (A_0 - A_0)\psi_1\rangle = 0.$$

This is a contradiction to (34).

Thus in both cases we get a contradiction and the proof of Theorem 6 is complete.

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Structure of solutions to the Emden equation on a geodesic ball in a sphere

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1 Introduction and main results

In this seminar we discussed the following problem

$$\begin{cases} \Delta_{\mathbf{S}^N} u + u^p = 0 & \text{in } B_{\theta_0}, \\ u > 0 & \text{in } B_{\theta_0}, \\ u + \kappa \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial B_{\theta_0}, \end{cases} \tag{1.1}$$

where $N \geq 3$, $\mathbf{S}^N = \{x \in \mathbf{R}^{N+1} \mid |x| = 1\}$, $\Delta_{\mathbf{S}^N}$ is the Laplace-Beltrami operator on $\mathbf{S}^N$, n is the outer unit normal vector to ∂B_{θ_0} and $\kappa \geq 0$. Here B_{θ_0} is a geodesic ball in $\mathbf{S}^N$ with its geodesic radius $\theta_0 \in (0, \pi)$, and its center is located at the north pole $P_n = (x_1, x_2, ..., x_{N+1}) = (0, 0, ..., 1)$.

Especially we focused on a *radial* solution to (1.1), that is, a solution u to (1.1) depending only on a geodesic distance from P_n. It is significant to consider a radial solution. In fact, in [3] and [5], it is proved that, under $p \leq p_* := (N+2)/(N-2)$ and $\kappa = 0$, any solution $u \in C^2(B_{\theta_0})$ to (1.1) is a radially symmetric. In addition the existence of a positive radial solution is corresponding the Sobolev imbedding $H_0^1(B_{\theta_0}) \to L^{p+1}(B_{\theta_0})$. In [1] and [2], the problem (1.1) is studied to investigate the existence of a function attaining the best constant of the Sobolev imbedding with $p = p_*$, that is, the critical Sobolev exponent. Thus we were also interested in a radial solution to (1.1), and hence we write (1.1) by using polar coordinates. Namely let

$$\begin{cases} x_1 = r \sin\theta \sin\varphi_1 \sin\varphi_2 ... \sin\varphi_{N-1} \\ x_2 = r \sin\theta \sin\varphi_1 \sin\varphi_2 ... \cos\varphi_{N-1} \\ x_3 = r \sin\theta \sin\varphi_1 \sin\varphi_2 ... \cos\varphi_{N-2} \\ \vdots \\ x_N = r \sin\theta \cos\varphi_1 \\ x_{N+1} = r \cos\theta \end{cases} \tag{1.2}$$

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with $r \geq 0$, $\theta, \varphi_i \in [0, \pi]$ ($i = 1, 2, ..., N-2$) and $\varphi \in [0, 2\pi]$. By (1.2), the Laplace–Beltrami operator $\Delta_{\mathbf{S}^N}$ is written by

$$\begin{aligned}\Delta_{\mathbf{S}^N} u =& \frac{1}{\sin^{N-1}} \frac{\partial}{\partial \theta} \left(\sin^{N-1} \theta \frac{\partial u}{\partial \theta} \right) \\ &+ \sum_{i=1}^{N-1} \frac{1}{\sin^2 \theta \sin^{N-i-1} \varphi_i \prod_{j=1}^{i-1} \sin^2 \varphi_j} \frac{\partial}{\partial \varphi_i} \left(\sin^{N-i-1} \varphi_i \frac{\partial u}{\partial \varphi_i} \right).\end{aligned}$$

Therefore a radial solution to (1.1) satisfies

$$\begin{cases} \dfrac{1}{\sin^{N-1} \theta} \left(u_\theta \sin^{N-1} \theta \right)_\theta + u^p = 0 & \text{for } \theta \in (0, \theta_0), \\ u(\theta) > 0 & \text{for } \theta \in (0, \theta_0), \\ u(\theta_0) + \kappa u_\theta(\theta_0) = 0. \end{cases} \tag{1.3}$$

In this talk, we spoke of structure of solutions to (1.3) with two cases, that is, the critical case $p = p_*$ and the supercritical case $p > p_*$. Hereafter a solution u to (1.3) converging to some constant as $\theta \to 0$ is said to be a *regular* solution. On the other hand, a solution u to (1.3) tends to $+\infty$ as $\theta \to 0$ is said to be a *singular* solution.

First we stated the critical case. Under $p = p_*$, there are preceding studies, e.g.,[1] and [2], and the following proposition is proved:

Proposition 1.1 *Assume $p = p_*$ and $\kappa = 0$. If $N = 3$, then the following statements hold:*

(i) *If $\theta_0 \in (0, \pi/2]$, then there exists no regular solution to* (1.3).

(ii) *If $\theta_0 \in (\pi/2, \pi)$, then there exists a singular classical solution to* (1.3).

On the other hand, if $N \geq 4$, then, for any $\theta_0 \in (0, \pi)$, a regular solution to (1.3) *exists.*

Thus, under the Dirichlet boundary condition, the structure of solutions to (1.3) is completely known. Moreover, from Proposition 1.1, we see that there is a difference between the cases $N = 3$ and $N \geq 4$. Under $N = 3$, the existence of solutions varies at $\theta_0 = \pi/2$. Bandle, Brillard and Flucher have proved that, in $N = 3$, the existence of solutions varies at some constant $\theta_c \in (0, \pi)$. After their study, Bandle and Peletier investigated this problem in detail, and they proved $\theta_c = \pi/2$.

We considered the above problem under a more general boundary condition. That is why we expect that the investigation provides us a comprehensive view to the structure of solutions to the Emden equation. Our results is as follows:

Theorem 1.1 (Theorem 1.1 in [6]) *Assume $p = p_*$. For* (1.3), *the following statements hold.*

(i) *Suppose that $0 \leq \kappa \leq 1/2$. If θ_0 satisfies*

$$\frac{1}{2} \operatorname{Arcsin} 2\kappa \leq \theta_0 \leq \frac{1}{2} (\pi - \operatorname{Arcsin} 2\kappa), \tag{C}$$

then (1.3) *has no regular or singular solution. On the other hand, if θ_0 does not satisfy (C), then, for each θ_0,* (1.3) *has a unique regular solution and infinitely many singular solutions.*

(ii) *Suppose that* $\kappa > 1/2$. *Then, for any* θ_0, (1.3) *has a unique regular solution and infinitely many of singular solutions.*

By Theorem 1.1, we strictly obtain the information concerning the existence of solutions to (1.3) with $N = 3$. Moreover we proved the existence of a singular solution as well as a regular solution.

Second we stated the supercritical case $p > p_*$. In this case, we assume $\kappa = 0$, that is, we only consider the Dirichlet boundary condition. Because of the continuity of solutions to (1.3) concerning a parameter, it seems that there exists a solution to (1.3) with $p > p_*$ for some θ_0. In fact, for p sufficiently near p_*, there exists a regular solution:

Theorem 1.2 *Assume* $\kappa = 0$, *and* $\theta_0 \in (\pi/2, \pi)$ $(N = 3)$ *or* $\theta_0 \in (0, \pi)$ $(N \geq 4)$. *Then there exists some* $\epsilon_0(\theta_0) > 0$ *such that, for any* $\epsilon \in (0, \epsilon_0)$, *there exists at least two regular solutions to* (1.3) *with* $p = p_* + \epsilon$.

Theorem 1.2 implies that, for a perturbation $p = p_* + \epsilon$, a new solution appears. The above result only explains the sufficiently near critical case, and it seems difficult to investigate the existence of solutions to (1.3) for any $p > p_*$, and we do not obtain such a result yet. However we can investigate the nonexistence of solutions for sufficiently large $p > p_*$, and the result is as follows:

Theorem 1.3 *Under the same assumptions as in Theorem* 1.2, *there exists some* $p_c(\theta_0) > p_*$ *such that, for any* $p > p_c$, *there exists no regular or singular solution to* (1.3).

Theorems 1.2 and 1.3 are quite different from results of the Emden equation on a ball in the Euclidean space $\mathbf{R}^N$. It seems that the difference derive from the metric of $\mathbf{S}^N$, and we are required to investigate the problem in detail.

2 Ideas of proofs

In this section we explain ideas used in proofs of Theorems 1.1–1.3. Our methods of proofs are owing to Yanagida and Yotsutani's studies [8], [9]. First we transform (1.3) to the exterior problem. Namely we define

$$\rho := \frac{\kappa}{\sin^{N-1}\theta_0}, \tag{2.1}$$

$$\tau := \int_\theta^{\theta_0} \frac{d\psi}{\sin^{N-1}\psi} + \rho, \tag{2.2}$$

and, by using (2.1) and (2.2), a new function

$$w(\tau) := \frac{u(\theta)}{\tau}$$

is defined. Here we remark that, from (2.2), τ attains ρ as $\theta = \theta_0$, and $\tau \to +\infty$ as $\theta \to 0$. For w defined above, the next lemma holds:

Lemma 2.1 *The function $w = u/\tau$ satisfies*

$$\begin{cases} \dfrac{1}{\tau^2}\left(\tau^2 w_\tau\right)_\tau + K(\tau)w^p(\tau) = 0 & \text{for } \tau \in (\rho, +\infty), \\ w(\rho) = \beta, \\ w_\tau(\rho) = 0, \end{cases} \tag{2.3}$$

where $\beta := \alpha \sin^2 \theta_0$ with $\alpha := -u_\theta(\theta_0)$. Here $K(\tau)$ is defined as

$$K(\tau) := \tau^{p-1} \sin^{2N-2} \theta.$$

Conversely if $w \in C^2(\rho, +\infty)$ is a positive solution to (2.3)*, then $u = \tau w$ is a solution to* (1.3)*.*

Next we will investigate the structure of solutions to (2.3) instead of that of (1.3). The following lemma implies that positive solutions to (2.3) is classified into two types:

Lemma 2.2 *If a solution w to* (2.3) *satisfies $w > 0$ on $(\rho, +\infty)$, then $\tau w(\tau)$ is non-decreasing for $\tau \in (\rho, +\infty)$.*

Namely if w is a positive solution to (2.3), then $\lim_{\tau\to+\infty} \tau w(\tau) = const.$ or τw tends to $+\infty$ as $\tau \to +\infty$. We define these types as follows:

Definition 2.1 (i) *A solution w to* (2.3) *is said to be a rapidly decaying solution if $w > 0$ on $[\rho, +\infty)$ and $\tau w(\tau)$ converges to some positive constant as $\tau \to +\infty$.*

(ii) *A solution w to* (2.3) *is said to be a slowly decaying solution if $w > 0$ on $[\rho, +\infty)$ and $\tau w(\tau) \to +\infty$ as $\tau \to +\infty$.*

(iii) *A solution w to* (2.3) *is said to be a crossing solution if w has a zero in $(\rho, +\infty)$.*

Moreover we see that three types of solutions are corresponding to the behavior of the function

$$P(\tau; w) := \frac{1}{2}\tau^2 w_\tau\{\tau w_\tau + w\} + \frac{\tau^3}{p} K(\tau) w^p.$$

Kawano, Yanagida and Yotsutani [4] investigated the relation between types defined in Definition 2.1 and the behavior of P. From their results, the following statements holds:

Lemma 2.3 (1) *$\tau w \to const.$ as $\tau \to +\infty$ if and only if $\lim_{\tau\to+\infty} P(\tau; w) = 0$,*

(2) *$\tau w \to +\infty$ as $\tau \to +\infty$ if and only if $\lim_{\tau\to+\infty} P(\tau; w) < 0$,*

(3) *w attains 0 for some $\tau_1 \in (\rho, +\infty)$ if and only if $\lim_{\tau\to+\infty} P(\tau; w) > 0$.*

From the above arguments, it suffices to investigate the behavior of P as $\tau \to +\infty$. Since P contains the unknown function w, and it is not easy to investigate the behavior of P. Yanagida and Yotsutani's idea is to use the next functions

$$G(\tau) := \frac{1}{p+1}\left\{\tau^3 K(\tau) - \frac{1}{2}(p+1)\int_\rho^\tau s^2 K(s)ds\right\},$$
$$H(\tau) := \frac{1}{p+1}\left\{\tau^{2-p} K(\tau) - \frac{1}{2}(p+1)\int_\tau^{+\infty} s^{1-p} K(s)ds\right\}.$$

For G and H, it holds that

$$\frac{d}{d\tau}P(\tau;w) = G_\tau(\tau)w^{p+1}(\tau),$$

and

$$G_\tau(\tau) = \frac{\tau^{(p+1)/2}}{p+1}(\tau^{-\frac{p-5}{2}}L)_\tau = \tau^{p+1}H_\tau(\tau).$$

We see that the behavior of P depends on that of G. Functions G and H plays a important role when we investigate the behavior of P. Before stating the result, we define

$$\begin{aligned}\tau_G &:= \inf\{\tau \in [\rho,+\infty) \mid G(\tau) < 0\},\\ \tau_H &:= \sup\{\tau \in [\rho,+\infty) \mid H(\tau) < 0\}.\end{aligned}$$

Here we define $\tau_G = +\infty$ if $G(\tau) \geq 0$ on $(\rho,+\infty)$ and $\tau_H = \rho$ if $H(\tau) \geq 0$ on $(\rho,+\infty)$. By using two value τ_G and τ_H, the next statements holds:

Proposition 2.1 *Under $p = p_*$, the following statements holds.*

(i) *If $\tau_G = +\infty$, then the structure of solutions to* (2.3) *is of type C: $w(\tau;\beta)$ is a crossing solution for any $\beta > 0$.*

(ii) *If $\rho = 0$ and $\tau_H = 0$, then the structure of solutions to* (2.3) *is of type S: $w(\tau;\beta)$ is a slowly decaying solution for any $\beta > 0$.*

(iii) *If $\rho < \tau_H \leq \tau_G < +\infty$, then the structure of solutions to* (2.3) *is of type M: there exists a constant $\beta_* > 0$ such that $w(\tau;\beta)$ is a slowly decaying solution for $\beta \in (0,\beta_*)$, $w(\tau;\beta_*)$ is a rapidly decaying solution, and $w(\tau;\beta)$ is a crossing solution for $\beta \in (\beta_*,+\infty)$.*

(iv) *If $0 < \rho = \tau_H \leq \tau_G < +\infty$, then the structure of solutions to* (2.3) *is of type M.*

(We newly proved Proposition 2.1 (iv) in [6]. The case (iv) is not proved in [8] and [9] yet.) Theorem 1.1 is a direct result of Proposition 2.1, and, for strict calculation, see Section 4 in [6].

Finally we states the case $p > p_*$. In this case we also use the function P, that is, since

$$\begin{aligned}\frac{d}{d\tau}P(\tau;w) &= G_\tau(\tau)w^{p+1}(\tau),\\ G_\tau(\tau) &= \frac{1}{p+1}r(\tau,p)\tau^{p+1}\sin^{2N-2}\theta,\\ r(\tau,p) &= \frac{p+3}{2} - (2N-2)\tau\sin^{N-2}\theta\cos\theta,\end{aligned}$$

we see that $P(\tau;w) \to +\infty$ as $\tau \to +\infty$ when p is sufficiently large. In fact if p is sufficiently large, then $r(\tau,p) > 0$ for any τ. Hence $dP/d\tau > 0$, and we obtain Theorem 1.3. On the other hand, if $p = p_* + \epsilon$ (ϵ is sufficiently small), then the behavior of P is complicated, and it is difficult to investigate the behavior of that (in this case, we cannot apply Proposition 2.1). However, for small initial data β, the following lemma is known:

Lemma 2.4 (Theorem 3 in [9]) *If* $\liminf_{\tau\to+\infty} G_\tau(\tau) > 0$*, then there exists* $\beta_c > 0$ *such that, for any* $\beta \in (0, \beta_c)$*,* $w(\tau; \beta)$ *is a crossing solution.*

If we assume that there exists only one rapidly decaying solution to (2.3) with $p = p_* + \epsilon$, then the assumption is inconsistent with Lemma 2.4, and therefore we can prove Theorem 1.2 (strict arguments are stated in [7]).

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ON WELL-POSEDNESS FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH POWER NONLINEARITY IN FRACTIONAL ORDER SOBOLEV SPACES

HARUNORI UCHIZONO AND TAKESHI WADA

Abstract. We study the well-posedness for the nonlinear Schrödinger equation (NLS)

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^{p-1}u$$

in $\mathbb{R}^{1+n}$, where $p > 1, \lambda \in \mathbb{C}$, and prove that (NLS) is locally well-posed in H^s if $2 < s < 4$ and $s/2 < p < 1 + 4/(n-2s)_+$. To obtain good lower bound for p, we systematically use Strichartz type estimates in fractional order Besov spaces for time variable.

1. Introduction

This paper is a survey of our recent result [15]. We consider the Cauchy Problem for the nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = f(u), \tag{1}$$

$$u(0) = \phi, \tag{2}$$

where $u : \mathbb{R}^{1+n} \to \mathbb{C}$ is the unknown function, $f(u) = \lambda|u|^{p-1}u$ with $p > 1, \lambda \in \mathbb{C}$. Introducing the propagator $U(t) = \exp(it\Delta/2)$ and the retarded potential $Gg(t) = \int_0^t U(t-\tau)g(\tau)d\tau$, we can convert the problem (1)-(2) to the equivalent integral equation

$$u(t) = U(t)\phi - i\left(Gf(u)\right)(t). \tag{3}$$

The solvability of (1)-(2) has been studied by many authors, see e.g. [1,3,5–8,10–13]. The problem (1)-(2) is said to be locally well-posed in H^s if (3) has a unique local (in time) solution $u \in C([0,T]; H^s)$ for any $\phi \in H^s$ and the flow mapping $\phi \mapsto u$ is a continuous mapping from H^s to $C([0,T]; H^s)$. Here T need to be taken uniformly in some neighborhood of arbitrarily fixed $\phi \in H^s$. If $0 \le s < n/2$, the local solvability of (3) has been established for $p_0(s) < p < 1 + 4/(n-2s)$, where $p_0(s) = 1$ for $s \le 2, s-1$ for $2 < s < 4$ and $s-2$ for $s \ge 4$; if $s \ge n/2$, (3) is locally solvable for $p_0(s) < p < \infty$. In some cases, we need auxiliary spaces of Strichartz type (see [9]). The lower bound $p_0(s)$ mentioned above is due to [11]. This result was proved for $s = 1$ by [5, 6], $s = 0$ by [12], and $s = 2$ by [13] provided that $\lambda \in \mathbb{R}$, mainly by the use of L^p-L^q estimate and the regularization technique. Kato [7, 8] systematically used the Strichartz estimate and gave an alternative proof of solvability for $s = 0, 1, 2$. His proof is also applicable for the

Topics: Nonlinear Differential Equations and Evolution Equations, Partial Differential Equations and their Applications case $\lambda \notin \mathbb{R}$. Cazenave-Weissler [3] proved the result above for $s \notin \mathbb{Z}$ under the additional assumption $p > [s]+1$, and this can be lowered to $p > s$ by the method of Ginibre-Ozawa-Velo [4]. Pecher [11] used fractional regularity spaces of Besov type for time variable and proved the result for $p > p_0(s)$.

In the preceding results referred above, the natural upper bound $p < 4/(n-2s)$ comes from the scale invariance of (1), whereas the lower bound $p > p_0(s)$ comes from the finite (at most p-times) differentiability of the nonlinear term $f(u)$. Indeed, Pecher [11] principally estimate the equation in $H^1_q(B^{s-2-\epsilon}_{r,2})$ when $2 < s < 4$, and in $H^2_q(B^{s-4-\epsilon}_{r,2})$ when $s \geq 4$, by which we would need $p > p_0(s)$. However, this condition does not seem to be natural since $p_0(4-0) > p_0(4+0)$. Taking account of the property that for Schrödinger equation, one time derivative corresponds to two space derivatives, the optimal lower bound for $2 < s < 4$ should be $p > s/2$, which linearly connects $p_0(2)$ and $p_0(4)$. Actually, by the systematical use of fractional order Besov spaces for time variable, we can obtain the desired result, to be stated in §2.

2. Main result

Theorem 1. *Let $n \geq 5, 2 < s < \min(4, n/2)$ and $s/2 < p < 1 + 4/(n-2s)$. Let*

$$\left(\frac{n}{2} - s\right)\frac{p-1}{p+1} < \frac{2}{q} = \delta(r) \equiv \frac{n}{2} - \frac{n}{r} < \min\left\{\frac{n}{2} - s; \frac{n}{2}\cdot\frac{p-1}{p+1}; \frac{2}{p+1}\right\}.$$

Then for any $\phi \in H^s$, there exists $T = T(\|\phi\|_{H^s})$ and (3) has a unique solution u in

$$X = C([0,T]; H^s) \cap L^q(0,T; B^s_{r,q}) \cap B^{s/2}_{q,2}(0,T; L^r).$$

Moreover, the flow mapping $\phi \mapsto u$ is a continuous mapping from H^s to X.

We remark that in the preceding we have assumed $s < n/2$, which requires $n \geq 5$ in our theorem, simply because we describe the results (and the proof of the theorem) in a unified manner. If $s > n/2$, we can obtain similar results more easily because $H^s \subset L^\infty$. Especially, we can prove the analogous result to our theorem under the assumption $n \geq 1$, $2 < s < 4$ and $s/2 < p < 1 + 4/(n-2s)_+$. If $s \geq n/2$, we should choose q, r so that

$$0 < \frac{2}{q} = \delta(r) < \min\left\{\frac{n}{2}\cdot\frac{p-1}{p+1}; \frac{2}{p+1}\right\}.$$

We can prove Theorem 1 by contraction mapping principle. For the proof, see [15]. One of the key estimate in the proof is the following version of Strichartz type estimates.

Lemma 1. *Let $s > 0, 0 < \theta_- < \theta < \theta_+ < 1$ and let $0 < 2/q = \delta(r) < 1$. Then we have the following:*
(i) *if $\phi \in H^s$, then $U(\cdot)\phi \in C(H^s) \cap L^q(B^s_{r,2}) \cap B^{s/2}_{q,2}(L^r)$ with the estimate*

$$\|U(\cdot)\phi\|_{L^\infty(H^s) \cap L^q(B^s_{r,2}) \cap B^{s/2}_{q,2}(L^r)} \leq C\|\phi\|_{H^s};$$

(ii) *if $f \in B^\theta_{q',2}(L^{r'}) \cap \bigcap_\pm L^{q_*(\theta_\pm)}(L^{r_*(\theta_\pm)})$, then $Gf \in C(H^{2\theta})$ with the estimate*

$$\|Gf\|_{L^\infty(H^{2\theta})} \leq C\|f\|_{B^\theta_{q',2}(L^{r'})} + C\max_\pm \|f\|_{L^{q_*(\theta_\pm)}(L^{r_*(\theta_\pm)})},$$

where $1/q_*(\theta) = (1-\theta)/q'$ *and* $1/r_*(\theta) = (1-\theta)/r' + \theta/2$;
(iii) *if* $f \in B^{\theta}_{q',2}(L^{r'}) \cap \bigcap_{\pm} L^{\bar{q}(\theta_\pm)}(L^{r_*(\theta_\pm)})$, *then* $Gf \in L^q(B^{2\theta}_{r,q}) \cap B^{\theta}_{q,2}(L^r)$ *with the estimate*

$$\|Gf\|_{L^q(B^{2\theta}_{r,q}) \cap B^{\theta}_{q,2}(L^r)} \le C\|f\|_{B^{\theta}_{q',2}(L^{r'})} + C \max_{\pm} \|f\|_{L^{\bar{q}(\theta_\pm)}(L^{\bar{r}(\theta_\pm)})},$$

where $1/\bar{q}(\theta) = (1-\theta)/q' + \theta/q$ *and* $1/\bar{r}(\theta) = (1-\theta)/r' + \theta/r$.

This lemma is first proved by Pecher [11] and refined in our papers [14, 15].

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Existence and stability of stationary solutions to a multidimensional SKT cross-diffusion equation

On the occasion of 80th birthday of Professor Hiroki Tanabe

Shoji Yotsutani (Ryukoku University)

1 Introduction

This is a joint project with Yuan Lou (Ohio State University), Wei.-Ming Ni (University of Minnesota and East China Normal University) concerning mathematical analysis, and Masaharu Nagayama (Hokkaido University), Tatsuki Mori (Ryukoku University) concerning numerical computation.

In an attempt to model segregation phenomena in population dynamics, Shigesada, Kawasaki and Teramoto [7] in 1979 incorporated the inter-competition system. In particular, the following system was proposed

$$\begin{cases} u_t = \Delta[(d_1 + \rho_{11}u + \rho_{12}v)u] + u(a_1 - b_1u - c_1v), & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta[(d_2 + \rho_{21}u + \rho_{22}v)v] + v(a_2 - b_2u - c_2v), & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain $R^N (N \geq 1)$ with smooth boundary $\partial\Omega$. Here u and v represent the densities of two competing species. The constants a_j, b_j, c_j and d_j $(j = 1, 2)$ are all positive, where a_1, a_2 denote the intrinsic growth rates of these two species, b_1 and c_2 account for intra-specific competitions while b_2, c_1 account for inter-specific competitions, and d_1, d_2 are their diffusion rates. The constants ρ_{11}, ρ_{22} represent intra-specific population pressures, also known as self-diffusion rates, and ρ_{12}, ρ_{21} are the coefficients of inter-specific population pressures, also known as cross-diffusion rates.

For convenience, we set $A := a_1/a_2,\ B := b_1/b_2,\ C := c_1/c_2$. If $B < C$, we call it the strong competition case and $B > C$ the weak competition case.

If $\rho_{11} = \rho_{12} = \rho_{21} = \rho_{22} = 0$, then (1.1) is the classical Lotka-Volterra competition diffusion system with Neumann boundary condition

$$\begin{cases} u_t = d_1\Delta u + u(a_1 - b_1u - c_1v), & \text{in } \Omega \times (0,\infty), \\ v_t = d_2\Delta v + v(a_2 - b_2u - c_2v), & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & \text{in } \Omega. \end{cases} \tag{1.2}$$

It is well known that in the "weak competition" case, i.e.

$$B > A > C,$$

the constant steady state $(u_*, v_*) = \left(\frac{a_1c_2 - a_2c_1}{b_1c_2 - b_2c_1}, \frac{b_1a_2 - b_2a_1}{b_1c_2 - b_2c_1}\right)$ is globally asymptotically stable regardless of the diffusion rates d_1 and d_2. This implies, in particular, that no nonconstant steady state can exist for any diffusion rates d_1, d_2.

On the other hand, it seems not entirely reasonable to add just diffusions to models in population dynamics, since individuals do not move around completely randomly. In particular, while modeling segregation phenomena for two competing species one must take into account the cross-diffusion pressures

$$\begin{cases} u_t = \Delta[(d_1 + \rho_{12}v)u] + u(a_1 - b_1u - c_1v), & \text{in } \Omega \times (0,\infty), \\ v_t = \Delta[(d_2 + \rho_{21}u)v] + v(a_2 - b_2u - c_2v), & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & \text{in } \Omega. \end{cases} \tag{1.3}$$

Mimura and his collaborators started mathematical analysis around 1980 (see, e.g. Mimura [4]). Considerable work has been done concerning the global existence of solutions to systems (1.3) under various hypotheses. A priori estimates are crucial to obtain the global existence. As for recent progress including stationary problems, see Ni [5], Ni [6], Yagi[9] and Yamada [10].

2 Limiting equation

The following two theorem are due to Lou-Ni [1], [2].

Theorem 2.1 *([2]) Suppose for simplicity that $\rho_{21} = 0$. Suppose further that $B \neq A \neq C$, $n \leq 3$ and $\frac{a_2}{d_2} \neq \lambda_k$ for any $k \geq 1$, where λ_k is the kth eigenvalue of $-\Delta$ on Ω with zero Neumann boundary data. Let (u_j, v_j) be a nonconstant steady state solution of (1.3) with $\rho_{12} = \rho_{12,j}$. Then by passing to a subsequence if necessary, either (i) of (ii) holds as $\rho_{12,j} \to \infty$:*

(i) $(u_j, \frac{\rho_{12,j}}{d_1}v_j) \to (u,v)$ uniformly, $u > 0$, $v > 0$, and

$$\begin{cases} d_1\Delta[(1+v)u] + u(a_1 - b_1u) = 0 & \text{in } \Omega, \\ d_2\Delta v + v(a_2 - b_2u) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

(ii) $(u_j, v_j) \to (\frac{\tau}{v}, v)$ *uniformly,* τ *is a positive constant,* $v > 0$*, and*

$$\begin{cases} \displaystyle\int_\Omega \frac{\tau}{v}\left(a_1 - b_1\frac{\tau}{v} - c_1 v\right) \mathrm{d}x = 0, & \\ d_2\Delta v + v(a_2 - c_2 v) - b_2\tau = 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Their proofs of obtaining the above limiting equations are quite hard and lengthy. The most important step in the proof is to obtain a priori bounds on steady states of (1.3) that are independent of ρ_{12}.

It seems from numerical computations that solutions of the case (i) is not directly related with stable solutions of the original equation with sufficiently large ρ_{12}. However, we observe numerically that solutions of the case (ii) is closely related with the original equation with sufficiently large ρ_{12}.

Thus, we will concentrate on the case (ii). Now, we consider the 1-dimensional case with $\Omega = (0, 1)$. The limiting equation becomes as follows:

$$\begin{cases} \displaystyle\int_0^1 \frac{1}{v}\left(a_1 - b_1\frac{\tau}{v}\right) \mathrm{d}x - c_1 = 0, \\ d_2 v_{xx} + v\left(a_2 - b_2\frac{\tau}{v} - c_2 v\right) = 0, \ \text{ in } (0, 1), \\ v_x(0) = v_x(1) = 0, \\ v > 0, \ \text{ in } (0, 1). \end{cases} \tag{2.2}$$

Let us consider about a time-dependent limiting equation as $\rho_{12} \to \infty$ under the condition $\rho_{21} = 0$. Time-dependent limiting equation is the following: Unknown functions are $\tau(t)$, $v(x, t)$, and

$$\begin{cases} \dfrac{d}{dt}\left(\displaystyle\int_\Omega \frac{\tau}{v}\, dx\right) = \displaystyle\int_\Omega \frac{\tau}{v}\left(a_1 - b_1\frac{\tau}{v} - c_1 v\right) \mathrm{d}x, & \\ \dfrac{\partial v}{\partial t} = d_2\Delta v + v(a_2 - c_2 v) - b_2\tau & \text{in } \Omega, \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

This is formally derived by rewriting the first equation as

$$u_t = \ \rho_{12}\ \Delta\left[\left(\frac{d_1}{\rho_{12}} + v\right) u\right] + u(a_1 - b_1 u - c_1 v)$$

and

$$\frac{d}{dt}\left(\int_\Omega u dx\right) = \int_\Omega u(a_1 - b_1 u - c_1 v) dx.$$

3 Structure and stability in 1-dimensional case

Due to the scaling and reflection properties of solutions to autonomous ordinary differential equations, all solutions to the (2.2) are obtained by several reflections and a suitable re-scaling from solutions of the following system:

$$\begin{cases} \displaystyle\int_0^1 \frac{1}{v}\left(a_1 - b_1\frac{\tau}{v}\right) \mathrm{d}x - c_1 = 0, \\ d_2 v_{xx} + v\left(a_2 - b_2\dfrac{\tau}{v} - c_2 v\right) = 0 \ \text{ in } (0,1), \\ v_x(0) = v_x(1) = 0, \\ v > 0, \ \text{ and } \ v_x > 0, \ \text{ in } (0,1). \end{cases} \tag{3.1}$$

Now, we will discuss about the structure of stationary solutions and their stability.

This system (3.1) consists of a nonlinear elliptic equation and an integral constraint. As far as existence and non-existence in one dimensional domain are concerned, Lou-Ni-Yotsutani [3] obtained nearly complete knowledge. They also obtained the precise qualitative behavior of solutions to this limiting system as the diffusion rate varies.

Their basic approach is to convert the problem of solving the system to a problem of solving its "representation" in a different parameter space. This is first done *without* the integral constraint, and then they use the integral constraint to find the "solution curve" in the new parameter space. This turns out to be a powerful method as it gives fairly precise information about the solutions.

We have recently made clear the remained delicate parts due to the explicit representation by elliptic functions.

We summarized the structure of solutions of (3.1). We concentrate on the case

$$B < C \quad \text{(strong competition case)}.$$

The following two theorem are due to [3].

Theorem 3.1 (Existence) *Suppose that $B < C$. If*

$$\max\left\{0, \frac{B+C-2A}{C-B}\right\} \frac{a_2}{\pi^2} < d_2 < \frac{a_2}{\pi^2},$$

then there exists a solution $(v(x), \tau)$ of (3.1).

Theorem 3.2 (Nonexistence) *Suppose that $B < C$.*

(i) If $d_2 \geq \dfrac{a_2}{\pi^2}$, then there exists no solution of (3.1).

(iii) If $A < B$, there exists no solution of (3.1).

(iii) If $B \leq A < \frac{B+C}{2}$, then there exists a $d_2^* = d_2^*(A, B, C, a_2) > 0$ such that there exists no solution of (3.1) for $d_2 \in (0, d_2^*]$.

We see that the above theorem is sharp by the following theorems. The existence region depending on the the ratio C/B. The situation drastically changes at $C/B = 7/3$.

Theorem 3.3 *Suppose that $B < C \leq 7B/3$. (3.1) has a solution $(v(x), \tau)$ if and only if d_2 satisfies*

$$\max\left\{0, \frac{B+C-2A}{C-B}\right\} \frac{a_2}{\pi^2} < d_2 < \frac{a_2}{\pi^2}.$$

Moreover, the solution is unique.

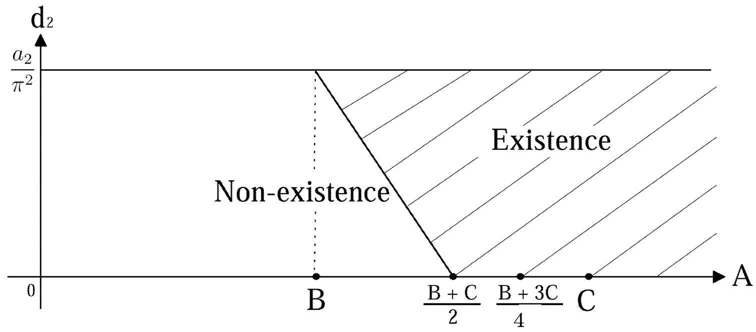

Figure 3.1: Case $B < C \leq 7B/3$

Theorem 3.4 *Suppose that $7B/3 < C$. (3.1) has the unique solution $(v(x), \tau)$ if*

$$\max\left\{0, \frac{B+C-2A}{C-B}\right\} \frac{a_2}{\pi^2} < d_2 < \frac{a_2}{\pi^2}.$$

Moreover, there exists the only one connected non-empty open set D with

$$D \subset \left\{(A, d_2) : B < A < \frac{B+C}{2},\ 0 < d_2 < \left\{\frac{B+C-2A}{C-B}\right\} \frac{a_2}{\pi^2}\right\}$$

such that (3.1) has exactly two solutions $(v(x), \tau)$ if and only if $d_2 \in D$.

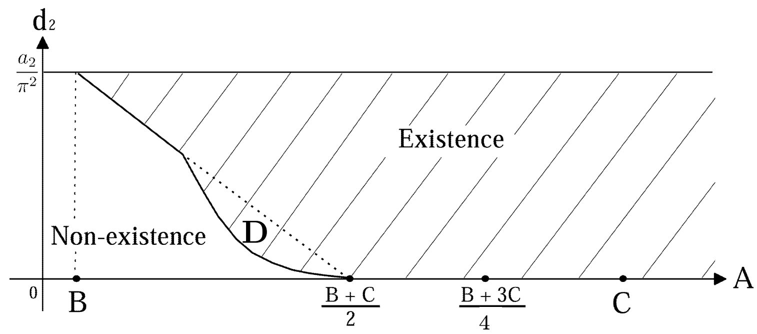

Figure 3.2: Case $7B/3 < C$

The following theorems in [3] give the shape of solutions to (3.1) as $d_2 \uparrow a_2/\pi^2$.

Theorem 3.5 (Shape of solutions as $d_2 \uparrow a_2/\pi^2$) *Suppose that $B < C$. Let $(v(x, d_2), \tau(d_2))$ be solutions of (3.1). If $A \geq B$, then*

$$v(x; d_2) \to 0, \qquad \frac{v(x; d_2) - v(0; d_2)}{v(1; d_2) - v(0; d_2)} \to \frac{1 - \cos(\pi x)}{2},$$

$$\frac{\tau(d_2)}{v(x; d_2)} \to \frac{a_2}{b_2} \cdot \frac{1}{1 - \sqrt{1 - \frac{B}{A}\cos(\pi x)}}$$

uniformly on $[0, 1]$ as $d_2 \uparrow a_2/\pi^2$.

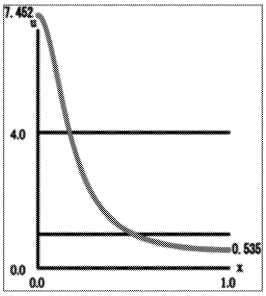

Figure 3.3: u as $d_2 \uparrow a_2/\pi^2$

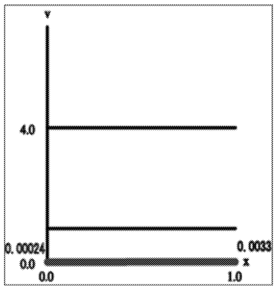

Figure 3.4: v as $d_2 \uparrow a_2/\pi^2$

The following theorems in [3] give the shape of solutions to (3.1) as $d_2 \downarrow 0$. A new number $(B + 3C)/4$ appears. The shape is drastically change at $A = (B + 3C)/4$

Theorem 3.6 (Shape of solutions as $d_2 \to 0$ for $A < \frac{B+3C}{4}$) *Suppose that $B \neq C$. Let $(v(x, d_2), \tau(d_2))$ be solutions of (3.1). If $A < \frac{B+3C}{4}$ and $B < C$, then*

$$v(0; d_2) \to 2 \cdot \frac{a_2}{c_2} \cdot \frac{\frac{B+3C}{4} - A}{C - B}, \qquad v(x; d_2) \to \frac{a_2}{c_2} \cdot \frac{A - B}{C - B} \quad \text{for } x > 0,$$

$$\frac{\tau(d_2)}{v(0; d_2)} \to \frac{a_2}{2c_2} \cdot \frac{C - A}{C - B} \cdot \frac{A - B}{\frac{B+3C}{4} - A}, \qquad \frac{\tau(d_2)}{v(x; d_2)} \to \frac{a_2}{b_2} \cdot \frac{C - A}{C - B} \quad \text{for } x > 0,$$

as $d_2 \downarrow 0$.

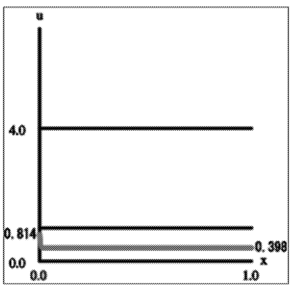

Figure 3.5: u for $A \leq (B + 3C)/4$

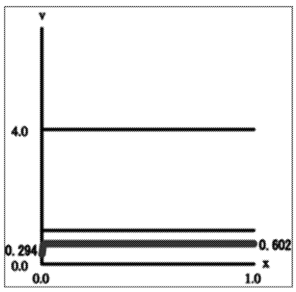

Figure 3.6: v for $A \leq (B + 3C)/4$

Theorem 3.7 (Shape of solutions as $d_2 \to 0$ for $A \geq \frac{B+3C}{4}$) *Suppose that* $B \neq C$. *Let* $(v(x, d_2), \tau(d_2))$ *be solutions of (3.1). If* $B < C$ *and* $A \geq \frac{B+3C}{4}$, *then*

$$v(0; d_2) \to 0, \qquad v(x; d_2) \to \frac{3a_2}{4c_2} \quad \text{for } x > 0,$$
$$\frac{\tau(d_2)}{v(0; d_2)} \to \infty, \qquad \frac{\tau(d_2)}{v(x; d_2)} \to \frac{a_2}{4c_2} \quad \text{for } x > 0, \text{ as } d_2 \to 0.$$

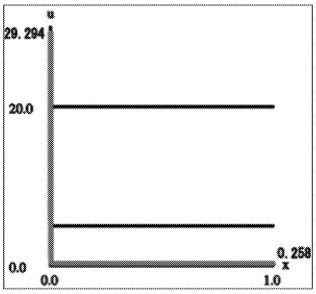

Figure 3.7: u for $(B + 3C)/4 < A$

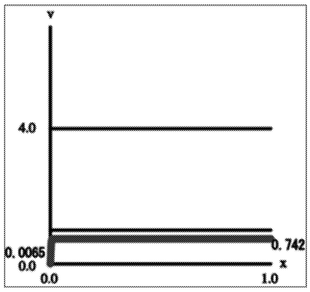

Figure 3.8: v for $(B + 3C)/4 < A$

4 Stability in one-dimensional problem

Let us consider the stability of stationary solutions, and multi-dimensional solutions with their stability.

The following Figure 4.1 shows numerical results for

$$d_1 = 1, \quad d_2 = *, \quad r = 700,000$$
$$a_2 = *, \quad b2 = 1, \quad c2 = 2.$$
$$a_2 = 1, \quad b2 = 1, \quad c2 = 1.$$

We note that $C < 7B/3$, $(B + C)/2 = 1.5$ and $(B + 3C)/4 = 1.75$.

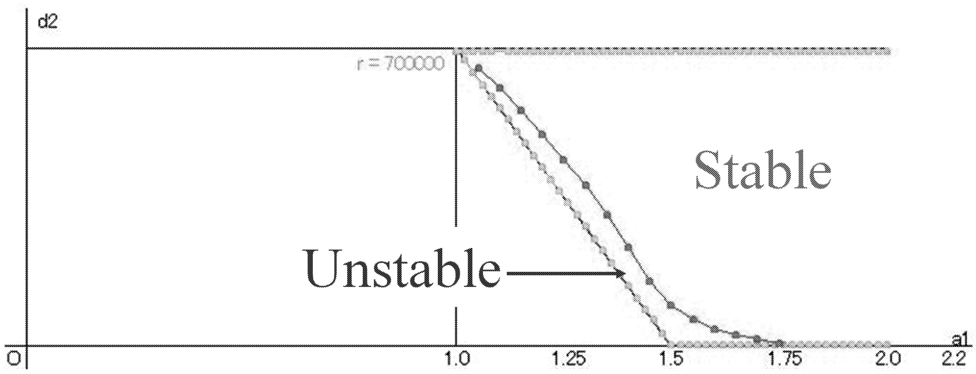

Figure 4.1: Stability and instability

Wu[8] gave a proof of instability for

d_2 sufficiently small with $(B+C)/2 < A < (B+4C)/4$

in one-dimensional case. Recently, she have also given a proof of stability for

$d_2(< a_2/\pi^2)$ sufficiently close to a_2/π^2 with $(B+C)/2 < A < (B+4C)/4$

in one-dimensional case.

5 Multi-dimensional problem

We have done various numerical computations for the case Ω is rectangles in 2-dimensional space. It seems that the structure of stable stationary solutions is essentially very similar to 1-dimensional case, though there are much varieties of shape of solutions in 2-dimensional case than in one-dimensional case.

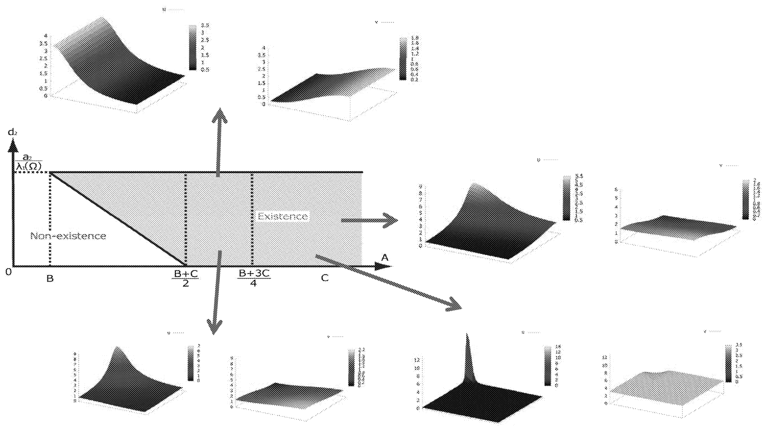

Figure 5.1: 2D global

Now, we will state some mathematical results. We prepare notations. Let

$$\lambda_0 = 0 \ < \lambda_1 \le \lambda_2 \le \cdots$$

$$\varphi_0 = const., \quad \varphi_1, \quad \varphi_2, \quad \cdots .$$

be eigen values and corresponding eigen functions of $-\Delta$ in $\Omega \subset R^N$ with Neumann boundary.

Theorem 5.1 *Suppose that $N \le 3$ and λ_1 be a simple eigen values with an eigen function φ_1. Then, there exists exactly two positive non-constant solutions (v_-, τ_-) and (v_+, τ_+) of (2.1) for d_2 sufficiently close to a_2/λ_1 with $d_2 < a_2/\lambda_1$*

Moreover,

$$\begin{aligned}&\tau \to 0,\\ &\frac{\tau_\pm(d_2)}{v_\pm(x;d_2)} \to \frac{a_2}{b_2}\cdot\frac{1}{1+\mu_\pm\varphi_1(x)}\end{aligned}$$

as $d_2 \uparrow a_2/\lambda_1$, *where* μ_-, μ_+ $(\mu_- < 0 < \mu_+)$ *are solutions of*

$$\frac{\int_\Omega \left(1+\mu\ \varphi_1(x)\right)^{-2} dx}{\int_\Omega \left(1+\mu\ \varphi_1(x)\right)^{-1} dx} = \frac{A}{B}.$$

Remark. *The set* $\{(v_-,\tau_-),(v_+,\tau_+)\}$ *is uniquely determined though there is a freedom to pick up* φ_1. The condition $N \le 3$ comes from Harnack's inequality in our proof.

Remark. *For* $N = 1$, $\Omega = (0,1)$, *it is easy to see that*

$$\lambda_1 = \pi^2, \quad \varphi_1(x) = \cos\pi x, \quad \frac{1}{1-\mu^2} = \frac{A}{B}, \quad \mu_\pm = \pm\sqrt{1-\frac{B}{A}}.$$

Remark. *For* $N = 2$, $\Omega = (0,1)\times(0,\ell)$ *with* $0 < \ell < 1$, *it is easy to see that*

$$\lambda_1 = \pi^2, \quad \varphi_1(x,y) = \cos\pi x, \quad \frac{1}{1-\mu^2} = \frac{A}{B}, \quad \mu_\pm = \pm\sqrt{1-\frac{B}{A}}.$$

Theorem 5.2 *Suppose that* $N \le 3$ *and* λ_1 *be a simple eigen values. Then,* (v_-,τ_-) *and* (v_+,τ_+) *defined by Theorem 5.1 are asymptotically stable for* d_2 *sufficiently close to* a_2/λ_1 *with* $d_2 < a_2/\lambda_1$.

The following general lemma plays crucial role to prove Theorems 5.1 and 5.2.

Lemma 5.3 *Suppose that* $N \ge 1$ *and* φ_1 *be eigen values corresponding to* λ_1. *Let* $g(\mu)$ *be defined by*

$$g(\mu) := \frac{\int_\Omega \left(1+\mu\ \varphi_1(x)\right)^{-2} dx}{\int_\Omega \left(1+\mu\ \varphi_1(x)\right)^{-1} dx}$$

for $\mu \in (-1/max_{\bar\Omega}\ \varphi_1, -1/min_{\bar\Omega}\ \varphi_1)$. *Then*

$$\frac{dg(\mu)}{d\mu} = \begin{cases} + & for\ \mu > 0,\\ 0 & for\ \mu = 0,\\ - & for\ \mu < 0.\end{cases}$$

Moreover, for $N \le 4$,

$$\begin{cases} g(\mu) \to \infty & as \quad \mu \uparrow \mu_+,\\ g(\mu) \to \infty & as \quad \mu \downarrow \mu_-.\end{cases}$$

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Different degrees of reaction rates can block interfacial dynamics in reaction-diffusion systems

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1 Introduction

We point out some remarks on singulrar limits of reaction-diffusion systems in this article which is based on the joint work with H. Monobe, H. Murakawa and H. Ninomiya [4].

Some reaction-diffusion systems with huge parameters are often reduced to free boundary problems as their singular limits when the parameters tend to infinity. Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. For any positive number T we set $Q_T = \Omega \times (0,T)$. Hilhorst-Hout-Peletier [1, 2] investigated a simple reaction-diffusion system with a huge parameter k

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - kuv, \\ \dfrac{\partial v}{\partial t} = -kuv \end{cases} \tag{1}$$

which describes a "rapid reaction" between a diffusive reactant and a non-diffusive one. Assuming that the initial values of u and v are non-negative, they derived the singular limit of an initial-boundary value problem in Q_T for (1) as $k \to \infty$. Their results are summarized as follows: the solution (u_k, v_k) of their initial-boundary value problem for (1) in Q_T posseses its singular limit (u_*, v_*) as $k \to \infty$ such that $u_* v_* \equiv 0$ in Q_T; therefore, when we use the notation

$$\begin{aligned} &\Omega_u(t) = \{x \in \Omega |\, u_*(x,t) > 0\}, \quad \Omega_v(t) = \{x \in \Omega |\, v_*(x,t) > 0\}, \\ &\Gamma(t) = \Omega \setminus (\Omega_u(t) \cup \Omega_v(t)) = \{x \in \Omega |\, u_*(x,t) = v_*(x,t) = 0\}, \end{aligned}$$

the region $\Omega_u(t)$ and the region $\Omega_v(t)$ are divided by an "interface" $\Gamma(t)$; moreover u_* satisfies the one-phase Stefan problem

$$\begin{cases} \dfrac{\partial u_*}{\partial t} = \Delta u_* \qquad \text{in } \Omega_u(t), \\ v_*\big|_{\Gamma(t)+0n} V_n = -\dfrac{\partial u_*}{\partial n}\Big|_{\Gamma(t)-0n}, \quad u_*\big|_{\Gamma(t)} = 0. \end{cases} \tag{2}$$

in a weak sense; in particular, if $\Gamma(t)$ is a smooth, closed and orientable hypersurface, and if u_* is smooth on $\bigcup_{t \in [0,T]} \left(\overline{\Omega_u(t)} \times \{t\}\right)$, and also if the boundary value of v_* on $\partial\Omega_v(t)$ is well-defined at each $t \in [0,T]$, then (2) holds true in the classical sense. Here n is the unit normal vector to $\Gamma(t)$ oriented from $\Omega_u(t)$ to $\Omega_v(t)$, and V_n is the velocity of $\Gamma(t)$ in the direction of n. They also proved in [3] that the singular limit of a reaction-"degenerated diffusion" system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta(u^2) - kuv, \\ \dfrac{\partial v}{\partial t} = -kuv \end{cases} \tag{3}$$

as $k \to \infty$ becomes a free boundary problem with no propagation of the interface: the solution (u_k, v_k) of an initial-boundary value problem in Q_T for (3) possesses its singular limit (u_*, v_*) as $k \to \infty$ such that $u_* v_* \equiv 0$ in Q_T; u_* is a weak solution of

$$\begin{cases} \dfrac{\partial u_*}{\partial t} = \Delta({u_*}^2) \qquad \text{in } \Omega_{u_0}, \\ V_n = 0, \quad u_*\big|_{\Gamma_0} = 0, \end{cases} \tag{4}$$

where Ω_{u_0}, Γ_0 and V_n are respectively defined similarly to $\Omega_u(t)$, $\Gamma(t)$ and V_n which are given above, however Ω_{u_0} and Γ_0 cannot propagate as time goes on. The singular limit (4) obtained from (3) is much different from the porous media equation in that the support of $u_*(\cdot, t)$ in (4) cannot propagate at all. On the other hand Nakaki-Murakawa [5] indicated that another reaction-"degenerate diffusion" system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta(u^2) - 2ku^{\frac{3}{2}}v^2, \\ \dfrac{\partial v}{\partial t} = -kuv \end{cases} \tag{5}$$

with a huge parameter k in Ω becomes a good approximation to a porous media equation

$$\frac{\partial u_*}{\partial t} = \Delta({u_*}^2) \qquad \text{in } \Omega, \tag{6}$$

where the support of $u_*(\cdot, t)$ does propagate with a positive speed. Let (u_k, v_k) be the solution of an initial-boundary value problem in Q_T for (5). When k is large enough in (5), $u_k v_k$ almost vanishes in Q_T, and a "transition layer" of the profile of $v_k(\cdot, t)$, together with a "corner layer" of the profile of $u_k(\cdot, t)$, appears in a thin region of Ω. They showed in [5] that the transition layer of $v_k(\cdot, t)$ well approximates the moving boundary of the support of $u_*(\cdot, t)$ by using the Barenblatt solution for (6). The consumption rate kuv of u in (3) is much greater than the consumption rate $2ku^{\frac{3}{2}}v^2$ of u in (5) when u is very small. Thus the blocking of the propagating front of the "region of diffusive u" by the great consumption of u due to the rapid reaction with v in the corner (resp. transition) layer of u (resp. v) seems to arise easier in (3) than in (5). However, the reason why the exponents in the reaction rates of (5) bring about exactly the propagation speed of the interface appearing in the porous media equation (6) has not been clarified at all.

Taking account of these results, we will investigate the singular limits of initial-boundary value problems in Q_T for reaction-diffusion systems

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta(u^m) - ku^p v^q, \\ \dfrac{\partial v}{\partial t} = -ku^r v^s \end{cases} \tag{7}$$

as $k \to \infty$. Let (u_k, v_k) be the solution of an initial-boundary value problem in Q_T for (7) and let $(u_*, v_*) = \lim\limits_{k\to\infty}(u_k, v_k)$. We expect a similar situation to above the results: $u_* v_* \equiv 0$ in Q_T; namely the region $\Omega_u(t) = \{x \in \Omega |\, u_*(x,t) > 0\}$ and the region $\Omega_v(t) = \{x \in \Omega |\, v_*(x,t) > 0\}$ would be divided by an interface $\Gamma(t) = \Omega \setminus (\Omega_u(t) \cup \Omega_v(t)) = \{x \in \Omega |\, u_*(x,t) = v_*(x,t) = 0\}$. Then a question naturally arises: which values of the exponents p, q, r and s can block the propagation of the interface $\Gamma(t)$? To answer this question, we will consider the situation where $m = q = s = 1$ and $p < r$ as a first step.

2 Results

For simplicity we assume that $p = 1$ and $r \geq 2$ in (7); so we investigate the singular limit of

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - kuv & \text{in } Q_T, \\ \dfrac{\partial v}{\partial t} = -ku^r v & \text{in } Q_T \end{cases} \tag{8}$$

as $k \to \infty$. We impose the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0, T] \tag{9}$$

on u and the following conditions on our initial datum (u_0, v_0):

(I1) $u_0 \in C(\overline{\Omega})$, $v_0 \in L^\infty(\Omega)$;

(I2) $u_0 \geq 0$, $v_0 \geq 0$ in Ω, $\Omega_{u_0} = \{x \in \Omega |\, u_0(x) > 0\} \neq \phi$, $\Omega_{v_0} = \{x \in \Omega |\, v_0(x) > 0\} \neq \phi$;

(I3) $\Omega_{u_0} \cap \Omega_{v_0} = \phi$.

We can obtain the following a priori estimates for the solution of our initial-boundary value problem for (8).

Theorem 1. *Let (u_k, v_k) be the solution of (8)(9) with the initial value (u_0, v_0) for each $k > 0$. Then*

(i) $0 \leq u_k(x,t) \leq \|u_0\|_\infty$, $0 \leq v_k(x,t) \leq \|v_0\|_\infty$ *in* Q_T;

(ii) $\left\{\iint_{Q_T} k u_k v_k \, dxdt\right\}_{k>0}$ *is bounded.*;

(iii) $\{u_k\}_{k>0}$ *and* $\{v_k\}_{k>0}$ *are pre-compact in* $L^2(Q_T)$;

(iv) $\{u_k\}_{k>0}$ *and* $\{{u_k}^r\}_{k>0}$ *are bounded in* $L^2\left(0,T;\, H^1(\Omega)\right)$;

(v) $\left\{\dfrac{\partial v_k}{\partial t}\right\}_{k>0}$ *is bounded in* $H^{-1}\left(0,T;\, L^2(\Omega)\right) \left(= \left\{H_0^1\left(0,T;\, L^2(\Omega)\right)\right\}^*\right)$;

(vi) $\left\{{u_k}^{r-2}|\nabla u_*|^2\right\}_{k>0}$ *is bounded in* $H^{-1}(Q_T) \left(= \left\{H_0^1(Q_T)\right\}^*\right)$.

Corollary 2. *Let (u_k, v_k) be the solution of (8)(9) with the initial value (u_0, v_0) for each $k > 0$. Then there exist functions $u_* \in L^\infty(Q_T) \cap L^2\left(0,T;\, H^1(\Omega)\right)$, $v_* \in L^\infty(Q_T)$ and a distribution $\omega_* \in H^{-1}(Q_T)$ such that*

$$\begin{cases} u_k \longrightarrow u_* & \textit{strongly in } L^2(Q_T),\ \textit{a.e. in } Q_T,\ \textit{weakly in } L^2\left(0,T;\, H^1(\Omega)\right); \\ v_k \longrightarrow v_* & \textit{strongly in } L^2(Q_T),\ \textit{a.e. in } Q_T; \\ {u_k}^r \longrightarrow {u_*}^r & \textit{strongly in } L^2(Q_T),\ \textit{weakly in } L^2\left(0,T;\, H^1(\Omega)\right); \\ \dfrac{\partial v_k}{\partial t} \longrightarrow \dfrac{\partial v_*}{\partial t} & \textit{weakly in } H^{-1}\left(0,T;\, L^2(\Omega)\right); \\ {u_k}^{r-2}|\nabla u_k|^2 \longrightarrow \omega_* & \textit{weakly in } H^{-1}(Q_T) \end{cases}$$

subsequentially as $k = k_j \to \infty$. Moreover (u_*, v_*, ω_*) satisfies

$$0 \leq u_* \leq \|u_0\|_\infty, \quad 0 \leq v_* \leq \|v_0\|_\infty, \quad u_* v_* \equiv 0, \quad \frac{\partial v_*}{\partial t} \leq 0, \quad \omega_* \geq 0 \qquad \text{in } Q_T$$

and

$$\iint_{Q_T} \left\{ -\left(\frac{{u_*}^r}{r} - v_*\right)\zeta_t + {u_*}^{r-1}\nabla u_* \cdot \nabla\zeta \right\} dxdt + (r-1)\ {}_{H^{-1}(Q_T)}\langle \omega_*, \zeta \rangle_{H_0^1(Q_T)} = 0 \tag{10}$$

for any $\zeta \in H_0^1(Q_T)$.

For the limit functions u_* and v_* obtained in Corollary 2 we set

$$\begin{array}{l}\Omega_u(t) = \{x \in \Omega |\, u_*(x,t) > 0\}, \quad \Omega_v(t) = \{x \in \Omega |\, v_*(x,t) > 0\}, \\ \Gamma(t) = \Omega \setminus (\Omega_u(t) \cup \Omega_v(t)) = \{x \in \Omega |\, u_*(x,t) = v_*(x,t) = 0\}\end{array}$$

at each $t \in [0,T]$. The fact $u_* v_* \equiv 0$ implies that

$$\Omega_u(t) \cap \Omega_v(t) = \phi, \qquad t \in [0,T].$$

We can rewrite the weak form (10) as a free boundary problem under the following assumptions on the smoothness of u_*, ω_* and v_0:

(A1) v_0 is continuous on $\overline{\Omega_{v_0}}$, and $\inf\limits_{\Omega_{v_0}} v_0 > 0$;

(A2) $\Gamma(t)$ is a smooth, closed and orientable hypersurface in $\mathbb{R}^N$ satisfying $\Gamma(t) \cap \partial\Omega \equiv \phi$ at each $t \in [0,T]$;

(A3) $\Gamma(t)$ smoothly moves with a normal velocity V_n from $\Omega_u(t)$ to $\Omega_v(t)$;

(A4) u_* is continuous in $\overline{Q_T}$;

(A5) u_* is a smooth on $\bigcup\limits_{t \,\in\, [0,T]} \left(\overline{\Omega_u(t)} \times \{t\}\right)$;

(A6) $\omega_* \in L^1_{\mathrm{loc}}(Q_T)$.

Theorem 3. *Let* (u_k, v_k) *be the solution of* (8)(9) *with the initial value* (u_0, v_0) *for each* $k > 0$. *Assume* (A1)-(A6). *Then*

$$V_n \equiv 0 \ \textit{on} \bigcup_{t \,\in\, [0,T]} \left(\overline{\Gamma(t)} \times \{t\}\right) \tag{11}$$

$$\left(\textit{i.e.,} \quad \Omega_u(t) \equiv \Omega_{u_0},\ \Omega_v(t) \equiv \Omega_{v_0},\ \Gamma(t) \equiv \Gamma_0 := \Omega \setminus (\Omega_{u_0} \cup \Omega_{v_0})\right) \tag{12}$$

and

$$\omega_* = \begin{cases} {u_*}^{r-2} |\nabla u_*|^2 & \textit{in } \Omega_{u_0} \times (0,T], \\ 0 & \textit{in } \Omega_{v_0} \times (0,T] \end{cases}$$

hold true. Moreover (u_*, v_*) *satisfies* (9) *and*

$$\begin{cases} \dfrac{\partial u_*}{\partial t} = \Delta u_* & \textit{in } \Omega_{u_0} \times (0,T], \\ u_* = 0 & \textit{on } \Gamma_0 \times (0,T], \\ u_*|_{t=0} = u_0 & \textit{in } \Omega_{u_0}; \end{cases}$$

$$v_* = v_0 \qquad \textit{in } \Omega_{v_0} \times (0,T].$$

In particular, the subsequential convergence as $k = k_j \to \infty$ *in Corollary 2 is replaced by the convergence as* $k \to \infty$.

Remark Among the assumptions (A1)-(A6) it seems that (A4) and (A6) might be removable; however we have not succeeded in removing them yet.

Here we omit the proofs of Theorems 1 and 3 which will be given in [4].

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Asymptotic behavior of epidemic models governed by logistic growth

Dedicated to Professor HIROKI TANABE on his eightieth birthday

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Abstract. In this paper, we consider an SIR epidemic model with delays in which population growth is subject to logistic growth in absence of disease. The force of infection with a discrete delay is given by a separable nonlinear incidence rate. Under the monotonicity conditions, we investigate asymptotic stability of the trivial equilibrium, the disease-free equilibrium and the endemic equilibrium. By constructing a Lyapunov functional, we establish the global stability of the disease-free equilibrium if and only if the basic reproduction number is less than or equal to one. Moreover, by investigating the location of roots of the associated characteristic equations, we prove that there exists a critical length of delay such that the endemic equilibrium is locally asymptotically stable when the delay is less than the value.

1 Introduction

In order to investigate the spread of infectious diseases, many authors have formulated various epidemic models and the stability of equilibria has also been extensively studied (see [1-5] and the references therein). From an epidemiological point of view, it is important to investigate the population dynamics of the disease transmission. Recently, based on an SIR (Susceptible-Infected-Recovered) epidemic model, Wang et al. [4] considered the asymptotic behavior of the following delayed epidemic model in which population growth is subject to logistic growth in absence of disease:

$$\begin{cases} \dfrac{dS(t)}{dt} = r\left(1 - \dfrac{S(t)}{K}\right)S(t) - \beta S(t)I(t-\tau), \\ \dfrac{dI(t)}{dt} = \beta S(t)I(t-\tau) - (\mu_1 + \gamma)I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - \mu_2 R(t). \end{cases} \tag{1.1}$$

$S(t)$, $I(t)$ and $R(t)$ denote the fractions of susceptible, infective and recovered host individuals at time t, respectively. In system (1.1), it is assumed that the population growth in susceptible host individuals is governed by the logistic growth with a carrying capacity $K > 0$ as well as intrinsic birth rate constant $r > 0$. $\beta > 0$ is the average number of constants per infective per unit time and $\tau \geq 0$ is the incubation time, $\mu_1 > 0$ and $\mu_2 > 0$ represent the death rates of infective and recovered individuals, respectively. $\gamma > 0$ represents the recovery rate of infective individuals.

Wang et al. [4] obtained stability results of equilibria of (1.1) in terms of the basic reproduction number R_0: the disease-free equilibrium is globally asymptotically stable if $R_0 < 1$ while a unique endemic equilibrium can be unstable if $R_0 > 1$. More precisely, if $1 < R_0 \leq 3$, then the endemic equilibrium is asymptotically stable for any delay τ and if $R_0 > 3$, then there exists a critical length of delay such that the endemic equilibrium is asymptotically stable for delay which is less than the value while it is unstable for delay which is greater than the value. It is also shown that Hopf bifurcation at the endemic equilibrium occurs when the delay crosses a sequence of critical values.

On the other hand, since nonlinearity in the incidence rates has been observed in disease transmission dynamics, it has been suggested that the standard bilinear incidence rate shall be modified into a nonlinear incidence rate by some authors (see, e.g., [1, 3]). In this paper, we replace the incidence rate in (1.1) by a nonlinear incidence rate of the form $F(S(t))G(I(t-\tau))$. Throughout the paper, it is assumed that the functions F and G are continuous on $[0, +\infty)$ and continuously differentiable on $(0, +\infty)$ satisfying the following hypotheses:

(H1) $F(S)$ is strictly monotone increasing on $[0, +\infty)$ with $F(0) = 0$,

(H2) $G(I)$ is strictly monotone increasing on $[0, +\infty)$ with $G(0) = 0$,

(H3) $I/G(I)$ is monotone increasing on $(0, +\infty)$ with $\lim_{I \to +0} I/G(I) = 1$.

Then we obtain the following system:

$$\begin{cases} \dfrac{dS(t)}{dt} = r\left(1 - \dfrac{S(t)}{K}\right)S(t) - F(S(t))G(I(t-\tau)), \\ \dfrac{dI(t)}{dt} = F(S(t))G(I(t-\tau)) - (\mu_1 + \gamma)I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - \mu_2 R(t). \end{cases} \tag{1.2}$$

The functions F and G include some special incidence rates. For instance, if $F(S) = \beta S$ with $\beta > 0$ and $G(I) = I$, then the incidence rate is used in Wang et al. [4] and if $F(S) = \frac{\beta S}{1+\alpha S}$ with $\alpha, \beta > 0$ and $G(I) = I$, then the incidence rate, describing saturated effects of the prevalence of infectious diseases, is used in Zhang et al. [5].

The initial conditions of system (1.2) take the following form

$$\begin{cases} S(\theta) = \phi_1(\theta), \;\; I(\theta) = \phi_2(\theta), \;\; R(\theta) = \phi_3(\theta), \\ \phi_i(\theta) \geq 0, \;\; \theta \in [-\tau, 0], \;\; \phi_i(0) > 0, \;\; \phi_i \in C([-\tau, 0], \mathbb{R}_{+0}), \;\; i = 1, 2, 3, \end{cases} \tag{1.3}$$

where $\mathbb{R}_{+0} = \{x \in \mathbb{R} | x \geq 0\}$. By the fundamental theory of functional differential equations, system (1.2) has a unique positive solution $(S(t), I(t), R(t))$ satisfying

$$\limsup_{t \to +\infty} (S(t) + I(t) + R(t)) \leq \frac{(r + \underline{\mu})K}{\underline{\mu}}, \tag{1.4}$$

where $\underline{\mu} = \min(\mu_1, \mu_2)$. We define the basic reproduction number by

$$R_0 = \frac{F(K)}{\mu_1 + \gamma}. \tag{1.5}$$

In this paper we analyze the stability of equilibria by investigating location of the roots of associated characteristic equation and constructing a Lyapunov functional. System (1.2) always has a trivial equilibrium $E_0 = (0, 0, 0)$ and a disease-free equilibrium $E_1 = (K, 0, 0)$. If $R_0 > 1$, then system (1.2) has an endemic equilibrium $E_* = (S^*, I^*, R^*)$ with $S^* > 0$, $I^* > 0$ and $R^* > 0$ (see Lemma 3.1).

The organization of this paper is as follows. In Section 2, we investigate the stability of the trivial equilibrium and the disease-free equilibrium. In Section 3, for $R_0 > 1$, we investigate unique existence of the endemic equilibrium of system (1.2) exists. Moreover, we investigate the delay effect concerning the local asymptotic stability of endemic equilibrium. Finally, in Section 4, we introduce an example of our model to offer some corollaries.

