BRS Symmetry and Unitarity

in Kaluza-Klein Theory

—— Towards the Realistic Unified Theory ——

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Abstract

We first review fundamental ideas of Kaluza-Klein theory and discuss various problems in constructing a realistic unified theory of elementary particles and gravitation based on the Kaluza-Klein theory. In the main part of this thesis we show i) The four dimensional BRS symmetry can be obtained correctly through the Kaluza-Klein dimensional reduction from the higher dimensional BRS symmetry, and the physical S-matrix unitarity can be established in the whole quantum Kaluza-Klein theory including both massless and massive modes. ii) The gyromagnetic g-factor of massive fields in the Kaluza-Klein theory is unity, which is one of the special features of massive fields in the Kaluza-Klein theory.

Contents

§1. Introduction and Histrical background

It is known that there exist four types of interactions in nature. They are the strong interaction, the electromagnetic interaction, the weak interaction and the gravitational interaction. These four types of interactions have different sources and have different intermediate Bosons. Also they have different range of interaction and the strength of them is full of variety. We observe that these interactions present various aspects in the real world.

However these four interactions are not completely different from each other and have some universal properties. For example about the gravitational interaction and the electromagnetic interaction the strength is in proportion to inverse square of the distance, the mass of the intermediate Boson is zero and there are general invariance and gauge invariance respectively in the Einstein theory and the Maxwell theory. If we consider that these four interactions have different origin, we will have too many elementary particles, since there must be many kinds of intermediate Bosons and matter fields. Therefore various attempts have been tried to construct a unified theory of interaction under the belief that the fundamental structure of the nature must be simple and the complexity that is seen in the real world is no more than the result of combination of a few fundamental constituents*).

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*) The author has the following personal opinion about unification of interactions. Consider for example Millikan's experiment which determins the magnitude of elementary charge. In this experiment oil drops with charge are put in such an electromagnetic field that can be cancelled with the gravitational force. Let us make the situation simpler. Consider two kinds of interactions, A and B, and a point particle which is a common source of these two interactions. The reason why this point particle does not split into two parts, the source part of the interaction A and the source part of the interaction B, should be the following. The interactions A and B have the same origin in a certain sense or there is a "glue" force connecting these two parts. Even if the latter is the case, however, then we must explain why the "glue" force can connect A and B. We will have to consider "glue", A and B have the same origin or will have to introduce "glue of glue" ····· and so on. Hence interactions in nature must have infinite hierarchical structure or must be unified alternatively. However physics should clarify in what level the unification of interactions occur and what kinds of constituents are in the same level. It is quite probable that we have to consider a certain kind of composite model to explain quarks, leptons etc. as found states of the fundamental constituents. We treat this problem in §3 again.

Since Einstein constructed the theory of general relativity in 1915[1], many attempts have been made to unify the gravitational interaction and the electromagnetic interaction, since only these two interactions were known in those days. The most famous attempts of them may be the Weyl theory (1918)[2] and the Kaluza-Klein theory (1921, 1926)[3]. In the Weyle theory the electromagnetic field is described by the new degree of freedom which is introduced by the transformation (gauge transformation) of the magnitude of the line element ds^2 . In the Kaluza-Klein theory the electromagnetic field is described by the new degrees of freedom in the metric tensor which is introduced by extending the four dimensional space-time to the five dimensional space-time. Although these theories succeeded to some extent, they did not come to be studied so much, after the existence of the weak and the strong interactions was found. On the other hand it became possible to describe the electromagnetic and the weak interactions in a unified manner on the basis of the gauge theory (Yang-Mills theory)[4] which was discovered in 1954. This is the Weinberg-Salam theory (1967)[5]. In the Weinberg-Salam theory the direct product group $SU(2) \times U(1)$ is taken for the non-Abelian gauge group of Yang-Mills field, where $SU(2)$ implies the weak interaction and $U(1)$ implies the electromagnetic interaction. Extending the gauge group into larger groups than that of Weinberg-Salam theory many physicists tried to construct ground unified theories, GUTs[6], in which the strong interaction is contained as well as the weak and the electromagnetic interactions. The supersymmetric theories[7] which treat Bosons and Fermions symmetrically were constructed as extended theories of gauge theories, and supergravity[8] which is the local supersymmetric theory began to be studied since 1976. In the research of supergravity it was noticed that the extended supergravity can be obtained easily from the higher dimensional simple supergravity.[9] In this context the Kaluza-Klein theory was rivived and has been studied actively again.[10] This Kaluza-Klein theory is not the original five dimensional theory but the extended $4 + D$ ($D \geq 2$) dimensional theory[11] which can unify the Einstein theory and the Yang-Mills theory. Though the extra dimensional space was considered as merely a mathematical tool when the original Kaluza-Klein theory was proposed, it is now considered as a physical object in recent literatures. The background of this change of interpretation is the idea of so called spontaneous compactification[12] that the direct product of four dimensional space-time and compact manifold can be obtained as the ground state solution of the field equation in the higher dimensional space-time. So as to explain that the extra space is not observed in nature the magnitude of that space is considered to be of the order of the Planck length. Hence the effect of the extra space could be observed only by the massive modes as heavy as the Planck mass.[13]

