Impulsive control of symmetric Markov processes
and quasi-variational inequalities

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By introducing the notion of impulsive control of a diffusion process A. Bensoussan - J.L. Lions ([1]) showed that if the solution of a quasi-variational inequality has sufficient regularity (twice differentiability and continuity), it turns out to be a pay-off function. Furthermore they constructed the optimal strategy out of the solution. But the regularity problems remained open. On the other hand M. Robin ([7]) has set up an impulsive control problem of a general Markov process with a Feller transition semi-group and has constructed the optimal strategy out of the pay-off function which was characterized however in terms of the semi-group rather than the generater of the basic Markov process. As for the characterization by means of the quasi-variational inequality the regularity of the solution was still assumed in order to identify the solution with the pay-off like that of Bensoussan-Lions. Regularity problems of elliptic or parabolic quasi-variational inequalities have been studied by L.A. Cafarelli - A. Friedman and others (cf. [2],[5]) under the condition that the diffusion and drift coefficients have sufficient regularity. Cafarelli-Friedmans' work combined with Robin's establishes completely the relationship between impulsive control problems and quasi-variational inequalities with respect to nice diffusion processes.

Our objective is to extend this relationship to general symmetric Markov processes associated with regular Dirichlet spaces. We prove that the pay-off is characterized by the weak (maximum) solution of the quasi-variational inequality defined on the Dirichlet space (Theorem 2 in §2). Since we assume only that the Dirichlet space is regular, Theorem 2 establishes the relationship for a wide class of processes. It applies as well to symmetric diffusion process with measurable coefficients and symmetric Markov processes with non local generators (cf. [4]).

Our approach is more potential theoretic than others and accordingly the regularity questions can be dispensed with. Indeed we use potential theories of Dirichlet spaces and Markov processes developed in [4]. The same method has been used in [6] to establish the relationship between variational inequalities and optimal stoppings and in [8] to include stopping games.

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§1. Quasi-variational inequalities on regular Dirichlet spaces

Let $m(dx)$ be a non-negative Radon everywhere dense measure on a locally compact Hausdorff space S with countable base. Suppose that $(\mathcal{F},\mathcal{E})$ is a regular Dirichlet space relative to $L^2(dm)$:

i) $\mathcal{F}$ is a dense linear subspace of $L^2(dm)$,

ii) $\mathcal{E}$ is a symmetric bilinear form on $\mathcal{F}\times\mathcal{F}$,

iii) $\mathcal{F}$ is closed with respect to $\mathcal{E}_1$-norm, where $\mathcal{E}_1(u,v) = \mathcal{E}(u,v) + (u,v)$, (u,v) denoting inner product of $L^2(dm)$,

iv) unit contraction operates, that is, if $v = (0\vee u)\wedge 1$, $u \in \mathcal{F}$, then $v \in \mathcal{F}$ and $\mathcal{E}(v,v) \leqq \mathcal{E}(u,u)$,

v) $\mathcal{F}\cap C_0(S)$ is dense in $\mathcal{F}$ with $\mathcal{E}_1$-norm as well as in $C_0(S)$ with uniform norm, $C_0(S)$ denoting the space of all continuous functions on S with compact support.

Definition 1.1. The capacity of a subset of S is defined as follows: for open set $A \subset S$

$$\mathrm{Cap}(A) = \begin{cases} \inf\{\mathcal{E}_1(u,u);\ u \in L_A\} & \text{if } L_A \neq \phi, \\ \infty & \text{otherwise,} \end{cases}$$

where $L_A = \{u \in \mathcal{F}\ ;\ u \geqq 1 \ \text{m-a.e. on } A\}$ and for general set $B \subset S$

$$\mathrm{Cap}(B) = \inf\{\mathrm{cap}(A) : B \subset A,\ A \text{ is open}\}.$$

Definition 1.2. A subset B of S with $\mathrm{Cap}(B) = 0$ is called almost polar and "Quasi-everywhere" or "q.e." will mean "except for an almost polar set".

Let $S_\Delta = S \cup \Delta$ be the one point compactification of S.

When S is already compact, Δ is regarded as an isolated point. Any function on S is extended to a function on $S \cup \Delta$ by setting $f(\Delta) = 0$.

Definition 1.3. A function f defined q.e. on S is said to be quasi-continuous (in the restricted sense) provided that for each $\varepsilon > 0$ there exists an open set $G \subset S$ such that $\mathrm{cap}(G) < \varepsilon$ and $f|_{S-G}$ ($f|_{S_\Delta - G}$ respectively) is continuous.

It is known that each $u \in \mathcal{F}$ admits a quasi-continuous modification $\tilde{u}$ in the restricted sense in the case that $(\mathcal{F}, \mathcal{E})$ is a regular Dirichlet space: $u = \tilde{u}$ m-a.e. and $\tilde{u}$ is quasi-continuous (cf. [4]). Hereafter $\tilde{u}$ denote a quasi-continuous modification of $u \in \mathcal{F}$.

Let $\nu(dx)$ be a given non-negative Radon measure of finite energy integral, that is, there exists for each $\alpha > 0$ a unique function $U_\alpha \nu \in \mathcal{F}$ such that

$$(1.1) \quad \mathcal{E}_\alpha(U_\alpha \nu, v) = \int_S v(x)\nu(dx) \quad \text{for each } v \in \mathcal{F} \cap C_0(S).$$

Suppose that M is a operator defined on $\tilde{\mathcal{F}}$ such that

(M.1) Mu is a Borel function for any $u \in \tilde{\mathcal{F}}$,

(M.2) $Mu_1(x) \leqq Mu_2(x)$ $\forall x$ if $u_1(x) \leqq u_2(x)$ q.e.,

(M.3) $Mu(x) \geqq 0$ $\forall x$ if $u(x) \geqq 0$ q.e. and

(M.4) $\lim_{n\to\infty} Mu_n(x) = Mu(x)$ $\forall x$ if $u_n(x) \downarrow u(x)$ q.e.

We consider the following quasi-variational inequality:

$$(1.2) \quad \begin{cases} \mathcal{E}_\alpha(u, v-u) \geqq \langle \nu, \tilde{v} - \tilde{u} \rangle & \forall \tilde{v} \leqq M\tilde{u} \text{ q.e.} \\ \tilde{u} \leqq M\tilde{u} \text{ q.e.} \end{cases}$$

Theorem 1. The above quasi-variational inequality (QVI) (1.2) has the maximal solution.

Put $u_0 = U_\alpha \nu$ and $V_1 = \{ v \in \mathcal{F} ; \tilde{v} \leq M\tilde{u}_0 \quad \text{q.e.}\}$, then we have the unique solution of the following variational inequality (VI) (1.3):

$$(1.3)\begin{cases} \mathcal{E}_\alpha(u, v-u) \geq \langle \nu, \tilde{v}-\tilde{u} \rangle & \forall v \in V_1 \\ u \in V_1, & \end{cases}$$

because (1.3) is equivalent to

$$(1.4)\begin{cases} \mathcal{E}_\alpha(u-U_\alpha\nu, u-U_\alpha\nu) \leq \mathcal{E}_\alpha(v-U_\alpha\nu, v-U_\alpha\nu) & \forall v \in V_1 \\ u \in V_1 & \end{cases}$$

and V_1 is the closed convex subset of Hilbert space $(\mathcal{F}, \mathcal{E}_\alpha)$. Let us denote the solution by u_1. In the same way we can inductively take the solution u_n of the VI:

$$(1.5)\begin{cases} \mathcal{E}_\alpha(u, v-u) \geq \langle \nu, \tilde{v}-\tilde{u} \rangle & \forall v \in V_n \\ u \in V_n, & \end{cases}$$

$$V_n = \{ v \in \mathcal{F} ; \tilde{v} \leq M\tilde{u}_{n-1} \quad \text{q.e.}\} \quad \text{for each } n.$$

At first we note the properties of the solution u_n of the VI (1.5).

