With Compliments of the Author

Fourier transforms on the motion groups

By Keisaku KUMAHARA

Reprinted from
JOURNAL OF THE MATHEMATICAL SOCIETY OF JAPAN,
Vol. 28, No. 1, January 1976, pp. 18-32

J. Math. Soc. Japan
Vol. 28, No. 1, 1976

Fourier transforms on the motion groups

By Keisaku KUMAHARA

(Received May 25, 1974)

§ 1. Introduction.

The purpose of the present paper is to characterize the images of some function spaces on the motion groups by the Fourier transform.

Let K be a connected compact Lie group acting on a finite dimensional real vector space V as a linear group. Let G be the semidirect product of V and K, i.e. G is the group comprised of all pairs (x, k) $(x \in V, k \in K)$ with the direct product topology, multiplication being given by $(x_1, k_1)(x_2, k_2) = (x_1 + k_1 x_2, k_1 k_2)$. G is called the motion group.

Let $\hat{V}$ be the dual space of V. For any $\xi \in \hat{V}$ we denote by U^ξ the induced representation of G by the unitary representation $x \mapsto e^{i<\xi, x>}$, $(i = \sqrt{-1})$ of the normal abelian subgroup V. U^ξ is not irreducible. Any irreducible unitary representation of G is, however, contained in U^ξ for some $\xi \in \hat{V}$ as an irreducible component. Let E be a function space on G. We define the Fourier transform T_f of $f \in E$ by $T_f(\xi) = \int_G f(g) U_g^\xi dg$. If f is integrable, this transform has meaning and T_f is a bounded operator valued function on $\hat{V}$.

The Plancherel formula for G (L_2-theory) was given by A. Kleppner and R. Lipsman ([1], Theorem 4.4). Let $C_c^\infty(G)$ be the space of all infinitely differentiable functions with compact support on G. Let $\mathcal{S}(G)$ be the space of all infinitely differentiable and rapidly decreasing functions on G. In this paper we consider these two cases $E = C_c^\infty(G)$ (the Paley-Wiener theorem) and $E = \mathcal{S}(G)$. Then $T_f(\xi)$ is an integral operator on $L_2(K)$ for any $f \in E$ and $\xi \in \hat{V}$ and its kernel function is given by $\kappa_f(\xi; k_1, k_2) = \int_V f(k_1 x, k_1 k_2^{-1}) e^{i<\xi, x>} dx$, $(k_1, k_2 \in K)$. When K is the identity group, κ_f is the ordinary Fourier transform on Euclidean space V. We call κ_f the scalar Fourier transform of f. Let $\tilde{E}$ and $\hat{E}$ be the images of E by the scalar Fourier transform and Fourier transform, respectively. The characterization of $\tilde{E}$ can be accomplished by the ordinary arguments of the classical Fourier analysis. To study the mapping $\kappa_f \mapsto T_f$ from $\tilde{E}$ to $\hat{E}$ we use an auxiliary theorem which can be proved using the representation theory of compact groups.

We can assume that there exists a K-invariant inner product on V. Therefore, we can assume beforehand that K is a connected subgroup of $SO(n)$, where n is the dimension of V. If $K=\{1\}$, $G=V\cong \boldsymbol{R}^n$. If $K=SO(n)$, G is the Euclidean motion group. We state another example. Let G_0 be a connected noncompact semisimple Lie group with finite centre and K be a maximal compact subgroup of G_0. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G_0, where $\mathfrak{k}$ is the subalgebra corresponding to K. Then K operates on $\mathfrak{p}$ via the adjoint representation. If $V=\mathfrak{p}$, G is called the Cartan motion group. If G_0 is the Lorentz group $SO_0(n, 1)$, G is again the Euclidean motion group. K. Okamoto and the author proved the Paley-Wiener theorem for the Euclidean motion group in [3]. M. Sugiura determined the space $\mathcal{S}(G)^{\wedge}$ for the Euclidean motion group ([5]).

This paper, in the first presented form, was entitled as "Fourier Transforms on the Cartan Motion Group", and treated the Cartan motion group case only. The author was suggested by the referee to represent in this more general form. The author is very thankful to the referee for his remarks of great value. The short summary for the Cartan motion group case is in [2].

The author would like to express his sincere gratitude to Professors O. Takenouchi and K. Okamoto who have encouraged him with kind advices. He also would like to express his thanks to Professor M. Sugiura who suggested the generalization of our problems.

§ 2. Scalar Fourier transform.

Let $(\ ,\ )$ be a K-invariant inner product in V. K also operates on $\hat{V}$ via the contragredient of the action on V. In V and $\hat{V}$ we can define the K-invariant measure which are induced by the above inner product. We normalize these measures by multiplying $(2\pi)^{-n/2}$, $(n=\dim V)$, and denote them by dx and $d\xi$, respectively. Let dk be the Haar measure on K normalized such as the total measure equals to 1. Then $dg=dxdk$ is the normalized Haar measure on G.

Let $\mathfrak{H}=L_2(K)$ be the space of all square integrable functions on K. The representation U^{ξ} induced by $\xi\in\hat{V}$ is realized on $\mathfrak{H}$ as follows; for $g=(x, k)\in G$

$$(U_g^{\xi}F)(k_1)=e^{i\langle\xi, k_1^{-1}x\rangle}F(k^{-1}k_1)\,, \qquad (F\in\mathfrak{H},\ k_1\in K)\,.$$

Hence if $f\in L_1(G)$, we have

$$\begin{aligned}(T_f(\xi)F)(k_1)&=\int_G f(g)(U_g^{\xi}F)(k_1)dg\\ &=\int_K \kappa_f(\xi\,;\,k_1,\,k_2)F(k_2)dk_2\,,\end{aligned}$$

where $F\in\mathfrak{H}$, $k_1\in K$ and

$$\kappa_f(\xi\,;\,k_1,\,k_2)=\int_V f(k_1x,\,k_1k_2^{-1})e^{i<\xi,x>}dx\,.$$

1) $E=C_c^\infty(G)$. Let $|x|=(x,\,x)^{1/2}$. We define a compact subset $\Omega(a)$ of G for any positive number a by $\Omega(a)=\{(x,\,k)\in G\,;\,|x|\leqq a\}$. We denote by $\hat{V}^c$ the complexification of $\hat{V}$. We extend naturally the K-action on $\hat{V}$ to the K-action on $\hat{V}^c$.