If the extra space really exists, it seems that the quantization has to be carried out in the higher dimensional space-time. On that occasion we must examine the consistency between the four dimensional quantum theory and the higher dimensional quantum theory. As is well known the Faddeev-Popov ghost fields[14] must be introduced to assure the physical S-matrix unitarity in the quantization of gauge fields such as the gravitational field or the Yang-Mills fields. In order for us to establish the unitarity the BRS symmetry[15] plays a crucial role.[16] In this thesis first we shall show that the four dimensional BRS symmetry can be obtained correctly through the Kaluza-Klein dimensional reduction from the higher dimensional BRS symmetry.[17][18][19][20] Next we shall derive the BRS symmetry of massive tensor fields through the dimensional reduction technique and establish the physical S-matrix unitarity in the whole quantum Kaluza-Klein theory including both massless and massive modes.[21] On the other hand the massive modes should be examined for the possible test of the Kaluza-Klein theory. Therefore it may be meaningful to find some special features of them. In this thesis we shall also show that the gyromagnetic g-factor of massive fields in the Kaluza-Klein theory is unity, which is one of these special features.[22]

In §2 we review the fundamental ideas of the Kaluza-Klein theory. First we explain the five dimensional theory and in the second place we illustrate the 4 + D ($D \geq 2$) dimensional theory.

In §3 we overview problems in building a realistic model. The problem of cosmological constant and the stability problems are also treated in this section.

In §4 we treat the problems in quantization of the Kaluza-Klein theory. We examine the consistency between the four dimensional BRS symmetry and the higher dimensional BRS symmetry and also examine that of the extended BRS symmetry. Constructing a unitary model of massive spin 2 fields through the dimensional reduction technique we establish the physical S-matrix unitarity of the whole quantum Kaluza-Klein theory.

In §5 we consider the problem how to test the Kaluza-Klein theory and also make comments on other problems in this theory. In particular we compute the gyromagnetic g-factor of massive fields in the Kaluza-Klein theory.

§6 is devoted to summary.

§2. Fundamental ideas in the Kaluza-Klein theory

We review the five dimensional theory in §2-1 and the 4 + D (D ≥ 2) dimensional theory in §2-2.

§2-1 The Kaluza-Klein theory in the 5-dimensional space-time

2-1-1 The Kaluza-Klein ansatz

In the Einstein theory of general relativity the gravitational field is considered as a geometrical object under the principle of general relativity and the principle of equivalence. In order to treat also the electromagnetic field geometrically let us consider the theory of general relativity in the five dimensional space-time. Take the five dimensional Einstein Lagrangian that is

$$\mathcal{L} = \sqrt{-\overset{(5)}{g}}\ \overset{(5)}{R} \tag{2-1}$$

(cf. the appendix A). If we impose no restriction on the theory, we have merely the five dimensional gravitational theory and we cannot tell which degrees of freedom correspond to the electromagnetic field. Let us assume the following ansatz on the five dimensional metric tensor $\overset{(5)}{g}_{MN}$:

$$\overset{(5)}{g}_{MN}(x, x_5) = \begin{pmatrix} g_{\mu\nu}(x) + e^2\ell^2 A_\mu(x) A_\nu(x) & e\ell^2 A_\mu(x) \\ e\ell^2 A_\nu(x) & \ell^2 \end{pmatrix} \tag{2.2}$$

with M,N = 0, 1, 2, 3, 5 and μ, ν = 0, 1, 2, 3. This assumption is called the ansatz of dimensional reduction.*)

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*) This ansatz is based on the idea that the five dimensional geometry is the direct product ($M^4 \times S^1$) of Minkowski space-time (M^4) and a circle (S^1) as we can subsequently see.