Lemma. The above u_n has the following properties

(i) $U_\alpha\nu - u_n$ is an α-almost excessive function and the unique element which minimizes its α-energy integral in the closed convex subset $U_\alpha\nu - V_n$ of $(\mathcal{F}, \mathcal{E}_\alpha)$:

$$(1.6) \quad \mathcal{E}_\alpha(U_\alpha\nu-u_n, U_\alpha\nu-u_n) \leq \mathcal{E}_\alpha(U_\alpha\nu-v, U_\alpha\nu-v) \quad \forall v \in V_n$$

for each n,

(ii) (1.7) $u_n \leq u_{n-1}$ m-a.e. for each n,

(iii) (1.8) $u_n \geq 0$ m-a.e. for each n and

(iv) $\{u_n\}$ is a $\mathcal{E}_\alpha$-Cauchy sequence.

Proof of Lemma. (i) Since u_n is the solution of (1.5) it satisfies the following inequality:

$$(1.9)\qquad \mathcal{E}_\alpha(u_n-U_\alpha\nu, v-u_n) \geqq 0 \quad \forall v \in V_n.$$

If $w \geqq 0$, $\in \mathcal{F}$, then $u_n - w \in V_n$. Therefore it holds that

$$(1.10)\qquad \mathcal{E}_\alpha(U_\alpha\nu-u_n, w) \geqq 0 \quad \forall w \geqq 0 \text{ m-a.e.}, \quad w \in \mathcal{F}.$$

that is $U_\alpha\nu - u_n$ is α-almost excessive because (1.10) is equivalent to

$$(1.11)\qquad u_n \geqq 0, \quad e^{-\alpha t}T_t u_n \leqq u_n \quad \text{m-a.e.}, \quad \forall t > 0.$$

Here T_t is the L^2-Markov semigroup corresponding to Dirichlet form $\mathcal{E}$ (cf. [4]). The latter half of (i) follows directly if V_1 in (1.4) is replaced by V_n.

(ii) Inquality (1.7) with $n = 1$ is obvious $U_\alpha\nu - u_1$ is α-almost excessive and $U_\alpha\nu = u_0$. Assume that it holds for n, then $M\widetilde{u}_n \leqq M\widetilde{u}_{n-1}$ $\forall x$. Therefore $\widetilde{u}_{n+1} \leqq M\widetilde{u}_{n-1}$ q.e.. Since $\widetilde{u}_n \leqq M\widetilde{u}_{n-1}$ q.e. by definition we have $\widetilde{u}_n \vee \widetilde{u}_{n+1} \leqq M\widetilde{u}_{n-1}$ q.e.. On the other hand $U_\alpha\nu - u_n \vee u_{n+1} = (U_\alpha\nu - u_n) \wedge (U_\alpha\nu - u_{n+1})$ is α-almost excessive because both $U_\alpha\nu - u_{n+1}$ and $U_\alpha\nu - u_n$ are α-almost excessive. So it follows that

$$(1.12)\qquad \mathcal{E}_\alpha(U_\alpha\nu-u_n\vee u_{n+1}, U_\alpha\nu-u_n\vee u_{n+1}) \leqq \mathcal{E}_\alpha(U_\alpha\nu-u_n, U_\alpha\nu-u_n)$$

from $U_\alpha\nu - u_n \geqq U_\alpha\nu - u_n \vee u_{n+1}$. By (i) of present Lemma we conclude that $u_n \vee u_{n+1} = u_n$, that is, $u_{n+1} \leqq u_n$ m-a.e..

(iii) Since $\widetilde{U_\alpha\nu} \geqq 0$ q.e. we have $M\widetilde{u}_0 \geqq 0$ $\forall x$. Furthermore $\widetilde{u}_1 \leqq M\widetilde{u}_0$ q.e. by definition, so we have $\widetilde{u}_1 \vee 0 \leqq M\widetilde{u}_0$ q.e.. Both $U_\alpha\nu - u_1$ and $U_\alpha\nu$ being α-almost excessive, $U_\alpha\nu - u_1 \vee 0 = (U_\alpha\nu - u_1) \wedge U_\alpha\nu$ is α-almost excessive. Therefore it follows that

$$(1.13)\quad \mathcal{E}_\alpha(U_\alpha\nu-u_1\vee 0,U_\alpha\nu-u_1\vee 0) \leqq \mathcal{E}_\alpha(U_\alpha\nu-u_1,U_\alpha\nu-u_1)$$

from $U_\alpha\nu - u_1\vee 0 \leqq U_\alpha\nu - u_1$ m-a.e.. It implies that $u_1\vee 0 = u_1$, that is $u_1 \geqq 0$ m-a.e.. We can inductively show $u_n \geqq 0$ m-a.e. by similar argument.

(iv) Since $U_\alpha\nu - u_n \leqq U_\alpha\nu - u_m$ m-a.e., $n \leqq m$, and $U_\alpha\nu - u_n \leqq U_\alpha\nu$ m-a.e. for each n by (ii) and (iii) it holds that

$$(1.14)\quad \mathcal{E}_\alpha(U_\alpha\nu-u_n,U_\alpha\nu-u_n) \leqq \mathcal{E}_\alpha(U_\alpha\nu-u_m,U_\alpha\nu-u_m) \leqq \mathcal{E}_\alpha(U_\alpha\nu,U_\alpha\nu)$$

for each $n \leqq m$. Therefore $\mathcal{E}_\alpha(U_\alpha\nu-u_n,U_\alpha\nu-u_n)$ monotonously increases to a finite number. Since $w_n = U_\alpha\nu - u_n$ is α-almost excessive

$$0 \leqq \mathcal{E}_\alpha(w_n-w_m,w_n-w_m) = \mathcal{E}_\alpha(w_n,w_n) - 2\mathcal{E}_\alpha(w_n,w_m) + \mathcal{E}_\alpha(w_m,w_m)$$

$$\leqq \mathcal{E}_\alpha(w_m,w_m) - \mathcal{E}_\alpha(w_n,w_n), \qquad n \leqq m.$$

Hence w_n is a $\mathcal{E}_\alpha$-cauchy sequence, so u_n is also.