2 Stability of the disease-free equilibrium

In this section, we analyze the stability of the trivial equilibrium E_0. By constructing a Lyapunov functional, we further establish the global asymptotic stability of the disease-free equilibrium E_1 for $R_0 \le 1$. At an arbitrary equilibrium $(\hat{S}, \hat{I}, \hat{R})$ of (1.2), the characteristic equation is given by

$$(\lambda + \mu_2)\left[\left\{\lambda + F'(\hat{S})G(\hat{I}) - r\left(1 - \frac{2\hat{S}}{K}\right)\right\}(\lambda + \mu_1 + \gamma - F(\hat{S})G'(\hat{I})e^{-\lambda\tau}) \right.$$
$$\left. + F(\hat{S})G'(\hat{I})e^{-\lambda\tau}F'(\hat{S})G(\hat{I})\right] = 0. \tag{2.1}$$

Theorem 2.1. *The trivial equilibrium E_0 of system (1.2) is always unstable.*

Proof. For $(\hat{S}, \hat{I}, \hat{R}) = (0, 0, 0)$ the characteristic equation (2.1) becomes as follows.

$$(\lambda + \mu_2)(\lambda - r)(\lambda + \mu_1 + \gamma) = 0. \tag{2.2}$$

Since (2.2) has a positive root $\lambda = r$, E_0 is unstable. □

Constructing a Lyapunov functional, we prove that the global asymptotic stability of the disease-free equilibrium E_1 is determined by the basic reproduction number R_0.

Theorem 2.2. *The disease-free equilibrium E_1 of system (1.2) is globally asymptotically stable if and only if $R_0 \le 1$ and it is unstable if and only if $R_0 > 1$.*

Proof. First we assume $R_0 \le 1$. We define a Lyapunov functional by

$$V(t) = \int_K^{S(t)} \left(1 - \frac{F(K)}{F(s)}\right) ds + I(t) + F(K)\int_{t-\tau}^t G(I(s))ds, \tag{2.3}$$

where $g(x) = x - 1 - \ln x \ge g(1) = 0$ for $x > 0$. Then the time derivative of $V(t)$ along the solution of (1.2) becomes as follows.

$$\begin{aligned}
\frac{dV(t)}{dt} &= \left(1 - \frac{F(K)}{F(S(t))}\right)\left\{r\left(1 - \frac{S(t)}{K}\right)S(t) - F(S(t))G(I(t-\tau))\right\} \\
&\quad + F(S(t))G(I(t-\tau)) - (\mu_1 + \gamma)I(t) + F(K)(G(I(t)) - G(I(t-\tau))) \\
&= -\frac{rS(t)}{KF(S(t))}(F(S(t)) - F(K))(S(t) - K) + F(K)G(I(t)) - (\mu_1 + \gamma)I(t). \\
&= -\frac{rS(t)}{KF(S(t))}(F(S(t)) - F(K))(S(t) - K) + F(K)\left(\frac{G(I(t))}{I(t)} - \frac{1}{R_0}\right)I(t).
\end{aligned}$$

Since the hypothesis (H3) yields $0 < \frac{G(I)}{I} \leq 1$ for $I > 0$, we obtain

$$\frac{dV(t)}{dt} \leq -\frac{rS(t)}{KF(S(t))}(F(S(t)) - F(K))(S(t) - K) + F(K)\left(1 - \frac{1}{R_0}\right)I(t). \tag{2.4}$$

By the hypothesis (H1), $(F(S(t)) - F(K))(S(t) - K) \geq 0$ with equality if and only if $S(t) = K$. For the case $R_0 < 1$, we obtain $\frac{dV(t)}{dt} \leq 0$ with equality if and only if $S(t) = K$ and $I(t) = 0$. For the case $R_0 = 1$, we obtain $\frac{dV(t)}{dt} \leq 0$ with equality if and only if $S(t) = K$. By Lyapunov-LaSalle asymptotic stability theorem, we have $\lim_{t\to+\infty} S(t) = K$ if $R_0 \leq 1$. By the first and third equations of (1.2), $\lim_{t\to+\infty} S(t) = K$ implies $\lim_{t\to+\infty} I(t) = 0$ and $\lim_{t\to+\infty} R(t) = 0$. Since it follows that E_1 is uniformly stable from the relation $V(t) \geq \int_K^{S(t)} \left(1 - \frac{F(K)}{F(s)}\right) ds + I(t)$, E_1 is globally asymptotically stable.

Second we assume $R_0 > 1$. For $(\hat{S}, \hat{I}, \hat{R}) = (K, 0, 0)$, the characteristic equation (2.1) becomes as follows.

$$(\lambda + \mu_2)(\lambda + r)(\lambda + \mu_1 + \gamma - F(K)e^{-\lambda\tau}) = 0. \tag{2.5}$$

One can see that $\lambda = -\mu_2$ and $\lambda = -r$ are negative real roots of (2.5). Moreover, (2.5) has roots of

$$p(\lambda) := \lambda + \mu_1 + \gamma - F(K)e^{-\lambda\tau} = 0.$$

Since $p(0) = (\mu_1 + \gamma)(1 - R_0) < 0$ and $\lim_{\lambda\to+\infty} p(\lambda) = +\infty$, we conclude that $p(\lambda) = 0$ has at least one positive root. Hence E_1 is unstable. The proof is complete. $\square$

3 Stability of the endemic equilibrium

In this section, we establish local asymptotic stability of the endemic equilibrium E_* for $R_0 > 1$ by investigating location of the roots of the characteristic equation.

3.1 Unique existence

In this subsection, we give the result on the unique existence of the endemic equilibrium E_* for $R_0 > 1$.

Lemma 3.1. *If $R_0 > 1$, then system (1.2) has an endemic equilibrium $E_* = (S^*, I^*, R^*)$ satisfying the following equations:*

$$\begin{cases} r\left(1 - \dfrac{S^*}{K}\right)S^* - F(S^*)G(I^*) = 0, \\ F(S^*)G(I^*) - (\mu_1 + \gamma)I^* = 0, \\ \gamma I^* - \mu_2 R^* = 0. \end{cases} \tag{3.1}$$

Moreover, if $R_0 > 1$ and

$$F'(S) - \frac{F(S)}{S} \geq 0 \text{ for all } S \in (0, K), \tag{3.2}$$

then the endemic equilibrium E_* is unique.

Proof. At a fixed point of (S, I, R) of system (1.2), the following equalitions hold.

$$r\left(1 - \frac{S}{K}\right)S - (\mu_1 + \gamma)I = 0,\ F(S)G(I) - (\mu_1 + \gamma)I = 0,\ \gamma I - \mu_2 R = 0. \tag{3.3}$$

Substituting the first equation of (3.3) into the second equation of (3.3), we consider the following equation:

$$H(S) := F(S) - (\mu_1 + \gamma)\frac{\frac{r}{\mu_1+\gamma}(1 - \frac{S}{K})S}{G(\frac{r}{\mu_1+\gamma}(1 - \frac{S}{K})S)} = 0.$$

By the hypotheses (H1) and (H3), we obtain

$$\lim_{S\to+0} H(S) = -(\mu_1 + \gamma) < 0,\ \lim_{S\to K-0} H(S) = F(K) - (\mu_1 + \gamma) = (\mu_1 + \gamma)(R_0 - 1) > 0,$$

which implies that there exists a positive root $S = S^* < K$ such that $H(S) = 0$. By the first and third equations of (3.3), we have $I^* = \frac{r}{\mu_1+\gamma}(1 - \frac{S^*}{K})S^* > 0$, and $R^* = \frac{\gamma}{\mu_2(\mu_1+\gamma)}r(1 - \frac{S^*}{K})S^* > 0$. Hence, we obtain the first part of this lemma.

Next, under the condition (3.2), we prove that the function H is strictly monotone increasing on $(0, K)$. We define

$$L(S) := \frac{r}{\mu_1 + \gamma}\left(1 - \frac{S}{K}\right)S.$$

By the relation $r(1 - \frac{2S}{K}) = r(1 - \frac{S}{K}) - \frac{rS}{K} = \frac{F(S)G(L(S))}{S} - \frac{rS}{K}$ and $\frac{d}{dI}(\frac{I}{G(I)})|_{I=L(S)} \geq 0$ for all $S \in (0, K)$, we obtain

$$\begin{aligned}
H'(S) =& F'(S) - (\mu_1 + \gamma)\frac{dL(S)}{dS} \cdot \frac{d}{dI}\left(\frac{I}{G(I)}\right)\Big|_{I=L(S)}\\
=& F'(S) - r\left(1 - \frac{2S}{K}\right) \cdot \frac{d}{dI}\left(\frac{I}{G(I)}\right)\Big|_{I=L(S)}\\
=& F'(S) - \left(\frac{F(S)G(L(S))}{S} - \frac{rS}{K}\right) \cdot \frac{d}{dI}\left(\frac{I}{G(I)}\right)\Big|_{I=L(S)}\\
\geq& F'(S) - \frac{F(S)G(L(S))}{S} \cdot \frac{d}{dI}\left(\frac{I}{G(I)}\right)\Big|_{I=L(S)}
\end{aligned}$$

for all $S \in (0, K)$. In addition, since $G(L(S))\frac{d}{dI}(\frac{I}{G(I)})|_{I=L(S)} = 1 - \frac{L(S)G'(L(S))}{G(L(S))}$ and $G'(L(S)) > 0$ for all $S \in (0, K)$ by the hypothesis (H2), it follows from the condition (3.2) that

$$\begin{aligned}
H'(S) \geq& F'(S) - \frac{F(S)}{S}\left(1 - \frac{L(S)G'(L(S))}{G(L(S))}\right)\\
>& F'(S) - \frac{F(S)}{S} \geq 0
\end{aligned}$$

for all $S \in (0, K)$. This implies that there exists a unique positive root $S = S^* < K$ such that $H(S) = 0$. Hence, we obtain the second part of this lemma. The proof is complete. □

Proposition 3.1. *The functions such that $\frac{F(S)}{S}$ is monotone increasing on $(0, +\infty)$ satisfy the condition (3.2). For example, the function $F(S) = \beta S^p$ with $p \geq 1$ satisfies (3.2).*

3.2 Local asymptotic stability

In this subsection, we investigate local asymptotic stability of the endemic equilibrium $E_* = (S^*, I^*, R^*)$ for system (1.2). Let us assume that $R_0 > 1$ holds. For $(\hat{S}, \hat{I}, \hat{R}) = (S^*, I^*, R^*)$ the characteristic roots of (2.1) are the root $\lambda = -\mu_2$ and the roots of

$$\lambda^2 + a\lambda + b - e^{-\lambda\tau}(c\lambda + d) = 0 \tag{3.4}$$

with

$$\begin{aligned} a =& \frac{F(S^*)G(I^*)}{I^*} + \left(F'(S^*) - \frac{F(S^*)}{S^*}\right)G(I^*) + \frac{rS^*}{K}, \\ b =& \frac{F(S^*)G(I^*)}{I^*}\left\{\left(F'(S^*) - \frac{F(S^*)}{S^*}\right)G(I^*) + \frac{rS^*}{K}\right\}, \\ c =& F(S^*)G'(I^*), \\ d =& F(S^*)G'(I^*)\left(-\frac{F(S^*)G(I^*)}{S^*} + \frac{rS^*}{K}\right). \end{aligned}$$

First we prove that all the roots of (3.4) have negative real part for $\tau = 0$.

Proposition 3.2. *Assume $R_0 > 1$. If the condition* (3.2) *holds, then all the roots of (3.4) have negative real part for $\tau = 0$.*

Proof. When $\tau = 0$, (3.4) yields

$$\lambda^2 + (a - c)\lambda + (b - d) = 0. \tag{3.5}$$

Noting from the hypotheses (H2) and (H3) that $G'(I^*) > 0$ and $G(I^*) - I^*G'(I^*) \geq 0$, we have

$$a - c = F(S^*)\left(\frac{G(I^*)}{I^*} - G'(I^*)\right) + \left(F'(S^*) - \frac{F(S^*)}{S^*}\right)G(I^*) + \frac{rS^*}{K} > 0$$

and

$$\begin{aligned} &b - d \\ =& \frac{F(S^*)G(I^*)}{I^*}\left\{\left(F'(S^*) - \frac{F(S^*)}{S^*}\right)G(I^*) + \frac{rS^*}{K}\right\} + F(S^*)G'(I^*)\left(\frac{F(S^*)G(I^*)}{S^*} - \frac{rS^*}{K}\right) \\ =& \frac{rS^*F(S^*)}{K}\left(\frac{G(I^*)}{I^*} - G'(I^*)\right) + \frac{F(S^*)(G(I^*))^2}{I^*}\left(F'(S^*) - \frac{F(S^*)}{S^*}\right) + \frac{(F(S^*))^2G'(I^*)G(I^*)}{S^*} \\ >& 0, \end{aligned}$$

which implies that all the roots of (3.5) have negative real part. The proof is complete. □

Next we consider the case $F(S) = \beta S$. Then, by Lemma 3.1, system (1.2) has a unique endemic equilibrium $E_* = (S^*, I^*, R^*)$. Let us define

$$\overline{R}_0 = 2\frac{I^*}{G(I^*)} + \frac{1}{G'(I^*)}. \tag{3.6}$$

Then we prove that $R_0 = \overline{R}_0$ is a threshold condition which determines the existence of purely imaginary roots of (3.4) for $\tau > 0$. The following proposition is an extension of the stability results for the case $G(I) = I$ in Wang et al. [4].

Proposition 3.3. *Assume $R_0 > 1$. Then the following statement holds true.*

(i) *If $R_0 \le \overline{R}_0$, then all the roots of (3.4) have negative real part for any $\tau > 0$.*

(ii) *If $\overline{R}_0 < R_0$, then there exists a monotone increasing sequence $\{\tau_n\}_{n=0}^{\infty}$ with $\tau_0 > 0$ such that (3.4) has a pair of imaginary roots for $\tau = \tau_n$ $(n = 0, 1, \cdots)$.*

Proof. From Proposition 3.2, all the roots of equation (3.4) have negative real part for sufficiently small τ. Suppose that $\lambda = i\omega$, $\omega > 0$ is a root of (3.4). Substituting $\lambda = i\omega$ into the characteristic equation (3.4) yields equations, which split into its real and imaginary parts as follows:

$$\begin{cases} -\omega^2 + b = d\cos\omega\tau + c\omega\sin\omega\tau, \\ a\omega = c\omega\cos\omega\tau - d\sin\omega\tau. \end{cases} \tag{3.7}$$

Squaring and adding both equations in (3.7), we have

$$\omega^4 + (a^2 - 2b - c^2)\omega^2 + (b+d)(b-d) = 0. \tag{3.8}$$

By the relations $r(1 - \frac{S^*}{K}) = \beta G(I^*)$, $R_0 = \frac{KI^*}{S^*G(I^*)}$ and

$$2S^*G'(I^*) + \frac{K}{R_0} = \frac{2KI^*G'(I^*)}{R_0G(I^*)} + \frac{K}{R_0} = \frac{KG'(I^*)}{R_0}\left(2\frac{I^*}{G(I^*)} + \frac{1}{G'(I^*)}\right) = \frac{\overline{R}_0KG'(I^*)}{R_0},$$

we obtain

$$\begin{aligned} a^2 - 2b - c^2 =& \left(\frac{\beta G(I^*)}{I^*} + \frac{r}{K}\right)^2 (S^*)^2 - \frac{\beta G(I^*)}{I^*}\frac{2r}{K}(S^*)^2 - (\beta S^*)^2 G'(I^*)^2 \\ =& (S^*)^2\left\{\left(\frac{\beta G(I^*)}{I^*}\right)^2 - (\beta G'(I^*))^2 + \left(\frac{r}{K}\right)^2\right\} \end{aligned}$$

and

$$\begin{aligned} b + d =& \frac{\beta S^*G(I^*)}{I^*}\frac{rS^*}{K} + \beta S^*G'(I^*)\left(-\beta G(I^*) + \frac{rS^*}{K}\right) \\ =& \frac{\beta S^*G(I^*)}{I^*}\frac{rS^*}{K} + \beta S^*G'(I^*)\left(-r + \frac{2rS^*}{K}\right) \\ =& \frac{r\beta S^*}{K}\left(2S^*G'(I^*) + \frac{K}{R_0}\right) - r\beta S^*G'(I^*) \\ =& \frac{r\beta S^*G'(I^*)}{R_0}(\overline{R}_0 - R_0). \end{aligned}$$

First we assume $R_0 \le \overline{R}_0$. Then we have $a^2 - 2b - c^2 > 0$ and $(b+d)(b-d) \ge 0$, that is, there is no positive real ω satisfying (3.8). This leads to a contradiction and all the roots of (3.4) have negative real part for any $\tau \ge 0$. Hence we obtain the first part of this proposition.

Second we assume $\overline{R}_0 < R_0$. Then it follows from the relations $a^2 - 2b - c^2 > 0$ and $(b+d)(b-d) < 0$ that there is a unique positive real ω_0 satisfying (3.8), where

$$\omega_0 = \left\{\frac{-(a^2 - 2b - c^2) + \sqrt{(a^2 - 2b - c^2)^2 - 4(b+d)(b-d)}}{2}\right\}^{\frac{1}{2}}.$$

Noting from (3.7) that $\lambda = -i\omega_0$ is also a root of (3.4), this implies that (3.8) has a single pair of purely imaginary roots $\pm i\omega_0$. By the relation

$$(ac-d)\omega_0^2 + bd = (c^2\omega_0^2 + d^2)\cos\omega_0\tau,$$

τ_n corresponding to ω_0 can be obtained as follows:

$$\tau_n = \frac{1}{\omega_0}\arccos\frac{(ac-d)\omega_0^2 + bd}{c^2\omega_0^2 + d^2} + \frac{2n\pi}{\omega_0}, \quad n = 0, 1, 2, \cdots.$$

Hence we obtain the second part of this proposition. The proof is complete. □

The following proposition indicates that a conjugate pair of the characteristic roots $\lambda = \pm i\omega_0$ of (2.1) cross the imaginary axis from the left half complex plane to the right half complex plane when τ crosses τ_n $(n = 0, 1, \cdots)$ if $1 < \overline{R}_0 < R_0$.

Proposition 3.4. *Assume $R_0 > 1$. If $\overline{R}_0 < R_0$, then the transversality condition:*

$$\frac{d\mathrm{Re}(\lambda(\tau))}{d\tau}\Big|_{\tau=\tau_n} > 0$$

holds for $n = 0, 1, \cdots$.

Proof. Differentiating (3.4) with respect to τ, we obtain

$$(2\lambda + a)\frac{d\lambda}{d\tau} = \{e^{-\lambda\tau}c - \tau e^{-\lambda\tau}(c\lambda + d)\}\frac{d\lambda}{d\tau} - \lambda e^{-\lambda\tau}(c\lambda + d),$$

that is,

$$\begin{aligned}
\left(\frac{d\lambda}{d\tau}\right)^{-1} =& \frac{(2\lambda + a) - e^{-\lambda\tau}c + \tau e^{-\lambda\tau}(c\lambda + d)}{-\lambda e^{-\lambda\tau}(c\lambda + d)} \\
=& \frac{2\lambda + a}{-\lambda e^{-\lambda\tau}(c\lambda + d)} + \frac{c}{\lambda(c\lambda + d)} - \frac{\tau}{\lambda} \\
=& -\frac{\lambda(2\lambda + a)}{\lambda^2(\lambda^2 + a\lambda + b)} + \frac{c\lambda}{\lambda^2(c\lambda + d)} - \frac{\tau}{\lambda} \\
=& -\frac{(\lambda^2 + a\lambda + b) + \lambda^2 - b}{\lambda^2(\lambda^2 + a\lambda + b)} + \frac{(c\lambda + d) - d}{\lambda^2(c\lambda + d)} - \frac{\tau}{\lambda} \\
=& -\frac{\lambda^2 - b}{\lambda^2(\lambda^2 + a\lambda + b)} + \frac{-d}{\lambda^2(c\lambda + d)} - \frac{\tau}{\lambda}.
\end{aligned}$$

By the relation

$$\frac{d\lambda}{d\tau} = \frac{d\mathrm{Re}(\lambda)}{d\tau} + i\frac{d\mathrm{Im}(\lambda)}{d\tau} = \left\{\left(\frac{d\mathrm{Re}(\lambda)}{d\tau}\right)^2 + \left(\frac{d\mathrm{Im}(\lambda)}{d\tau}\right)^2\right\}\left(\frac{d\mathrm{Re}(\lambda)}{d\tau} - i\frac{d\mathrm{Im}(\lambda)}{d\tau}\right)^{-1},$$

we have $\frac{d\mathrm{Re}(\lambda)}{d\tau} = \mathrm{Re}(\frac{d\lambda}{d\tau})^{-1}\{(\frac{d\mathrm{Re}(\lambda)}{d\tau})^2 + (\frac{d\mathrm{Im}(\lambda)}{d\tau})^2\}$ and

$$\begin{aligned}
\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Big|_{\tau=\tau_n} =& \frac{(-\omega_0^2 - b)(b - \omega_0^2)}{\omega_0^2\{(b - \omega_0^2)^2 + a^2\omega_0^2\}} + \frac{d^2}{\omega_0^2(c^2\omega_0^2 + d^2)} \\
=& \frac{\omega_0^4 - b^2 + d^2}{\omega_0^2(c^2\omega_0^2 + d^2)} \\
=& \frac{\omega_0^4 - (b - d)(b + d)}{\omega_0^2(c^2\omega_0^2 + d^2)} > 0.
\end{aligned}$$

Hence we obtain $\frac{d\mathrm{Re}(\lambda)}{d\tau}|_{\tau=\tau_n} > 0$ for $n = 0, 1, \cdots$. The proof is complete. □

By Proposition 3.2 and the first part of Proposition 3.3, all the roots of (3.4) have negative real part for any $\tau \geq 0$ if $1 < R_0 \leq \overline{R}_0$. By Proposition 3.2, the second part of Proposition 3.3 and Proposition 3.4, all the roots of (3.4) have negative real part for $0 \leq \tau < \tau_0$ and there exists at least 2 roots having positive real part for $\tau > \tau_0$ if $1 < \overline{R}_0 < R_0$. We then establish the stability condition for the endemic equilibrium as follows.

Theorem 3.1 (Enatsu et al. [2, Theorem 3.2]). *Assume $R_0 > 1$. Then the following statement holds true.*

(i) *If $R_0 \leq \overline{R}_0$, then the endemic equilibrium E_* of system (1.2) is locally asymptotically stable for any $\tau \geq 0$.*

(ii) *If $\overline{R}_0 < R_0$, then the endemic equilibrium E_* of system (1.2) is locally asymptotically stable for $0 \leq \tau < \tau_0$ and it is unstable for $\tau > \tau_0$.*

Remark 3.1. System (1.2) undergoes Hopf bifurcation at the endemic equilibrium E_* when τ crosses τ_n $(n = 0, 1, \cdots)$ for $1 < \overline{R}_0 < R_0$.

4 Example

In this section, we consider the following model:

$$\begin{cases} \dfrac{dS(t)}{dt} = r\left(1 - \dfrac{S(t)}{K}\right)S(t) - \beta S(t)\dfrac{I(t-\tau)}{1+\alpha I(t-\tau)}, \\ \dfrac{dI(t)}{dt} = \beta S(t)\dfrac{I(t-\tau)}{1+\alpha I(t-\tau)} - (\mu_1 + \gamma)I(t), \\ \dfrac{dR(t)}{dt} = \gamma I(t) - \mu_2 R(t) \end{cases} \tag{4.1}$$

with $\alpha \geq 0$. One can see that system (4.1) always has the trivial equilibrium E_0 and the disease-free equilibrium E_1. Applying Theorems 2.1 and 2.2, we obtain the following results:

Corollary 4.1. *The trivial equilibrium E_0 of system (4.1) is always unstable.*

Corollary 4.2. *The disease-free equilibrium E_1 of system (4.1) is globally asymptotically stable if and only if $R_0 \leq 1$ and it is unstable if and only if $R_0 > 1$.*

Since $G(I) = \frac{I}{1+\alpha I}$ satisfies the hypothesis (H3), system (4.1) has a unique endemic equilibrium $E_* = (S^*, I^*, R^*)$ if and only if $R_0 > 1$. In particular, the second component of I^* becomes

$$I^* = \frac{K(\alpha r - \beta) - 2\alpha r(\mu_1 + \gamma) + \sqrt{K^2(\alpha r - \beta)^2 + 4K\alpha r\beta(\mu_1 + \gamma)}}{2\alpha^2 r(\mu_1 + \gamma)} > 0.$$

Applying Theorem 3.1, we obtain the following result:

Corollary 4.3 (Enatsu et al. [2, Corollary 4.3]). *Assume* $R_0 > 1$*. Then the following statement holds true.*

(i) *If* $R_0 \leq \overline{R}_0$*, then the endemic equilibrium* E_* *of system (4.1) is locally asymptotically stable for any* $\tau \geq 0$*.*

(ii) *If* $\overline{R}_0 < R_0$*, then the endemic equilibrium* E_* *of system (4.1) is locally asymptotically stable for* $0 \leq \tau < \tau_0$ *and it is unstable for* $\tau > \tau_0$*.*

By Corollary 4.3, $R_0 = 1$ is a threshold condition which determines stability of the disease-free equilibrium and the existence of the endemic equilibrium. Moreover, if $R_0 > 1$ then $R_0 = \overline{R}_0$ is a threshold condition which determines delay-dependent stability or delay-independent stability for the endemic equilibrium (see, for details, Enatsu et al. [2, Section 4 and Appendix A]).

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Asymptotic behavior of solutions for free boundary problems related to an ecological model

dedicated to Professor Hiroki Tanabe on the occasion of his 80th birthday

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1 Introduction

The spreading of invasive or new species has been a main topic in mathematical ecology. Many researchers have studied the problem from various aspects. See, for example Shigesada and Kawasaki [11], for detailed information. We consider, in this article, a new mathematical model which has been proposed by Du and Lin [3]. It is described as a free boundary problem for diffusive logistic equation:

$$\begin{cases} u_t - du_{xx} = u(a-bu), & t>0,\ 0<x<h(t), \\ u_x(t,0)=0,\ u(t,h(t))=0, & t>0, \\ h'(t) = -\mu u_x(t,h(t)), & t>0, \\ h(0)=h_0,\ u(0,x)=u_0(x), & 0\le x\le h_0, \end{cases} \tag{1.1}$$

where μ, h_0, d, a and b are given positive numbers. Initial data satisfies $u_0 \in C^2(0,h_0)$, $u_0'(0) = u(h_0) = 0$ and $u_0(x) > 0$ in $[0,h_0)$. An unknown quantity $u = u(t,x)$ is a population density of invasive or new species which occupies one dimensional region, $(0,h(t))$. The right-hand side of the habitat $x = h(t)$ is called free boundary which means a spreading front of the species. Moreover, the dynamical behavior of the free boundary is determined by Stefan-like condition, $h'(t) = -\mu u_x(t,h(t))$. This implies that spreading speed of the species is proportional to the population pressure at the free boundary.

Du and Lin [3] have obtained the existence and uniqueness of global solutions for (1.1) and studied their asymptotic behavior as $t\to\infty$. In particular, the asymptotic behavior is divided into two cases:

(i) Spreading: $\lim_{t\to\infty} h(t) = \infty$ and $\lim_{t\to\infty} u(t,x) = \dfrac{a}{b}$ uniformly in any compact subset of $[0,\infty)$;

(ii) Vanishing: $\lim_{t\to\infty} h(t) \le \dfrac{\pi}{2}\sqrt{\dfrac{d}{a}}$ and $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(0,h(t))} = 0$.

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This result is called in [3] the dichotomy theorem, where case (i) implies that the species success to spread to a new environment, while case (ii) implies that the species must vanish eventually.

We are concerned with more realistic environments and seek radially symmetric solutions of a free boundary problem in higher space dimension. Our free boundary problem is given by

$$(\mathrm{FBP}) \qquad \begin{cases} u_t - du_{rr} - \dfrac{(N-1)d}{r}u_r = uf(u), & t>0,\ R<r<h(t), \\ u(t,R)=0,\ u(t,h(t))=0, & t>0, \\ h'(t) = -\mu u_r(t,h(t)), & t>0, \\ h(0)=h_0,\ u(0,r)=u_0(r), & R \le r \le h_0, \end{cases}$$

where $r=|x|$ ($x\in\mathbb{R}^N$, $N\ge 1$) and μ, h_0, d and R are positive constants. Initial data (u_0,h_0) satisfies

$$u_0 \in C^2(R,h_0) \ \text{ with } \ u_0(R)=u_0(h_0)=0 \ \text{ and } \ u_0>0 \ \text{ in } (R,h_0).$$

Moreover, the nonlinear function in the diffusion equation is assumed to satisfy

$$f\in C^1(\mathbb{R}) \quad \text{and} \quad f(u)<0 \ \text{ for } \ u>1. \tag{1.2}$$

Differently from the problem discussed in [3], our problem (FBP) allows more general nonlinearity in the diffusion equation and has Dirichlet boundary conditions on both fixed and free boundaries. This condition means, from an ecological view-point, that species inhabit an annular domain $\{x\in\mathbb{R}^N \mid R<|x|<h(t)\}$, but a region $\{x\in\mathbb{R}^N \mid |x|\le R\}$ is a hostile environment for the species.

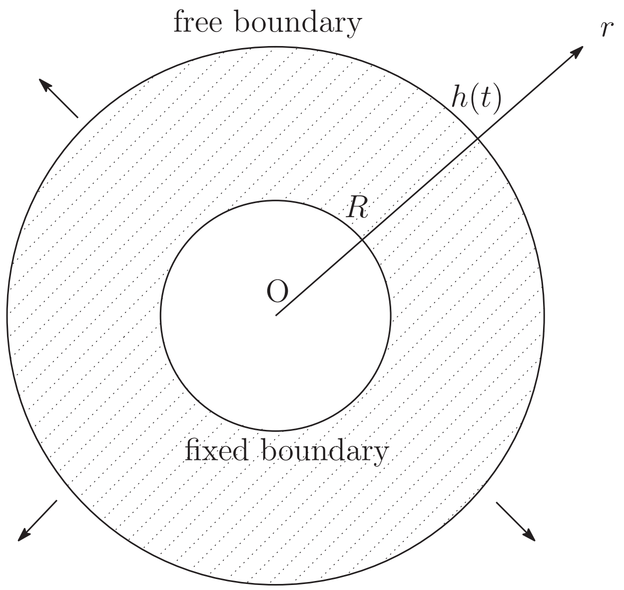

Figure 1. habitat of species

The main purpose of this paper is as follows:

(i) Present recent results on global existence and asymptotic properties of solutions for (FBP).

(ii) Find underlying principles to determine spreading or vanishing of species.

(iii) Construct a dichotomy theorem in the radially symmetric case and compare the theorem with that in one-dimensional case.

We have obtained a global existence and uniqueness theorem for (FBP).

Theorem 1.1. *The free boundary problem* (FBP) *has a unique solution* (u, h) *satisfying*

$$0 < u(t, r) \le C_1, \quad 0 < h'(t) \le \mu C_2 \quad \textit{for} \ \ t \ge 0, \ R < r < h(t),$$

where C_1 *and* C_2 *are positive constants depending only on* $\|u_0\|_{C(R,h_0)}$ *and* $\|u_0\|_{C^1(R,h_0)}$, *respectively.*

By this theorem, we find that the free boundary is strictly increasing with respect to t; so the limit of $h(t)$ exists and it may be a finite number or equal to infinity.

We define spreading and vanishing of species under general situations as follows.

Definition 1.1. Let (u, h) be any solution of (FBP).

(I) *Spreading of species* is the case when

$$\lim_{t\to\infty} h(t) = \infty \quad \text{and} \quad \liminf_{t\to\infty} \|u(t,\cdot)\|_{C(R,h(t))} > 0;$$

(II) *Vanishing of species* is the case when

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{C(R,h(t))} = 0.$$

One of sufficient conditions for the spreading property of (FBP) satisfying

$$\liminf_{t\to\infty} u(t,r) > 0 \ \ \text{for} \ \ R < r < \infty$$

is given by the following proposition.

Proposition 1.1. *Let* q *be a positive solution of*

$$\text{(IP)} \qquad \begin{cases} dq_{rr} + \dfrac{(N-1)d}{r} q_r + qf(q) = 0, \quad R < r < l, \\ q(R) = q(l) = 0 \end{cases}$$

with a positive number $l > R$. *Then, the solution* (u, h) *of* (FBP) *with initial data* (q, l) *satisfies*

(i) $\lim_{t\to\infty} h(t) = \infty$;

(ii) $u_t(t,r) \geq 0$ *for* $t > 0,\ R < r < h(t)$;

(iii) $\lim_{t\to\infty} u(t,r) = v^*(r)$: *uniformly in any compact subset of* $[R,\infty)$,

where v^* *is a minimal positive solution of*

$$\text{(SP)} \qquad \begin{cases} dv_{rr} + \dfrac{(N-1)d}{r} v_r + vf(v) = 0, & R < r < \infty, \\ v(R) = 0 \end{cases}$$

which satisfies $v^*(r) \geq q(r)$ *in* $[R,l]$.

The following proposition is a vanishing property.

Proposition 1.2. *Let* (u,h) *be any solution of* (FBP). *If* $\lim_{t\to\infty} h(t) < \infty$, *then*

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{C(R,h(t))} = 0.$$

We omit here the proofs of Theorem 1.1, Propositions 1.1 and 1.2. The proofs of the results in one-dimensional case can be found in Kaneko and Yamada [6], where we have referred to some properties in Tanabe [13] to investigate the asymptotic behavior of solutions. Note that the results can be naturally extended to the radially symmetric case.

2 Asymptotic behavior

We assume that the nonlinear function in (FBP) satisfies $f \in C^1(\mathbb{R})$ and

$$f(u) > 0 \text{ for } 0 \leq u < 1,\ f(u) < 0 \text{ for } u > 1,\ f(1) = 0 \text{ and } f'(u) \leq 0 \text{ for } u \geq 0.$$

It is a kind of nonlinearities satisfying (1.2); so we can obtain global existence and asymptotic properties of solutions by Theorem 1.1, Propositions 1.1 and 1.2.

2.1 Spreading and vanishing in one-dimensional case

Let $N = 1$ and $R = 0$ in (FBP), (IP) and (SP). We replace r with x. Then, the free boundary problem for a reaction-diffusion equation is given by

$$\text{(P)} \qquad \begin{cases} u_t - du_{xx} = uf(u), & t > 0,\ 0 < x < h(t), \\ u(t,0) = 0,\ u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ h(0) = h_0,\ u(0,x) = u_0(x), & 0 \leq x \leq h_0, \end{cases}$$

where μ, h_0 and d are positive constants and initial data (u_0, h_0) satisfies

$$u_0 \in C^2(0, h_0) \ \text{ with } \ u_0(0) = u_0(h_0) = 0 \ \text{ and } \ u_0 > 0 \ \text{ in } (0, h_0).$$

We present some recent results obtained in Kaneko, Oeda and Yamada [5]. The following theorem is a dichotomy theorem which means that the asymptotic behavior of solutions for (P) is divided into two cases.

Theorem 2.1. *Let* (u, h) *be any solution of* (P)*. Then, either* (I) *or* (II) *holds true*:

(I) *Spreading* : $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t, x) = v^*(x)$ *uniformly in any compact subset of* $[0, \infty)$*, where* $v^*(x)$ *is a unique positive solution of* (SP);

(II) *Vanishing* : $\lim_{t\to\infty} h(t) \le \pi\sqrt{\dfrac{d}{f(0)}}$ *and* $\lim_{t\to\infty} \|u(t, \cdot)\|_{C(0,h(t))} = 0.$

We will also show some sufficient conditions for spreading and vanishing.

Theorem 2.2. *Let* (u, h) *be any solution of* (P)*. Then, the following results hold true*:

(i) *Suppose* $h_0 \ge \pi\sqrt{d/f(0)}$*. Then spreading occurs.*

(ii) *Suppose* $h_0 < \pi\sqrt{d/f(0)}$*.*

(a) *There exists a positive function* w *in* $[0, h_0]$ *such that, if* $u_0(x) \le w(x)$ *in* $[0, h_0]$*, then vanishing occurs and* $\|u(t, \cdot)\|_{C(0,h(t))} = O(e^{-\beta t})$ *for some* $\beta > 0$ *as* $t \to \infty$*.*

(b) *If*

$$\int_0^{h_0} x u_0(x)\, dx \ge \frac{d}{2\mu}\Big(\frac{\pi^2 d}{f(0)} - h_0^2\Big) \max\{1, \|u_0\|_{C(0,h_0)}\},$$

then spreading occurs.

2.2 Spreading and vanishing in radially symmetric case

First we prepare some results of the elliptic boundary value problem (IP) and a corresponding eigenvalue problem:

$$\text{(EP)} \quad \begin{cases} d\phi_{rr} + \dfrac{(N-1)d}{r}\phi_r + \lambda\phi = 0, & R < r < l, \\ \phi(R) = \phi(l) = 0. \end{cases}$$

Here l is a given positive number. By Proposition 1.1, when (IP) has a positive solution $q(r; l)$, one can show that the solution for (FBP) with initial data (q, l) satisfies spreading property.

Proposition 2.1. *The following results hold true*:

(i) *If* $f(0) > \lambda_1$, *then* (IP) *has a unique positive solution* $q(x)$;

(ii) *If* $f(0) \leq \lambda_1$, *then* $q \equiv 0$ *is a unique solution of* (IP),

where $\lambda_1 = \lambda_1(R, d, l)$ *is the least eigenvalue of* (EP).

Regard $\lambda_1(R, d, l)$ as a function of l. It is well known that $\lambda_1(R, d, l)$ is continuous and decreasing with respect to l. Hence

$$\lim_{l \to R+0} \lambda_1(l) = +\infty \quad \text{and} \quad \lim_{l \to +\infty} \lambda_1(l) = 0.$$

It follows that, for any given R, d and f, there exists a positive number $R^* = R^*(R, d, f(0))$ such that

$$f(0) = \lambda_1(R^*) \quad \text{and} \quad f(0) > \lambda_1(l) \ \text{ for } \ l > R^*.$$

For example, in one-dimensional case, $R^*(R, d, f(0))$ is given by $\pi\sqrt{d/f(0)}+R$. This number R^* plays an important role to study the asymptotic behavior of solutions. In particular, Proposition 2.1 is rewritten to a more convenient style (c.f. Cantrell and Cosner [1]).

Proposition 2.2. *The following results hold true*:

(i) *If* $l > R^*$, *then* (IP) *has a unique positive solution* $q(x)$;

(ii) *If* $l \leq R^*$, *then* $q \equiv 0$ *is a unique solution of* (IP).

The following result is a dichotomy theorem for (FBP).

Theorem 2.3. *Let* (u, h) *be any solution of* (FBP). *Then, either* (I) *or* (II) *holds true*:

(I) *Spreading* : $\lim_{t\to\infty} h(t) = \infty$ *and* $\lim_{t\to\infty} u(t, r) = v^*(r)$ *uniformly in any compact subset of* $[R, \infty)$, *where* $v^*(r)$ *is a unique positive solution of* (SP);

(II) *Vanishing* : $\lim_{t\to\infty} h(t) \leq R^*$ *and* $\lim_{t\to\infty} \|u(t, \cdot)\|_{C(R,h(t))} = 0$.

The following theorem gives some sufficient conditions for spreading and vanishing.

Theorem 2.4. *Let* (u, h) *be any solution of* (FBP).

(i) *Suppose* $h_0 \geq R^*$. *Then spreading occurs.*

(ii) *Suppose* $h_0 < R^*$. *Then there exists a positive function* w *in* $[R, h_0]$ *such that, if* $0 \leq u_0(r) \leq w(r)$ *in* $[R, h_0]$, *then vanishing occurs. Moreover, it holds that* $\|u(t, \cdot)\|_{C[R,h(t)]} = O(e^{-\beta t})$ *for some* $\beta > 0$ *as* $t \to \infty$.

2.3 Proofs of main results

We will prove Theorem 2.3. The proof will be accomplished by using Propositions 2.3 and 2.4.

Proposition 2.3. *Let* (u, h) *be any solution of* (FBP). *If* $\lim_{t\to\infty} h(t) < \infty$*, then*

$$\lim_{t\to\infty} h(t) \le R^* \quad \text{and} \quad \lim_{t\to\infty} \|u(t,\cdot)\|_{C(R,h(t))} = 0.$$

Proof. Proposition 1.2 shows that, if $\lim_{t\to\infty} h(t) < \infty$, then

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{C(R,h(t))} = 0. \tag{2.1}$$

Hence, it suffices to prove $\lim_{t\to\infty} h(t) \le R^*$. Otherwise, there exists $T > 0$ such that $h(T) > R^*$. Take $l = h(T)$ and let $w = w(t, r)$ be the solution of the problem:

$$\begin{cases} w_t - dw_{rr} - \dfrac{(N-1)d}{r} w_r = wf(w), & t > 0,\ R < r < l, \\ w(t, R) = 0,\ w(t, l) = 0, & t > 0, \\ w(T, r) = u(T, r), & R < r < l. \end{cases}$$

Then, the comparison principle (see Protter and Weinberger [9] or Smoller [12]) shows

$$u(t, r) \ge w(t, r) \quad \text{for} \quad t \ge T,\ R < r < l.$$

Moreover, it holds that $\lim_{t\to\infty} w(t, r) = q(r)$ for $R < r < l$, where $q(r)$ is a positive solution of (IP). Hence

$$\liminf_{t\to\infty} u(t, r) \ge q(r) > 0 \quad \text{for} \quad R < r < l.$$

This contradicts (2.1) and the free boundary must satisfy $\lim_{t\to\infty} h(t) \le R^*$. □

Proposition 2.4. *Let* (u, h) *be any solution of* (FBP). *If* $\lim_{t\to\infty} h(t) = \infty$*, then*

$$\lim_{t\to\infty} u(t, r) = v^*(r) \quad \text{uniformly in any compact subset of } [R, \infty),$$

where $v^*(r)$ *is a unique solution of* (SP).

Proof. We will first construct a suitable upper solution for the free boundary problem. Define $M = \max\{1, \|u_0\|_{C(R,h_0)}\}$. Let $\overline{u}(t, r)$ be the solution of

$$\begin{cases} \overline{u}_t - d\overline{u}_{rr} - \dfrac{(N-1)d}{r} \overline{u}_r = \overline{u} f(\overline{u}), & t > 0,\ r > R, \\ \overline{u}(t, R) = 0, & t > 0, \\ \overline{u}(0, r) = M, & r > R. \end{cases}$$

Then, $v \equiv M$ is regarded as an upper solution of (SP). Hence $\overline{u}(t,\cdot)$ is decreasing and satisfies $\lim_{t\to\infty} \overline{u}(t,r) = v^*(r)$ uniformly in any compact subset of $[R,\infty)$ (see Sattinger [10]). Note that $u_0(r) \leq M$ in $[R,h_0]$. Then, the comparison principle proves

$$u(t,r) \leq \overline{u}(t,r) \quad \text{for} \ \ t>0,\ R<r<h(t).$$

Letting $t \to \infty$ implies

$$\limsup_{t\to\infty} u(t,r) \leq \lim_{t\to\infty} \overline{u}(t,r) = v^*(r) \quad \text{for} \ \ R<r<\infty.$$

On the other hand, for any positive number $l > R^*$, one can take $T > 0$ such that $h(T) = l$. In the same way of the proof of Proposition 2.3, we obtain

$$\liminf_{t\to\infty} u(t,r) \geq q(r;l) \quad \text{for} \quad R<r<l.$$

Moreover, we get

$$\lim_{l\to\infty} q(r;l) = v^*(r) \quad \text{for} \quad R<r<\infty.$$

Hence,

$$\liminf_{t\to\infty} u(t,r) \geq v^*(r) \quad \text{for} \quad R<r<l.$$

As a result, it holds that

$$\lim_{t\to\infty} u(t,r) = v^*(r) \ \ \text{uniformly in any compact subset of} \ (R,\infty).$$

The proof is complete. □

3 Concluding remarks

Spreading and vanishing for the asymptotic behavior of solutions are characteristic of this free boundary model. A Similar dichotomy theorem also holds true for free boundary problems with other nonlinear terms like bistable nonlinearities which satisfy

$$\begin{aligned} &f(u)<0 \ \ \text{for} \ \ 0\leq u<c \ \ \text{and} \ \ u>1, \ \ f(u)>0 \ \ \text{for} \ \ c<u<1, \\ &f(c)=f(1)=0, \ \ f'(c)>0 \ \ \text{and} \ \ f'(1)<0 \ \ \text{with} \ \ \int_0^1 uf(u) \ \ du>0. \end{aligned}$$

Kaneko, Oeda and Yamada [5] have proved that, in this free boundary model,

- the population vanishes, that is,

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{C(0,h(t))} = 0$$

  if and only if $\lim_{t\to\infty} h(t) < \infty$.

- when vanishing occurs, the population decreases exponentially to 0 in large time.

We should refer to some related free boundary problems. Du and Lou [4] have studied a one-dimensional free boundary problem with free boundaries in both left and right boundaries. Du and Guo [2] have investigated a logistic free boundary problem in multi-dimensional ball and extended their dichotomy results to the higher dimensional case. Two species models with a free boundary condition have been studied by Mimura, Yamada and Yotsutani [8] and Lin [7].

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Turing's instability and pattern transitions in a nonlinear differential equation

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Abstract

Motivated to study the pattern formations of solutions' level sets, which are seen in many nonlinear reaction-diffusion equations from chemistry, physics, biology, etc, we research necessary or sufficient conditions on the equation

$$du/dt \in -\partial\varphi(u(t)) + g_\infty, \quad t > 0,$$

for Turing's instability, aftereffects of a kind of momentary decomposition, or transitions of level set patterns. Here $\partial\varphi$ denotes a subdifferential operator defined in a real Hilbert space H and $g_\infty \in H$.

It is shown that for pattern transitions the relation $g_\infty \in \overline{R(\partial\varphi)}$ is necessary, while to get Turing's instability the relation $g_\infty \notin R(\partial\varphi)$ is needed. If $g_\infty \in \overline{R(\partial\varphi)} \setminus R(\partial\varphi)$ and φ satisfies $\varphi(rx) = |r|^p\varphi(x)$ for some $p > 1$, then the solutions behave in aftereffects of the momentary decomposition and show pattern transitions.

Key Words: pattern formation, reaction-diffusion equation, Turing's instability, asymptotic behavior of solutions, subdifferential operator.

1 Introduction

This paper's motive is to research essential or sufficient conditions on the reaction-diffusion equations of the form

$$\frac{\partial u}{\partial t}(x,t) \; = \; \triangle u + f(u,v,\alpha), \qquad \frac{\partial v}{\partial t}(x,t) \; = \; \varepsilon\triangle v + g(u,v,\beta), \tag{1.1}$$

or

$$\frac{\partial u}{\partial t}(x,t) = \triangle u(x,t) + f(u(x,t),x) \tag{1.2}$$

for the pattern formations of level sets of the solution u, which are seen in many situations of physics, chemistry, biology etc.

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In many cases, solution $v(x,t)$ of (1.1) converges to some $v_\infty(x)$ as $t \to \infty$. Putting $f_\infty(u,x) = f(u, v_\infty(x), \beta)$ implies the single equation (1.2) with $f = f_\infty$. Hence, in this paper, we are concerned with only (1.2).

Solution u of the form of (1.1) or (1.2) many times satisfies the following properties.

(P1) (instability) $u(.,t)$ grows up as $t \to \infty$.

(P2) (asymptotic disappearance of movement) $(\partial/\partial t)u(.,t)$ decreases and vanishes to 0 as $t \to \infty$.

(P3) (pattern transitions) There is a sequence $\{t_i\} \subset (0, \infty)$ such that for each $i \neq j$, the patterns of level sets of $u(.,t_i)$ and $u(.,t_j)$ are enough different from each other. Hence, in the case where $u(.,t_i) \in \mathrm{L}^2(\Omega)$, there is $\delta > 0$ such that $|(u(.,t_i),\ u(.,t_j))| \le (1-\delta)\|u(.,t_i)\|\|u(.,t_j)\|$ holds for $i \neq j$.

Although (P1) means the instability of u, the both term $\triangle$ and f of (1.2) often satisfy the stability as below:

$$\text{Every solution } v(x,t) \text{ of } \frac{\partial v}{\partial t} = \triangle v \text{ converges to 0 as } t \to \infty.$$

$$\text{Every solution } w(x,t) \text{ of } \frac{\partial w}{\partial t} = f(w,x) \text{ converges to some } w_\infty(x) \text{ as } t \to \infty.$$

To research necessary or sufficient conditions for the properties (P1)-(P3), we put $\tilde{u}(x,t) = u(x,t) - w_\infty(x)$. Then, by (1.2),

$$\frac{\partial \tilde{u}}{\partial t}(x,t) = \triangle \tilde{u}(x,t) + f(\tilde{u} + w_\infty, x) + \triangle w_\infty(x), \qquad (x,t) \in \Omega \times [0,\infty). \tag{1.3}$$

In this paper, we consider (1.3) to be an ordinary differential equation in $L^2(\Omega)$ or in a real Hilbert space H. Under the condition that $f(.,x)$ is nondecreasing for each x, it is usually possible to put $-\partial\varphi(\tilde{u})$ in stead of $\triangle\tilde{u} + f(\tilde{u} + w_\infty, .)$ in (1.3). We also take $g_\infty \in H$ in stead of $\triangle w_\infty$. Hence

$$\frac{d\tilde{u}}{dt}(t) \in -\partial\varphi(\tilde{u}(t)) + g_\infty, \qquad t \in [0,\infty). \tag{1.4}$$

Here $\partial\varphi$ denotes a subdifferential operator defined in H.

We show the following.

(a) Any solution $\tilde{u}$ of (1.4) satisfies (P1), more precisely $\|\tilde{u}(t)\| \to \infty$ as $t \to \infty$, if and only if $g_\infty \notin R(\partial\varphi)$. Thus, the relation $\Delta w_\infty \notin R(-\Delta \cdot - f(\cdot + w_\infty))$ seems essential for solutions of (1.3) to satisfy (P1).

(b) If a solution $\tilde{u}$ of (1.4) satisfies (P3), then $\tilde{u}$ needs to satisfy (P2).

(c) A solution of (1.4) satisfies (P2) if and only if $g_\infty \in \overline{R(\partial\varphi)}$. This suggests that the relation $\Delta w_\infty \in \overline{R(-\Delta \cdot -f(\cdot + w_\infty))}$ is close to a essential condition for solutions of (1.3) to fulfill (P2).

(d) For all solutions of (1.4) one gets (P3) together with (P1), (P2) under the conditions that $g_\infty \in \overline{R(\partial\varphi)} \setminus R(\partial\varphi)$ and $\varphi(rx) = |r|^p \varphi(x),\ r \in \mathbf{R}$. Hence, the combination of these conditions seems close to a sufficient condition for solutions of (1.3) to satisfy all of (P1)-(P3).

2 Results

Let H be a real Hilbert space with norm $\|.\|$ and inner product $(.,.)$, and $\varphi : H \to (-\infty,\ \infty]$ be a proper lower semi continuous (l.s.c.) convex functional. The set $D(\varphi) \equiv \{v \in H : \varphi(v) < \infty\}$ is called the effective domain of φ. The subdifferential operator $\partial\varphi$ of φ is defined as below:

$$\partial\varphi(v) = \{f \in H :\ \varphi(w) \geq \varphi(v) + (f,\ w - v),\ \forall w \in D(\varphi)\},$$

$$D(\partial\varphi) = \{v \in D(\varphi) :\ \partial\varphi(v) \neq \phi\}.$$

Under the assumption that φ is a proper l.s.c convex functional, $\partial\varphi$ is known to be a maximal monotone operator defined in H. (e.g., [3], [4])

Let $g_\infty \in H$. We research the asymptotic behaviors of solutions of

$$\frac{d}{dt}u(t) \in -\partial\varphi(u(t)) + g_\infty, \quad t \geq 0. \tag{2.1}$$

Here $u(.)$ is called a solution of (2.1) if and only if $u(.)$ belongs to $W^{1,2}_{\mathrm{loc}}((0,\infty) : H)$ and satisfies the relation (2.1) for almost all $t > 0$. For each $u_0 \in \overline{D(\partial\varphi)}$, there is an unique solution $u \in W^{1,2}_{\mathrm{loc}}((0,\infty) : H) \cap C([0,\infty) : H)$ of (2.1) satisfying $u(0) = u_0$.

It is known that each solution of (2.1) satisfies the equation

$$\frac{d^+}{dt}u(t) = -(\partial\varphi(u(t)) - g_\infty)^0, \qquad \forall t > 0,$$

where $(\partial\varphi(x) - g_\infty)^0$ denotes the minimum norm point of $\partial\varphi(x) - g_\infty$, that is, $(\partial\varphi(x) - g_\infty)^0 \in \partial\varphi(x) - g_\infty$ and $\|(\partial\varphi(x) - g_\infty)^0\| = \min\{\|y\| : y \in \partial\varphi(x) - g_\infty\}$. (e.g., [3], [4])

In the following, $u'(t)$ denotes $(d^+/dt)u(t)$.

Proposition 2.1 *For arbitrary $\{t_n\}$ with $0 = t_0 < t_1 < \cdots < t_n < \cdots$, let $\{U_n\}$ be the approximate solution of (2.1) such that*

$$U_n' \in -\partial\varphi(U_n) + g_\infty, \qquad U_n' = \frac{U_n - U_{n-1}}{\Delta t_n}, \quad \Delta t_n = t_n - t_{n-1}. \tag{2.2}$$

Then, one has

$$\|U'_{n+1}\| \le \left(\frac{U'_{n+1}}{\|U'_{n+1}\|}, \ \frac{U'_n}{\|U'_n\|} \right) \|U'_n\|. \tag{2.3}$$

Corollary 2.1 *For any solution u of (2.1), $\|u'(t)\|$ is nonincreasing.*

Remark 2.1 *The result of Corollary 2.1 is well known by other proofs.*

Estimate (2.3) in Proposition 2.1 means that the speed $\|U'_{n+1}\|$ has to be smaller than the speed $\|U'_n\|$ so much as the directions of $U'_{n+1}/\|U'_{n+1}\|$ and $U'_n/\|U'_n\|$ are different from each other. This suggests that if a solution u of (2.1) satisfies (P3) in Section 1, then $\|u'(t)\|$ converges to 0, or, (P2) holds. We will see that this expectation is true by Corollary 2.3 in below.

Our first theorem below shows essential conditions on g_∞ for the asymptotic behaviors (P1), (P2).

Theorem 2.1 *Let u be an arbitrary solution of (2.1).*

(i) If $g_\infty \in R(\partial\varphi)$, then the orbit $\cup_{t>0} u(t)$ is bounded, the value $\varphi(u(t)) - (g_\infty, u(t))$ converges to $\min_H \{\varphi(.) - (g_\infty,\ .)\}$, $\|u'(t)\| \downarrow 0$, and $u(t)$ converges weakly to a point of $(\partial\varphi)^{-1}(g_\infty)$ as $t \to \infty$.

(ii) Let $g_\infty \notin R(\partial\varphi)$. Then, $\lim_{t\to\infty} \|u(t)\| = \infty$ and the following hold.

(ii-1) In case of $g_\infty \in \overline{R(\partial\varphi)} \setminus R(\partial\varphi)$, $\|u'(t)\| \downarrow 0$ as $t \to \infty$.

(ii-2) If $g_\infty \notin \overline{R(\partial\varphi)}$, then $\|u'(t) - h\| \to 0$ as $t \to \infty$, where $h = (I - \mathrm{Proj}_{\overline{R(\partial\varphi)}})\, g_\infty$.

Corollary 2.2 *For each solution u of (2.1), $u'(t)$ converges strongly to $(I - \mathrm{Proj}_{\overline{R(\partial\varphi)}}) g_\infty$ as $t \to \infty$.*

Corollary 2.1 impleis that $u(t)/t$ converges strongly to $(I - \mathrm{Proj}_{\overline{R(\partial\varphi)}}) g_\infty$ as $t \to \infty$. Thus, if $(I - \mathrm{Proj}_{\overline{R(\partial\varphi)}}) g_\infty = h \neq 0$, then $u(t)/\|u(t)\|$ converges strongly to $h/\|h\|$. This means that the level sets patterns of $u(t)$ converges that of h as $t \to \infty$, or, u does not satisfy (P3). Hence, we have the following Corollary.

Corollary 2.3 *Suppose that a solution of (2.1) satisfies (P3). Then, $g_\infty \in \overline{D(\partial\varphi)}$ holds and (P2) satisfied for all solutions of (2.1).*

Remark 2.2 *Assertion (i) of Theorem 2.1 is a simple application of well known results. (e.g., [5],[8])*

Remark 2.3 *To get $g_\infty \notin R(\partial\varphi)$, $\partial\varphi$ needs to be not coercive. In fact, $\partial\varphi$ is coercive if and only if $R(\partial\varphi) = H$.*

We see by Theorem 2.1 that an arbitrary solution u satisfies the instability (P1), more precisely $\|u(t)\| \to \infty$ as $t \to \infty$, if and only if $g_\infty \notin \overline{R(\partial\varphi)}$.

One might wish that solutions satisfy all of (P1)-(P3) in every case of $g_\infty \in \overline{R(\partial\varphi)} \setminus R(\partial\varphi)$. However, some additional condition to $g_\infty \in \overline{R(\partial\varphi)} \setminus R(\partial\varphi)$ is needed for (P3). In fact, there is an example as below.

Example Fix any $z \in H \setminus \{0\}$ and put

$$\varphi(x) = \frac{1}{(x,\ z)}, \qquad D(\varphi) = \{x :\ (x,\ z) > 0\}.$$

Then, since $R(\partial\varphi) = \{rz;\ r < 0\}$, one has (i) $0 \in \overline{R(\partial\varphi)} \setminus R(\partial\varphi)$; and (ii) every solution u of (2.1) with $g_\infty = 0$ is described as $u(t) = u(0) + \lambda(t)z$ satisfying $\lambda(t) \to \infty$ as $t \to \infty$, hence $u(t)/\|u(t)\| \to z/\|z\|$. Thus u does not satisfy (P3).

To get the pattern transitions (P3) for each solution u of (2.1), we suppose that, for some $p > 1$, φ satisfies

$$\varphi(rx) = |r|^p \varphi(x), \qquad r \in \mathbf{R},\ x \in D(\varphi). \tag{2.4}$$

One notes that, in general, the convexity of φ implies $p \geq 1$. If $p = 1$, then $R(\partial\varphi)$ is closed, or $\overline{R(\partial\varphi)} \setminus R(\partial\varphi)$ is empty.

Fixing any $t > 0$ and $f \in \partial\varphi(u(t))$ and putting $P_{L(t,f)} = \mathrm{Proj}_{L(t,f)},\ L(t,f) = \{\lambda f : \lambda \in \mathbf{R}\}$, we consider the decomposition

$$-\partial\varphi(\cdot) + g_\infty = \{-\partial\varphi(\cdot) + P_{L(t,f)} g_\infty\} + (I - P_{L(t,f)}) g_\infty. \tag{2.5}$$

Here $(I - P_{L(t,f)})g_\infty$ is orthogonal to $\lambda^{p-1} f \in \partial\varphi(\lambda u(t)),\ \forall\lambda \in \mathbf{R}$.

In the case where $g_\infty \in \overline{R(\partial\varphi)}$ and $f = -u'(t) + g_\infty \in \partial\varphi(u(t))$, one sees by Theorem 2.1 that $\|(I - P_{L(t,f)})g_\infty\| \leq \|g_\infty - f\| = \|u'(t)\| \downarrow 0$.

Let $\lambda_0 = \lambda_0(f)$ be such that

$$P_{L(t,f)} g_\infty \in \partial\varphi(\lambda_0 u(t)). \tag{2.6}$$

Then, $\lambda_0 u(t)$ is a stable fixed point of the first term $-\partial\varphi(\cdot) + P_{L(t,f)} g_\infty$ in (2.5).

Theorem 2.2 *(aftereffects of decompositions (2.5)) Suppose that φ satisfies (2.4) for some $p > 1$. Then, for each $t > 0$ and $f \in \partial\varphi(u(t))$,*

$$-\alpha(((I - P_{L(t,f)})g_\infty,\ u(\tau) - u(t))) \leq \left(\frac{f}{\|f\|},\ u(\tau) - u(t)\right) \leq \beta(((I - P_{L(t,f)})g_\infty,\ u(\tau) - u(t))), \qquad \tau > t,$$

where $\alpha,\ \beta$ are continuous functions satisfying

$$\beta(\rho) \leq \frac{\rho}{\|f - P_{L(t,f)} g_\infty\|},\ \forall\rho > 0, \quad \alpha(\rho) \approx (a_0 + \rho)^{1/2},\ \rho \text{ is not large,}$$

$$\alpha(\rho) \approx b_0 \rho^{1/p}, \quad \beta(\rho) \approx b_0 \rho^{1/p}, \qquad \rho \gg 1.$$

In the case where φ satisfies (2.4) and $g_\infty \in \overline{R(\partial\varphi)} \setminus R(\partial\varphi)$, Theorem 2.2 means that $u(\tau) - u(t)$ is almost orthogonal to every $f \in \partial\varphi(u(t))$ if $\tau \gg t$ such that $\|u(\tau)\|$ is sufficiently large. On the other hand, for $f = -u'(t) + g_\infty \in \partial\varphi(u(t))$, (ii-1) of Theorem 2.1 implies $\|u'(t)\| = \|f - g_\infty\| \downarrow 0$. Thus, every solution u of (2.1) has the following property;

(A) $\quad u(\tau) - u(t)$ ($\approx u(\tau)$), $\forall\tau \gg \forall t \gg 1$, is almost orthogonal to g_∞.

Here one notes $g_\infty \neq 0$, since $g_\infty \notin R(\partial\varphi)$ and $0 \in \partial\varphi(0) \subset R(\partial\varphi)$ by (2.4).

This property (A) is very different from the case where $\partial\varphi \equiv 0$, because each solution v of (2.1) with $\partial\varphi \equiv 0$ satisfies

(B) $\quad v'(\tau) = g_\infty, \qquad \forall\tau > 0.$

In the following theorem, we assume a generalized condition of (2.4) on φ and get all of (P1)-(P3) for every solution of (2.1).

Theorem 2.3 *(pattern transitions) Suppose that $g_\infty \in \overline{R(\partial\varphi)} \setminus R(\partial\varphi)$ and that, for all $z \in D(\varphi)$ with $\|z\| = 1$, φ satisfies either (i) or (ii) as below.*

(i) $\exists\varepsilon_z > 0$, $\exists R_z > 0$, $\exists k_z : [R_z, \infty) \to (0, \infty)$ *satisfying* $\lim_{r\to\infty} k_z(r)/r = \infty$ *and*
$$\varphi(ry) \geq k_z(r)\varphi(R_z y) > 0, \quad \forall r \geq R_z,\ \forall y \in \{\|y\| = 1,\ \|y - z\| < \varepsilon_z\}.$$

(ii) $\varphi(rz) = 0$, $\forall r \in \mathbf{R}$.

Here $\inf_H \varphi = 0$ *is assumed without loss of generality.*

Then, for any solution $u(t)$ of (2.1), the omega limit set of $u(t)/\|u(t)\|$ is empty. Consequently, there are $\{T_i\}$ with $T_i \uparrow \infty$ and $\delta > 0$ such that

$$(u(T_i),\ u(T_j)) < (1 - \delta)\|u(T_j)\|\|u(T_i)\|, \qquad i \neq j.$$

Example The following φ satisfies the condition (i) of Theorem 2.3 with $R_z = 1$ and $k_z(r) = r^{\min\{p,q\}}$ for each z.
$$\varphi(v) = \int_\Omega a(x)|\nabla v(x)|^p dx + \int_\Omega b(x)|v(x)|^q dx, \quad \text{where } a(x),\ b(x) \geq 0,\ p, q > 1.$$

To end this talk, we consider the cases of $g_\infty \in R(\partial\varphi)$. As is mentioned in (i) of Theorem 2.1, any solution of (2.1) converges weakly to a point of $(\partial\varphi)^{-1}(g_\infty)$ as $t \to \infty$ if and only if $g_\infty \in R(\partial\varphi)$. Concerning this fact, J. B. Billon [1] gives an abstract example of φ on ℓ^2 which satisfies (i) $0 \in R(\partial\varphi)$, and (ii) a solution u of (2.1) with $g_\infty = 0$ converges weakly to $0 \in \ell^2$ but does not converge strongly, thus, for each $t > 0$, $\lim_{\tau\to\infty}(u(\tau), u(t)) = 0$ and $\inf_t \|u(t)\| > 0$. Hence this solution u satisfies (P3) together with (P2) but (P1). However, it seems that no concrete differential equation of the form (2.1) is known to have solutions satisfying such asymptotic weak and not strong convergence.

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Control problems for semilinear differential equations with local Lipschitz continuity

(In honor of Professor Hiroki Tanabe's 80th birthday)

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1 Introduction

Let H and V be real Hilbert spaces such that V is a dense subspace in H. Let U be a Banach space of control variables. In this paper, we are concerned with the global existence of solution and the approximate controllability for the following abstract neutral functional differential system in a Hilbert space H:

$$\begin{cases} \frac{d}{dt}[(x(t)+(Bx)(t)] = Ax(t) + f(t,x(t)) + (Cu)(t), \quad t \in (0,T], \\ x(0) = x_0, \quad (Bx)(0) = y_0, \end{cases} \tag{1.1}$$

where A is an operator associated with a sesquilinear form on $V \times V$ satisfying Gårding's inequality, f is a nonlinear mapping of $[0,T] \times V$ into H satisfying the local Lipschitz continuity, $B : L^2(0,T;V) \to L^2(0,T;H)$ and $C : L^2(0,T;U) \to L^2(0,T;H)$ are appropriate bounded linear mapping.

Recently, the existence of solutions for mild solutions for neutral differential equations with state-dependence delay has been recently studied in the literature in [1] and references therein. As for partial neutral integro-differential equations, we refer to [2]. However there are few papers treating the regularity and controllability for the systems with local Lipschipz continuity, we can just find a recent article Wang [3] in case semilinear systems.

In thia paper, we construct some results on the regularity of solutions and the approximate controllability for neutral functional differential equations with unbounded principal operators in Hilbert spaces. In order to establish the controllability of the neutral equations, we first consider the existence and regularity of solutions of the neutral control system by using fractional power of operators and the local Lipschtiz continuity of nonlinear term. Our purpose is to obtain the existence of solutions and the approximate controllability for neutral functional differential control systems without using many of the strong restrictions considering in the previous literature.

2 preliminaries

If H is identified with its dual space we may write $V \subset H \subset V^*$ densely and the corresponding injections are continuous. The norm on V, H and V^* will be denoted by $||\cdot||$, $|\cdot|$ and $||\cdot||_*$, respectively. For brevity, we may regard that

$$||u||_* \leq |u| \leq ||u||, \quad \forall u \in V. \tag{2.1}$$

Let $a(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re } a(u,u) \geq \delta ||u||^2, \quad \delta > 0. \tag{2.2}$$

Let A be the operator associated with this sesquilinear form: $(Au, v) = a(u, v)$ for any $u,\ v \in V$. Then A is a bounded linear operator from V to V^* by the Lax-Milgram Theorem. The realization of A in H which is the restriction of A to $D(A) = \{u \in V : Au \in H\}$ is also denoted by A. From (2.2) we may think that there exists a constant $C_0 > 0$ such that

$$||u|| \leq C_0 ||u||_{D(A)}^{1/2} |u|^{1/2}. \tag{2.3}$$

Thus we have the following sequence:

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*, \tag{2.4}$$

where each space is dense in the next one and continuous injection.

Lemma 2.1. *With the notations (2.3), (2.4), we have*

$$(V, V^*)_{1/2,2} = H, \quad (D(A), H)_{1/2,2} = V,$$

where $(V, V^*)_{1/2,2}$ denotes the real interpolation space between V and V^*(Section 1.3.3 of [6]).

It is also well known that A generates an analytic semigroup $S(t)$ in both H and V^*. By virtue of (2.2), we have that $0 \in \rho(A)$ the closed half plane $\{\lambda : \text{Re}\,\lambda \geq 0\}$ is contained in the resolvent set of A. In this case, we can define the fractional power $A^\alpha (\alpha > 0)$ of A and collect some simple properties of the fractional power of A.