(cf. §2-1-4 and §2-1-6) In eq. (2.2) e and ℓ are parameters which denote the charge and the size of the fifth dimensional space, respectively. The fields $g_{\mu\nu}(x)$, $A_\mu(x)$ are functions of $x^0 \sim x^3$ and do not depend on x^5. We interpret that $g_{\mu\nu}(x)$ is the four dimensional gravitational field and $A_\mu(x)$ is the electromagnetic field. This interpretation is based on the following reason. The five dimensional connection $\overset{(5)}{\Gamma}{}^P{}_{MN}$ is calculated from the ansatz (2-2) as

$$
\begin{aligned}
&\overset{(5)}{\Gamma}{}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \frac{e^2\ell^2}{2}(A_\mu F_\nu{}^\lambda + A_\nu F_\mu{}^\lambda) \qquad ; \\
&\overset{(5)}{\Gamma}{}^\lambda_{\mu 5} = \frac{e\ell^2}{2} F_\mu{}^\lambda \quad , \quad \overset{(5)}{\Gamma}{}^\lambda_{55} = 0 \qquad , \\
&\overset{(5)}{\Gamma}{}^5_{\mu\nu} = -2A_\rho \Gamma^\rho_{\mu\nu} - \frac{e^3\ell^2}{2} A^\rho (A_\mu F_{\nu\rho} + A_\nu F_{\mu\rho}) + \frac{e}{2}(\partial_\mu A_\nu + \partial_\nu A_\mu) , \qquad (2\text{-}3) \\
&\overset{(5)}{\Gamma}{}^5_{\mu 5} = \frac{e^2\ell^2}{2} A^\rho F_{\rho\mu} \quad , \quad \overset{(5)}{\Gamma}{}^5_{55} = 0 \qquad ,
\end{aligned}
$$

with $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. The five dimensional Ricci tensor $\overset{(5)}{R}_{MN}$ reads

$$
\begin{aligned}
\overset{(5)}{R}_{\mu\nu} = {} & R_{\mu\nu} + \frac{e^2\ell^2}{2}(A_\mu \partial_\lambda F_\nu{}^\lambda + A_\nu \partial_\lambda F_\mu{}^\lambda - F_\mu{}^\lambda F_{\nu\lambda}) + \frac{e^4\ell^4}{4} A_\mu A_\nu F_{\rho\sigma} F^{\rho\sigma} \\
& + \frac{e^2\ell^2}{2}\{\Gamma^\lambda_{\lambda\tau}(A_\mu F_\nu{}^\tau + A_\nu F_\mu{}^\tau) - F_\lambda{}^\tau (A_\mu \Gamma^\lambda_{\nu\tau} + A_\nu \Gamma^\lambda_{\mu\tau})\} \quad , \\
\overset{(5)}{R}_{\mu 5} = {} & \frac{e\ell^2}{2}\partial_\lambda F_\mu{}^\lambda + \frac{e\ell^2}{2}(F_\mu{}^\tau \Gamma^\lambda_{\lambda\tau} - F_\tau{}^\lambda \Gamma^\tau_{\lambda\mu}) + \frac{e^3\ell^4}{4} A_\mu F_{\rho\sigma} F^{\rho\sigma} \quad , \qquad (2.4) \\
\overset{(5)}{R}_{55} = {} & \frac{e^2\ell^4}{4} F_{\rho\sigma} F^{\rho\sigma} \quad .
\end{aligned}
$$

We obtain the scalar curvature $\overset{(5)}{R}$:

$$\overset{(5)}{R} = R - \frac{e^2\ell^2}{4} F_{\mu\nu}F^{\mu\nu} \quad , \qquad (2\text{-}5)$$

and the Lagrangian $\mathcal{L}$:

$$\mathcal{L} = \sqrt{-\overset{(5)}{g}}\,\overset{(5)}{R} = \ell\sqrt{-g}\left(R - \frac{e^2\ell^2}{4} F_{\mu\nu}F^{\mu\nu}\right) \quad , \qquad (2\text{-}6)$$

Here we find that the first term is the Einstein Lagrangian and the second term is the Maxwell Lagrangian.[3] It is a remarkable fact that we have the correct sign of the Maxwell Lagrangian relatively to that of the Einstein Lagrangian. Hence we may be able to regard $g_{\mu\nu}(x)$ as the gravitational field and $A_\mu(x)$ as the electromagnetic field.