Proof of Theorem 1. As the result of (ii) and (iv) of Lemma there exists u such that $\mathcal{E}_\alpha(u_n-u,u_n-u) \to 0$ and $\widetilde{u}_n \downarrow \widetilde{u}$ q.e.. We can now prove that this function u is a solution of the quasi-variational inequality (1.2). We at first note that it follows that

$$\mathcal{E}_\alpha(U_\alpha\nu-u_n,U_\alpha\nu-u_n) \leqq \mathcal{E}_\alpha(U_\alpha\nu-v,U_\alpha\nu-v) \quad \forall\widetilde{v} \leqq M\widetilde{u} = \lim_{n\to\infty} M\widetilde{u}_n \quad \text{q.e.}$$

from (1.6) because $\widetilde{u}_n \downarrow \widetilde{u}$ q.e. implies $M\widetilde{u}_n \downarrow M\widetilde{u}$. Therefore it holds that

$$(1.15)\quad \mathcal{E}_\alpha(U_\alpha\nu-u,U_\alpha\nu-u) \leqq \mathcal{E}_\alpha(U_\alpha\nu-v,U_\alpha\nu-v), \quad \forall\widetilde{v} \leqq M\widetilde{u} \quad \text{q.e.}$$

since $\mathcal{E}_\alpha(u_n-u,u_n-u) \to 0$. On the other hand, since $\widetilde{u} \leqq \widetilde{u}_n \leqq M\widetilde{u}_{n-1}$

q.e. for each n we have

(1.16) $\tilde{u} \leqq \lim_{n\to\infty} M\tilde{u}_n = M\tilde{u}$ q.e..

(1.15) with (1.16) is equivalent to the QVI (1.2).

Now we are going to prove that the above solution u of QVI (1.2) is the maximal one. Take another solution w of the QVI

$$\begin{cases} \mathcal{E}_\alpha(w,v-w) \geqq \langle \nu, \tilde{v}-\tilde{w} \rangle & \forall \tilde{v} \leqq M\tilde{w} \quad \text{q.e.} \\ \tilde{w} \leqq M\tilde{w} \quad \text{q.e.} \end{cases}$$

In the same way as Lemma we can see $U_\alpha\nu - w$ is α-excessive, so $U_\alpha\nu \geqq w$. Therefore $M\widetilde{U_\alpha\nu} \geqq \tilde{w}$ q.e.. That is $w \in V_1$. Since $U_\alpha\nu - u_1 \vee w = (U_\alpha\nu - u_1)\wedge(U_\alpha\nu - w)$ is α-almost excessive and $U_\alpha\nu - u_1\vee w \leqq U_\alpha\nu - u_1$ it holds that

(1.17) $\mathcal{E}_\alpha(U_\alpha\nu - u_1\vee w, U_\alpha\nu - u_1\vee w) \leqq \mathcal{E}_\alpha(U_\alpha\nu - u_1, U_\alpha\nu - u_1)$.

Hence we have $u_1 \geqq w$ by similar argument as (iii) of Lemma. In the same way we can inductively see $u_n \geqq w$ for each n, which implies $u \geqq w$.

§2. Impulsive control of symmetric Markov processes

Let $X = \{\Omega, \mathcal{B}, \mathcal{B}_t, P_x, X_t, \theta_t\}$ be an m-symmetric standard Markov process of function space type with the state space S. We assume that its Dirichlet space $(\mathcal{F}, \mathcal{E})$ is regular. We are now going to repeat Robin's construction of controlled process (cf. [7]) with a little modification and set up an impulsive control problem.

Consider the infinite product space $\Omega_\infty = \Omega\times\Omega\times\Omega\times\cdots$ and define its sub-σ-fields by

$$(2.1)\quad \mathcal{B}_t^n = \pi_n^{-1}(\mathcal{B}_t)^{\otimes n}$$

where π_n is the projection from Ω_∞ to the n-th product $(\Omega)^n$. $\mathcal{B}^n$ is similarly defined. For $\underline{\underline{\omega}} = (\omega_1, \omega_2, \cdots\cdots\cdots) \in \Omega_\infty$, we let

$$(2.2)\quad (\theta_{n,t}\underline{\underline{\omega}})(s) = (\theta_t\omega_1(s), \cdots\cdots\cdots\cdots, \theta_t\omega_n(s))$$
$$= (\omega_1(t+s), \cdots\cdots\cdots\cdots, \omega_n(t+s)).$$

We note that, if $\sigma(\underline{\underline{\omega}})$ is a $\mathcal{B}^n$-measurable function on Ω_∞, then $\sigma(\underline{\underline{\omega}}) = \tilde{\sigma}(\omega_1, \omega_2, \cdots\cdots, \omega_n)$, $\tilde{\sigma}$ being a $(\mathcal{B})^{\otimes n}$-measurable function on $(\Omega)^n$. Such an identification of σ and $\tilde{\sigma}$ will be made below without mentioning explicitly. It is further noticed that P_x for each $x \in S$ can be regarded as a probability measure on $(\Omega_\infty, \mathcal{B}^1)$.

A family of subsets $\{\Gamma_x\}_{x\in S}$ of S is called <u>admissible</u> if the following condition (Γ) is satisfied:

(Γ) if $x_n \to x$, $x_n, x \in S$ and $y_n \in \Gamma_{x_n}$, then there exist $y \in \Gamma_x$ and $\{y_{n_k}\} \subset \{y_n\}$ such that $y_{n_k} \to y$.

A sequence $v = \{(\tau_i, \xi_i)_{i=1}^{\infty}\}$ of the pairs of random variables τ_i and ξ_i on Ω_∞ is called an admissible control if the following conditions (v.1)～(v.3) are satisfied for a given admissible $\{\Gamma_x\}$:

(v.1) τ_i is a $\mathcal{B}_t^i$-stopping time such that $\tau_i \leqq \tau_{i+1}$ for each i and $\lim_{i\to\infty} \tau_i = \infty$

(v.2) ξ_i is $\Gamma_{X_{\tau_i}(\omega_i)}$-valued $\mathcal{B}_{\tau_i}^i$-measurable random variable for each i

(v.3) for each N with $\mathrm{Cap}(N) = 0$ there exists $\tilde{N} \supset N$ with $\mathrm{Cap}(\tilde{N}) = 0$ such that $P_x^i(\xi_i \in \tilde{N}) = 0$, $x \in S - \tilde{N}$ for each i, where P_x^i is a probability measure on $(\Omega_\infty, \mathcal{B}^i)$ specified below.

The set of all admissible controls are denoted by $\underline{\underline{V}}$. Let us define, for $y \in S$, an element $\delta_y \in \Omega$ by

(2.3) $\qquad \delta_y(t) = y \quad \forall t \geqq 0$

and denote by ε_{δ_y} the probability measure on $(\Omega, \mathcal{B})$ which is concentrated on δ_y.