Lemma 2.2. *(a) A^α is a closed operator with its domain dense.*
(b) If $0 < \alpha < \beta$, then $D(A^\alpha) \supset D(A^\beta)$.
(c) For any $T > 0$, there exists a posive constant C_α such that the following inequalities hold for all $t > 0$([7, Lemma 3.6.2]):

$$||A^\alpha S(t)||_{\mathcal{L}(H)} \leq \frac{C_\alpha}{t^\alpha}, \quad ||A^\alpha S(t)||_{\mathcal{L}(V,H)} \leq \frac{C_\alpha}{t^{3\alpha/2}}\,. \tag{2.5}$$

Let the solution spaces $\mathcal{W}(T)$ and $\mathcal{W}_1(T)$ of strong solutions be defined by

$$\mathcal{W}(T) = L^2(0,T; D(A)) \cap W^{1,2}(0,T; H),\ \mathcal{W}_1(T) = L^2(0,T; V) \cap W^{1,2}(0,T; V^*).$$

Here, we note that by using interpolation theory, there exists a constant $M_1 > 0$ such that

$$||x||_{C([0,T];V)} \leq M_1 ||x||_{\mathcal{W}(T)}, \quad ||x||_{C([0,T];H)} \leq M_1 ||x||_{\mathcal{W}_1(T)}. \tag{2.6}$$

By a simple calculation, we obtain the following.

Lemma 2.3. *For every $k \in L^2(0, T; H)$, let $x(t) = \int_0^t S(t-s)k(s)ds$ for $0 \le t \le T$. Then there exists a constant C_2 such that such that*

$$||x||_{L^2(0,T;V)} \le C_2\sqrt{T}||k||_{L^2(0,T;H)}. \tag{2.7}$$

3 Neutral differential equations

Consider the following abstract neutral functional differential system:

$$\begin{cases} \frac{d}{dt}[(x(t) + (Bx)(t)] = Ax(t) + f(t, x(t)) + k(t), \quad t \in (0, T], \\ x(0) = x_0, \quad (Bx)(0) = y_0. \end{cases} \tag{3.1}$$

Then we will show that the initial value problem (3.1) has a solution by solving the integral equation:

$$x(t) = S(t)[x_0 + y_0] - (Bx)(t) + \int_0^t AS(t-s)Bx(s)ds + \int_0^t S(t-s)\{f(s, x(s)) + k(s)\}ds. \tag{3.2}$$

Now we give the basic assumptions on the system (3.1)

Assumption (B). Let $B : L^2(0, T; V) \to L^2(0, T; H)$ be a bounded linear mapping such that there exists constants $\beta > 2/3$ and $L > 0$ such that

$$||A^\beta Bx||_{L^2(0,T;H)} \le L||x||_{L^2(0,T;V)}, \quad \forall x \in L^2(0, T; V).$$

Assumption (F). f is a nonlinear mapping of $[0, T] \times V$ into H satisfying following:

(i) There exists a function $L_1 : \mathbb{R}_+ \to \mathbb{R}$ such that for $||x|| \le r$ and $||y|| \le r$,

$$|f(t, x) - f(t, y)| \le L_1(r)||x - y||, \quad t \in [0, T].$$

(ii) The inequality

$$|f(t, x)| \le L_1(r)(||x|| + 1)$$

holds For every $t \in [0, T]$ and $x \in V$.

Let us rewrite $(Fx)(t) = f(t, x(t))$ for each $x \in L^2(0, T; V)$. From now on, we establish the following results on the solvability of the equation (3.1).

Theorem 3.1. *Let Assumptions (B) and (F) be satisfied. Assume that $x_0 \in H$, $k \in L^2(0, T; V^*)$ for $T > 0$. Then, there exists a solution x of the equation (3.1) such that*

$$x \in \mathcal{W}_1(T) \equiv L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H).$$

Moreover, there is a constant C_3 independent of x_0 and the forcing term k such that

$$||x||_{\mathcal{W}_1(T)} \le C_3(1 + |x_0| + ||k||_{L^2(0,T;V^*)}). \tag{3.3}$$

One of the main useful tools is the following Sadvoskii's fixed point theorem.

Lemma 3.1. *Suppose that Σ is a closed convex subset of a Banach space X. Assume that K_1 and K_2 are mappings from Σ into X such that the following conditions are satisfied:*
(i) $(K_1+K_2)(\Sigma)\subset\Sigma$,
(ii) K_1 is a completely continuous mapping,
(iii) K_2 is a contraction mapping.
Then the operator K_1+K_2 has a fixed point in Σ.

Proof of Theorem.
Let $r_0=2(C_1|x_0+y_0|+r_0M_0L)$, where C_1 is constant satisfying

$$||x||_{\mathcal{W}(T)}\le C_1(||x_0||+||k||_{L^2(0,T;H)}). \tag{3.4}$$

Let $\gamma=\max\{1/2,(3\beta-2)^{1/2}\}$, choose $0<T_1<T$ such that

$$T_1^{\gamma}\big[\{C_2L_1(r_0)(r_0+1)+C_2||k||_{L^2(0,T_1;V)}\}+(3\beta-2)^{-1/2}r_0LC_{1-\beta}\big]\le C_1|x_0+y_0|+r_0M_0L, \tag{3.5}$$

where C_2 is constant in (2.7) and

$$\hat{M}\equiv T_1^{\gamma}\big\{C_2L_1(r_0)+(3\beta-2)^{-1/2}C_{1-\beta}L\big\}<1. \tag{3.6}$$

Define a mapping $J:L^2(0,T_1;V)\to L^2(0,T_1;V)$ as

$$\begin{aligned}(Jx)(t)=&S(t)(x_0+y_0)-(Bx)(t)\\ &+\int_0^t AS(t-s)(Bx)(s)ds+\int_0^t S(t-s)\{f(s,x(s))+k(s)\}ds.\end{aligned}$$

It will be shown that the operator J has a fixed point in the space $L^2(0,T_1;V)$. By assumptions (B) and (F), it is easily seen that J is continuous from $C([0,T_1];H)$ into itself. Let

$$\Sigma=\{x\in L^2(0,T_1;V):||x||_{L^2(0,T_1;V)}\le r_0,\ x(0)=x_0\},$$

which is a bounded closed subset of $L^2(0,T_1;V)$. By (2.5), (2.6) and Assumption (B) we have

$$||Bx||_{L^2(0,T_1;V)}\le||A^{-\beta}||_{\mathcal{L}(H,V)}||A^{\beta}Bx||_{L^2(0,T_1;H)}\le r_0M_0L. \tag{3.7}$$

By virtue of (2.7), for $0<t<T_1$, it holds

$$\begin{aligned}&||\int_0^t S(t-s)\{f(s,x(s))+k(s)\}ds||_{L^2(0,T_1;V)}\le C_2\sqrt{T_1}||Fx+k||_{L^2(0,T_1;H)}\\ &\le C_2\sqrt{T_1}\{L_1(r_0)(r_0+1)+||k||_{L^2(0,T_1;V)}\}.\end{aligned} \tag{3.8}$$

Since (2.5) and Assumption (F) the following inequality holds:

$$||AS(t-s)Bx(s)||=||A^{1-\beta}S(t-s)A^{\beta}Bx(s)||\le\frac{C_{1-\beta}}{(t-s)^{3(1-\beta)/2}}r_0L,$$

there holds

$$||\int_0^t AS(t-s)Bx(s)ds||_{L^2(0,T_1;V)}\le(3\beta-2)^{-1/2}r_0LC_{1-\beta}T_1^{\sqrt{3\beta-2}}. \tag{3.9}$$

Therefore, from (3.4), (3.6)-(3.9) it follows that

$$||Jx||_{L^2(0,T_1;V)} \le C_1|x_0+y_0| + r_0M_0L$$
$$+ T_1^\gamma\big[\{C_2L_1(r_0)(r_0+1) + C_2||k||_{L^2(0,T_1;V)}\} + (3\beta-2)^{-1/2}r_0LC_{1-\beta}\big] \le r_0,$$

and hence J maps Σ into Σ. Define mapping $J = K_1 + K_2$ on $L^2(0,T_1;V)$ by the formula

$$(K_1x)(t) = -(Bx)(t)$$
$$(K_2x)(t) = S(t)(x_0+y_0) + \int_0^t AS(t-s)(Bx)(s)ds + \int_0^t S(t-s)\{f(s,x(s)) + k(s)\}ds.$$

We can now employ Lemma 3.1 with Σ. Assume that a sequence $\{x_n\}$ of $L^2(0,T_1;V)$ converges weakly to an element $x_\infty \in L^2(0,T_1;V)$, i.e., $w - \lim_{n\to\infty} x_n = x_\infty$. Then we will show that

$$\lim_{n\to\infty} ||K_1x_n - K_1x_\infty|| = 0, \tag{3.10}$$

which is equivalent to the completely continuity of K_1 since $L^2(0,T_1;V)$ is reflexive. For a fixed $t \in [0,T_1]$, let $x_t^*(x) = (K_1x)(t)$ for every $x \in L^2(0,T_1;V)$. Then $x_t^* \in L^2(0,T_1;V^*)$ and we have $\lim_{n\to\infty} x_t^*(x_n) = x_t^*(x_\infty)$ since $w - \lim_{n\to\infty} x_n = x_\infty$. Hence,

$$\lim_{n\to\infty}(K_1x_n)(t) = (K_1x_\infty)(t), \quad t \in [0,T_1].$$

By (2.5), (2.6) and Assumption (B) we have $||(K_1x)(t)|| \le ||A^{-\beta}||_{\mathcal{L}(H,V)}||A^\beta Bx||_{L^2(0,T_1;H)} \le \infty$. Therefore, by Lebesgue's dominated convergence theorem it holds $\lim_{n\to\infty} ||K_1x_n||_{L^2(0,T_1;V)} = ||K_1x_\infty||_{L^2(0,T_1;V)}$. Since $L^2(0,T_1;V)$ is a Hilbert space, it holds (3.10).

Next, we prove that K_2 is a contraction mapping on Σ. Indeed, for every x_1 and $x_2 \in \Sigma$, by similar to (3.9) and (3.10), we have

$$||K_2x_1 - K_2x_2||_{L^2(0,T_1;V)} \le T_1^\gamma\big\{C_2L_1(r_0) + (3\beta-2)^{-1/2}C_{1-\beta}L\big\}||x_1-x_2||_{L^2(0,T_1;V)}.$$

So by virtue of the condition (3.6) the contraction mapping principle gives that the solution of (3.1) exists uniquely in $[0,T_1]$. So by virtue of the condition (3.6), K_2 is contractive. Thus, Lemma 3.1 gives that the equation of (3.1) has a solution in $\mathcal{W}_1(T_1)$.

From now on we establish a variation of constant formula (3.3) of solution of (3.1). Let x be a solution of (3.1) and $x_0 \in H$. Then we have that from (3.7)-(3.10) it follows that

$$||x||_{L^2(0,T_1;V)} \le C_1|x_0+y_0| + r_0M_0L + T_1^\gamma\big[\{C_2L_1(r_0)(||x||_{L^2(0,T_1;V^*)}+1)$$
$$+ C_2||k||_{L^2(0,T_1;V^*)}\} + (3\beta-2)^{-1/2}C_{1-\beta}L||x||_{L^2(0,T_1;V)}\big]$$

Taking into account (3.6), there exists a constant C_3 such that

$$||x||_{L^2(0,T_1;V)} \le (1-\hat{M})^{-1}\big[C_1|x_0+y_0| + r_0M_0L$$
$$+ T_1^\gamma\{C_2L_1(r_0) + C_2||k||_{L^2(0,T_1;V^*)}\}\big] \le C_3(1+|x_0|+||k||_{L^2(0,T_1;V^*)})$$

which obtain the inequality (3.3). Since the conditions (3.5) and (3.6) are independent of initial value and by (2.6)

$$|x(T_1)| \leq ||x||_{C([0,T_1;H])} \leq M_1||x||_{\mathcal{W}_1(T)},$$

by repeating the above process, the solution can be extended to the interval $[0, T]$. $\square$

From the following result, we obtain that the solution mapping is continuous, which is useful for physical applications of the given equation. The proof is immediately obtained from Theorem 3.1.

Theorem 3.2. *Let Assumptions (B) and (F) be satisfied and* $(x_0, y_0, k) \in H \times H \times L^2(0, T; V^*)$*. Then the solution* x *of the equation (3.1) belongs to* $x \in \mathcal{W}_1(T) \equiv L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$ *and the mapping*

$$H \times H \times L^2(0, T; V^*) \ni (x_0, y_0, k) \mapsto x \in \mathcal{W}_1(T)$$

is continuous.

For $k \in L^2(0, T; V^*)$ let x_k be the solution of equation (3.1) with k instead of Bu. Here, we remark that if V is compactly embedded in H by assumption, the embedding $\mathcal{W}_1(T) \subset L^2(0, T; H)$ is compact in view of Theorem 2 of Aubin [9]. So we can prove the following result from Theorem 3.1.

Theorem 3.3. *Let us assume that the embedding* $V \subset H$ *is compact. For* $k \in L^2(0, T; V^*)$ *let* x_k *be the solution of equation (3.1). Then the mapping* $k \mapsto x_k$ *is compact from* $L^2(0, T; V^*)$ *to* $L^2(0, T; H)$*. Moreover, if we define the operator* $\mathcal{F}$ *by* $\mathcal{F}(k) = f(\cdot, x_k)$*, then* $\mathcal{F}$ *is also a compact mapping from* $L^2(0, T; V^*)$ *to* $L^2(0, T; H)$*.*

4 Approximate Controllability

4.1 Newtral control systems

In this section, we show that the controllability of the corresponding linear equation is extended to the nonlinear differential equation. Let U be a Banach space of control variables. Here C is a linear bounded operator from $L^2(0, T; U)$ to $L^2(0, T; H)$, which is called a controller. For $x \in L^2(0, T; H)$ we set

$$(Bx)(t) = \int_0^t N(t-s)x(s)ds,$$

where $N : [0, \infty) \to \mathcal{L}(H, V)$ is strongly continuous. Then it is immediately seen that $Bx \in C([0, T]; V)$ and hence $AS(s)(Bx)(s) = AS(s)(Bx)(s)$ for $0 \leq s \leq T$ because $D(A) = V$. Since $t \to N(t)$ is strong continuous, by the uniform boundedness principle there exists a constant M_N such that for any $T > 0$,

$$\sup_{t \in [0,T]} ||AN(t)||_{\mathcal{L}(H,V^*)} \leq M_N. \tag{4.1}$$

Let $x(T; B, f, u)$ be a state value of the system (1.1) at time T corresponding to the operator B, the nonlinear term f and the control u. We note that $S(\cdot)$ is the analytic semigroup generated by $-A$. In view of Theorem 4.1,

$$||x(\cdot; B, f, u)||_{\mathcal{W}_1(T)} \leq C_3(|x_0| + ||C||_{\mathcal{L}(U,H)}||u||_{L^2(0,T;U)}). \tag{4.3}$$

We define the reachable sets for the system (1.1) as follows:

$$R(T) = \{x(T; B, f, u) : u \in L^2(0, T; U)\}, \ L(T) = \{x(T; 0, 0, u) : u \in L^2(0, T; U)\}.$$

Definition 4.1. *The system (1.1) is said to be approximately controllable on* $[0, T]$ *if for every desired final state* $z_T \in H$ *and* $\epsilon > 0$ *there exists a control function* $u \in L^2(0, T; U)$ *such that the solution* $x(T; B, f, u)$ *of (1.1) satisfies* $|x(T; f, u) - z_T| < \epsilon$*, that is,* $\overline{R_T(f)} = H$ *where* $\overline{R(T)}$ *is the closure of* $R(T)$ *in* H*.*

We define the linear operator $\hat{S}$ from $L^2(0, T; H)$ to H by

$$\hat{S}p = \int_0^T S(T - s)p(s)ds, \quad \forall p \in L^2(0, T; H)$$

We need the following hypothesis:

Assumption (S). (i) For any $\varepsilon > 0$ and $p \in L^2(0, T; H)$ there exists a $u \in L^2(0, T; U)$ such that

$$|\hat{S}p - \hat{S}Cu| < \varepsilon, \ ||Cu||_{L^2(0,t;H)} \leq q_1||p||_{L^2(0,t;H)}, \quad 0 \leq t \leq T.$$

where q_1 is a constant independent of p.
(ii) f is a nonlinear mapping of $[0, T] \times H$ into H satisfying following:
There exists a function $L_1 : \mathbb{R}_+ \to \mathbb{R}$ such that

$$|f(t, x) - f(t, y)| \leq L_1(r)|x - y|, \quad t \in [0, T]$$

hold for $|x| \leq r$ and $|y| \leq r$.

By virtue of the condition (i) of Assumption (S) we note that $AS(t - s)Bx = S(t - s)ABx$ for each $x \in V$. Therefore, the system (1.1) is approximately controllable on $[0, T]$ if for any $\varepsilon > 0$ and $z_T \in H$ there exists a control $u \in L^2(0, T; U)$ such that

$$||S(T)(x_0 + y_0) - (Bx)(T) + \hat{S}\{ABx + Fx + Cu\}) - z_T|| < \varepsilon,$$

where $(Fx)(t) = f(t, x(t))$ for $t \geq 0$. Throughout this section, Invoking (4.3), we can choose a constant r_1 such that

$$r_1 > C_3(|x_0| + ||C||_{\mathcal{L}(U,H)}||u||_{L^2(0,T;U)}). \tag{4.4}$$

The proof of the following lemma is obtained by using Gronwall's inequality,.

Lemma 4.1. *Let* u_1 *and* u_2 *be in* $L^2(0, T; U)$*. Then under the assumption (S), we have that for* $0 \leq t \leq T$*,*

$$|x(t; B, f, u_1) - x(t; B, f, u_2)| \leq Me^{M_2}\sqrt{t}||Cu_1 - Cu_2||_{L^2(0,T;H)},$$

where $M_2 = e^{M(M_NT + L_1(r_1))}$

Thanks to Lemma 4.1, the following theorem is obtained from [10, Theorem 4.1].

Theorem 4.1. *Under the assumptions (S), the system (1.1) is approximately controllable on* $[0, T]$*.*

4.2 Semilinear control systems($B \equiv 0$)

Let

$$N = \{p \in L^2(0,T;H) : \int_0^T S(T-s)p(s)ds = 0\}$$

and denote by $N^\perp$ be the orthogonal complement of N in $L^2(0,T;H)$. We denote the range of the operator C by H_C. We need the following assumption:

Assumption (A). For each $p \in L^2(0,T;H)$ there exists an element $q \in \overline{H}_C$ such that

$$\int_0^T S(T-s)p(s)ds = \int_0^T S(T-s)q(s)ds,$$

that is, $L^2(0,T;H) = \overline{H}_C + N$, where $\overline{H}_C$ is the closure of H_C in $L^2(0,T;H)$.

Here, we remark that under Assumption (A) it is known that $\overline{R_T(0)} = H$ as in [4].

Theorem 4.2. *Under Assumptions (F) in Section 3 and (A), and assuming in addition*

$$\limsup_{r\to\infty} (r - \sqrt{T}\sup\{L(s) : |s| \le r\}) = \infty, \tag{4.5}$$

we have

$$R_T(0) \subset \overline{R_T(f)}.$$

Therefore, if the linear system (1.1) with $f \equiv 0$ and $B \equiv 0$ is approximately controllable at time T, then so is the nonlinear system (1.1).

Proof. It will be shown that $R_T(0) \subset \overline{R_T(f)}^V$, where $\overline{R_T(f)}^V$ is the closure of $R_T(f)$ in V. For $u \in N^\perp$, let Pu be the unique minimum norm element of $\{u+N\} \cap \overline{H}_C$. Then the proof of Lemma 1 of Naito [4] can be applied to show that P is a linear and continuous operator from $N^\perp$ to $\overline{H}_C$. Let $\tilde{Y} = L^2(0,T;H)/N$ be the quotient space and the norm of a coset $\tilde{u} = u + N \in \tilde{Y}$ is defined of $||\tilde{u}|| = \inf\{|u+f| : f \in N\}$.

We define by Q the isometric isomorphism from $\tilde{Y}$ onto $N^\perp$, that is, $Q\tilde{u}$ is the minimum norm element in $\tilde{u} = \{u + f : f \in N\}$. Let

$$\tilde{\mathcal{F}}\tilde{u} = \mathcal{F}(PQ\tilde{u}) + N, \quad \forall \tilde{u} \in \tilde{Y}.$$

Then $\tilde{\mathcal{F}}$ is a compact mapping from $\tilde{Y}$ to itself by Theorem 3.1. If $(x_0, k) \in V \times L^2(0,T;H)$, we know $y \in \mathcal{W}(T) \subset C([0,T];V)$ by (2.6). Let

$$\eta = \int_0^T S(T-s)(Cv)(s)ds \in R_T(0).$$

We are going to show that for every $\epsilon > 0$ there exists w such that

$$||\eta - x(T;f,w)|| \le \epsilon.$$

Put $z = Cv$ and $r_1 = ||C||||v||_{L^2(0,T;U)}$. Then it follows that

$$\tilde{z} = z + N \in V_{r_1} = \{\tilde{x} \in \tilde{Y} : ||\tilde{x}||_{\tilde{Y}} \le r_1\}.$$

From (3.3), noting that $||y_k||_{L^2(0,T;V)} \leq C_3(1+||x_0||+||k||_{L^2(0,T;H)})$, we choose a constant $r > 0$ such that

$$r \geq C_3(1 + ||x_0|| + ||k||_{L^2(0,T;H)}).$$

Then it holds that

$$||\mathcal{F}(k)||_{L^2(0,T;H)} \leq L(r)\sqrt{T}, \quad ||\tilde{\mathcal{F}}(\tilde{k})||_{\tilde{Y}} \leq L(r)\sqrt{T}.$$

Let

$$\mathcal{L}(r) = \sup\{L(s) : |s| \leq r\}.$$

Then by the assumption (4.5), there exists $r > 0$ such that

$$\mathcal{L}(r)\sqrt{T} + r_1 < r. \tag{4.6}$$

Define an operator J from $\tilde{Y}$ to itself as

$$J(\tilde{u}) = \tilde{z} - \tilde{\mathcal{F}}\tilde{u}, \quad \tilde{u} \in \tilde{Y}.$$

Then since $\tilde{z} \in V_{r_1}$ and from (4.6) it follows that

$$||J\tilde{u}|| \leq ||\tilde{z}|| + ||\tilde{\mathcal{F}}\tilde{u}|| \leq r_1 + L(r)\sqrt{T} \leq r_1 + \mathcal{L}(r)\sqrt{T} < r.$$

Hence, J maps bounded closed set V_r into itself. It follows from the Schauder fixed point theorem that there exists a fixed point $\tilde{u}$ of J in V_r, that is, it holds

$$\tilde{z} = \tilde{\mathcal{F}}\tilde{u} + \tilde{u}.$$

Put $u = Q\tilde{u}$ and $u_C = PQ\tilde{u}$. Then we have that $u_C = Pu$ and $u - u_C = u - Pu \in N$. Hence

$$\tilde{z} = \mathcal{F}(u_C) + u + N = \mathcal{F}(u_C) + u_C + N.$$

Therefore,

$$\eta = \int_0^T S(T-s)(\mathcal{F}(u_C)(s) + u_C(s))ds = \int_0^T S(T-s)(f(s, y_{u_C}) + u_C(s))ds.$$

Since $u_C \in \overline{H}_C$, there exists a sequence $\{v_n\} \in L^2(0,T;U)$ such that $Cv_n \mapsto u_C$ in $L^2(0,T;H)$. Then by Theorem 3.2 we have that $x(\cdot; f, v_n) \mapsto y_{u_C}$ in $L^2(0,T;D(A)) \cap W^{1,2}(0,T;H)$, and hence $x(T; f, v_n) \mapsto y_{u_C}(T) = \eta$ in V. Thus we conclude $\eta \in \overline{R_T(f)}^V$. $\square$

Corollary 4.1. *Under Assumptions (A) and (F), and assuming in addition that $f(\cdot, \cdot)$ is continuous and uniformly bounded, we have*

$$R_T(0) \subset \overline{R_T(f)}.$$

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Identification problem for degenerate parabolic equations

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1 Introduction

This paper is concerned with the following identification problem

$$\frac{d}{dt}Mu(t) + Lu(t) = f(t)z + h(t), \quad 0 \leqslant t \leqslant T, \tag{1.1}$$

$$Mu(0) = Mu_0, \tag{1.2}$$

$$\Phi[Mu(t)] = g(t), \quad 0 \leqslant t \leqslant T. \tag{1.3}$$

Here L is the realization in $L^2(\Omega)$ of a second order strongly elliptic linear differential operator with the Dirichlet boundary condition, and M is the multiplication operator by a nonnegative function $m \in L^\infty(\Omega)$: $Mu = mu$ and $\Phi \in L^2(\Omega)^*$. The coefficients of L are assumed to be sufficiently smooth and Ω is a bounded open set of $\mathbb{R}^n$ with smooth boundary. The problem is to seek for u and f from known values of z, h, u_0 and g.

It is assumed that the sesquilinear form $a(\cdot,\cdot)$ associated with L satisfies

$$\operatorname{Re} a(u,u) \geqslant c_0 \|u\|_{H_0^1(\Omega)}, \quad \forall u \in H_0^1(\Omega)$$

for some positive constant c_0. Hence $0 \in \rho(L)$. Set $A = LM^{-1}$. Then A is multivalued unless $m > 0$ a.e., and

$$D(A) = MD(L) = \{mu;\, u \in D(L)\},$$
$$Ay = \{Lu;\, y = mu,\, u \in D(L)\} \text{ for } y \in D(A).$$

$D(A)$ is a Banach space with norm $\|y\|_{D(A)} = \inf_{f\in Ay} \|f\|_{L^2(\Omega)}$. If we introduce the new unknown variable $y(t) = Mu(t)$, problem (1.1)-(1.3) is transformed to

$$\begin{cases} \dfrac{d}{dt}y(t) + Ay(t) \ni f(t)z + h(t), & 0 \leqslant t \leqslant T, \\ y(0) = y_0, & \\ \Phi[y(t)] = g(t), & 0 \leqslant t \leqslant T, \end{cases} \tag{1.4}$$

where $y_0 = mu_0$.

The following result is an extension of Theorem 4.2 of A. Favini, A. Lorenzi and H. Tanabe [1] to the case where A is multivalued and its proof will be published in a forthcoming paper:

Let A be a possibly multivalued linear operator in a complex Banach space X such that

$$\rho(A) \supset \Sigma_\alpha = \{\lambda \in \mathbb{C};\, \operatorname{Re}\lambda \leqslant c(1+|\lambda|)^\alpha\}, \tag{1.5}$$

and the following inequality holds for $\lambda \in \Sigma_\alpha$

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X,X)} \leqslant c(1+|\lambda|)^{-\beta}, \tag{1.6}$$

where α, β and c are positive constants such that $\beta \leqslant \alpha \leqslant 1$ and $2\alpha + \beta > 2$. Let for $0 < \theta < 1$

$$X_A^\theta = \{u \in X;\, \sup_{0<t<\infty} t^\theta \|u - t(t+A)^{-1}u\|_X < \infty\}. \tag{1.7}$$

Theorem A *Suppose* $(0 \leqslant)3 - 2\alpha - \beta < \theta < 1$ *and*

$$y_0 \in D(A), \quad Ay_0 \cap (X, D(A))_{\theta,\infty} \neq \emptyset, \tag{1.8}$$

$$z \in (X, D(A))_{\theta,\infty}, \tag{1.9}$$

$$h \in C([0,T]; X) \cap B([0,T]; (X, D(A))_{\theta,\infty}), \tag{1.10}$$

$$g \in C^1([0,T]; \mathbb{C}), \quad \Phi[y_0] = g(0), \tag{1.11}$$

$$\Phi[z] \neq 0. \tag{1.12}$$

Then problem (1.4) *admits a unique solution* (y, f) *such that*

$$y \in C^1([0,T]; X), \quad f \in C([0,T]; \mathbb{C}), \tag{1.13}$$

$$y' - f(\cdot)z - h \in C^{(2\alpha+\beta+\theta-3)/\alpha}([0,T]; X) \cap B([0,T]; X_A^{(2\alpha+\beta+\theta-3)/\alpha}), \tag{1.14}$$

where $B([0,T]; Y)$ *is the set of all bounded (not necessarily measurable) functions defined in* $[0,T]$ *with values in* Y *for any Banach space* Y.

In the case of problem (1.1)-(1.3) $X = L^2(\Omega)$, $A = LM^{-1}$, and it is established in Chapter III of A. Favini and A. Yagi [3] and Theorem 3.3 of A. Favini, A. Lorenzi, H. Tanabe and A. Yagi [2] that $\alpha = 1$ and

$$\beta = 1/2 \qquad \text{if } m \in L^\infty(\Omega),$$

$$\beta = (2-\rho)^{-1} (> 1/2) \qquad \text{if } m \text{ is } \rho\text{-regular, } 0 < \rho < 1,$$

where m is said to be *ρ-regular* if $m \in C^1(\overline{\Omega})$ and $\exists c > 0 \ \ |\nabla m| \leqslant cm^\rho$. The condition $3 - 2\alpha - \beta < \theta < 1$ becomes $\beta + \theta > 1$. Therefore if we apply Theorem A with $\theta = 1/2$, it is necessary that $\beta > 1/2$, and so m has to be ρ-regular.

Suppose $u_0 \in D(L)$ and $Lu_0 \in (L^2(\Omega), D(A))_{1/2,\infty}$. Then $y_0 = mu_0 (= Mu_0) \in D(A)$ and $Lu_0 \in Ay_0$. Hence $Ay_0 \cap (L^2(\Omega), D(A))_{1/2,\infty} \neq \emptyset$, and assumption (1.8) is satisfied with $\theta = 1/2$. Suppose further that m is ρ-regular and (1.9)-(1.12) are satisfied with $\theta = 1/2$. Then problem (1.4) has a unique solution (y, f). If we define a function u by

$$u(t) = L^{-1}(f(t)z + h(t) - y'(t)),$$

then

$$Lu(t) = f(t)z + h(t) - y'(t) \in Ay(t) = LM^{-1}y(t). \tag{1.15}$$

This implies $u(t) \in M^{-1}y(t)$, since L is invertible. Hence $y(t) = Mu(t)$. Substitution of this in the first equality of (1.15) yields (1.1). Therefore the pair (u, f) is a solution to (1.1)-(1.3). Substituting $\alpha = 1$, $\beta = (2-\rho)^{-1}$ and $\theta = 1/2$ in (1.14) and noting $Lu = f(\cdot)z + h - y'$ one obtains

$$Lu \in C^{\rho/[2(2-\rho)]}([0,T]; L^2(\Omega)) \cap B([0,T]; L^2(\Omega)_A^{\rho/[2(2-\rho)]}).$$

Note that $0 < \rho/[2(2-\rho)] < 1/2$. Thus the following theorem is obtained.

Theorem B *Suppose that* m *is* ρ*-regular for some* $\rho \in (0,1)$*, and*

$$u_0 \in D(L), \quad Lu_0 \in (L^2(\Omega), D(A))_{1/2,\infty},$$

$$z \in (L^2(\Omega), D(A))_{1/2,\infty},$$

$$h \in C([0,T]; L^2(\Omega)) \cap B([0,T]; (L^2(\Omega), D(A))_{1/2,\infty}),$$

$$g \in C^1([0,T]; \mathbb{C}), \quad \Phi[mu_0] = g(0),$$

$$\Phi[z] \neq 0.$$

Then problem (1.1)-(1.3) *admits a unique solution* (u, f) *such that*

$$Mu \in C^1([0,T]; L^2(\Omega)), \quad f \in C([0,T]; \mathbb{C}),$$

$$Lu \in C^{\rho/[2(2-\rho)]}([0,T]; L^2(\Omega)) \cap B([0,T]; L^2(\Omega)_A^{\rho/[2(2-\rho)]}), \quad .$$

This is a result obtained by applying the general theory.

Let $\widetilde{L}$ be the operator defined by

$$a(u,v) = (\widetilde{L}u, v), \quad u, v \in H_0^1(\Omega).$$

Then $\widetilde{L} \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ and $L \subset \widetilde{L}$. Set $\widetilde{A} = \widetilde{L}M^{-1}$. Then

$$\begin{aligned} &D(\widetilde{A}) = MD(\widetilde{L}) = MH_0^1(\Omega) = \{mu;\, u \in H_0^1(\Omega)\}, \\ &\widetilde{A}y = \{\widetilde{L}u;\, y = mu,\, u \in H_0^1(\Omega)\} \text{ for } y \in D(\widetilde{A}). \end{aligned}$$

$D(\widetilde{A})$ is a Banach space with norm $\|y\|_{D(\widetilde{A})} = \inf_{\phi \in \widetilde{A}y} \|\phi\|_{H^{-1}(\Omega)}$ for $y \in D(\widetilde{A})$.

It was shown in the book A. Favini and A. Yagi [3] that the following inequalities hold for $\lambda \in \Sigma = \{\lambda \in \mathbb{C};\, \operatorname{Re}\lambda \leqslant c(1+|\lambda|)\}$:

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))} \leqslant C_0(1+|\lambda|)^{-1/2}, \tag{1.16}$$

$$\|(\lambda - \widetilde{A})^{-1}\|_{\mathcal{L}(H^{-1}(\Omega), H^{-1}(\Omega))} \leqslant C_0(1+|\lambda|)^{-1}, \tag{1.17}$$

$$\|(\lambda - \widetilde{A})^{-1}\|_{\mathcal{L}(H^{-1}(\Omega), L^2(\Omega))} \leqslant C_0(1+|\lambda|)^{-1/2}, \tag{1.18}$$

and hence $-A$ and $-\widetilde{A}$ generate C^∞-semigroups e^{-tA} and $e^{-t\widetilde{A}}$ in $L^2(\Omega)$ and $H^{-1}(\Omega)$ respectively such that for $0 < t < \infty$

$$\|e^{-tA}\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))} \leqslant C_0 t^{-1/2}, \tag{1.19}$$

$$\|e^{-t\widetilde{A}}\|_{\mathcal{L}(H^{-1}(\Omega), H^{-1}(\Omega))} \leqslant C_0, \tag{1.20}$$

$$\|e^{-t\widetilde{A}}\|_{\mathcal{L}(H^{-1}(\Omega), L^2(\Omega))} \leqslant C_0 t^{-1/2}, \tag{1.21}$$

where C_0 is some positive constant. It is known that $e^{-tA}u \to u$ in $L^2(\Omega)$ as $t \to 0$ for $u \in D(A)$. By virtue of (1.20) it holds that $e^{-t\widetilde{A}}u \to u$ in $H^{-1}(\Omega)$ as $t \to 0$ if u belongs to the closure of $D(\widetilde{A})$ in $H^{-1}(\Omega)$ just as in the nondegenerate case $m \equiv 1$.

Let $u \in X_A^\theta$, and $u_0(t) = u - t(t+A)^{-1}u$, $u_1(t) = t(t+A)^{-1}u$, $t > 0$. Then $u = u_0(t) + u_1(t)$, and

$$\sup_{0<t<\infty} t^\theta \|u_0(t)\|_X = \sup_{0<t<\infty} t^\theta \|u - t(t+A)^{-1}u\|_X < \infty,$$

by the definition of X_A^θ. Since $A(t+A)^{-1}u \ni u - t(t+A)^{-1}u$, one has $Au_1(t) = tA(t+A)^{-1}u \ni t(u - t(t+A)^{-1}u)$. Hence

$$\sup_{0<t<\infty} t^{\theta-1}\|u_1(t)\|_{D(A)} \leqslant \sup_{0<t<\infty} t^{\theta-1}\|t(u - t(t+A)^{-1}u)\|_X = \sup_{0<t<\infty} t^\theta \|u - t(t+A)^{-1}u\|_X < \infty.$$

Therefore $u \in (X, D(A))_{\theta,\infty}$. Thus it has been proved that

$$X_A^\theta \subset (X, D(A))_{\theta,\infty}, \quad 0 < \theta < 1. \tag{1.22}$$

Suppose $u \in D(\widetilde{A})$, and $\phi \in \widetilde{A}u$. Then

$$u = (t+\widetilde{A})^{-1}(tu + \phi) = t(t+A)^{-1}u + (t+\widetilde{A})^{-1}\phi.$$

Hence with the aid of (1.18)

$$t^{1/2}\|u - t(t+A)^{-1}u\|_{L^2(\Omega)} = t^{1/2}\|(t+\widetilde{A})^{-1}\phi\|_{L^2(\Omega)} \leqslant C_0\|\phi\|_{H^{-1}(\Omega)}.$$

This means $u \in L^2(\Omega)_A^{1/2}$, and we have proved $D(\widetilde{A}) \subset L^2(\Omega)_A^{1/2}$. By combining this with (1.22) the following inclusion relation is obtained:

$$D(\widetilde{A}) \subset L^2(\Omega)_A^{1/2} \subset (L^2(\Omega), D(A))_{1/2,\infty}. \tag{1.23}$$

The object of this paper is to show that if we choose $D(\widetilde{A})$ instead of $(L^2(\Omega), D(A))_{1/2,\infty}$, we can obtain better estimates without assuming the ρ-regularity of m.

An analogous results are obtained also when the boundary condition is of Robin type.

2 Main result

Theorem 2.1 *Suppose that*

$$u_0 \in D(L), \;\; Lu_0 \in D(\widetilde{A}), \tag{2.1}$$

$$z \in D(\widetilde{A}), \tag{2.2}$$

$$h \in C([0,T]; L^2(\Omega)) \cap B([0,T]; D(\widetilde{A})), \tag{2.3}$$

$$g \in C^1([0,T]; \mathbb{C}), \quad \Phi[mu_0] = g(0), \tag{2.4}$$

$$\Phi[z] \neq 0. \tag{2.5}$$

Then there exists a unique pair of functions (u, f) such that

$$\begin{gathered} Mu \in C^1([0,T]; L^2(\Omega)), \quad f \in C([0,T]; \mathbb{C}), \\ Lu \in C^{1/2}([0,T]; L^2(\Omega)) \cap B([0,T]; L^2(\Omega)^{1/2}_A), \\ d(Mu)/dt \in C([0,T]; L^2(\Omega)) \cap B([0,T]; L^2(\Omega)^{1/2}_A), \end{gathered} \tag{2.6}$$

and (1.1)-(1.3) *holds.*

In the proof of this theorem we use the following proposition whose proof is given in a forthcoming paper.

Proposition 2.1 *Suppose that*

$$h \in C([0,T]; L^2(\Omega)) \cap B([0,T]; D(\widetilde{A})).$$

Then $\int_0^t e^{-(t-s)A} h(s)ds$ is differentible in $L^2(\Omega)$, and

$$\frac{d}{dt}\int_0^t e^{-(t-s)A} h(s)ds = h(t) + \int_0^t \frac{\partial}{\partial t} e^{-(t-s)A} h(s)ds, \tag{2.7}$$

$$\left\| \int_0^t \frac{\partial}{\partial t} e^{-(t-s)A} h(s)ds \right\|_{L^2(\Omega)} \leqslant 2C_0 t^{1/2} \|h\|_{B([0,T]; D(\widetilde{A}))}. \tag{2.8}$$

Furthermore, the function $t \to \int_0^t \frac{\partial}{\partial t} e^{-(t-s)A} h(s)ds$ belongs to $C^{1/2}([0,T]; L^2(\Omega)) \cap B([0,T]; L^2(\Omega)^{1/2}_A)$.

Lemma 2.1 *For $v \in D(\widetilde{A})$ one has*

$$\left\| \frac{d}{dt} e^{-tA} v \right\|_{L^2(\Omega)} \leqslant C_0 t^{-1/2} \|v\|_{D(\widetilde{A})}, \tag{2.9}$$

$$\|e^{-tA}v - e^{-sA}v\|_{L^2(\Omega)} \leqslant 2C_0 (t-s)^{1/2} \|v\|_{D(\widetilde{A})}, \tag{2.10}$$

$$\|e^{-tA}v - v\|_{L^2(\Omega)} \to 0 \text{ as } t \to 0. \tag{2.11}$$

Proof. If $v \in D(\widetilde{A})$, there exists an element $\phi \in H^{-1}(\Omega)$ such that $\phi \in \widetilde{A}v$. Hence in view of (1.18)

$$\left\| \frac{d}{dt} e^{-tA} v \right\|_{L^2(\Omega)} = \left\| \frac{d}{dt} e^{-t\widetilde{A}} \widetilde{A}^{-1} \phi \right\|_{L^2(\Omega)} = \|e^{-t\widetilde{A}}\phi\|_{L^2(\Omega)} \leqslant C_0 t^{-1/2} \|\phi\|_{H^{-1}(\Omega)},$$

which implies (2.9). With the aid of (2.9) one observes for $0 < s < t$

$$\begin{aligned} \|e^{-tA}v - e^{-sA}v\|_{L^2(\Omega)} &= \left\| \int_s^t \frac{d}{d\sigma} e^{-\sigma A} v d\sigma \right\|_{L^2(\Omega)} \\ &\leqslant C_0 \int_s^t \sigma^{-1/2} d\sigma \|v\|_{D(\widetilde{A})} \leqslant 2C_0 (t-s)^{1/2} \|v\|_{D(\widetilde{A})}. \end{aligned}$$

Hence (2.10) is established and $\lim_{t\to 0} e^{-tA}v$ exists in $L^2(\Omega)$. Since $e^{-tA}v = e^{-t\widetilde{A}}v \to v$ in $H^{-1}(\Omega)$, it follows that $\lim_{t\to 0} e^{-tA}v = v$. $\square$

From (1.1) and (1.3) it follows that

$$g'(t) + \Phi[Lu(t)] = f(t)\Phi[z] + \Phi[h(t)].$$

Dividing both sides by $\Phi[z]$ one gets

$$f(t) = \chi g'(t) + \chi\Phi[Lu(t)] - \chi\Phi[h(t)],$$

where $\chi = \Phi[z]^{-1}$. Substitution of this in (1.3) yields the equation to be satified by u:

$$\frac{d}{dt}Mu(t) + Lu(t) = \chi g'(t)z + \chi\Phi[Lu(t)]z - \chi\Phi[h(t)]z + h(t). \tag{2.12}$$

Set

$$y_0 = mu_0. \tag{2.13}$$

For the time being we make formal calculations. Suppose that there exists a solution $u \in C([0,T]; D(L))$ to the following integral equation

$$\begin{aligned} Lu(t) = &-\frac{d}{dt}e^{-tA}y_0 - \chi\int_0^t g'(s)\frac{\partial}{\partial t}e^{-(t-s)A}zds \\ &-\chi\int_0^t \Phi[Lu(s)]\frac{\partial}{\partial t}e^{-(t-s)A}zds - \int_0^t \frac{\partial}{\partial t}e^{-(t-s)A}h(s)ds, \end{aligned} \tag{2.14}$$

Applying $A^{-1} = ML^{-1}$ to both sides of (2.14) and noting that

$$A^{-1}\frac{d}{dt}e^{-tA} = \frac{d}{dt}e^{-tA}A^{-1} = -e^{-tA} \tag{2.15}$$

one obtains

$$Mu(t) = e^{-tA}y_0 + \chi\int_0^t g'(s)e^{-(t-s)A}zds + \chi\int_0^t \Phi[Lu(s)]e^{-(t-s)A}zds + \int_0^t e^{-(t-s)A}h(s)ds.$$

By differentiztion one obtains using that $e^{-tA}z \to z$ as $t \to 0$ in view of Lemma 2.1

$$\begin{aligned} \frac{d}{dt}Mu(t) = &\frac{d}{dt}e^{-tA}y_0 + \chi g'(t)z + \chi\int_0^t g'(s)\frac{\partial}{\partial t}e^{-(t-s)A}zds \\ &+\chi\Phi[Lu(t)]z + \chi\int_0^t \Phi[Lu(s)]\frac{\partial}{\partial t}e^{-(t-s)A}zds + \frac{d}{dt}\int_0^t e^{-(t-s)A}h(s)ds. \end{aligned} \tag{2.16}$$

Addition of (2.14) and (2.16) yields (2.12) in view of (2.7). Consequently the problem is reduced to solving (2.14). Let u_1 be the function defined by

$$Lu_1(t) = -\frac{d}{dt}e^{-tA}y_0 - \chi\int_0^t g'(s)\frac{\partial}{\partial t}e^{-(t-s)A}zds - \int_0^t \frac{\partial}{\partial t}e^{-(t-s)A}h(s)ds. \tag{2.17}$$

Then the integral equation (2.14) is rewritten as

$$Lu(t) = Lu_1(t) - \chi\int_0^t \Phi[Lu(s)]\frac{\partial}{\partial t}e^{-(t-s)A}zds, \tag{2.18}$$

Since in view of (2.13) $y_0 = mu_0 = A^{-1}Lu_0$, one gets using (2.15)

$$-\frac{d}{dt}e^{-tA}y_0 = -\frac{d}{dt}e^{-tA}A^{-1}Lu_0 = e^{-tA}Lu_0. \tag{2.19}$$

Since $Lu_0 \in D(\widetilde{A})$ by assumption (2.1), the first term of the right hand side of (2.17) belongs to $C^{1/2}([0,T]; L^2(\Omega))$ in view of (2.19) and Lemma 2.1, (2.10). Let $\phi \in \widetilde{A}Lu_0$. Then $e^{-tA}Lu_0 = e^{-tA}\widetilde{A}^{-1}\phi = \widetilde{A}^{-1}e^{-t\widetilde{A}}\phi$. This implies $\widetilde{A}e^{-tA}Lu_0 \ni e^{-t\widetilde{A}}\phi$. Hence

$$\|e^{-tA}Lu_0\|_{D(\widetilde{A})} \leqslant \|e^{-t\widetilde{A}}\phi\|_{H^{-1}(\Omega)} \leqslant C_0\|\phi\|_{H^{-1}(\Omega)},$$

where we used (1.20). Therefore the first term of the right hand side of (2.17) belongs to $B([0,T];D(\widetilde{A}))$, and hence to $B([0,T];L^2(\Omega)_A^{1/2})$ in view of the first inclusion relation of (1.23). The last term of (2.17) belongs to $C^{1/2}([0,T];L^2(\Omega)) \cap B([0,T];L^2(\Omega)_A^{1/2})$ in view of Proposition 2.1. Clearly $g'(\cdot)z \in C([0,T];L^2(\Omega)) \cap B([0,T];D(\widetilde{A}))$. Hence applying this proposition to $g'(\cdot)z$ instead of h we see that the second term of the right hand side of (2.17) also belongs to $C^{1/2}([0,T];L^2(\Omega)) \cap B([0,T];L^2(\Omega)_A^{1/2})$. Therefore

$$Lu_1 \in C^{1/2}([0,T];L^2(\Omega)) \cap B([0,T];L^2(\Omega)_A^{1/2}). \tag{2.20}$$

The integral equation (2.18) is solved by successive approximation. Let

$$Lu_{n+1}(t) = Lu_1(t) - \chi \int_0^t \Phi[Lu_n(s)] \frac{\partial}{\partial t} e^{-(t-s)A} z ds, \quad n = 1, 2, 3, \dots .$$

By virtue of (2.9) the following inequalities hold for $n = 2, 3, \dots$

$$\|Lu_{n+1}(t) - Lu_n(t)\|_{L^2(\Omega)} = \left\| \chi \int_0^t \Phi[Lu_n(s) - Lu_{n-1}(s)] \frac{\partial}{\partial t} e^{-(t-s)A} z ds \right\|_{L^2(\Omega)}$$

$$\leqslant C_0 |\chi| \|\Phi\| \int_0^t \|Lu_n(s) - Lu_{n-1}(s)\|_{L^2(\Omega)} (t-s)^{-1/2} ds \|z\|_{D(\widetilde{A})}.$$

By induction it can be shown without difficulty that

$$\|Lu_{n+1}(t) - Lu_n(t)\|_{L^2(\Omega)} \leqslant 2C_2 (C_0 |\chi| \|\Phi\|)^n \frac{\pi^{(n-1)/2}}{n\Gamma(n/2)} \sqrt{\pi} t^{n/2} \|z\|^n_{D(\widetilde{A})}, \quad n = 2, 3, \dots ,$$

where C_2 is a constant such that

$$\|Lu_1(t)\| \leqslant C_2, \quad 0 \leqslant t \leqslant T.$$

Consequently the sequence $\{u_n\}$ tends to a function u satisfying (2.18) in $C([0,T];D(L))$. Applying Proposition 2.1 to $\Phi[Lu(\cdot)]z$ we see that the second term of the right hand side of (2.18) belongs to $C^{1/2}([0,T];L^2(\Omega)) \cap B([0,T];L^2(\Omega)_A^{1/2})$. Therefore

$$Lu \in C^{1/2}([0,T];L^2(\Omega)) \cap B([0,T];L^2(\Omega)_A^{1/2}).$$

Other regularity properties of u listed in the statement of the theorem are obvious. Consequently the proof of the thoerem is complete.

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DEGENERATE DIFFERENTIAL EQUATIONS OF PARABOLIC TYPE AND INVERSE PROBLEMS

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Abstract

Some identification problems for degenerate linear differential equations in Banach spaces are studied by reducing them to related direct problems. The abstract results are applied to treat some inverse problems for partial differential equations.

1 Introduction

In this paper a general method to solve inverse problems for degenerate differential equations in Banach spaces is described.

It basically consists in reducing the inverse problem to a direct problem whose operator-coefficients are perturbations of some operator-coefficients of the given equation.

More precisely, the strategy for solving the inverse problem to determine the pair $(y, f) \in C\big([0, r]; D(L)\big) \times C\big([0, r]; \mathbb{C}\big)$ satisfying the possibly degenerate initial-value problem

$$\frac{d}{dt}\big(My(t)\big) + Ly(t) = f(t)z + h(t), \qquad 0 \le t \le r, \tag{1.1}$$

$$(My)(0) = My_0, \qquad y_0 \in D(L), \tag{1.2}$$

$$\Phi\big[My(t)\big] = g(t), \qquad 0 \le t \le r, \tag{1.3}$$

under the assumptions

(i) L and M are closed linear operators acting on the complex Banach space X, the domain $D(L)$ of L is contained in $D(M)$, $0 \in \rho(L)$,

(ii) L and M satisfy the weak parabolicity condition $\zeta M + L$ has a bounded inverse for any $\zeta \in \Sigma_\alpha$, where

$$\Sigma_\alpha = \big\{\zeta \in \mathbb{C}\colon\ \operatorname{Re}\zeta \ge -C\big(1 + |\operatorname{Im}\zeta|\big)^\alpha\big\}, \qquad C > 0$$

and

$$\big\|M(\zeta M + L)^{-1}\big\|_{\mathcal{L}(X)} \le C'\big(1 + |\zeta|\big)^{-\beta}, \qquad \forall \zeta \in \Sigma_\alpha, \quad 0 < \beta \le \alpha \le 1,$$

(iii) $h \in C\big([0, r]; X\big)$, $\Phi \in X^*$, $g \in C^1\big([0, r]; \mathbb{C}\big)$ and the compatibility relation

$$g(0) = \Phi[My_0] \tag{1.4}$$

holds,

is as follows. Notice that the parabolicity assumptions (ii) comes from the monograph [12] from Favini and Yagi.

Applying Φ to (1.1) and using the additional information (1.3), we get that necessarily f satisfies

$$g'(t) = -\Phi\big[Ly(t)\big] + f(t)\Phi[z] + \Phi\big[h(t)\big]; \qquad 0 \le t \le r,$$

therefore, if $\Phi[z] \ne 0$, then $f(t)$ is given by

$$f(t) = \frac{1}{\Phi[z]}\left\{g'(t) + \Phi\big[Ly(t)\big] - \Phi\big[h(t)\big]\right\}. \tag{1.5}$$

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If we substitute (1.5) in (1.1), this yields the direct problem

$$\begin{aligned}&\frac{d}{dt}\big(My(t)\big) = -Ly(t) - L_1 y(t) - \frac{1}{\Phi[z]}\Phi\big[h(t)\big]z + \frac{g'(t)}{\Phi[z]}\,z + h(t), \qquad 0 \le t \le r,\\ &(My)(0) = My_0,\end{aligned} \tag{1.6}$$

where the operator L_1 is defined by

$$D(L_1) = D(L), \qquad L_1 y := -\frac{\Phi[Ly]}{\Phi[z]}\,z, \quad y \in D(L_1).$$

Hence the direct problem (1.6), together with $f(t)$ furnished by (1.5) is equivalent to (1.1)∼(1.3).

Such a strategy has been already used in the paper [11] from A. Favini and G. Marinoschi in order to treat identification problems for degenerate equations of hyperbolic type. One can solve problem (1.6), after having shown that the pair $(L + L_1, M)$ satisfies the same resolvent estimate as in (ii), by using either regularity in time of the data or spatial regularity of them.

Concerning space regularity, a first approach can be found in a paper [10] from Favini, Lorenzi, Marinoschi, Tanabe. In that paper, the assumption $L + L_1$ to have a bounded inverse (and thus to be closed) was introduced to simplify the treatment, but it is not so obvious.

A main aim in this paper is to cover this possible gap. The assumptions on the given elements y_0, z will be concerned with the intermediate spaces

$$X_A^\theta = \Big\{u \in X\colon \sup_{t>0}(1+t)^\theta \left\|A^\circ(tI + A)^{-1}u\right\|_X = \|u\|_{X_A^\theta} < \infty\Big\} \tag{1.7}$$

where $A^\circ(tI + A)^{-1}$ means $I - t(tI + A)^{-1}$, A being the multivalued linear operator LM^{-1}, $D(A) = M\big(D(L)\big)$.

Recently, in russian literature, see papers from G.A. Baskakov and his co-authors, such operators are also named linear relations. The choice of the spaces X_A^θ compels us to some restrictions on θ, that can be avoided provided that, according to a very recent paper [8] by Favini, Lorenzi, Tanabe we use the interpolation spaces $\big(X, D(A)\big)_{\theta,\infty}$ instead. On the other hand a characterization of these interpolation spaces as in Triebel's monograph [14] for $\alpha = \beta = 1$ appears very difficult.

We note that maximal regularity in time for degenerate differential equations and its applications to inverse problems was investigated in the paper [3] from Favini, Lorenzi, Tanabe and very recently in [1], [2] from Favini and Favaron.

The contents of the paper are as follows. In Section 2 we recall some perturbation results from [10]. Section 3 and 4 concern solvability of (1.1)∼(1.3) under maximal regularity in space and in time, respectively. In Section 5 a related inverse problem for the equation $My' = -Ly + f(t)z + h(t)$ is considered. Section 6 is devoted to applications to PDEs and integro-differential equations.

At last, we want to thank very much professor Giovanni Dore for his important help and useful discussions and remarks.

2 Perturbation results

The result that follows furnishes an extension to well known statements concerning sectorial operators, cfr. Lunardi [13]. We refer to Favini, Lorenzi, Marinoschi, Tanabe [10].

Theorem 2.1. *Let M, L, L_1 be closed linear operators in the complex Banach space X, with $D(L_1) = D(L) \subseteq D(M)$, $0 \in \rho(L)$, $0 < \beta \le \alpha \le 1$,*

$$\left\|M(\zeta M + L)^{-1}x\right\|_X \le C(1 + |\zeta|)^{-\beta}\|x\|_X; \qquad \zeta \in \Sigma_\alpha, \quad x \in X. \tag{2.1}$$

If, in addition, $L_1 \in \mathcal{L}\big(D(L), X_A^\theta\big)$, where A is the multivalued operator LM^{-1} and $1 - \beta < \theta < 1$, then

$$\left\|M(\zeta M + L + L_1)^{-1}\right\|_{\mathcal{L}(X)} \le C(1 + |\zeta|)^{-\beta}; \qquad \zeta \in \Sigma_\alpha, \quad |\zeta| \text{ large.} \tag{2.2}$$

Notice that A satisfies the resolvent estimate

$$\big\|(\zeta I + A)^{-1}\big\|_{\mathcal{L}(X)} \leq C(1+|\zeta|)^{-\beta}, \qquad \zeta \in \Sigma_\alpha.$$

If $\alpha = \beta = 1$, then X_A^θ coincides with $\big(X, D(A)\big)_{\theta,\infty}$, $0 < \theta < 1$. Recall (cfr. [11]) that

$$A^\circ(\zeta I + A)^{-1} = (\zeta A^{-1} + I)^{-1} = I - \zeta(\zeta I + A)^{-1}.$$

Then $D(A)$ is a Banach space under the graph norm

$$\|x\|_{D(A)} := \inf_{y \in Ax} \|y\|_X.$$

We recall the following existence and uniqueness result from Favini, Lorenzi, Marinoschi, Tanabe [10].

We need to introduce a further notation to this purpose. If Y is a Banach space, $B\big([0,r];Y\big)$ denotes the space of all bounded Y-valued functions f on $[0,r]$ with

$$\|f\|_{B([0,r];Y)} = \sup_{0 \leq t \leq r} \|f(t)\|_Y.$$

We have

Theorem 2.2. *Suppose L and M satisfy (2.1), $0 < \beta \leq \alpha \leq 1$, $y_0 \in D(L)$, $Ly_0 \in X_A^\theta$ where $2-\alpha-\beta < \theta < 1$, $\alpha+\beta > 1$, $f \in B\big([0,r];X_A^\theta\big) \cap C\big([0,r];X\big)$. Then the problem*

$$\frac{d}{dt}\big(My(t)\big) + Ly(t) = f(t), \qquad 0 \leq t \leq r, \tag{2.3}$$

$$(My)(0) = My_0, \tag{2.4}$$

admits a unique strict solution $y \in C\big([0,r];D(L)\big)$ with the spatial regularity

$$Ly, (My)' \in B\big([0,r];X_A^{\theta-(2-\alpha-\beta)}\big) \cap C\big([0,r];X\big).$$

If, in addition, $2\alpha+\beta > 2$ and $3-2\alpha-\beta < \theta < 1$, then y enjoys the time regularity

$$Ly \in C^{(2\alpha+\beta+\theta-3)/\alpha}\big([0,r];X\big).$$

3 A first identification problem

After the change of variable $y(t) = e^{kt}w(t)$, $k > 0$, problem (1.1)$\sim$(1.3) becomes

$$\begin{aligned} \frac{d}{dt}\big(Mw(t)\big) + (kM+L)w(t) &= e^{-kt}f(t)z + e^{-kt}h(t), \qquad 0 \leq t \leq r, \qquad (3.1)\\ &= f_1(t)z + h_1(t), \\ (Mw)(0) &= My_0, \qquad y_0 \in D(L), \qquad (3.2)\\ \Phi\big[Mw(t)\big] &= e^{-kt}g(t), \qquad 0 \leq t \leq r, \qquad (3.3)\\ &= g_1(t), \end{aligned}$$

Applying Φ to both members in (3.1), and taking into account (3.3), we get

$$g_1'(t) + \Phi\big[(kM+L)w(t)\big] = f_1(t)\Phi[z] + \Phi\big[h_1(t)\big].$$

Since $\Phi[z] \neq 0$, then

$$f_1(t) = \frac{g_1'(t) + \Phi\big[(kM+L)w(t)\big] - \Phi\big[h_1(t)\big]}{\Phi[z]}$$

and thus (3.1) becomes

$$\begin{aligned} &\frac{d}{dt}\big(w(t)\big) + (kM+L+L_1)w(t) = \frac{g_1'(t)}{\Phi[z]}z - \frac{\Phi\big[h_1(t)\big]}{\Phi[z]}z + h_1(t), \qquad 0 \leq t \leq r, \\ &(Mw)(0) = My_0, \end{aligned} \tag{3.4}$$

where L_1 is the operator from $D(L_1) = D(L)$ into X defined by

$$L_1 w = -\frac{\Phi\big[(kM+L)w\big]}{\Phi[z]}\, z.$$

Assume $z \in X_A^\theta$, $\theta > 1-\beta$, and observe that

$$\begin{aligned} kM + L + L_1 &= (kM+L)\big(I + (kM+L)^{-1}L_1\big) \\ &= (kM+L)\big(I + L^{-1}(kML^{-1}+I)^{-1}L_1\big) \\ &= (kM+L)\big(I + L^{-1}A^\circ(kI+A)^{-1}L_1\big). \end{aligned}$$

It is known (see [12], p. 49) that

$$\big\|A^\circ(kI+A)^{-1}f\big\|_X \le C(1+k)^{1-\beta-\theta}\|f\|_{X_A^\theta}$$

and hence, if k is large enough,

$$\big\|L^{-1}A^\circ(kI+A)^{-1}L_1 f\big\|_{D(L)} \le q\|f\|_{D(L)},$$

with $0<q<1$. It follows that $I + L^{-1}A^\circ(kI+A)^{-1}L_1$ has a bounded inverse form $D(L)$ to itself. Since $kM+L$ has a bounded inverse from X into $D(L)$, we conclude that $kM+L+L_1$ has a bounded inverse from X into $D(L)$ and also from X into itself. Moreover

$$\Big\|\big(I+L^{-1}A^\circ(kI+A)^{-1}L_1\big)^{-1}(kM+L)^{-1}f\Big\|_{D(L)} \le C\,\big\|L(kM+L)^{-1}f\big\|_X \le C(1+k)^{1-\beta}\|f\|_X.$$

Hence $kM+L+L_1$ has a bounded inverse in $\mathcal{L}(X)$ as declared in Theorem 2.1,

$$\Big\|M\big((\zeta+k)M+L+L_1\big)^{-1}\Big\|_{\mathcal{L}(X)} \le C\big(1+|\zeta|\big)^{-\beta}, \qquad \operatorname{Re}\zeta \ge -C\big(1+|\operatorname{Im}\zeta|\big)^{\alpha}.$$

It follows that Theorem 2.2 applies with $kM+L+L_1$ instead of L.

Denote by $\tilde{A}$ the multivalued linear operator $(kM+L+L_1)M^{-1}$ and by A_1 the multivalued operator $(kM+L)M^{-1} = kI + A$.

We need two lemmas as follows.

Lemma 3.1. *Let $\theta\in(0,1)$. Then $X_{A_1}^\theta = X_A^\theta$.*

Proof. First of all, we observe that since A is supposed to have a bounded inverse, the space X_A^θ has an equivalent norm $\|x\|_X + \sup_{t>k}(1+t)^\theta\,\big\|A^\circ(tI+A)^{-1}x\big\|_X$. Write

$$\begin{aligned} \big(tA^{-1}+I\big)^{-1} - \big(t(kI+A)^{-1}+I\big)^{-1} &= \big(tA^{-1}+I\big)^{-1}\big[t(kI+A)^{-1} - tA^{-1}\big]\big(t(kI+A)^{-1}+I\big)^{-1} \\ &= -k\big(tA^{-1}+I\big)^{-1}A^{-1}t(kI+A)^{-1}\big(t(kI+A)^{-1}+I\big)^{-1}. \end{aligned}$$

Observe that if $(kI+A)^{-1}=0$ then $(tI+A)^{-1}=0$ for any $t\in\rho(-A)$, cfr. [11]. p. 23. Therefore

$$\begin{aligned} &\big(tA^{-1}+I\big)^{-1} - \big(t(kI+A)^{-1}+I\big)^{-1} = \\ &= -k(kI+A)^{-1}\big(t(kI+A)^{-1}+I\big)^{-1} + k\big(tA^{-1}+I\big)^{-1}(kI+A)^{-1}\big(t(kI+A)^{-1}+I\big)^{-1} \\ &= -k(kI+A)^{-1}\big(t(kI+A)^{-1}+I\big)^{-1} + k\big(I - t(tI+A)^{-1}\big)(kI+A)^{-1}\big(t(kI+A)^{-1}+I\big)^{-1} \\ &= -kt(tI+A)^{-1}(kI+A)^{-1}\big(t(kI+A)^{-1}+I\big)^{-1} \\ &= -k\,\frac{t}{t-k}\big[(kI+A)^{-1} - (tI+A)^{-1}\big]\big(t(kI+A)^{-1}+I\big)^{-1}. \end{aligned}$$

Therefore

$$\sup_{t>k+1}(1+t)^\theta\,\big\|A^\circ(tI+A)^{-1}f\big\|_X \le C\sup_{t>k+1}(1+t)^\theta\,\big\|(A+kI)^\circ(tI+kI+A)^{-1}f\big\|_X.$$

Thus $X_{A+kI}^\theta \hookrightarrow X_A^\theta$. Exchanging the role of A and $A+kI$, the embedding $X_A^\theta \hookrightarrow X_{A+kI}^\theta$ holds too. This proves the assertion. □

Lemma 3.2. *If $1-\beta<\theta<1$, then*

$$X_A^\theta \hookrightarrow X_{\tilde{A}}^{\theta+\beta-1}.$$

Proof. We recall that if $A_1=(kM+L)M^{-1}$, then $X_{A_1}^\theta = X_A^\theta$. We have for any $f\in X$

$$\begin{aligned}\tilde{A}^\circ(tI+\tilde{A})^{-1}f - A_1^\circ(tI+A_1)^{-1}f &= (t\tilde{A}^{-1}+I)^{-1}f-(tA_1^{-1}+I)^{-1}f\\ &= (t\tilde{A}^{-1}+I)^{-1}t(A_1^{-1}-\tilde{A}^{-1})(tA_1^{-1}+I)^{-1}f\end{aligned}$$

(using $\tilde{A}^{-1}=T=M(kM+L+L_1)^{-1}$, $A_1^{-1}=S=M(kM+L)^{-1}$)

$$\begin{aligned}&= -(tT+I)^{-1}t(T-S)(tS+I)^{-1}f\\ &= -(tT+I)^{-1}tM\big[(kM+L+L_1)^{-1}-(kM+L)^{-1}\big](tS+I)^{-1}f\\ &= (tT+I)^{-1}tM(kM+L+L_1)^{-1}L_1(kM+L)^{-1}(tS+I)^{-1}f\\ &= L_1L^{-1}L(kM+L)^{-1}(tS+I)^{-1}f-(tT+I)^{-1}L_1(kM+L)^{-1}(tS+I)^{-1}f.\end{aligned}$$

Since $L_1L^{-1}\in\mathcal{L}(X)$, we conclude that $\|u\|_{X_{\tilde{A}}^\theta}\le \|u\|_{X_{A_1}^{\theta+1-\beta}}$.

Writing $\sigma=\theta+1-\beta$, the continuous embedding $X_A^\sigma\hookrightarrow X_{\tilde{A}}^{\sigma+\beta-1}$, $\sigma\in(1-\beta,1)$ is proved. Analogously, one sees that $X_{\tilde{A}}^\sigma\hookrightarrow X_A^{\sigma+\beta-1}$, $\sigma\in(1-\beta,1)$. □

We are now in a position to solve problem (3.4). Take $z\in X_A^\theta$ $(\subseteq X_{\tilde{A}}^{\theta+\beta-1})$, $\theta>1-\beta$. Then if

$$\frac{g_1'(\cdot)}{\Phi[z]}\,z-\frac{\Phi\big[h_1(\cdot)\big]}{\Phi[z]}\,z+h_1(\cdot)\in B\big([0,r];X_A^\theta\big)\cap C\big([0,r];X\big), \tag{3.5}$$

problem (3.4) admits a unique strict solution w such that (see Theorem 2.2 with $kM+L+L_1$ instead of L)

$$\begin{aligned}(Mw)',(kM+L+L_1)w\in B\big([0,r];X_{\tilde{A}}^{\theta-1+\beta-(2-\alpha-\beta)}\big)&\cap C\big([0,r];X\big)\\ = B\big([0,r];X_{\tilde{A}}^{\theta+\alpha+2\beta-3}\big)&\cap C\big([0,r];X\big),\end{aligned}$$

$\theta>3-\alpha-2\beta$, $\alpha+2\beta>2$, provided that $(kM+L+L_1)y_0\in X_A^\theta$. Moreover, if $2\alpha+\beta>2$ and $3-2\alpha-\beta<\theta-1+\beta$, then $(kM+L+L_1)w\in C^{2\alpha+2\beta+\theta-4}\big([0,r];X\big)$. On the other hand,

$$L(tM+L)^{-1}My_0=t^{-1}L(tM+L)^{-1}(tM+L-L)y_0=t^{-1}Ly_0-t^{-1}L(tM+L)^{-1}Ly_0$$

guarantees that $\sup_{t>0}(1+t)^\theta\left\|L(tM+L)^{-1}My_0\right\|_X<\infty$. Moreover

$$\begin{aligned}\sup_{t>0}(1+t)^\theta\left\|L(tM+L)^{-1}L_1y_0\right\|_X &= \sup_{t>0}(1+t)^\theta\left\|L(tM+L)^{-1}\frac{\Phi\big[(kM+L)y_0\big]}{\Phi[z]}\,z\right\|_X\\ &= \frac{\big|\Phi\big[(kM+L)y_0\big]\big|}{|\Phi[z]|}\sup_{t>0}(1+t)^\theta\left\|L(tM+L)^{-1}z\right\|_X<\infty,\end{aligned}$$

since $z\in X_A^\theta$. Thus $(kM+L+L_1)y_0\in X_A^\theta$ reduces to $Ly_0\in X_A^\theta$. Since $z\in X_A^\theta$, (3.5) reduces to $h\in B\big([0,r];X_A^\theta\big)\cap C\big([0,r];X\big)$. Note also that $(2\alpha+\beta)-(\alpha+2\beta)=\alpha-\beta\ge 0$.