2-1-2 Interpretation of the extra dimensional space

Let us determine the parameter ℓ which expresses the size of the fifth dimensional space. We assume that the fifth dimensional space is compact and in particular it is a circle S^1 with $0 \leq x_5 < 2\pi$. To obtain the correct four dimensional Einstein-Maxwell action we choose the normalization of the five dimensional action to be

$$S = \frac{1}{32\pi^2 G\ell}\int d^4x \int_0^{2\pi} dx_5 \sqrt{-\overset{(5)}{g}}\,\overset{(5)}{R} \quad , \qquad (2.7)$$

where G is the Newtonian constant. Employing eq. (2-6) we obtain

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,R - \left(\frac{e^2\ell^2}{16\pi G}\right)\times\frac{1}{4}\int d^4x\sqrt{-g}\,F_{\mu\nu}F^{\mu\nu} \quad . \qquad (2.8)$$

In order to have the standard four dimensional Maxwell action it is necessary to put $\frac{e^2 \ell^2}{16\pi G} = 1$ i.e. $\ell = \sqrt{\frac{16\pi G}{e^2}}$ or $\frac{1}{\ell} = \frac{\sqrt{\alpha}}{2} m_{pl}$, where $\alpha = \frac{e^2}{4\pi}$ and m_{pl} are the fine structure constant and the Planck mass ($\sim 10^{19}$ GeV), respectively. Therefore, if α is the order of unity,*) the size of the fifth dimensional space is as small as the Planck length. This makes it possible to consider the extra dimensional space as a physical object in spite of the fact that our real world is observed to be a four dimensional space-time. Therefore we can make the interpretation that the extra dimensional space does exist but cannot be observed as long as the super high energy experiment or cosmological observation with the energy of the Planck mass is not performed.

2-1-3 Gauge transformation

In this section we examine how the gauge transformation of the electromagnetic field is explained in this Kaluza-Klein theory. The ansatz of dimensional reduction (2-2) is nothing but an assumption of a special dependence on the coordinates x^μ, x^5 of the five dimensional metric tensor. (cf. §2-1-4) If we perform a general transformation in the five dimensional space-time, this ansatz cannot be preserved. In other words this ansatz does not respect the five dimensional principle of general relativity. Let us look for the special five dimensional coordinate transformation that preserves the ansatz (2-2). We write the five dimensional coordinate transformation as

$$x^M \to x'^M = x^M + \delta x^M \quad , \quad \delta x^M \equiv \epsilon^M \tag{2-9}$$

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*) To determine the magnitude of the parameter e we must consider the electro-magnetic interaction of charged matter fields. (cf. eqs. (2-27,38)

As $\overset{(5)}{g}_{MN}$ is a tensor, it is transformed like

$$\begin{aligned}\delta \overset{(5)}{g}_{MN}(x) &\equiv \overset{(5)}{g}{}'_{MN}(x) - \overset{(5)}{g}_{MN}(x) \\ &= -\partial_M \epsilon^P \cdot \overset{(5)}{g}_{PN} - \partial_N \epsilon^P \cdot \overset{(5)}{g}_{MP} - \epsilon^P \partial_P \overset{(5)}{g}_{MN}\end{aligned} \tag{2-10}$$

We are searching for such ϵ^M that preserves the ansatz that is

$$\overset{(5)}{g}{}'(x', x'_5) = \begin{pmatrix} g'_{\mu\nu}(x') + e^2 \ell^2 A'_\mu(x') A'_\nu(x') & e\ell^2 A'_\mu(x') \\ e\ell^2 A'_\nu(x') & \ell^2 \end{pmatrix}$$

We see that the infinitesimal change of $\overset{(5)}{g}_{55}$ is

$$\delta \overset{(5)}{g}_{55} = -\partial_5 \epsilon^P \cdot \overset{(5)}{g}_{P5} - \partial_5 \epsilon^P \cdot \overset{(5)}{g}_{5P} - \epsilon^P \partial_P \overset{(5)}{g}_{55} = -2\partial_5 \epsilon^P \cdot \overset{(5)}{g}_{P5} \quad .$$