For a given $v = \{(\tau_i, \xi_i)_{i=1}^{\infty}\} \in \underline{\underline{V}}$, we are interested in the process $X_t(\omega_1)$ governed by P_x up to time $\tau_1(\omega_1)$. $X_t(\omega_1)$ is stopped at time τ_1 and then our interest is switched to the process $X_{\tau_1(\omega_1)+t}(\omega_2)$, $t \geqq 0$, governed by $P_{\xi_1(\omega_1)}$ up to time $\tau_2(\omega_1, \omega_2)$ and so forth. To formulate such a process, we construct probability measures P_x^n on $(\Omega_\infty, \mathcal{B}^n)$, $n = 1, 2, \cdots$, as follows:

First let

$$P_x^1 = P_x \qquad \text{on} \quad (\Omega_\infty, \mathcal{B}^1)$$

We can construct a probability measure P_x^2 on $(\Omega_\infty, \mathcal{B}^2)$ such that

$$(2.4)\begin{cases} P_x^2 = P_x^1 \quad \text{on} \quad \mathcal{B}^1_{\tau_1} \quad (\subset \mathcal{B}^1) \\ P_x^2(\theta^{-1}_{2,\tau_1} B \mid \mathcal{B}^1_{\tau_1}) = \varepsilon_{\delta_{X_{\tau_1}}} \otimes P_{\xi_1}(B) \qquad P_x^1 - \text{a.s.} \quad \text{on} \ \{\tau_1 < +\infty\} \end{cases}$$

for each $B \in \mathcal{B}^2$. Then the process $X_{\tau_1+t}(\omega_2)$, $t \geqq 0$, is Markovian with respect to $(\mathcal{B}^2_{\tau_1+t}, P_x^2)$ under the condition $\mathcal{B}^1_{\tau_1}$. We define the probability measure P_x^{n+1} on $(\Omega_\infty, \mathcal{B}^{n+1})$ inductively by

$$(2.5)\begin{cases} P_x^{n+1} = P_x^n \qquad \text{on} \quad \mathcal{B}^n_{\tau_n} \quad (\subset \mathcal{B}^n) \\ P_x^{n+1}(\theta^{-1}_{n+1,\tau_n} B \mid \mathcal{B}^n_{\tau_n}) = \varepsilon_{\delta_{X_{\tau_1}(\omega_1)}} \otimes \cdots\cdots \otimes \varepsilon_{\delta_{X_{\tau_n}(\omega_n)}} \otimes P_{\xi_n}(B) \\ \qquad\qquad P_x^n - \text{a.s.} \quad \text{on} \quad \{\tau_n < +\infty\} \end{cases}$$

where $B \in \mathcal{B}^{n+1}$.

We are now in a position to formulate our main theorem. Consider the Dirichlet space $(\mathcal{F}, \mathcal{E})$ associated with the process X. We suppose that a non-negative Radon measure $\nu(dx)$ of finite energy integral and non-negative continuous function $k(x,\xi)$, $x,\xi \in S$, are given which are to define a pay-off function. It is known that a non-negative continuous additive functional $A_t(\omega)$ on X corresponds to $\nu(dx)$:

$$(2.6) \qquad E_x[\int_0^\infty e^{-\alpha s} dA_s] = U_\alpha \nu \quad \text{q.e.}$$

(cf. [4]). Let for $\underline{\underline{\omega}} = (\omega_1, \omega_2, \cdots\cdots\cdots) \in \Omega_\infty$

$$(2.7) \qquad \underline{\underline{A}}_t = \begin{cases} A_t(\omega_1), & 0 \leqq t \leqq \tau_1 \\ \underline{\underline{A}}_{\tau_1} + A_{t-\tau_1}(\theta_{\tau_1}\omega_2), & \tau_1 < t \leqq \tau_2 \\ \underline{\underline{A}}_{\tau_{n-1}} + A_{t-\tau_{n-1}}(\theta_{\tau_{n-1}}\omega_n), & \tau_{n-1} < t \leqq \tau_n \end{cases}$$

and

$$(2.8) \qquad y_t(\underline{\underline{\omega}}) = X_t(\omega_{k+1}) \quad \text{if} \quad t \in [\tau_k, \tau_{k+1}).$$

We can now define the pay-off function $u^*(x)$ by

$$(2.9) \qquad u^*(x) = \inf_{v \in \underline{\underline{V}}} J_x(v)$$

$$(2.10) \qquad J_x(v) = \lim_{n \to \infty} J_x^n(v)$$

$$(2.11) \qquad J_x^n(v) = E_x^n[\int_0^{\tau_n} e^{-\alpha t} d\underline{\underline{A}}_t + \sum_{i=1}^n e^{-\alpha \tau_i} k(X_{\tau_i}(\omega_i), \xi_i)].$$

We then introduce the operator M by

$$(2.12) \qquad M\phi(x) = \underset{y \in \Gamma_x}{\text{q-essinf}} \{\phi(y) + k(x,y)\}$$

$$\equiv \sup \{c: \text{Cap}\{y \in \Gamma_x; \phi(y) + k(x,y) < c\} = 0\}$$

for $\phi \in \mathcal{F}$. The fact that this operator M satisfies (M.1)~(M.4) will be shown later (§3). Recall that Theorem 1 then guarantees the existence of the maximum solution of the QVI (1.2) associated with the present data $(\mathcal{F}, \mathcal{E})$, ν and M.

Theorem 2. The pay-off function $u^*(x)$ defined by (2.9) is a quasi-continuous modification of the maximum solution u of the QVI (1.2) corresponding to the above $(\mathcal{F}, \mathcal{E})$, ν and M.

Remark. We note that if $\nu(dx) = f(x)dm$ with a Borel function f in $L^2(dm)$, $J_x^n(v)$ is written as

$$(2.13) \qquad J_x^n(v) = E_x^n[\int_0^{\tau_n} e^{-\alpha t} f(y_t)dt + \sum_{i=1}^{n} e^{-\alpha \tau_i} k(X_{\tau_n}(\omega_n), \xi_n)]$$

In the next section we study the operator M defined by (2.12). All assumptions and notations in section 2 are assumed through the following sections.

§3. Operator M

Definition 3.1. A sequence $\{F_k\}$ of closed sets such that $F_k \uparrow$ and $\mathrm{Cap}(S-F_k) \downarrow 0$, $k \to \infty$ is called a nest on S. A nest $\{F_k\}$ is said to be (m)-regular if for each k $m(U(x)\cap F_k) \not\downarrow 0$ for any $x \in F_k$ and any open neighborhood $U(x)$ of x.

Let Q be a countable family of quasi-continuous function in the restricted sense on S. Then it is known that there exists a regular nest $\{F_k\}$ on S such that $u|_{F_k \cup \Delta}$ is continuous for each k for any function $u \in Q$.

Lemma 3.1. For any function $\phi \in \tilde{\mathcal{F}}$ $M\phi$ is a Borel function and has the following representation:

$$(3.1) \qquad M\phi(x) = \liminf_{n\to\infty} \inf_{y\in\Gamma_x^n \cap F_n} \{\phi(y) + k(x,y)\}$$

where $\{F_n\}$ is a regular nest and Γ_x^n is a subset of S which satisfies (Γ).