LEMMA 1. *A function $\kappa(\xi\,;\,k_1,\,k_2)$ on $\hat{V}\times K\times K$ is the scalar Fourier transform of $f\in C_c^\infty(G)$ such that* supp $(f)\subset\Omega(a)$ $(a>0)$ *if and only if it satisfies the following conditions:*

(i) *$\kappa(\xi\,;\,k_1,\,k_2)$ can be extended to a C^∞ function on $\hat{V}^c\times K\times K$ and $\kappa(\zeta\,;\,k_1,\,k_2)$ $(\zeta\in\hat{V}^c)$ is entire analytic with respect to ζ for each $k_1,\,k_2\in K$.*

(ii) *For any K-invariant polynomial function $p(\zeta)$ on $\hat{V}^c$ and for any right invariant differential operators $y,\,y'$ on K there exists a constant $C_p^{y,y'}\geqq 0$ such that*

$$|p(\zeta)y_{k_1}y'_{k_2}\kappa(\zeta\,;\,k_1,\,k_2)|\leqq C_p^{y,y'}\exp a|\mathrm{Im}\,\zeta|$$

for any $k_1,\,k_2\in K$.

(iii) *For any $k\in K$*

$$\kappa(k\zeta\,;\,k_1,\,k_2)=\kappa(\zeta\,;\,k_1k,\,k_2k)\,,\qquad(\zeta\in\hat{V}^c,\,k_1,\,k_2\in K)\,.$$

PROOF. Let $f\in C_c^\infty(G)$ and supp $(f)\subset\Omega(a)$. For $\zeta\in\hat{V}^c$ we define the scalar Fourier-Laplace transform of f by

$$\kappa_f(\zeta\,;\,k_1,\,k_2)=\int_V f(k_1x,\,k_1k_2^{-1})e^{i<\zeta,x>}dx\,.$$

It is easy to see that the scalar Fourier-Laplace transform is a holomorphic extension of the scalar Fourier transform to $\hat{V}^c\times K\times K$ satisfying (i)$\sim$(iii). Conversely, for a function $\kappa(\zeta\,;\,k_1,\,k_2)$ on $\hat{V}^c\times K\times K$ satisfying (i)$\sim$(iii) we define a function f on G by

$$f(g)=\int_{\hat{V}}\kappa(\xi\,;\,1,\,k^{-1})e^{-i<\xi,x>}d\xi\,,\tag{2.1}$$

where $g=(x,\,k)\in G$. Then we have that $f\in C_c^\infty(G)$, supp $(f)\subset\Omega(a)$ and $\kappa(\xi\,;\,k_1,\,k_2)=\kappa_f(\xi\,;\,k_1,\,k_2)$ for all $\xi\in\hat{V}$ and $k_1,\,k_2\in K$ by the classical Paley-Wiener theorem.

2) $E=\mathcal{S}(G)$. Let $\mathfrak{k}$ be the Lie algebra of K. We denote by $U(\mathfrak{k}^c)$ the universal enveloping algebra of the complexification $\mathfrak{k}^c$ of $\mathfrak{k}$. Let $Y_1,\cdots,Y_\delta$ $(\delta=\dim K)$ be a fixed basis of $\mathfrak{k}$. Then the set $y(m)=\{Y_1^{m_1}\cdots Y_\delta^{m_\delta}\,;\,m=(m_1,\cdots,m_\delta)\in N^\delta\}$ forms a basis of $U(\mathfrak{k}^c)$ by the Birkhoff-Witt theorem. We regard any element of $U(\mathfrak{k}^c)$ as a right invariant differential operator on K. Then it operates on $\mathfrak{H}=L_2(K)$ in the sense of distributions. The Lie algebra of G is $V+\mathfrak{k}$ and $[V,\,V]=\{0\}$, $[\mathfrak{k},\,V]\subset V$ and $[\mathfrak{k},\,\mathfrak{k}]\subset\mathfrak{k}$. We consider that $U(\mathfrak{k}^c)$ is a subalgebra of the universal enveloping algebra of the complexification of

the Lie algebra of G. We denote by λ and μ the left and the right regular representations of G, respectively, and also denote by the same notations the corresponding representations of the universal enveloping algebra on the space of C^{∞}-vectors. The bracket product $[Y, x]$ of $Y \in \mathfrak{k}$ and $x \in V$ is the differential of the K-action on V.

Let $v_1, \cdots, v_n$ be an orthonomal basis of V with respect to the K-invariant inner product $(\,,)$. And let $w_1, \cdots, w_n$ be its dual basis of $\hat{V}$. The inner product $(\,,)$ induces the K-invariant inner product of $\hat{V}$. If $x=\sum_{j=1}^{n} x_j v_j \in V$ and $\xi=\sum_{j=1}^{n} \xi_j w_j \in \hat{V}$, then $|x|^2=(x, x)=\sum_{j=1}^{n} x_j^2$ and $|\xi|^2=\sum_{j=1}^{n} \xi_j^2$. Making use of the coordinate systems with respect to these bases, we define differential operators D_x^{α} on V and D_{ξ}^{α} on $\hat{V}$ for any $\alpha=(\alpha_1, \cdots, \alpha_n) \in N^n$ by

$$D_x^{\alpha}=\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

and

$$D_{\xi}^{\alpha}=\left(\frac{\partial}{\partial \xi_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial \xi_n}\right)^{\alpha_n},$$

respectively. For $f \in C^{\infty}(G)$ we have $\frac{\partial}{\partial x_j} f(x, k)=\lambda(-v_j) f(x, k)$ for all $j=1, \cdots, n$.

Let $\mathcal{S}=\mathcal{S}(G)$ be the set of all those functions f on G satisfying the following conditions:

(i) f is of class C^{∞},

(ii) for any $\alpha \in N^n$, $\beta \in N$ and $m, m' \in N^{\delta}$ there exists a constant $C_{\alpha,\beta}^{m,m'} \geqq 0$ such that

$$|(1+|x|^2)^{\beta}(D_x^{\alpha} \lambda(y(m)) \mu(y(m')) f)(x, k)| \leqq C_{\alpha,\beta}^{m,m'}$$

for all $(x, k) \in G$.

Such functions are called rapidly decreasing.

LEMMA 2. *$\mathcal{S}$ is closed with respect to the applications of D_x^{α} and $\lambda(y)$, $\mu(y')$ for all $\alpha \in N^n$ and $y, y' \in U(\mathfrak{k}^c)$.*

PROOF. Because the left regular representation and the right regular representation commute, we have $\lambda(y)\mu(y')=\mu(y')\lambda(y)$ for all $y, y' \in U(\mathfrak{k}^c)$ and $D_x^{\alpha}\mu(y)=\mu(y)D_x^{\alpha}$ for all $y \in U(\mathfrak{k}^c)$ and $\alpha \in N^n$. Therefore we only have to see that $\lambda(y)D_x^{\alpha}f \in \mathcal{S}$ for all $f \in \mathcal{S}$ and for all $y \in U(\mathfrak{k}^c)$, $\alpha \in N^n$. For $Y \in \mathfrak{k}$ and $1 \leqq j \leqq n$ we have

$$\begin{aligned}
\lambda(Y)\frac{\partial}{\partial x_j} f &= \lambda(Y)\lambda(-v_j) f \\
&= \lambda(-v_j)\lambda(Y) f+\lambda([Y, -v_j]) f \\
&= \lambda(-v_j)\lambda(Y) f+\sum_{q=1}^{n}([Y, v_j], v_q)\lambda(-v_q) f
\end{aligned}$$

$$=\frac{\partial}{\partial x_j}\lambda(Y)f+\sum_{q=1}^{n}([Y, v_j], v_q)\frac{\partial}{\partial x_q}f.$$

Therefore $\lambda(Y)\dfrac{\partial}{\partial x_j}f\in\mathcal{S}$ by the condition (ii) of $\mathcal{S}$. And hence we obtain $\lambda(y)D_x{}^\alpha f\in\mathcal{S}$ for all $y\in U(\mathfrak{k}^c)$ and $\alpha\in N^n$. q. e. d.