On the other hand the consistency of the ansatz demands $\delta \overset{(5)}{g}_{55} = \delta \ell^2 = 0$. Therefore it is needed that $\partial_5 \epsilon^P = 0$ i.e. ϵ^P does not depend on x^5 . In this case we obtain from eqs. (2-10) and (2-2) that

$$\delta g_{\mu\nu} = -\partial_\mu \epsilon^\lambda \cdot g_{\lambda\nu} - \partial_\nu \epsilon^\lambda \cdot g_{\mu\lambda} - \epsilon^\lambda \partial_\lambda g_{\mu\nu} \quad , \tag{2-11}$$

$$\delta A_\mu = -\partial_\mu \epsilon^\lambda \cdot A_\lambda - \epsilon^\lambda \partial_\lambda A_\lambda - \frac{1}{e} \partial_\mu \epsilon^5 \quad , \tag{2-12}$$

$$\delta g_{55} = 0 \quad . \tag{2-13}$$

The equation (2-11) implies the four dimensional coordinate transformation of $g_{\mu\nu}$, and the last term of (2-12) implies the gauge transformation of A_μ . We have seen that the gauge transformation of the vector potential A_μ

is described by a special five dimensional coordinate transformation of the metric tensor.

2-1-4 Extended ansatz including a scalar field

In this section we examine the meaning of the ansatz (2-2). For notational simplicity we take the system of unit: $l = 1$. Rewrite the general five dimensional metric tensor as

$$\begin{aligned} \overset{(5)}{g}_{\mu\nu}(x,x_5) - \overset{(5)}{g}_{5\mu}(x,x_5)\,\overset{(5)}{g}_{5\nu}(x,x_5) &\equiv g_{\mu\nu}(x,x_5) , \\ \overset{(5)}{g}_{5\mu}(x,x_5) &\equiv e A_\mu(x,x_5) , \\ \overset{(5)}{g}_{55}(x,x_5) &\equiv \varphi^2(x,x_5) . \end{aligned} \tag{2-14}$$

That is

$$\overset{(5)}{g}_{MN} = \begin{pmatrix} g_{\mu\nu} + e^2 A_\mu A_\nu & e A_\mu \\ e A_\nu & \varphi^2 \end{pmatrix} .$$

We notice that the ansatz (2-2) is equal to the definition (2-14) with the restriction that $g_{\mu\nu}$ and A_μ do not depend on x_5 and $\varphi = 1$, in other words, $\overset{(5)}{g}_{\mu\nu}$ and $\overset{(5)}{g}_{5\mu}$ do not depend on x_5 and $\overset{(5)}{g}_{55} = 1$.

Let us extend the ansatz (2-2) in order for $\overset{(5)}{g}_{55}$ to depend on x^μ. This is equivalent to the assumption that each component of $\overset{(5)}{g}_{MN}$ does not depend on x_5 but depends on x^μ. It is convenient to work with the fünfbein $\overset{(5)}{E}_M{}^A$ which is more fundamental than the metric tensor $\overset{(5)}{g}_{MN}$. The ansatz (2-2) can be rewritten with the fünfbein as

$$\overset{(5)}{E}{}_M{}^A(x,x_5) = \begin{pmatrix} E_\mu{}^\alpha(x) & eA_\mu(x) \\ 0 & 1 \end{pmatrix} , \tag{2-15}$$

$$\overset{(5)}{g}_{MN} = \overset{(5)}{E}{}_M{}^A \overset{(5)}{E}{}_N{}^B \eta_{AB} \quad , \quad \eta_{AB} \equiv diag(-++++) \quad ,$$

where M , μ are the indices of the world coordinate and A , α are the indices of the inertial frame. Let us assume the following extended ansatz:

$$\overset{(5)}{E}{}_M{}^A(x,x_5) = \begin{pmatrix} \varphi^r(x) E_\mu{}^\alpha(x) & e\,\varphi^s(x) A_\mu(x) \\ 0 & \varphi^t(x) \end{pmatrix} . \tag{2-16}$$

In this case the metric tensor reads

$$\overset{(5)}{g}_{MN}(x,x_5) = \begin{pmatrix} \varphi^{2r} g_{\mu\nu} + e^2 \varphi^{2s} A_\mu A_\nu & e\,\varphi^{s+t} A_\mu \\ e\,\varphi^{s+t} A_\nu & \varphi^{2t} \end{pmatrix} . \tag{2-17}$$