Proof of Lemma 3.1. It holds that by definition

$$\mathrm{Cap}\,\{y \in \Gamma_x;\ \phi(y) + k(x,y) < M\phi(x) - \varepsilon\} = 0$$

for any $\varepsilon > 0$. Take a regular nest $\{F_n\}$ such that $\phi|_{F_n\cup\Delta}$ is continuous for each n. Put

$$N_x^n = \{y \in \Gamma_x;\ \text{there exists a open neighborhood } U_y \text{ such that } \mathrm{Cap}(U_y \cap F_n \cap \Gamma_x) = 0\}$$

and define

$$\Gamma_x^n = \Gamma_x \cap \Big(\bigcup_{y \in N_x^n} U_y \Big)^c$$

by above U_y. Then it is obvious that Γ_x^n satisfies (Γ) because $\big(\bigcup_{y \in N_x^n} U_y \big)^c$ is closed. Since $\phi(\cdot)$ and $k(x,\cdot)$ are continuous on $\Gamma_x^n \cap F_n$ it follows that

$$\phi(y) + k(x,y) \geqq M\phi(x) - \varepsilon \qquad \forall y \in \Gamma_x^n \cap F_n$$

from

$$\mathrm{Cap}\{y \in \Gamma_x^n \cap F_n;\ \phi(y) + k(x,y) < M\phi(x) - \varepsilon\} = 0$$

Therefore

$$\liminf_{n\to\infty} \inf_{y \in \Gamma_x^n \cap F_n} \{\phi(y) + k(x,y)\} \geqq M\phi(x).$$

In order to get converse inequality put

$$c = \lim_{n\to\infty} \inf_{y \in \Gamma_x^n \cap F_n} \{\phi(y) + k(x,y)\},$$

then

$$\mathrm{Cap}\{y \in \Gamma_x;\ \phi(y) + k(x,y) < c\} = \mathrm{Cap}\{\Gamma_x \cap (\bigcup_n F_n);\ \phi(y) + k(x,y) < c\}$$

$$\leqq \sum_{n=1}^{\infty} \mathrm{Cap}\{y \in \Gamma_x \cap F_n;\ \phi(y) + k(x,y) < c\}$$

$$= \sum_{n=1}^{\infty} \mathrm{Cap}\{y \in \Gamma_x^n \cap F_n;\ \phi(y) + k(x,y) < c\}.$$

Hence $c \leq M\phi(x)$. Now (3.1) has been proved. On the other hand, since $\inf_{y \in \Gamma_x^n \cap F_n} \{\phi(y) + k(x,y)\}$ is a lower semi-continuous function according to the following lemma, we have the conclusion that $M\phi(x)$ is a Borel function.

Lemma 3.2. For any $\phi \in \tilde{\mathcal{F}}$

$$M_n\phi(x) = \inf_{y \in \Gamma_x^n \cap F_n} \{\phi(y) + k(x,y)\}$$

is a lower semi-continuous function and has a measurable selection for each n.

This lemma is a trivial modification of Theorem A in §5, Chap. 2 of [3]. Because $\Gamma_x^n \cap F_n$ also satisfies (Γ) and $\phi(\cdot)$ and $k(x,\cdot)$ are continuous on F_n.

Lemma 3.3. The operator M defined by (2.12) satisfies (M.1) $\sim$ (M.4).

Proof of Lemma 3.3. (M.1) has been proved in Lemma 3.1. (M.2) and (M.3) are obvious. As to (M.4) it is easily seen that

$$\lim_{n\to\infty} Mu_n(x) \geq Mu(x).$$

On the other hand

$$Mu_n(x) \leqq u_n(y) + k(x,y) \qquad \forall y \in \Gamma_x^m \cap F_m,$$

so we have

$$\lim_{n\to\infty} Mu_n(x) \leqq u(y) + k(x,y) \qquad \forall y \in \Gamma_x^m \cap F_m$$

for each m. Then it holds that

$$\lim_{n\to\infty} Mu_n(x) \leqq \lim_{m\to\infty} \inf_{y \in \Gamma_x^m \cap F_m} \{u(y) + k(x,y)\}$$

$$= Mu(x).$$

§4. Optimal stoppings of Markov processes

We prepare for the proof of Theorem 2 some lemmas on optimal stoppings of Markov processes with which regular Dirichlet spaces are associated.

Let ψ_n be a given Borel function and s_n be the unique solution of the following variational inequality:

$$(4.1)\quad \begin{cases} \mathcal{E}_\alpha(s_n, v-s_n) \geqq \langle\nu, \tilde{v}-\tilde{s}_n\rangle & \forall v \in \mathcal{F},\ \tilde{v} \leqq \psi_n \text{ q.e.} \\ s_n \in \mathcal{F},\ \tilde{s}_n \leqq \psi_n \text{ q.e.} \end{cases}$$

for each n.

Lemma 4.1. Suppose that $\psi_n(x) \downarrow \psi(x) \geq 0$ $\forall x$, then $\mathcal{E}_\alpha(s_n-s, s_n-s) \to 0$ where s is the unique solution of

$$(4.2)\quad \begin{cases} \mathcal{E}_\alpha(s, v-s) \geqq \langle\nu, \tilde{v}-\tilde{s}\rangle & v \in \mathcal{F},\ \tilde{v} \leqq \psi \text{ q.e.} \\ s \in \mathcal{F},\ \tilde{s} \leqq \psi \text{ q.e.} \end{cases}$$

Proof of Lemma 4.1. In a similar way as the proof of Theorem 1 we can easily show that $s_n \geqq s_{n+1}$, $s_n \geqq 0$ and $U_\alpha\nu - s_n$ is an α-almost excessive function for each n (cf. Lemma in §1). Therefore we have

$$\mathcal{E}_\alpha(U_\alpha\nu-s_n, U_\alpha\nu-s_n) \leqq \mathcal{E}_\alpha(U_\alpha\nu-s_m, U_\alpha\nu-s_m) \leqq \mathcal{E}_\alpha(U_\alpha\nu, U_\alpha\nu) \qquad n \leqq m.$$

So there exists $s_0 \in \mathcal{F}$ such that $\mathcal{E}_\alpha(s_n-s_0, s_n-s_0) \to 0$. Furthermore s_0 satisfies

$$(4.3)\quad \begin{cases} \mathcal{E}_\alpha(U_\alpha\nu-s_0, U_\alpha\nu-s_0) \leqq \mathcal{E}_\alpha(U_\alpha\nu-v, U_\alpha\nu-v) & \forall \tilde{v} \leqq \lim_{n\to\infty} \psi_n = \psi \text{ q.e.} \\ \tilde{s}_0 \leqq \psi \text{ q.e.} \end{cases}$$

which is equivalent to (4.2). Hence we conclude $s_0 = s$ because of uniqueness of the solution of (4.2).

Lemma 4.2. Put

$$t_n(x) = \inf_{\tau} E_x\left[\int_0^{\tau} e^{-\alpha s} dA_s + e^{-\alpha\tau} M_n\phi(X_\tau)\right] \quad \text{q.e.}$$

where $\phi \in \tilde{\mathcal{F}}$, then t_n is a quasi-continuous modification of the solution s_n of the variational inequality (4.1) for each n in which ψ_n is considered $M_n\phi$. Furthermore there exists an optimal stopping time.