In view of Lemma 3.2 we deduce that if $\alpha+2\beta>2$, then

$$\frac{d}{dt}(Mw)(t),(kM+L+L_1)w(t)\in X_{\tilde{A}}^{\theta+\alpha+2\beta-3}\hookrightarrow X_A^{\theta+\alpha+3\beta-4},$$

$\theta>4-\alpha-3\beta$, $\alpha+3\beta>3$.

At last, if $\alpha+\beta>3/2$, so that $2\alpha+\beta>2$ and $\theta>4-2\alpha-2\beta=2(2-\alpha-\beta)$, then $(kM+L+L_1)w\in C^{2\alpha+2\beta-4+\theta}\big([0,r];X\big)$ and hence $Lw\in C^{2\alpha+2\beta-4+\theta}\big([0,r];X\big)$. On the other hand, $\alpha+3\beta-4-(2\alpha+2\beta-4)=-\alpha+\beta\le 0$. It follows that if $\alpha+3\beta>3$, $\alpha+\beta>3k$, $\theta\in(4-\alpha-3\beta,1)$, the unique solution w of (3.4) possesses the additional regularities $\frac{d}{dt}(Mw)(t),(kM+L+L_1)w(t)\in X_A^{\theta+\alpha+3\beta-4}$, $Lw\in C^{\theta+2(\alpha+\beta-2)}\big([0,r];X\big)$.

We can establish the result concerning problem (1.1)∼(1.3) as follows

Theorem 3.3. *Suppose* $0 < \beta \le \alpha \le 1$*,* $\alpha + 3\beta > 3$*,* $4 - \alpha - 3\beta < \theta < 1$*,* $y_0 \in D(L)$*,* $Ly_0 \in X_A^\theta$*,* $h \in C\big([0,r];X\big) \cap B\big([0,r];X_A^\theta\big)$*,* $z \in X_A^\theta$*,* $\Phi \in X^*$*,* $\Phi[z] \neq 0$*,* $g \in C^1\big([0,r];\mathbb{C}\big)$*,* $g(0) = \Phi[My_0]$*.*

Then problem (1.1)$\sim$(1.3) *admits a unique solution* $(y,f) \in C\big([0,r];D(L)\big) \times C\big([0,r];\mathbb{C}\big)$*.*

Very recently a different approach was used for solving problem (1.1)$\sim$(1.3) in Favini, Lorenzi, Tanabe [9]. The starting point is the following existence and uniqueness result.

Proposition 3.4. *Suppose that* $2\alpha+\beta+\theta > 3$ *(*$2\alpha+\beta > 2$*),* $y_0 \in D(L)$*,* $Ly_0 \in \big(X, D(A)\big)_{\theta,\infty}$*,* $A = LM^{-1}$*,* $\|y\|_{D(A)} := \inf\big\{\|Lu\|_X : y = Mu, u \in D(L)\big\}$*,* $f \in C\big([0,r];X\big) \cap B\big([0,r];\big(X,D(A)\big)_{\theta,\infty}\big)$*. Then there exists a unique solution* y *to problem*

$$\begin{aligned} &\frac{d}{dt}\big(My(t)\big) + Ly(t) = f(t), \qquad 0 \le t \le r,\\ &(My)(0) = My_0, \end{aligned}$$

such that $My \in C^1\big([0,r];X\big)$*,* $Ly \in C^{(2\alpha+\beta-3+\theta)/\alpha}\big([0,r];X\big) \cap B\big([0,r];X_A^{(2\alpha+\beta-3+\theta)/\alpha}\big)$*.*

Proof. Let $u_0 = My_0$, then $u_0 \in D(A)$ and by hypothesis $Ly_0 \in Au_0 \cap \big(X, D(A)\big)_{\theta,\infty}$.

One can apply an existence and uniqueness result for $u' - f \in Au$, $u(0) = u_0$ in [9] according to which there is a unique solution u to this initial problem for the inclusion $u' - f \in Au$. Set $y(t) = L^{-1}\big(f(t) - u'(t)\big)$. Then $Ly(t) = f(t) - u'(t) \in Au(t) = LM^{-1}u(t)$. Since L is invertible we have $y(t) \in M^{-1}u(t)$, i.e. $My(t) = u(t)$ and thus $Ly(t) = f(t) - \frac{d}{dt}My(t)$.

Moreover $(My)(0) = u_0 = My_0$. □

By using a fixed point argument, the identification result is established as follows.

Theorem 3.5. *Let* $2\alpha + \beta > 2$ *and suppose* $3 - 2\alpha - \beta < \theta < 1$*,* $y_0 \in D(L)$*,* $g \in C^1\big([0,r];\mathbb{C}\big)$*,* $\Phi \in X^*$*,* $\Phi[My_0] = g(0)$*,* $z \in \big(X, D(A)\big)_{\theta,\infty}$*,* $\Phi[z] \neq 0$*,* $h \in C\big([0,r];X\big) \cap B\big([0,r];\big(X,D(A)\big)_{\theta,\infty}\big)$*. Then there is a unique solution* (y,f) *to* (1.1)$\sim$(1.3) *such that* $My \in C^1\big([0,r];X\big)$*,* $f \in C\big([0,r];\mathbb{C}\big)$*,* $(My)' - f(\cdot)z - h(\cdot) \in C^{(2\alpha+\beta-3+\theta)/\alpha}\big([0,r];X\big) \cap B\big([0,r];X_A^{(2\alpha+\beta-3+\theta)/\alpha}\big)$*.*

The proof of Theorem 3.5 is an immediate consequence of Theorem 4.1 in [9], after transforming our problem to the one to determine (x,f), $x \in C^1\big([0,r];X\big)$, $f \in C\big([0,r];\mathbb{C}\big)$, satisfying

$$\begin{aligned} &x'(t) - f(t)z - h(t) \in Ax(t),\\ &x(0) = My_0,\\ &\Phi\big[x(t)\big] = g(t). \end{aligned}$$

4 Maximal regularity in time

We want to solve problem (3.4) under maximal time regularity assumptions, by using [3], Theorem 7.2.

Suppose $z \in X_A^\theta$, $\theta > 1 - \beta$, so that $X_A^\theta \hookrightarrow X_{\tilde{A}}^{\theta-1+\beta}$, and take $h \in C^\theta\big([0,r];X\big)$, $g \in C^{1+\theta}\big([0,r];\mathbb{C}\big)$, so that h_1 and g_1 have the same regularity.

Compute

$$h_1(0) + \frac{\Phi\big[h_1(0)\big]}{\Phi[z]}\,z + \frac{g_1'(0)}{\Phi[z]}\,z - (kM + L + L_1)y_0,$$

where

$$L_1 y_0 = -\frac{\Phi\big[(kM+L)y_0\big]}{\Phi[z]}\,z.$$

Suppose $h(0) - Ly_0 \in X_A^\theta \subset X_{\tilde{A}}^{\theta-(1-\beta)}$. Then, by Theorem 7.2 in [3], problem (3.4) admits a unique strict solution $w \in C\big([0,r], D(L)\big)$, $(kM + L + L_1)w \in C\big([0,r], X\big)$, such that

$$(Mw)', (kM + L + L_1)w \in C^{\theta-1+\beta+\alpha+\beta-2}\big([0,r], X\big) = C^{\theta+\alpha+2\beta-3}\big([0,r], X\big),$$

$(2 - \alpha - \beta) + (1 - \beta) = 3 - \alpha - 2\beta < \theta < 1$, $\alpha + 2\beta > 2$,

This yields a result as follows:

Theorem 4.1. *Suppose $\alpha + 2\beta > 2$ and let $3 - \alpha - 2\beta < \theta < 1$. If $\Phi \in X^*$, $\Phi[z] \neq 0$, $\Phi[My_0] = g(0)$, $g \in C^{1+\theta}\big([0,r];\mathbb{C}\big)$, $h \in C^\theta\big([0,r];X\big)$, then inverse problem* (1.1)$\sim$(1.3) *admits a unique strict solution $(y,f) \in C^{\theta+\alpha+2\beta-3}\big([0,r];D(L)\big) \times C^{\theta+\alpha+2\beta-3}\big([0,r];\mathbb{C}\big)$.*

A refinement of Theorem 4.1 comes from [2], Theorem 6.3 and can be established as follows:

Theorem 4.2. *Suppose $3\alpha + 4\beta > 6$ (so that $5\alpha + 2\beta > 6$)). Take $f(0) - Ly_0 \in X_A^{\varphi_0}$, $\varphi_0 \in (6-3\alpha-3\beta,\beta)$, $z \in X_A^{\varphi_1}$, $\varphi_1 \in (6-3\alpha-3\beta,1)$, $f \in C^{\mu_0}\big([0,r],X\big)$, $\mu_0 \in \big[(6-3\alpha-3\beta)/(2\alpha),1\big)$, $g \in C^{1+\mu_1}\big([0,r],\mathbb{C}\big)$, $\mu_1 \in \big((3-2\alpha-\beta)/\alpha,1\big)$, $0 \in \rho(L)$, $\Phi \in X^*$, $\Phi[z] \neq 0$, $g(0) = \Phi[My_0]$. Let $\gamma_i = \varphi_i + \beta - 1$, $i = 0,1$, $\gamma = \min_{i=0,1}\gamma_i$, $\tau = \min\big\{\mu, (\alpha+\beta+\gamma-2)/\alpha\big\}$, where $\mu = \min_{i=0,1}\mu_i$. Let*

$$I_{\alpha,\beta,\tau} = \begin{cases} \big((3-2\alpha-\beta)/\alpha,\tau\big], & \text{if } \tau \in \big((3-2\alpha-\beta)/\alpha,1/2\big), \\ \big((3-2\alpha-\beta)/\alpha,1/2\big), & \text{if } \tau \in [1/2,1). \end{cases}$$

Then, for every fixed $\delta \in I_{\alpha,\beta,\tau}$, the degenerate identification problem (1.1)$\sim$(1.3) *admits a unique solution (y,f) such that $y \in C^\delta\big([0,r],D(L)\big)$, $My \in C^{1+\delta}\big([0,r],X\big)$, $y(0) = y_0$, $f \in C^\delta\big([0,r],\mathbb{C}\big)$.*

It is to be noticed the same regularity in the class C^δ of y and f.

5 A latter identification problem

Let us consider the problem to find the pair $(y,f) \in C^1\big([0,r];X\big) \times C\big([0,r];\mathbb{C}\big)$ such that

$$My'(t) = -Ly(t) + f(t)z + h(t), \qquad 0 \le t \le r, \tag{5.1}$$

$$y(0) = y_0, \tag{5.2}$$

$$\Phi\big[My(t)\big] = g(t), \qquad 0 \le t \le r. \tag{5.3}$$

It is well known that further compatibility relations must be introduced. A first remark says us that necessarily

$$g'(0) = -\Phi[Ly_0] + f(0)\Phi[z] + \Phi\big[h(0)\big],$$

so that, if $\Phi[z] \neq 0$, then

$$f(0) = \frac{1}{\Phi[z]}\big\{g'(0) + \Phi[Ly_0] - \Phi\big[h(0)\big]\big\}.$$

In particular, if $y_1 = y'(0)$, then necessarily

$$My_1 = -Ly_0 + f(0)z + h(0) \tag{5.4}$$

for such a $f(0)$. Deriving both the members of (5.1) we get

$$\frac{d}{dt}\big(My'(t)\big) = -Ly'(t) + f'(t)z + h'(t), \qquad 0 \le t \le r, \tag{5.5}$$

$$(My')(0) = -Ly_0 + f(0)z + h(0) = My_1, \tag{5.6}$$

$$\Phi\big[My'(t)\big] = g'(t), \qquad 0 \le t \le r. \tag{5.7}$$

This inverse problem can be solved by using the previous results and for sake of brevity we confine to use Theorem 3.5.

Therefore, if $2\alpha + \beta > 2$, $3 - 2\alpha - \beta < \theta < 1$, $y_1 \in D(L)$, $g \in C^2\big([0,r];\mathbb{C}\big)$, $\Phi \in X^*$, $\Phi[z] \neq 0$, $z \in \big(X,D(A)\big)_{\theta,\infty}$, $h \in C^1\big([0,r];X\big)$, $h' \in B\big([0,r];\big(X,D(A)\big)_{\theta,\infty}\big)$, then the identification problem

$$\frac{d}{dt}\big(M\xi(t)\big) = -L\xi(t) + f_1(t)z + h'(t), \qquad 0 \le t \le r, \tag{5.8}$$

$$(M\xi)(0) = My_1, \tag{5.9}$$

$$\Phi\big[M\xi(t)\big] = g'(t), \qquad 0 \le t \le r, \tag{5.10}$$

admits a unique strict solution $(\xi, f_1) \in C\big([0,r]; D(L)\big) \times C\big([0,r]; \mathbb{C}\big)$ such that $M\xi \in C^1\big([0,r]; X\big)$ and $(M\xi)' - f_1(\cdot)z - h'(\cdot) \in C^{(2\alpha+\beta-3+\theta)/\alpha}\big([0,r]; X\big) \cap B\big([0,r]; X_A^{(2\alpha+\beta-3+\theta)/\alpha}\big)$.

Integrating (5.8) on $(0,t)$, we get

$$M\xi(t) - My_1 = -L\int_0^t \xi(s)\,ds + \int_0^t f_1(s)\,ds\,z + h(t) - h(0).$$

Now, write $f_1(s) = f'(s)$, with $f(0) = \dfrac{g'(0) + \Phi[Ly_0] - \Phi\big[h(0)\big]}{\Phi[z]}$. Thus we obtain

$$M\xi(t) - My_1 = -L\left[\int_0^t \xi(s)\,ds + y_0\right] + Ly_0 + f(t)z - f(0)z + h(t) - h(0).$$

But $My_1 = -Ly_0 + f(0)z + h(0)$. Hence $y(t) = y_0 + \int_0^t \xi(s)\,ds$ satisfies problem (5.1), (5.2). Moreover

$$\Phi\big[My'(t)\big] = \frac{d}{dt}\Phi\big[My(t)\big] = \Phi\big[M\xi(t)\big] = g'(t)$$

and thus $\int_0^t \frac{d}{dt}\big(My(s)\big)\,ds = My(t) - My_0$ implies that

$$\int_0^t \Phi\big[My'(s)\big]\,ds = \Phi\big[My(t)\big] - \Phi[My_0] = \int_0^t g'(s)\,ds = g(t) - g(0).$$

But $g(0) = \Phi[My_0]$, therefore $\Phi\big[My(t)\big] = g(t)$, as desired.

We have established the result as follows.

Theorem 5.1. *Let $2\alpha + \beta > 2$, $3 - 2\alpha - \beta < \theta < 1$. Let $y_1 \in D(L)$ such that*

$$My_1 = -Ly_0 + \frac{g'(0) + \Phi[Ly_0] - \Phi\big[h(0)\big]}{\Phi[z]}\,z + h(0),$$

with $y_0 \in D(L)$, $z \in \big(X, D(A)\big)_{\theta,\infty}$, $\Phi \in X^*$, $\Phi[My_0] = g(0)$, $\Phi[z] \neq 0$, $g \in C^2\big([0,r]; \mathbb{C}\big)$, $h \in C^1\big([0,r]; X\big)$, $h' \in B\big([0,r]; \big(X, D(A)\big)_{\theta,\infty}\big)$.

Then problem (5.1)~(5.3) admits a unique strict solution $(y, f) \in C^1\big([0,r]; D(L)\big) \times C^1\big([0,r]; \mathbb{C}\big)$ such that $My' \in C^1\big([0,r]; X\big)$,

$$(My')' - f'(\cdot)z - h'(\cdot) \in C^{(2\alpha+\beta-3+\theta)/\alpha}\big([0,r]; X\big) \cap B\big([0,r]; X_A^{(2\alpha+\beta-3+\theta)/\alpha}\big).$$

6 Examples and applications

6.1 Degenerate integrodifferential equations

In a complex Banach space X, consider the integrodifferential inverse problem to recover $(y, f) \in C\big([0,r]; D(L)\big) \times C\big([0,r]; \mathbb{C}\big)$ such that

$$\frac{d}{dt}\big(My(t)\big) + Ly(t) = \int_0^t K(t-s)L_1 y(s)\,ds + f(t)z + h(t), \qquad 0 \le t \le r, \tag{6.1}$$

$$(My)(0) = My_0, \tag{6.2}$$

$$\Phi\big[My(t)\big] = g(t), \qquad 0 \le t \le r, \tag{6.3}$$

with $\Phi[My_0] = g(0)$, under the type of hypotheses made in Sections 2, 3.

Applying Φ to both members of (6.1), we see that necessarily, if $\Phi[z] \neq 0$, and after the change of variable $y(t) = e^{kt}x(t)$, we get

$$\begin{aligned}&\frac{d}{dt}\big(Mx(t)\big) + (kM + L)x(t) - \frac{\Phi\big[(kM+L)x(t)\big]}{\Phi[z]}\,z \\ &= \int_0^t K_1(t-s)L_1 x(s)\,ds - \int_0^t K_1(t-s)\frac{\Phi\big[L_1 x(s)\big]}{\Phi[z]}\,z\,ds + \frac{g_1'(t)}{\Phi[z]}\,z - \frac{\Phi\big[h_1(t)\big]}{\Phi[z]}\,z + h_1(t),\end{aligned} \tag{6.4}$$

where $g_1(t) = e^{-kt}g(t)$, $h_1(t) = e^{-kt}h(t)$, $K_1(t) = e^{-kt}K(t)$,

$$f_1(t) = \frac{g_1'(t) + \Phi\big[(kM+L)x(t)\big] - \int_0^t K_1(t-s)\Phi\big[L_1x(s)\big]\,ds - \Phi\big[h_1(t)\big]}{\Phi[z]},$$

$$(Mx)(0) = My_0. \tag{6.5}$$

(6.4), (6.5) is a direct problem. Introduce the operator L_2 by

$$D(L_2) = D(L_1), \qquad L_2x = L_1x - \frac{\Phi[L_1x]}{\Phi[z]}\,z.$$

We must require that L_2 is a closed operator in X.

To this end, suppose $\big\|(\zeta I + L_1)^{-1}\big\|_{\mathcal{L}(X)} \le C\big(1+|\zeta|\big)^{-\beta_1}$, $\operatorname{Re}\zeta \ge -C\big(1+|\operatorname{Im}\zeta|\big)^{\alpha_1}$, $0 < \beta_1 \le \alpha_1 \le 1$. If $B_1x = -\frac{\Phi[L_1x]}{\Phi[z]}\,z$, assume $z \in X^\sigma_{L_1}$, $\beta_1 < \sigma < 1$. Then $\zeta I + L_1 + B_1 = (\zeta I + L_1)\big(I + (\zeta I+L_1)^{-1}B_1\big)$, $B_1 \in \mathcal{L}\big(D(L_1), X^\sigma_{L_1}\big)$, $\big\|L_1(\zeta I+L_1)^{-1}\big\|_{\mathcal{L}(X^\sigma_{L_1},X)} \le C\big(1+|\zeta|\big)^{1-\beta_1-\sigma}$. Therefore $\big(I + (\zeta I + L_1)^{-1}B_1\big)^{-1} \in \mathcal{L}\big(D(L_1)\big)$ and

$$(\zeta I + L_1 + B_1)^{-1} = \big(I + (\zeta I + L_1)^{-1}B_1\big)^{-1}(\zeta I + L_1)^{-1} \in \mathcal{L}\big(X, D(L_1)\big) \subset \mathcal{L}(X).$$

Therefore $\zeta I + L_1 + B_1$ has a bounded inverse and hence it is closed. But then $L_1 + B_1$ is closed too.

At a first step, we want to apply Favini, Lorenzi, Tanabe [4], Theorem 2.1. p. 469.

Suppose $D(L) \subset D(M) \cap D(L_1)$, $z \in X^\theta_A \cap X^\sigma_{L_1}$, $\theta > 1-\beta$, where

$$\big\|(\zeta I + L_1)^{-1}\big\|_{\mathcal{L}(X)} \le C\big(1+|\zeta|\big)^{-\beta_1}, \qquad \operatorname{Re}\zeta \ge -C\big(1+|\operatorname{Im}\zeta|\big)^{\alpha_1}, \tag{6.6}$$

$0 < \beta_1 \le \alpha_1 \le 1$, $\sigma > 1-\beta_1$. Then $z \in X^{\theta-1+\beta}_{\tilde{A}}$.

Take $g \in C^{\theta+\beta}\big([0,r];\mathbb{C}\big)$, $h \in C^\theta\big([0,r];X\big) \subseteq C^{\theta-1+\beta}\big([0,r];X\big)$, $K \in C^\theta\big([0,r];\mathbb{C}\big)$, $h(0) - (kM+L)y_0 + \frac{\Phi[(kM+L)y_0]}{\Phi[z]}\,z \in \mathcal{R}(T)$, i.e. $h(0) - Ly_0 \in \mathcal{R}(T) = D(A)$, $z \in D(A)$.

Then (6.1)∼(6.3) has a unique strict solution

$$\begin{aligned}(y,f) &\in C^{\theta-1+\beta-2+\alpha+\beta}\big([0,r];D(L)\big) \times C^{\theta-3+2\beta+\alpha}\big([0,r];\mathbb{C}\big)\\ &= C^{\theta-3+2\beta+\alpha}\big([0,r];D(L)\big) \times C^{\theta-3+2\beta+\alpha}\big([0,r];\mathbb{C}\big),\end{aligned}$$

$3-2\beta-\alpha < \theta < 1$, $\alpha+2\beta > 2$.

Summing up, we have:

Theorem 6.1. *Let $\alpha+2\beta > 2$, $3-2\beta-\alpha < \theta < 1$ and suppose $D(L) \subset D(M) \cap D(L_1)$ where the closed linear operator L_1 satisfies* (6.6). *Let $z \in D(A) \cap X^\sigma_{L_1}$, $\sigma > 1-\beta_1$. Suppose $g \in C^{\theta+\beta}\big([0,r];\mathbb{C}\big)$, $h \in C^\theta\big([0,r];X\big)$, $K \in C^\theta\big([0,r];\mathbb{C}\big)$, $h(0) - Ly_0 \in \mathcal{R}(T) = D(A)$, $\Phi[z] \ne 0$.*

Then problem (6.1)∼(6.3) *admits a unique strict solution*

$$(y,f) = C^{\theta-3+2\beta+\alpha}\big([0,r];D(L)\big) \times C^{\theta-3+2\beta+\alpha}\big([0,r];\mathbb{C}\big).$$

If we allow to α and β more restrictive conditions, we can obtain less restrictive assumptions on z, taking into account the following particular case of Theorem. 5.13 in Favaron, Favini [1], that we recall as a lemma.

Lemma 6.2. *Suppose $0 < \beta \le \alpha \le 1$ and (ii) to hold with $5\alpha+2\beta > 6$. Suppose $K \in C^s\big([0,r];\mathbb{C}\big)$, $s \in \big((3-2\alpha-\beta)/\alpha, 1\big)$, $f(0) - Ly_0 \in Y_\gamma \in \big\{\big(X, D(A)\big)_{\gamma,\infty}, X^\gamma_A\big\}$, with $\gamma \in (5-3\alpha-2\beta, 1)$. Let $\tau = \max\big\{s, (\alpha+\beta+\gamma-2)/\alpha\big\}$ and introduce the interval $I_{\alpha,\beta,\tau}$ by*

$$I_{\alpha,\beta,\tau} = \begin{cases} \big((3-2\alpha-\beta)/\alpha, \tau\big), & \text{if } \tau \in \big((3-2\alpha-\beta)/\alpha, 1/2\big),\\ \big((3-2\alpha-\beta)/\alpha, 1/2\big), & \text{if } \tau \in [1/2, 1). \end{cases}$$

Then for every fixed $\delta \in I_{\alpha,\beta,\tau}$*, problem*

$$\frac{d}{dt}\big(My(t)\big) + Ly(t) = (K * L_1)(y)(t) + f(t), \quad t \in [0,r], \qquad (My)(0) = My_0, \tag{6.7}$$

admits a unique strict solution $y \in C^\delta\big([0,r];D(L)\big)$ *such that* $y(0) = y_0$ *and* $(My)' \in C^\delta\big([0,r];X\big)$*, provided that* $f \in C^\mu\big([0,r];X\big)$*,* $\mu \in \big[\delta + (3-2\alpha-\beta)/\alpha, 1\big)$*. As a consequence, we have the result on* (6.1)$\sim$(6.3) *as follows.*

Theorem 6.3. *Suppose* $0 < \beta \le \alpha \le 1$*,* $3\alpha + 3\beta > 5$*,* $\theta \in \big(6 - 3(\alpha+\beta), 1\big)$*,* $z \in X_A^\theta$*,* $\Phi \in X^*$*,* $\Phi[z] \neq 0$*,* $y_0 \in D(L)$*,* $h(0) - Ly_0 \in X_A^\theta \cup \big(X, D(A)\big)_{\theta,\infty}$*. Let* $K \in C^s\big([0,r];\mathbb{C}\big)$ *and let* $\tau^* = \max\big\{s, (\alpha+2\beta+\theta-3)/\alpha\big\}$*. Let* I_{α,β,τ^*} *be the interval introduced previously. Then for every fixed* $\delta \in I_{\alpha,\beta,\tau^*}$*, for all* $g \in C^{1+\mu}\big([0,r];\mathbb{C}\big)$*,* $h \in C^\mu\big([0,r];X\big)$*,* $\mu \in \big[\delta + (3-2\alpha-\beta)/\alpha, 1\big)$*, problem* (6.4)*,* (6.5) *admits a unique strict solution* $x \in C^\delta\big([0,r];D(L)\big)$*,* $(Mx)' \in C^\delta\big([0,r];X\big)$*.*

Proof. All is reduced to solve the initial value problem

$$\begin{aligned}
&\frac{d}{dt}\big(Mx(t)\big) + (kM + L + L_1)x(t) \\
&\qquad = \int_0^t K_1(t-s)L_2x(s)\,ds + \frac{g_1'(t)}{\Phi[z]}\,z - \frac{\Phi\big[h_1(t)\big]}{\Phi[z]}\,z + h_1(t), \qquad 0 \le t \le r, \\
&(Mx)(0) = My_0
\end{aligned}$$

Since $K \in C^s\big([0,r];\mathbb{C}\big)$, $(\alpha+2\beta-3)/\alpha < s < 1$, $g \in C^{1+\mu}\big([0,r];\mathbb{C}\big)$, $h \in C^\mu\big([0,r];X\big)$, $z \in X_A^\theta \hookrightarrow X_{\tilde{A}}^{\theta-1+\beta}$, then

$$\frac{g'(0)}{\Phi[z]}\,z - \frac{\Phi\big[h_1(0)\big]}{\Phi[z]}\,z + h_1(0) - (kM + L + L_1)y_0 \in X_A^\theta \subset X_{\tilde{A}}^{\theta-1+\beta}.$$

$\theta - 1 + \beta$ ($= \gamma$ in the Lemma) $\in (5 - 3\alpha - 2\beta, 1) \iff \theta \in \big(6 - 3(\alpha+\beta), 2-\beta\big)$, $3\alpha + 2\beta > 4$.

Moreover, since $X_A^\theta \hookrightarrow \big(X, D(A)\big)_{\theta,\infty}$ and $\big(X, D(A)\big)_{\theta,\infty} = \big(X, D(\tilde{A})\big)_{\theta,\infty}$,

$$\frac{g_1'(0)}{\Phi[z]}\,z - \frac{\Phi\big[h_1(0)\big]}{\Phi[z]}\,z + h_1(0) - (kM + L + L_1)y_0 \in \big(X, D(A)\big)_{\theta,\infty}.$$

We conclude that

$$\frac{g_1'(0)}{\Phi[z]}\,z - \frac{\Phi\big[h_1(0)\big]}{\Phi[z]}\,z + h_1(0) - (kM + L + L_1)y_0 \in \big(X, D(A)\big)_{\theta,\infty} \cup X_A^\theta \subseteq \big(X, D(\tilde{A})\big)_{\theta,\infty} \cup X_{\tilde{A}}^{\theta-1+\beta}.$$

Recall that, in our case, γ in Lemma 6.2 is $\theta - 1 + \beta$, so that

$$\tau^* = \max\big\{s, (\alpha + \beta + \theta - 1 + \beta - 2)/\alpha\big\} = \max\big\{s, (\alpha + 2\beta + \theta - 3)/\alpha\big\};$$

I_{α,β,τ^*} is introduced correspondingly as in Lemma 6.2. Hence we deduce that, for every fixed $\delta \in I_{\alpha,\beta,\tau^*}$, problem (6.4), (6.5) admits a unique strict solution $x \in C^\delta\big([0,r];D(L)\big)$ such that $(Mx)' \in C^\delta\big([0,r];X\big)$, provided that

$$\frac{g_1'(\cdot)}{\Phi[z]}\,z - \frac{\Phi\big[h_1(\cdot)\big]}{\Phi[z]}\,z + h_1(\cdot) \in C^\mu\big([0,r];X\big),$$

$\mu \in \big[\delta + (3-2\alpha-3\beta)/\alpha, 1\big)$, i.e. just $g \in C^{1+\mu}\big([0,r];\mathbb{C}\big)$, $h \in C^\mu\big([0,r];X\big)$. □

As a consequence, let $0 < \beta \le \alpha \le 1$, $3\alpha + 4\beta > 6$, $\theta \in \big(6 - 3(\alpha+\beta), \beta\big)$, $z \in X_A^\theta$, $\Phi \in X^*$, $\Phi[z] \neq 0$, $y_0 \in D(L)$, $h(0) - Ly_0 \in X_A^\theta$ or $h(0) - Ly_0 \in \big(X, D(A)\big)_{\theta,\infty}$, $K \in C^s\big([0,r];\mathbb{C}\big)$, $s \in \big[(3-2\alpha-\beta)/\alpha, 1\big)$. Fixed $\delta \in I_{\alpha,\beta,\tau^*}$, for all $g \in C^{1+\mu}\big([0,r];\mathbb{C}\big)$, $h \in C^\mu\big([0,r];X\big)$, where $\mu \in \big[\delta + (3-2\alpha-\beta)/\alpha, 1\big)$, problem (6.1)$\sim$(6.3) admits a unique solution (y, f) such that $Ly \in C^\delta\big([0,r];X\big)$, $(My)' \in C^\delta\big([0,r];X\big)$ and $f \in C^\delta\big([0,r];\mathbb{C}\big)$.

6.2 Examples of applications to PDE

Let Ω be a bounded open set in $\mathbb{R}^n$ with a C^2 boundary $\partial\Omega$, $1 < p < \infty$. Let

$$\mathcal{L} = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial}{\partial x_i}\right) + \sum_{i=1}^{n} a_i(x)\frac{\partial}{\partial x_i} + a_0(x),$$

be a second order differential operator such that a_{ij}, a_i are real-valued functions such that

$$a_{ij}, \frac{\partial a_{ij}}{\partial x_j}, a_i, \frac{\partial a_i}{\partial x_i}, a_0 \in C(\overline{\Omega}), \qquad i,j = 1,\dots,n,$$

$\big(a_{ij}(x)\big)$ is a positive definite symmetric matrix for each $x \in \overline{\Omega}$, $a_0(x) \geq C_1$, $a_0(x) - \sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i} \geq C_1$, for $x \in \overline{\Omega}$, with some positive constant C_1.

Let b^0 be a real-valued function belonging to $C(\partial\Omega)$, such that, for $x \in \partial\Omega$, $b^0(x) \geq 0$, $b^0(x) + \sum_{i=1}^{n} a_i(x)\nu_i(x) \geq 0$ when $\nu = (\nu_1,\dots,\nu_n)$ is the outer normal vector to $\partial\Omega$.

The operator L_p is the realization of $\mathcal{L}$ in $L^p(\Omega)$, with Robin boundary conditions, defined by

$$D(L_p) = \left\{u \in W^{2,p}(\Omega) \colon \sum_{i,j=1}^{n} a_{ij}(\cdot)\nu_j \frac{\partial u}{\partial x_j} + b^0(\cdot)u = 0 \text{ on } \partial\Omega\right\},$$

$$L_p u = \mathcal{L}u, \qquad \text{for } u \in D(L_p).$$

If $m(x)$ is a non-negative bounded measurable function on $\overline{\Omega}$, let us denote by M_p the operator of multiplication by $m(\cdot)$ in $L^p(\Omega)$.

We assume that $m \in C^1(\overline{\Omega})$ and satisfies $|\nabla m(x)| \leq C^0 m(x)^\rho$, $x \in \overline{\Omega}$, where C^0 is a positive constant, ρ satisfies $2 - p < \rho < 1$.

In the forthcoming paper [8] from Favini, Lorenzi, Tanabe it is shown that

$$\begin{aligned}
&\big\|M_p(\zeta M_p + L_p)^{-1}\big\|_{\mathcal{L}(L^p(\Omega))} \leq C_{1,p}\big(|\zeta| + 1\big)^{-(2-\rho)^{-1}}, && \text{if } p \leq 2,\\
&\big\|M_p(\zeta M_p + L_p)^{-1}\big\|_{\mathcal{L}(L^p(\Omega))} \leq C_{2,p}\big(|\zeta| + 1\big)^{-2[p(2-\rho)]^{-1}}, && \text{if } p > 2,
\end{aligned}$$

without the additional assumptions in Favini, Lorenzi, Tanabe, Yagi [5], [6].

Notice that if $m(\cdot) \in C^1(\overline{\Omega})$ and $k \in \mathbb{N}$, then

$$\big|\nabla\big(m(x)^k\big)\big| = km(x)^{k-1}\,|\nabla m(x)| = k\big[m(x)^k\big]^{1-1/k}\,|\nabla m(x)| \leq C\big[m(x)^k\big]^{1-1/k},$$

and $1 - 1/k \to 1$ as $k \to \infty$.

Then all our abstract results in the previous Sections apply to the identification problem

$$\begin{aligned}
&\frac{d}{dt}\big((M_p y)(t,x)\big) + (L_p y)(t,x) = f(t)z(x) + h(t,x), && t \in [0,r], \quad x \in \Omega,\\
&(M_p y)(0,x) = m(x)y_0(x), && x \in \Omega,\\
&\int_\Omega \eta(x)(M_p y)(t,x)\,dx = g(t), && t \in [0,r],
\end{aligned}$$

where $\eta \in L^{p'}(\Omega)$, $1/p + 1/p' = 1$.

Analogously, if we take the operator $\tilde{L}_p$ as previously, but with Dirichlet boundary conditions in L^p and $L_1 = \Delta + c(\cdot)$, where $c(x) \leq 0$ is continuous on $\overline{\Omega}$, we can handle the inverse problem (see [7], Favini, Lorenzi, Tanabe)

$$\begin{aligned}
&\frac{d}{dt}\big((M_p y)(t,x)\big) + (L_p y)(t,x) = \int_0^t K(t-s)(L_1 y)(s,x)\,ds + f(t)z(x) + h(t,x), && t \in [0,r], \quad x \in \Omega,\\
&(M_p y)(0,x) = m(x)y_0(x), && x \in \Omega,\\
&\int_\Omega \eta(x)(M_p y)(t,x)\,dx = g(t), && t \in [0,r],
\end{aligned}$$

In this case, z is to be taken in $X_A^\theta \cap \big(L^p(\Omega), W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega)\big)_{\sigma,\infty}$.

The details are left to the reader.

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A variational approach to gradient flows

Goro Akagi[1]

Dedicated to Professor Hiroki Tanabe on the occasion of his 80th birthday

Abstract

This note is concerned with a variational approach to gradient flows of nonconvex energies in Hilbert spaces H. We employ the *Weighted Energy-Dissipation* (*WED* for short) *functional*, which consists of dissipation functional and energy functional with exponential weight in time and which is defined for each orbit $u : [0,T] \to H$ satisfying initial condition. In this approach, gradient flows will be obtained as a limit of minimizers u_ε of WED functionals $\mathcal{W}_\varepsilon$ as a parameter ε goes to zero.

This note is based on a joint work [1] with Ulisse Stefanelli (IMATI-CNR, Italy).

1 Variational formulation of gradient flows

Gradient flow is a major principle in the descriptions of various sorts of phenomena (e.g., phase transition), and there have been a large number of contributions from numerous points of view. On the other hand, *variational principle* is also another major principle, and particularly, it would be the most universal principle of physics. Gradient flows could be regarded as a principle describing more transitional phase and they are not necessarily formulated as a variational principle in a natural way. However, in this study, we are making an attempt to pursue (natural) variational principles to describe gradient flows.

Let H be a Hilbert space and let $E : H \to \mathbb{R}$ be an energy functional. Gradient flows $u : [0,T] \to H$ of E are generated by the evolution equation

$$u'(t) = -\mathrm{d}E(u(t)) \quad \text{in } H, \quad t \in (0,T), \tag{1}$$

where $u' = \mathrm{d}u/\mathrm{d}t$ and $\mathrm{d}E$ denotes a functional derivative of E in a proper sense. Let us give a typical example below.

Example 1.1 (Allen-Cahn equation). Set $H = L^2(\Omega)$ with a domain $\Omega \subset \mathbb{R}^N$ and define

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x + \int_\Omega W(u(x))\mathrm{d}x \quad \text{for } u \in D(E) \subset H$$

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with a double-well potential $W : \mathbb{R} \to \mathbb{R}$ and $D(E) = \{u \in H^1_0(\Omega) \colon W(u(\cdot)) \in L^1(\Omega)\}$. Then the evolution equation (1) corresponds to the Allen-Cahn equation

$$\partial_t u - \Delta u + W'(u) = 0 \quad \text{in } \Omega \times (0, T)$$

with the homogeneous Dirichlet boundary condition. □

On the other hand, equilibrium states ϕ of the evolution equation (1) are formulated in a variational fashion,

$$\mathrm{d}E(\phi) = 0.$$

In this note, we shall present a variational formulation for gradient flows. Such variational formulations can be realized in a couple of manners such as

(i) time-discretization

(ii) Brézis-Ekeland's principle (see [5, 6])

(iii) *Weighted Energy-Dissipation* (*WED* for short) *functional*

Example 1.2 (Time-discretization of gradient systems)**.** One can incrementally obtain a next step u_n from the previous step u_{n-1} by solving the semi-discretized problem for (1),

$$\frac{u_n - u_{n-1}}{h} = -\mathrm{d}E(u_n),$$

which is an Euler-Lagrange equation of the functional

$$I_n(w) := \frac{1}{2}|w|^2_H + hE(w) - (u_{n-1}, w)_H \quad \text{for } w \in H. \quad \square$$

Example 1.3 (Brézis-Ekeland's variational principle [5, 6])**.** Let ϕ be a proper lower semicontinuous convex functional on a Hilbert space H. Then Brézis and Ekeland found the following relation,

$$u'(t) + \partial\phi(u(t)) \ni 0, \quad u(0) = u_0 \quad \text{iff} \quad J(u) = \inf_{D(J)} J = 0,$$

where J is a functional on $L^2(0, T; H)$ given by

$$J(u) := \int_0^T \Big(\phi(u(t)) + \phi^*(-u'(t))\Big)\mathrm{d}t + \frac{1}{2}|u(T)|^2_H - \frac{1}{2}|u_0|^2_H$$

with the domain $D(J) := \{u \in W^{1,2}(0, T; H) \colon u(0) = u_0,\ \phi(u(\cdot)), \phi^*(-u'(\cdot)) \in L^1(0, T)\}$, where ϕ^* is the convex conjugate of ϕ defined by

$$\phi^*(v) := \sup_{u \in H} \{(v, u)_H - \phi(u)\} \quad \text{for } v \in H. \quad \square$$

In this note, we address ourselves to the third approach based on the Weighted Energy-Dissipation (WED) functional. Let us briefly exhibit an overview of this approach. Let $E : H \to \mathbb{R}$ be a *convex* energy and consider the Cauchy problem,

$$\begin{cases} u'(t) = -\mathrm{d}E(u(t)) \quad \text{in } H, \quad 0 < t < T, \\ u(0) = u_0. \end{cases} \tag{2}$$

An WED functional is defined for (2) as follows:

$$\mathcal{W}_\varepsilon(u) := \int_0^T e^{-t/\varepsilon} \left(\frac{\varepsilon}{2} |u'(t)|_H^2 + E(u(t)) \right) \mathrm{d}t$$

for $u \in \mathcal{H} := L^2(0, T; H)$ satisfying $u(0) = u_0$. The minimization approach using WED functionals is formulated as follows: Let u_ε be a *unique* (by convexity) minimizer of $\mathcal{W}_\varepsilon(u)$ subject to $u(0) = u_0$. Then the minimizer u_ε approximates a gradient flow u of E for $\varepsilon > 0$ sufficiently small (more precisely, $u_\varepsilon \to u$ strongly in $C([0, T]; H)$ as $\varepsilon \to 0$). In [14], the convergence of minimizers has already been proved for convex, lower semicontinuous energy functionals in a subdifferential framework.

Example 1.4 (WED approach to the heat equation)**.** As a simplest example, let us treat the heat equation,

$$(\text{Heat}) \begin{cases} \partial_t u - \Delta u = 0 \quad \text{in } Q := \Omega \times (0, T), \\ u|_{\partial\Omega} = 0, \quad u(\cdot, 0) = u_0. \end{cases}$$

For each $\varepsilon > 0$, let us define

$$\mathcal{W}_\varepsilon(u) := \iint_Q e^{-t/\varepsilon} \left(\frac{\varepsilon}{2} |\partial_t u|^2 + \frac{1}{2} |\nabla u|^2 \right) \mathrm{d}x\mathrm{d}t$$

for $u \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ with $u(\cdot, 0) = u_0$. Then the Euler-Lagrange equation $\mathrm{d}\mathcal{W}_\varepsilon(u) = 0$ provides elliptic-in-time regularizations of (Heat):

$$(\text{Heat})_\varepsilon \begin{cases} -\varepsilon \partial_t^2 u + \partial_t u - \Delta u = 0 \quad \text{in } Q, \\ u|_{\partial\Omega} = 0, \quad u(\cdot, 0) = u_0, \quad \partial_t u(\cdot, T) = 0. \end{cases}$$

Indeed, for smooth test functions ϕ satisfying $\phi(0) = 0$ (from initial constraint), we observe that

$$\begin{aligned}
&(\mathrm{d}\mathcal{W}_\varepsilon(u), \phi)_{L^2(Q)} = \iint_Q e^{-t/\varepsilon} \left(\varepsilon \partial_t u \partial_t \phi + \nabla u \cdot \nabla \phi \right) \mathrm{d}x\mathrm{d}t \\
&= \int_\Omega \varepsilon e^{-t/\varepsilon} \partial_t u \phi \, \mathrm{d}x \Big|_{t=0}^{t=T} - \iint_Q \varepsilon \partial_t \left(e^{-t/\varepsilon} \partial_t u \right) \phi \, \mathrm{d}x\mathrm{d}t \\
&\quad - \iint_Q e^{-t/\varepsilon} \Delta u \phi \, \mathrm{d}x\mathrm{d}t \\
&= \int_\Omega \varepsilon e^{-T/\varepsilon} \partial_t u(T) \phi(T) \, \mathrm{d}x - \iint_Q \varepsilon \left(e^{-t/\varepsilon} \partial_t^2 u - \frac{1}{\varepsilon} e^{-t/\varepsilon} \partial_t u \right) \phi \, \mathrm{d}x\mathrm{d}t \\
&\quad - \iint_Q e^{-t/\varepsilon} \Delta u \phi \, \mathrm{d}x\mathrm{d}t \\
&= \int_\Omega \varepsilon e^{-T/\varepsilon} \partial_t u(T) \phi(T) \, \mathrm{d}x + \iint_Q e^{-t/\varepsilon} \left(-\varepsilon \partial_t^2 u + \partial_t u - \Delta u \right) \phi \, \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

From the arbitrariness of $\phi(T)$ and ϕ, we obtain $(\mathrm{Heat})_\varepsilon$. Then $u_\varepsilon \to u$ strongly in $C([0,T];L^2(\Omega))$ as $\varepsilon \to 0$ and u solves (Heat). This fact is well known as elliptic-in-time regularizations technique (see, e.g., [12]). $\square$

Let us briefly review previous studies on this topic. Ilmanen [9] introduced *translative functional*, which is based on a similar idea to WED functionals, to prove the partial regularity of Brakke mean curvature flow of varifolds. Hirano [8] also employed functionals similar to WED functionals to verify the existence of periodic solutions for nonlinear evolution equations. The term "WED functional" was first introduced by Mielke-Ortiz and Conti-Ortiz in [13] and [7], where rate-independent systems are studied. Moreover, Mielke-Stefanelli [14] provided an WED approach to gradient flows of the form $u' + \partial\phi(u) \ni f$ in a Hilbert space H for convex energies ϕ and proved the convergence of minimizers as $\varepsilon \to 0$. Furthermore, Akagi-Stefanelli [3, 2] also presented an WED formulation for generalized gradient flows, $\mathrm{d}\psi(u')+\partial\phi(u) \ni 0$ with convex dissipation functional ψ and convex energy ϕ in a Banach space setting. Recently, Rossi et al [18] established an WED approach to metric gradient flows. On the other hand, the WED approach is also applicable to other types of problems, e.g., wave equations and Lagrange systems (see [11]). As for semilinear wave equations, the convergence of minimizers of the corresponding WED functionals is known as one of De Giorgi's conjectures. Stefanelli [20] first treated this issue and Serra-Tilli [19] (almost) completely proved the conjecture. Here we emphasize that all these works are done for convex (or λ-convex) energies. The aim of this note is to extend the WED framework to gradient flows for *non-convex energies*.

2 Main results

Let H be a Hilbert space and let $E : H \to (-\infty,\infty]$ be a *non-convex* energy of the form

$$E(u) := \varphi^1(u) - \varphi^2(u) \quad \text{for } \ u \in D(E) := D(\varphi^1) \cap D(\varphi^2)$$

with proper lower semicontinuous convex functionals $\varphi^1, \varphi^2 : H \to [0,\infty]$ and *effective domains* $D(\varphi^i) := \{u \in H : \varphi^i(u) < \infty\}$.

Let us consider gradient flows $u : [0,T] \to H$ of E generated by

$$(\mathrm{GF}) \begin{cases} u'(t) + \partial\varphi^1(u(t)) - \partial\varphi^2(u(t)) = 0 \ \text{ in } H, \quad 0 < t < T, \\ u(0) = u_0, \end{cases}$$

where $\partial\varphi^i : H \to H$ is the *subdifferential operator* of φ^i $(i = 1, 2)$ defined by

$$\partial\varphi^i(u) := \{\xi \in H : \varphi^i(v) - \varphi^i(u) \geq (\xi, v-u)_H \quad \forall v \in D(\varphi^i)\} \quad \text{for } \ u \in D(\varphi^i)$$

with the domain $D(\partial\varphi^i) := \{u \in D(\varphi^i) : \partial\varphi^i(u) \neq \emptyset\}$ (throughout this paper, we assume that $\partial\varphi^i$ is single-valued). We refer the reader to [15, 10, 16, 17] for the existence of solutions and their large-time behaviors for the Cauchy problem (GF).

We define the WED functional $\mathcal{W}_\varepsilon$ for (GF) as follows:

$$\mathcal{W}_\varepsilon(u) := \begin{cases} \displaystyle\int_0^T e^{-t/\varepsilon}\left(\frac{\varepsilon}{2}|u'(t)|_H^2 + E(u(t))\right) \mathrm{d}t \\ \qquad\qquad \text{if} \quad u \in W^{1,2}(0,T;H),\ E(u(\cdot)) \in L^1(0,T), \\ \qquad\qquad\qquad u(0) = u_0,\ u(t) \in D(E) \text{ for a.e. } t \in (0,T), \\ \infty \qquad\quad \text{else} \end{cases}$$

for $u \in \mathcal{H} := L^2(0,T;H)$. The target issues of the present note are to prove the following items:

- the existence of minimizers u_ε of $\mathcal{W}_\varepsilon$ for any $\varepsilon > 0$,
- the convergence of minimizers u_ε to a limit u as $\varepsilon \to \infty$,
- the limit u is a solution of (GF).

Denote by $\mathcal{L}$ the set of non-increasing functions in $\mathbb{R}$ and let us introduce the following assumptions:

(A0) φ^1, φ^2 are proper, lower semicontinuous and convex in H. Moreover, $\partial\varphi^1$ and $\partial\varphi^2$ are single-valued.

(A1) There exist a Banach space X compactly embedded in H and $\ell_1 \in \mathcal{L}$ such that

$$|u|_X \le \ell_1(|u|_H)\left(\varphi^1(u) + 1\right) \quad \forall u \in D(\varphi^1).$$

(A2) There exist $k_1 \in (0,1)$ and $C_1 \ge 0$ such that

$$\varphi^2(u) \le k_1\varphi^1(u) + C_1 \quad \forall u \in D(\varphi^1).$$

(A3) There exist $k_2 \in (0,1)$ and $\ell_2 \in \mathcal{L}$ such that

$$|\partial\varphi^2(u)|_H^2 \le k_2|\partial\varphi^1(u)|_H^2 + \ell_2(|u|_H)\left(\varphi^1(u) + 1\right) \quad \forall u \in D(\partial\varphi^1).$$

Remark 2.1 (Meaning of assumptions)**.** (i) Roughly speaking, (A1) means the compactness in H of sublevel sets $[\varphi^1 \le \lambda]$ given by

$$[\varphi^1 \le \lambda] := \left\{u \in H \colon \varphi^1(u) \le \lambda\right\} \quad \text{for} \ \ \lambda \in \mathbb{R}$$

of φ^1.

(ii) By (A2), the energy functional E is bounded from below. Indeed,

$$E(u) = \varphi^1(u) - \varphi^2(u) \overset{\text{(A2)}}{\ge} (1-k_1)\varphi^1(u) - C_1 \quad \forall u \in D(\varphi^1).$$

Moreover by (A1), sublevels $[E \le \lambda]$ of E are also precompact in H.

(iii) (A3) means that the anti-monotone part $-\partial\varphi^2$ is dominated by the monotone part $\partial\varphi^1$. $\square$

Now our main result reads,

Theorem 2.2 (WED formulation of (GF))**.** *Suppose that* (A0)–(A3) *hold and* $u_0 \in D(\partial\varphi^1)$*. Then it holds that*:

(i) *For each* $\varepsilon > 0$*,* $\mathcal{W}_\varepsilon$ *admits a global minimizer* u_ε *over* $\mathcal{H}$*.*

Let u_ε *be a global or local minimizer of* $\mathcal{W}_\varepsilon$*.*

(ii) *Then* u_ε *belongs to* $W^{2,2}(0,T;H)$ *and solves*

$$\begin{cases} -\varepsilon u_\varepsilon''(t) + u_\varepsilon'(t) + \partial\varphi^1(u_\varepsilon(t)) - \partial\varphi^2(u_\varepsilon(t)) = 0 \quad \text{in } H, \quad 0 < t < T, \\ u_\varepsilon(0) = u_0, \quad u_\varepsilon'(T) = 0. \end{cases}$$

(iii) *There exists* $u \in W^{1,2}(0,T;H)$ *such that, up to a subsequence,*

$$\begin{aligned} u_\varepsilon \to u \qquad & \text{strongly in } C([0,T];H), \\ & \text{weakly in } W^{1,2}(0,T;H) \qquad \text{as } \ \varepsilon \to 0. \end{aligned}$$

Moreover, the limit u *solves* (GF)*.*

Remark 2.3 (Assertions (i) and (ii) are not obvious)**.** The existence of minimizers of $\mathcal{W}_\varepsilon$ is not obvious, as $\mathcal{W}_\varepsilon$ might not be lower semicontinuous in $\mathcal{H}$. Moreover, from the non-smoothness of $\mathcal{W}_\varepsilon$, it would be difficult to directly calculate $\mathrm{d}\mathcal{W}_\varepsilon$. Hence (ii) is also not obvious.

3 Application to Allen-Cahn equations

Let us consider the following Allen-Cahn equation:

$$\text{(AC)} \begin{cases} \partial_t u - \Delta u + W'(u) = 0 \ \text{ in } Q := \Omega \times (0,T), \\ u|_{\partial\Omega} = 0, \quad u(\cdot, 0) = u_0, \end{cases}$$

where $W'(u) = |u|^{m-2}u - |u|^{q-2}u$ $\ (1 < q < m < \infty \ )$ and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain.

Set $H = L^2(\Omega)$ and an energy functional,

$$E(u) := \varphi^1(u) - \varphi^2(u)$$

with convex parts φ^1, $\varphi^2 : H \to [0,\infty]$ given by

$$\varphi^1(u) := \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2 \mathrm{d}x + \dfrac{1}{m}\displaystyle\int_\Omega |u|^m \mathrm{d}x & \text{if } \ u \in H_0^1(\Omega) \cap L^m(\Omega), \\ \infty & \text{else,} \end{cases}$$

$$\varphi^2(u) := \begin{cases} \dfrac{1}{q}\displaystyle\int_\Omega |u|^q \mathrm{d}x & \text{if } \ u \in L^q(\Omega), \\ \infty & \text{else.} \end{cases}$$

Then (AC) is reduced into the Cauchy problem,

$$u'(t) + \partial\varphi^1(u(t)) - \partial\varphi^2(u(t)) = 0 \ \text{ in } H, \quad 0 < t < T, \quad u(0) = u_0.$$

Let us briefly check (A0)–(A3) for this setting. (A0) follows immediately from well-known facts. (A1) with $X = H_0^1(\Omega) \cap L^m(\Omega)$ follows from Rellich-Kondrachov's compactness lemma. As for (A2), since $q < m$, for any $k_1 > 0$ there exists $C_1 \geq 0$ such that

$$\varphi^2(u) = \frac{1}{q}\int_\Omega |u|^q \mathrm{d}x \leq \frac{k_1}{m}\int_\Omega |u|^m \mathrm{d}x + C_1 \leq k_1\varphi^1(u) + C_1.$$

Concerning (A3), for any $k_2 > 0$ one can take $C_2 \geq 0$ such that

$$\begin{aligned}|\partial\varphi^2(u)|_H^2 = \left||u|^{q-2}u\right|_{L^2}^2 &= \int_\Omega |u|^{2(q-1)}\mathrm{d}x \\ &\leq k_2\int_\Omega |u|^{2(m-1)}\mathrm{d}x + C_2 \\ &\leq k_2|\partial\varphi^1(u)|_H^2 + C_2.\end{aligned}$$

Therefore Theorem 2.2 is applicable to (AC).

4 Outline of proof

In this section, we give an outline of proof for Theorem 2.2.

Step 1. Construction of a solution of (EL) by minimization of approximated WED functionals.

Define approximated functionals for $\mathcal{W}_{\varepsilon,\lambda}$ by

$$\mathcal{W}_{\varepsilon,\lambda}(u) := \int_0^T e^{-t/\varepsilon}\left(\frac{\varepsilon}{2}|u'(t)|_H^2 + \varphi^1(u(t)) - \varphi^2_\lambda(u(t))\right)\mathrm{d}t$$

for $u \in W^{1,2}(0,T;H)$ satisfying $\varphi^1(u(\cdot)) \in L^1(0,T)$ and $u(0) = u_0$. Here φ^2_λ stands for the Moreau-Yosida regularization of φ^2 (see, e.g., [4]). Since φ^2_λ is Fréchet differentiable in H, the WED functionals $\mathcal{W}_{\varepsilon,\lambda}$ admit minimizers u_λ, and moreover, u_λ solves

$$(\mathrm{EL})_\lambda \begin{cases} -\varepsilon u_\lambda''(t) + u_\lambda'(t) + \partial\varphi^1(u_\lambda(t)) - \partial\varphi^2_\lambda(u_\lambda(t)) = 0 \ \text{ in } H, \quad 0 < t < T, \\ u_\lambda(0) = u_0, \quad u_\lambda'(T) = 0. \end{cases}$$

Finally, we prove that $u_\lambda \to u$ strongly in $C([0,T];H)$ as $\lambda \to 0$ and u solves (EL).

Step 2. The solution u minimizes $\mathcal{W}_\varepsilon$.

Since u_λ minimizes $\mathcal{W}_{\varepsilon,\lambda}$, we see that

$$\mathcal{W}_{\varepsilon,\lambda}(u_\lambda) \le \mathcal{W}_{\varepsilon,\lambda}(v) \quad \forall v \in \mathcal{H}.$$

Noting that $\varphi^2_\lambda(u) \le \varphi^2(u)$ and $u_\lambda \to u$, we have

$$\mathcal{W}_{\varepsilon,\lambda}(u_\lambda) \ge \mathcal{W}_\varepsilon(u_\lambda) \quad \text{and} \quad \liminf_{\lambda \to 0} \mathcal{W}_\varepsilon(u_\lambda) \ge \mathcal{W}_\varepsilon(u).$$

Moreover, it also follows that

$$\mathcal{W}_{\varepsilon,\lambda}(v) \to \mathcal{W}_\varepsilon(v) \quad \text{as} \ \ \lambda \to 0.$$

Combining these facts, we deduce that

$$\mathcal{W}_\varepsilon(u) \le \mathcal{W}_\varepsilon(v) \quad \forall v \in \mathcal{H}.$$

Therefore u minimizes $\mathcal{W}_\varepsilon$. Thus the assertion (i) of Theorem 2.2 is proved.

Step 3. Every minimizer u_ε of $\mathcal{W}_\varepsilon$ solves (EL).

In case the minimizer of $\mathcal{W}_\varepsilon$ is unique, u_ε coincides with the solution of (EL) obtained in Step 1.

In case u_ε is one of global minimizers, for $\delta > 0$ let us set

$$\hat{\mathcal{W}}_\varepsilon(u) := \mathcal{W}_\varepsilon(u) + \frac{1}{2\delta} \int_0^T e^{-t/\varepsilon} |u(t) - u_\varepsilon(t)|_H^2 \, \mathrm{d}t.$$

Then u_ε becomes a unique global minimizer of $\hat{\mathcal{W}}_\varepsilon$. Therefore u_ε solves the Euler-Lagrange equation for $\hat{\mathcal{W}}_\varepsilon$,

$$-\varepsilon u'' + u' + \partial\varphi^1(u) - \partial\varphi^2(u) + \frac{1}{\delta}(u - u_\varepsilon) = 0.$$

Hence u_ε solves (EL) as well (to this end, we should repeat the preceding argument for (1) with a forcing term $f(t)$ and it is possible).

In case u_ε is a local minimizer, one can similarly obtain the conclusion. Thus (ii) is proved.

Step 4. Convergence of minimizers u_ε as $\varepsilon \to 0$

Here we only show how to obtain uniform estimates for u_ε. For simplicity, we omit the subscript ε. Multiply (EL) by u' to get

$$-\varepsilon(u'', u')_H + |u'|_H^2 + \frac{\mathrm{d}}{\mathrm{d}t}\varphi^1(u) - \frac{\mathrm{d}}{\mathrm{d}t}\varphi^2(u) = 0.$$

Noting $(u'', u')_H = (1/2)(\mathrm{d}/\mathrm{d}t)|u'|_H^2$ and integrating it over $(0, T)$, we deduce that

$$\frac{\varepsilon}{2}|u'(0)|_H^2 + \int_0^T |u'(t)|_H^2 \mathrm{d}t + E(u(T)) = \frac{\varepsilon}{2}|u'(T)|_H^2 + E(u_0).$$

Here we note that $E(u(T)) \geq (1-k_1)\varphi^1(u(T)) - C_1$ by (A1) and $u'(T) = 0$. Hence we get

$$\varepsilon|u'(0)|_H^2 + \int_0^T |u'(t)|_H^2 \mathrm{d}t \leq C.$$

Moreover, integrating (EL) $\times u'(t)$ over $(0,t)$, we have

$$\begin{aligned}\frac{\varepsilon}{2}|u'(0)|_H^2 + \int_0^t |u'(\tau)|_H^2 \mathrm{d}\tau + (1-k_1)\varphi^1(u(t)) \\ \leq C_1 + \frac{\varepsilon}{2}|u'(t)|_H^2 + E(u_0).\end{aligned}$$

Integrate both sides over $(0,T)$ again. We obtain

$$(1-k_1)\int_0^T \varphi^1(u(t))\mathrm{d}t \leq C + \frac{\varepsilon}{2}\int_0^T |u'(t)|_H^2 \mathrm{d}t \leq C.$$

We can also derive a maximal regularity type estimate of the form

$$\begin{aligned}&\int_0^T |\varepsilon u''(t)|_H^2 \mathrm{d}t + \int_0^T |u_\varepsilon'(t)|_H^2 \mathrm{d}t + \int_0^T |\partial\varphi^1(u(t))|_H^2 \mathrm{d}t \\ &\leq \int_0^T |\partial\varphi^2(u(t))|_H^2 \mathrm{d}t + C\left(\varphi^1(u_0) + \varepsilon|u_\varepsilon'(0)|_H|\partial\varphi^1(u_0)|_H\right).\end{aligned}$$

Then by (A3),

$$\int_0^T |\partial\varphi^2(u)|_H^2 \mathrm{d}t \leq k_2\int_0^T |\partial\varphi^1(u)|_H^2 \mathrm{d}t + C\left(\int_0^T \varphi^1(u)\mathrm{d}t + 1\right).$$

Hence

$$\int_0^T |\varepsilon u''(t)|_H^2 \mathrm{d}t + \int_0^T |\partial\varphi^1(u(t))|_H^2 \mathrm{d}t \leq C.$$

Combining these estimates with monotone technique, one can prove the convergence of minimizers u_ε as $\varepsilon \to 0$ and the identification of the limit.

5 Final remarks

We close this paper with the following remarks.

(i) WED formalism enables us to apply variational tools such as direct method, other critical point theories and Γ-convergence theory to analyze gradient flows.

(ii) WED formalism provides numerical schemes for gradient flows. Actually, one may obtain approximate solutions by directly minimizing WED functionals.

(iii) As for further generalizations, one may consider the cases with energies unbounded from below, relaxation of (A0)–(A3), smoothing effect, Banach space framework.

(iv) Moreover, one can also apply Theorem 2.2 to other types of PDEs such as semilinear (resp., p-Laplace) heat equations with sublinear (resp., subprincipal) nonlinearity.

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Boundary Controllability of Nonlocal Diffusion Equations

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Dedicated to Professor Hiroki Tanabe on the occasion of his eightieth birthday.

Abstract

This paper studies a boundary controllability problem for the control system described by diffusion equation with nonlocal terms. It is shown that the associated nonlocal operator generates a C_0-semigroup in L^2-space. The deformation formula method is extended for nonlocal diffusion equations, and the structural properties of semigroups are investigated. Applying the method to the original system, the existence of Riesz basis is proved and the exact representation of semigroups is shown. Based on the representation it is proved that the boundary control system is controllable in any finite time.

1. Introduction

In this paper, we study a boundary control system described by the diffusion equation with Volterra integral and nonlocal terms both in state and boudary condition:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - p(x)u + \displaystyle\int_0^x f(x,y)u(t,y)dy + g(x)u(t,0), \quad t>0, \quad x \in (0,1), \\ -\dfrac{\partial u(t,0)}{\partial x} + hu(t,0) = 0, \\ \dfrac{\partial u(t,1)}{\partial x} + ju(t,1) + \displaystyle\int_0^1 \gamma(y)u(t,y)dy = U(t), \quad t>0, \\ u(0,x) = u_0(x), \quad x \in [0,1], \end{cases} \tag{1.1}$$

where $p, g, \gamma \in C[0,1], f \in C(\overline{D}), D = \{(x,y) : 0 < x < y < 1\}, h, j \in \mathbb{R}, u_0 \in L^2(0,1)$, and $U(t) \in \mathbb{R}$ is a boundary control input at the boundary point $x = 1$. Throughout this paper we suppose that all coefficients $p(x), g(x), \gamma(x), f(x,y), h, j$ in (1.1) are real valued. As for the control systems described by heat equations without nonlocal terms, many studies on control problems have been done since old times (cf. Curtain and Zwart [1]). Recently, in the study of boundary feedback stabilization of unstable heat equations with nonlocal terms, Krstic and Smyshlyaev [9] have constructed the integral kernel function such that the integral transformation converts the solution of nonlocal heat equation to the solution pf a simple heat equation with nonlocal boundary condition. The transformation is considered a generalization of the deformation formula due to Suzuki [10]

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which is used in the study of inverse problems for heat equations. We shall extend the deformation formula method to nonlocal equation (1.1), and investigate the structural properties of solution semigroups. Then, applying the method to the original system, the spectral properties of operators such as the existence of Riesz basis are given. Thus, it is verified that the solution semigroup generates an analytic semigroup, and the exact representation of solutions for the boundary control system (1.1) is given. Based on the representation, it is shown that the boundary control system (1.1) is approximately controllable in any finite time.

2. Solution semigroup and deformation formula

First we state the semigroup treatment of free system (1.1) with null control $U(t) \equiv 0$. Let $L^2(0,1)$ be the complex Hilbert space with inner product defined by $\langle \varphi, \psi \rangle_{L^2} := \int_0^1 \varphi(x)\overline{\psi(x)}dx$ for $\varphi, \psi \in L^2(0,1)$. The norm of $L^2(0,1)$ is denoted by $\| \cdot \|_{L^2}$. Let $H^2(0,1)$ be the Sobolev space of order 2.

In what follows we suppose that $p, g \in C^1[0,1], \gamma \in C[0,1], f \in C^1(\overline{D}),\ h, j \in \mathbb{R}$ in (1.1). We introduce the space of coefficients X by

$$X := C^1[0,1] \times C^1(\overline{D}) \times C^1[0,1] \times C[0,1] \times \mathbb{R}^2. \tag{2.1}$$

For each

$$\mathcal{P} = (p, f, g, \gamma, h, j) \in X, \tag{2.2}$$

we define the operator $A_{\mathcal{P}} : D(A_{\mathcal{P}}) \subset L^2(0,1) \to L^2(0,1)$ by

$$\begin{cases} (A_{\mathcal{P}}\varphi)(x) = \dfrac{d^2\varphi(x)}{dx^2} - p(x)\varphi(x) + \displaystyle\int_0^x f(x,y)\varphi(y)dy + g(x)\varphi(0), \\ \hfill a.e.\ x \in [0,1], \\ D(A_{\mathcal{P}}) = \Big\{ \varphi \in H^2(0,1); -\varphi'(0) + h\varphi(0) = 0, \\ \hfill \varphi'(1) + j\varphi(1) + \displaystyle\int_0^1 \gamma(y)\varphi(y)dy = 0 \Big\}. \end{cases} \tag{2.3}$$

It is verified that $A_{\mathcal{P}}$ is a densely defined closed linear operator in $L^2(0,1)$. We shall show that $A_{\mathcal{P}}$ generates a C_0-semigroup $e^{tA_{\mathcal{P}}}$ on $L^2(0,1)$. For the purpose, we calculate the adjoint operator $A_{\mathcal{P}}^*$ of $A_{\mathcal{P}}$.

Proposition 2.1 The adjoint operator $A_{\mathcal{P}}^*$ of $A_{\mathcal{P}}$ is given by

$$\begin{cases} (A_{\mathcal{P}}^*\psi)(x) = \dfrac{d^2\psi(x)}{dx^2} - p(x)\psi(x) + \displaystyle\int_x^1 f(y,x)\psi(y)dy - \gamma(x)\psi(1), \\ \hfill a.e.\ x \in [0,1], \\ D(A_{\mathcal{P}}^*) = \Big\{ \psi \in H^2(0,1)\,; -\psi'(0) + h\psi(0) - \displaystyle\int_0^1 g(y)\psi(y)dy = 0, \\ \hfill \psi'(1) + j\psi(1) = 0 \Big\}. \end{cases} \tag{2.4}$$

By showing the estimates

$$\begin{cases} \operatorname{Re}\langle A_{\mathcal{P}}\varphi, \varphi \rangle_{L^2} \le \omega \|\varphi\|_{L^2}^2, & \forall \varphi \in D(A), \\ \operatorname{Re}\langle A_{\mathcal{P}}^*\psi, \psi \rangle_{L^2} \le \omega \|\psi\|_{L^2}^2, & \forall \psi \in D(A^*) \end{cases} \tag{2.5}$$

for some $\omega \in \mathbb{R}$, we can prove that the operator $A_{\mathcal{P}}$ generates a C_0-semigroup (cf. Curtain and Zwart [1], Corollary 2.2.3.).