Provided we assume (2-16) in place of (2-2), the Lagrangian $\mathcal{L} = \sqrt{-\overset{(5)}{g}}\,\overset{(5)}{R}$ becomes

$$\begin{aligned} \mathcal{L} = \sqrt{-g}\,[\, & \varphi^{2r+t} R \\ & - \frac{e^2}{4}\{ \varphi^{2s+t} F_{\mu\nu} F^{\mu\nu} + 2(s-t)\varphi^{2s+t-1} F^{\mu\nu}(A_\nu \partial_\mu \varphi - A_\mu \partial_\nu \varphi) \\ & \quad + (s-t)^2 \varphi^{2s+t-2} (A_\nu \partial_\mu \varphi - A_\mu \partial_\nu \varphi)^2 \} \\ & + 6r(r+t)\varphi^{2r+t-2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \quad] . \end{aligned} \tag{2-18}$$

To obtain this equation we have used the fact that the metric tensor (2-17) is obtained by replacing $\overset{(5)}{g}_{MN}$ in the ansatz (2-2) by $\varphi^{2t}\,\overset{(5)}{g}_{MN}$ and also by redefining $A_\mu \to A_\mu \varphi^{s-t}$, $g_{\mu\nu} \to g_{\mu\nu} \varphi^{2r-2t}$, and moreover we have used the formula (A-24) of the conformal transformation in Appendix A. We notice that the coupling between the electromagnetic field A_μ and the scalar field φ becomes simple, if we set $s-t=0$ and $2s+t=1$. The Einstein Lagrangian can be obtained by setting $2r+t=0$. Then we have

$$\mathcal{L}=\sqrt{-g}\left[\,R-\frac{e^2}{4}\varphi F_{\mu\nu}F^{\mu\nu}-\frac{1}{6}g^{\mu\nu}\frac{\partial_\mu\varphi\partial_\nu\varphi}{\varphi^2}\,\right] \quad . \tag{2-19}$$

In an alternative case that $s-t=0$ and $2r+t=1$ we have

$$\mathcal{L}=\sqrt{-g}\left[\,\varphi R-\frac{e^2}{4}\varphi^{3s}F_{\mu\nu}F^{\mu\nu}+\frac{3}{2}(1-s^2)g^{\mu\nu}\frac{\partial_\mu\varphi\partial_\nu\varphi}{\varphi^2}\right] \quad . \tag{2-20}$$

In this case the scalar field can be identified with the Jordan-Brans-Dicke field[23].

As we have seen above, if we allow $\overset{(5)}{g}_{55}$ to depend on x^μ, we have a scalar field which couples with the gravitational field and the electromagnetic field. This scalar field can be used to plausibly explain the contraction of the extra dimensional space (cf. §3-6).

2-1-5 Harmonic expansion and massive modes

As we have seen in the previous section, the ansatz of dimensional reduction (2-2) is to restrict the dependence on the coordinates x^μ, x^5 of the five dimensional metric tensor $\overset{(5)}{g}_{MN}$. Now let us examine small perturbation which depends on x^5 around the ansatz (2-2). We will have a series of massive modes as follows.[13)22)] Let us decompose the five dimensional metric tensor $\overset{(5)}{g}_{MN}$ into the Kaluza-Klein background $\bar{g}_{MN}$ and perturbation h_{MN} as

$$\overset{(5)}{g}_{MN}(x,x_5) = \bar{g}_{MN}(x) + h_{MN}(x,x_5) \qquad , \quad (2\text{-}21)$$

with $\bar{g}_{MN}(x) = \begin{pmatrix} g_{\mu\nu}(x) + e^2\ell^2 A_\mu(x)A_\nu(x) & e\ell^2 A_\mu(x) \\ e\ell^2 A_\nu(x) & \ell^2 \end{pmatrix}$.

Since plhysical particles should be defined in the local inertial frame of the five dimensional space-time, we take this inertial frame. Define the field h_{AB} as

$h_{AB} \equiv \bar{E}_A{}^M \bar{E}_B{}^N h_{MN}$. Now the harmonic expansion (Fourier expansion) of the field h_{AB} can be written as

$$h_{AB}(x,x_5) = h^{(0)}_{AB}(x) + \sum_{n=1}^{\infty}\left\{ e^{inx_5} h^{(n)}_{AB}(x) + h.c. \right\} \quad . \quad (2\text{-}22)$$