Proof of Lemma 4.2. Since $U_\alpha\nu - s_n$ is α-almost excessive by similar argument as Lemma 1.1 there corresponds a non-negative Radon measure μ_n of finite energy integral such that

$$\mathcal{E}_\alpha(U_\alpha\nu - s_n, v) = \int \mu_n(dx)\tilde{v}(x) \qquad \forall v \in \mathcal{F}.$$

Therefore it follows that

$$(4.4) \quad \begin{cases} \int \mu_n(dx)(\tilde{s}_n(x) - \tilde{v}(x)) \geqq 0 & \forall \tilde{v} \leqq M_n\phi \quad \text{q.e.}, \quad v \in \mathcal{F} \\ \tilde{s}_n \leqq M_n\phi \quad \text{q.e.}, \quad s_n \in \mathcal{F} \end{cases}$$

from (4.1) with $\psi_n = M_n\phi$. Put

$$(4.5) \qquad L_n = \{x \in \bigcup_{k=1}^{\infty} F_k ;\ \tilde{s}_n(x) < M_n\phi(x)\},$$

where $\{F_k\}$ is a regular nest corresponding to the family of quasi-continuous functions $\{\tilde{s}_n\}$. Take an arbitrary point $x_0 \in L_n$, then $x_0 \in F_{k_0}$ for some k_0. On the other hand, Since $M_n\phi(x)$ is a lower semi-continuous function there exists a sequence of continuous functions $c_j^n(x)$ such that $c_j^n(x) \uparrow M_n\phi(x)$, $j \to \infty$, $\forall x$. Therefore $c_{j_0}^n(x_0) > s_n(x_0)$ for sufficiently large j_0, which implies that there exists a neighborhood $U(x_0)$ of x_0 such that

$$s_n(x) < c_{j_0}^n(x) \qquad \forall x \in F_{k_0} \cap U(x_0).$$

Accordingly there exists a neighborhood $V(x_0)$ and $v_n \in \mathcal{F} \cap C_0(S)$ such that

$$\overline{V}(x_0) \subset U(x_0),$$

$$\text{Supp } v_n \subset U(x_0), \quad v_n(x) > 0 \quad \text{on} \quad V(x_0)$$

and

$$\tilde{s}_n(x) + v_n(x) \leqq M_n\phi(x)$$

because the Dilichlet space $(\mathcal{F},\mathcal{E})$ is regular. Therefore

$$-\int \mu_n(dx)v_n(x) \geqq 0$$

which implies $\mu_n(V(x_0)) = 0$. Since $x_0 \in L_n$ is arbitrary we conclude that

$$(4.6) \quad \mu_n(L_n) = 0.$$

Next, we have

$$(4.7) \quad \tilde{s}_n(x) \leqq M_n\phi(x) \quad \text{q.e.}$$

On the other hand let $S - N$ be a defining set of the additive functional A_t (cf. [4]) and put $\tau_n = \inf\{t;\ X_t \in L_n^c \cap \{S-N\}\}$, then

$$(4.8) \quad P_x(X_{\tau_n} \in L_n^c \cap (S-N)) = 1 \qquad x \in (\bigcup_{n=1}^{\infty} F_n) \cap (S-N)$$

for the benifit of lower semi-continuity of $M_n\phi$ and quasi-continuity of s_n.

From (4.6), (4.7) and (4.8) in addition to the fact that there corresponds a non-negative additive functional A_t^n to the α-almost excessive function $U_\alpha\nu - s_n$ such that

$$\widetilde{U_\alpha\nu - s_n} = E_x\Big[\int_0^\infty e^{-\alpha s} dA_s^n\Big] \quad \text{q.e.}$$

our present lemma follows in the same way as Theorem in [6].

Lemma 4.3. Put

$$(4.9) \quad t(x) = \inf_\tau E_x\Big[\int_0^\tau e^{-\alpha s} dA_s + e^{-\alpha\tau} M\phi(X_\tau)\Big],$$

then $t(x)$ is a quasi-continuous modification of the solution of the variational inequality (4.2) in which ψ is considered as $M\phi(x)$.

Proof of lemma 4.3. Let $s(x)$ be the solution of (4.2) with $\psi = M\phi$, then $U_\alpha \nu - s$ is α-almost excessive and there corresponds non-negative continuous additive functional A_t^0 such that

$$(4.10) \quad \widetilde{U_\alpha \nu(x) - s(x)} = E_x\Big[\int_0^\infty e^{-\alpha t} dA_t^0\Big] \quad \text{q.e..}$$

On the other hand we have

$$(4.11) \quad \tilde{s}(x) \leqq M\phi(x) \quad \text{q.e..}$$

From (4.10) and (4.11) it follows that

$$\begin{aligned}
\tilde{s}(x) &= E_x\Big[\int_0^\infty e^{-\alpha t} dA_t\Big] - E_x\Big[\int_0^\infty e^{-\alpha t} dA_t^0\Big] \\
&= E_x\Big[\int_0^\tau e^{-\alpha t} dA_t\Big] - E_x\Big[\int_0^\tau e^{-\alpha t} dA_t^0\Big] + E_x\Big[e^{-\alpha\tau}\tilde{s}(X_\tau)\Big] \\
&\leqq E_x\Big[\int_0^\tau e^{-\alpha t} dA_t + e^{-\alpha\tau}\tilde{s}(X_\tau)\Big] \\
&\leqq E_x\Big[\int_0^\tau e^{-\alpha t} dA_t + e^{-\alpha\tau} M\phi(X_\tau)\Big] \quad \text{q.e.,}
\end{aligned}$$

for any stopping time τ. Therefore it holds that

$$(4.12) \quad \tilde{s}(x) \leqq t(x) \quad \text{q.e..}$$

Now it is clear that

$$t(x) \leqq \inf_\tau E_x\Big[\int_0^\tau e^{-\alpha t} dA_t + e^{-\alpha\tau} M_n\phi(X_\tau)\Big] = \tilde{s}_n(x) \quad \text{q.e..}$$

Since $\mathcal{E}_\alpha(s_n - s, s_n - s) \to 0$ by lemma 4.1 we obtain $\tilde{s}_n(x) \downarrow \tilde{s}(x)$ q.e.. Hence

$$(4.13) \quad t(x) \leqq \tilde{s}(x) \quad \text{q.e.}$$

(4.12) and (4.13) give our conclusion.

§5. Proof of Theorem 2

Now we are going to prove Theorem 2. Let us introduce the set $\underline{\underline{V}}_n$ of admissible controls which have n jump times at most:

$$(5.1) \qquad \underline{\underline{V}}_n = \{ v \in \underline{\underline{V}};\ \tau_{n+1}(\underline{\underline{\omega}}) = \infty \}$$

for each n. Put

$$(5.2) \quad u_n^*(x) = \inf_{v \in \underline{\underline{V}}_n} E_x^{n+1}\Big[\int_0^{\tau_{n+1}} e^{-\alpha s} d\underline{\underline{A}}_s + \sum_{i=1}^{n} e^{-\alpha \tau_i} k(X_{\tau_i}(\omega_i), \xi_i)\Big]$$

and

$$(5.3) \quad w_n(x) = \inf_{\tau} E_x\Big[\int_0^{\tau} e^{-\alpha s} dA_s + e^{-\alpha\tau} M w_{n-1}(X_\tau)\Big]$$

for each n where $w_0(x) = \widetilde{U_\alpha \nu}(x)$.

Theorem 2 is a consequence of the following two propositions.

Proposition 5.1. $W_n(x)$ is a quasi-continuous modification of the solution u_n of the variational inequality (1.5).

Proposition 5.2. It holds that

$$(5.4) \qquad w_n(x) = u_n^*(x) \quad \text{q.e..}$$

Proposition 5.1 is a direct consequence of Lemma 4.3. For the proof of Proposition 5.2 we prepare the following two lemmas.