Theorem 2.1 The operator $A_{\mathcal{P}}$ defined by (2.3) generates a C_0-semigroup $e^{tA_{\mathcal{P}}}$ on $L^2(0,1)$.

We can prove the unique existence of $k(x,y)$ such that it transforms the semigroup $e^{tA_{\mathcal{P}}}$ associated with (1.1) to other semigroup $e^{tA_{\mathcal{Q}}}$ associated with nonlocal diffusion equation

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial x^2} - P(x)v + \displaystyle\int_0^x F(x,y)v(t,y)dy + G(x)v(t,0), \quad t>0, \quad x \in (0,1), \\ -\dfrac{\partial v(t,0)}{\partial x} + Hv(t,0) = 0 \\ \dfrac{\partial v(t,1)}{\partial x} + Jv(t,1) + \displaystyle\int_0^1 \Gamma(y)v(t,y)dy = 0, \quad t>0, \\ v(0,x) = v_0(x), \quad \text{a.e. } x \in [0,1], \end{cases} \tag{2.6}$$

where $\mathcal{Q} = (P,F,G,\Gamma,H,J) \in X$ and P,G,F,H are arbitrarily given parameters and J,Γ are prescribed parameters.

Proposition 2.2 Let $p,g \in C^1[0,1]$, $f \in C^1(\overline{D})$, $h \in \mathbb{R}$, and let $P,G \in C^1[0,1]$, $F \in C^1(\overline{D})$, $H \in \mathbb{R}$. Then there exists a unique solution $k(x,y) \in C^2(\overline{D})$ of the hyperbolic partial integro-differential equation

$$\begin{cases} k_{xx}(x,y) - k_{yy}(x,y) = (P(x)-p(y))k(x,y) + \displaystyle\int_y^x k(x,\xi)f(\xi,y)d\xi \\ \qquad\qquad - \displaystyle\int_y^x k(\xi,y)F(x,\xi)d\xi + f(x,y) - F(x,y), \quad (x,y) \in D, \\ k_y(x,0) = hk(x,0) - g(x) - \displaystyle\int_0^x k(x,y)g(y)dy + G(x), \quad x \in [0,1], \\ k(x,x) = (H-h) + \dfrac{1}{2}\displaystyle\int_0^x (P(y)-p(y))dy, \quad x \in [0,1]. \end{cases} \tag{2.7}$$

The resolvent kernel of $k(x,y)$ is given by the following proposition. For a proof, see Miller [4].

Proposition 2.3 Let $k(x,y) \in C^2(\overline{D})$ be the kernel function in Proposition 2.2, and let $\psi \in L^2(0,1)$. Then the Volterra integral equation

$$\varphi(x) + \int_0^x k(x,y)\varphi(y)dy = \psi(x), \quad \text{a.e. } x \in [0,1], \tag{2.8}$$

admits a unique solution $\varphi(x) \in L^2(0,1)$ given by

$$\varphi(x) = \psi(x) + \int_0^x r(x,y)\psi(y)dy, \quad \text{a.e. } x \in [0,1], \tag{2.9}$$

where the resolvent kernel $r \in C^2(\overline{D})$ in (2.4) is defined by the solution of

$$r(x,y) = -k(x,y) - \int_y^x r(x,\xi)k(\xi,y)d\xi, \quad (x,y) \in \overline{D}. \tag{2.10}$$

The following theorem gives a parabolic deformation formula (cf. Suzuki [10], Nakagiri [7]).

Theorem 2.2 Let $(p, f, g, \gamma, h, j) \in X$ in (1.1). For any $P, G \in C^1[0,1]$, $F \in C^1(\overline{D})$, $H \in \mathbb{R}$, we define J and $\Gamma(y)$ by

$$J := j - (H - h) - \int_0^1 (P(y) - p(y))dy \tag{2.11}$$

and

$$\Gamma(y) = \gamma(y) + (hr(1, y) + r_x(1, y)) + \int_y^1 \gamma(\xi)r(\xi, y)d\xi, \quad y \in [0, 1], \tag{2.12}$$

where $r(x, y)$ is the resolvent kernel for $k(x, y)$ in Proposition 2.3. Then by the parabolic deformation formula

$$v(t, x) = u(t, x) + \int_0^x k(x, y)u(t, y)dy, \quad x \in [0, 1], \tag{2.13}$$

the solution $u(t, x)$ of (1.1) with $U(t) \equiv 0$ is transformed to the solution $v(t, x)$ of (2.6) with the initial value

$$v_0(x) = u_0(x) + \int_0^x k(x, y)u_0(y)dy, \quad \text{a.e. } x \in [0, 1]. \tag{2.14}$$

3. Spectral properties of a generator $A_{\mathcal{P}}$

In this section we state basic spectral properties of a generator $A_{\mathcal{P}}$.

Definition 3.1 (i) φ is called a generalized eigenvector of A for an eigenvalue λ if and only if $(\lambda - A)^m\varphi = 0$ for some natural number $m \in \mathbb{N}$.
(ii) The set of generalized eigenvectors $\{\varphi_n\}_{n=1}^\infty$ is said to be a Riesz basis in $L^2(0, 1)$ if and only if any $\varphi \in L^2(0, 1)$ has a unique expansion $\varphi = \sum_{n=1}^\infty c_n\varphi_n$ with $c_n \in \mathbb{C}, n \in \mathbb{N}$ and there exists a $C > 0$ independent of φ such that

$$C^{-1}\sum_{n=1}^\infty |c_n|^2 \le \|\varphi_n\|_{L^2} \le C\sum_{n=1}^\infty |c_n|^2. \tag{3.1}$$

Theorem 3.1 (i) $\sigma(A_{\mathcal{P}})$ consists entirely of countable isolated eigenvalues with finite algebraic multiplicities. Set $\sigma(A_{\mathcal{P}}) = \{\lambda_n\}_{n=1}^\infty$. We denote by $m_n \in \mathbb{N}$ the algebraic multiplicity of each eigenvalue $\lambda_n \in \sigma(A_{\mathcal{P}})$. Then $\sigma(A_{\mathcal{P}})$ is devided as

$$\sigma(A_{\mathcal{P}}) = \Sigma_0 \cup \Sigma_1, \quad \Sigma_0 \cap \Sigma_1 = \emptyset \tag{3.2}$$

such that Σ_0 is a finite set of all eigenvalues λ_n whose multiplicities $m_n \ge 2$, and $\Sigma_1 = \sigma(A_{\mathcal{P}}) \setminus \Sigma_0$. Let $M = \sum_{\lambda_n \in \Sigma_0} m_n$. If we re-order Σ_1 by $\Sigma_1 = \{\lambda_{k_n}\}_{n=1}^\infty$ suitably, then we have $\lambda_{k_n} = \lambda_{n+M}$ for sufficiently large n and

$$\lambda_{k_n} = -(n + M)^2\pi^2 + (h + j) + \frac{1}{2}\int_0^1 p(s)ds + O(\frac{1}{n}) \quad \text{as } n \to \infty. \tag{3.3}$$

(ii) For each $\lambda_n \in \sigma(A_{\mathcal{P}})$, let $P_n = P_n(\mathcal{P})$ be the eigen-projection

$$P_n = \frac{1}{2\pi i}\int_{C_n} (\mu - A_{\mathcal{P}})^{-1} d\mu, \tag{3.4}$$

where C_n is a sufficiently small circle with center λ_n. Then the generalized eigenspace $P_n L^2(0,1) := \mathcal{M}_n = \mathcal{M}_n(\mathcal{P})$ is given by

$$\mathcal{M}_n = \mathrm{Ker}\,(\lambda_n - A_{\mathcal{P}})^{m_n}, \quad \dim \mathcal{M}_n = m_n. \tag{3.5}$$

Further, the space $\mathcal{M}_n$ is represented by

$$\mathcal{M}_n = \mathrm{Span}\,\{\varphi_{n_j} = \varphi_{n_j}(\mathcal{P}) : j = 1, \cdots, m_n\}, \tag{3.6}$$

where the generalized eigenvectors φ_{n_j} are defined inductively by

$$(\lambda_n - A_{\mathcal{P}})\varphi_{n_1} = 0, \quad (\lambda_n - A_{\mathcal{P}})\varphi_{n_{j+1}} = -\varphi_{n_j}, \quad j = 1, \cdots, m_n - 1. \tag{3.7}$$

(iii) The set of all generalized eigenvectors

$$\{\varphi_{n_j} = \varphi_{n_j}(\mathcal{P}) : n = 1, 2, \cdots, \; j = 1, \cdots, m_n\} \tag{3.8}$$

of $A_{\mathcal{P}}$ forms a Riesz basis in $L^2(0,1)$.
(iv) The spectrum of the adjoint operator $A^*_{\mathcal{P}}$ is given by $\sigma(A^*_{\mathcal{P}}) = \{\overline{\lambda_n}\}_{n=1}^{\infty}$, and the set of all generalized eigenvectors of $A^*_{\mathcal{P}}$ is represented by

$$\{\psi^*_{n_j} = \psi^*_{n_j}(\mathcal{P}) : n = 1, 2, \cdots, \; j = 1, \cdots, m_n\}, \tag{3.9}$$

where

$$(\overline{\lambda_n} - A^*_{\mathcal{P}})\psi^*_{n_1} = 0, \quad (\overline{\lambda_n} - A^*_{\mathcal{P}})\psi^*_{n_{j+1}} = -\psi^*_{n_j}, \quad j = 1, \cdots, m_n - 1. \tag{3.10}$$

(v) Suppose that $\{\varphi_{n_j}, \psi^*_{n_j}\}$ are normalized as

$$\langle \varphi_{n_j}, \psi^*_{n_j}\rangle = 1, \quad \langle \varphi_{n_j}, \psi^*_{m_l}\rangle = 0, \quad n_j \neq m_l. \tag{3.11}$$

Then the following two generalized Fourie expansions hold:

$$\varphi = \sum_{n=1}^{\infty}\sum_{j=1}^{m_n} \langle \varphi, \psi^*_{n_j}\rangle \varphi_{n_j} = \sum_{n=1}^{\infty}\sum_{j=1}^{m_n} \langle \varphi, \varphi_{n_j}\rangle \psi^*_{n_j}, \quad \forall \varphi \in L^2(0,1). \tag{3.12}$$

Theorem 3.2 (i) For each $\mathcal{P} = (p, f, g, \gamma, h, j) \in X$, the semigroup $e^{tA_{\mathcal{P}}}$ is analytic. Further, there exists a constant $M_1 > 0$ such that

$$\|e^{tA_{\mathcal{P}}}\|_{\mathcal{L}(L^2)} \leq M_1 e^{\lambda_0 t}, \quad \forall t \geq 0, \tag{3.13}$$

where $\lambda_0 = \max\{\mathrm{Re}\,\lambda_n : n = 1, 2, \cdots\}$.
(ii) The semigroup $e^{tA_{\mathcal{P}}}$ has the expansion

$$e^{tA_{\mathcal{P}}}\varphi = \sum_{n=1}^{\infty} e^{\lambda_n t} \sum_{j=1}^{m_n} \Big(\sum_{l=0}^{j-1} \frac{t^l}{l!} \langle \varphi, \psi^*_{n_{j-l}}\rangle \Big) \varphi_{n_j}, \quad \forall \varphi \in L^2(0,1). \tag{3.14}$$

(iii) The generator $A_{\mathcal{P}}$ has the representation

$$
\begin{aligned}
A_{\mathcal{P}}\varphi &= \sum_{n=1}^{\infty}\left(\sum_{j=1}^{m_n}\lambda_n\langle\varphi,\psi_{n_j}^*\rangle\varphi_{n_j}+\sum_{j=2}^{m_n}\langle\varphi,\psi_{n_{j-1}}^*\rangle\varphi_{n_j}\right), \quad \forall\varphi\in D(A_{\mathcal{P}}),\\
D(A_{\mathcal{P}}) &= \Bigg\{\varphi\in L^2(0,1): \sum_{n=1}^{\infty}\left(\sum_{j=1}^{m_n}|\lambda_n|^2|\langle\varphi,\varphi_{n_j}^*\rangle|^2\right.\\
&\qquad \left.+2\mathrm{Re}\sum_{j=2}^{m_n}\lambda_n\langle\varphi,\varphi_{n_{j-1}}^*\rangle\overline{\langle\varphi,\varphi_{n_j}^*\rangle}+\sum_{j=2}^{m_n}|\langle\varphi,\varphi_{n_{j-1}}^*\rangle|^2\right)<\infty\Bigg\}. \quad (3.15)
\end{aligned}
$$

(iv) The resolvents $R(\lambda;A_{\mathcal{P}}) := (\lambda - A_{\mathcal{P}})^{-1}$ of $A_{\mathcal{P}}$ is given by

$$
R(\lambda;A_{\mathcal{P}})\varphi=\sum_{n=1}^{\infty}\sum_{j=1}^{m_n}\left(\sum_{l=0}^{j-1}\frac{1}{(\lambda-\lambda_n)^{l+1}}\langle\psi,\varphi_{n_{j-l}}^*\rangle\right)\varphi_{n_j}, \quad \forall\lambda\in\mathbb{C}\setminus\sigma(A_{\mathcal{P}}). \quad (3.16)
$$

Similar results in Theorem 3.2 hold true for the adjoint operator $A_{\mathcal{P}}^*$.

4. Structural properties of semigroups

In this section, based on the deformation formula (2.8) in Theorem 2.1, we develop the structural study for semigroups associated with nonlocal diffusion equations.

Let $\mathcal{P} = (p, f, g, \gamma, h, j) \in X$ be given. For any $P, G \in C^1[0,1]$, $F \in C^1(\overline{D})$, $H \in \mathbb{R}$, we define J and $\Gamma(y)$ by (2.11) and (2.12), respectively. Now, we put

$$
\mathcal{Q} = (P, F, G, \Gamma, H, J) \in X. \quad (4.1)
$$

For notational convenience, we denote

$$
\begin{aligned}
S_{\mathcal{P}}(t) &:= e^{tA_{\mathcal{P}}}, \quad \mathcal{P} = (p, f, g, \gamma, h, j),\\
S_{\mathcal{Q}}(t) &:= e^{tA_{\mathcal{Q}}}, \quad \mathcal{Q} = (P, F, G, \Gamma, H, J). \quad (4.2)
\end{aligned}
$$

Let $k(x,y)$ be the deformation kernel in Proposition 2.2. Define the operator $K : L^2(0,1) \to L^2(0,1)$ by

$$
[K\varphi](x) = \varphi(x) + \int_0^x k(x,y)\varphi(y)dy \quad a.e.\ \ x\in(0,1), \quad \forall\varphi\in L^2(0,1). \quad (4.3)
$$

Proposition 4.1 The operator $K = K(\mathcal{P},\mathcal{Q}) : L^2(0,1) \to L^2(0,1)$ has a bounded inverse $K^{-1} = K(\mathcal{P},\mathcal{Q})^{-1} : L^2(0,1) \to L^2(0,1)$ given by

$$
[K^{-1}\varphi](x) = \varphi(x) + \int_0^x r(x,y)\varphi(y)dy, \quad a.e.\ \ x\in(0,1), \quad \forall\varphi\in L^2(0,1), \quad (4.4)
$$

where $r(x,y)$ is the resolvent kernel for $k(x,y)$.

The following theorem gives a refinement of Theorem 2.1 and its consequences.

Theorem 4.1 (i) The semigroups $S_{\mathcal{P}}(t)$ and $S_{\mathcal{Q}}(t)$ are intertwing, i.e.

$$KS_{\mathcal{P}}(t) = S_{\mathcal{Q}}(t)K, \quad \forall t \geq 0. \tag{4.5}$$

(ii) Th operators $A_{\mathcal{P}}$ and $A_{\mathcal{Q}}$ are intertwing, i.e.

$$KD(A_{\mathcal{P}}) \subset D(A_{\mathcal{Q}}) \quad \text{and} \quad KA_{\mathcal{P}} = A_{\mathcal{Q}}K. \tag{4.6}$$

(iii) The spectra of $A_{\mathcal{P}}$ and $A_{\mathcal{Q}}$ are identical, i.e.

$$\sigma(A_{\mathcal{P}}) = \sigma(A_{\mathcal{Q}}) = \{\lambda_n\}_{n=1}^{\infty}. \tag{4.7}$$

(iv) For each $\lambda_n \in \sigma(A_{\mathcal{P}})$,

$$\mathrm{Ker}\,(\lambda_n - A_{\mathcal{Q}})^j = K\,\mathrm{Ker}\,(\lambda_n - A_{\mathcal{P}})^j, \quad j = 1, \cdots, m_n. \tag{4.8}$$

In particular, the generalized eigenspaces of $A_{\mathcal{P}}$ and $A_{\mathcal{Q}}$ for λ_n are related as

$$\begin{aligned} \mathcal{M}_n(\mathcal{Q}) &= \mathrm{Ker}\,(\lambda_n - A_{\mathcal{Q}})^{m_n} = K\mathcal{M}_n(\mathcal{P}) \\ &= \mathrm{Span}\,\{K\varphi_{n_j}(\mathcal{P}) : j = 1, \cdots, m_n\}, \end{aligned} \tag{4.9}$$

where $\varphi_{n_j}(\mathcal{P})$ are generalized eigenvectors of $A_{\mathcal{P}}$ for λ_n.
(v) The resolvents of $A_{\mathcal{P}}$ and $A_{\mathcal{Q}}$ are identical, i.e.

$$\rho(A_{\mathcal{P}}) = \rho(A_{\mathcal{Q}}) = \mathbb{C} \setminus \sigma(A_{\mathcal{P}}). \tag{4.10}$$

Further, the resolvents $R(\lambda; A_{\mathcal{P}})$ and $R(\lambda; A_{\mathcal{Q}})$ satisfy

$$K\,R(\lambda; A_{\mathcal{P}}) = R(\lambda; A_{\mathcal{Q}})\,K, \quad \forall \lambda \in \rho(A_{\mathcal{P}}). \tag{4.11}$$

Next, for the adjoint operators of $A_{\mathcal{P}}$ and $A_{\mathcal{Q}}$, we give corresponding results to Theorem 4.1. The adjoint operator $K^* : L^2(0,1) \to L^2(0,1)$ of K is given by

$$[K^*\psi](x) = \psi(x) + \int_x^1 k(y,x)\psi(y)dy \quad a.e.\ \ x \in (0,1), \quad \forall \psi \in L^2(0,1). \tag{4.12}$$

For $\mathcal{Q} = (P, F, G, \Gamma, H, J) \in X$, the adjoint operator $A_{\mathcal{Q}}^*$ of $A_{\mathcal{Q}}$ is given by

$$\begin{cases} (A_{\mathcal{Q}}^*\psi)(x) = \dfrac{d^2\psi(x)}{dx^2} - P(x)\varphi(x) + \displaystyle\int_x^1 F(y,x)\psi(y)dy - \Gamma(x)\psi(1), \\ \hfill a.e.\ x \in [0,1], \\ D(A_{\mathcal{Q}}^*) = \Big\{ \psi \in H^2(0,1)\,; -\psi'(0) + H\psi(0) - \displaystyle\int_0^1 G(y)\psi(y)dy = 0, \\ \hfill \psi'(1) + J\psi(1) = 0 \Big\}. \end{cases} \tag{4.13}$$

As is well-known in Kato [3], the operators $A_{\mathcal{P}}^*$ and $A_{\mathcal{Q}}^*$ generate adjoint semigroups

$$S_{\mathcal{P}}^*(t) := e^{tA_{\mathcal{P}}^*} = (e^{tA_{\mathcal{P}}})^*, \quad S_{\mathcal{Q}}^*(t) := e^{tA_{\mathcal{Q}}^*} = (e^{tA_{\mathcal{Q}}})^*, \tag{4.14}$$

which are analytic.

Theorem 4.2 (i) The semigroups $S^*_{\mathcal{P}}(t)$ and $S^*_{\mathcal{Q}}(t)$ are intertwing, i.e.

$$S^*_{\mathcal{P}}(t)K^* = K^*S^*_{\mathcal{Q}}(t), \quad \forall t \geq 0. \tag{4.15}$$

(ii) Th operators $A_{\mathcal{P}}$ and $A_{\mathcal{Q}}$ are intertwing, i.e.

$$K^*D(A^*_{\mathcal{Q}}) \subset D(A^*_{\mathcal{P}}) \quad \text{and} \quad A^*_{\mathcal{P}}K^* = K^*A^*_{\mathcal{Q}}. \tag{4.16}$$

(iii) The spectra of $A^*_{\mathcal{P}}$ and $A^*_{\mathcal{Q}}$ are identical, i.e.

$$\sigma(A^*_{\mathcal{P}}) = \sigma(A^*_{\mathcal{Q}}) = \{\overline{\lambda_n}\}_{n=1}^{\infty}. \tag{4.17}$$

(iv) For each $\overline{\lambda_n} \in \sigma(A^*_{\mathcal{P}})$,

$$\mathrm{Ker}\,(\overline{\lambda_n} - A^*_{\mathcal{P}})^j = K^*\mathrm{Ker}\,(\overline{\lambda_n} - A^*_{\mathcal{Q}})^j, \quad j = 1, \cdots, m_n \tag{4.18}$$

In particular, the generalized eigenspaces of $A^*_{\mathcal{P}}$ and $A^*_{\mathcal{Q}}$ for $\overline{\lambda_n}$ are related as

$$\begin{aligned} \mathcal{M}^*_n(\mathcal{P}) &= \mathrm{Ker}\,(\overline{\lambda_n} - A^*_{\mathcal{P}})^{m_n} = K^*\mathcal{M}^*_n(\mathcal{Q}) \\ &= \mathrm{Span}\,\{K^*\psi^*_{n_j}(\mathcal{Q}) : j = 1, \cdots, m_n\}, \end{aligned} \tag{4.19}$$

where $\psi^*_{n_j}(\mathcal{Q})$ are generalized eigenvectors of $A^*_{\mathcal{Q}}$ for $\overline{\lambda_n}$.
(v) The resolvents of $A^*_{\mathcal{P}}$ and $A^*_{\mathcal{Q}}$ are identical, i.e.

$$\rho(A^*_{\mathcal{P}}) = \rho(A^*_{\mathcal{Q}}) = \mathbb{C} \setminus \sigma(A^*_{\mathcal{P}}). \tag{4.20}$$

Further, the resolvents $R(\lambda; A^*_{\mathcal{P}})$ and $R(\lambda; A^*_{\mathcal{Q}})$ satisfy

$$R(\lambda; A^*_{\mathcal{P}})K^* = K^*R(\lambda; A^*_{\mathcal{Q}}), \quad \forall \lambda \in \rho(A^*_{\mathcal{P}}). \tag{4.21}$$

These Theorem 4.1 and Theorem 4.2 imply that the operators K and K^* are the structural operators which connect two generators $A_{\mathcal{P}}$, $A_{\mathcal{Q}}$, and two adjoint generators $A^*_{\mathcal{P}}$, $A^*_{\mathcal{Q}}$, respectively. For related further arguments on structural operators, we refer to Nakagiri [5], Nakagiri and Tanabe [6].

Now, we take a special operator $A_{\mathcal{O}}$ which is connected by $A_{\mathcal{P}}$. That is, we consider transformation of parameters $\mathcal{P} = (p, f, g, \gamma, h, j) \to \mathcal{Q} = \mathcal{O} := (0, 0, 0, \Gamma^0, 0, J^0)$ in Theorem 4.1. Let $k^0(x, y)$ be the solution of (2.7) with

$$P(x) \equiv 0, \quad F(x, y) \equiv 0, \quad G(x) \equiv 0, \quad H = 0. \tag{4.22}$$

The resolvent kernel for $k^0(x, y)$ is denoted by $r^0(x, y)$. Then J^0 and Γ^0 are given by

$$J^0 := (h + j) + \frac{1}{2}\int_0^1 p(y)dy, \tag{4.23}$$

$$\Gamma^0(y) := \gamma(y) + (r^0_x(1, y) + jr^0(1, y)) + \int_y^1 \gamma(\xi)r^0(\xi, y)d\xi, \quad y \in [0, 1], \tag{4.24}$$

respectively. Thus, the proof of Theorem 3.1 is reduced to the analysis of the special eigenvalue problem $A_{\mathcal{O}}\phi = \lambda\phi$, i.e.,

$$\begin{cases} \dfrac{d^2\phi(x)}{dx^2} = \lambda\phi(x), \quad x \in [0, 1] \\ -\dfrac{d\phi(0)}{dx} = 0, \quad \dfrac{d\phi(1)}{dx} + J^0\phi(1) + \displaystyle\int_0^1 \Gamma^0(y)\phi(y)dy = 0. \end{cases} \tag{4.25}$$

5. Boundary controllability

In this final section we study a boundary controllability problem for the control system (1.1). For related results on reactor diffusion equations, we refer to Winkin, Dochain and Ligarius [13] and Sano and Nakagiri [8]. By applying the parabolic theory (cf. Tanabe [11]), the control system (1.1) admits a unique solution $u(t,x)$ in $0 \le x \le 1,\ 0 \le t < \infty$ in the following sense:

(i) For each $t \ge 0$, $u(t,\cdot) \in L^2(0,1)$ and if $t > 0$, the both distributive derivatives $\frac{\partial u}{\partial x}(t,\cdot)$ and $\frac{\partial^2 u}{\partial x^2}(t,\cdot)$ exist and lie in $L^2(0,1)$.

(ii) The function $u(t,\cdot):\ [0,\infty) \to L^2(0,1)$ is continuous on $[0,\infty)$ and continuously differentiable on $(0,\infty)$ with respect to the norm $\|\cdot\|_{L^2}$ of $L^2(0,1)$. In particular the initial condition in (1.1) is satisfied in the sense that $\lim_{t\to 0+} \|u(t,\cdot) - u_0\|_{L^2} = 0$.

(iii) For each $t > 0$ the first differential equation in (1.1) is satisfied for almost all $x \in [0,1]$, $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ being defined as in (i) while $\frac{\partial u}{\partial t}$ is defined as in (ii).

(iv) The boundary conditions in (1.1) are satisfied for $t > 0$.

Using the generalized Fourie expansion (3.14) and applying the divergence theorem to the expansion of solutions as in Fattorini and Russel [2], we can verify that the unique solution of (1.1), in the above sense, is represented by

$$
\begin{aligned}
u(t,x) \;&=\; \sum_{n=1}^{\infty} e^{\lambda_n t} \sum_{j=1}^{m_n} \Big(\sum_{l=0}^{j-1} \frac{t^l}{l!} \langle u_0, \psi^*_{n_{j-l}} \rangle \Big) \varphi_{n_j}(x) \\
&\quad - \int_0^t \left\{ \sum_{n=0}^{\infty} e^{-\overline{\lambda_n}(t-s)} \sum_{j=1}^{m_n} \Big(\sum_{l=0}^{j-1} \frac{(t-s)^l}{l!} \psi^*_{n_{j-l}}(1) \Big) \varphi_{n_j}(x) \right\} U(s)ds \\
&\equiv\; u(u_0;t,x) + u(U;t,x), \quad t > 0, \quad x \in [0,1].
\end{aligned} \tag{5.1}
$$

In the representation (5.1) we remark that

$$
\psi^*_{n_l}(1) \neq 0, \quad \psi^*_{n_2}(1) = \cdots = \psi^*_{n_{m_n}}(1) = 0, \quad \forall n = 1,2,\cdots. \tag{5.2}
$$

The attainable subspace of system (1.1) at time $t > 0$ is defined by

$$
\mathcal{A}(t) := \Big\{ \varphi \in L^2(0,1)\,;\, \exists U \in L^2(0,t;\mathbb{R}) \quad \text{s.t.} \quad \varphi = u(U;t,\cdot) \Big\}. \tag{5.3}
$$

Definition 5.1 (i) The control system (1.1) is said to be exactly controllable at time $t > 0$ if and only if $\mathcal{A}(t) = L^2(0,1)$.
(ii) The control system (1.1) is said to be approximately controllable at time $t > 0$ if and only if $\overline{\mathcal{A}(t)} = L^2(0,1)$, where $\overline{\mathcal{A}(t)}$ denotes the closure of set $\mathcal{A}(t)$ in $L^2(0,1)$.

For the exact controllability the following negative result holds.

Theorem 5.1 The boundary control system (1.1) can never be exactly reachable at any time $t > 0$.

This theorem follows from the compactness of analytic semigroup $e^{tA_{\mathcal{P}}}$ for $t > 0$ and the Baire's category theorem as in the proof of Triggiani [12].

For the approximate controllability the following positive result holds.

Theorem 5.2 The boundary control system (1.1) is approximately controllable at any time $t > 0$. That is, $\overline{\mathcal{A}(t)} = L^2(0, 1)$ holds for all $t > 0$.

The above theorem is proved by showing that the orthogonal complement $\{\mathcal{A}(t)\}^\perp$ equals $\{0\}$ for all $t > 0$ via (5.1) and (5.2).

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On the convergence of an area minimizing scheme for the anisotropic mean curvature flow

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1 Introduction

In this article we present the convergence of an area minimizing scheme for for the anisotropic mean curvature flow (AMCF for short) and its application to an approximation of the crystalline curvature flow (CCF) in the plane.

A family $\{\Gamma(t)\}_{t\geq 0}$ of hypersurfaces in $\mathbb{R}^N$ is called an AMCF provided that $\Gamma(t)$ evolves by the equation of the form

$$V = -\operatorname{div}\xi(\mathbf{n}) \quad \text{on } \Gamma(t), t > 0. \tag{1.1}$$

Here $\mathbf{n}$ is the Euclidean outer unit normal vector field of $\Gamma(t)$, the function $\gamma = \gamma(p)$ is the *surface energy density*, $\xi = \nabla_p\gamma := (\gamma_{p_1}, \cdots, \gamma_{p_N})$ is called the *Cahn-Hoffman vector*. The function γ is assumed to be convex. In particular, if $\gamma(p) = |p|$, then (1.1) is the usual mean curvature flow (MCF) equation:

$$V = -\operatorname{div}\mathbf{n} \quad \text{on } \Gamma(t),\ t > 0. \tag{1.2}$$

These equations arise in geometry, interface dynamics, crystal growth and image processing etc. Many people have been studying MCF, AMCF and CCF from various viewpoints. With relation to the applications mentioned above, numerical schemes have also been studied.

Among them, Chambolle [4] proposed an algorithm for MCF. His algorithm is described as follows: Let $E_0 \subset \mathbb{R}^N$ be a compact set and fix a time step $h > 0$. We choose a bounded domain $\Omega \subset \mathbb{R}^N$ including E_0 and take a function $w_0 \in L^2(\Omega) \cap BV(\Omega)$ as a unique minimizer of the functional $J_h(\cdot, E_0)$ defined by

$$J_h(v, E_0) := \begin{cases} \displaystyle\int_\Omega |Dv| + \frac{1}{2h}\|v - d_{E_0}\|^2_{L^2(\Omega)} & \text{if } v \in L^2(\Omega) \cap BV(\Omega), \\ +\infty & \text{if } v \in L^2(\Omega)\backslash BV(\Omega). \end{cases} \tag{1.3}$$

Here $\int_\Omega |Dv|$ is the total variation of v, Dv is the gradient of v in the sense of distribution, and $d(E_0) = d(\cdot, E_0)$ denotes the Euclidean signed distance function to ∂E_0, namely,

$$d(x, E_0) := \operatorname{dist}(x, E_0) - \operatorname{dist}(x, \mathbb{R}^N\backslash E_0) \quad \text{for } x \in \mathbb{R}^N. \tag{1.4}$$

Then we set

$$E_1 := \{w_0 \leq 0\}. \tag{1.5}$$

Throughout this paper we use the notations $\{f \geq \mu\} := \{x \in \mathbb{R}^N \mid f(x) \geq \mu\}$, $\{f \leq \mu\} := \{x \in \mathbb{R}^N \mid f(x) \leq \mu\}$ etc. Next we take a function $w_1 \in L^2(\Omega) \cap BV(\Omega)$ as a unique minimizer of the functional $J_h(\cdot, E_1)$ and define E_2 as the set in (1.5) with w_1 replacing w_0. Repeating this process, we have a sequence $\{E_k\}_{k=0,1,\ldots}$ of compact sets. We then set

$$E^h(t) := E_k \quad \text{for } t \in [kh, (k+1)h) \text{ and } k = 0, 1, \ldots \tag{1.6}$$

Sending $h \to 0$, we obtain a limit $\{E(t)\}_{t\geq 0}$ of $\{E^h(t)\}_{t\geq 0, h>0}$ and formally observe that $\{\Gamma(t) = \partial E(t)\}_{t\geq 0}$ is an MCF starting from $\Gamma(0)(= \partial E_0)$.

In this paper we extend Chambolle's algorithm to the AMCF by use of the elliptic differential inclusion:

$$w - h \operatorname{div} \partial_p \gamma(\nabla w) \ni d(E) \quad \text{in } \mathbb{R}^N. \tag{1.7}$$

(See section 3 below for the precise description of our algorithm.) Note that this is the Euler - Lagrange equation for such a variational problem as (1.3). This idea is essentially given by Caselles - Chambolle [3].

There are some papers studying anisotropic extensions of Chambolle's algorithm. See Bellettini - Caselles - Chambolle - Novaga [2], Caselles - Chambolle [3], Chambolle - Novaga [5], [6] and Eto - Giga - Ishii [8]. In these papers the convergences are proved in the sense of the Hausdorff distance and are locally uniform with respect to the time variable (except for [5]). As for the proofs of the convergences, the authors of [3], [2], [6] and [5] used some variational techniques. In [8] the authors applied some ideas from mathematical morphology, level set method and the theory of viscosity solutions.

The main purpose of this paper is to provide a different proof of the convergence of an anisotropic Chambolle's algorithm from those given in [2], [3], [5], [6] and [8]. Moreover, we apply our results to an approximation of the noncompact and nonconvex NLMCF.

The main idea is to employ the signed distance functions and the eikonal equations. This is motivated by Soner [16] and Goto - Ishii - Ogawa [12], in which they discussed, respectively, the convergence of Allen - Cahn equations and that of the Bence - Merriman - Osher algorithm for MCF. Consequently, under the nonfattening condition, we are able to show that the approximate flow by (1.7) converges to an AMCF in the sense of the Hausdorff distance and that it is locally uniform with respect to the time variable. Also we are able to apply our results to an approximation to CCF in the plane.

2 Preliminaries

2.1 Anisotropies and an elliptic differential inclusion

We make the following assumptions on γ.

(A1) $\gamma : \mathbb{R}^N \longrightarrow [0, +\infty)$: convex.

(A2) $\gamma(-p) = \gamma(p)$ and $\gamma(ap) = a\gamma(p)$ for all $p \in \mathbb{R}^N$ and $a > 0$.

(A3) $\Lambda^{-1}|p| \leq \gamma(p) \leq \Lambda|p|$ for all $p \in \mathbb{R}^N$ and some $\Lambda > 0$.

We easily see by (A1) - (A3) that γ Lipshcitz continuous in $\mathbb{R}^N$. Let $\partial_p\gamma(p)$ be the subdifferential of ζ at $p \in \mathbb{R}^N$:

$$\partial_p\gamma(p) := \{\xi \in \mathbb{R}^N \mid \langle \xi, q-p\rangle \leq \gamma(q) - \gamma(p) \text{ for all } q \in \mathbb{R}^N\}.$$

If γ is differentiable at p, then we write $\nabla_p\gamma(p)$ in place of $\partial_p\gamma(p)$. It follows from [8, Lemma 2.1] that $\partial_p\gamma(p) \subset \partial_p\gamma(0) \subset \mathrm{cl}\, B(0,\Lambda)$ for all $p \in \mathbb{R}^N$. Here and in the sequel, $B(x,r) := \{y \in \mathbb{R}^N \mid |y-x| < r\}$ for $x \in \mathbb{R}^N$ and $r > 0$ and $\mathrm{cl}\, A$ is the closure of $A \subset \mathbb{R}^N$.

We define the support function γ° of the convex set $\{\gamma \leq 1\}$ (often called Frank diagram for γ) by

$$\gamma^\circ(p) := \sup_{\gamma(q)\leq 1} \langle p, q\rangle.$$

We observe that γ° also satisfies (A1) - (A3) and Lipschitz continuity in $\mathbb{R}^N$.

In addition to (A1) - (A3), we assume some regularity on γ.

(A4) $\gamma \in C^2(\mathbb{R}^N\backslash\{0\})$, $\nabla_p^2\gamma^2 > O$ in $\mathbb{R}^N\backslash\{0\}$.

Remark 2.1. The second condition of (A4) is equivalent to the strict convexity of $\{\gamma \leq 1\}$ and that if ζ satisfies (A4), then γ° does so (cf. [14, Section 2.5] and [10, Remark 1.7.5]).

Assume that ∂E is smooth. The anisotropic mean curvature is defined as follows.

Definition 2.1. *Let E be an open set in $\mathbb{R}^N$ with the smooth boundary ∂E. Then the anisotropic mean curvature $\kappa_{\gamma^\circ}(x, E)$ of ∂E is defined by*

$$\kappa_{\gamma^\circ}(x,E) := -\mathrm{div}\,\nabla_p\gamma(\mathbf{n})(= -\mathrm{div}\,\xi(\mathbf{n}(x))) \quad \textit{for } x \in \partial E.$$

Next we introduce the anisotropic total variation. Let $\Omega \subset \mathbb{R}^N$ be an open set with Lipschitz boundary. Denote by $BV(\Omega)$ the space of all functions of bounded variation and by $BV_{loc}(\Omega)$ the class of all functions of locally bounded variation.

We define the anisotropic total variation of $u \in BV(\Omega)$ with respect to γ in Ω as

$$\int_\Omega \gamma(Du) := \sup\left\{ \int_\Omega u\, \mathrm{div}\,\varphi\, dx \;\middle|\; \varphi \in C_0^1(\Omega;\mathbb{R}^N),\ \gamma^\circ(\varphi) \leq 1 \text{ in } \Omega\right\}.$$

Set $X(\Omega) := \{z \in L^\infty(\Omega;\mathbb{R}^N) \mid \mathrm{div}\, z \in L^2(\Omega)\}$. For $w \in L^2(\Omega)\cap BV(\Omega)$ and $z \in X(\Omega)$, we define a functional on $C_0^1(\Omega)$ as

$$\int_\Omega (z, Dw)\psi := -\int_\Omega w\,\psi \mathrm{div}\, z\, dx - \int_\Omega w\langle z, \nabla\psi\rangle dx \quad \text{for } \psi \in C_0^1(\Omega). \tag{2.1}$$

We can extend this functional to a linear one on $C_0(\Omega)$. Hence (z, Dw) is a Radon measure. We recall Green's formula for $w \in L^2(\Omega) \cap BV(\Omega)$ and $z \in X(\Omega)$.

Theorem 2.1. ([1]) *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary. Let $w \in L^2(\Omega) \cap BV(\Omega)$ and $z \in X(\Omega)$. Then there exists $[z\cdot\mathbf{n}] \in L^\infty(\partial\Omega)$ such that $\|[z\cdot\mathbf{n}]\|_{L^\infty(\partial\Omega)} \leq \|z\|_{L^\infty(\Omega)}$ and*

$$\int_\Omega w\,\mathrm{div}\, z dx + \int_\Omega (z, Dw) = \int_{\partial\Omega} [z\cdot\mathbf{n}]w\, d\mathcal{H}^{N-1},$$

where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure. In the case $\Omega = \mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N} w \operatorname{div} z dx + \int_{\mathbb{R}^N} (z, Dw) = 0$$

for all $w \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$ and $z \in X(\mathbb{R}^N)$.

We briefly review some results on solutions of an elliptic differential inclusion:

$$w - h \operatorname{div} \partial_p \gamma(\nabla w) \ni g \quad \text{in } \mathbb{R}^N, \tag{2.2}$$

where $g \in L^2_{loc}(\mathbb{R}^N)$ and $h > 0$.

We give the definition of weak solutions of (2.2).

Definition 2.2. *We say that $w \in L^2_{loc}(\mathbb{R}^N) \cap BV_{loc}(\mathbb{R}^N)$ is a weak solution of (2.2) provided that there exists $z \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$, $\operatorname{div} z \in L^2_{loc}(\mathbb{R}^N)$ such that*

(1) *$z \in \partial\gamma(\nabla w)$ a.e. in $\mathbb{R}^N$,*

(2) *$(z, Dw) = \gamma(Dw)$ locally as measures in $\mathbb{R}^N$,*

(3) *$w - h \operatorname{div} z = g$ in $\mathcal{D}'(\mathbb{R}^N)$.*

The existence, uniqueness and regularity of solutions of (2.2) are stated as follows.

Theorem 2.2. (cf. [3] and [8]) *Assume (A1) - (A3). For any $g \in L^2_{loc}(\mathbb{R}^N)$, (2.2) admits a unique weak solution. Moreover, a weak solution w of (2.2) is Lipschitz continuous in $\mathbb{R}^N$ and $|\nabla w| \leq 1$ for a.e. in $\mathbb{R}^N$ and all $h > 0$.*

2.2 Generalized AMCF

Assume that γ satisfies (A1) - (A4). The level set equation for (1.1) is the following:

$$u_t - |\nabla u| \operatorname{div} \xi(\nabla u) = 0 \quad \text{in } (0, T) \times \mathbb{R}^N. \tag{2.3}$$

Notice that $\operatorname{div} \xi(\nabla u) = \operatorname{tr}(\nabla_p^2 \gamma(\nabla u) \nabla^2 u)$ if $\nabla u \neq 0$.

We give the definition of viscosity solutions of (2.3). Let U be a subset of a metric space (X, ρ) and let f be a function on U. The upper (resp., lower) semicontinuous envelope f^* (resp., f_*) is defined as follows: For each $x \in \overline{U}$,

$$f^*(x) := \limsup_{y \in U, \rho(y,x) \to 0} f(y), \; f_*(x) := \liminf_{y \in U, \rho(y,x) \to 0} f(y). \tag{2.4}$$

Definition 2.3. *Let $u : [0, T) \times \mathbb{R}^N \longrightarrow \mathbb{R}$.*

(1) *We say that u is a viscosity subsolution (resp., supersolution) of (2.3) provided that $u^*(t, x) < +\infty$ (resp., $u_*(t, x) > -\infty$) for all $(t, x) \in [0, T) \times \mathbb{R}^N$ and for any $\phi \in C^\infty((0, T) \times \mathbb{R}^N)$, if $u^* - \phi$ takes a local maximum (resp., minimum) at $(\hat{t}, \hat{x})$, then*

$$\phi_t(\hat{t}, \hat{x}) - |\nabla \phi(\hat{t}, \hat{x})| \operatorname{div} \xi(\nabla \varphi(\hat{t}, \hat{x})) \leq 0 \; (resp., \geq 0) \text{ if } \nabla \varphi(\hat{t}, \hat{x}) \neq 0,$$
$$\phi_t(\hat{t}, \hat{x}) \leq 0 \; (resp., \geq 0) \text{ if } \nabla \varphi(\hat{t}, \hat{x}) = 0 \text{ and } \nabla^2 \varphi(\hat{t}, \hat{x}) = O.$$

(2) *We say that u is a viscosity solution of (2.3) if u is a viscosity sub- and super-solution of (2.3).*

A family $\{\Gamma(t)\}_{t\geq 0}$ of hypersurfaces in $\mathbb{R}^N$ is called a generalized AMCF (or a generalized motion by (1.1)) if $\Gamma(t) = \{u(t,\cdot) = 0\}$, where u is a viscosity solution of (2.3). We refer to [10] for the theory of generalized motion of surface evolution equations including (1.1).

In sections 4 and 5 we use the notion of distance solutions for AMCF developed by [15]. Let $\{\Gamma(t)\}_{t\geq 0}$ be a family of hypersurfaces and $E(t)$ a closed set such that $\Gamma(t) = \partial E(t)$. Let $d = d(t,\cdot)$ be the signed distance function to $\Gamma(t)$ given by (1.4) with $E_0 = E(t)$.

Definition 2.4. *We say that $\{\Gamma(t)\}_{t\geq 0}$ is a distance solution of (1.1) provided that $d \wedge 0$ and $d \vee 0$ are, respectively, a viscosity subsolution and a viscosity supersolution of (2.3).*

Remark 2.2. In section 5 we will discuss an approximation of CCF and not assume (A4). Then $-\mathrm{div}\,\xi(\mathbf{n})$ in (1.1) is not defined in the classical sense. However, in two dimensional case it can be regarded as the crystalline curvature due to [17], [13] etc., more generally as the nonlocal curvature due to [9]. In [9] the authors develop the theory of the generalized motion by nonlocal curvature including CCF.

3 An anisotropic version of Chambolle's algorithm

An anisotropic version of Chambolle's algorithm is stated in the following way.

Fix $E_0 \in \mathcal{C}(\mathbb{R}^N)$. Let $w(E_0) := w(\cdot, E_0)$ be a weak solution of (1.7) with $E = E_0$. We then define a new set E_1 by

$$E_1 := \{w(\cdot, E_0) \leq 0\}.$$

Notice by Theorem 2.2 that $E_1 \in \mathcal{C}(\mathbb{R}^N)$. Let $w(E_1)$ be a weak solution of (1.7) with $E = E_1$. Again we define a new set E_2 by

$$E_2 := \{w(\cdot, E_1) \leq 0\}.$$

Repeating this process, we have a sequence $\{E_k\}_{k=0}^{[T/h]}$ of closed subsets of $\mathbb{R}^N$. Set

$$E^h(t) := E_{[t/h]} \quad \text{for } t \geq 0. \tag{3.1}$$

Letting $h \to 0$, we obtain a limit flow $\{E(t)\}_{t\geq 0}$ of $\{E^h(t)\}_{t\geq 0, h>0}$ and formally observe that $\partial E(t)$ is an AMCF starting from ∂E_0.

4 Convergence

In this section we assume (A1) - (A4) and formally show the convergences of $\{d^h\}_{h>0}$, $\{w^h)\}_{h>0}$ and $\{E^h(t)\}_{t\geq 0, h>0}$. We also establish that $\{\Gamma(t) = \partial E(t)\}_{t\geq 0}$ is a distance solution of (1.1).

For $E_0 \in \mathcal{C}(\mathbb{R}^N)$ let $\{E^h(t)\}_{t\geq 0, h>0}$, $\{d(E^h(t))\}_{t\geq 0, h>0}$, and $\{w(E^h(t))\}_{t\geq 0, h>0}$ be defined in the previous section. Set

$$d^h(t,x) := d(x, E^h(t)),\ w^h(t,x) := w(x, E^h(t)) \quad \text{for } t \in [0,T) \text{ and } x \in \mathbb{R}^N.$$

We mention our strategy to prove the convergence of our scheme. Since $w^h(t,\cdot)$ satisfies

$$w^h(t,\cdot) - h\,\mathrm{div}\,\partial_p\gamma(\nabla w^h(t,\cdot)) \ni d^h(t,\cdot) \quad \text{in } \mathbb{R}^N,$$

in a weak sense, letting $h \to 0$, we get $\lim_{h\to 0} w^h(t,x) = \lim_{h\to 0} d^h(t,x)$ at least formally. By this observation we compute the limit of $\{d^h\}_{h>0}$ as $h \to 0$.

We observe that for each $t \in [0,T)$, $d^h(t,\cdot)$ satisfies

$$|\nabla d^h| - 1 = 0 \quad \text{in } \{d^h(t,\cdot) > 0\}, -|\nabla d^h| + 1 = 0 \quad \text{in } \{d^h(t,\cdot) < 0\}, \tag{4.1}$$

in the sense of viscosity solutions. Then setting

$$\overline{d}(t,x) := \limsup_{(h,s,y)\to(0,t,x)} d^h(s,y),\ \underline{d}(t,x) := \liminf_{(h,s,y)\to(0,t,x)} d^h(s,y), \tag{4.2}$$

we can verify by the stability of viscosity solutions that $\rho(=\overline{d},\underline{d})$ is a viscosity solution of

$$|\nabla\rho| - 1 = 0 \quad \text{in } \{\rho(t,\cdot) > 0\}, -|\nabla\rho| + 1 = 0 \quad \text{in } \{\rho(t,\cdot) < 0\},$$

Besides, it is seen from the barrier construction argument that

$$\overline{d}(0,\cdot) = \underline{d}(0,\cdot) = d(\cdot, E_0) \quad \text{in } \mathbb{R}^N.$$

We impose an important assumption: Set $\Gamma(t) := \{\underline{d}(t,\cdot) \le 0 \le \overline{d}(t,\cdot)\}$.

$$\Gamma(t) \neq \emptyset,\ \Gamma(t) = \partial\{\overline{d}(t,\cdot) < 0\} = \partial\{\underline{d}(t,\cdot) > 0\} \quad \text{for all } t \in [0,T). \tag{4.3}$$

Then we observe that the map $t \mapsto \Gamma(t)$ is continuous in $[0,T)$ in the sense that

$$\lim_{s\to t} d_H(\Gamma(s),\Gamma(t)) = 0 \quad \text{for each } t \in [0,T), \tag{4.4}$$

where d_H is the Hausdorff distance defined by

$$d_H(A,B) := \max\left\{\sup_{x\in A} \mathrm{dist}\,(x,B), \sup_{x\in B} \mathrm{dist}\,(x,A)\right\} \quad \text{for } A,B \subset \mathbb{R}^N.$$

Hence we have the convergence of $\{d^h\}_{h>0}$. Let $d = d(t,\cdot)$ be the signed distance function to $\Gamma(t)$ given by (1.4) with $E_0 = E(t)$. Note that d is continuous in $[0,T)\times\mathbb{R}^N$ under the assumption (4.3) because of (4.4) and Lipschitz continuity of $d(t,\cdot)$ for all $t \in [0,T)$.

Theorem 4.1. *Assume (A1) - (A4) and (4.3). Then* $\overline{d} = \underline{d} = d$ *in* $[0,T)\times\mathbb{R}^N$. *Thus* $\{d^h\}_{h>0}$ *converges to* d *as* $h \to 0$ *locally uniformly in* $[0,T)\times\mathbb{R}^N$. *Moreover,* $\partial E^h(t)$ *converges to* $\Gamma(t)$ *as* $h \to 0$ *in the sense of the Hausdorff distance, locally uniformly in* $[0,T)$.

The formula $\lim_{h\to 0} w^h(t,x) = \lim_{h\to 0} d^h(t,x)$ can be obtained as follows. First, we remark that any weak solution of (2.2) is a minimizer of the associated variational problem.

Proposition 4.1. ([3, Proposition 3.1]) *Let* $g \in L^2_{loc}(\mathbb{R}^N)$ *and* $w \in L^2_{loc}(\mathbb{R}^N) \cap BV_{loc}(\mathbb{R}^N)$. *The following assertions are equivalent.*

(1) *w is a weak solution of (2.2).*

(2) *For each $r > 0$, w satisfies*

$$\begin{aligned}\int_{B(0,r)} \gamma(Dw) + \frac{1}{2h}\|w-g\|^2_{L^2(B(0,r))} \quad \le \quad & \int_{B(0,r)} \gamma(Dv) + \frac{1}{2h}\|v-g\|^2_{L^2(B(0,r))} \\ & + \int_{\partial B(0,r)} \gamma(\mathbf{n}(B(0,r)))|v-w|d\mathcal{H}^{N-1}\end{aligned}$$

for all $v \in L^2(B(0,r)) \cap BV(B(0,r))$. where $\mathcal{H}^{N-1}$ denotes the $(N-1)$ dimensional Hausdorff measure.

Applying this proposition with $g = v = d^h(t,\cdot)$ and $w = w^h(t,\cdot)$, we get

$$\frac{1}{2h}\|w^h(t,\cdot)-d^h(t,\cdot)\|^2_{L^2(B(0,r))} \le \int_{B(0,r)} \gamma(\nabla d^h(t,\cdot)) + \int_{\partial B(0,r)} \gamma(\mathbf{n})|d^h(t,\cdot)-w^h(t,\cdot)|d\mathcal{H}^{N-1}$$

It is seen by (A3) and the fact $|\nabla d^h(t,\cdot)| = 1$ for a.e. in $\mathbb{R}^N$ that the first term of the right-hand side of this inequality is uniformly bounded for $h > 0$. Since we can observe that the second term is also uniformly bounded for $h > 0$, we have

$$\sup_{t\in[0,T)} \|w^h(t,\cdot) - d^h(t,\cdot)\|_{L^2(B(0,r))} \le C\sqrt{h},$$

where $C > 0$ is independent of $h > 0$. Moreover, note that $\{d^h(t,\cdot)\}_{t\ge 0,h>0}$ and $\{w^h(t,\cdot)\}_{t\ge 0,h>0}$ are equi-Lipschitz continuous in $\mathbb{R}^N$ (cf. Theorem 2.2). Hence combining these facts, we obtain $\overline{w} = \overline{d}$ and $\underline{w} = \underline{d}$ in $[0,T)\times\mathbb{R}^N$. Here $\overline{w}$, $\underline{w}$ is defined by (4.2) with w^h replacing d^h. These formulae and Theorem 4.1 yield $\lim_{h\to 0} w^h(t,x) = \lim_{h\to 0} d^h(t,x)$ and the convergence of $\{w^h(t,\cdot)\}_{t\ge 0,h>0}$.

Theorem 4.2. *Assume (A1) - (A4) and (4.3). Then $\{w^h\}_{h>0}$ converges to d as $h \to 0$ locally uniformly in $[0,T)\times\mathbb{R}^N$.*

Now we show that $\Gamma(t)$ is an AMCF. For simplicity we assume that $\lim_{h\to 0} w^h = d^h$ in the $C^{1,2}$ sense. We get from (1.7) with $w = w^h(t,\cdot)$ and $E_0 = E^h(t)$

$$\frac{w^h(t,\cdot) - d(\cdot, E^h(t))}{h} \in \operatorname{div}\partial_p\gamma(\nabla w^h(t,\cdot)) \quad \text{on } \partial E^h(t). \tag{4.5}$$

Recall that $E^h(t)$ is given by

$$E^h(t) = E_{[t/h]} = \{w(\cdot, E_{[t/h]-1}) \le 0\} = \{w^h(t-h,\cdot) \le 0\}.$$

Hence $w^h(t-h,\cdot) = 0$ on $\partial E^h(t)$. Since $d(\cdot, E^h(t)) = 0$ and $|\nabla d(\cdot, E^h(t))| = 1$ on $\partial E^h(t)$, we obtain from (4.5)

$$\frac{w^h(t,\cdot) - w^h(t-h,\cdot)}{h} = \operatorname{div}\xi(\nabla w^h(t,\cdot)) \quad \text{on } \partial E^h(t).$$

Sending $h \to 0$, we have

$$d_t = \operatorname{div} \xi(\nabla d) \quad \text{on } \Gamma(t),\ t > 0.$$

This equation is nothing but (1.1) because $d_t = -V$ and $\nabla d = \mathbf{n}$.

The above arguments are justfied in the sense of a distance solution, mentioned at the end of subsection 2.2.

Theorem 4.3. *Assume (A1) - (A4) and (4.3). Then $\{\Gamma(t)\}_{t\geq 0}$ is a distance solution of (1.1).*

5 An application to CCF

The purpose of this section is to apply the results in section 4 to an approximation for CCF.

Fix $n(\geq 2) \in \mathbb{N}$. Let $\theta_i := i\pi/n$ and let $q_i := (\cos\theta_i, \sin\theta_i)$. Define $\gamma(p) := \max_{1\leq i\leq 2n}\langle q_i, p\rangle$ for $p \in \mathbb{R}^2$. Then this γ satisfies (A1) - (A3), but not (A4). In this case $\operatorname{div}(\mathbf{n})$ cannot be defined in the classical sense, as mentioned in Remark 2.2. Hence we rewrite (1.1) as follows:

$$V = -\text{“}\operatorname{div} \xi(\mathbf{n})\text{''} = 0 \quad \text{on } \Gamma(t),\ t > 0. \tag{5.1}$$

Here $\Gamma(t)$ is a simple and closed curve in $\mathbb{R}^2$ and “$\operatorname{div}\xi(\mathbf{n})$'' is interpreted as the crystalline curvature (cf. [13], [18]). The family $\{\Gamma(t)\}_{t\geq 0}$ evolving by (5.1) is is often called a crystalline curvature flow (CCF).

The level set equation for (5.1) is given by

$$u_t - |\nabla u|\,\text{“}\operatorname{div}\xi(\nabla u)\text{''} \quad \text{in } (0,T)\times\mathbb{R}^2. \tag{5.2}$$

The generalized CCF $\{\Gamma(t)\}_{t\geq 0}$ (or generalized motion by (5.1)) is defined by $\Gamma(t) := \{u(t,\cdot) = 0\}$ for each $t \in [0,T)$. Here u is a viscosity solution of (5.2). We use the results in [9] to show the convergence of our scheme to a generalized CCF, although we omit the detail.

For our purpose we approximate γ by smooth functions. By [11, Lemma 2.5] there is a sequence $\{\gamma_\tau\}_{\tau>0}$ satisfying (A1) - (A4) and

$$\gamma_\tau \longrightarrow \gamma \quad \text{as } \tau \to 0 \text{ locally uniformly in } \mathbb{R}^2, \tag{5.3}$$

$$\frac{1}{2\Lambda}|p| \leq \gamma_\tau(p) \leq 2\Lambda|p| \quad \text{for } p \in \mathbb{R}^2 \text{ and } \tau > 0. \tag{5.4}$$

We use $\{\gamma_\tau\}_{\tau>0}$ to construct approximate sequences: Fix a compact set $E_0 \subset \mathbb{R}^N$ and set $E_0^\tau := E_0$. Let $w^\tau(E_0)$ be a weak solution of (1.7) with $\gamma = \gamma_\tau$ and $E_0 := E_0^\tau$. Then we define a new set $E_1^\tau := \{w^\tau(E_0) \leq 0\}$. Next take $w^\tau(E_1)$ as a weak soltuion of (1.7) with $\gamma = \gamma_\tau$ and $E_0 := E_1^\tau$. Define a new set $E_2^\tau := \{w^\tau(E_1) \leq 0\}$. Repeating the process, we have sequences $\{E_k^\tau\}_{k=0,1,\dots}$, $\{d(E_k^\tau)\}_{k=0,1,\dots}$ and $\{w^\tau(E_k^\tau)\}_{k=0,1,\dots}$.

For $t \geq 0$ and $x \in \mathbb{R}^N$, set

$$E^{\tau,h}(t) := E^\tau_{[t/h]},\ d^{\tau,h}(t,x) := d(x, E^{\tau,h}(t)),\ w^{\tau,h}(t,x) := w^\tau(x, E^{\tau,h}(t)).$$

Define

$$\overline{\rho}(t,x) := \limsup_{(\tau,h,s,y)\to(0,0,t,x)} d^{\tau,h}(s,y), \ \underline{\rho}(t,x) := \liminf_{(\tau,h,s,y)\to(0,0,t,x)} d^{\tau,h}(s,y), \tag{5.5}$$

and $\Gamma(t) := \{\underline{\rho}(t,\cdot) \le 0 \le \overline{\rho}(t,\cdot)\}$. Similar arguments to those before Theorem 4.1 yield the following theorem. Let $d = d(t,\cdot)$ be the signed distance function to $\Gamma(t)$ given by (1.4) with $E_0 = E(t)$

Theorem 5.1. *Assume (A1) - (A4) and*

$$\Gamma(t) \neq \emptyset \text{ and } \Gamma(t) = \partial\{\overline{\rho}(t,\cdot) < 0\} = \partial\{\underline{\rho}(t,\cdot) > 0\} \quad \text{for all } t \in [0,T). \tag{5.6}$$

Then $\overline{d} = \underline{d} = d$ in $[0,T)\times\mathbb{R}^N$. Thus $\{d^{\tau,h}\}_{\tau,h>0}$ converges to d as τ, $h \to 0$ locally uniformly in $[0,T)\times\mathbb{R}^N$. Moreover, $\partial E^{\tau,h}(t)$ converges to $\Gamma(t)$ as τ, $h\to 0$ in the sense of the Hausdorff distance, locally uniformly in $[0,T)$.

Thanks to (5.4), we directly apply Proposition 4.1 with $\gamma = \gamma_\tau$ to get

$$\sup_{t\in[0,T),\tau>0} \|w^{\tau,h}(t,\cdot) - d^{\tau,h}(t,\cdot)\|_{L^2(B(0,r))} \le C\sqrt{h},$$

where $C > 0$ is independent of τ, $h > 0$. Since we observe that $|\nabla d^{\tau,h}(t,\cdot)| = 1$ and $|w^{\tau,h}(t,\cdot)| \le 1$ a.e. in $\mathbb{R}^N$ for all $t \in [0,T)$, combining these facts, we have the convergence of $\{w^{\tau,h}\}_{\tau,h>0}$.

Theorem 5.2. *Assume (A1) - (A4) and (5.6). Then $\{w^{\tau,h}\}_{\tau,h>0}$ converges to d as τ, $h \to 0$ locally uniformly in $[0,T)\times\mathbb{R}^N$.*

The characterization of $\{\Gamma(t)\}_{t\ge0}$ is shown by using the results due to [15] and [9].

Theorem 5.3. *Assume (A1) - (A4) and (5.6). Then $\{\Gamma(t)\}_{t\ge0}$ is a distance solution of (5.1). In other words, $d\wedge 0$ and $d \vee 0$ are, respectively, a viscosity subsolution and a viscosity supersolution of (5.2).*

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BEHAVIOR OF SOLUTIONS TO SOME MATHEMATICAL MODEL FOR ANGIOGENESIS

DOAN DUY HAI AND ATSUSHI YAGI

1. Introduction

We are concerned with the Cauchy problem for a mathematical model of tumor-induced angiogenesis:

$$
(1.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = a\Delta u - \mu\nabla\cdot[u(\sigma-u)\nabla\chi_1(\rho)] & \\ \qquad\qquad -\nu\nabla\cdot[u(\sigma-u)\nabla\chi_2(\eta)] + fu(\sigma-u) & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial \rho}{\partial t} = b\Delta\rho - \alpha u\rho + gu & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial \eta}{\partial t} = c\Delta\eta - h\eta - \beta u\eta + \varphi(x) & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \rho}{\partial n} = \dfrac{\partial \eta}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0)=u_0(x),\ \ \rho(x,0)=\rho_0(x),\ \ \eta(x,0)=\eta_0(x) & \text{in } \Omega, \end{cases}
$$

in a bounded domain Ω in $\mathbb{R}^3$. Here, unknown functions $u(x,t), \rho(x,t)$ and $\eta(x,t)$ denote the densities of endothelial-cells, the concentration of fibronectin and the concentration of TAF (tumor angiogenesis factor), respectively, at a position x of an organism Ω and at time $t \geq 0$. Cells diffuse in Ω with diffusion constant $a > 0$. Cells have a directed mobility in responce to fibronectin gradients which is called haptotaxis. Haptotaxis is described by the nonlinear advection term $-\mu\nabla\cdot[u\nabla\rho]$ with flow rate $\mu > 0$. Similarly, cells have a directed mobility in responce to TAF gradients called chemotaxis. Chemotaxis is described by $-\nu\nabla\cdot[u\nabla\log(1+\theta\eta)]$ with flow rate $\nu > 0$ and some constant $\theta > 0$. In the equation of fibronectin, the term gu denotes production due to cells with rate $g > 0$ and conversely the term $-\alpha u\rho$ denotes uptake by cells themselves with rate $\alpha > 0$. In the equation of TAF, the term $-\beta u\rho$ denotes uptake by cells.

In 1998, Anderson-Chaplain [1] has presented a mathematical model for describing the process of tumor-induced angiogenesis. We, however, intend to modify the model equations of [1] into the form (1.1) in the view points:

(1) The diffusion for both fibronectin and ATF is considered with diffusion constant $b > 0$ and $c > 0$, respectively.
(2) The proliferation of endothelial-cells are considered. The growth term is assumed to be given by $fu(\sigma-u)$ with saturation density $\sigma > 0$ and constant $f > 0$.
(3) Saturation takes place in the effects of not only proliferation but also advection. So, the advection terms take the forms $-\mu\nabla\cdot[u(\sigma-u)\nabla\chi_1(\rho)]$ and $-\nu\nabla\cdot[u(\sigma-u)\nabla\chi_2(\eta)]$, respectively, with suitable sensitivity functions $\chi_1(\cdot)$ and $\chi_2(\cdot)$.

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(4) Constant supply of TAF due to tumor and natural decline are considered. Supplying rate is given by $\varphi(x) \geq 0$ and the decling constant is denoted by $h > 0$.

As for validity of these modifications, we refer the reader to the papers [1, 4, 5].

This paper is devoted to studying longtime behavior of solutions to (1.1). First, we will construct a dynamical system generated by (1.1) and prove that, unless $u_0 \not\equiv 0$, the solution $(u(t), \rho(t), \eta(t))$ converges as $t \to \infty$ to the stationary solution $(\sigma, \frac{g}{\alpha}, \overline{\eta}(x))$, where $\overline{\eta}(x)$ is a nonnegative function satisfying $-c\Delta\overline{\eta} + (h+\beta\sigma)\overline{\eta} = \varphi(x)$ in Ω with boundary conditions $\frac{\partial\overline{\eta}}{\partial n} = 0$ on $\partial\Omega$. Second, we will exhibit some numerical results which enlighten very complex profiles in a short time range.