We topologize $\mathcal{S}$ by the system of semi-norms of the form

$$\gamma_{\alpha,\beta}^{m,m'}(f)=\sup_{(x,k)\in G}|(1+|x|^2)^\beta(D_x{}^\alpha\lambda(y(m))\mu(y(m'))f)(x, k)|,$$

where $\alpha\in N^n$, $\beta\in N$ and $m, m'\in N^\delta$.

PROPOSITION 1. *$\mathcal{S}$ is a Fréchet space.*

PROOF. It is easy to see that $\mathcal{S}$ is a locally convex topological vector space by the topology defined above. The topology is defined by a system of countable semi-norms. As is easily seen, $\mathcal{S}$ is a Hausdorff space. Hence $\mathcal{S}$ is metrizable. Using Lemma 2, we have the sequentially completeness of $\mathcal{S}$. Hence $\mathcal{S}$ is complete. q. e. d.

Let $\tilde{\mathcal{S}}$ be the set of those functions $\kappa(\xi; k_1, k_2)$ on $\hat{V}\times K\times K$ satisfying the following conditions:

(i)$^\sim$ $\kappa(\xi; k_1, k_2)$ is a C^∞ function on $\hat{V}\times K\times K$,

(ii)$^\sim$ for any $\alpha\in N^n$, $\beta\in N$ and $m, m'\in N^\delta$ there exists a constant $C_{\alpha,\beta}^{m,m'}$ such that

$$|(1+|\xi|^2)^\beta(D_\xi{}^\alpha y(m)_{k_1}y(m')_{k_2}\kappa)(\xi; k_1, k_2)|\leqq C_{\alpha,\beta}^{m,m'}$$

for all $(\xi, k_1, k_2)\in\hat{V}\times K\times K$,

(iii)$^\sim$ for any $k\in K$

$$\kappa(k\xi; k_1, k_2)=\kappa(\xi; k_1k, k_2k), \qquad (\xi\in\hat{V},\ k_1, k_2\in K).$$

We topologize $\tilde{\mathcal{S}}$ by the system of semi-norms of the form

$$\tilde{\gamma}_{\alpha,\beta}^{m,m'}(\kappa)=\sup_{(\xi,k_1,k_2)}|(1+|\xi|^2)^\beta(D_\xi{}^\alpha y(m)_{k_1}y(m')_{k_2}\kappa)(\xi; k_1, k_2)|,$$

where $\alpha\in N^n$, $\beta\in N$ and $m, m'\in N^\delta$. Then we have the following proposition.

PROPOSITION 2. *$\tilde{\mathcal{S}}$ is a Fréchet space.*

Now we prove the space $\tilde{\mathcal{S}}$ is the image of the space $\mathcal{S}$ by the scalar Fourier transform.

LEMMA 3. *The scalar Fourier transform $f\to\kappa_f$ is a topological isomorphism from $\mathcal{S}$ onto $\tilde{\mathcal{S}}$.*

PROOF. Let $f\in\mathcal{S}$. We put $|\alpha|=\alpha_1+\cdots+\alpha_n$ for $\alpha=(\alpha_1,\cdots,\alpha_n)\in N^n$. If $x=\sum_{j=1}^{n}x_jv_j\in V$, we have

$$|x_j|\leqq 1+|x_j|^2\leqq 1+|x|^2 \tag{2.2}$$

for any $j=1, \cdots, n$. By the condition (ii) of the definition of $\mathcal{S}$ and (2.2) we have, for any $\alpha \in N^n$,

$$\begin{aligned}|f(k_1x, k_1k_2^{-1})D_\xi^\alpha e^{i<\xi,x>}| &= |(ix_1)^{\alpha_1}\cdots(ix_n)^{\alpha_n}f(k_1x, k_1k_2^{-1})e^{i<\xi,x>}| \\ &= |x_1|^{\alpha_1}\cdots|x_n|^{\alpha_n}|f(k_1x, k_1k_2^{-1})| \\ &\leqq (1+|x|^2)^{|\alpha|}|f(k_1x, k_1k_2^{-1})| \\ &= (1+|k_1x|^2)^{|\alpha|}|f(k_1x, k_1k_2^{-1})| \\ &\leqq C^{0,0}_{0,|\alpha|+n}(1+|k_1x|^2)^{-n} \\ &= C^{0,0}_{0,|\alpha|+n}(1+|x|^2)^{-n}.\end{aligned}$$

This is integrable on V. Hence $\kappa_f(\xi; k_1, k_2)$ is infinitely differentiable with respect to ξ for any k_1 and k_2. On the other hand, for any $Y, Y' \in \mathfrak{k}$ we obtain

$$Y_{k_1}Y'_{k_2}(f(k_1x, k_1k_2^{-1})) = (\lambda(-Y)\mu(-Y')f)(k_1x, k_1k_2^{-1})$$

for all $x \in V$, where Y_{k_1} (resp. Y'_{k_2}) denote the differential by the right invariant vector field Y (resp. Y') with respect to k_1 (resp. k_2). By this fact and the condition (ii) of the definition of $\mathcal{S}$, $\kappa_f(\xi; k_1, k_2)$ is infinitely differentiable with respect to k_1 and k_2. Because $f=0$ at $x=\infty$ and $\left(1-\sum_{j=1}^n\left(\frac{\partial}{\partial x_j}\right)^2\right)^\beta$ is a K-invariant differential operator on V for any $\beta \in N$, we have

$$(1+|\xi|^2)^\beta\kappa_f(\xi; k_1, k_2) = \int_V\left(\left(1-\sum_{j=1}^n\left(\frac{\partial}{\partial x_j}\right)^2\right)^\beta f\right)(k_1x, k_1k_2^{-1})e^{i<\xi,x>}dx,$$

using the integration by parts. If we expand $\left(1-\sum_{j=1}^n\left(\frac{\partial}{\partial x_j}\right)^2\right)^\beta$, we can find positive numbers $c_1, \cdots, c_\nu$ and $\alpha(1), \cdots, \alpha(\nu) \in N^n$ and $\beta(1), \cdots, \beta(\nu) \in N$ such that

$$\tilde{\gamma}^{m,m'}_{\alpha,\beta}(\kappa_f) \leqq \sum_{j=1}^{\nu} c_j \gamma^{m,m'}_{\alpha(j),\beta(j)}(f).$$