Here $h^{(0)}_{AB}(x)$ is the massless mode and $h^{(n)}_{AB}(x)$ $(n \neq 0)$ are the massive modes, as we shall see below. So that we may see this, let us write down the equation of motion which is in the first order of the field h_{AB} . The corresponding bilinear terms of h_{AB} in the five dimensional Einstein Lagrangian $\mathcal{L} = \sqrt{-\overset{(5)}{g}}\,\overset{(5)}{R}$ are (cf. eq. (A-22))

$$\mathcal{L}_{(2)} = \sqrt{-\bar{g}}\,\Big[\tfrac{1}{2}\bar{\nabla}_A h_B{}^C \bar{\nabla}_C h^{AB} - \tfrac{1}{2}\bar{\nabla}_A h_B{}^B \bar{\nabla}_C h^{AC} + \tfrac{1}{4}\bar{\nabla}_A h_B{}^B \bar{\nabla}^A h_C{}^C - \tfrac{1}{4}\bar{\nabla}_A h_{BC} \bar{\nabla}^A h^{BC} \Big] \quad , \quad (2\text{-}23)$$

where $\bar{\nabla}_A \equiv \bar{E}_A{}^M \bar{\nabla}_M$ and $\bar{\nabla}_M$ is the covariant derivative with respect to the background $\bar{g}_{MN}$. So as to extract the physical modes we impose the unitary gauge condition $h^{(n)}_{5A} = 0 \ (n \neq 0)$. To make the calculation simpler we use the weak field approximation that the background four dimensional space-time is flat ($\Gamma \sim 0$) and the electromagnetic field is weak $A^2 \sim 0$, $\partial F \sim 0$. Then the coefficient of e^{inx_5} in the linearized field equation*) is written as

$$D_\gamma D^\gamma h^{(n)}_{\alpha\beta} - \frac{n^2}{\ell^2} h^{(n)}_{\alpha\beta} - 2D_{(\alpha} D^\gamma h^{(n)}_{\beta)\gamma} + 2ien F^\gamma{}_\alpha h^{(n)}_{\beta)\gamma}$$
$$+ D_{(\alpha} D_{\beta)} h^{(n)} + \eta_{\alpha\beta} (D^\gamma D^\delta h^{(n)}_{\gamma\delta} - D_\gamma D^\gamma h^{(n)} + \frac{n^2}{\ell^2} h^{(n)}) = 0 , \tag{2-24}$$

$$D^\gamma h^{(n)}_{\alpha\gamma} - \frac{ie\ell^2}{n} F^{\gamma\delta} D_\gamma h^{(n)}_{\alpha\delta} = 0 , \tag{2-25}$$

$$h^{(n)} \equiv h^{(n)}{}_\alpha{}^\alpha = 0 , \tag{2-26}$$

with $D_\rho \equiv \partial_\rho - ienA_\rho$. Here the symmetrization means e.g. $D_{(\alpha} D_{\beta)} = \frac{1}{2}(D_\alpha D_\beta + D_\beta D_\alpha)$. Using (2-25) and (2-26) we rewrite the equation (2-24) as

$$D_\gamma D^\gamma h^{(n)}_{\alpha\beta} - \frac{n^2}{\ell^2} h^{(n)}_{\alpha\beta} + 2ien F^\gamma{}_{(\alpha} h^{(n)}_{\beta)\gamma}$$
$$- \frac{2ie\ell^2}{n} F^{\gamma\delta} D_{(\alpha|} D_\gamma h^{(n)}_{|\beta)\delta} = 0 . \tag{2-27}$$

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*) The equation (2-24) happens to be identical to that appearing in ref. 24), where Velo and Zwanzinger claimed that six degrees of freedom are contained in this equation. However in our case the other two constraints (2-25), (2-26) allow us to describe a spin two particle correctly.

The first term is the kinetic term including the minimal coupling with the electromagnetic field, the second term is the mass term and the rest terms are nonminimal interactions with the electromagnetic field. If we define the charge of the massive tensor field $h^{(n)}_{\alpha\beta}$ to be q, the minimal coupling D_ρ should be written as $\partial_\rho - i\, q A_\rho$. Hence we notice $q = en$, that is the charge is multiplicative of e with an integer n. From eq. (2-27) we find that the mass of the field $h^{(n)}_{\alpha\beta}$ is $m = \frac{|n|}{\ell} = |n| \cdot \frac{\sqrt{\alpha}}{2} m_{Pl}$. Namely the mass of the tensor field $h^{(n)}_{\alpha\beta}$ is in the order of the Planck mass. Therefore it is possible to explain that the reason why such massive tensor fields are not observed in nature is because the mass of them is too heavy than the energy in present experiments.