Lemma 5.3. It holds that

(5.5) $$w_n(x) = \lim_{k_1\uparrow\infty} \cdots\cdots \lim_{k_n\uparrow\infty} w^n_{k_n\cdots k_1}(x) \quad \text{q.e.}$$

where

(5.6) $$w^n_{k_n\cdots k_1}(x) = \inf_\tau E_x\left[\int_0^\tau e^{-\alpha s}dA_s + e^{-\alpha\tau}M_{k_n} w^{n-1}_{k_{n-1}\cdots k_1}(X_\tau)\right]$$
$n=2,3,\cdots\cdots$

and

(5.7) $$w^1_{k_1}(x) = \inf_\tau E_x\left[\int_0^\tau e^{-\alpha s}dA_s + e^{-\alpha\tau}M_{k_1}\widetilde{U_\alpha}\nu(X_\tau)\right].$$

Proof of Lemma 5.3. Because of Lemma 4.1 it follows that

(5.8) $$w^1_{k_1}(x) \downarrow w_1(x) \quad \text{q.e.,} \quad k_1\uparrow\infty$$

from $M_{k_1}\widetilde{U_\alpha}\nu(x) \downarrow M\widetilde{U_\alpha}\nu(x)$ $\forall x$, $k_1\uparrow\infty$, in the same way as the the proof of Lemma 4.3. Let us assume that

(5.9) $$w_{n-1}(x) = \lim_{k_1\uparrow\infty} \cdots\cdots \lim_{k_n\uparrow\infty} w^{n-1}_{k_{n-1}\cdots k_1}(x) \quad \text{q.e..}$$

Then it follows that

(5.10) $$\lim_{k_n\uparrow\infty} w^n_{k_n\cdots k_1}(x) = \inf_\tau E_x\left[\int_0^\tau e^{-\alpha s}dA_s + e^{-\alpha\tau}Mw^{n-1}_{k_{n-1}\cdots k_1}(X_\tau)\right]$$

from $M_{k_n} w^{n-1}_{k_{n-1}\cdots k_1}(x) \downarrow Mw^{n-1}_{k_{n-1}\cdots k_1}(x)$ $\forall x$, $k_n\uparrow\infty$,

in the same way as above. On the other hand it holds that

$$(5.11) \qquad Mw_{n-1}(x) = \lim_{k_1 \uparrow \infty} \cdots\cdots \lim_{k_{n-1} \uparrow \infty} Mw^{n-1}_{k_{n-1} \cdots k_1}(x) \qquad \forall x$$

by our assumption and the property of M. Making use of Lemma 4.1 we obtain our present lemma from (5.10) and (5.11).

Lemma 5.4. Let $\overline{X} = (\overline{\Omega}, \overline{\mathcal{B}}, \overline{\mathcal{B}}_t, \overline{P}_x, \overline{X}_t)$ be a m-symmetric Markov process associated with a regular Dirichlet space $(\mathcal{F}, \mathcal{E})$ and

$$H(M\phi;x) = \inf_{\tau} \overline{E}_x[\int_0^{\tau} e^{-\alpha s} d\overline{A}_s + e^{-\alpha\tau} M\phi(\overline{X}_\tau)] \qquad \phi \in \widetilde{\mathcal{F}},$$

then it holds that

$$\overline{E}_x[\int_\sigma^{\tau} e^{-\alpha s} d\overline{A}_s + e^{-\alpha\tau} M\phi(\overline{X}_\tau) \mid \overline{\mathcal{B}}_\sigma] \geqq e^{-\alpha\sigma} H(M\phi;\overline{X}_\sigma)$$

for any stopping time σ, τ such that $\sigma \leqq \tau$. Here $\overline{A}_t$ is an additive functional of $\overline{X}$ corresponding to the Radon measure $\nu(dx)$ of finite energy integral.

Proof. At first we note that $H(M\phi;x) \in \widetilde{\mathcal{F}}$, $U_\alpha\nu(x) - H(M\phi;x)$ is α-almost excessive and $H(M\phi;x) \leqq M\phi(x)$ q.e. by Lemma 4.3. Therefore $e^{-\alpha t}\{U_\alpha\nu(\overline{X}_t) - H(M\phi;\overline{X}_t)\}$ is a $(\overline{P}_x, \overline{\mathcal{B}}_t)$ supermartingale for q.e. x. Hence

$$\overline{E}_x[\, e^{-\alpha\tau}\{U_\alpha\nu(\overline{X}_\tau) - H(M\phi;\overline{X}_\tau)\} \mid \overline{\mathcal{B}}_\sigma]$$

$$\leqq \; e^{-\alpha\sigma}\{U_\alpha\nu(\overline{X}_\sigma) - H(M\phi;\overline{X}_\sigma)\} \qquad \overline{P}_x\text{ - a.s.} \quad \text{q.e. } x.$$

So we have

$$\bar{E}_x[\int_\sigma^\tau e^{-\alpha s}d\bar{A}_s + e^{-\alpha\tau}H(M\phi;\bar{X}_\tau)\,|\,\bar{\mathcal{B}}_\sigma] \geqq e^{-\alpha\sigma}H(M\phi;\bar{X}_\sigma).$$

Accordingly it follows that

$$\bar{E}_x[\int_\sigma^\tau e^{-\alpha s}d\bar{A}_s + e^{-\alpha\tau}M\phi(\bar{X}_\tau)\,|\,\bar{\mathcal{B}}_\sigma] \geqq e^{-\alpha\sigma}H(M\phi;\bar{X}_\sigma)$$

from $H(M\phi;x) \leqq M\phi(x)$ q.e.

Proof of Proposition 5.2. Let $v \in \underline{\underline{V}}_n$, $v = \{(\tau_k,\xi_k)_{k=1,2,\cdots,n}$ $\tau_{n+1} = \infty\}$, then

$$J_x^n(v) = E_x^{n+1}[\int_0^\infty e^{-\alpha s}d\underline{\underline{A}}_s + \sum_{i=1}^n e^{-\alpha\tau_i}\, k(X_{\tau_i}(\omega_i),\xi_i)]$$

$$= E_x^{n+1}[\int_0^{\tau_n} e^{-\alpha s}d\underline{\underline{A}}_s + \sum_{i=1}^n e^{-\alpha\tau_i}\, k(X_{\tau_i}(\omega_i),\xi_i) + e^{-\alpha\tau_n}\int_0^\infty e^{-\alpha s}dA_s(\theta_{\tau_n}\omega_{n+1}$$

$$= E_x^{n+1}[\int_0^{\tau_n} e^{-\alpha s}d\underline{\underline{A}}_s + \sum_{i=1}^n e^{-\alpha\tau_i}\, k(X_{\tau_i}(\omega_i),\xi_i) + e^{-\alpha\tau_n}\, E_{\xi_n}[\int_0^\infty e^{-\alpha s}dA_s]]$$

$$\geqq E_x^n[\int_0^{\tau_n} e^{-\alpha s}d\underline{\underline{A}}_s + \sum_{i=1}^{n-1} e^{-\alpha\tau_i}\, k(X_{\tau_i}(\omega_i),\xi_i) + e^{-\alpha\tau_n}\, M\widetilde{U_\alpha\nu}(X_{\tau_n}(\omega_n))]$$

$$= E_x^n[\int_0^{\tau_{n-1}} e^{-\alpha s}d\underline{\underline{A}}_s + \sum_{i=1}^{n-1} e^{-\alpha\tau_i}\, k(X_{\tau_i}(\omega_i),\xi_i)$$

$$+ E_x^n[\int_{\tau_{n-1}}^{\tau_n} e^{-\alpha s}dA_{s-\tau_{n-1}}(\theta_{\tau_{n-1}}\omega_n) + e^{-\alpha\tau_n}\, M\widetilde{U_\alpha\nu}(X_{\tau_n}(\omega_n))\,|\,\mathcal{B}^{n-1}_{\tau_{n-1}}]$$