It is easy to see that $\kappa_f(k\xi; k_1, k_2) = \kappa_f(\xi; k_1k, k_2k)$ for all $k \in K$. Thus we proved that $\kappa_f \in \tilde{\mathcal{S}}$ and the mapping $f \mapsto \kappa_f$ is continuous from $\mathcal{S}$ to $\tilde{\mathcal{S}}$. As for the case of $C_c^\infty(G)$, for any $\kappa(\xi; k_1, k_2) \in \tilde{\mathcal{S}}$ we put

$$f(x, k) = \int_{\hat{V}} \kappa(\xi; 1, k^{-1})e^{-i<\xi,x>}d\xi. \tag{2.3}$$

If we can see that $f \in \mathcal{S}$, we have $\kappa_f(\xi; k_1, k_2) = \kappa(\xi; k_1, k_2)$ without difficulty.

By the conditions (i)$^{\sim}$$\sim$(iii)$^{\sim}$ we can prove $f\in C^{\infty}(G)$ in similar way. We can also prove that for any $\alpha\in N^n$, $\beta\in N$ and $m, m'\in N^{\delta}$ there exist positive numbers $c_1, \cdots, c_\nu$ and $\alpha(1), \cdots, \alpha(\nu)\in N^n$ and $\beta(1), \cdots, \beta(\nu)\in N$ such that

$$\gamma_{\alpha,\beta}^{m,m'}(f)\leqq \sum_{j=1}^{\nu} c_j \tilde{\gamma}_{\alpha(j),\beta(j)}^{m,m'}(\kappa)\,.$$

Hence $f\in \mathcal{S}$ and the mapping $\kappa\mapsto f$ defined by (2.3) is a continuous mapping from $\tilde{\mathcal{S}}$ to $\mathcal{S}$. As stated above, this mapping is the inverse of the mapping $f\mapsto\kappa_f$. Thus we proved Lemma 3.

§ 3. Auxiliary theorem on compact Lie groups.

Let K be a compact connected Lie group and $\mathfrak{k}$ be its Lie algebra. Let T be a maximal torus subgroup of K and $\mathfrak{t}$ the corresponding subalgebra of $\mathfrak{k}$. Let r and r' be the ranks of $\mathfrak{k}$ and of the derived subalgebra $[\mathfrak{k}, \mathfrak{k}]$ of $\mathfrak{k}$, respectively. We fix an $\mathrm{Ad}(K)$-invariant positive definite inner product (Y, Y') on $\mathfrak{k}$. We define the norm by $|Y|=(Y, Y)^{1/2}$. Let $Y_1, \cdots, Y_\delta$ ($\delta=\dim K$) be a basis of $\mathfrak{k}$ and $g_{ij}=(Y_i, Y_j)$ and $(g^{ij})=(g_{ij})^{-1}$. The element $\Delta=-\sum_{i,j=1}^{\delta} g^{ij}Y_iY_j$ in the universal enveloping algebra of $\mathfrak{k}$ is the Casimir operator of $\mathfrak{k}$. We regard Δ as a differential operator on K.

We put $\Gamma=\{H\in\mathfrak{t}\,;\,\exp_K H=1\}$. For any $\lambda\in\sqrt{-1}\,\hat{\mathfrak{t}}$ we denote by $H_\lambda\in\mathfrak{t}$ the element defined by $\langle\lambda, H\rangle=\sqrt{-1}\,(H_\lambda, H)$ $(H\in\mathfrak{t})$. We identify λ and H_λ. Let I be the set of all K-integral forms on $\mathfrak{t}$;

$$I=\{\lambda\in\mathfrak{t}\,;\,(\lambda, H)\in 2\pi \boldsymbol{Z} \quad \text{for all} \quad H\in\Gamma\}\,.$$

Let us fix a lexicographic order in $\mathfrak{t}$. Let P be the set of all positive roots with respect to this order. Then P consists of $(\delta-r)/2$ elements. The simple root system in P consists of r' elements, say $\alpha_1, \cdots, \alpha_{r'}$. The set

$$\mathcal{F}=\{\lambda\in I\,;\,(\lambda, \alpha_i)\geqq 0, \quad 1\leqq i\leqq r'\}$$

is the set of all dominant K-integral forms on $\mathfrak{t}$. Let $\hat{K}$ be the set of all equivalence classes of irreducible unitary representations of K. For any irreducible unitary representation τ of K we denote by $[\tau]$ the equivalence class which contains τ. For each $\lambda\in\mathcal{F}$ we denote by τ^λ a representative of $[\tau^\lambda]\in\hat{K}$ which is a matricial representation of K with the highest weight λ. Then we have the bijection between $\mathcal{F}$ and $\hat{K}$ by the mapping $\lambda\mapsto[\tau^\lambda]$. Let $d(\lambda)$ be the degree of τ^λ.

Let dk be the Haar measure on K normalized such as the total measure equals 1. Put $\mathfrak{H}=L_2(K)$. We consider the set $\{\phi_j\}_{j\in J}$ of functions on K

such that $\phi_j=d(\lambda)^{1/2}\tau^{\lambda}_{pq}$ for some $\lambda\in\mathcal{F}$ and $p, q=1, \cdots, d(\lambda)$ $(\tau^{\lambda}(k)=(\tau^{\lambda}_{pq}(k)))$. That is, $\{\phi_j\}_{j\in J}$ is the complete orthonormal basis of $\mathfrak{H}$ owing to the Peter-Weyl theorem. We denote by J_λ for $\lambda\in\mathcal{F}$ the set of $j\in J$ such that $\phi_j=d(\lambda)^{1/2}\tau^{\lambda}_{pq}$ for some $p, q=1, \cdots, d(\lambda)$.

We put $\rho=(1/2)\sum_{\alpha\in P}\alpha$. Then the following lemma is well known (see [4]).

LEMMA 4. (i) (*Weyl's dimension formula*) *For every* $\lambda\in\mathcal{F}$

$$d(\lambda)=\prod_{\alpha\in P}\frac{(\lambda+\rho, \alpha)}{(\rho, \alpha)}. \tag{3.1}$$

If K *is abelian, we understand that the right hand side of* (3.1) *expresses* 1.