We consider (1.1) in a three-dimensional, $\mathcal{C}^2$ or convex bounded domain Ω. The supplying function $\varphi(x)$ is such that

$$0 \leq \varphi \in L_2(\Omega). \tag{1.2}$$

The sensitivity functions $\chi_i(\cdot)\,(i = 1, 2)$ are a real valued $\mathcal{C}^1$ function on $[0, \infty)$, i.e.,

$$\chi_i \in \mathcal{C}^1([0,\infty);\mathbb{R}), \qquad i = 1, 2. \tag{1.3}$$

2. Abstract Formulation

Let us formulate (1.1) as the Cauchy problem for a semilinear abstract evolution equation

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), \qquad 0 < t < \infty, \\ U(0) = U_0 \end{cases} \tag{2.1}$$

in a Banach space X. As X we set

$$X = \{{}^t(u, \rho, \eta);\ u \in H^1(\Omega)',\ \rho \in L_2(\Omega),\ \eta \in L_2(\Omega)\}. \tag{2.2}$$

Here, $A = \operatorname{diag}\{A_1, A_2, A_3\}$ is a diagonal operator matrix of X, where $A_i\,(i = 1, 2, 3)$ are a realization of Laplace operator under the homogeneous Neumann boundary conditions on $\partial\Omega$ such that $A_1 = -a\Delta + 1$, $A_2 = -b\Delta + 1$ and $A_3 = -c\Delta + h$, respectively. More precisely, A_1 is a sectorial operator acting in $H^1(\Omega)'$ with domain $\mathcal{D}(A_1) = H^1(\Omega)$; meanwhile, $A_i\,(i = 2, 3)$ are a self-adjoint operator of $L_2(\Omega)$ with domain $\mathcal{D}(A_i) = H^2_N(\Omega)$. Consequently, the domain of A is given by

$$\mathcal{D}(A) = \{{}^t(u, \rho, \eta);\ u \in H^1(\Omega),\ \rho \in H^2_N(\Omega),\ \eta \in H^2_N(\Omega)\}.$$

The operator $F(U)$ is a nonlinear operator of X of the form

$$F(U) = F\begin{pmatrix} u \\ \rho \\ \eta \end{pmatrix} = \begin{pmatrix} F_1(U) + u[1 + f(\sigma - u)] \\ \rho - \alpha u\rho + gu \\ -\beta u\eta + \varphi(x) \end{pmatrix},$$

where

$$\begin{aligned} F_1(U) = -\mu\nabla\cdot[\Psi(\operatorname{Re} u)\Psi(\sigma - \operatorname{Re} u)\nabla\chi_1(\operatorname{Re}\rho)] \\ - \nu\nabla\cdot[\Psi(\operatorname{Re} u)\Psi(\sigma - \operatorname{Re} u)\nabla\chi_2(\operatorname{Re}\eta)]. \end{aligned} \tag{2.3}$$

Here, $\Psi(u)$ denotes a cutoff function defined for $-\infty < u < \infty$ such that $\Psi(u) \equiv 0$ if $-\infty < u < 0$, $\Psi(u) = u$ if $0 \leq u \leq \sigma$ if $0 \leq u \leq \sigma$, $\Psi(u) \equiv \sigma$ if $\sigma < u < \infty$. And for $i = 1, 2$, $\chi_i(\cdot)$ are assumed to be extended on the whole real line $(-\infty, \infty)$ as a real valued $\mathcal{C}^1$ function (remember (1.3)). As for the domain of F, we set

$$\mathcal{D}(F) = \{{}^t(u, \rho, \eta);\ u \in L_4(\Omega),\ \rho \in H_N^{\frac{7}{4}}(\Omega),\ \eta \in H_N^{\frac{7}{4}}(\Omega)\}.$$

Since $H_N^{\frac{7}{4}}(\Omega) \subset \mathcal{C}(\overline{\Omega})$, it is easily obtained that

$$\|F_1(U) - F_1(\widetilde{U})\|_{(H^1)'} \leq p(\|A^{\frac{7}{8}}U\|_X + \|A^{\frac{7}{8}}\widetilde{U}\|_X)\|A^{\frac{7}{8}}(U - \widetilde{U})\|_X, \qquad U, \widetilde{U} \in \mathcal{D}(A^{\frac{7}{8}}),$$

where $p(\cdot)$ denotes some suitable increasing function which is determined from $\chi_i(\cdot)\,(i = 1, 2)$. So it is similar for $F(U)$, i.e.,

$$\|F(U) - F(\widetilde{U})\|_X \leq p(\|A^{\frac{7}{8}}U\|_X + \|A^{\frac{7}{8}}\widetilde{U}\|_X)\|A^{\frac{7}{8}}(U - \widetilde{U})\|_X, \qquad U, \widetilde{U} \in \mathcal{D}(A^{\frac{7}{8}}). \tag{2.4}$$

We are then led to introduce the space of initial functions by

$$\begin{aligned} K = \{U_0 = {}^t(u_0, \rho_0, \eta_0);\ & 0 \leq u_0 \leq \sigma,\ u_0 \in H^{\frac{3}{4}}(\Omega), \\ & 0 \leq \rho_0 \leq \tfrac{g}{\alpha},\ \rho_0 \in H_N^{\frac{7}{4}}(\Omega),\ 0 \leq \eta_0 \in H_N^{\frac{7}{4}}(\Omega)\}. \end{aligned} \tag{2.5}$$

This space is a subset of $\mathcal{D}(A^{\frac{7}{8}})$. We here apply the theory of semilinear abstract parabolic evolution equations to conclude local existence of solutions. The following theorem is a direct consequence of [6, Theorem 4.1].

Theorem 2.1. *For any $U_0 \in K$, (2.1) possesses a unique local solution in the function space:*

$$U \in \mathcal{C}((0, T_{U_0}]; \mathcal{D}(A)) \cap \mathcal{C}([0, T_{U_0}]; \mathcal{D}(A^{\frac{7}{8}})) \cap \mathcal{C}^1((0, T_{U_0}]; X). \tag{2.6}$$

Furthermore, $U(t)$ satisfies

$$0 \leq u(t) \leq \sigma,\ \ 0 \leq \rho(t) \leq \tfrac{g}{\alpha},\ \ \eta(t) \geq 0 \qquad \textit{for every } 0 \leq t \leq T_{U_0}. \tag{2.7}$$

3. Dynamical System

In this section, we shall construct global solutions for (1.1) and the dynamical system generated by (1.1). We begin with proving *a priori* estimates for the local solutions. Let $U(t)$ denote a local solution of (2.1) in the function space:

$$U \in \mathcal{C}((0, T_U]; \mathcal{D}(A)) \cap \mathcal{C}([0, T_U] : \mathcal{D}(A^{\frac{7}{8}})) \cap \mathcal{C}^1((0, T_U]; X) \tag{3.1}$$

with values in K, i.e.,

$$0 \leq u(t) \leq \sigma,\ \ 0 \leq \rho(t) \leq \tfrac{g}{\alpha},\ \ \eta(t) \geq 0 \quad \text{for every } 0 \leq t \leq T_U, \tag{3.2}$$

where $[0, T_U]$ is the interval on which $U(t)$ is defined.

Proposition 3.1. *Let $U_0 \in K$ and let $U(t)$ be any local solution to (2.1) in the function space (3.1) with values in K. Then, there exists an increasing function $p(\cdot)$ such that*

$$\|A^{\frac{7}{8}}U(t)\|_X \leq p(\|A^{\frac{7}{8}}U_0\|_X), \qquad 0 \leq t \leq T_U. \tag{3.3}$$

Proof. The proof is divided into several steps.

Step 1. The second equation of (1.1) is written as the abstract equation in the space $L_2(\Omega)$. Then, $\rho(t)$ is represented by

$$\rho(t) = e^{-tA_2}\rho_0 + \int_0^t e^{-(t-s)A_2}\{\rho(s) + u(s)[g - \alpha\rho(s)]\}ds,$$

where e^{-tA_2} is the analytic semigroup generated by $-A_2$ on $L_2(\Omega)$. As $A_2 \geq 1$, we have $\|e^{-tA_2}\|_{L_2} \leq e^{-t}$. Operating $A_2^{\frac{7}{8}}$ to this equality and estimating its norm, we obatin that

$$\|A_2^{\frac{7}{8}}\rho(t)\|_{L_2} \leq e^{-t}\|A_2^{\frac{7}{8}}\rho_0\|_{L_2} + C\int_0^t (t-s)^{-\frac{7}{8}}e^{-(t-s)}\|\rho(s) + u(s)[g - \alpha\rho(s)]\|_{L_2}ds.$$

On account of (3.2), we conclude that

$$\|A_2^{\frac{7}{8}}\rho(t)\|_{L_2} \leq e^{-t}\|A_2^{\frac{7}{8}}\rho_0\|_{L_2} + C, \qquad 0 \leq t \leq T_U. \tag{3.4}$$

Step 2. Multiply the third equation of (1.1) by η and integrate the product in Ω. Then,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \eta^2 dx + c\int_\Omega |\nabla\eta|^2 dx + h\int_\Omega \eta^2 dx &= \int_\Omega (-\beta u\eta^2 + \varphi\eta)dx \\ &\leq \frac{h}{2}\|\eta\|_{L^2}^2 + \frac{1}{2h}\|\varphi\|_{L_2}^2.\end{aligned}$$

Therefore, on account of (1.2),

$$\frac{d}{dt}\int_\Omega \eta^2 dx + 2c\int_\Omega |\nabla\eta|^2 dx + h\int_\Omega \eta^2 dx \leq C.$$

Hence,

$$\|\eta(t)\|_{L_2}^2 \leq e^{-ht}\|\eta_0\|_{L_2}^2 + C, \qquad 0 \leq t \leq T_U. \tag{3.5}$$

Step 3. The third equation of (1.1) is regarded as the abstract equation in the space $L_2(\Omega)$. Then, $\eta(t)$ is represented by

$$\eta(t) = e^{-tA_3}\eta_0 + \int_0^t e^{-(t-s)A_3}[-\beta u(s)\eta(s) + \varphi]ds.$$

Furthermore,

$$A_3^{\frac{7}{8}}\eta(t) = e^{-tA_3}A_3^{\frac{7}{8}}\eta_0 + \int_0^t A_3^{\frac{7}{8}}e^{-(t-s)A_3}[-\beta u(s)\eta(s) + \varphi]ds.$$

As $A_3 \geq h$, the analytic semigroup e^{-tA_3} generated by $-A_3$ on $L_2(\Omega)$ is estimated by $\|e^{-tA_3}\|_{L_2} \leq e^{-ht}$. Hence, in view of (1.2), (3.2) and (3.5), we obtain that

$$\|A_3^{\frac{7}{8}}\eta(t)\|_{L_2} \leq e^{-ht}\|A_3^{\frac{7}{8}}\eta_0\|_{L_2} + C(\|\eta_0\|_{L_2} + 1), \qquad 0 \leq t \leq T_U. \tag{3.6}$$

Step 4. From the first equation of (1.1) we have a representation for $u(t)$ of the form

$$u(t) = e^{-tA_1}u_0 + \int_0^t e^{-(t-s)A_1}\{F_1(U(s)) + u(s)[1 + f(\sigma - u(s))]\}ds,$$

using the analytic semigroup e^{-tA_1} generated by $-A_1$ on $H^1(\Omega)'$. Furthermore,

$$A_1^{\frac{7}{8}}u(t) = e^{-tA_1}A_1^{\frac{7}{8}}u_0 + \int_0^t A_1^{\frac{7}{8}}e^{-(t-s)A_1}\{F_1(U(s)) + u(s)[1 + f(\sigma - u(s)]\}ds.$$

Since the spectrum $\sigma(A_1)$ of A_1 is contained in the half plane $\{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda \geq 1\}$, we have $\|e^{-tA_1}\|_{(H^1)'} \leq Ce^{-t}$. On account of this fact, we observe that

$$\begin{aligned}\|A_1^{\frac{7}{8}}u(t)\|_{(H^1)'} &\leq Ce^{-t}\|A_1^{\frac{7}{8}}u_0\|_{(H^1)'} \\ &\quad + C\int_0^t (t-s)^{-\frac{7}{8}}e^{-(t-s)}\{\|F_1(U(s))\|_{(H^1)'} + \|u(s)[1 + f(\sigma - u(s))]\|_{(H^1)'}\}ds.\end{aligned}$$

Here, due to (1.3),

$$\begin{aligned}\|F_1(U)\|_{(H^1)'} &\leq C[\|\nabla\chi_1(\rho)\|_{L_2} + \|\nabla\chi_2(\eta)\|_{L_2}] \\ &\leq C[\|\chi_1'(\rho)\nabla\rho\|_{L_2} + \|\chi_2'(\eta)\nabla\eta\|_{L_2}] \\ &\leq p(\|\rho\|_{H^{\frac{7}{4}}} + \|\eta\|_{H^{\frac{7}{4}}}),\end{aligned}$$

$p(\cdot)$ being some suitable increasing function. While, we already know from (3.4) and (3.5) that $\|\rho(t)\|_{H^{\frac{7}{4}}} + \|\eta(t)\|_{H^{\frac{7}{4}}} \leq C(\|A_2^{\frac{7}{8}}\rho_0\|_{L_2} + \|A_3^{\frac{7}{8}}\eta_0\|_{L_2} + 1)$. Hence, we conclude that

$$\|A_1^{\frac{7}{8}}u(t)\|_{(H^1)'} \leq Ce^{-\delta t}\|A_1^{\frac{7}{8}}u_0\|_{(H^1)'} + p(\|A_2^{\frac{7}{8}}\rho_0\|_{L_2} + \|A_3^{\frac{7}{8}}\eta_0\|_{L_2}), \qquad 0 \leq t \leq T_U, \tag{3.7}$$

with some exponent $\delta > 0$.

We have thus accomplished the proof of (3.3). □

As an immediate consequence of Proposition 3.1 we have the global existence. For any $U_0 \in K$, (2.1) possesses a unique global solution in the function space:

$$U \in \mathcal{C}((0,\infty);\mathcal{D}(A)) \cap \mathcal{C}([0,\infty);\mathcal{D}(A^{\frac{7}{8}})) \cap \mathcal{C}^1((0,\infty);X). \tag{3.8}$$

with values $U(t) \in K$ for all $0 \leq t < \infty$.

In the meantime, it is possible to show the smoothing estimate

$$\|AU(t)\|_X \leq C_{U_0}(t^{-\frac{1}{8}} + 1), \qquad 0 < t < \infty, \tag{3.9}$$

C_{U_0} being determined by the norm $\|A^{\frac{7}{8}}U_0\|_X$.

Let next us construct a dynamical system generated by (2.1). For $U_0 \in K$, let $U(t;U_0)$ denote the global solution to (2.1) belonging to (3.8). Put for $0 \leq t < \infty$, $S(t)U_0 = U(t;U_0)$. Then, $S(t)$ is a nonlinear operator acting on K. It is also seen by the methods described in [6, Subsection 6.5.1] that the mapping $G\colon [0,\infty) \times K \to K$ defined by $G(t,U_0) = S(t)U_0$ is continuous, here K is regarded as a metric space equipped with the distance $d(U_0,V_0) = \|A^{\frac{7}{8}}(U_0 - V_0)\|_X$.

Theorem 3.1. (2.1) *(and hance* (1.1)*) generates a dynamical system* $(S(t),K,\mathcal{D}(A^{\frac{7}{8}}))$*, where* K *is a phase space and* $\mathcal{D}(A^{\frac{7}{8}}) \equiv \mathcal{D}_{\frac{7}{8}}$ *is a universal space.*

4. Convergence of $S(t)U_0$ as $t \to \infty$

Let us study longtime behavior of trajectories of $(S(t), K, \mathcal{D}_{\frac{7}{8}})$. Let $\overline{U} = {}^t(\overline{u}, \overline{\rho}, \overline{\eta}) \in K$ be a point such that $\overline{u} \equiv \sigma$, $\overline{\rho} \equiv \frac{g}{\alpha}$, and $\overline{\eta} \in \mathcal{D}(A_3)$ is a unique solution to

$$-c\Delta\eta + (h + \beta\sigma)\eta = \varphi(x). \tag{4.1}$$

Since $\overline{U} \in \mathcal{D}(A)$ is a stationary solution to (2.1), this point is clearly an equilibrium. Our goal is then to show that, for every $U_0 \in K$, unless $u_0 \equiv 0$, $S(t)U_0$ converges to $\overline{U}$ in $\mathcal{D}_{\frac{7}{8}}$ as $t \to \infty$.

Let $U_0 \in K$ and let $S(t)U_0 = {}^t(u(t), \rho(t), \eta(t))$. We notice from (3.9) that $u(t)$ satisfies a uniform bounded estimate

$$\|u(t)\|_{H^1} \le C_{U_0}, \qquad 1 \le t < \infty. \tag{4.2}$$

Our first goal is then to observe that, as $t \to \infty$, $\|u(t)[\sigma - u(t)]\|_{L_1} \to 0$.

Proposition 4.1. *It holds that* $\int_0^\infty \|u(s)[\sigma - u(s)]\|_{L_1} ds \le f^{-1}\sigma|\Omega|$.

Proof. Integrate the first equation of (1.1) in Ω. Then, we have

$$\frac{d}{dt}\int_\Omega u\,dx = f\int_\Omega u(\sigma - u)\,dx. \tag{4.3}$$

Since $0 \le u \le \sigma$, we observe that

$$0 \le \frac{d}{dt}\int_\Omega u\,dx \le \frac{f\sigma^2}{4}|\Omega|. \tag{4.4}$$

In particular, $\|u(t)\|_{L_1}$ is monotone increasing as $t \to \infty$.

Integrate further (4.3) in t. Then,

$$\|u(T)\|_{L_1} - \|u_0\|_{L_1} = f\int_0^T \int_\Omega u(\sigma - u)\,dxdt.$$

Since $\|u(T)\|_{L_1} \le \sigma|\Omega|$, the desired estimate is verified. $\square$

Proposition 4.2. *It holds that* $\left|\frac{d}{dt}\|u(t)[\sigma - u(t)]\|_{L_1}\right| \le C_{U_0}$ *for every* $1 \le t < \infty$.

Proof. As $0 \le u \le \sigma$, we see that

$$\frac{d}{dt}\|u(t)[\sigma - u(t)]\|_{L_1} = \sigma\frac{d}{dt}\int_\Omega u\,dx - \frac{d}{dt}\int_\Omega u^2 dx.$$

Meanwhile, from the first equation of (1.1) we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + a\int_\Omega |\nabla u|^2 dx$$
$$= \int_\Omega u(\sigma - u)[\mu\nabla\chi_1(\rho) + \nu\nabla\chi_2(\eta)] \cdot \nabla u\,dx + f\int_\Omega u^2(\sigma - u)\,dx.$$

So, (4.2) yields that $\left|\frac{d}{dt}\int_\Omega u^2 dx\right| \le C_{U_0}$. Hence, the desired estimate is verified by this and (4.4). $\square$

Two Propositions 4.1 and 4.2 naturally yield that, as $t \to \infty$,

$$\|u(t)[\sigma - u(t)]\|_{L_1} \to 0. \tag{4.5}$$

Proposition 4.3. *If* $\|u_0\|_{L_1} > 0$, *then* $u(t)$ *converges to* σ *in* $L_2(\Omega)$.

Proof. Consider any time sequence $t_n \uparrow \infty$. It then suffices to prove that this sequence contains a subsequence $t_{n'} \uparrow \infty$ such that $u(t_{n'}) \to \sigma$ in $L_2(\Omega)$.

Since the bounded balls of $H^1(\Omega)$ is weakly sequentially compact and since (4.2) is valid, the sequence $u(t_n)$ contains a subsequence which is weakly convergent in $H^1(\Omega)$. Since $H^1(\Omega)$ is compactly embedded in $L_2(\Omega)$, $u(t_n)$ contains a subsequence which is convergent in $L_2(\Omega)$. Moreover, (3.2) implies that $u(t_n)$ contains a subsequence which is convergent in $L_\infty(\Omega)$ with w*-topology. These mean that one can extract a subsequence $u(t_{n'})$ of $u(t_n)$ which converges to u_∞ in $L_2(\Omega)$ and that u_∞ belongs to the intersection $H^1(\Omega) \cap L_\infty(\Omega)$. Since $u(t_{n'})[\sigma - u(t_{n'})]$ converges to $u_\infty(\sigma - u_\infty)$ in $L_1(\Omega)$, too, we observe by (4.5) that $u_\infty(\sigma - u_\infty) = 0$, namely, u_∞ equals to either 0 or σ almost everywhere. In addition, we observe that $u_\infty(\sigma - u_\infty) \in H^1(\Omega)$ with $0 = \nabla[u_\infty(\sigma - u_\infty)] = (\sigma - 2u_\infty)\nabla u_\infty$, hence $\nabla u_\infty = 0$ almost everywhere. Consequently, u_∞ is a homogeneous function, i.e., either $u_\infty = 0$ almost everywhere or $u_\infty = \sigma$ almost everywhere. But, as shown above, $\|u(t)\|_{L_1}$ is monotone increasing; therefore, $\|u_0\|_{L_1} > 0$ implies $\|u_\infty\|_{L_1} > 0$; hence, u_∞ coincides with σ almost everywhere. □

Proposition 4.4. *If $\|u_0\|_{L_1} > 0$, then $\rho(t)$ converges to $\frac{g}{\alpha}$ in $L_2(\Omega)$.*

Proof. Writing the second equation of (1.1) as

$$\frac{\partial}{\partial t}(\rho - \tfrac{g}{\alpha}) = b\Delta(\rho - \tfrac{g}{\alpha}) - \alpha\sigma(\rho - \tfrac{g}{\alpha}) - \alpha(u - \sigma)(\rho - \tfrac{g}{\alpha}), \tag{4.6}$$

we multiply this one by $(\rho - \frac{g}{\alpha})$ and integrate the product in Ω. Then,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\rho - \tfrac{g}{\alpha})^2 dx + b\int_\Omega |\nabla(\rho - \tfrac{g}{\alpha})|^2 dx + \alpha\sigma\int_\Omega (\rho - \tfrac{g}{\alpha})^2 dx = \alpha\int_\Omega (\sigma - u)(\rho - \tfrac{g}{\alpha})^2 dx.$$

Therefore,

$$\frac{d}{dt}\int_\Omega (\rho - \tfrac{g}{\alpha})^2 dx + 2\alpha\sigma\int_\Omega (\rho - \tfrac{g}{\alpha})^2 dx \le C\|u(t) - \sigma\|_{L_1}.$$

Solving this differential inequality on $[T, \infty)$, we have

$$\begin{aligned}\|\rho(t) - \tfrac{g}{\alpha}\|_{L_2}^2 &\le e^{-2\alpha\sigma(t-T)}\|\rho(T) - \tfrac{g}{\alpha}\|_{L_2}^2 + C\int_T^t e^{-2\alpha\sigma(t-s)}\|u(s) - \sigma\|_{L_1} ds \\ &\le e^{-2\alpha\sigma(t-T)}\|\rho(T) - \tfrac{g}{\alpha}\|_{L_2}^2 + C\sup_{T \le s < \infty}\|u(s) - \sigma\|_{L_1}.\end{aligned}$$

Therefore,

$$\limsup_{t\to\infty}\|\rho(t) - \tfrac{g}{\alpha}\|_{L_2}^2 \le C\sup_{T \le s < \infty}\|u(s) - \sigma\|_{L_1}.$$

Meanwhile, in view of Proposition 4.3, the right hand side can be arbitrarily small as $T \uparrow \infty$. Hence, the assertion of proposition is proved. □

Proposition 4.5. *If $\|u_0\|_{L_1} > 0$, then $\eta(t)$ converges to $\overline{\eta}$ in $L_2(\Omega)$.*

Proof. Since $\overline{\eta}$ satisfies (4.1), we can write the equation of η in the form

$$\frac{\partial}{\partial t}(\eta - \overline{\eta}) = c\Delta(\eta - \overline{\eta}) - (h + \beta\sigma)(\eta - \overline{\eta}) - \beta(u - \sigma)\eta. \tag{4.7}$$

Multiply this equation by $\eta - \overline{\eta}$ and integrate the product in Ω. Then,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\eta-\overline{\eta})^2dx + c\int_\Omega |\nabla(\eta-\overline{\eta})|^2dx + (h+\beta\sigma)\int_\Omega(\eta-\overline{\eta})^2dx$$
$$= \beta\int_\Omega (\sigma-u)\eta(\eta-\overline{\eta})dx.$$

We can then argue in an analogous way as in the proof of Proposition 4.4. □

In this way we arrive at the following theorem.

Theorem 4.1. *Let* $\|u_0\|_{L_1} > 0$. *As* $t\to\infty$, $S(t)U_0$ *converges to* $\overline{U}$ *in* $\mathcal{D}_{\frac{7}{8}}$.

5. Numerical Examples

In this section, we demonstrate the behavior of the angiogenesis model (1.1) by two computational examples. The values of parameters are the same in both examples and taken from [1]:

$$a = 3.5\times 10^{-4},\ b = 10^{-5},\ c = 1.0,\ \mu = 0.75, \nu = 4.0,$$
$$\sigma = 1.0,\ f = 0.1,\ \alpha = 0.1,\ g = 0.05,\ h = 0.05,\ \beta = 1.0,$$

the computational domain Ω is $[0,1]^2$. The initial values of endothelial cell density and of fibronectin concentration in two examples are given as in Figure 1. The differences

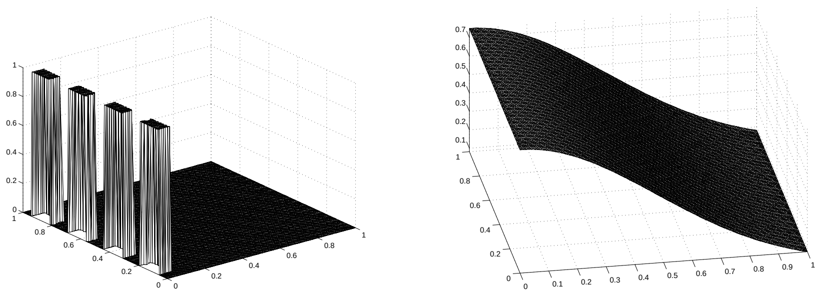

Figure 1. Initial values of endothelial cell density (left) and of fibronectin concentration (right).

between two examples are about initial value of TAF concentration and TAF supplying rate. In the first example, the tumor size is supposed to be small and therefore the TAF supplying source is also restricted in a small area of domain Ω. We increased the tumor size and then the TAF supplying source in the second example. The TAF supplying rates are given respectively:

$$\varphi(x) = \begin{cases} 0.1 & \text{if } \sqrt{(1-x_1)^2+(0.5-x_2)^2} \leqslant 0.1, \\ 0 & \text{otherwise,}\end{cases} \quad \text{in the first example,}$$

$$\varphi(x) = \begin{cases} 0.1 & \text{if } \sqrt{(1-x_1)^2+(0.5-x_2)^2} \leqslant 0.4, \\ 0 & \text{otherwise,}\end{cases} \quad \text{in the second example.}$$

The initial values of TAF concentration are plotted as in Figure 2. For computing, we employed the discontinuous Galerkin method [2] for spatial discretization and the Rosenbrock strong stability–preserving method [3] for temporal discretization.

If the tumor's size is small as in the first example, the numerical results in Figure 3 showed that the blood vessel developed toward the tumor and then it stopped before approaching the tumor. It can be explained that the attraction generated by the chemotaxis in this case is equivalent to the one generated by the haptotaxis. The appearance of haptotaxis also slowed down the development of the blood vessel.

But when the tumor's size becomes bigger as in the second example, the strength of the attraction generated by the chemotaxis is superior to the one generated by the haptotaxis. In this case the development of blood vessel is not only up to the tumor but also much faster than the one in the first example. This process is shown in Figure 4.

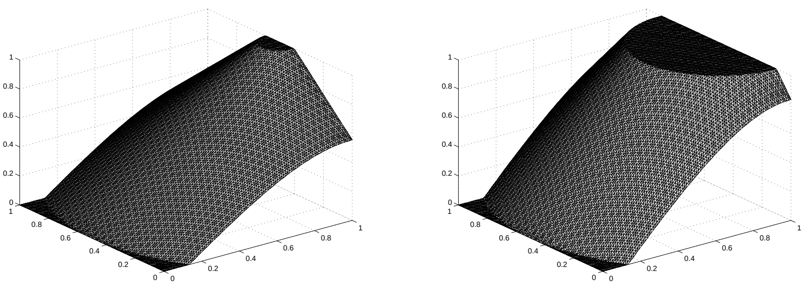

FIGURE 2. Initial values of TAF concetrations in the first (left) and the second (right) example.

FIGURE 3. The development of endothelial cell in case of small size tumor. From the left to right, there are density of endothelial cell at $t = 2.0$, $t = 12.0$, $t = 72.0$, and $t = 100.0$. The light regions represent high density of endothelial cell.

FIGURE 4. The development of endothelial cell in case of big size tumor. From the left to right, there are density of endothelial cell at $t = 2.0$, $t = 6.0$ and $t = 12.0$. The light regions represent high density of endothelial cell.

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Linear approximation of a system of quasilinear hyperbolic equations having linear growth energy functional

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1 Introduction and our main result

In this article we investigate linear application of a system of quasilinear hyperbolic equations. In the sequel $\boldsymbol{R}^{nN}$ denotes the set of all N by n matrices with real elements. Let f be a real valued function defined on $\boldsymbol{R}^{nN}$ and let ε be a positive number. Suppose that u satisfies

$$\frac{\partial^2 u^i}{\partial t^2}(t,x) - \sum_{\alpha=1}^{n} \frac{\partial}{\partial x^\alpha}\{f_{p^i_\alpha}(\nabla u(t,x))\} = 0, \quad i = 1, 2, \ldots, N, \tag{1}$$

in a bounded domain Ω with $u(0,x) = \varepsilon u_0(x), u_t(0,x) = \varepsilon v_0(x)$ and

$$u(t,x) = 0, \qquad x \in \partial\Omega. \tag{2}$$

Our final destination is to show that, as $\varepsilon \to 0$, $u^\varepsilon := \varepsilon^{-1} u$ converges to a weak solution to linearlized equation for (1).

In the case that $N = 1$ and $f(p) = \sqrt{1+|p|^2}$, Equation (1) is

$$\frac{\partial^2 u}{\partial t^2}(t,x) - \sum_{\alpha=1}^{n} \frac{\partial}{\partial x^\alpha}\{(1+|\nabla u(t,x)|^2)^{-1/2}\frac{\partial u}{\partial x^\alpha}\} = 0, \qquad x \in \Omega, \tag{3}$$

which is in [3, 4, 5] referred to as an equation of motion of vibrating membrane. Up to now neither existence nor uniqueness of a solution to (3) is obtained. In [3, 4, 5] we only have that *a sequence of approximate solutions to* (3) *converges to a function u in an appropriate function space*, and that, *if u satisfies the energy conservation law, it is a weak solution to* (3). Instead in [6] linear approximation for (3) is investigated.

For (1), existence and uniqueness are also very difficult problems, but linear approximation seems to be attackable and hence in this article we try it. In this article we suppose that f is linear growth and quasiconvex, more precisely,

(A1) there exist constants m and M such that

$$m|p| \le f(p) \le M(1+|p|) \tag{4}$$

(A2) for each bounded domain $D \subset \boldsymbol{R}^n$, for each $p_0 \in \boldsymbol{R}^{nN}$, and for each $\varphi \in [W_0^{1,\infty}(D)]^N$

$$\frac{1}{\mathcal{L}^n(D)}\int_D f(p_0 + \nabla\varphi(x))dx \ge f(p).$$

The energy functional for the operator $-\sum_{\alpha=1}^{n} \frac{\partial}{\partial x^\alpha}\{f_{p^i_\alpha}(\nabla u(t,x))\}$ is the functional

$$u \mapsto \int_\Omega f(\nabla u(x))dx.$$

When u belongs to $[L^1(\Omega)]^N \setminus [W^{1,1}(\Omega)]^N$, the value of this functional is infinity. But it is not lower semicontinuous and thus we should introduce relaxation. The relaxed functional of the above functional in the $[L^1(\Omega)]^N$ norm, which is denoted by J, is finite for $u = (u^1, u^2, \ldots, u^N) \in [BV(\Omega)]^N$ and is expressed as

$$J(u, \Omega) = \int_\Omega f(\nabla u(x))dx + \int_\Omega f_\infty(\frac{dD^s u}{d|D^s u|})d|D^s u|, \tag{5}$$

where $Du = D^a u + D^s u$ (absolutely continuous part and singular part with respect to $\mathcal{L}^n$), $D^a u = \mathcal{L}^n \llcorner \nabla u$, and $f_\infty(p)$ is defined as, for $p \in \boldsymbol{R}^n$,

$$f_\infty(p) = \limsup_{\rho \to 0} f(\frac{p}{\rho})\rho \tag{6}$$

(see, for example, [1, Theorem 5.47]).

Similarly to the scalar case the most appropriate weak formulation of Dirichlet condition (2) is to replace $J(u, \Omega)$ with $J(u, \overline{\Omega})$. The functional $J(u, \overline{\Omega})$ is expressed as

$$J(u, \overline{\Omega}) = J(u, \Omega) + \int_{\partial\Omega} f_\infty(\gamma u \times \vec{n})d\mathcal{H}^{n-1}, \tag{7}$$

where $\vec{n}$ denotes the inward pointing unit normal to $\partial\Omega$ and $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure.

In this article we further suppose that

(A3) $f \in C^2(\boldsymbol{R}^{nN})$ and f_{pp} is bounded in $\boldsymbol{R}^{nN}$

(A4) there exist positive constants α and C such that $0 < \alpha \leq 1$ and

$$|f_{pp}(p) - f_{pp}(0)| \leq C|p|^\alpha$$

whenever $|p| \leq 1$.

(A5) there exist a positive constant c_0 such that, for each $\varphi \in W_0^{1,\infty}(\Omega)$,

$$\int_\Omega [f(\nabla\varphi) - f(0)]dx \geq c_0 \int_\Omega G(\nabla\varphi)dx,$$

where $G(p) \sim |p|^2$ when $|p| << 1$ and $G(p) \sim |p|$ when $|p| >> 1$.

Remark. Assumption (A5) means a kind of strictness of quasiconvexity at the origin. We remark that, without loss of generality, we may suppose that $G(p) = \dfrac{|p|^2}{\sqrt{1+|p|^2}+1}$, and in the sequel we let G denote this function.

Example. We define $L \subset \boldsymbol{R}^4$ (regarded as the set of all 2 by 2 matrices) as

$$L = \{\begin{pmatrix} t & -s \\ s & t \end{pmatrix}; s,\ t \in \boldsymbol{R}\}.$$

Let K be an arbitrary closed subset of L and we define $f_0(p) = \mathrm{dist}(p, K)$. Then it is proved in [9] that the quasiconvexification $f_1 := Qf_0$ satisfies $f_1(p) >$ for any $p \in \boldsymbol{R}^4 \setminus K$.

Now we put $f_2(p) = (\rho_\sigma * f_1)(p)$, where ρ_σ denotes the standard mollifier, and we define $f(p) = f_2(p) + a\sqrt{1+|p|^2}$. Then, if K is not convex and a is sufficiently small, f satisfies all assumptions (A1)–(A5) and f is not convex.

Suppose that u is a solution to (1) with $u(0,x) = \varepsilon u_0(x), u_t(0,x) = \varepsilon v_0(x)$ and (2). Then u^ε is a weak solution to

$$\frac{\partial^2 u^i}{\partial t^2}(t,x) - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\{\frac{1}{\varepsilon} f_{p_\alpha^i}(\varepsilon \nabla u(t,x))\} = 0, \quad x \in \Omega, \quad i = 1,2,\ldots,N, \tag{8}$$

with

$$u(0,x) = u_0(x), \qquad \frac{\partial u}{\partial t}(0,x) = v_0(x), \qquad x \in \Omega, \tag{9}$$

and (2). A weak solution to this equation is defined as in the following way. We put

$$J_\varepsilon(u,\Omega) = \frac{1}{\varepsilon^2} J(\varepsilon u, \Omega) - f(0)\mathcal{L}^n(\Omega).$$

Note that J_ε is the relaxed functional of $u \mapsto \int_\Omega f_\varepsilon(\nabla u(x))dx$, where $f_\varepsilon(p) = \frac{1}{\varepsilon^2}(f(\varepsilon p) - f(0))$. In [4, 5] a weak solution to (3) is defined as a weak solution to the evolution equation $u_{tt} + \partial(\sqrt{1+|Du|^2}) \ni 0$. In our problem, since J_ε is not convex, this definition is not available. However, since f is quasiconvex (and thus f is rank-one convex), the functional

$$L^2(\Omega) \cap BV(\Omega) \ni v \mapsto J(u^1, \ldots, u^{j-1}, v, u^{j+1}, \ldots, u^N, \overline{\Omega})$$

is convex for each j. Hence we are able to define a weak solution to (8) with (9), (2) as a weak solution to $u_{tt}^j + \partial_{u^j} J(u, \overline{\Omega})$ $(j = 1,2,\ldots,N)$: supposing that $u_0 \in [L^2(\Omega) \cap BV(\Omega)]^N$ and $v_0 \in [L^2(\Omega)]^N$ and putting

$$\mathcal{X} = \{\phi \in L^\infty((0,T); L^2(\Omega) \cap BV(\Omega)); \phi_t \in L^2((0,T) \times \Omega)\},$$

we define

Definition 1 A function u is said to be a solution to (8) with (9), (2) if and only if

i) $u \in [L^\infty((0,T); BV(\Omega))]^N, \quad u_t \in [L^2((0,T) \times \Omega)]^N$

ii) $\text{s-}\lim_{t \searrow 0} u(t,x) = u_0(x)$ in $L^2(\Omega)$

iii) for each $T > 0$ and for any $\phi \in C_0^0([0,T); L^2(\Omega)) \cap \mathcal{X}$, and any $j = 1, \ldots, N$,

$$\begin{aligned}\int_0^T \{J_\varepsilon(u^1, \ldots, u^{j-1}, u^j + \phi, u^{j+1}, \ldots, u^N, \overline{\Omega}) - J_\varepsilon(u, \overline{\Omega})\}dt \\ \geq \int_0^T \int_\Omega u_t^j \phi_t dx dt + \int_\Omega v_0^j(x)\phi(0,x)dx.\end{aligned}$$

Remark. Replacing J_ε with J, we are able to define a weak solution to (1) with (9), (2).

Let us put $f_{p_\alpha^i p_\beta^j}(0) = a_{ij}^{\alpha\beta}$ and define a linear operator

$$Lu = {}^t(\sum_{j=1}^{N} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} a_{ij}^{\alpha\beta} \frac{\partial^2 u^j}{\partial x^\alpha \partial x^\beta}; i = 1,2,\ldots,N).$$

Now we have

Proposition 1 *L is strongly elliptic, namely, there exists a constant m_0 such that*

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} a_{ij}^{\alpha\beta} \xi_\alpha \xi_\beta \eta^i \eta^j \geq m_0 |\xi|^2 |\eta|^2$$

for each $\xi \in \boldsymbol{R}^n$ and each $\eta \in \boldsymbol{R}^N$.

. *Proof.* By (A5), for each $\varphi \in W_0^{1,\infty}(\Omega)$ and each $\varepsilon > 0$,

$$\frac{1}{\varepsilon^2}\int_\Omega [f(\varepsilon\nabla\varphi) - f(0)]dx \geq c_0\int_\Omega G(\nabla\varphi)dx = c_0\int_\Omega \frac{|\nabla\varphi|^2}{\sqrt{1+\varepsilon^2|\nabla\varphi|^2}+1}dx. \tag{10}$$

Since f is of C^2 class, we have by the Taylor expansion

$$\frac{1}{\varepsilon^2}\int_\Omega [f(\varepsilon\nabla\varphi) - f(0)]dx = \frac{1}{\varepsilon^2}\int_\Omega [\varepsilon f_p(0)\nabla\varphi + \frac{1}{2}\int_0^1 < f_{pp}(\theta\varepsilon\nabla\varphi)\nabla\varphi, \nabla\varphi > d\theta]dx, \tag{11}$$

where $< f_{pp}(q)p, p >= \sum_{i=1}^N\sum_{j=1}^N\sum_{\alpha=1}^n\sum_{\beta=1}^n f_{p^i_\alpha p^j_\beta}(q)p^i_\alpha p^j_\beta$. Divergence theorem implies $\int_\Omega f_p(0)\nabla\varphi dx = 0$. Thus we have by (10) and (11)

$$\frac{1}{2}\int_\Omega\int_0^1 < f_{pp}(\theta\varepsilon\nabla\varphi)\nabla\varphi, \nabla\varphi > d\theta dx \geq c_0\int_\Omega \frac{|\nabla\varphi|^2}{\sqrt{1+\varepsilon^2|\nabla\varphi|^2}+1}dx.$$

Letting $\varepsilon \to 0$, we have

$$\int_\Omega < f_{pp}(0)\nabla\varphi, \nabla\varphi > dx \geq c_0\int_\Omega |\nabla\varphi|^2 dx,$$

which is equivalent to that L is strongly elliptic. Q.E.D.

By well-known uniqueness result of linear hyperbolic equations we have the following by Proposition 1.

Corollary 1 *Suppose that $u_0 \in [W_0^{1,2}(\Omega)]^N$ and $v_0 \in [L^2(\Omega)]^N$. A solution to linearized equation*

$$\begin{cases} u_{tt} - Lu = 0, & (t,x) \in (0,T)\times\Omega) \\ u(0,x) = u_0(x), & x \in \Omega \\ u_t(0,x) = v_0(x), & x \in \Omega \\ u(t,x) = 0, & x \in \partial\Omega. \end{cases} \tag{12}$$

is unique in $L^\infty((0,T); W_0^{1,2}(\Omega)) \cap W^{1,2}((0,T)\times\Omega)$ for each $T > 0$.

Our main theorem is as follows:

Theorem 1 *Let T be a positive number. Suppose that $u_0 \in W_0^{1,2}(\Omega)$ and $v_0 \in L^2(\Omega)$ and that u^ε is a weak solution to* (8) with (9), (2) *in $(0,T)\times\Omega$. We further suppose u^ε satisfies energy inequality: for $\mathcal{L}^1$-a.e. $t \in (0,T)$,*

$$\frac{1}{2}\int_\Omega |u_t^\varepsilon(t,x)|^2 dx + J_\varepsilon(u^\varepsilon(t,\cdot),\overline{\Omega}) \leq \frac{1}{2}\int_\Omega |v_0|^2 dx + J_\varepsilon(u_0,\overline{\Omega}). \tag{13}$$

Then there exists a function u such that

1). $\{\| u_t^\varepsilon \|_{L^\infty((0,T);L^2(\Omega))}\}$ *is uniformly bounded with respect to ε*

2). $\{\| u^\varepsilon \|_{L^\infty((0,T);L^2(\Omega)\cap BV(\Omega))}\}$ *is uniformly bounded with respect to ε*

3). *u^ε converges to u as $\varepsilon \to 0$ weakly star in $L^\infty((0,T);L^2(\Omega))$*

4). *u_t^ε converges to u_t as $\varepsilon \to 0$ weakly star in $L^\infty((0,T);L^2(\Omega))$*

5). *u^ε converges to u as $\varepsilon \to 0$ strongly in $L^p((0,T)\times\Omega)$ for each $1 \leq p < 1^*$*

6). *for $\mathcal{L}^1$-a.e. $t \in (0,T)$, $Du^\varepsilon(t,\cdot)$ converges to $Du(t,\cdot)$ as $\varepsilon \to 0$ in the sense of distributions*

7). *$u \in L^\infty((0,T); W_0^{1,2}(\Omega)) \cap W^{1,2}((0,T)\times\Omega)$*

8). *u is a weak solution to* (12) *with* (9), (2).

2 1st step of the proof of Theorem 1

Lemma 1 *For each* $v \in [BV(\Omega)]^N$,

$$J_\varepsilon(v, \overline{\Omega}) \geq c_0\Big(\int_\Omega \frac{|\nabla v|^2}{\sqrt{1+\varepsilon^2|\nabla v|^2}+1}dx + \frac{1}{\varepsilon}|D^s v|(\overline{\Omega})\Big).$$

Before the proof of this lemma we introduce some notations. Let μ be a $\boldsymbol{R}^m$ valued Radon measure. Then we write its total variation as $|\mu|$ and the Radon-Nikodym derivative of μ with respect to $|\mu|$ as $\vec{\mu}$. In particular, $\mu = |\mu| \llcorner \vec{\mu}$. For $v \in [BV(\Omega)]^N$ we define an $\boldsymbol{R}^{nN+1}$ valued Radon measure μ_v by

$$\mu_v = {}^t(Dv, \mathcal{L}^n).$$

For an open set $A \subset \Omega$, total variation $|\mu_v|$ is given by

$$|\mu_v|(A) = \sup\{\int_\Omega (g_0 + v\,\mathrm{div} g)dx; (g_0, g) \in C^1(\Omega, \boldsymbol{R}^{nN+1}),\ |g_0|^2 + |g|^2 \leq 1\}$$

In this article, for the sake of simplicity, we write $S_+^{nN+1} = S_+$:

$$S_+ = \{\vec{s} = (s^1, \cdots, s^{nN+1}) \in S^{nN}; s^{nN+1} > 0\}.$$

We also write

$$S_0 = \{\vec{s} = (s^1, \cdots, s^{nN+1}) \in S^n; s^{nN+1} = 0\}. \tag{14}$$

Then $\overline{S}_+ = S_+ \cup S_0$. Given a Radon measure λ in $\overline{\Omega} \times \overline{S}_+$, we let $|\lambda|$ denote a Radon measure on $\overline{\Omega}$ defined by

$$|\lambda|(A) = \lambda(A \times \overline{S}_+) \qquad \text{for a Borel set } A \subset \overline{\Omega}.$$

Clearly this notation is an analogy with that of a total variations of a vector valued Radon measure. In particular, letting λ be a Radon measure in $\overline{\Omega} \times \overline{S}_+$ defined as, for a BV function $v \in [BV(\Omega)]^N$,

$$\int_{\overline{\Omega}\times\overline{S}_+} \beta(x, \vec{s})d\lambda = \int_{\overline{\Omega}} \beta(x, \vec{\mu_v}(x))d|\mu_v| \qquad (\beta \in C^0(\overline{\Omega} \times \overline{S}_+)), \tag{15}$$

then we have $|\lambda| = |\mu_v|$. For each Radon measure λ in $\overline{\Omega} \times \overline{S}_+$, there exists a probability Radon measure $\nu_{\lambda,x}$ on $\overline{S}_+$ for $|\lambda|$-a.e. $x \in \overline{\Omega}$ such that

$$\int_{\overline{\Omega}\times\overline{S}_+} \beta(x, \vec{s})d\lambda = \int_{\overline{\Omega}}\Big(\int_{\overline{S}_+} \beta(x, \vec{s})d\nu_{\lambda,x}\Big)d|\lambda| \qquad (\beta \in C^0(\overline{\Omega} \times \overline{S}_+))$$

(for example, Theorem 10 of page 14 of [2]). Using these notations, we often write $\lambda = |\lambda| \otimes \nu_{\lambda,x}$. In particular, if λ is as in (15), then $\lambda = |\mu_v| \otimes \delta_{\vec{\mu}_v(x)}$.

In the proof of Lemma 1 we use the following theorem (compare to Proposition 3 of [7]).

Theorem 2 *Suppose that* $\sqrt{1+|Du_k|^2}(\overline{\Omega}) \to \sqrt{1+|Du|^2}(\overline{\Omega})$. *Then* $|\mu_{u_k}| \otimes \delta_{\vec{\mu}_{u_k}} \rightharpoonup |\mu_u| \otimes \delta_{\vec{\mu}_u}$ *in the sense of Radon measures in* $\overline{\Omega} \times \overline{S}_+$.

Proof of Lemma 1. For each $v \in [BV(\Omega)]^N$ there exists a sequence $\varphi_k \in C^1(\Omega)$ such that $\varphi_k \to v$ strongly in $[L^1(\Omega)]^N$ and

$$\lim_{k\to\infty} \sqrt{1+|D\varphi_k|^2}(\overline{\Omega}) = \sqrt{1+|Dv|^2}(\overline{\Omega})$$

Then by Theorem 2 and upper semicontinuity of f_ε^*

$$
\begin{aligned}
(16)\qquad \limsup_{k\to\infty} J_\varepsilon(\varphi_k,\overline{\Omega}) &= \limsup_{k\to\infty}\int_{\overline{\Omega}\times\overline{S}_+} f_\varepsilon^*(\vec{s})d|\mu_{\varphi_k}|\otimes\delta_{\mu_{\vec{\varphi}_k}} \\
&\le \int_{\overline{\Omega}\times\overline{S}_+} f_\varepsilon^*(\vec{s})d|\mu_v|\otimes\delta_{\vec{\mu_v}} = J_\varepsilon(v,\overline{\Omega}).
\end{aligned}
$$

Now we have by (A5)

$$
\begin{aligned}
(17)\qquad J_\varepsilon(\varphi_k,\overline{\Omega}) &\ge \int_\Omega \frac{1}{\varepsilon^2}(f(\varepsilon\nabla\varphi_k)-f(0))dx \ge \frac{c_0}{\varepsilon^2}\int_\Omega G(\varepsilon\nabla\varphi_k)dx \\
&= \frac{c_0}{\varepsilon^2}\int_\Omega \frac{|\varepsilon\nabla\varphi_k|^2}{\sqrt{1+\varepsilon^2|\nabla\varphi_k|^2}+1}dx = c_0\int_\Omega \frac{|\nabla\varphi_k|^2}{\sqrt{1+\varepsilon^2|\nabla\varphi_k|^2}+1}dx
\end{aligned}
$$

Since $\dfrac{|\nabla\varphi_k|^2}{\sqrt{1+\varepsilon^2|\nabla\varphi_k|^2}+1} = \dfrac{1}{\varepsilon^2}(\sqrt{1+\varepsilon^2|\nabla\varphi_k|^2}-1)$, we have by the lower semicontinuity

$$
\begin{aligned}
(18)\qquad \liminf_{k\to\infty}\int_\Omega \frac{|\nabla\varphi_k|^2}{\sqrt{1+\varepsilon^2|\nabla\varphi_k|^2}+1}dx &\ge \frac{1}{\varepsilon^2}\int_\Omega(\sqrt{1+\varepsilon^2|Dv|^2}-1) \\
&= \int_\Omega \frac{|\nabla v|^2}{\sqrt{1+\varepsilon^2|\nabla v|^2}+1}dx + \frac{1}{\varepsilon}|D^s v|(\Omega).
\end{aligned}
$$

Combining (16), (17), (18), we obtain the assertion. Q.E.D.

Proposition 2 *There exists a function u such that, up to a subsequence, Assertions* 1) $\sim$ 6) *of Theorem 1 hold. Furthermore the function u satisfies*

a). $u \in L^\infty((0,T); BV(\Omega)\cap L^2(\Omega))$

b). $\text{s-}\lim_{t\searrow 0} u(t) = u_0$ *in* $L^2(\Omega)$

Proof. By (A3) we have

$$
(19)\qquad f_\varepsilon(p) = \frac{1}{\varepsilon}f_p(0):p + \int_0^1 < f_{pp}(\theta\varepsilon p)p, p > d\theta
$$

and furthermore there exists a constant C_1 such that $|f_{pp}(p)| \le C_1$. Since

$$
\int_\Omega f_p(0)\nabla u_0(x)dx = 0,
$$

we find

$$
(20)\qquad J_\varepsilon(u_0,\overline{\Omega}) \le C_1\int_\Omega |\nabla u_0(x)|^2 dx.
$$

Thus Assertion 1) immediately follows from (13). Since the function $\varepsilon \mapsto \varepsilon^{-2}(\sqrt{1+\varepsilon^2|p|^2}-1)$ is decreasing, we have by Lemma 1

$$
J_\varepsilon(u^\varepsilon,\overline{\Omega}) \ge \int_\Omega \frac{1}{\varepsilon^2}(\sqrt{1+\varepsilon^2|Du^\varepsilon|^2}-1) \ge \int_\Omega(\sqrt{1+|Du^\varepsilon|^2}-1)
$$

Thus it also follows from (13) and (20) that $\{\|\,|Du^\varepsilon|(\Omega)\,\|_{L^\infty(0,T)}\}$ is uniformly bounded with respect to ε. Then Assertion 2) follows from Assertion 1) because

$$
u^\varepsilon(t,x) = u_0(x) + \int_0^t u_t^\varepsilon(s,x)ds.
$$

Passing to a subsequence if necessary, we have Assertions 3) and 4) by Assertions 2) and 1), respectively. By Sobolev's theorem $BV(\Omega) \subset L^p(\Omega)$ compactly for each $1 \le p < 1^*$. Then in the same way as in the proof of [3, Proposition 5.1], passing to a subsequence if necessary, we obtain Assertion 5). Assertions 3) and 5) imply $u \in L^\infty((0,\infty); BV(\Omega) \cap L^2(\Omega))$. Assertion 6) follows from 5). Assertion b) is obtained in the same way as in the proof of [8, Theorem 4.1]. Q.E.D.

Rests are proofs of Assertions 7) and 8).

3 Key propositions (2nd step of the proof of Theorem 1)

The key of the proof of Theorem 1 is the following two propositions. Proofs of them are essentially the same as those of Lemma 4.4 and Proposition 4.2 of [6], which are mentioned in terms of varifolds. In this article, taking account of its importance, we present the proof of Proposition 3, and the proof of Proposition 4 is omitted.

Proposition 3 $|D^s u(t,\cdot)|(\overline{\Omega}) = 0$

Remark that this proposition implies, in particular, $\gamma u = 0$.

Proposition 4 $u \in [L^\infty((0,T); W^{1,2}(\Omega))]^N$

Proposition 3 implies that the distributional derivative Du coincides with ∇u and hence $u(t,\cdot) \in W_0^{1,1}(\Omega)$ for $\mathcal{L}^1$-a.e. t. Thus, combining these two propositions, we have Assertion 7).

Let u^ε and u be as in Proposition 2. Then there are one parameter families of $\boldsymbol{R}^{nN+1}$-valued Radon measures $\mu_{u^\varepsilon(t,\cdot)}$, $\mu_{u(t,\cdot)}$ in $\overline{\Omega}$, which are in the sequel simply denoted by μ_t^ε, μ_t, respectively. By Theorem 1 2), which is proved in Proposition 2, there exists a constant K which is independent of ε such that

$$\operatorname*{ess.\,sup}_{t>0} |\mu_t^\varepsilon|(\overline{\Omega}) \le K. \tag{21}$$

Thus, for any $\beta \in C^0(\overline{\Omega} \times \overline{S}_+)$,

$$\operatorname*{ess.\,sup}_{t>0} \left| \int_{\overline{\Omega}\times\overline{S}_+} \beta(x,\vec{s}) d|\mu_t^\varepsilon| \otimes \delta_{\vec{\mu}_t^\varepsilon(x)} \right| \le K \sup |\beta|. \tag{22}$$

By the use of (22) and standard compactness argument we obtain the following lemma (see, for example, [3, Proposition 4.3]).

Lemma 2 *There exists a subsequence of $\{\varepsilon\}$ (still denoted by $\{\varepsilon\}$) and a one parameter family of Radon measures λ_t in $\overline{\Omega} \times \overline{S}_+$, $t \in (0,\infty)$, such that, for each $\psi \in L^1(0,\infty)$ and $\beta \in C^0(\overline{\Omega} \times \overline{S}_+)$,*

$$\lim_{\varepsilon \to 0} \int_0^\infty \psi(t) \int_{\overline{\Omega}\times\overline{S}_+} \beta(x,\vec{s}) d|\mu_t^\varepsilon| \otimes \delta_{\vec{\mu}_t^\varepsilon(x)} dt = \int_0^\infty \psi(t) \int_{\overline{\Omega}\times\overline{S}_+} \beta(x,\vec{s}) d\lambda_t dt.$$

Proposition 5 *Then it holds that*

1). $\mu_t = |\lambda| \llcorner \int_{\overline{S}_+} \vec{s} d\nu_{\lambda_t,x}$

2). $|\lambda_t|(A) \ge |\mu_t|(A)$ *for each Borel set* $A \subset \Omega$

3). $|\lambda_t|(A) = \int_A D_{|\mu_t|}|\lambda_t|(x) d|\mu_t| + (|\lambda_t| \llcorner Z)(A)$ *for* $A \subset \Omega$, *where* $D_{|\mu_t|}|\lambda_t|$ *is the derivative of* $|\lambda_t|$ *with respect to* $|\mu_t|$ *and* Z *is the* $|\mu_t|$*-null set defined by* $Z = \{x; D_{|\mu_t|}|\lambda_t|(x) = \infty\}$

4). $\int_{\overline{S}_+} \vec{s}d\nu_{\lambda_t,x} = 0$ *for* $|\lambda_t| \llcorner Z$*-a.e.* x

5). spt $\nu_{\lambda_t,x} \subset S_0$ *for* $|\lambda_t| \llcorner Z$*-a.e.* x*, where* S_0 *is as in (14)*

6). $|D^s u(t,\cdot)|(\Omega) \leq \lambda_t(\Omega \times S_0)$.

The proof of this proposition is essentially same as that of [6, Proposition 3.3]. Thus we omit it.

Let us put $\lambda_t^\varepsilon = |\mu_t^\varepsilon| \otimes \delta_{\vec{\mu}_t^\varepsilon}$. As a collorary of Lemma 2 we have

Lemma 3 *Put* $\overline{\lambda}^\varepsilon = \int_0^T \lambda_t^\varepsilon dt$ *and* $\overline{\lambda} = \int_0^T \lambda_t dt$*. Then* $\overline{\lambda}^\varepsilon \stackrel{*}{\rightharpoonup} \overline{\lambda}$ *in the sense of Radon measures in* $\overline{\Omega} \times \overline{S}_+$*.*

Proof. Letting $\psi = \chi_{[0,T]}$, the characteristic function of $[0,T]$, in Lemma 2, we immediately have the conclusion. Q.E.D.

Proof of Proposition 3. If we have $\lambda_t(\overline{\Omega} \times S_0) = 0$ for $\mathcal{L}^1$-a.e. t, then the conclusion immediately follows from Lemma 5 6), and, since $\lambda_t \geq 0$, this follows if we have

$$\overline{\lambda}(\overline{\Omega} \times S_0) = 0. \tag{23}$$

Hence we prove (23).

By the definition of μ_t^ε we immediately obtain

$$\int_\Omega \frac{|\nabla u^\varepsilon|^2}{\sqrt{1+\varepsilon^2|\nabla u^\varepsilon|^2}+1}dx + \frac{1}{\varepsilon}|Du^\varepsilon|(\overline{\Omega}) = \int_{\overline{\Omega}\times\overline{S}_+} \frac{|\vec{s}'|^2}{\sqrt{(s^{nN+1})^2+\varepsilon^2|\vec{s}'|^2}+s^{nN+1}}d\lambda_t^\varepsilon, \tag{24}$$

the left hand side of which is estimated from above by $C_2(\frac{1}{2}\int_\Omega |v_0|^2dx + \int_\Omega |\nabla u_0(x)|^2dx)$ in (28). On the other hand, given $\sigma > 0$, we have

$$\int_{\overline{\Omega}\times\overline{S}_+\cap\{s^{nN+1}<\sigma\}} \frac{|\vec{s}'|^2}{\sqrt{(s^{nN+1})^2+\varepsilon^2|\vec{s}'|^2}+s^{nN+1}}d\lambda_t^\varepsilon \geq \frac{1-\sigma^2}{\sqrt{\sigma^2+\varepsilon^2}+\sigma}\lambda_t^\varepsilon(\overline{\Omega}\times\overline{S}_+\cap\{s^{nN+1}<\sigma\}).$$

Integrating from 0 to T, we have by (28) and (24)

$$\overline{\lambda}^\varepsilon(\overline{\Omega}\times\overline{S}_+\cap\{s^{nN+1}<\sigma\}) \leq \frac{\sqrt{\sigma^2+\varepsilon^2}+\sigma}{1-\sigma^2}C_2(\frac{1}{2}\int_\Omega |v_0|^2dx + \int_\Omega |\nabla u_0(x)|^2dx)T.$$

Thus, letting $\varepsilon \to 0$, we have by Lemma 3 and the lower semicontinuity of Radon measures

$$\overline{\lambda}(\overline{\Omega}\times\overline{S}_+\cap\{s^{nN+1}<\sigma\}) \leq \frac{2\sigma}{1-\sigma^2}C_2(\frac{1}{2}\int_\Omega |v_0|^2dx + \int_\Omega |\nabla u_0(x)|^2dx)T. \tag{25}$$

Letting $\sigma \to 0$, we have (23). Q.E.D.

4 Completion of the proof of Theorem 1

Now the rest is the proof of 8). We already have (9) and (2) in Proposition 2. Hence we only have to show u satisfies (12) in the weak sense. Noting

$$\int_{\{|\nabla u^\varepsilon|<1/\varepsilon\}} \frac{|\nabla u^\varepsilon|^2}{\sqrt{1+\varepsilon^2|\nabla u^\varepsilon|^2}+1} dx \geq \frac{1}{\sqrt{2}+1} \int_{\{|\nabla u^\varepsilon|<1/\varepsilon\}} |\nabla u^\varepsilon|^2 dx,$$

we have by (28)

$$\int_{\{|\nabla u^\varepsilon|<1/\varepsilon\}} |\nabla u^\varepsilon|^2 dx \leq C_2(\sqrt{2}+1)(\frac{1}{2}\int_\Omega |v_0|^2 dx + \int_\Omega |\nabla u_0(x)|^2 dx). \tag{26}$$

Let $\phi \in [C_0^1([0,T)\times\Omega)]^N$. Then, since the functional $u \mapsto J(u^1,\ldots,u^{j-1},u,u^{j+1},\ldots,u^N)$ is convex, we have by Definition 1 iii)

$$\int_\Omega (f_\varepsilon)_{p^j}(\nabla u_\varepsilon)\nabla\phi^j dx = \int_0^T \int_\Omega u_t^j \phi_t dx dt + \int_\Omega v_0^j(x)\phi(0,x)dx$$

for each $j = 1, 2, \ldots, N$, namely,

$$\int_\Omega \frac{1}{\varepsilon} f_p(\varepsilon\nabla u_\varepsilon) : \nabla\phi dx = \int_0^T \int_\Omega u_t\phi_t dx dt + \int_\Omega v_0(x)\phi(0,x)dx. \tag{27}$$

Now, for each $\phi \in [C_0^1([0,T)\times\Omega)]^N$

$$\begin{aligned}\int_\Omega \frac{1}{\varepsilon} f_p(\varepsilon\nabla u^\varepsilon) : \nabla\phi dx &= \int_\Omega (\frac{1}{\varepsilon} f_p(0) : \nabla\phi + \int_0^1 < f_{pp}(\varepsilon\theta\nabla u^\varepsilon)\nabla u^\varepsilon, \nabla\phi > d\theta)dx \\ &= \int_\Omega \int_0^1 < f_{pp}(\varepsilon\theta\nabla u^\varepsilon)\nabla u^\varepsilon, \nabla\phi > d\theta dx.\end{aligned}$$

Now

$$\begin{aligned}&\int_{\{|\nabla u^\varepsilon|<1/\varepsilon\}} \int_0^1 < f_{pp}(\varepsilon\theta\nabla u^\varepsilon)\nabla u^\varepsilon, \nabla\phi > d\theta dx \\ = &\int_{\{|\nabla u^\varepsilon|<1/\varepsilon\}} < f_{pp}(0)\nabla u^\varepsilon, \nabla\phi > dx + \int_0^1 < [f_{pp}(\varepsilon\theta\nabla u^\varepsilon) - f_{pp}(0)]\nabla u^\varepsilon, \nabla\phi > d\theta dx \\ =: &\ I + II.\end{aligned}$$

Lemma 4 $\displaystyle\int_{\{|\nabla u^\varepsilon|\geq 1/\varepsilon\}} |\nabla u^\varepsilon(x)|dx + |D^s u^\varepsilon|(\overline{\Omega}) \to 0$ *as* $\varepsilon \to 0$.

The proof of this lemma is the same as that of [6, Lemma A.1], thus we omit it. But we note that in the proof we have

$$\int_\Omega \frac{|\nabla u^\varepsilon|^2}{\sqrt{1+\varepsilon^2|\nabla u^\varepsilon|^2}+1} dx + \frac{1}{\varepsilon}|D^s u^\varepsilon|(\overline{\Omega}) \leq C_2(\frac{1}{2}\int_\Omega |v_0|^2 dx + \int_\Omega |\nabla u_0(x)|^2 dx). \tag{28}$$

By Lemma 4 we have

$$\begin{aligned}I &= \int_\Omega < f_{pp}(0)dDu^\varepsilon, \nabla\phi > \\ &\qquad -(\int_{\{|\nabla u^\varepsilon|\geq 1/\varepsilon\}} < f_{pp}(0)\nabla u^\varepsilon, \nabla\phi > dx + \int_\Omega < f_{pp}(0)dD^s u^\varepsilon, \nabla\phi >) \\ &\to \int_\Omega < f_{pp}(0)dDu, \nabla\phi > (= \int_\Omega \sum_{i=1}^N \sum_{j=1}^N \sum_{\alpha=1}^n \sum_{\beta=1}^n a_{ij}^{\alpha\beta} \frac{\partial u^j}{\partial x^\beta} \frac{\partial \phi^i}{\partial x^\alpha} dx).\end{aligned}$$

Next, since (A4) implies

$$|[f_{pp}(\varepsilon\theta\nabla u^\varepsilon) - f_{pp}(0)]\nabla u^\varepsilon| \leq \text{Const. } \varepsilon^\alpha |\nabla u^\varepsilon|^{\alpha+1},$$

we have by (26)

$$\begin{aligned}
|II| &\leq \text{Const. } \varepsilon^\alpha \sup|\nabla\phi| \int_{\{|\nabla u^\varepsilon|<1/\varepsilon\}} |\nabla u^\varepsilon|^{\alpha+1} dx \\
&\leq \text{Const. } \varepsilon^\alpha \sup|\nabla\phi| \mathcal{L}^n(\Omega)^{\frac{1-\alpha}{2}} \Big(\int_{\{|\nabla u^\varepsilon|<1/\varepsilon\}} |\nabla u^\varepsilon|^2 dx\Big)^{\frac{\alpha+1}{2}} \\
&\leq \text{Const. } \varepsilon^\alpha \sup|\nabla\phi| \mathcal{L}^n(\Omega)^{\frac{1-\alpha}{2}} \{C_2(\sqrt{2}+1)(\frac{1}{2}\int_\Omega |v_0|^2 dx + \int_\Omega |\nabla u_0(x)|^2 dx)\}^{\frac{\alpha+1}{2}} \to 0
\end{aligned}$$

as $\varepsilon \to 0$. Hence, letting $\varepsilon \to 0$, we have by (27)

$$\int_0^T \{-\int_\Omega u_t \phi_t(t,x) dx + \int_\Omega \sum_{i=1}^N \sum_{j=1}^N \sum_{\alpha=1}^n \sum_{\beta=1}^n a_{ij}^{\alpha\beta} \frac{\partial u^j}{\partial x^\beta} \frac{\partial \phi^i}{\partial x^\alpha} dx\} dt = \int_\Omega v_0(x)\phi(0,x) dx,$$

which means u satisfies (12) in a weak sense.

Finally the uniqueness of a solution to (12) (Corollary 1) implies the rest of the subsequence has another subsequence that converges to the same function u. Thus we do not have to subtract a subsequence. This completes the proof of Theorem 1.

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REFINED PROPERTIES OF EVOLUTION OPERATOR UNDER KATO-TANABE CONDITIONS

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1. Introduction

We are concerned with the Cauchy problem for a linear abstract parabolic evolution equation

$$\begin{cases} \dfrac{dU}{dt} + A(t)U = F(t), \qquad 0 < t \leq T, \\ U(0) = U_0, \end{cases} \tag{1.1}$$

in a Banach space X (see [3, 4, 7]). Here, $A(t)$, $0 \leq t \leq T$, is a family of densely defined, closed linear operators acting in X and each $-A(t)$ is assumed to be the generator of an analytic semigroup on X, more precisely to satisfy the conditions (2.1) and (2.2).

Treatments for (1.1) are quite different depending on the nature of varying of the domains $\mathcal{D}(A(t))$ of $A(t)$ with respect to the temporal variable t. The case when $\mathcal{D}(A(t))$ vary in a temperate manner (including that of constant domains, i.e., $\mathcal{D}(A(t)) \equiv \mathcal{D}$) was handled in author's monograph [7, Chapter 3]. In the case when $\mathcal{D}(A(t))$ vary completely, one can treat Problem (1.1) under two different conditions; one is Tanabe's condition [2] (i.e., [8, (2.3)-(2.4)]) and the other Kato-Tanabe's condition [1] (i.e., (2.3)-(2.4) below). This paper is concerned with the Kato-Tanabe's condition. Under that, Kato-Tanabe [1] has constructed an evolution operator $U(t,s)$ which plays a role of the fundamental solution for (1.1), see Theorems 3.1 and 3.2. This paper then shows several refined properties of the $U(t,s)$ which were not seen in [1].

Among others, we shall prove the uniform estimate $\|A(t)^\theta U(t,s)A(s)^{-\theta}\|_{\mathcal{L}(X)} \leq C$, $0 \leq s \leq t \leq T$, for all exponents $0 \leq \theta \leq 1$. This property is indeed one of the important properties of the evolution operator for (1.1). Under the Tanabe's condition, this uniform estimate was already seen in [5]. We shall employ the techniques of using integral equations of Volterra type as done in author's old paper [6] in which Problem (1.1) of completely variable domains of $A(t)$ was treated under more general assumptions than both of Tanabe and Kato-Tanabe.

The author has recently shown in [8] that the Tanabe's condition implies an extra spatial regularity $\|A(t)^\theta U(t,s)\|_{\mathcal{L}(X)} \leq C(t-s)^{-\theta}$ for some suitable exponent $\theta > 1$ that played a crucial role in establishing the Hölder type maximal regularity of solutions for (1.1) in the paper. In the present case, however, the condition (2.3)-(2.4) does not seem to imply such an extra spatial regularity, see Remark 4.1.