$$\geqq E_x^n[\int_0^{\tau_{n-1}} e^{-\alpha s}d\underline{\underline{A}}_s + \sum_{i=1}^{n-1} e^{-\alpha\tau_i}\, k(X_{\tau_i},\xi_i) + e^{-\alpha\tau_{n-1}}\, H(M\widetilde{U_\alpha\nu};X_{\tau_{n-1}})]$$

$$= E_x^{n-1}\Big[\int_0^{\tau_{n-1}} e^{-\alpha s} d\underline{\underline{A}}_s + \sum_{i=1}^{n-1} e^{-\alpha\tau_i} k(X_{\tau_i},\xi_i) + e^{-\alpha\tau_{n-1}} w_1(X_{\tau_{n-1}})\Big]$$

$$\geqq E_x^1\Big[\int_0^{\tau_1} e^{-\alpha s} dA_s + e^{-\alpha\tau_1} k(X_{\tau_1}, \xi_1) + e^{-\alpha\tau_1} w_{n-1}(X_{\tau_1})\Big]$$

$$\geqq E_x^1\Big[\int_0^{\tau_1} e^{-\alpha s} dA_s + e^{-\alpha\tau_1} Mw_{n-1}(X_{\tau_1})\Big] \geqq w_n(x) \quad \text{q.e.}$$

In order to get the converse inequality take a sequence $\{\tilde{\tau}_j\}_{j=1,2,\cdots n}$ of stopping times each of which minimizes

$$E_x\Big[\int_0^{\tau} e^{-\alpha s} dA_s + M_{k_j} w^{j-1}_{k_{j-1}\cdots k_1}(X_\tau)\Big].$$

Furthermore take a sequence of functions $y_{k_j}(x)$, $j=1,2,\cdots,n$ such that

$$M_{k_j} w^{j-1}_{k_{j-1}\cdots k_1}(x) = w^{j-1}_{k_{j-1}\cdots k_1}(y_{k_j}(x)) + k(x, y_{k_j}(x))$$

Put

$$\hat{\tau}_i = \hat{\tau}_{i-1} + \tilde{\tau}_{n+1-i}(\theta_{\tau_{i-1}}\omega_i), \quad \hat{\tau}_1 = \tilde{\tau}_n(\omega_1)$$

and

$$\hat{\xi}_i = y_{k_{n+1-i}}(X_{\hat{\tau}_i}(\omega_i)).$$

Then $\hat{v} = \{(\hat{\tau}_i, \hat{\xi}_i)_{i=1,2,\cdots,n},\ \hat{\tau}_{i+1} = \infty\} \in \underline{\underline{V}}_n$

and

$$w^n_{k_n\cdots k_1}(x) = E_x[\int_0^{\tilde{\tau}_n} e^{-\alpha s}dA_s + e^{-\alpha\tilde{\tau}_n} M_{k_n} w^{n-1}_{k_{n-1}\cdots k_1}(X_{\tau_n})]$$

$$= E_x[\int_0^{\tilde{\tau}_n} e^{-\alpha s}dA_s + e^{-\alpha\tilde{\tau}_n}\{w^{n-1}_{k_{n-1}\cdots k_1}(y_{k_n}(X_{\tilde{\tau}_n})) + k(X_{\tilde{\tau}_n}, y_{k_n}(X_{\tilde{\tau}_n}))\}]$$

$$= E^1_x[\int_0^{\hat{\tau}_1} e^{-\alpha s}dA_s + e^{-\alpha\hat{\tau}_1} k(X_{\hat{\tau}_1}, y_{k_n}(X_{\hat{\tau}_1}))$$

$$+ e^{-\alpha\hat{\tau}_1} E_{\hat{\xi}_1}[\int_0^{\tilde{\tau}_{n-1}} e^{-\alpha s}dA_s + e^{-\alpha\tilde{\tau}_{n-1}} M_{k_{n-1}} w^{n-2}_{k_{n-2}\cdots k_1}(X_{\tilde{\tau}_{n-1}})$$

$$= E^{n+1}_x[\int_0^{\infty} e^{-\alpha s}d\underline{\underline{A}}_s + \sum_{i=1}^{n} e^{-\alpha\hat{\tau}_i} k(X_{\hat{\tau}_i}, \hat{\xi}_i)]$$

$$\geqq u^*_n(x) \quad \text{q.e..}$$

Therefore making use of Lemma 5.3 we conclude that

$$w_n(x) \geqq u^*_n(x) \quad \text{q.e..}$$

Proof of Theorem 2. By Propositions 5.1 and 5.2 u^*_n is a quasi-continuous modification of the solution u_n of the variational inequality (1.5) for each n. Since u_n converges to the maximum solution u of the QVI (1.2) in $\mathcal{E}_\alpha$ - norm: $\mathcal{E}_\alpha(u_n-u, u_n-u) \to 0$, $n \to \infty$ we have

$$u^*_n(x) \to u(x) \quad \text{q.e.,}\ n \to \infty$$

taking a sub-sequence if necessary. On the other hand it holds that

$$u^*_n(x) \downarrow u^*(x) \quad \text{q.e.,}\ n \to \infty$$

by the next lemma. This completes the proof of Theorem 2.

Lemma 5.5. It holds that $u_N^*(x) \downarrow u^*(x)$ q.e., $N \to \infty$.

Proof of Lemma 5.5. For each $\varepsilon > 0$ there exists $v = v(x) = \{(\tau_i,\xi_i)_{i=1}^{\infty}\} \in \underline{\underline{V}}$ such that

$$u^*(x) \geqq \lim_{n\to\infty} E_x^n[\int_0^{\tau_n} e^{-\alpha s} d\underline{\underline{A}}_s + \sum_{i=1}^{n} e^{-\alpha\tau_i} k(X_{\tau_i},\xi_i)] - \varepsilon$$

So for any N it holds that

$$u^*(x) \geqq E_x^N[\int_0^{\tau_N} e^{-\alpha s} d\underline{\underline{A}}_s + \sum_{i=1}^{N} e^{-\alpha\tau_i} k(X_{\tau_i},\xi_i)] - \varepsilon$$

Put $v^N = \{(\tau_i,\xi_i)_{i=1,2,\cdots N}, \tau_{N+1} = \infty\}$, then $v^N \in \underline{\underline{V}}_N$. Therefore from

$$E_x^{N+1}[\int_0^{\tau_{N+1}} e^{-\alpha s} d\underline{\underline{A}}_s + \sum_{i=1}^{N+1} e^{-\alpha\tau_i} k(X_{\tau_i},\xi_i)] \geqq u_N^*(x)$$

and $u_N^*(x) \geqq u^*(x)$ it follows that

$$|u_N^*(x) - u^*(x)| \leqq E_x^{N+1}[\int_{\tau_N}^{\infty} e^{-\alpha s} d\underline{\underline{A}}_s] + 2\varepsilon.$$

Since $\tau_N \to \infty$ as $N \to \infty$ we obtain

$$\lim_{N\to\infty} u_N^*(x) = u^*(x) \quad \text{q.e.}.$$

$u_N^*(x) \geqq u_{N+1}^*(x)$ q.e. is obvious.

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