(ii) *For every* $\lambda\in\mathcal{F}$ *and* $j\in J_\lambda$ *we have*

$$(\Delta+|\rho|^2)\phi_j=|\lambda+\rho|^2\phi_j. \tag{3.2}$$

(iii) *The Dirichlet series*

$$Z(s)=\sum_{\lambda\in\mathcal{F}}|\lambda+\rho|^{-s}$$

(*in the case* K *is abelian* $Z(s)=\sum_{\lambda\in\mathcal{F}-\{0\}}|\lambda|^{-s}$) *converges if* $s>r$.

The Casimir operator Δ is a formally selfadjoint differential operator.

THEOREM 1. *Let* T *be a bounded operator on* $\mathfrak{H}$. *Then* T *has a* C^∞ *kernel function if and only if it leaves the space* $C^\infty(K)$ *stable and for any* $l, m\in N$ *there exists a constant* $C^{l,m}$ *such that*

$$\|\Delta^l T\Delta^m\|\leqq C^{l,m}. \tag{3.3}$$

From this theorem we have immediately the following corollary.

COROLLARY. *Let* T *be a bounded operator on* $\mathfrak{H}$ *such that for any* $l, m\in N$ *there exists a constant* $C^{l,m}$ *satisfying* (3.3). *Then* T *is of the trace class.*

PROOF OF THEOREM 1. Let $\kappa(k_1, k_2)$ be the kernel function of T, i.e.

$$(TF)(k_1)=\int_K\kappa(k_1, k_2)F(k_2)dk_2.$$

And assume that $\kappa(k_1, k_2)$ is of class C^∞ on $K\times K$. Then

$$(\Delta^l T\Delta^m F)(k_1)=\Delta^l\int_K\kappa(k_1, k_2)(\Delta^m F)(k_2)dk_2$$

$$=\int_K(\Delta^l_{k_1}\Delta^m_{k_2}\kappa)(k_1, k_2)dk_2.$$

Hence

$$\|\Delta^l T\Delta^m\|\leqq C^{l,m},$$

where $C^{l,m}=\left(\int_K\int_K|\Delta^l_{k_1}\Delta^m_{k_2}\kappa(k_1, k_2)|^2dk_1dk_2\right)^{1/2}$.

Conversely, we assume that there exists a constant as in Theorem. Then we can prove that there exists a constant $C_1^{l,m}$ such that

$$\|(\Delta+|\rho|^2)^m T(\Delta+|\rho|^2)^l\| \leqq C_1^{l,m}.$$

We put

$$\kappa(k_1, k_2) = \sum_{i,j\in J} (T\phi_j, \phi_i)\phi_i(k_1)\overline{\phi_j(k_2)}. \tag{3.4}$$

First we show that the series in the right hand side of (3.4) converges absolutely when K is not abelian. If $j\in J_\lambda$, we have $|\phi_j(k)| \leqq d(\lambda)^{1/2}$ because $|\tau_{pq}^\lambda(k)| \leqq 1$. By (3.1)

$$\begin{aligned} d(\lambda) &\leqq \prod_{\alpha\in P} |\lambda+\rho||\alpha|(\rho, \alpha)^{-1} \\ &= |\lambda+\rho|^{(\delta-r)/2} \prod_{\alpha\in P} |\alpha|(\rho, \alpha)^{-1}. \end{aligned}$$

On the other hand by (3.2)

$$\phi_j = |\lambda+\rho|^{-2l}(\Delta+|\rho|^2)^l \phi_j$$

for $l=0, 1, 2, \cdots$. Therefore, for $j\in J_\lambda$ and $i\in J_{\lambda'}$, we have

$$\begin{aligned} |(T\phi_j, \phi_i)| &\leqq |\lambda+\rho|^{-2l}|\lambda'+\rho|^{-2m}|(T(\Delta+|\rho|^2)^l\phi_j, (\Delta+|\rho|^2)^m\phi_i)| \\ &= |\lambda+\rho|^{-2l}|\lambda'+\rho|^{-2m}|((\Delta+|\rho|^2)^m T(\Delta+|\rho|^2)^l\phi_j, \phi_i)| \\ &\leqq |\lambda+\rho|^{-2l}|\lambda'+\rho|^{-2m}\|(\Delta+|\rho|^2)^m T(\Delta+|\rho|^2)^l\| \\ &\leqq C_1^{l,m}|\lambda+\rho|^{-2l}|\lambda'+\rho|^{-2m}. \end{aligned}$$

Hence

$$\begin{aligned} &\sum_{j\in J_\lambda}\sum_{i\in J_{\lambda'}} |(T\phi_j, \phi_i)\phi_i(k_1)\overline{\phi_j(k_2)}| \\ &\quad \leqq \sum_{j\in J_\lambda}\sum_{i\in J_{\lambda'}} C_1^{l,m}|\lambda+\rho|^{-2l}|\lambda'+\rho|^{-2m} d(\lambda)^{1/2} d(\lambda')^{1/2} \\ &\quad = C_1^{l,m}|\lambda+\rho|^{-2l}|\lambda'+\rho|^{-2m} d(\lambda)^{5/2} d(\lambda')^{5/2}. \end{aligned}$$

If we take $l=m$,

$$\begin{aligned} &\sum_{i,j\in J} |(T\phi_j, \phi_i)\phi_i(k_1)\overline{\phi_j(k_2)}| \\ &\quad \leqq C_1^{l,l} \sum_{\lambda,\lambda'\in\mathcal{F}} |\lambda+\rho|^{-2l}|\lambda'+\rho|^{-2l} d(\lambda)^{5/2} d(\lambda')^{5/2} \\ &\quad = C_1^{l,l}\Big(\sum_{\lambda\in\mathcal{F}} |\lambda+\rho|^{-2l} d(\lambda)^{5/2}\Big)^2 \\ &\quad \leqq C_1^{l,l} \prod_{\alpha\in P} |\alpha|^5(\rho, \alpha)^{-5}\Big(\sum_{\lambda\in\mathcal{F}} |\lambda+\rho|^{-2l+5(\delta-r)/4}\Big)^2 \\ &\quad = C_1^{l,l} \prod_{\alpha\in P} |\alpha|^5(\rho, \alpha)^{-5} Z(2l-5(\delta-r)/4)^2. \end{aligned} \tag{3.5}$$

Then by Lemma 4 (iii) this has a finite value for $l>(5\delta-r)/8$. When K is abelian, $d(\lambda)=1$ and $|\phi_j(k)|=1$ for all $j\in J$. So we have

$$\sum_{i,j\in J}|(T\phi_j, \phi_i)\phi_i(k_1)\overline{\phi_j(k_2)}|$$
$$\leqq C_1^{l,l}(\sum_{\lambda\neq 0}|\lambda|^{-2l})^2+C_1^{l,0}\sum_{\lambda\neq 0}|\lambda|^{-2l}+C_1^{0,l}\sum_{\lambda\neq 0}|\lambda|^{-2l}+C_1^{0,0}$$
$$=C_1^{l,l}Z(2l)^2+(C_1^{l,0}+C_1^{0,l})Z(2l)+C_1^{0,0}. \qquad (3.6)$$