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2. Structural Assumptions

Let X be a Banach space with norm $\|\cdot\|$. Consider a family of densely defined, closed linear operators $A(t)$, $0 \le t \le T$, acting in X. We assume for each $A(t)$ that its spectrum $\sigma(A(t))$ is contained in a fixed sectorial open domain

$$\sigma(A(t)) \subset \Sigma = \{\lambda \in \mathbb{C};\ |\arg \lambda| < \omega\}, \tag{2.1}$$

where $0 < \omega < \frac{\pi}{2}$, and the resolvent $(\lambda - A(t))^{-1}$ satisfies

$$\|(\lambda - A(t))^{-1}\|_{\mathcal{L}(X)} \le \frac{M}{|\lambda|}, \qquad \lambda \notin \Sigma,\ 0 \le t \le T, \tag{2.2}$$

with some constant $M > 0$. As $\sigma(A(t))$ is a closed set, (2.1) implicitly means that $0 \notin \sigma(A(t))$, namely, $A(t)^{-1}$ is a bounded operator on X. We further assume that $A(t)^{-1}$ is strongly, continuously differentiable on X for $0 \le t \le T$ and that the derivative satisfies the declining estimate

$$\left\| A(t)(\lambda - A(t))^{-1} \frac{dA(t)^{-1}}{dt} A(t)(\lambda - A(t))^{-1} \right\|_{\mathcal{L}(X)} \le \frac{N}{|\lambda|^{\nu}}, \quad \lambda \notin \Sigma,\ 0 \le t \le T, \tag{2.3}$$

with some exponent $0 < \nu \le 1$ and a constant $N > 0$, and the Hölder condition

$$\left\| \frac{dA(t)^{-1}}{dt} - \frac{dA(s)^{-1}}{ds} \right\|_{\mathcal{L}(X)} \le L|t-s|^{\rho}, \qquad 0 \le t,\ s \le T, \tag{2.4}$$

with some exponent $0 < \rho \le 1$ and a constant $L > 0$.

We next notice some immediate consequences from the structural assumptions (2.1), (2.2), (2.3) and (2.4).

Firstly, since $A(t)^{-1}$ is continuously differentiable, we have $\sup_{0\le t \le T} \left\| \frac{d}{dt} A(t)^{-1} \right\|_{\mathcal{L}(X)} < \infty$. From $A(t)^{-1} = A(0)^{-1} + \int_0^t \frac{d}{ds} A(s)^{-1} ds$, we observe that

$$\|A(t)^{-1}\|_{\mathcal{L}(X)} \le D, \qquad 0 \le t \le T,$$

with some constant $D > 0$. Therefore, there exists some constant $\delta > 0$ such that

$$\{\lambda \in \mathbb{C};\ |\lambda| \le \delta\} \subset \rho(A(t)), \qquad 0 \le t \le T. \tag{2.5}$$

Secondly, (2.1) and (2.2) yield that each $-A(t)$ generates an analytic semigroup $e^{-\tau A(t)}$, $0 \le \tau < \infty$, on X. And for $\tau > 0$, the semigroup is given by the Dunford integral

$$e^{-\tau A(t)} = \frac{1}{2\pi i} \int_{\Gamma} e^{-\tau \lambda} (\lambda - A(t))^{-1} d\lambda, \qquad 0 \le t \le T, \tag{2.6}$$

in the space $\mathcal{L}(X)$, where Γ is an infinite integral contour lying in $\rho(A(t))$ and encircling $\sigma(A(t))$ anticlockwise. Furthermore, for $0 < \theta < \infty$, let $A(t)^{\theta}$ be the fractional power of $A(t)$. Then, for $\tau > 0$,

$$A(t)^{\theta} e^{-\tau A(t)} = \frac{1}{2\pi i} \int_{\Gamma^{\delta}} \lambda^{\theta} e^{-\tau\lambda} (\lambda - A(t))^{-1} d\lambda, \qquad 0 \le t \le T,\ 0 < \theta < \infty, \tag{2.7}$$

where $\Gamma^{\delta} = \Gamma^{\delta 1} \cup \Gamma^{\delta 2}$ such that $\Gamma^{\delta 1} : \lambda = \delta e^{i\theta}$, $|\theta| \le \omega$, and $\Gamma^{\delta 2} : \lambda = r e^{\pm i\omega}$, $\delta \le r < \infty$, with the δ appearing in (2.5). It is known that, for $0 \le \theta \le 2$,

$$\|A(t)^{\theta} e^{-\tau A(t)}\|_{\mathcal{L}(X)} \le C\tau^{-\theta}, \qquad 0 \le t \le T,\ \tau > 0. \tag{2.8}$$

Thirdly, the differentiability of $A(t)^{-1}$ implies that the resolvent $(\lambda - A(t))^{-1}$, $\lambda \notin \Sigma$, is also strongly differentiable for $0 \le t \le T$ with the derivative

$$\frac{\partial}{\partial t}(\lambda - A(t))^{-1} = -A(t)(\lambda - A(t))^{-1}\frac{dA(t)^{-1}}{dt}A(t)(\lambda - A(t))^{-1}. \tag{2.9}$$

So, (2.3) directly yields that

$$\left\|\frac{\partial}{\partial t}(\lambda - A(t))^{-1}\right\|_{\mathcal{L}(X)} \le N|\lambda|^{-\nu}, \qquad 0 \le t \le T,\ \lambda \in \Gamma. \tag{2.10}$$

Finally, let us verify that a family of the Yosida approximations $A_n(t)$ also satisfies the same structural assumptions. For $n = 1, 2, 3, \dots$, $A_n(t)$ is defined by

$$A_n(t) = nA(t)(n + A(t))^{-1}, \qquad 0 \le t \le T, \tag{2.11}$$

and is called the Yosida approximation of $A(t)$. Obviously, each $A_n(t)$ is a bounded operator on X. It is easy to see that each $A_n(t)$ satisfies (2.1) with the same angle ω and also satisfies the estimate

$$\|(\lambda - A_n(t))^{-1}\|_{\mathcal{L}(X)} \le \frac{\widetilde{M}}{|\lambda|}, \qquad \lambda \notin \Sigma,\ 0 \le t \le T, \tag{2.12}$$

with some constant $\widetilde{M}$ independent of n. It follows from (2.11) that $A_n(t)^{-1} = n^{-1} + A(t)^{-1}$. Therefore $\frac{dA_n(t)^{-1}}{dt} = \frac{dA(t)^{-1}}{dt}$. This then yields that it holds true for $A_n(t)$, too, that

$$\left\|A_n(t)(\lambda - A_n(t))^{-1}\frac{dA_n(t)^{-1}}{dt}A_n(t)(\lambda - A_n(t))^{-1}\right\|_{\mathcal{L}(X)} \le \frac{\widetilde{N}}{|\lambda|^{\nu}}, \tag{2.13}$$

with some constant $\widetilde{N}$ independent of n. It is clear that the derivative $\frac{dA_n(t)^{-1}}{dt}$ satisfies the same condition (2.4). Thus, the family of $A_n(t)$ is verified to satisfy the similar conditions as (2.1), (2.2), (2.3) and (2.4) with the same angle ω and the same exponents μ, ν as $A(t)$, $\widetilde{M}$ and $\widetilde{N}$ being independent of n. In addition, it is also verified that, as $n \to \infty$,

$$(\lambda - A_n(t))^{-1} \to (\lambda - A(t))^{-1} \qquad \text{strongly on } X \text{ for } \lambda \notin \Sigma. \tag{2.14}$$

Throughout the paper, we shall denote by C a universal constant which is determined only by ω, μ, ν, M and N. So, it may change from occurrence to occurrence.

Remark 2.1. We constructed in [6] an evolution operator for Problem (1.1) under somewhat general assumption [6, (1.2)] than the (2.3)-(2.4) above. Similar results of the present paper can be proved even under such a weaker condition. □

3. Evolution Operator

This section is devoted to reviewing construction of an evolution operator for $A(t)$. We will argue along the similar method as for the proof of [6, Theorem 1].

For $n = 1, 2, 3, \dots$, let $A_n(t)$ be the Yosida approximation of $A(t)$. For $A_n(t)$, the evolution operator $U_n(t, s)$, $0 \le s \le t \le T$, is uniquely constructed. Indeed, $U_n(t, s)$ satisfies $\frac{\partial}{\partial t}U_n(t, s) = -A_n(t)U_n(t, s)$ and $\frac{\partial}{\partial s}U_n(t, s) = U_n(t, s)A_n(s)$ for $0 \le s \le t \le T$. Furthermore, $U_n(t, s)$ has the generalized semigroup property: $U_n(t, r)U_n(r, s) = U_n(t, s)$ for $0 \le s \le r \le t \le T$ and $U_n(s, s) = I$ for $0 \le s \le T$.

3.1. **Integral equations for $U_n(t,s)$.** It is possible to combine $U_n(t,s)$ with the semigroup $e^{-(t-s)A_n(s)}$ by an integral equation. In fact,

$$\begin{aligned} U_n(t,s) - e^{-(t-s)A_n(s)} &= \int_s^t \frac{\partial}{\partial \tau}[e^{-(t-\tau)A_n(\tau)}U_n(\tau,s)]d\tau \\ &= \int_s^t \left[\frac{\partial}{\partial \tau}e^{-(t-\tau)A_n(\tau)} - e^{-(t-\tau)A_n(\tau)}A_n(\tau)\right] U_n(\tau,s)d\tau. \end{aligned}$$

Hence,

$$U_n(t,s) = e^{-(t-s)A_n(s)} + \int_s^t P_n(t,\tau)U_n(\tau,s)d\tau \tag{3.1}$$

with

$$P_n(t,s) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right) e^{-(t-s)A_n(s)}, \qquad 0 \le s \le t \le T. \tag{3.2}$$

Similarly,

$$\begin{aligned} U_n(t,s) - e^{-(t-s)A_n(t)} &= -\int_s^t U_n(t,\tau)\left[A_n(\tau)e^{-(\tau-s)A_n(\tau)} + \frac{\partial}{\partial \tau}e^{-(\tau-s)A_n(\tau)}\right] d\tau \\ &= -\int_s^t U_n(t,\tau)\left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial s}\right) e^{-(\tau-s)A_n(\tau)}d\tau \end{aligned}$$

with

$$Q_n(t,s) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right) e^{-(t-s)A_n(t)}, \qquad 0 \le s \le t \le T. \tag{3.3}$$

Operate $A_n(t)$ to this equality and put $W_n(t,s) = A_n(t)U_n(t,s) - A_n(t)e^{-(t-s)A_n(t)}$, $0 \le s \le t \le T$, then we obtain that

$$W_n(t,s) = R_n(t,s) - \int_s^t W_n(t,\tau)Q_n(\tau,s)d\tau, \tag{3.4}$$

where $R_n(t,s)$ is given by

$$R_n(t,s) = -\int_s^t A_n(t)e^{-(t-\tau)A_n(t)}Q_n(\tau,s)d\tau. \tag{3.5}$$

3.2. **Convergence of $U_n(t,s)$.** We next show that $U_n(t,s)$ and $W_n(t,s)$ are strongly convergent as $n \to \infty$ with the aid of (2.14). To this end we shall use the dominate convergence theorems announced in [7].

Consider first the equation (3.1). From (2.6) and (3.2) it follows that

$$P_n(t,s) = \frac{1}{2\pi i}\int_\Gamma e^{-(t-s)\lambda}\frac{\partial}{\partial s}(\lambda - A_n(s))^{-1}d\lambda.$$

Therefore, (2.10) ($A(t)$ being replaced with $A_n(t)$) provides that

$$\|P_n(t,s)\|_{\mathcal{L}(X)} \le C(t-s)^{\nu-1}, \qquad 0 \le s < t \le T. \tag{3.6}$$

In the meantime, by (2.12) and (2.14), as $n \to \infty$, $P_n(t,s)$ converges to the operator

$$P(t,s) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right) e^{-(t-s)A(s)}, \qquad 0 \le s < t \le T,$$

strongly on X. According to [7, Theorem 1.31], these provide that $U_n(t,s)$ satisfies the uniform estimate

$$\|U_n(t,s)\|_{\mathcal{L}(X)} \le C, \qquad 0 \le s \le t \le T, \tag{3.7}$$

and converges to a bounded operator $U(t,s)$ strongly on X for $0 \le s \le t \le T$. The limit $U(t,s)$ is characterized as a solution to the integral equation

$$U(t,s) = e^{-(t-s)A(s)} + \int_s^t P(t,\tau)U(\tau,s)d\tau. \tag{3.8}$$

Consider next the equation (3.4). From (3.3),

$$Q_n(t,s) = \frac{1}{2\pi i}\int_\Gamma e^{-(t-s)\lambda}\frac{\partial}{\partial t}(\lambda - A_n(t))^{-1}d\lambda. \tag{3.9}$$

Then, by the same reason as for $P_n(t,s)$, $Q_n(t,s)$ also satisfies the uniform estimates

$$\|Q_n(t,s)\|_{\mathcal{L}(X)} \le C(t-s)^{\nu-1}, \qquad 0 \le 0 \le s < t \le T, \tag{3.10}$$

and to converge to the operator

$$Q(t,s) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)e^{-(t-s)A(t)}, \qquad 0 \le s < t \le T, \tag{3.11}$$

strongly on X.

It is, however, more delicate to verify such a uniform estimate for $R_n(t,s)$. We shall do this by the following proposition.

Proposition 3.1. *$R_n(t,s)$ satisfies the estimates*

$$\|R_n(t,s)\|_{\mathcal{L}(X)} \le C[(t-s)^{\nu-1} + (t-s)^{\rho-1}], \qquad 0 \le s < t \le T, \tag{3.12}$$

and, as $n \to \infty$, converges to a bounded operator $R(t,s)$, $0 \le s < t \le T$, strongly on X.

Proof. We first divide the integral in (3.5) into

$$R_n(t,s) = -\left(\int_r^t + \int_s^r\right)A_n(t)e^{-(t-\tau)A_n(t)}Q_n(\tau,s)d\tau = R_n^1(t,s) + R_n^2(t,s),$$

where $r = \frac{t+s}{2}$. Then, in view of (2.8) and (3.10), we have

$$\|R_n^2(t,s)\|_{\mathcal{L}(X)} \le C\int_s^r (t-\tau)^{-1}(\tau-s)^{\nu-1}d\tau \le C(t-s)^{\nu-1}, \qquad 0 \le s < t \le T.$$

So, it suffies to estimate $R_n^1(t,s)$. We here write

$$\begin{aligned} Q_n(\tau,s) &= \frac{1}{2\pi i}\int_\Gamma e^{-(\tau-s)\lambda}\left[\frac{\partial}{\partial\tau}(\lambda - A_n(\tau))^{-1} - \frac{\partial}{\partial t}(\lambda - A_n(t))^{-1}\right]d\lambda \\ &\quad + \frac{1}{2\pi i}\int_\Gamma e^{-(\tau-s)\lambda}\frac{\partial}{\partial t}(\lambda - A_n(t))^{-1}d\lambda = Q_n^1(t,\tau,s) + Q_n^2(t,\tau,s). \end{aligned}$$

The assumptions (2.3) and (2.4) yield the following lemma.

Lemma 3.1. *It holds true that*

$$\left\|\frac{\partial}{\partial\tau}(\lambda - A_n(\tau))^{-1} - \frac{\partial}{\partial t}(\lambda - A_n(t))^{-1}\right\|_{\mathcal{L}(X)} \le C(|t-\tau||\lambda|^{1-\nu} + |t-\tau|^\rho).$$

Proof of Lemma. We observe that

$$A_n(\tau)(\lambda - A_n(\tau))^{-1} - A_n(t)(\lambda - A_n(t))^{-1} = [-1 + \lambda(\lambda - A_n(\tau))^{-1}] \\ - [-1 + \lambda(\lambda - A_n(t))^{-1}] = -\lambda \int_\tau^t \frac{\partial}{\partial \sigma}(\lambda - A_n(\sigma))^{-1} d\sigma.$$

Therefore, it is obtained by (2.13) that

$$\|A_n(\tau)(\lambda - A_n(\tau))^{-1} - A_n(t)(\lambda - A_n(t))^{-1}\|_{\mathcal{L}(X)} \leq C|t-\tau||\lambda|^{1-\nu}.$$

The desired result then follows from (2.9). □

By this lemma, we have

$$\|Q_n^1(t,\tau,s)\|_{\mathcal{L}(X)} \leq C(|t-\tau||\tau - s|^{\nu-2} + |t-\tau|^\rho|\tau - s|^{-1}).$$

Hence,

$$\left\| \int_r^t A_n(t) e^{-(t-\tau)A_n(t)} Q_n^1(t,\tau,s) d\tau \right\|_{\mathcal{L}(X)} \leq C[(t-s)^{\nu-1} + (t-s)^{\rho-1}].$$

As for $Q_n^2(t,\tau,s)$, since $A(t)(\lambda - A(t))^{-1} = -1 + \lambda(\lambda - A(t))^{-1}$, we can write

$$Q_n^2(t,\tau,s) = \frac{dA_n(t)^{-1}}{dt} A_n(t) e^{-(\tau - s)A_n(t)} \\ + A_n(t)^{-1} \frac{1}{2\pi i} \int_\Gamma e^{-(\tau-s)\lambda} \lambda \frac{\partial}{\partial t}(\lambda - A_n(t))^{-1} d\lambda = Q_n^{21}(t,\tau,s) + Q_n^{22}(t,\tau,s).$$

According to [6, (2.11)],

$$\int_r^t A_n(t) e^{-(t-\tau)A_n(t)} Q_n^{21}(t,\tau,s) d\tau = -Q_n(t,r) e^{-(r-s)A_n(t)}.$$

Meanwhile, it is easy to see that

$$\left\| \int_r^t A_n(t) e^{-(t-\tau)A_n(t)} Q_n^{22}(t,\tau,s) d\tau \right\|_{\mathcal{L}(X)} \leq C(t-s)^{\nu-1}.$$

Hence, we have proved (3.12).

Strong convergence is now obvious. Indeed, $R_n(t,s)$ is observed to converge to the operator

$$R(t,s) = R^2(t,s) + Q(t,r)e^{-(r-s)A(t)} - \int_r^t A(t) e^{-(t-\tau)A(t)} [Q^1(t,\tau,s) + Q^{22}(t,\tau,s)] d\tau \tag{3.13}$$

strongly on X for $0 \leq s < t \leq T$. Of course, (3.12) holds true for the operators $A(t)$. □

It is ready to apply the dominate convergence [7, Theorem 1.32] to $W_n(t,s)$. We then conclude that $W_n(t,s)$ satisfies the uniform estimate

$$\|W_n(t,s)\|_{\mathcal{L}(X)} \leq C[(t-s)^{\nu-1} + (t-s)^{\rho-1}], \qquad 0 \leq s < t \leq T, \tag{3.14}$$

and converges to a bounded operator $W(t,s)$ strongly on X. As before, $W(t,s)$ is characterized as a solution to the integral equation

$$W(t,s) = R(t,s) - \int_s^t W(t,\tau) Q(\tau,s) d\tau. \tag{3.15}$$

Furthermore, $W(t,s)$ equals to $A(t)U(t,s) - A(t)e^{-(t-s)A(t)}$.

In this way, we can arrive at the following theorem (for the detailed proof, see [6]).

Theorem 3.1. *Under* (2.1), (2.2), (2.3) *and* (2.4), *there exists a unique family of bounded operators* $U(t,s)$ *on* X *defined for* $0 \le s \le t \le T$ *with the following properties: 1)* $U(t,s)$ *has the semigroup property; 2)* $U(t,s)$ *is strongly continuous for* $0 \le s \le t \le T$ *with the estimate* $\|U(t,s)\|_{\mathcal{L}(X)} \le C$*; 3)* $A(t)U(t,s)$ *is strongly continuous for* $0 \le s < t \le T$ *with the estimate* $\|A(t)U(t,s)\|_{\mathcal{L}(X)} \le C(t-s)^{-1}$*; and 4)* $U(t,s)$ *is strongly differentiable for* $t > s$ *with the derivative* $\frac{\partial}{\partial t}U(t,s) = -A(t)U(t,s)$.

3.3. **Cauchy problem.** The evolution operator $U(t,s)$ provides a unique solution to Problem (1.1). Let $0 < \beta \le 1$ and let F belong to the space

$$F \in \mathcal{F}^{\beta,\sigma}((0,T];X), \qquad 0 < \sigma < \beta. \tag{3.16}$$

For the definition of $\mathcal{F}^{\beta,\sigma}((0,T];X)$, see [7, 8].

Theorem 3.2. *Under* (2.1), (2.2), (2.3) *and* (2.4), *let* F *satisfy* (3.16) *and let* U_0 *be in* X. *Then,* (1.1) *possesses a unique solution* U *in the function space:*

$$U \in \mathcal{C}([0,T];X) \cap \mathcal{C}^1((0,T];X), \;\; A(t)U \in \mathcal{C}((0,T];X).$$

Moreover, U *is necessarily given by*

$$U(t) = U(t,0)U_0 + \int_0^t U(t,\tau)F(\tau)d\tau, \qquad 0 \le t \le T.$$

As the proof of this theorem is quite similar to that of [8, Theorem 3.2], we may omit it.

4. Refined properties of $U(t,s)$

Let (2.1), (2.2), (2.3) and (2.4) be satisfied, and let $U(t,s)$ be the evolution operator for $A(t)$ constructed by Theorem 3.1. Let us investigate more refined properties of $U(t,s)$.

For $0 \le \theta \le 1$, it holds true that

$$\|A(t)^\theta U(t,s)\|_{\mathcal{L}(X)} \le C(t-s)^{-\theta}, \qquad 0 \le s < t \le T. \tag{4.1}$$

For $0 \le \theta < 1$, $U(t,s)A(s)^\theta$ admits a bounded extension on X and its extension (denoted again by $U(t,s)A(s)^\theta$) satisfies the estimate

$$\|U(t,s)A(s)^\theta\|_{\mathcal{L}(X)} \le C_\theta(t-s)^{-\theta}, \qquad 0 \le s < t \le T. \tag{4.2}$$

For $0 \le \theta \le 1$,

$$\|A(t)^\theta U(t,s)A(s)^{-\theta}\|_{\mathcal{L}(X)} \le C, \qquad 0 \le s \le t \le T. \tag{4.3}$$

For the difference of $U(t,s)$ and $e^{-(t-s)A(s)}$, we have

$$\|U(t,s) - e^{-(t-s)A(s)}\|_{\mathcal{L}(X)} \le C(t-s)^\nu, \qquad 0 \le s \le t \le T. \tag{4.4}$$

Similarly, for the difference of $U(t,s)$ and $e^{-(t-s)A(t)}$,

$$\|U(t,s) - e^{-(t-s)A(t)}\|_{\mathcal{L}(X)} \le C(t-s)^\nu, \qquad 0 \le s \le t \le T, \tag{4.5}$$

$$\|A(t)[U(t,s) - e^{-(t-s)A(t)}]\|_{\mathcal{L}(X)} \le C[(t-s)^{\nu-1} + (t-s)^{\rho-1}], \qquad 0 \le s < t \le T. \tag{4.6}$$

Remark 4.1. As mentioned in Introduction, it is shown in [8] that the Tanabe's condition imples the extra spatial regularity that (4.1) holds true for some θ exceeding 1. But under (2.3)-(2.4), that may not be the case. The point is whether the operator $R(t,s)$ given by (3.13) satisfies the condition $\mathcal{R}(R(t,s)) \subset \mathcal{D}(A(t)^\theta)$ for sme $\theta > 0$ or not. All the members in the right hand side of (3.13) except $Q(t,r)e^{-(r-s)A(t)}$ are certainly verified to satisfy this, but the definition (3.9) seems not to allow the operator $Q(t,r)$ (and hence $Q(t,r)e^{-(r-s)A(t)}$) to enjoy this property. □

Let us here describe the proof of these properties step by step.

For $\theta = 0,\ 1$, (4.1) is already seen by Theorem 3.1. Therefore, for general $0 < \theta < 1$, it can be verified by the moment inequality applied to $A(t)^\theta$.

For $0 \le \theta < 1$, it follows from (3.1) that

$$U_n(t,s)A_n(s)^\theta = A_n(s)^\theta e^{-(t-s)A_n(s)} + \int_s^t P_n(t,\tau)U_n(\tau,s)A_n(s)^\theta d\tau.$$

Then, $U_n(t,s)A_n(s)^\theta$ is shown to converge strongly to a bounded operator of X. Hence, $U(t,s)A(s)^\theta$ has a bounded extension on X with the estimate (4.2). Its extension is also denoted by $U(t,s)A(s)^\theta$.

Let us next prove (4.3). When $0 \le \theta < 1$, its proof is immediate; to the contrary, when $\theta = 1$, it is rather complicated. First, consider the case when $0 \le \theta < 1$. Operating $A(t)^{\theta-1}$ to (3.15) from the left hand side, we have

$$A(t)^{\theta-1}W(t,s) = A(t)^{\theta-1}R(t,s) - \int_s^t A(t)^{\theta-1}W(t,\tau)Q(\tau,s)d\tau. \tag{4.7}$$

Since

$$A(t)^{\theta-1}R(t,s) = -\int_s^t A(t)^\theta e^{-(t-\tau)A(t)}Q(\tau,s)d\tau,$$

it follows by (2.8) and (3.10) that

$$\|A(t)^{\theta-1}R(t,s)\|_{\mathcal{L}(X)} \le C\int_s^t (t-\tau)^{-\theta}(\tau-s)^{\nu-1}d\tau \le C(t-s)^{\nu-\theta}.$$

Regarding $A(t)^{\theta-1}W(t,s)$ as a solution of (4.7), we obtain that $\|A(t)^{\theta-1}W(t,s)\|_{\mathcal{L}(X)} \le C(t-s)^{\nu-\theta}$.

Now operate $A(s)^{-\theta}$ to (4.7) from the right hand side. Then,

$$A(t)^{\theta-1}W(t,s)A(s)^{-\theta} = A(t)^{\theta-1}R(t,s)A(s)^{-\theta} - \int_s^t A(t)^{\theta-1}W(t,\tau)Q(\tau,s)A(s)^{-\theta}d\tau.$$

Note that $Q(\tau,s)A(s)^{-\theta} = Q(\tau,s)[A(s)^{-\theta} - A(\tau)^{-\theta}] + Q(\tau,s)A(\tau)^{-\theta}$. Then, in view of Lemma 4.1 below, we see that

$$\|Q(\tau,s)[A(s)^{-\theta} - A(\tau)^{-\theta}]\|_{\mathcal{L}(X)} \le C(\tau-s)^{\theta+\nu-1}.$$

In addition, from (3.9) ($A_n(t)$ being replaced with $A(t)$) and (3.11),

$$\begin{aligned}\|Q(\tau,s)A(\tau)^{-\theta}\|_{\mathcal{L}(X)} &\le C\int_\Gamma e^{-(\tau-s)\operatorname{Re}\lambda}\left\|\frac{\partial}{\partial\tau}(\lambda - A(\tau))^{-1}\cdot A(\tau)^{-\theta}\right\|_{\mathcal{L}(X)}|d\lambda| \\ &\le C\int_\Gamma |\lambda|^{-\theta}e^{-(\tau-s)\operatorname{Re}\lambda}|d\lambda| \le C(\tau-s)^{\theta-1}.\end{aligned}$$

Hence, $\|Q(\tau,s)A(s)^{-\theta}\|_{\mathcal{L}(X)} \le C(\tau-s)^{\theta-1}$. Similarly, $\|A(t)^{\theta-1}R(t,s)A(s)^{-\theta}\|_{\mathcal{L}(X)} \le C$. We thus conclude that

$$\|A(t)^{\theta-1}W(t,s)A(s)^{-\theta}\|_{\mathcal{L}(X)} \le C + C\int_s^t (t-\tau)^{\nu-\theta}(\tau-s)^{\theta-1}d\tau \le C.$$

Then, (4.3) is verified from Lemma 4.2.

Let us verify (4.3) for $\theta = 1$. From (3.15) we have

$$W(t,s)A(s)^{-1} = R(t,s)A(s)^{-1} - \int_s^t W(t,\tau)Q(\tau,s)A(s)^{-1}d\tau. \tag{4.8}$$

We already know that $\|Q(\tau,s)A(s)^{-1}\|_{\mathcal{L}(X)} \le C$. So, our goal is to show the same estimate for $R(t,s)A(s)^{-1}$. To this end, however, we have to use (3.13). Since

$$R^2(t,s)A(s)^{-1} = -\int_s^r A(t)e^{-(t-\tau)A(t)}Q(\tau,s)A(s)^{-1}d\tau,$$

it is clear that $\|R^2(t,s)A(s)^{-1}\|_{\mathcal{L}(X)} \le C$. Writing

$$Q(t,r)e^{-(r-s)A(t)}A(s)^{-1} = Q(t,r)e^{-(r-s)A(t)}[A(s)^{-1} - A(t)^{-1}] + Q(t,r)A(t)^{-1}e^{-(r-s)A(t)},$$

we observe that $\|Q(t,r)e^{-(r-s)A(t)}A(s)^{-1}\|_{\mathcal{L}(X)} \le C$. For estimating the integral term containing $Q^1(t,\tau,s)$, we use the following estimate

$$\left\|\left[\frac{\partial}{\partial\tau}(\lambda - A(\tau))^{-1} - \frac{\partial}{\partial t}(\lambda - A(t))^{-1}\right]A(\tau)^{-1}\right\|_{\mathcal{L}(X)} \le C(|t-\tau||\lambda|^{-\nu} + |t-\tau|^{\rho}|\lambda|^{-1} + |t-\tau|)$$

which can readily be verified in the same way as Lemma 3.1. As a result, we have

$$\left\|\int_r^t A(t)e^{-(t-\tau)A(t)}Q^1(t,\tau,s)A(s)^{-1}d\tau\right\|_{\mathcal{L}(X)} \le C.$$

It is the same for the integral term containing $Q^{22}(t,\tau,s)$, i.e.,

$$\left\|\int_r^t A(t)e^{-(t-\tau)A(t)}Q^{22}(t,\tau,s)A(s)^{-1}d\tau\right\|_{\mathcal{L}(X)} \le C.$$

Hence, we have shown that $\|R(t,s)A(s)^{-1}\|_{\mathcal{L}(X)} \le C$.

It then follows from (4.8) that $\|W(t,s)A(s)^{-1}\|_{\mathcal{L}(X)} \le C$. The desired estimate (4.3) for $\theta = 1$ is now obtained by Lemma 4.2.

Finally, (4.4) is verified from (3.8) in view of (3.6), $A_n(t)$ being replaced with $A(t)$. Similarly, (4.5) is shown by operating $A(t)^{-1}$ to (3.15) from the left hand side, and (4.6) is also verified from (3.15) in view of (3.12), $A_n(t)$ being replaced with $A(t)$.

Lemma 4.1. *For $0 \le \theta \le 1$,*

$$\|A(t)^{-\theta} - A(s)^{-\theta}\|_{\mathcal{L}(X)} \le C|t-s|^{\theta}, \qquad 0 \le t,\ s \le T. \tag{4.9}$$

Proof of Lemma. Obviously it suffices to consider the case when $0 < \theta < 1$. Dividing 1 as $1 = (1-\theta) + \theta$, we obtain by (2.2) and (2.3) that

$$\|(\lambda - A(t))^{-1} - (\lambda - A(s))^{-1}\|_{\mathcal{L}(X)} \le C|\lambda|^{\theta-1}(|\lambda|^{-\nu}|t-s|)^{\theta}, \quad \lambda \in \Gamma.$$

Then,

$$\begin{aligned}\|A(t)^{-\theta} - A(s)^{-\theta}\|_{\mathcal{L}(X)} &\leq \frac{1}{2\pi}\int_{\Gamma} |\lambda|^{-\theta}\|(\lambda - A(t))^{-1} - (\lambda - A(s))^{-1}\|_{\mathcal{L}(X)} \\ &\leq C\int_{\Gamma} |\lambda|^{-\theta\nu-1}|d\lambda|\,|t-s|^{\theta} \leq C|t-s|^{\theta}.\end{aligned}$$

□

Lemma 4.2. *For* $0 \leq \theta \leq 1$,

$$\|[A(t)^{\theta}e^{-\tau A(t)} - A(s)^{\theta}e^{-\tau A(s)}]A(s)^{-\theta}\|_{\mathcal{L}(X)} \leq C\tau^{-1}|t-s|.$$

Proof of Lemma. From (2.7) we write

$$\begin{aligned}&[A(t)^{\theta}e^{-\tau A(t)} - A(s)^{\theta}e^{-\tau A(s)}]A(s)^{-\theta} \\ &\quad = -\frac{1}{2\pi i}\int_{\Gamma^{\delta}} \lambda^{\theta}e^{-\tau\lambda}A(t)(\lambda - A(t))^{-1}[A(t)^{-1} - A(s)^{-1}]A(s)^{1-\theta}(\lambda - A(s))^{-1}d\lambda.\end{aligned}$$

Then it follows that

$$\|[A(t)^{\theta}e^{-\tau A(t)} - A(s)^{\theta}e^{-\tau A(s)}]A(s)^{-\theta}\|_{\mathcal{L}(X)} \leq C|t-s|\int_{\Gamma^{\delta}} e^{-\tau \mathrm{Re}\,\lambda}|d\lambda| \leq C|t-s|\tau^{-1}.$$

□

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Part II: Reports by Other Contributors

Analyticity for C_0-semigroups generated by elliptic operators in L^p

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Abstract. Analytic continuation of the C_0-semigroup $\{e^{-zA}\}$ on $L^p(\mathbb{R}^N)$ generated by a second order elliptic operator $-A$ is investigated, where A is formally defined as $Au = -\operatorname{div}(a\nabla u) + (F \cdot \nabla)u + Vu$ with lower order coefficients having singularities at infinity or at the origin. The result extends the sector of analyticity for the contraction semigroup determined in Metafune et al. [**23**] and [**24**].

1. Introduction

In this paper we deal with general second order elliptic operators of the form

$$(Au)(x) := -\operatorname{div}(a(x)\nabla u(x)) + (F(x) \cdot \nabla)u(x) + V(x)u(x), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \in \mathbb{N}$, $a \in C^1 \cap W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^{N\times N})$, $F \in C^1(\Omega; \mathbb{R}^N)$ and $V \in L^\infty_{\text{loc}}(\Omega; \mathbb{R})$ and the choice of $\Omega = \mathbb{R}^N$ or $\Omega = \mathbb{R}^N \setminus \{0\}$ depends on the location of the singularities of F and V. As a differential expression A may be said to be symmetric or nonsymmetric, respectively, if $F = 0$ or $F \neq 0$. Under the assumption on the triplet (a, F, V) specified below we want to discuss the maximal sector of analyticity for the semigroups $\{e^{-zA_{p,\max}}\}$ and $\{e^{-zA_p}\}$ on $L^p = L^p(\mathbb{R}^N)$ $(1 < p < \infty)$ generated by $-A_{p,\max}$ and $-A_p$, respectively, defined as

$$A_{p,\max}u := Au, \quad D(A_{p,\max}) := \{u \in L^p \cap W^{2,p}_{\text{loc}}(\mathbb{R}^N);\ Au \in L^p\}, \tag{1.2}$$

$$A_p u := Au, \quad D(A_p) := \{u \in W^{2,p}(\mathbb{R}^N);\ (F \cdot \nabla)u, Vu \in L^p\}. \tag{1.3}$$

In particular, if $A = -\Delta$ and G_z is the Gaussian kernel, then $D(A_p) = W^{2,p}(\mathbb{R}^N)$ $(= D(A_{p,\max}))$ and the C_0-semigroup $\{e^{-zA_p}\} = \{e^{z\Delta}\}$ on L^p is explicitly given by

$$(e^{z\Delta} f)(x) = (G_z * f)(x), \quad z \in \mathbb{C}_+ := \{z \in \mathbb{C};\ \operatorname{Re} z > 0\}, \tag{1.4}$$

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with $\|e^{z\Delta}\|_{L^p} \le (\sin\varepsilon)^{-N/2}$ for $z \in \overline{\Sigma}(\pi/2-\varepsilon)$. Here $\overline{\Sigma}(\psi)$ is the closure of an open sector

$$\Sigma(\psi) := \{z \in \mathbb{C}\setminus\{0\};\ |\arg z| < \psi\}. \tag{1.5}$$

Thus the maximal domain of analyticity for $\{e^{z\Delta}\}$ is nothing but $\mathbb{C}_+ = \Sigma(\pi/2)$. The generalization from $-\Delta$ to general A is divided into two cases, symmetric and nonsymmetric, which had started by Stein and Kato, respectively, in the 1960's. Namely, if $p \ne 2$, then Stein [**31**, p. 67, Theorem 1] observed through his interpolation theorem that general **symmetric** diffusion semigroups on L^p can be extended to an analytic contraction semigroup in the sector $\Sigma(\frac{\pi}{2}(1-|1-\frac{2}{p}|))$. Later, Henry [**14**, p. 32] established the optimal angle of contractivity for $\{e^{z\Delta}\}$ as

$$\overline{\Sigma}(\tilde{\omega}_p) = \overline{\Sigma}\Big(\frac{\pi}{2} - \omega_p\Big), \quad \tilde{\omega}_p := \tan^{-1}\frac{2\sqrt{p-1}}{|p-2|}, \quad \omega_p := \tan^{-1}\frac{|p-2|}{2\sqrt{p-1}} \tag{1.6}$$

(more generally, for the symmetric case see Pazy [**28**, Theorem 7.3.6], Bakry [**4**], Okazawa [**25**] and Liskevich-Perel'muter [**18**], for the nonsymmetric case see the next paragraph, and for optimality see also Voigt [**33**]). (1.4) and (1.6) shows that $e^{z\Delta}$ with $p \ne 2$ is *non-contractive* on $\mathbb{C}_+ \setminus \overline{\Sigma}(\tilde{\omega}_p)$ as a simplest case. Nevertheless, the maximal domain $\mathbb{C}_+$ is stable under perturbation by a real-valued potential V. In fact, the contraction semigroup $\{e^{t(\Delta-V)}; t \ge 0\}$ generated by the (negative) Schrödinger operator $\Delta - V$ has an analytic continuation $e^{z(\Delta-V)}$ onto the *maximal* domain $\mathbb{C}_+$. This is conjectured by Kato [**17**, Part D, Remark (e)] and later solved by Ouhabaz [**27**] by introducing Gaussian estimates (see also Arendt [**1**] and Hieber [**15**]). Incidentally, it was noted by Okazawa [**26**, Theorem 3.3] that $\|e^{z(\Delta-V)}\|_{L^p} \le 1$ in the same sector as (1.6). That is, $e^{z(\Delta-V)}$ is also non-contractive on $\mathbb{C}_+ \setminus \overline{\Sigma}(\tilde{\omega}_p)$. In this connection it is worth noticing that if $|\theta| \le \omega_p = \pi/2 - \tilde{\omega}_p$, then $\{\exp(te^{i\theta}(\Delta - V)); t \ge 0\}$ is a contraction semigroup on L^p. This is equivalent to the m-sectoriality of $-\Delta + V$ in L^p in the sense of Goldstein [**13**, Definition 1.5.8]:

$$|\mathrm{Im}\,((-\Delta + V)u, |u|^{p-2}u)| \le (\tan\omega_p)\mathrm{Re}\,((-\Delta + V + sV)u, |u|^{p-2}u).$$

Sometimes the term "regular m-accretivity" (see Tanabe [**32**, Section 2.2]) is employed instead of m-sectoriality because there is another notion of sectoriality (see Engel-Nagel [**8**, Section II.4a]). Therefore it is worth noticing that A is m-**sectorial** of type $S(\tan\omega)$ in L^p if and only if $-A$ is the generator of an **analytic contraction** semigroup $\{e^{-zA};\ z \in \overline{\Sigma}(\tilde{\omega}) = \overline{\Sigma}(\pi/2-\omega)\}$ on L^p (see [**13**, Theorem 1.5.9 and Proposition 1.3.9]).

The simplest case (where $N = 1$ and $p = 2$) of nonsymmetric A with bounded and continuous coefficients is stated in Kato [**16**, Example V.3.34] and then the general case (where $N \in \mathbb{N}$ and $1 < p < \infty$) is dealt with in Fattorini [**9**, Theorem 4.9.1] (see also [**10**]), Lunardi [**20**, Theorem 3.1.3] and more recently in Chill-Fašangová-Metafune-Pallara [**7**] (with nonsymmetric diffusion matrix a). Later, these results are completely extended to the non-contractive case in Arendt-ter Elst [**2**, Theorem 5.3] by utilizing Gaussian estimates; in particular, they established that if the diffusion a is symmetric and real-valued, then e^{-zA_p} admits a non-contractive analytic continuation onto the maximal domain $\mathbb{C}_+$. In this connection Sobol-Vogt [**30**] and Liskevich-Sobol-Vogt [**19**] show that the diffusion semigroup associated with the formal operator A (without mentioning the domain explicitly) can be extended to an analytic semigroup in a p-independent sector.

On the other hand, if the lower order terms have singularities (i.e., F and/or V are not bounded), then $A_{p,\max}$ does not in general coincide with A_p. These cases are also investigated intensively in the last decade (for bounded diffusion see, e.g., a pioneering work by Cannarsa-Vespri [**5**] and Metafune-Prüss-Rhandi-Schnaubelt [**24**], for unbounded diffusion see, e.g., Metafune-Pallara-Prüss-Schnaubelt [**23**], Fornaro-Lorenzi [**11**] and Giuli-Gozzi-Monte-Vespri [**12**]). In particular, [**24**] and [**23**] established that A_p is m-sectorial of type $S(\tan\omega)$ with $\omega > \omega_p$ in the respective case where $\Omega = \mathbb{R}^N$ and $\Omega = \mathbb{R}^N \setminus \{0\}$. This means that the sector of analyticity for $\{e^{-zA_p}\}$ is smaller than the sector (1.6):

$$\overline{\Sigma}(\pi/2-\omega) \subset \overline{\Sigma}(\pi/2-\omega_p).$$

Recently, it is shown by Sobajima [**29**] that when $\Omega = \mathbb{R}^N$, $-A_{p,\max}$ generates an analytic contraction semigroup $\{e^{-zA_{p,\max}}\}$ on essentially the same sector as in [**23**] and [**24**] under weaker assumption. To weaken the assumption the key found by Sobajima is an identity (see [**29**, Section 1]) which plays a crucial role in proving the m-sectoriality of $A_{p,\max}$.

The previous works mentioned above can be divided into two major cases (where A is symmetric or not) as in the following two tables:

Coefficients of A		Generation of analytic (quasi-)**contraction** semigroup	**Noncontractive** analytic continuation
a	V		
(δ_{jk})	0	Henry [**14**]	Before 19th century
bounded	0	Pazy [**28**], Okazawa [**25**]	Ouhabaz [**27**]
(δ_{jk})	locally singular	Kato [**17**], Okazawa [**26**]	
bounded	locally singular	M-P-P-S [**23**]	Incomplete **I**
diffusion semigroup		Stein [**31**], Bakry [**4**], Liskevich-Perel'muter [**18**], Voigt [**33**]	Ouhabaz [**27**] L-S-V [**19**]

Table 1: Known results for symmetric case ($Au = -\mathrm{div}(a\nabla u) + Vu$)

Coefficients of A			Generation of analytic (quasi-)**contraction** semigroup	**Non-contractive** analytic continuation
a	F	V		
bounded	bounded	bounded	Kato [**16**], Fattorini [**9**], [**10**], Lunardi [**20**], C-F-M-P [**7**]	Arendt-ter Elst [**2**]
bounded	singular at infnity	singular at infnity	Cannarsa-Vespri [**5**], M-P-R-S [**24**]	Incomplete **II**
bounded	locally singular	locally singular	M-P-P-S [**23**]	
singular at infinity	locally singular	locally singular	M-P-P-S [**23**], Fornaro-Lorenzi [**11**], G-G-M-V [**12**]	unknown
diffusion semigroup			Sobol-Vogt [**30**]	L-S-V [**19**]

Table 2: Known results for nonsymmetric case ($Au = -\mathrm{div}(a\nabla u) + F\cdot\nabla u + Vu$)

Two incomplete cases **I** and **II** in Table 1 and 2 are partially studied. In fact, the case **I** can be discussed by the result in [**27**] in which it is assumed that $\{e^{A_{p,\max}}\}$ admits a Gaussian estimate, while Gaussian estimates in the case **II** is already proved by in

Arendt-Metafune-Pallara [**3**] when F, V have singularities at infinity. In both cases the coincidence of $A_{p,\max}$ and A_p is still open; note that this is false if V is strongly singular. In particular, a p-independence of the sectors of analyticity for $\{e^{-tA_{p,\max}}\}$ and $\{e^{-tA_p}\}$ in the case **II** is dealt with in [**3**], but the *maximality* is not. In other words, in both cases **I** and **II** there seems to be no previous work for the maximal sector of analyticity corresponding to [**23**] and [**24**].

This paper is a resumé of the preprint [**22**] in which the case **II** (including case **I**) is completed. Namely, we have shown in [**22**] that when the lower order coefficients have singularities, both $\{e^{-zA_{p,\max}}\}$ and $\{e^{-zA_p}\}$ admit *non-contractive* analytic continuations to a certain sector which is independent of p and bigger than the sector described in [**23**] and [**29**] (see Theorems 2.1 ($\Omega = \mathbb{R}^N$) and 2.2 ($\Omega = \mathbb{R}^N \setminus \{0\}$) in Section 2 and an outline of their proofs in Section 3). The full notes [**22**] will be published elsewhere.

2. Main result

2.1. Basic assumption

Now we present the basic assumption on the triplet (a, F, V) defining $A_{p,\max}$ and A_p [see (1.2) and (1.3)]. As stated in Introduction Ω stands for $\mathbb{R}^N$ or $\mathbb{R}^N \setminus \{0\}$.

(**H1**) ${}^t a = a \in C^1 \cap W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^{N\times N})$ and a is uniformly elliptic on $\mathbb{R}^N$, that is, there exists a constant $\nu > 0$ such that $\langle a(x)\xi, \xi\rangle \ge \nu|\xi|^2$ for $x \in \mathbb{R}^N, \xi \in \mathbb{C}^N$;

(**H2**) $F \in C^1(\Omega; \mathbb{R}^N)$, $V \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R})$ and there exist three constants $\beta \ge 0$, γ_1, $\gamma_\infty > 0$ and a **nonnegative** auxiliary function $U \in L^\infty_{\mathrm{loc}}(\Omega)$ such that

$$|\langle F(x), \xi\rangle| \le \beta U(x)^{\frac{1}{2}} \langle a(x)\xi, \xi\rangle^{\frac{1}{2}} \quad \text{a.a. } x \in \Omega,\ \xi \in \mathbb{C}^N, \tag{2.1}$$

$$V(x) - \mathrm{div} F(x) \ge \gamma_1 U(x) \quad \text{a.a. } x \in \Omega, \tag{2.2}$$

$$V(x) \ge \gamma_\infty U(x) \quad \text{a.a. } x \in \Omega; \tag{2.3}$$

(**H3**) the auxiliary function $U \ge 0$ in (**H2**) belongs to $C^1(\Omega; \mathbb{R})$ and there exists a constant $c_0 \ge k_0 := \max\{\gamma_1, \gamma_\infty\} > 0$ such that

$$V(x) \le c_0 U(x) \quad \text{a.a. } x \in \Omega \tag{2.4}$$

and U satisfies an **oscillation condition** with respect to the diffusion a, that is,

$$\lambda_0 := \lim_{c\to\infty} \Big[\sup_{x\in\Omega} \Big(\frac{\langle a(x)\nabla U(x), \nabla U(x)\rangle^{1/2}}{(U(x)+c)^{3/2}} \Big)\Big] < \infty. \tag{2.5}$$

This yields a working form of the oscillation condition: for every $\lambda > \lambda_0$ there exist a constant $C_\lambda > 0$ such that

$$\langle a(x)\nabla U(x), \nabla U(x)\rangle^{1/2} \le \lambda (U(x) + C_\lambda)^{3/2}, \quad x \in \Omega. \tag{2.6}$$

In particular, if $\Omega = \mathbb{R}^N \setminus \{0\}$ then $U(x)$ is assumed to tend to infinity as $x \to 0$.

Example 1 (Maeda-Okazawa [**21**]). In the simplest case $a_{jk} = \delta_{jk}$ it is possible to compute λ_0 for $U(x) := |x|^\alpha$ when $\alpha \notin (-2, 1]$.

(**i**) Let $U(x) := |x|^\alpha$ $(\alpha > 1)$. Then $U \in C^1(\mathbb{R}^N)$ and $\lambda_0 = \lim_{c\to 0}(\alpha c^{-1/2-1/\alpha}) = 0$.

(**ii**) Let $U(x) := |x|^{-\beta}$ $(\beta > 2)$. Then $U \in C^1(\mathbb{R}^N \setminus \{0\})$ and $\lambda_0 = 0$. The computation is similar as above. In particular, if $\beta = 2$, then $\lambda_0 = 2$.

Remark 1. Let $\lambda > \lambda_0$ and $C_\lambda > 0$ as in (2.6) and put

$$\tilde{U}(x) := U(x) + C_\lambda > 0, \quad \tilde{V}(x) := V(x) + k_0 C_\lambda > 0 \quad \text{on } \Omega,$$

where k_0 is as in condition (**H3**). Then $\tilde{U}$ plays the role of a **positive** auxiliary function for the new (formal) operator with modified potential

$$\tilde{A}u := Au + k_0 C_\lambda u = -\mathrm{div}(a\nabla u) + F\cdot\nabla u + \tilde{V}u.$$

In fact, the new triplet $(a, F, \tilde{V})$ satisfies the original inequalities (2.1)–(2.4) with $U(x)$ and $V(x)$ replaced with $\tilde{U}(x)$ and $\tilde{V}(x)$, respectively:

$$|\langle F(x), \xi\rangle| \le \beta(U(x) + C_\lambda)^{\frac{1}{2}}\langle a(x)\xi, \xi\rangle^{\frac{1}{2}}, \tag{2.1$'$}$$

$$[V(x) + k_0 C_\lambda] - \mathrm{div}F(x) \ge \gamma_1(U(x) + C_\lambda), \tag{2.2$'$}$$

$$V(x) + k_0 C_\lambda \ge \gamma_\infty(U(x) + C_\lambda), \tag{2.3$'$}$$

$$V(x) + k_0 C_\lambda \le c_0(U(x) + C_\lambda). \tag{2.4$'$}$$

Note further that (2.6) is also written in terms of $\tilde{U}$:

$$\langle a(x)\nabla\tilde{U}(x), \nabla\tilde{U}(x)\rangle^{1/2} \le \lambda\,\tilde{U}(x)^{3/2} \quad \text{on } \Omega. \tag{2.6$'$}$$

2.2. The operators with singularities at infinity

Now we are in a position to state the first theorem on analytic continuation for (analytic contraction) semigroups generated by the elliptic operators $A_{p,\max}$ and A_p [see (1.2) and (1.3)] under conditions (**H1**), (**H2**) and (**H3**) with $\Omega = \mathbb{R}^N$.

Theorem 2.1. *Assume that conditions* (**H1**) *and* (**H2**) *are satisfied with* $\Omega = \mathbb{R}^N$. *Then one has the following assertions*:

(**i**) *Let* $1 < q < \infty$. *Then* $A_{q,\max}$ *is m-sectorial in* L^q, *that is,* $\{e^{-zA_{q,\max}}\}$ *is an analytic* **contraction** *semigroup on* L^q *on the closed sector* $\overline{\Sigma}(\pi/2 - \tan^{-1} c_{q,\beta,\gamma})$, *where*

$$c_{q,\beta,\gamma} := \sqrt{\frac{(q-2)^2}{4(q-1)} + \frac{\beta^2}{4}\left(\frac{\gamma_1}{q} + \frac{\gamma_\infty}{q'}\right)^{-1}} \tag{2.7}$$

and q' *is the Hölder conjugate of* q. *Moreover,* $C_0^\infty(\mathbb{R}^N)$ *is a core for* $A_{q,\max}$.

(**ii**) *Let* $p \in (1, \infty)$ *be arbitrarily fixed. Then the semigroup* $\{e^{-zA_{p,\max}}\}$ *in assertion* (**i**) *admits an analytic continuation to the open sector* $\Sigma(\pi/2 - \tan^{-1} K_{\beta,\gamma})$, *where*

$$K_{\beta,\gamma} := \min\{c_{q,\beta,\gamma}\,;\ 1 < q < \infty\}. \tag{2.8}$$

Moreover, there exists a constant $\omega_0 > 0$ *such that* $\{e^{-z(\omega_0 + A_{p,\max})}\}$ *forms a* **bounded** *analytic semigroup on* L^p :

$$\|e^{-zA_{p,\max}}\|_{L^p} \le M_\varepsilon e^{\omega_0 \mathrm{Re}\, z} \quad \text{on } \overline{\Sigma}(\pi/2 - \tan^{-1} K_{\beta,\gamma} - \varepsilon). \tag{2.9}$$

Here the constant ω_0 depends only on N, $\|a_{jk}\|_{L^\infty(\mathbb{R}^N)}$ and $\|\nabla a_{jk}\|_{L^\infty(\mathbb{R}^N)}$, while the other constant $M_\varepsilon \geq 1$ depends only on ε, N, ν, β, γ_1, γ_∞ and $\|a_{jk}\|_{L^\infty(\mathbb{R}^N)}$.

(**iii**) *Assume further that* (**H3**) *is satisfied with $\Omega = \mathbb{R}^N$. If*

$$(p-1)\lambda_0\big[(\beta/p) + (\lambda_0/4)\big] < (\gamma_1/p) + (\gamma_\infty/p'), \tag{2.10}$$

then $A_{p,\max}$ has the so-called **separation property**: *for all $u \in D(A_{p,\max})$*

$$\|\mathrm{div}(a\nabla u)\|_{L^p} + \|(F\cdot\nabla)u\|_{L^p} + \|Vu\|_{L^p} \leq C\|(1+A_{p,\max})u\|_{L^p}, \tag{2.11}$$

that is, $D(A_{p,\max})$ coincides with $D(A_p) = W^{2,p}(\mathbb{R}^N)\cap D(F\cdot\nabla)\cap D(V)$, where $D(F\cdot\nabla) := \{u \in L^p \cap W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N); (F\cdot\nabla)u \in L^p\}$ and $D(V) := \{u \in L^p; Vu \in L^p\}$, and hence $\{e^{-zA_p}\}$ is analytic in $\Sigma(\pi/2 - \tan^{-1} K_{\beta,\gamma})$.

Here three remarks and an example to Theorem 2.1 are in order.

Remark 2. Assertion (**i**) is a particular case of [**29**, Theorem 1.3]; it is worth noticing that the sector of analyticity and *contraction property* for $\{e^{-zA_{p,\max}}\}$ is reduced to the positive real axis (that is, $\tan^{-1} c_{p,\beta,\gamma} \to \pi/2$) as p tends to 1 or to ∞.

Remark 3. Assertion (**ii**) suggests the possibility that $\{e^{-zA_{p,\max}}\}$ admits a *non-contractive* analytic continuation to a p-independent sector $\Sigma(\pi/2 - \tan^{-1} K_{\beta,\gamma})$ which is bigger than $\Sigma(\pi/2 - \tan^{-1} c_{q,\beta,\gamma})$, $1 < q < \infty$. Moreover, the constant $c_{2,\beta,\gamma}$ does not in general attain $\min\{c_{q,\beta,\gamma};\ 1 < q < \infty\}$ $(= K_{\beta,\gamma})$ if $\gamma_1 \neq \gamma_\infty$. This implies that if $\gamma_1 \neq \gamma_\infty$, then the sector derived in L^p (for some p) can be bigger than the one derived in L^2. In other words, we have $c_{2,\beta,\gamma} > K_{\beta,\gamma}$ and hence we may conclude that $\{e^{-tA_2}\}$ has a noncontractive analytic continuation to a wider sector in spite of the belief that the best property is held when $p = 2$. An example with $\gamma_1 \neq \gamma_\infty$ is also given later (see Example 3 in Subsection 2.3).

Remark 4. The proof in [**23**] is based on a perturbation technique with the separation property (2.11) under a setting similar to assertion (**iii**). Theorem 2.1 makes it clear that (2.11) is necessary only for the domain characterization of A_p.

Example 2. We consider a typical one-dimensional Ornstein-Uhlenbeck operator

$$(A_\mu v)(x) := -v''(x) + xv'(x)$$

in L^p_μ (the L^p-space with respect to the invariant measure $e^{-x^2/2}dx$). Chill-Fašangová-Metafune-Pallara [**6**] show that the C_0-semigroup on L^p_μ generated by $-A_\mu$ is analytic in the sector $\Sigma(\tilde\omega_p)$ and that the angle $\tilde\omega_p = \pi/2 - \omega_p$ of analyticity is optimal.

Here, applying Theorem 2.1 (**ii**), we give another derivation of their angle ω_p. Using the isometry $u \mapsto e^{-x^2/2p}u$, we can transform A_μ into A:

$$(Au)(x) := -u''(x) + p^{-1}(p-2)x\,u'(x) + [\,p^{-2}(p-1)x^2 - p^{-1}]u$$

in the usual space $L^p(\mathbb{R}^N)$. Thus we set $a(x) = 1$, $F(x) = (1-2/p)x$ and $V(x) = p^{-2}(p-1)x^2 - 1/p$. Taking $U(x) := x^2$, we see that the triplet $(a, F, V+1)$ satisfies conditions (**H1**) and (**H2**) with respective constants

$$\beta = p^{-1}|p-2|, \quad \gamma_1 = p^{-2}(p-1) = \gamma_\infty.$$

This leads us to the angle ω_p introduced in (1.6):

$$K_{\beta,\gamma} = \inf\Big\{\sqrt{(\tan\omega_q)^2 + (\tan\omega_p)^2}\,;\ 1 < q < \infty\Big\} = \tan\omega_p.$$

This shows that the domain of analyticity in this case cannot extend beyond $\Sigma(\pi/2 - \omega_p)$ in a form of sector with vertex at the origin. Moreover, $U(x)$ satisfies (2.4) and (2.5) in (**H3**) with $c_0 = 1$ and $\lambda_0 = 0$ (see Example 1), respectively. Hence A has a separation property (2.11).

2.3. The operators with local singularities

Next we state the second theorem on analytic continuation for (analytic contraction) semigroups generated by elliptic operators A_p [see (1.3)] under conditions (**H1**), (**H2**) and (**H3**) with $\Omega = \mathbb{R}^N \setminus \{0\}$ together with domain characterization of A_p.

Theorem 2.2. *Let $1 < p < \infty$. Assume that conditions* (**H1**)*,* (**H2**) *and* (**H3**) *are satisfied with $\Omega = \mathbb{R}^N \setminus \{0\}$. Let $K_{\beta,\gamma}$ be the constant determined by* (2.8)*. If* (2.10) *holds, then $\{e^{-zA_p}\}$ admits an analytic continuation to the sector $\Sigma(\pi/2 - \tan^{-1} K_{\beta,\gamma})$. In this case A_p has the separation property* (2.11)*.*

Example 3 (A case where $\gamma_1 \neq \gamma_\infty$). We consider the following operator

$$Au = -\Delta u + b|x|^{-2}(x\cdot\nabla)u + c|x|^{-2}u, \tag{2.12}$$

that is, (a, F, V) and Ω in our notation are given by

$$a_{jk}(x) := \delta_{jk}, \quad F(x) := b|x|^{-2}x, \quad V(x) := c|x|^{-2}, \quad \Omega = \mathbb{R}^N \setminus \{0\};$$

note that this operator has a singularities at the origin. Taking $U(x) := |x|^{-2}$ as an auxiliary funtion, we have $\gamma_1 \neq \gamma_\infty$ in condition (**H2**) if $N \neq 2$ and $b \neq 0$. In fact, we can see that the constants are given by $\beta = |b|$, $\gamma_1 = c - b(N-2)$ and $\gamma_\infty = c$. We also have $\lambda_0 = 2$ (see Example 1). Hence if b, c and p satisfy (2.10), that is, if

$$p - 1 + (2/p)|b| = (p-1)\lambda_0(\beta/p + \lambda_0/4) < (\gamma_1/p) + (\gamma_\infty/p') = c - b(N-2)/p$$

holds, then we can apply Theorem 2.2 to the operator A and hence the conclusion of Remark 3 implies that $c_{2,\beta,\gamma} > K_{\beta,\gamma}$.

3. Strategy of the proof of theorems

Here we give the rough description of the proof of Theorem 2.1.

Proof of Theorem 2.1. First show (**i**). Noting that (**H1**) and (**H2**) with $\Omega = \mathbb{R}^N$are satisfied, we can apply [**29**, Theorem 1.3] with the following auxiliary function Ψ_p to (a, F, V):

$$\Psi_p(x) := \big[(\gamma_1/q) + (\gamma_\infty/q')\big]U(x).$$

Thus we obtain (**i**).

Next we prove (**ii**). The key of this part is a Gaussian estimate for the C_0-semigroup $\{e^{-A_{p,\max}}\}$ in (**i**). This estimate is already established in [**3**]. That is, under the assumption in (**ii**), $\{e^{-tA_{2,\max}}\}$ admits a Gaussian estimate with **nonnegative** kernel $\{k_t\}$ satisfying

$$0 \le k_t(x,y) \le Ct^{-N/2}\exp\bigl(\omega_0 t - |x-y|^2/(bt)\bigr) \quad \text{a.a. } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where the constant ω_0 depends only on N, $\|a_{jk}\|_{L^\infty}$ and $\|\nabla a_{jk}\|_{L^\infty}$, while C, b depend only on N, ν, β, γ_1, γ_∞ and $\|a_{jk}\|_{L^\infty}$. $\{e^{-A_{p,\max}}\}$ is also represented by the kernel $\{k_t\}$. Now we put $p_0 \in (1,\infty)$ satisfying

$$c_{p_0,\beta,\gamma} = \min\{c_{q,\beta,\gamma}\,;\ 1<q<\infty\}\,(=: K_{\beta,\gamma}).$$

Applying [**15**] (or [**2**]) to $\{e^{-A_{p_0,\max}}\}$, we obtain that for every $1<p<\infty$, $\{e^{-A_{p,\max}}\}$ can be extended a bounded analytic C_0-semigroup on L^p in the sector $\Sigma(\pi/2 - \tan^{-1} K_{\beta,\gamma})$.

It remains to prove (**iii**) that $A_{p,\max}$ has a separation property (2.11). Set

$$k_p := (\gamma_1/p) + (\gamma_\infty/p') - (p-1)\lambda_0\bigl[(\beta/p) + (\lambda_0/4)\bigr] > 0.$$

In a way similar to that in [**23**, Lemma 2.3], it follows from (**H3**) with $\mathbb{R}^N$ and condition (2.10) that $k_p > 0$ and there exists a constant $C>0$ such that for every $u \in C_0^\infty(\mathbb{R}^N)$,

$$\|u\|_{W^{2,p}(\mathbb{R}^N)} + \|(F\cdot\nabla)u\|_{L^p} + \|Vu\|_{L^p} \le C(1+k_p^{-1})(\|u\|_{L^p} + \|Au\|_{L^p}).$$

where $C>0$ depends only on N, p, ν, β, and $\|a_{jk}\|_{W^{1,\infty}}$. This implies (2.11). This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. To apply Theorem 2.1, we approximate F, U and V as follows:

$$F_\delta(x) := \begin{cases} F(x)(1+\delta U(x))^{-2}, & x \ne 0,\\ 0, & x = 0,\end{cases}$$

$$U_\delta(x) := \begin{cases} U(x)(1+\delta U(x))^{-1}, & x \ne 0,\\ \delta^{-1}, & x = 0,\end{cases}$$

$$V_\delta(x) := \frac{V(x)}{1+\delta U(x)} + \gamma_1\frac{\delta U(x)^2}{(1+\delta U(x))^2} + 2\beta\lambda_0\frac{\delta(U(x)+C_\lambda)^2}{(1+\delta U(x))^3} \quad \text{a.a. } x \in \mathbb{R}^N$$

for $\delta>0$, where λ and C_λ are the constants in (2.6). Then it is worth noticing that (a, F_δ, V_δ) and U_δ satisfy (2.1)–(2.3) and (2.5) with $\Omega = \mathbb{R}^N$ and the respective original constants. Moreover, (a, F_δ, V_δ) and U_δ satisfy (2.4):

$$V_\delta(x) \le (c_0 + \gamma_1 + 2\beta\lambda_0)U_\delta(x),$$

Applying Theorem 2.1 (**iii**) to (a, F_δ, V_δ) (and U_δ), and letting $\delta \downarrow 0$, we obtain that $\{e^{-A_p}\}$ can be extended a bounded analytic C_0-semigroup on L^p in the sector $\Sigma(\pi/2 - \tan^{-1} K_{\beta,\gamma})$. We finish the proof. □

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Time-discretization approach to various parabolic systems associated with grain boundaries

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Dedicated to Professor Hiroki Tanabe on the Occasion of his 80th Birthday

1 Introduction

This paper is based on a recent collaboration with Professor Salvador Moll, University of Valencia, Spain (cf. [12]), which is communicated and supported by Professor José M. Mazón, University of Valencia, Spain.

Let $0 < T < \infty$ be a fixed constant, let $N \in \mathbb{N}$ be a fixed number, and let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Also, let us assume that the boundary $\partial\Omega$ of Ω is smooth if $N > 1$. On that basis, we denote by $\nu_{\partial\Omega}$ the unit outer normal vector on $\partial\Omega$, and we set $Q := (0,T) \times \Omega$ and $\Sigma := (0,T) \times \partial\Omega$.

Let $\nu > 0$ be a fixed small constant. In this paper, a coupled system of two parabolic initial-boundary value problems is considered. This system is denoted by $(\mathrm{S})_\nu$, and formally described as follows.

$(\mathrm{S})_\nu$:

$$\begin{cases} \eta_t - \Delta\eta + g(\eta) + \alpha'(\eta)\beta(\nabla\theta) = 0 \ \text{ in } Q, \\ \nabla\eta \cdot \nu_{\partial\Omega} = 0 \ \text{ on } \Sigma, \\ \eta(0,x) = \eta_0(x), \quad x \in \Omega; \end{cases} \tag{1.1}$$

$$\begin{cases} \alpha_0(\eta)\theta_t - \mathrm{div}\left(\alpha(\eta)D\beta(\nabla\theta) + \dfrac{\nu}{2}D[(\beta_0)^2](\nabla\theta)\right) = 0 \ \text{ in } Q, \\ \left(\alpha(\eta)D\beta(\nabla\theta) + \dfrac{\nu}{2}D[(\beta_0)^2](\nabla\theta)\right) \cdot \nu_{\partial\Omega} = 0 \ \text{ on } \Sigma, \\ \theta(0,x) = \theta_0(x), \quad x \in \Omega. \end{cases} \tag{1.2}$$

System $(\mathrm{S})_\nu$ is derived from the following energy functional, called "free energy":

$$\begin{aligned} [\eta,\theta] \in H^1(\Omega) \times H^1(\Omega) \quad \mapsto \quad \mathscr{F}_\nu(\eta,\theta) := {} & \frac{1}{2}\int_\Omega |\nabla\eta|^2\,dx + \int_\Omega \hat{g}(\eta)\,dx \\ & + \int_\Omega \alpha(\eta)\beta(\nabla\theta)\,dx + \frac{\nu}{2}\int_\Omega \beta_0(\nabla\theta)^2\,dx; \end{aligned} \tag{1.3}$$

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and initial-boundary value problems (1.1) and (1.2) correspond to the L^2-gradient flows of the unknowns η and θ, respectively. Here, α_0 is a positive-valued locally Lipschitz function of one-variable. α is a positive-valued C^2-convex function of one-variable, and α' is the differential of α. β_0 and β are given nonnegative-valued Lipschitz convex functions of N-variables, and $D\beta$ and $D[(\beta_0)^2]$ denote the differentials (subdifferentials) of β and $(\beta_0)^2$, respectively. g is a given locally Lipschitz function of one-variable, and $\hat{g}$ is a nonnegative primitive of g. η_0 and θ_0 are given initial data.

The functional $\mathscr{F}_\nu$ given in (1.3) is a generalized version of the free energy adopted in "Kobayashi-Warren-Carter model", which is a phase field model of planar grain boundary motion proposed by Kobayashi et al. [9, 10]. In the context, the unknowns $\eta = \eta(t, x)$ and $\theta = \theta(t, x)$ are supposed to be order parameters which indicate, respectively, "the orientation order" and "the orientation angle" at each $(t, x) \in Q$ in the crystal. The function g is a perturbation which is to constrain the value of η onto the range $[0, 1]$ of ratio, i.e. $0 \leq \eta \leq 1$ in Q.

In the original study of Kobayashi et al. [9, 10], the grain boundary is prescribed as a free boundary between the facet structures (or simply facets) in the crystal. In this regard, α_0 and α are supposed to activate the mobility of the grain boundary. β_0 and β are supposed to advance the presences of facets, and the both of these are settled as the Euclidean norm (in $\mathbb{R}^2$). Hence, in [9, 10], the diffusion term as in (1.2) is described in the following form of singular type:

$$-\mathrm{div}\left(\alpha(\eta)\frac{\nabla\theta}{|\nabla\theta|} + \nu\nabla\theta\right). \tag{1.4}$$

Under the above setting, the authors of [9, 10] provided some numerical data to confirm the appropriateness of their modelling method. Also, in recent years, the studies by mathematical theories have been developed by several mathematicians [4, 5, 6, 8]. The goals and objectives of these studies include the limiting observation of $(\mathrm{S})_\nu$ as $\nu \searrow 0$, but this objective have not been achieved yet, except for a few study examples [13, 14] concerned with the one-dimensional case of Ω.