Hence by Lemma 4 this has a finite value for $l>r/2$. Thus the series (3.4) converges absolutely and also uniformly. Therefore $\kappa(k_1, k_2)$ is a continuous function on $K\times K$ and the double Fourier coefficients of $\kappa(k_1, k_2)$ are $(T\phi_j, \phi_i)$ $(i, j\in J)$. In order to prove that $\kappa(k_1, k_2)$ is a C^∞ function it is enough to show that the Fourier coefficients are rapidly decreasing (see [4]), i.e. for any $l\in \boldsymbol{N}$ and $m\in \boldsymbol{N}$ there exists a constant $C^{l,m}$ such that

$$|\lambda+\rho|^{2l}|\lambda'+\rho|^{2m}|(T\phi_j, \phi_i)|\leqq C^{l,m}$$

for any $j\in J_\lambda$ and any $i\in J_{\lambda'}$. This is an immediate consequence of the conditions of the theorem from Lemma 4. q. e. d.

§ 4. Paley-Wiener theorem.

Let $\boldsymbol{B}(\mathfrak{H})$ be the Banach space of all bounded linear operators on $\mathfrak{H}$. Then the Fourier transform T_f of $f\in C_c^\infty(G)$ defined by

$$T_f(\xi)=\int_G f(g)U_g^\xi dg$$

is a $\boldsymbol{B}(\mathfrak{H})$-valued function on V. Let $\kappa_f(\zeta\,;k_1, k_2)$ be the scalar Fourier-Laplace transform. We define the Fourier-Laplace transform of f by

$$(T_f(\zeta)F)(k_1)=\int_K \kappa_f(\zeta\,;k_1, k_2)F(k_2)dk_2\,, \qquad (F\in\mathfrak{H})\,.$$

Then T_f is a $\boldsymbol{B}(\mathfrak{H})$-valued function on $\hat{V}^c$ by (ii) of Lemma 1. For each $\zeta\in\hat{V}^c$ and $g=(x, k)\in G$ we put

$$(U_g^\zeta F)(k_1)=e^{i<\zeta, k_1{}^{-1}x>}F(k^{-1}k_1)\,, \qquad (F\in\mathfrak{H})\,.$$

Then U^ζ is a bounded representation of G on $\mathfrak{H}$. And we have

$$T_f(\zeta)=\int_G f(g)U_g^\zeta dg\,.$$

Let R be the right regular representation of K. The following theorem is an analogue of the Paley-Wiener theorem.

THEOREM 2. *A $\boldsymbol{B}(\mathfrak{H})$-valued function T on $\hat{V}$ is the Fourier transform of $f\in C_c^\infty(G)$ such that* supp $(f)\subset\Omega(a)$ $(a>0)$ *if and only if it satisfies the following* *conditions:*

(I) *T can be extended to an entire analytic function on $\hat{V}^c$.*

(II) *For any $\zeta \in \hat{V}$, $T(\zeta)$ leaves the space $C^\infty(K)$ stable and for any K-invariant polynomial function p on $\hat{V}^c$ and for any $l, m \in \boldsymbol{N}$ there exists a constant $C_p^{l,m}$ such that*

$$\|p(\zeta)\Delta^l T(\zeta)\Delta^m\| \leqq C_p^{l,m} \exp a|\mathrm{Im}\,\zeta|.$$

(III) *For any $k \in K$*

$$T(k\zeta) = R_k T(\zeta) R_k^{-1}, \qquad (\zeta \in \hat{V}^c).$$

PROOF. Let $f \in C_c^\infty(G)$ and $\mathrm{supp}(f) \subset \Omega(a)$. For any $\alpha = (\alpha_1, \cdots, \alpha_n) \in \boldsymbol{N}^n$ we define an operator T_f^α by

$$(T_f^\alpha F)(k_1) = \int_V \int_K f(k_1 x, k) x_1^{\alpha_1} \cdots x_n^{\alpha_n} F(k^{-1}k_1) dx dk,$$

where $x = \sum_{j=1}^n x_j v_j$. By the inequality

$$\|T_f^\alpha\| \leqq a^{|\alpha|}\Big\{\int_K \Big(\int_V |f(x, k)|dx\Big)^2 dk\Big\}^{1/2}$$

we can see that for any fixed $\zeta = \sum_{j=1}^n \zeta_j w_j \in \hat{V}^c$ the series

$$\sum_{l=0}^\infty i^l \sum_{l=|\alpha|} \frac{|\alpha|\,!}{\alpha_1! \cdots \alpha_n!} T_f^\alpha \zeta_1^{\alpha_1} \cdots \zeta_n^{\alpha_n}$$

converges in the norm of $\boldsymbol{B}(\mathfrak{H})$ and equals $T_f(\zeta)$. Hence $T_f(\zeta)$ is entire analytic on $\hat{V}^c$.

Let us define a differential operator on $\hat{V}$ by

$$p(D) = p\Big(i\frac{\partial}{\partial x_1}, \cdots, i\frac{\partial}{\partial x_n}\Big).$$

Then the K-invariance of p and the standard arguments on the Fourier transform theory of the Euclidean space give us

$$p(\zeta)T_f(\zeta) = T_{p(D)f}(\zeta). \tag{4.1}$$

Because Δ is a two-sided invariant differential operator on K of the 2-nd order, we have

$$\Delta T_f(\zeta) = T_{\lambda(\Delta)f}(\zeta)$$

and

$$T_f(\zeta)\Delta = T_{\mu(\Delta)f}(\zeta).$$

Hence we obtain, for any l and m,

$$\Delta^l T_f(\zeta)\Delta^m = T_{\lambda(\Delta)^l \mu(\Delta)^m f}(\zeta). \tag{4.2}$$

By (4.1) and (4.2)

$$p(\zeta)\Delta^l T_f(\zeta)\Delta^m = T_{p(D)\lambda(\Delta)^l\mu(\Delta)^m f}(\zeta)\,.$$

Hence we have

$$\|p(\zeta)\Delta^l T_f(\zeta)\Delta^m\| \leqq e^{a|\mathrm{Im}\,\zeta|}\Big\{\int_K\Big(\int_V |(p(D)\lambda(\Delta)^l\mu(\Delta)^m f)(x, k)|\,dx\Big)^2 dk\Big\}^{1/2}.$$

If we put

$$C_p^{l,m} = \Big\{\int_K\Big(\int_V |(p(D)\lambda(\Delta)^l\mu(\Delta)^m f)(x, k)|\,dx\Big)^2 dk\Big\}^{1/2},$$

we have (II).

The property (III) is the immediate consequence of the property $\kappa_f(k\zeta\,;\,k_1, k_2) = \kappa_f(\zeta\,;\,k_1 k, k_2 k)$.

Conversely, let T be a $\boldsymbol{B}(\mathfrak{H})$-valued function on $\hat{V}$ which satisfies the conditions (I)$\sim$(III) of the theorem. Let $\{\phi_j\}_{j\in J}$ be the complete orthonormal basis of $\mathfrak{H}$ chosen in § 3. Then, by Theorem 1, $T(\zeta)$ has C^∞ kernel function $\kappa(\zeta\,;\,k_1, k_2)$ and it is given by

$$\kappa(\zeta\,;\,k_1, k_2) = \sum_{i,j\in J}(T(\zeta)\phi_j, \phi_i)\phi_i(k_1)\overline{\phi_j(k_2)}\,. \tag{4.3}$$

And the series in the right hand side converges uniformly on every compact set in $\hat{V}^c\times K\times K$. If we adopt the similar computations in § 3 to $p(\zeta)yT(\zeta)y'$ $(y, y'\in U(\mathfrak{k}^c))$, we can prove that there exists a constant $C_p^{y,y'}$ such that

$$|p(\zeta)y_{k_1}y'_{k_2}\kappa(\zeta\,;\,k_1, k_2)| \leqq C_p^{y,y'}\exp a|\mathrm{Im}\,\zeta|, \qquad (\zeta\in\hat{V}^c,\ k_1, k_2\in K)\,.$$

By the condition (III) and (4.3) give us

$$\kappa(k\zeta\,;\,k_1, k_2) = \kappa(\zeta\,;\,k_1 k, k_2 k) \qquad (\zeta\in\hat{V}^c,\ k, k_1, k_2\in K)\,.$$

Thus the kernel function $\kappa(\xi\,;\,k_1, k_2)$ of $T(\xi)$, $(\xi\in\hat{V})$, satisfies the conditions (i)$\sim$(iii) of Lemma 1. Therefore, $\kappa(\xi\,;\,k_1, k_2)$ is the scalar Fourier transform of a function $f\in C_c^\infty(G)$ such that $\mathrm{supp}\,(f)\subset\Omega(a)$. By the equality $\kappa(\xi\,;\,k_1, k_2) = \kappa_f(\xi\,;\,k_1, k_2)$ for any $\xi\in\hat{V}$ and for any $k_1, k_2\in K$ we have $T=T_f$. This completes the proof of the theorem. q. e. d.

REMARK. Let T be a $\boldsymbol{B}(\mathfrak{H})$-valued function on $\hat{V}$ as in the theorem. Then $T(\xi)$ is of trace class by (II) and Corollary to Theorem 1. And hence $T(\xi)U^{\xi}_{g^{-1}}$ is of trace class and its trace is

$$\begin{aligned}\mathrm{Tr}\,(T(\xi)U^{\xi}_{g^{-1}}) &= \mathrm{Tr}\,(U^{\xi}_{g^{-1}}T(\xi))\\ &= \int_K e^{-i\langle\xi, kk_1^{-1}x\rangle}\kappa(\xi\,;\,kk_1, k_1)dk_1\,.\end{aligned}$$

By the condition (ii) in Lemma 1, $\kappa(\xi\,;\,k_1, k_2)$ is a rapidly decreasing function with respect to ξ. So, by the condition (iii) in Lemma 1, we have

$$\int_{\hat{V}} \mathrm{Tr}\,(T(\xi)U^{\xi}_{g^{-1}})d\xi = \int_{\hat{V}}\int_K e^{-i<kk_1\xi,x>}\kappa(\xi\,;\,kk_1,\,k_1)d\xi dk_1$$
$$= \int_K \Big\{\int_{\hat{V}} e^{-i<\xi,x>}\kappa((kk_1)^{-1}\xi\,;\,kk_1,\,k_1)d\xi\Big\} dk_1$$
$$= \int_K \Big\{\int_{\hat{V}} e^{-i<\xi,x>}\kappa(\xi\,;\,1,\,k^{-1})d\xi\Big\} dk_1$$
$$= \int_{\hat{V}} e^{-i<\xi,x>}\kappa(\xi\,;\,1,\,k^{-1})d\xi\,.$$

Hence by (2.1) we have

$$f(g) = \int_{\hat{V}} \mathrm{Tr}\,(T(\xi)U^{\xi}_{g^{-1}})d\xi\,.$$

This is the inverse Fourier transform.

§ 5. Fourier transforms of rapidly decreasing functions.

Let $\hat{\mathcal{S}}$ be the set of all $\boldsymbol{B}(\mathfrak{H})$-valued functions T on $\hat{V}$ satisfying the following conditions:

(i)^ T is a $\boldsymbol{B}(\mathfrak{H})$-valued C^∞ function on $\hat{V}$,

(ii)^ for any $\alpha\in \boldsymbol{N}^n$, $\xi\in\hat{V}$, $D^{\alpha}_{\xi}T(\xi)$ leaves the space $C^\infty(K)$ stable and for any $\alpha\in\boldsymbol{N}^n$, $\beta\in\boldsymbol{N}$ and $m, m'\in\boldsymbol{N}^\delta$ there exists a constant $C^{m,m'}_{\alpha,\beta}$ such that

$$\|(1+|\xi|^2)^\beta y(m)D_\xi{}^\alpha T(\xi)y(m')\| \leqq C^{m,m'}_{\alpha,\beta}$$

for all $\xi\in\hat{V}$,

(iii)^ for any $k\in K$

$$T(k\xi) = R_k T(\xi) R_k{}^{-1}\,, \qquad (\xi\in\hat{V})\,.$$

We topologize $\hat{\mathcal{S}}$ by the system of semi-norms of the form

$$\hat{\gamma}^{m,m'}_{\alpha,\beta}(T) = \sup_{\xi\in\hat{V}} \|(1+|\xi|^2)^\beta y(m)D_\xi{}^\alpha T(\xi)y(m')\|\,,$$

where $\alpha\in\boldsymbol{N}^n$, $\beta\in\boldsymbol{N}$ and $m, m'\in\boldsymbol{N}^\delta$. Then we have the following proposition as in § 2.