Based on such background, we set the solvability of the generalized version $(\mathrm{S})_\nu$ of the Kobayashi-Warren-Carter model, as the theme of the Main Theorem. Then, the concepts of the generalization will be to enable the mathematical treatments of various problems, which are possibly appear in several useful situations such as the following.

(A) Reproduction of the anisotropy. The functions β_0 and β can be involved in the reproduction of the anisotropy in the crystalline structure. Then, the structural unit of crystal is to be characterized by a compact and origin-symmetric convex set $W \subset \mathbb{R}^N$, called "Wulff shape", and the both functions β_0 and β are to be settled as the gauge function of the Wulff shape W, i.e.:

$$\beta_0(\vartheta) = \beta(\vartheta) := \inf\left\{ \lambda \geq 0 \,\middle|\, \vartheta \in \lambda W \right\}, \quad \text{for any } \vartheta \in \mathbb{R}^N.$$

Incidentally, the setting (1.4) corresponds to the case when W coincides with the convex hull $\mathrm{co}(\mathbb{S}^{N-1})$ of the N-dimensional sphere $\mathbb{S}^{N-1}$.

(B) Approximations. The singularity as in (1.4) brings down the difficulties in the theoretical and numerical analyses of the Kobayashi-Warren-Carter model. In this light, the functions β_0 and β can be given as suitable approximations of the gauge functions, which can relax such singularity. For example, under (1.4), one of representative choices is to set that:

$$\beta_0(\vartheta) := |\vartheta| \ \text{ and } \ \beta(\vartheta) := \sqrt{\varepsilon^2 + |\vartheta|^2} \quad \text{for any } \vartheta \in \mathbb{R}^N,$$
$$\text{with a sufficiently small constant } \varepsilon > 0.$$

Note that the functions β_0 and β may be given separately, in general approximating situations.

The Main Theorem will be stated as the existence theorem for the system $(\mathrm{S})_\nu$, and the proof of the Main Theorem will be proceeded in accordance with the method of the time-discretization. Furthermore, as another consequence, it will be asserted that the time-discretization approach will be a uniform solution method for various problems associated with the Kobayashi-Warren-Carter model.

2 Statement of the the Main Theorem

First of all, let us confirm the assumptions for the given functions g, $\hat{g}$, α_0, α, β_0, β, η_0 and θ_0 associated with the system $(\mathrm{S})_\nu$.

(A1) $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a locally Lipschitz continuous function, such that:

$$g(-\infty, 0] \subset (-\infty, 0] \ \text{ and } \ g[1, \infty) \subset [0, \infty).$$

Also, g is supposed to have a nonnegative primitive $\hat{g} : \mathbb{R} \longrightarrow [0, \infty)$.

(A2) $\alpha_0 : \mathbb{R} \longrightarrow (0, \infty)$ is a locally Lipschitz function, and $\alpha : \mathbb{R} \longrightarrow (0, \infty)$ is a C^2-function, such that $\alpha'(0) = 0$ and $\alpha'' \geq 0$ on $\mathbb{R}$, where α' and α'' are the first differential and the second differential of α, respectively. Note that α turns out to be a nonnegative and convex function on $\mathbb{R}$, and:

$$\delta_\alpha := \min_{\tau \in [0,1]} \{\alpha_0(\tau), \alpha(\tau)\} > 0.$$

(A3) $\beta_0 : \mathbb{R}^N \longrightarrow [0, \infty)$ and $\beta : \mathbb{R}^N \longrightarrow [0, \infty)$ are Lipschitz continuous convex functions such that:

$$\beta_0(\vartheta) \geq \beta_0(0) \text{ and } \beta(\vartheta) \geq \beta(0), \text{ for any } \vartheta \in \mathbb{R}^N,$$

and there exist constants $\delta_\beta > 0$ and $c_\beta \geq 0$, such that:

$$\beta_0(\vartheta) \geq \delta_\beta |\vartheta| - c_\beta, \text{ for any } \vartheta \in \mathbb{R}^N.$$

(A4) The pair $[\eta_0, \theta_0]$ of initial data belongs to a class $D_0 \subset H^1(\Omega) \times H^1(\Omega)$, defined as:

$$D_0 := \left\{ \ [w, z] \in H^1(\Omega) \times H^1(\Omega) \ \middle| \ 0 \leq w \leq 1 \text{ a.e. in } \Omega \ \text{ and } \ z \in L^\infty(\Omega) \ \right\}.$$

Additionally, for the convenience of descriptions, we prepare the following notations.

Notation 2.1 (I) For any $w \in L^2(\Omega)$, let $\Phi_\nu(w; \cdot\,)$ be a proper l.s.c. and convex function on $L^2(\Omega)$, defined as:

$$z \in L^2(\Omega) \mapsto \Phi_\nu(w; z) := \begin{cases} \displaystyle\int_\Omega \alpha(w)\beta(\nabla z)\,dx + \frac{\nu}{2}\int_\Omega \beta_0(\nabla z)^2\,dx, & \text{if } z \in H^1(\Omega), \\ \infty, \ \text{otherwise}, & \end{cases}$$

and let $\partial\Phi_\nu(w; \cdot\,)$ be the L^2-subdifferential of $\Phi_\nu(w; \cdot\,)$.

(II) For any open interval $I \subset (0,T)$ and any $\xi \in L^2(I; L^2(\Omega))$, let $\hat{\Phi}_\nu(\xi; \cdot)_I$ be a proper l.s.c. and convex function on $L^2(I; L^2(\Omega))$, defined as:

$$\zeta \in L^2(I; L^2(\Omega)) \mapsto \hat{\Phi}_\nu(\xi; \zeta)_I := \begin{cases} \displaystyle\int_I \Phi_\nu(\xi(t); \zeta(t))\, dt, & \text{if } \zeta \in L^2(I; H^1(\Omega)), \\ \infty, & \text{otherwise.} \end{cases}$$

Now, the Main Theorem in this paper is stated as follows.

Main Theorem. *(Solvability of the system (S)$_\nu$) Under the assumptions (A1)-(A4), the system (S)$_\nu$ admits at least one solution $[\eta, \theta]$, in the sense of the following four items.*

(S1) $\eta \in W^{1,2}(0,T; L^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega))$, $0 \leq \eta \leq 1$ *a.e. in* Q; $\theta \in W^{1,2}(0,T; L^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)) \cap L^\infty(Q)$, $|\theta|_{L^\infty(Q)} \leq |\theta_0|_{L^\infty(\Omega)}$.

(S2) η *solves the following variational identity of parabolic type with* θ*:*

$$\int_\Omega \bigl(\eta_t(t) + g(\eta(t)) + \alpha'(\eta(t))\beta(\nabla\theta(t))\bigr)\, w\, dx + \int_\Omega \nabla\eta(t) \cdot \nabla w\, dx = 0, \qquad (2.1)$$
$$\text{for any } w \in H^1(\Omega) \text{ and a.e. } t \in (0,T).$$

(S3) θ *solves the following variational inequality of parabolic type with* η*:*

$$\int_\Omega \alpha_0(\eta(t))\theta_t(t)(\theta(t) - z)\, dx + \Phi_\nu(\eta(t); \theta(t)) \leq \Phi_\nu(\eta(t); z), \qquad (2.2)$$
$$\text{for any } z \in H^1(\Omega) \text{ and a.e. } t \in (0,T).$$

(S4) $[\eta(0), \theta(0)] = [\eta_0, \theta_0]$ *in* $L^2(\Omega) \times L^2(\Omega)$.

3 Approximation problem for the system (S)$_\nu$

As seen from (1.1)-(1.2), the system (S)$_\nu$ can be reformulated to the following system of evolution equations:

$$\begin{cases} \eta_t(t) - \Delta_N \eta(t) + g(\eta(t)) + \alpha'(\eta(t))\beta(\nabla\theta(t)) = 0 \text{ in } L^2(\Omega), \\ \alpha_0(\eta(t))\theta_t(t) + \partial\Phi_\nu(\eta(t); \theta(t)) \ni 0 \text{ in } L^2(\Omega), \end{cases} \quad \text{a.e. } t \in (0,T), \qquad (3.1)$$

where Δ_N is the operator of the Laplacian subject to the Neumann-zero boundary condition, i.e.

$$\Delta_N : w \in D_N := \bigl\{\ \tilde{w} \in H^2(\Omega) \ \big|\ \nabla\tilde{w} \cdot \nu_{\partial\Omega} = 0 \text{ a.e. on } \partial\Omega\ \bigr\} \mapsto \Delta w \in L^2(\Omega).$$

Based on this, we prepare an approximation index $h \in (0,1)$ of the time-step, and denote by $(\mathrm{AP})_h^{(\nu)}$ the time-discretization system for (3.1), formulated as follows.

$(\mathrm{AP})_h^{(\nu)}$:

$$\begin{cases} \dfrac{\eta_{h,i} - \eta_{h,i-1}}{h} - \Delta_N \eta_{h,i} + g(\eta_{h,i}) + \alpha'(\eta_{h,i})\beta(\nabla\theta_{h,i-1}) = 0 \text{ in } L^2(\Omega), & (3.2) \\ \alpha_0(\eta_{h,i}) \dfrac{\theta_{h,i} - \theta_{h,i-1}}{h} + \partial\Phi_\nu(\alpha(\eta_{h,i}); \theta_{h,i}) \ni 0 \text{ in } L^2(\Omega), & (3.3) \end{cases}$$

for $i = 1, 2, 3, \cdots$, subject to:

$$[\eta_{h,0}, \theta_{h,0}] := [\eta_0, \theta_0] \text{ in } L^2(\Omega) \times L^2(\Omega). \tag{3.4}$$

Here, for any $0 < h < 1$, we call a pair $[\{\eta_{h,i}\}, \{\theta_{h,i}\}] \subset L^2(\Omega) \times L^2(\Omega)$ of sequences a solution to $(\mathrm{AP})_h^{(\nu)}$, or simply an approximating solution, if and only if $\{\eta_{h,i} \,|\, i \in \mathbb{N}\} \subset H^2(\Omega)$, $\{\theta_{h,i} \,|\, i \in \mathbb{N}\} \subset H^1(\Omega)$, and for any $i \in \mathbb{N}$, the components $\eta_{h,i}$ and $\theta_{h,i}$ fulfill the respective elliptic type problems (3.2) and (3.3) with (3.4).

In this paper, the class $\{(\mathrm{AP})_h^{(\nu)} \,|\, h, \nu \in (0, 1]\}$ of the time-discretization systems is adopted as that of the approximation problems for $(\mathrm{S})_\nu$. With regard to each approximation problem, we can prove the following theorem.

Theorem 3.1 *(Solvability of the approximation system) Let us assume that* $0 < \nu \leq 1$, *and:*

$$0 < h \leq h_* := \frac{1}{1 + 3|g'|_{L^\infty(0,1)} + 2|g|^2_{C[0,1]}|\Omega| + 2|\alpha'|^2_{C[0,1]}}, \tag{3.5}$$

where $|\Omega|$ *is the* N*-dimensional Lebesgue measure of* Ω*. Then, the system* $(AP)_h^{(\nu)}$ *admits a unique solution* $[\{\eta_{h,i}\}, \{\theta_{h,i}\}]$*, such that:*

$$0 \leq \eta_{h,i} \leq 1 \text{ a.e. in } \Omega, \quad |\theta_{h,i}|_{L^\infty(\Omega)} \leq |\theta_{h,i-1}|_{L^\infty(\Omega)}, \quad \text{and} \tag{3.6}$$

$$\begin{aligned} &\frac{1}{2h}|\eta_{h,i} - \eta_{h,i-1}|^2_{L^2(\Omega)} + \frac{1}{h}\left|\sqrt{\alpha_0(\eta_{h,i})}(\theta_{h,i} - \theta_{h,i-1})\right|^2_{L^2(\Omega)} + \mathscr{F}_\nu(\eta_{h,i}, \theta_{h,i}) \\ &\qquad \leq \mathscr{F}_\nu(\eta_{h,i-1}, \theta_{h,i-1}), \ i = 1, 2, 3, \cdots. \end{aligned} \tag{3.7}$$

Proof. Note that (3.2) and (3.3) can be regarded as independent variational problems of elliptic types. Indeed, the problem (3.2) has a unique unknown variable $\eta_{h,i}$. Hence, after solving (3.2), we can restrict the unknown in (3.3) to only one variable $\theta_{h,i}$. Namely, for each step $i \in \mathbb{N}$, these problems can be solved in order of (3.2) and (3.3) by means of the usual variational method such as [3]. The property (3.6) can be deduced on the basis of the theory of T-monotonicity (cf. [2, 7]). Furthermore, the inequality (3.7) is obtained by multiplying the both sides of (3.2) and (3.3) by $(\eta_{h,i} - \eta_{h,i-1})$ and $(\theta_{h,i} - \theta_{h,i-1})$, respectively, and taking the sum of the results. Incidentally, the constraint (3.5) for h will be needed only for the discussions associated with $\{\eta_{h,i}\}$: the solvability of (3.2); the range constraint property as in (3.6); the derivation of the coefficient $\frac{1}{2h}$ at the head of (3.7). ∎

4 Proof of the Main Theorem

Let $0 < h_* \leq 1$ be the small constant given in (3.5), and for any $0 < h \leq h_*$, let $[\{\eta_{h,i}\}, \{\theta_{h,i}\}]$ be the solution to $(\mathrm{AP})_h^{(\nu)}$. On that basis, let us set:

$$\begin{cases} t_{h,i} := ih, \ i = 0, 1, 2, 3, \cdots, \\ \Delta_{h,i} := [t_{h,i-1}, t_{h,i}), \ i = 1, 2, 3, \cdots, \end{cases} \text{for any } 0 < h \leq h_*, \tag{4.1}$$

and let us construct sequences:

$$\begin{cases} \{[\overline{\eta}_h, \overline{\theta}_h] \,|\, 0 < h \leq h_*\} \subset L^\infty(0, T; H^2(\Omega)) \times L^\infty(0, T; H^1(\Omega)), \\ \{[\underline{\eta}_h, \underline{\theta}_h] \,|\, 0 < h \leq h_*\} \subset L^\infty(0, T; H^1(\Omega)) \times L^\infty(0, T; H^1(\Omega)), \\ \{[\widehat{\eta}_h, \widehat{\theta}_h] \,|\, 0 < h \leq h_*\} \subset W^{1,2}(0, T; H^1(\Omega)) \times W^{1,2}(0, T; H^1(\Omega)), \end{cases}$$

by using the following different kinds of time-interpolations:

$$\begin{cases} [\overline{\eta}_h(t), \overline{\theta}_h(t)] := [\eta_{h,i}, \theta_{h,i}] & \text{in } H^2(\Omega) \times H^1(\Omega), \\ [\underline{\eta}_h(t), \underline{\theta}_h(t)] := [\eta_{h,i-1}, \theta_{h,i-1}] & \text{in } H^1(\Omega) \times H^1(\Omega), \\ [\widehat{\eta}_h(t), \widehat{\theta}_h(t)] := \dfrac{t_{h,i} - t}{h}[\eta_{h,i-1}, \theta_{h,i-1}] + \dfrac{t - t_{h,i-1}}{h}[\eta_{h,i}, \theta_{h,i}] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } H^1(\Omega) \times H^1(\Omega), \end{cases} \tag{4.2}$$

$$\text{for all } 0 < h \leq h_* \text{ and all } t \in \Delta_{h,i},\ i = 1, 2, 3, \cdots.$$

Now, let us fix any $0 < T < \infty$, and let us set:

$$N_h^\circ(T) := \min \left\{ \ n^\circ \in \mathbb{N} \ \middle| \ n^\circ h \geq T \ \right\}, \quad \text{for any } 0 < h \leq h_*.$$

Then, the assumptions (A2)-(A3) and (3.6)-(3.7) of Theorem 3.1 enable us to see that:

$$\begin{cases} \bullet\ \{\overline{\eta}_h \,|\, 0 < h \leq h_*\} \text{ is bounded in } L^\infty(0, T; H^1(\Omega)), \\ \bullet\ \{\underline{\eta}_h \,|\, 0 < h \leq h_*\} \text{ is bounded in } L^\infty(0, T; H^1(\Omega)), \\ \bullet\ \{\widehat{\eta}_h \,|\, 0 < h \leq h_*\} \text{ is bounded in } W^{1,2}(0, T; L^2(\Omega)) \text{ and} \\ \quad \text{bounded in } L^\infty(0, T; H^1(\Omega)), \\ \bullet\ 0 \leq \overline{\eta}_h \leq 1, \ \ 0 \leq \underline{\eta}_h \leq 1 \ \text{ and } \ 0 \leq \widehat{\eta}_h \leq 1, \ \text{ a.e. in } Q, \\ \quad \text{for all } 0 < h \leq h_*; \end{cases}$$

$$\begin{cases} \bullet\ \{\overline{\theta}_h \,|\, 0 < h \leq h_*\} \text{ is bounded in } L^\infty(0, T; H^1(\Omega)), \\ \bullet\ \{\underline{\theta}_h \,|\, 0 < h \leq h_*\} \text{ is bounded in } L^\infty(0, T; H^1(\Omega)), \\ \bullet\ \{\widehat{\theta}_h \,|\, 0 < h \leq h_*\} \text{ is bounded in } W^{1,2}(0, T; L^2(\Omega)) \text{ and} \\ \quad \text{bounded in } L^\infty(0, T; H^1(\Omega)), \\ \bullet\ |\overline{\theta}_h|_{L^\infty(Q)} \leq |\theta_0|_{L^\infty(\Omega)}, \quad |\underline{\theta}_h|_{L^\infty(Q)} \leq |\theta_0|_{L^\infty(\Omega)} \quad \text{and} \\ \quad |\widehat{\theta}_h|_{L^\infty(Q)} \leq |\theta_0|_{L^\infty(\Omega)}, \ \text{ for all } 0 < h \leq h_*. \end{cases}$$

Therefore, applying the compactness theory of Aubin's type [15], we find a sequence $\{h_n \,|\, n \in \mathbb{N}\} \subset (0, h_*]$, a pair $[\eta, \theta] \in L^2(0, T; L^2(\Omega)) \times L^2(0, T; L^2(\Omega))$ of functions and a function $\xi_* \in L^2(0, T; L^2(\Omega))$, such that:

$$h_n \searrow 0 \ \text{ as } n \to \infty;$$

$$\begin{cases} \eta \in W^{1,2}(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)), \ \ 0 \leq \eta \leq 1 \text{ a.e. in } Q, \\ \theta \in W^{1,2}(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)), \ \ |\theta|_{L^\infty(Q)} \leq |\theta_0|_{L^\infty(\Omega)}, \end{cases} \tag{4.3}$$

$$\left.\begin{cases} \overline{\eta}_n := \overline{\eta}_{h_n} \to \eta \ \text{ and } \ \underline{\eta}_n := \underline{\eta}_{h_n} \to \eta \ \text{ in } L^\infty(I; L^2(\Omega)), \\ \quad \text{weakly-}* \text{ in } L^\infty(I; H^1(\Omega)), \text{ weakly-}* \text{ in } L^\infty(I \times \Omega), \text{ and} \\ \quad \text{the pointwise sense a.e. in } I \times \Omega, \\ \widehat{\eta}_n := \widehat{\eta}_{h_n} \to \eta \ \text{ in } C(\overline{I}; L^2(\Omega)), \text{ weakly in } W^{1,2}(I; L^2(\Omega)), \\ \quad \text{weakly-}* \text{ in } L^\infty(I; H^1(\Omega)), \text{ weakly-}* \text{ in } L^\infty(I \times \Omega), \text{ and} \\ \quad \text{the pointwise sense a.e. in } I \times \Omega, \end{cases}\right. \text{as } n \to \infty, \tag{4.4}$$

$$\begin{cases} \overline{\theta}_n := \overline{\theta}_{h_n} \to \theta \ \text{ and } \ \underline{\theta}_n := \underline{\theta}_{h_n} \to \theta \ \text{ in } L^\infty(I; L^2(\Omega)), \\ \qquad \text{weakly-}* \text{ in } L^\infty(I; H^1(\Omega)), \text{ weakly-}* \text{ in } L^\infty(I \times \Omega), \text{ and} \\ \qquad \text{the pointwise sense a.e. in } I \times \Omega, \\ \widehat{\theta}_n := \widehat{\theta}_{h_n} \to \theta \ \text{ in } C(\overline{I}; L^2(\Omega)), \text{ weakly in } W^{1,2}(I; L^2(\Omega)), \\ \qquad \text{weakly-}* \text{ in } L^\infty(I; H^1(\Omega)), \text{ weakly-}* \text{ in } L^\infty(I \times \Omega), \text{ and} \\ \qquad \text{the pointwise sense a.e. in } I \times \Omega, \end{cases} \quad \text{as } n \to \infty, \tag{4.5}$$

and

$$\alpha'(\overline{\eta}_n)\beta(\nabla \underline{\theta}_n) \to \xi_* \text{ weakly-}* \text{ in } L^\infty(I; L^2(\Omega)), \ \text{ as } n \to \infty, \tag{4.6}$$

for any open interval $I \subset (0, T)$.

Now, since:

$$[\widehat{\eta}_n(0), \widehat{\theta}_n(0)] = [\eta_0, \theta_0] \ \text{ in } L^2(\Omega) \times L^2(\Omega), \ n = 1, 2, 3, \cdots,$$

the limiting pair $[\eta, \theta]$ satisfies the condition (S4) as in the Main Theorem.

Next, note that the following two variational formulas are derived from the governing equations (3.2)-(3.3):

$$\begin{aligned} &\int_I \bigl((\widehat{\eta}_n)_t(t) + g(\overline{\eta}_n(t)), w\bigr)_{L^2(\Omega)} \, dt + \int_I \int_\Omega \nabla \overline{\eta}_n(t) \cdot \nabla w \, dx dt \\ &\qquad + \int_I \int_\Omega w \alpha'(\overline{\eta}_n(t))\beta(\nabla \underline{\theta}_n(t)) \, dx dt = 0, \end{aligned} \tag{4.7}$$

for any $n \in \mathbb{N}$, any open interval $I \subset (0, T)$ and any $w \in H^1(\Omega)$;

$$\begin{aligned} &\int_I \bigl(\alpha_0(\overline{\eta}_n(t))(\widehat{\theta}_n)_t(t), \overline{\theta}_n(t) - \psi(t)\bigr)_{L^2(\Omega)} \, dt \\ &\qquad + \hat{\Phi}_\nu(\alpha(\overline{\eta}_n); \overline{\theta}_n)_I \leq \hat{\Phi}_\nu(\alpha(\overline{\eta}_n); \psi)_I, \end{aligned} \tag{4.8}$$

for any $n \in \mathbb{N}$, any open interval $I \subset (0, T)$ and any $\psi \in L^2(I; H^1(\Omega))$.

Here, with (A2)-(A3) and (4.3)-(4.5) in mind, letting $n \to \infty$ in (4.8) yields that:

$$\begin{aligned} &\int_I \bigl(\alpha_0(\eta(t))(\theta)_t(t), \theta(t) - \psi(t)\bigr)_{L^2(\Omega)} \, dt + \hat{\Phi}_\nu(\alpha(\eta); \theta)_I \\ \leq \ &\lim_{n \to \infty} \int_I \bigl(\alpha_0(\overline{\eta}_n(t))(\widehat{\theta}_n)_t(t), \overline{\theta}_n(t) - \psi(t)\bigr)_{L^2(\Omega)} \, dt + \liminf_{n \to \infty} \hat{\Phi}_\nu(\alpha(\eta); \overline{\theta}_n)_I \\ = \ &\liminf_{n \to \infty} \left(\int_I \bigl(\alpha_0(\overline{\eta}_n(t))(\widehat{\theta}_n)_t(t), \overline{\theta}_n(t) - \psi(t)\bigr)_{L^2(\Omega)} \, dt + \hat{\Phi}_\nu(\alpha(\overline{\eta}_n); \overline{\theta}_n)_I \right) \\ &- \lim_{n \to \infty} \int_I \bigl(\alpha(\overline{\eta}_n(t)) - \alpha(\eta(t)), \beta(\nabla \overline{\theta}_n(t))\bigr)_{L^2(\Omega)} \, dt \\ \leq \ &\lim_{n \to \infty} \hat{\Phi}_\nu(\alpha(\overline{\eta}_n); \psi)_I = \hat{\Phi}_\nu(\alpha(\eta); \psi)_I, \end{aligned} \tag{4.9}$$

for any open interval $I \subset (0, T)$ and any $\psi \in L^2(I; H^1(\Omega))$.

This implies that the pair $[\eta, \theta]$ fulfills (S3) in the Main Theorem.

Next, for the component $\theta \in C([0, T]; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))$, let us prepare a sequence $\{\theta_n^\circ \,|\, n \in \mathbb{N}\} \subset C^\infty(\overline{Q})$, such that:

$$\theta_n^\circ \to \theta \ \text{ in } L^2(0, T; H^1(\Omega)) \ \text{ as } n \to \infty,$$

and let us set $\psi = \theta_n^\circ$ in (4.8). Then, with help from (A3) and (4.3)-(4.5), we have:

$$\begin{aligned}
&\hat{\Phi}_\nu(\alpha(\eta);\theta)_I \\
\leq\ & \liminf_{n\to\infty} \hat{\Phi}_\nu(\alpha(\eta);\overline{\theta}_n)_I + \lim_{n\to\infty} \int_I \big(\alpha(\overline{\eta}_n) - \alpha(\eta), \beta(\nabla\overline{\theta}_n)\big)_{L^2(\Omega)}\, dt \\
=\ & \liminf_{n\to\infty} \hat{\Phi}_\nu(\alpha(\overline{\eta}_n);\overline{\theta}_n)_I \leq \limsup_{n\to\infty} \hat{\Phi}_\nu(\alpha(\overline{\eta}_n);\overline{\theta}_n)_I \\
\leq\ & \limsup_{n\to\infty} \hat{\Phi}_\nu(\alpha(\overline{\eta}_n);\theta_n^\circ)_I - \lim_{n\to\infty} \int_I \big(\alpha_0(\overline{\eta}_n(t))(\widehat{\theta}_n)_t(t), \overline{\theta}_n(t) - \theta_n^\circ(t)\big)_{L^2(\Omega)}\, dt \\
=\ & \lim_{n\to\infty} \left(\int_I \big(\alpha(\overline{\eta}_n(t)), \beta(\nabla\theta_n^\circ(t))\big)_{L^2(\Omega)}\, dt + \frac{\nu}{2} \int_I |\beta_0(\nabla\theta_n^\circ(t))|^2_{L^2(\Omega)}\, dt \right) \\
=\ & \hat{\Phi}_\nu(\alpha(\eta);\theta)_I,
\end{aligned}$$

namely

$$\lim_{n\to\infty} \hat{\Phi}_\nu(\alpha(\overline{\eta}_n);\overline{\theta}_n)_I = \hat{\Phi}_\nu(\alpha(\eta);\theta)_I, \quad \text{for any open interval } I \subset (0,T). \tag{4.10}$$

Also:

$$\begin{aligned}
&\int_I \int_\Omega \alpha(\eta(t))\beta(\nabla\theta(t))\, dxdt \\
\leq\ & \liminf_{n\to\infty} \int_I \int_\Omega \alpha(\eta(t))\beta(\nabla\overline{\theta}_n(t))\, dxdt \\
&\quad + \lim_{n\to\infty} \int_I \big(\alpha(\overline{\eta}_n(t)) - \alpha(\eta(t)), \beta(\nabla\overline{\theta}_n(t))\big)_{L^2(\Omega)}\, dt \\
=\ & \liminf_{n\to\infty} \int_I \int_\Omega \alpha(\overline{\eta}_n)\beta(\nabla\overline{\theta}_n(t))\, dxdt \leq \limsup_{n\to\infty} \int_I \int_\Omega \alpha(\overline{\eta}_n)\beta(\nabla\overline{\theta}_n(t))\, dxdt \\
=\ & \lim_{n\to\infty} \hat{\Phi}_\nu(\alpha(\overline{\eta}_n);\overline{\theta}_n)_I - \frac{\nu}{2} \liminf_{n\to\infty} \int_I \int_\Omega \beta_0(\nabla\overline{\theta}_n(t))^2\, dxdt \\
\leq\ & \int_I \int_\Omega \alpha(\eta(t))\beta(\nabla\theta(t))\, dxdt,
\end{aligned}$$

and therefore:

$$\begin{aligned}
&\left| \int_I \int_\Omega \alpha(\overline{\eta}_n(t))\beta(\nabla\underline{\theta}_n(t))\, dxdt - \int_I \int_\Omega \alpha(\eta(t))\beta(\nabla\theta(t))\, dxdt \right| \\
\leq\ & \left| \int_I \int_\Omega \alpha(\overline{\eta}_n(t))\beta(\nabla\overline{\theta}_n(t))\, dxdt - \int_I \int_\Omega \alpha(\eta(t))\beta(\nabla\theta(t))\, dxdt \right| \\
&\quad + \left| \int_I \int_\Omega \alpha(\overline{\eta}_n(t))\beta(\nabla\overline{\theta}_n(t))\, dxdt - \int_I \int_\Omega \alpha(\overline{\eta}_n(t))\beta(\nabla\underline{\theta}_n(t))\, dxdt \right| \\
\leq\ & \left| \int_I \int_\Omega \alpha(\overline{\eta}_n(t))\beta(\nabla\overline{\theta}_n(t))\, dxdt - \int_I \int_\Omega \alpha(\eta(t))\beta(\nabla\theta(t))\, dxdt \right| \\
&+ \sup_{n\in\mathbb{N}} |\beta(\nabla\widehat{\theta}_n)|_{L^\infty(I;L^2(\Omega))} \Big(T|\alpha(\overline{\eta}_n) - \alpha(\underline{\eta}_n)|_{C(\overline{I};L^2(\Omega))} + 2h_n|\alpha|_{C[0,1]}|\Omega|^{1/2} \Big) \\
\to\ & 0 \ \text{ as } n \to \infty, \ \text{ for any open interval } I \subset (0,T).
\end{aligned} \tag{4.11}$$

Meanwhile, in the light of (A3) and (4.3)-(4.5),

$$\begin{aligned}
&\liminf_{n\to\infty} \int_A \alpha(\overline{\eta}_n)\beta(\nabla\underline{\theta}_n)\, d\mathscr{L}^{N+1} \\
\geq\ &\liminf_{n\to\infty} \int_A \alpha(\eta)\beta(\nabla\underline{\theta}_n)\, d\mathscr{L}^{N+1} \\
&\qquad + \lim_{n\to\infty} \int_A \bigl(\alpha(\overline{\eta}_n) - \alpha(\eta)\bigr)\, \beta(\nabla\underline{\theta}_n)\, d\mathscr{L}^{N+1} \\
\geq\ &\int_A \alpha(\eta)\beta(\nabla\theta)\, d\mathscr{L}^{N+1}, \qquad (4.12)\\
&\text{for any open interval } I \subset (0,T) \text{ and any open set } A \subset I \times \Omega,
\end{aligned}$$

where for any $d \in \mathbb{N}$, $\mathscr{L}^d$ denotes the d-dimensional Lebesgue measure. Taking into account (4.6), (4.11)-(4.12) and [1, Proposition 1.80], we infer that:

$$\begin{gathered}
\alpha(\overline{\eta}_n)\beta(\nabla\underline{\theta}_n) \to \alpha(\eta)\beta(\nabla\theta) \text{ weakly-}* \text{ in } L^\infty(I; L^2(\Omega)) \ \text{ as } n \to \infty, \\
\text{for any open interval } I \subset (0,T).
\end{gathered} \qquad (4.13)$$

Moreover, since the assumption (A2) and (4.4) lead to:

$$\begin{gathered}
\frac{w\alpha'(\overline{\eta}_n)}{\alpha(\overline{\eta}_n)} \to \frac{w\alpha'(\eta)}{\alpha(\eta)} \text{ in } L^2(I; L^2(\Omega)) \text{ as } n \to \infty, \\
\text{for any } w \in H^1(\Omega) \text{ and any open interval } I \subset (0,T),
\end{gathered}$$

we can derive from (4.13) that:

$$\begin{aligned}
\int_I \int_\Omega w\alpha'(\overline{\eta}_n(t))\beta(\nabla\underline{\theta}_n(t))\, dxdt &= \left(\frac{w\alpha'(\overline{\eta}_n)}{\alpha(\overline{\eta}_n)}, \alpha(\overline{\eta}_n)\beta(\nabla\underline{\theta}_n) \right)_{L^2(I;L^2(\Omega))} \\
\to \left(\frac{w\alpha'(\eta)}{\alpha(\eta)}, \alpha(\eta)\beta(\nabla\theta) \right)_{L^2(I;L^2(\Omega))} &= \int_I \int_\Omega w\alpha'(\eta(t))\beta(\nabla\theta(t))\, dxdt, \qquad (4.14)\\
\text{for any } w \in H^1(\Omega) \text{ and any open interval } I \subset (0,T).&
\end{aligned}$$

Owing to (4.3)-(4.5) and (4.14), we obtain the compatibility of the pair $[\eta, \theta]$ with (S2), by letting $n \to \infty$ in (4.7).

Finally, with (4.3) and (S2) in mind, the condition (S1) will be verified by means of the standard regularity theory of parabolic PDEs. ■

Remark 4.1 The line of arguments as in (4.9)-(4.10) are essentially rely on the fact that the sequence $\{\hat{\Phi}_\nu(\alpha(\overline{\eta}_n);\, \cdot\,)_I \,|\, n \in \mathbb{N}\}$ of convex functions converges to the convex function $\hat{\Phi}_\nu(\alpha(\eta);\, \cdot\,)_I$ on $L^2(I; L^2(\Omega))$, in the sense of Mosco [11]. The essence of this fact can be refer to the previous studies, such as [4, 5, 6, 8].

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An idea of the proof of the comparison principle of viscosity solutions for doubly nonlinear Hamilton-Jacobi equations

Naoki Yamada

1 Introduction

It is well known that a variational inequality with unilateral condition

$$\max\{F(x,u,Du,D^2u), u(x)-k\} = 0 \quad \text{in } \Omega$$

is represented as

$$F(x,u,Du,D^2u) + \partial\psi(u) \ni 0$$

by using the subdifferential $\partial\psi$ of a convex function

$$\psi(x) = \begin{cases} 0 & x \leqq k, \\ +\infty & \text{otherwise.} \end{cases}$$

Here, Du and D^2 represent the gradient vector and the Hessian matrices for u, respectively.

This equation is mainly formulated in a framework of Hilbert space as a typical example of subdiffrential operators.

The author had treated this inequality in the framework of viscosity solutions [6].

On the other hand, the doubly nonlinear equation

$$\partial\varphi(u_t(t)) + Au(t) \ni f(t)$$

is also considered in the theory of evolution equations ([4], [3], [2], [7], [5], [1]). Here, $\varphi : H \to]-\infty, +\infty]$ is a proper lower semicontinuous convex function in a Hilbert space H, $\partial\varphi$ is its subdifferential and A is a monotone operator.

Let $\varphi : \mathbb{R} \to]-\infty, +\infty]$ be a proper lower semicontinuous function on $\mathbb{R}$, and $\partial\varphi$ be its subdifferential. In this note we consider an equation

$$\partial\varphi(u_t(t,x)) + H(Du(t,x)) + u(t,x) \ni f(x) \qquad (t,x) \in (0,\infty)\times\mathbb{R}^n \text{ or } (0,\infty)\times\Omega$$

in the framework of viscosity solutions.

The main goal is to give a definition of viscosity solutions to this equation and show an idea of the proof of comparison principle.

Since we only present an idea of the proof, we assume that the solution is smooth. Also we do not pay the attention for the class of the solution such as increasing rate or the class of initial functions.

The author would like to express his sincere gratitude to Professor Hiroki Tanabe for his encouragement for many years.

2 Idea of the proof (parabolic case)

In this section we review the idea of the proof of comparison principle for parabolic equations:

$$u_t(t,x) + H(Du(t,x)) = f(x).$$

It is well known that these formal discussion is justified in the framework of viscosity solutions.

Let u be a subsolution and let v be a supersolution, that is u and v satisfy the inequality

$$\begin{aligned} u_t + H(Du) &\leqq f, \\ v_t + H(Dv) &\geqq f, \end{aligned}$$

for respectively.

Take $\varepsilon > 0$ and let $u^\varepsilon = u - \varepsilon t$. It holds

$$u^\varepsilon_t + H(Du^\varepsilon) + \varepsilon \leqq f.$$

We prove by contradiction. Assume that there exists a point (t_0, x_0) such that the inequality

$$\delta = (u^\varepsilon - v)(t_0, x_0) = \max_{t,x}(u^\varepsilon - v)(t,x) > 0$$

holds. By initial condition, it is known that $t_0 > 0$ and x_0 is an interior point. In the following we compute at (t_0, x_0).

It holds

$$u_t^\varepsilon - v_t \geqq 0, \quad Du^\varepsilon = Dv.$$

Then it follows

$$\begin{aligned} \varepsilon &\leqq f - u_t^\varepsilon - H(Du^\varepsilon) \\ &\leqq f - v_t - H(Dv) \\ &\leqq 0, \end{aligned}$$

which is a contradiction. □

3 Idea of the proof (doubly nonlinear case)

In this section we describe the idea of the proof of comparison principle for doubly nonlinear equations:

$$\partial\varphi(u_t) + H(Du) + u \ni f.$$

We also assume that the solution is smooth.

Definition (subsolution) : We say that a function $u(t,x)$ is a subsolution if it holds

$$\varphi(w) - \varphi(u_t) \geqq (f - H(Du) - u)(w - u_t)$$

for any $w \leqq u_t$ at every point (t,x).

Definition (supersolution) : We say that a function $v(x,t)$ is a supersolution if it holds

$$\varphi(w) - \varphi(v_t) \geqq (f - H(Dv) - v)(w - v_t)$$

for any $w \geqq v_t$ at every point (t,x).

Let $u(t,x)$ be a subsolution and $v(t,x)$ be a supersolution. Since we want to prove $u \leqq v$, we assume that

$$\delta = (u - v)(t_0, x_0) = \max_{t,x}(u - v)(t,x) > 0$$

happens at some point (t_0, x_0) and drive a contradiction. We assume that such (t_0, x_0) exists and $t_0 > 0$. We compute at (t_0, x_0) in the following. It holds

$$u_t - v_t \geqq 0, \quad Du = Dv.$$

We take $\varepsilon > 0$ arbitrary and fix it.

If we take $w = v_t - \varepsilon$ in the definition of subsolution, we have

$$\frac{\varphi(u_t) - \varphi(v_t - \varepsilon)}{u_t - v_t + \varepsilon} \leqq f - H(Du) - u.$$

If we take $w = u_t + \varepsilon$ in the definition of supersolution, we get

$$\frac{\varphi(u_t + \varepsilon) - \varphi(v_t)}{u_t - v_t + \varepsilon} \leqq f - H(Dv) - v.$$

Since $v_t - \varepsilon < v_t \leqq u_t < u_t + \varepsilon$, by a property of convex functions, it must be holds

$$\frac{\varphi(u_t) - \varphi(v_t - \varepsilon)}{u_t - v_t + \varepsilon} \leqq \frac{\varphi(u_t + \varepsilon) - \varphi(v_t)}{u_t - v_t + \varepsilon}.$$

However, since it holds

$$\begin{aligned}
\frac{\varphi(u_t) - \varphi(v_t - \varepsilon)}{u_t - v_t + \varepsilon} + u &\leqq f - H(Du) \\
&= f - H(Dv) \\
&\leqq \frac{\varphi(u_t + \varepsilon) - \varphi(v_t)}{u_t - v_t + \varepsilon} + v,
\end{aligned}$$

we get

$$\frac{\varphi(u_t) - \varphi(v_t - \varepsilon)}{u_t - v_t + \varepsilon} - \frac{\varphi(u_t + \varepsilon) - \varphi(v_t)}{u_t - v_t + \varepsilon} \leqq v - u = -\delta < 0,$$

which is a contradiction. □

In the usual parabolic equations, the comparison principle is stated for the equation

$$u_t + H(Du) = f.$$

In this case the term u_t acts as the term λu in the elliptic case, that is, it makes the equation strictly monotone.

On the other hand, in the case of doubly nonlinear equation, the term $\partial\varphi(u_t)$ is only monotone. This is the reason why we describe the idea for the equation

$$\partial\varphi(u_t) + H(Du) + u \ni f$$

not for the equation

$$\partial\varphi(u_t) + H(Du) \ni f.$$

4 Definition of viscosity solutions

We hope that the previous formal discussion will be justified under the following definition of viscosity solutions.

Definition (subsolution): We say that a continuous function $u(t, x)$ is a viscosity subsolution if it holds

$$\varphi(w) - \varphi(\tau) \geqq (f - H(p) - u)(w - \tau)$$

for any $(\tau, p) \in J_+^{1,1} u(t, x)$ and $w \leqq \tau$ at every point (t, x). □

Definition (supersolution): We say that a continuous function $v(x, t)$ is a viscosity supersolution if it holds

$$\varphi(w) - \varphi(\sigma) \geqq (f - H(q) - v)(w - \sigma)$$

for any for any $(\sigma, q) \in J_-^{1,1} u(t, x)$ and $w \geqq \sigma$ at every point (t, x). □

Definition (viscosity solution): We say that a continuous function $u(t, x)$ is a viscosity solution if u is both viscosity sub- and supersolution.

Here, $J_+^{1,1} u(t, x)$ and $J_-^{1,1} u(t, x)$ are upper and lower semijets, respectively.

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On Logistic Diffusion Equations with Nonlocal Effects

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1 Introduction

In this paper we discuss the following problem for logistic equations with diffusion and nonlocal effects:

$$\text{(P)} \qquad \begin{cases} u_t = d\Delta u + u\left(a - bu - \displaystyle\int_\Omega k(x,y)u(y,t)dy\right) & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, a, d are positive constants, b is a nonnegative constant, $k \in C(\overline{\Omega} \times \overline{\Omega})$ is a nonnegative function and u_0 is a nonnegative function. In (P), u denotes the population density of a certain species. Usually, the dynamics of the population density is governed by a logistic diffusion equation (without nonlocal terms). If $k \equiv 0$ in (P), it is well known that there exists a unique global solution u and that

$$\lim_{t\to\infty} u(\cdot,t) = \begin{cases} 0 & \text{uniformly in } \Omega \text{ if } 0 < a \le d\lambda_1, \\ \theta & \text{uniformly in } \Omega \text{ if } a > d\lambda_1, \end{cases}$$

where λ_1 is the principal eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary condition and θ is a unique positive stationary solution (which exists if and only if $a > d\lambda_1$). However, it is sometimes reasonable to take account of nonlocal effects since each individual species interacts either visually or by chemical means in a real world. So we will discuss a logistic diffusion equation by adding a nonlocal reaction term as in (P).

Our main purpose is to investigate the difference or similarity between local problems and nonlocal problems for logistic diffusion equations. In particular, we are interested in the following points:

(a) Existence and uniqueness of bounded global solutions for (P),
(b) Asymptotic behavior of global solutions as $t \to \infty$,
(c) Structure of positive solutions for the corresponding stationary problem:

$$\text{(SP)} \qquad \begin{cases} d\Delta u + u\left(a - bu - \displaystyle\int_\Omega k(x,y)u(y)dy\right) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

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For semilinear elliptic equations with nonlocal terms, there are a lo of works (see, e.g, [1], [2], [3], [6], [8]). In most papers, existence of positive solutions has been established with use of bifurcation theory or the Leray-Schauder degree theory. Here we will give a very elementary method to construct a positive stationary solution to (SP).

The contents of the present paper are as follows. In Section 2, we will show that (P) admits a unique global solution for any nonnegative initial data in a suitable class. Section 3 is devoted to the analysis of (SP). We will look for a positive solution of (SP) by a constructive manner. Finally, some remarks are given in section 4.

Notation. We denote by $L^p(\Omega)$ the space of measurable functions $u: \Omega \to R$ such that $|u(x)|^p$ is integrable over Ω with norm

$$\|u\|_p := \left\{ \int_\Omega |u(x)|^p \, dx \right\}^{1/p}.$$

For $p = 2$, we simply write $\|\cdot\|$ in place of $\|\cdot\|_2$. By $W^{k,p}(\Omega)$, we denote the Sobolev space of functions $u \to R$ such that u and its distributional derivatives up to order k belong to $L^p(\Omega)$. Its norm is defined by

$$\|u\|_{W^{k,p}}^p = \sum_{|\rho| \le k} \|D^\rho u\|_p^p,$$

where ρ denotes a multi-index for derivatives.

2 Existence of global solutions

We will discuss (P) in the framework of $L^p(\Omega)$ with $p > 1$. Define a closed linear operator A in $L^p(\Omega)$ by

$$Au = -d\Delta u \quad \text{with domain} \quad D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

Then it is well known that $-A$ generates an analytic semigroup $\{e^{-tA}\}_{t \ge 0}$ in $L^p(\Omega)$ (see, e.g., [9, 11]). Our problem (P) can be written as

$$\begin{cases} u_t + Au = f(u, \ell(u)), \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where

$$f(u, v) = u(a - bu - v) \qquad \text{with} \quad \ell(u) = \int_\Omega k(x, y) u(y) dy.$$

For (2.1) we can prove the following local existence theorem:

Theorem 2.1. *Let $p > \max\{1, N/2\}$. For any $u_0 \in L^p(\Omega)$, there exists a positive number T such that* (2.1) *has a unique solution u in the class*

$$u \in C([0, T]; L^p(\Omega)) \cap C((0, T]; W^{2,p}(\Omega)) \cap C^1((0, T]; L^p(\Omega)).$$

Proof. The proof is standard. The first procedure is to rewrite (2.1) in the form of integral equation

$$u(t) = e^{-tA} u_0 + \int_0^t e^{-(t-s)A} f(u(s), \ell(u(s))) ds. \tag{2.2}$$

The second procedure is to apply Banach's fixed point theorem to (2.2) in order to show the existence and uniqueness of a local solution. For details, see [9] or [11]. □

In what follows we assume

$$u_0 \in L^\infty(\Omega) \tag{2.3}$$

and establish the global existence theorem.

Theorem 2.2. *Let $p > \max\{1, N/2\}$ and assume* (2.3).
(i) *If $b > 0$, then* (2.1) *has a unique solution u in the class*

$$u \in C([0,\infty); L^p(\Omega)) \cap C((0,\infty); W^{2,p}(\Omega)) \cap C^1((0,\infty); L^p(\Omega)).$$

Moreover, u satisfies

$$0 \le u(x,t) \le \max\left\{\|u_0\|_\infty, \frac{a}{b}\right\}$$

for all $(x,t) \in \Omega \times [0,\infty)$.
(ii) *If $b = 0$, then* (2.1) *has a unique solution u in the same class as* (i)*. Moreover, if there exists a positive constant k_0 such that $k(x,y) \ge k_0$ for all $x, y \in \Omega$, then*

$$0 \le u(x,t) \le m$$

with a positive number m for all $(x,t) \in \Omega \times [0,\infty)$.

Proof. (i) Since $u_0 \ge 0$, it is easy to show by the maximum principle for parabolic equations (see [12]) that $u(\cdot,t) \ge 0$ as long as it exists. Therefore, u satisfies

$$u_t \le d\Delta u + u(a - bu) \qquad \text{in} \ \ \Omega \times [0,T),$$

where T is a maximal existence time. The comparison theorem for parabolic equations enables us to show that

$$u \le \max\left\{\|u_0\|_\infty, \frac{b}{a}\right\}$$

for $(x,t) \in \Omega \times [0,T)$. Hence we can conclude $T = \infty$ and obtain a required estimate.

(ii) We will show the uniform boundedness of the solution u in case $k \ge k_0$. Integrating the first equation of (P) leads to

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u(x,t)dx &= d\int_\Omega \Delta u(x,t) + a\int_\Omega u(x,t)dx - \int_\Omega u(x,t)\ell(u(t))dx \\
&= d\int_{\partial\Omega} \frac{\partial u}{\partial n}d\sigma + a\int_\Omega u(x,t)dx - \int_\Omega u(x,t)\left(\int_\Omega k(x,y)u(y,t)dy\right)dx \qquad (2.4)\\
&< a\int_\Omega u(x,t)dx - k_0\left(\int_\Omega u(x,t)dx\right)^2.
\end{aligned}$$

Here we have used $\partial u/\partial n|_{\partial\Omega} < 0$ by the strong maximum principle (see [12]). Solving differential inequality (2.4) we get

$$\int_\Omega u(x,t)dx \le \max\left\{\|u_0\|_1, \frac{a}{k_0}\right\}. \tag{2.5}$$

Since $|\ell(u)| \le k_\infty\|u\|_1$ with $k_\infty = \sup\{k(x,y);\ x,y \in \Omega\}$, we see

$$\|f(u,\ell(u))\|_1 = \|u(a - \ell(u))\|_1 \le a\|u\|_1 + k_\infty\|u\|_1^2;$$

so that it follows from (2.5) that

$$\sup_{t\ge0}\{\|f(u(t),\ell(u(t)))\|_1\} = m_1.$$

In order to derive uniform boundedness of $u(t)$, it is sufficient to use $L^p - L^q$ estimates for $\{e^{-tA}\}_{t\ge0}$ with $p,q \in [1,\infty]$ and follow the arguments developed in the work of Rothe [13]. So we omit the rest of the proof.

$\square$

3 Stationary positive solutions

In this section we will study (SP) associated with (P). In particular, we are interested in positive stationary solutions and look for them in the case

$$k(x,y) = p(x)q(y), \tag{3.1}$$

where $p, q(\not\equiv 0)$ are nonnegative continuous functions in $\overline{\Omega}$. So our problem is written as follows:

$$\begin{cases} d\Delta u + u\left(a - bu - p(x)\displaystyle\int_\Omega q(y)u(y)dy\right) = 0 & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega, \\ u > 0 & \text{in } \ \Omega, \end{cases} \tag{3.2}$$

where a, d are positive constants, b is a nonnegative number. Lots of authors (e.g., [1], [2], [3]. [6], [8]) have discussed the existence of positive solutions for semilinear elliptic equations with nonlocal terms by means of bifurcation theory, the Leray-Schauder degree theory and monotone methods. Among them, Corréa, Delgado and Suárez [2] have studied (3.2) in case $b = 0$ and obtained an interesting result.

Theorem 3.1. ([2]) *Assume that* $\Omega_0 := \mathrm{Int}\{x \in \Omega; p(x) = 0\}$ *is connected. Then* (3.2) *has a unique positive solution* u *if and only if*

$$\begin{cases} a \in (\lambda_{1,\Omega}, \infty) & \textit{in case } \ \Omega_0 = \emptyset, \\ a \in (\lambda_{1,\Omega}, \lambda_{1,\Omega_0}) & \textit{in case } \ \Omega_0 \neq \emptyset. \end{cases}$$

Here $\lambda_{1,D}$ *stands for the principal eigenvalue of the following eigenvalue problem*

$$-\Delta u = \lambda u \quad \textit{in } \ D \quad \textit{with } \ u = 0 \quad \textit{on } \ \partial D.$$

We will briefly explain the idea of the proof of Theorem 3.1. Let u be a positive solution of (3.2) with $b = 0$. If we put

$$\alpha = \int_\Omega q(x)u(x)dx, \tag{3.3}$$

we can rewrite (3.2) in the following form

$$\begin{cases} -d\Delta u + \alpha p(x)u = au & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega, \\ u > 0 & \text{in } \ \Omega. \end{cases} \tag{3.4}$$

Since u is a positive definite function, a must be identical with the principal eigenvalue of the following eigenvalue problem

$$-d\Delta u + \alpha p(x)u = \lambda u \quad \text{in } \ \Omega \quad \text{and } \ u = 0 \quad \text{on } \ \partial\Omega. \tag{3.5}$$

If we denote by $\lambda_1(\alpha p)$ the principal eigenvalue of (3.5), we have only to find α satisfying $\lambda_1(\alpha p) = a$.

It is well known that $\lambda_1(\alpha p)$ can be expressed by the following variational characterization

$$\lambda_1(\alpha p) = \inf\left\{d\int_\Omega |\nabla u|^2 dx + \alpha\int_\Omega p(x)u^2 dx; u \in H_0^1(\Omega), \|u\|_2 = 1\right\}. \tag{3.6}$$

It should be noted that $\lambda_1(\alpha p)$ has the following properties:

Lemma 3.1. *Let $p(\not\equiv 0)$ be a nonnegative continuous function in $\overline{\Omega}$ and assume that Ω_0 is connected. Then the following properties hold true.*
(i) *The mapping $\alpha \to \lambda_1(\alpha p)$ is continuous and strictly increasing for $\alpha \geq 0$.*
(ii) $\lim\limits_{\alpha\to 0} \lambda_1(\alpha p) = \lambda_1(0) = \lambda_{1,\Omega}$.
(iii) $\lim\limits_{\alpha\to\infty} = \begin{cases} \infty & \textit{in case } \ \Omega_0 = \emptyset, \\ \lambda_{1,\Omega_0} & \textit{in case } \ \Omega_0 \neq \emptyset. \end{cases}$

Proof. Assertions (i) and (ii) come from (3.6). For the proof of (iii), see López-Gómez [10]. □

In order to find a positive solution u of (3.2), it is sufficient to look for α^* satisfying $\lambda_1(\alpha^* p) = a$ for given a. Then u can be obtained as $u = c\varphi$ with positive constant c, where φ is a positive eigenfunction of (3.5) corresponding to $\lambda_1(\alpha^* p)$. In view of (3.3), positive constant c can be determined from

$$\alpha^* = c\int_\Omega q(x)\varphi(x)dx.$$

Therefore, it is easy to prove Theorem 3.1 if we use Lemma 3.1.

We now discuss the existence of positive solutions of (3.2) in case $b > 0$. Let u be a positive solution of (3.2). If we define α by (3.3), then the first equation of (3.2) can be written as

$$\begin{cases} -d\Delta u + \alpha p(x)u = u(a - bu) & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega. \end{cases} \tag{3.7}$$

Our strategy is to look for a positive solution $\theta(x : \alpha p)$ for (3.7) for each $\alpha \geq 0$ and determine α from

$$\alpha = \int_\Omega q(x)\theta(x; \alpha p)dx. \tag{3.8}$$

In place of (3.7) we will study the existence of positive solutions for the following auxiliary problem:

$$\begin{cases} -d\Delta u + m(x)u = u(a - bu) & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega, \\ u > 0 & \text{in } \ \Omega, \end{cases} \tag{3.9}$$

where a, b, d are positive constants and $m : \overline{\Omega} \to R$ is a nonnegative continuous function. We have the following result.

Proposition 3.1. *Let m be a nonnegative continuous function in $\overline{\Omega}$. Then* (3.9) *has a unique positive solution $\theta(x; m)$ if and only if $a > \lambda_1(m)$. Moreover, if $m_1 \geq m_2$ $(m_1 \not\equiv m_2)$, then $\theta(x; m_2) > \theta(x; m_1)$ for $x \in \Omega$.*

Proof. Since $\lambda_1(m)$ is the principal eigenvalue, one can choose a positive eigenfunction $\varphi(x; m)$ corresponding to $\lambda_1(m)$ such that

$$\max_{x\in\Omega} \varphi(x; m) = 1 \quad \text{and} \quad \varphi(x; m) > 0 \quad \text{in } \ \Omega.$$

If we set $u^*(x) = c_1$ with positive constant c_1 satisfying $c_1 \geq a/b$, then we see that u^* is a supersolution of (3.9). We next take

$$v_*(x) = \varepsilon\varphi(x; m) \quad \text{with positive constant } \ \varepsilon.$$

Then

$$-d\Delta v_* + v_*(m(x) - a + bv_*) = \epsilon\varphi(x;m)(\lambda_1(m) - a + b\varepsilon\varphi(x;m)).$$

Hence, if $a > \lambda_1(m)$, one can take a sufficiently small $\varepsilon > 0$ such that $b\varepsilon \le a - \lambda_1(m)$. In this case,

$$-d\Delta v_* + v_*(m(x) - a + bv_*) \le 0;$$

that is, v_* is a subsolution of (3.9). Thus we can construct a supersolution u^* and a subsolution v_* satisfying $u^* \ge v_*$. Hence it follows from the result of Sattinger [14] that (3.9) has a positive solution.

The proofs of the necessity part and the uniqueness of positive solutions are standard; so we omit them.

Finally, we will prove the order preserving property. Let $m_1 \ge m_2$; then $\theta(x;m_2)$ is a supersolution of (3.9) with $m = m_1$. Therefore

$$\theta(x;m_2) \ge \theta(x;m_1) \quad \text{in} \ \ \Omega.$$

Moreover, if we set $w(x) = \theta(x;m_2) - \theta(x;m_1)$, then w satisfies

$$\begin{cases} -d\Delta w + m_2 w + w\{b(\theta(x;m_1) + \theta(x;m_2)) - a\} \ge 0 & \text{in} \ \ \Omega, \\ w = 0 & \text{on} \ \ \partial\Omega. \end{cases}$$

Therefore, one can apply the strong maximum principle ([12]) to conclude $w > 0$ in Ω. □

We are ready to study (3.2) in case $b > 0$. It follows from Proposition 3.1 that (3.7) has a unique solution $\theta(x;\alpha p)$ if and only if

$$a > \lambda_1(\alpha p). \tag{3.10}$$

Here we should recall basic properties of $\lambda_1(\alpha p)$ as a function of α (see Lemma 3.1).

In what follows, assume

$$a > d\lambda_{1,\Omega}. \tag{3.11}$$

Then it is possible to find a unique $\bar{\alpha} > 0$ satisfying $a = \lambda_1(\bar{\alpha}p)$ in case $\Omega_0 = \emptyset$. In case $\Omega_0 \ne \emptyset$, if we additionally assume $a < d\lambda_{1,\Omega_0}$; then it is also possible to find $\bar{\alpha}$ which satisfies the same property as above. When a satisfies $a \ge d\lambda_{1,\Omega_0}$ in case $\Omega_0 \ne \emptyset$, we set $\bar{\alpha} = \infty$. Then we see that (3.10) is equivalent to

$$0 \le \alpha < \bar{\alpha} \tag{3.12}$$

and that, if α satisfies (3.12), then (3.7) has a unique positive solution $\theta(x;\alpha p)$.

Lemma 3.2. *The mapping $\alpha \to \theta(x;\alpha p)$ is of class C^1 from $[0,\bar{\alpha})$ to $C(\overline{\Omega})$ and strictly decreasing. Moreover, it satisfies the following properties:*

(i) $\lim_{\alpha\to 0} \theta(\cdot;\alpha p) = \theta_0$ *uniformly in Ω, where θ_0 is a unique positive solution of*

$$d\Delta\theta + \theta(a - b\theta) = 0 \quad \textit{in} \ \ \Omega \quad \textit{and} \ \ \theta = 0 \quad \textit{on} \ \ \partial\Omega.$$

(ii) $$\lim_{\alpha\to\bar{\alpha}} \theta(\cdot;\alpha p) = \begin{cases} 0 & \textit{uniformly in} \ \ \Omega \quad \textit{if} \ \ \bar{\alpha} < \infty, \\ \theta_\infty & \textit{uniformly in} \ \ \Omega \quad \textit{if} \ \ \bar{\alpha} = \infty. \end{cases}$$

Here θ_∞ is a function satisfying $\theta_\infty \equiv 0$ in $\Omega \setminus \Omega_0$ and

$$\begin{cases} d\Delta\theta_\infty + \theta_\infty(a - b\theta_\infty) = 0 & \textit{in} \ \ \Omega_0, \\ \theta_\infty = 0 & \textit{on} \ \ \partial\Omega_0, \\ \theta_\infty > 0 & \textit{in} \ \ \Omega_0. \end{cases}$$

Before giving the proof of Lemma 3.2 we will prove the solvability of (3.2).

Theorem 3.2. *Let $a > d\lambda_{1,\Omega}$. Then* (3.2) *has a unique positive solution u^*.*

Remark 3.1. *It is easy to show that* (3.2) *has no positive solution for $a \leq d\lambda_{1,\Omega}$.*

Proof. Since $\theta(x; \alpha p)$ is a positive solution of (3.7) for $0 \leq \alpha < \bar{\alpha}$, we see, in view of (3.3), that $\theta(x; \alpha p)$ is a positive solution of (3.2) if and only if α satisfies (3.8). Denote the right-hand side of (3.8) by $F(\alpha)$. It follows from Lemma 3.2 that $F(\alpha)$ is strictly decreasing for $\alpha \in [0, \bar{\alpha}]$ and satisfies

$$F(0) = \int_\Omega q(x)\theta_0(x)dx > 0$$

and

$$\begin{cases} F(\bar{\alpha}) = 0 & \text{in case } \ \bar{\alpha} < \infty, \\ \lim\limits_{\alpha\to\infty} F(\alpha) = \displaystyle\int_{\Omega_0} q(x)\theta_\infty(x)dx & \text{in case } \ \bar{\alpha} = \infty. \end{cases}$$

Therefore, it is easy to find a unique α^* satisfying $\alpha^* = F(\alpha^*)$ in both cases $\bar{\alpha} < \infty$ and $\bar{\alpha} = \infty$. Clearly, $\theta(x; \alpha^* p)$ becomes a unique positive solution of (3.2). □

Proof of Lemma 3.2. Observe that $\theta(x; \alpha p)$ satisfies

$$-d\Delta\theta(x; \alpha p) + \alpha p(x)\theta(x; \alpha p) + \theta(x; \alpha p)(b\theta(x; \alpha p) - a) = 0 \quad \text{in } \ \Omega$$

with $\theta(x; \alpha p) = 0$ on $\partial\Omega$. Differentiation of the above equation with respect to α leads us to

$$-d\Delta w + \alpha p(x)w + (2b\theta(x; \alpha p) - a)w = -p(x)\theta(x; \alpha p) \quad \text{in } \ \Omega \quad \text{and} \quad w = 0 \quad \text{on } \ \partial\Omega$$

with $w(x) = (\partial/\partial\alpha)\theta(x; \alpha p)$. We should recall that $-d\Delta + \alpha p(x) + 2b\theta(x; \alpha p) - a$ is an invertible and order-preserving operator from $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to $L^p(\Omega)$ (see, e.g., [15, Lemma 1.1]). Therefore, the implicit function theorem assures to show

$$\frac{\partial\theta(\alpha p)}{\partial\alpha} = -\{-d\Delta + \alpha p(x) + 2\theta(x; \alpha p) - a\}^{-1}(p\theta(\alpha p)) < 0 \quad \text{in } \ \Omega.$$

Thus $\alpha \to \theta(x; \alpha p)$ is strictly decreasing.

It is easy to see $\theta(0) = \theta_0$ and $\theta(\bar{\alpha}p) = 0$ in case $\bar{\alpha} < \infty$.

It remains to study $\lim_{\alpha\to\infty} \theta(\alpha p)$ in case $\bar{\alpha} = \infty$. Since $\theta(\alpha p)$ is positive and strictly decreasing with respect to α, there exists a nonnegative function θ_∞ such that

$$\lim_{\alpha\to\infty} \theta(\alpha p) = \theta_\infty \quad \text{pointwise in } \ \Omega. \tag{3.13}$$

Take any $\varphi \in C_0^\infty(\Omega)$; then it holds that

$$-d\int_\Omega \theta(x; \alpha p)\Delta\varphi dx + \alpha\int_\Omega p(x)\theta(x; \alpha p)\varphi dx = \int_\Omega \theta(x; \alpha p)(a - b\theta(x; \alpha p))dx. \tag{3.14}$$

Since $p(x) = 0$ in Ω_0, we see from (3.14) that

$$\int_{\Omega\setminus\Omega_0} p(x)\theta(x; \alpha p)\varphi dx = \frac{1}{\alpha}\left\{d\int_\Omega \theta(x; \alpha p)\Delta\varphi dx + \int_\Omega \theta(x; \alpha p)(a - b\theta(x; \alpha p))dx\right\}. \tag{3.15}$$

Making use of the uniform boundedness of $\theta(x; \alpha p)$ for $\alpha \geq 0$ and letting $\alpha \to \infty$ in (3.15) one can derive

$$\int_{\Omega\setminus\Omega_0} p(x)\theta_\infty(x)\varphi dx = 0 \quad \text{for any } \ \varphi \in C_0^\infty(\Omega).$$

Therefore, $\theta_\infty(x) = 0$ for $x \in \Omega \setminus \Omega_0$.

We next take any $\varphi \in C_0^\infty(\Omega_0)$ and define $\tilde{\varphi} \in C_0^\infty(\Omega)$ by $\tilde{\varphi}(x) = \varphi(x)$ if $x \in \Omega_0$ and $\tilde{\varphi}(x) = 0$ if $x \in \Omega \setminus \Omega_0$. Setting $\varphi = \tilde{\varphi}$ in (3.14) leads to

$$-d \int_{\Omega_0} \theta(x;\alpha p)\Delta\varphi dx = \int_{\Omega_0} \theta(x;\alpha p)(a - b\theta(x;\alpha p)\varphi dx.$$

Letting $\alpha \to \infty$ in the above identity we get

$$-d \int_{\Omega_0} \theta_\infty \Delta\varphi dx = \int_{\Omega_0} \theta_\infty(a - b\theta_\infty)\varphi dx;$$

which implies

$$\begin{cases} -d\Delta\theta_\infty = \theta_\infty(a - b\theta_\infty) & \text{in } \Omega, \\ \theta_\infty = 0 & \text{on } \partial\Omega. \end{cases}$$

It should be noted by elliptic regularity theory that θ_∞ becomes continuous in Ω. Therefore, one can conclude from Dini's theorem that the convergence in (3.13) is uniform. Thus the proof is complete.

4 Concluding remarks

4.1 Stability of stationary solution

In the previous section, we have shown in Theorem 3.2 that (3.2) has a unique positive solution u^*. Then it is a very important problem to study the stability of u^*. The spectral problem for the linearized operator around $u = u^*$ is given by

$$\begin{cases} -d\Delta v + a_1(x)v + p(x)u^*(x) \int_\Omega q(y)v(y)dy = \sigma v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where

$$a_1(x) = 2bu^*(x) - a + p(x) \int_\Omega q(y)u^*(y)dy.$$

The above linearized operator is not self-adjoint; so that the spectral problem may have complex eigenvalues. Moreover, we do not know if the Krein-Rutman theorem holds for (4.1) or not. So it is difficult to get satisfactory information on the spectrum for (4.1). (Note that Theorem 2.1 in [2] is not applicable to (4.1).)

In general, it is a delicate and difficult problem to study the eigenvalues for the operator with nonlocal terms, see, e.g., [4], [5, 6, 7].

Finally, it should be noted that, if a is regarded as a bifurcation parameter in (3.2), then the local bifurcation theory assures the existence and uniqueness of bifurcating positive solutions of (3.2) if a ($> d\lambda_{1,\Omega}$) is very close to $d\lambda_{1,\Omega}$. We can also show that such bifurcating positive solutions are asymptotically stable when a is very close to $d\lambda_{1,\Omega}$. So we have a conjecture that u^* is asymptotically stable for every $a > d\lambda_{1,\Omega}$.

4.2 Positive solutions for general case

Our method of analysis is applicable for more general class of equations with diffusion and nonlocal effects:

$$u_t = d\Delta u + u(f(u) - p(x) \int_\Omega q(y)g(u(y,t))dy),$$

where $f(u)$ is a deceasing and locally Lipschitz continuous function such that $f(0) > 0$ and $g(v)$ is an increasing, positive and locally Lipschitz continuous function for $v > 0$.

In Section 3, we have discussed the stationary problem in a case when k has a special form (3.1). Taking account of nonlocal effects it is also important to study the stationary problem in case k has the following form

$$k(x, y) = \rho(x - y),$$

where ρ is nonnegative and continuous function. For this problem, we can also apply the bifurcation theory by regarding a as a bifurcation parameter. So it is also possible to show that, for each $a > d\lambda_{1,\Omega}$

$$\begin{cases} d\Delta u + u\left(a - bu - \displaystyle\int_\Omega \rho(x - y)u(y)dy\right) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least one positive solution. We will discuss this fact elsewhere.

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