PROPOSITION 3. *$\hat{\mathcal{S}}$ is a Fréchet space.*

Let us prove the space $\hat{\mathcal{S}}$ is the image of the space $\mathcal{S}$ by the Fourier transform. Let κ_f and T_f be the scalar Fourier transform and the Fourier transform of $f\in\mathcal{S}$, respectively. Then

$$(T_f(\xi)F)(k_1) = \int_K \kappa_f(\xi\,;\,k_1,\,k_2)F(k_2)dk_2\,, \qquad (F\in\mathfrak{H})\,.$$

LEMMA 5. *The mapping $\kappa_f\to T_f$ gives a topological isomorphism from $\tilde{\mathcal{S}}$ onto $\hat{\mathcal{S}}$.*

PROOF. For any $\alpha\in\boldsymbol{N}^n$ we obtain

$$\int_K\left(\int_K|(D_\xi{}^\alpha\kappa_f)(\xi\,;k_1,k_2)F(k_2)|dk_2\right)^2dk_1$$
$$\leqq\int_K\int_K|(D_\xi{}^\alpha\kappa_f)(\xi\,;k_1,k_2)|^2dk_1dk_2\|F\|^2$$
$$\leqq(C_{\alpha,0}^{0,0})^2\|F\|^2\,,\qquad(F\in\mathfrak{H})\,.$$

Hence, by the completeness of $\boldsymbol{B}(\mathfrak{H})$, T_f is a C^∞ $\boldsymbol{B}(\mathfrak{H})$-valued function on $\hat{V}$. For any $\alpha\in\boldsymbol{N}^n$, $\beta\in\boldsymbol{N}$ and $m, m'\in\boldsymbol{N}^\delta$ we have

$$\|(1+|\xi|^2)^\beta y(m)D_\xi{}^\alpha T_f(\xi)y(m')\|$$
$$\leqq\left(\int_K\int_K|(1+|\xi|^2)^\beta y(m)_{k_1}y(m')^*{}_{k_2}D_\xi{}^\alpha\kappa_f(\xi\,;k_1,k_2)|^2dk_1dk_2\right)^{1/2}.$$

The right hand side is dominated by a linear combination of $\tilde{\gamma}$ semi-norms. The condition (iii)^ is the immediate consequence of the condition (iii)~. Therefore $T_f\in\hat{\mathcal{S}}$ and the mapping $\kappa_f\mapsto T_f$ is continuous. Conversely, we assume that $T\in\hat{\mathcal{S}}$. Then, by the condition (ii)^, for any $l, l'\in\boldsymbol{N}$ there exists a constant $C^{l,l'}$ such that

$$\|\Delta^l T(\xi)\Delta^{l'}\|\leqq C^{l,l'}\,.$$

From Theorem 1 $T(\xi)$ has a C^∞ (with respect to k_1, k_2) kernel function $\kappa(\xi\,;k_1,k_2)$ which is defined by

$$\kappa(\xi\,;k_1,k_2)=\sum_{i,j\in J}(T(\xi)\phi_j,\phi_i)\phi_i(k_1)\overline{\phi_j(k_2)}\,.\tag{5.1}$$

By (ii)^ we can prove that for any $\alpha\in\boldsymbol{N}^n$, $m, m'\in\boldsymbol{N}$ and $l, l'\in\boldsymbol{N}$ there exists a constant C such that

$$\|\Delta^l(1+|\xi|^2)^\beta y(m)D_\xi{}^\alpha T(\xi)y(m')\Delta^{l'}\|\leqq C\,.\tag{5.2}$$

Hence the series

$$\sum_{i,j\in J}((1+|\xi|^2)^\beta y(m)D_\xi{}^\alpha T(\xi)y(m')\phi_j,\phi_i)\phi_i(k_1)\overline{\phi_j(k_2)}$$

converges absolutely and uniformly with respect to ξ. If we take $m=m'=0$, we have the infinitely differentiability of $\kappa(\xi\,;k_1,k_2)$ with respect to ξ. We denote by L the left regular representation of K. Then we have

$$\sum_{i,j\in J}(T(\xi)\phi_j,\phi_i)\phi_i(\exp_K tY\cdot k_1)\overline{\phi_j(\exp_K t'Y'\cdot k_2)}$$
$$=\sum_{i,j\in J}(L_{\exp_K(-tY)}T(\xi)L_{\exp_K(t'Y')}\phi_j,\phi_i)\phi_i(k_1)\overline{\phi_i(k_2)}$$

for all $Y, Y'\in\mathfrak{k}$ and $t, t\in\boldsymbol{R}$ (see [3], page 87 for a similar computation). Therefore, we obtain

$$Y_{k_1}Y'_{k_2}\kappa(\xi\,;k_1,k_2)=\sum_{i,j\in J}(YT(\xi)(-Y')\phi_j,\phi_i)\phi_i(k_1)\overline{\phi_j(k_2)}\,.$$

So we have

$$y(m)_{k_1}y(m')_{k_2}D_\xi^\alpha\kappa(\xi\,;k_1,k_2)=\sum_{i,j\in J}\pm(y(m)^*D_\xi^\alpha T(\xi)y(m')\phi_j,\ \phi_i)\phi_i(k_1)\overline{\phi_j(k_2)}\,. \quad (5.3)$$

As a constant C in (5.2) we can take a linear combination of $\hat{\gamma}$ semi-norms of T. Hence, by (3.5), (3.6) and (5.3), the value of

$$|(1+|\xi|^2)^\beta D_\xi^\alpha y(m)_{k_1}y(m')_{k_2}\kappa(\xi\,;k_1,k_2)|$$

is dominated by a linear combination of $\hat{\gamma}$ semi-norms of T. We can obtain (iii)$^\sim$ from (iii)$^\wedge$ as in the proof of Theorem 2. Thus we have $\kappa(\xi\,;k_1,k_2)\in\tilde{\mathcal{S}}$ and the mapping $T\to\kappa$ defined by (5.1) is a continuous mapping from $\hat{\mathcal{S}}$ to $\tilde{\mathcal{S}}$. Because

$$(T(\xi)F)(k_1)=\int_K\kappa(\xi\,;k_1,k_2)F(k_2)dk_2$$

for $F\in\mathfrak{H}$, this mapping is an isomorphism. q.e.d.

By Lemma 3 and Lemma 5 we have the following theorem.

THEOREM 3. *The Fourier transform $f\to T_f$ is a topological isomorphism from $\mathcal{S}$ onto $\hat{\mathcal{S}}$.*

References

[1] A. Kleppner and R. Lipsman, The Plancherel formula for group extensions, Ann. Sci. École Norm. Sup., 4^e serie, 5 (1972), 459–516.
[2] K. Kumahara, Fourier transforms on the Cartan motion group, Proc. Japan Acad., 50 (1974), 433–435.
[3] K. Kumahara and K. Okamoto, An analogue of the Paley-Wiener theorem for the euclidean motion group, Osaka J. Math., 10 (1973), 77–92.
[4] M. Sugiura, Fourier series of smooth functions on compact Lie groups, Osaka J. Math., 8 (1971), 33–47.
[5] M. Sugiura, Unitary Representations and Harmonic Analysis, Wiley, 1975.

Keisaku KUMAHARA
Department of Mathematics
Faculty of General Education
Tottori University
Koyama-cho, Tottori
Japan