By the way we notice that both the charge and the mass include the common integer n. This n comes from the differential operator ∂_5 in the field equation. At first sight it is curious that the charge and the mass come from the same origin, but this seems natural in the following consideration. Since in the Kaluza-Klein theory we start from the five dimensional theory of general relativity, the electromagnetic field is also based on the curvature of the (five dimensional) space-time. Hence it is natural that the charge which is the source of the electromagnetic field has the same origin as the mass which is the source of the gravitational field.

2-1-6 Ground state solution and the meaning of the Kaluza-Klein ansatz

In the previous section we have examined the excited modes in the background of the Kaluza-Klein ansatz. However the ansatz (2-2) is not a solution of the five dimensional Einstein equation, because eq. (2-4) shows $\overset{(5)}{R}_{55} = \frac{e^2 \ell^4}{4} F_{\rho\sigma} F^{\rho\sigma} \neq 0$ and this contradicts with the Einstein equation $\overset{(5)}{R}_{MN} = 0$. In this section let us examine what the ansatz (2-2) really means. Consider the following ground state solution of the five dimensional Einstein equation:

$$\bar{g}_{MN} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \ell^2 \end{pmatrix} , \tag{2-28}$$

which trivially satisfies the five dimensional Einstein equation $\overset{(5)}{R}_{MN} = 0$. Here the fifth space is assumed to be a circle with $0 \le x_5 < 2\pi$. In the same way as the previous section we decompose the field $\overset{(5)}{g}_{MN}$ into the ground state $\bar{g}_{MN}$ and perturbation h_{MN} ,

$$\overset{(5)}{g}_{MN}(x, x_5) = \bar{g}_{MN} + h_{MN}(x, x_5) . \tag{2-29}$$

Suppose we perform the harmonic expansion of h_{AB} in a similar way to the equation (2-22), we find that $h^{(0)}_{AB}$ is a massless field and $h^{(n)}_{AB}$ $(n \neq 0)$ are massive fields. Adding the zero mode to the ground state $\bar{g}_{MN}$ we have

$$\overset{(5)}{g}_{MN}(x, x_5) \simeq \begin{pmatrix} \eta_{\mu\nu} + h^{(0)}_{\mu\nu}(x) & h^{(0)}_{\mu 5}(x) \\ h^{(0)}_{5\nu}(x) & \ell^2 + h^{(0)}_{55}(x) \end{pmatrix} . \tag{2-30}$$

This expression is equivalent to the one when we assume that each component of $\overset{(5)}{g}_{MN}$ does not depend on x^5 but depends on x^μ . In this case, as we examined in the section 2-1-4, we obtain the Kaluza-Klein Lagrangian including a scalar field from the five dimensional Einstein Lagrangian. In other words we find that the Kaluza-Klein ansatz with a scalar field is nothing but the zero mode of the harmonic expansion around the ground state solution and the Kaluza-Klein Lagrangian is the low energy effective Lagrangian which contains only the zero mode.[13)]

2-1-7 Spinor field

So far we have treated Bose fields only, but there are also Fermion fields in nature. In this section we explain how to treat a spinor field in the Kaluza-Klein framework.[13)22)] (cf. §3-1)

In order to treat a spinor field we have only to add the following Dirac Lagrangian[*)] by hand to the Einstein Lagrangian in the five dimensional space-time:

$$\mathcal{L}_\psi = i\sqrt{-\overset{(5)}{g}(x,x_5)}\;\overset{(5)}{\bar{\psi}}(x,x_5)\,\overset{(5)}{\Gamma}{}^A\,\overset{(5)}{E}{}_A{}^M(x,x_5)\,\overset{(5)}{\nabla}_M\,\overset{(5)}{\psi}(x,x_5) \quad , \tag{2-31}$$

$$\overset{(5)}{\nabla}_M \equiv \partial_M + \frac{1}{8}\overset{(5)}{B}_{M[AB]}\,[\overset{(5)}{\Gamma}{}^A, \overset{(5)}{\Gamma}{}^B] \quad , \tag{2-32}$$

$$\begin{aligned}\overset{(5)}{B}_{A[BC]} \equiv{}& \overset{(5)}{E}{}_{[A}{}^M \overset{(5)}{E}{}_{B]}{}^N \partial_M \overset{(5)}{E}{}_{NC} - \overset{(5)}{E}{}_{[B}{}^M \overset{(5)}{E}{}_{C]}{}^N \partial_M \overset{(5)}{E}{}_{NA} \\ &+ \overset{(5)}{E}{}_{[C}{}^M \overset{(5)}{E}{}_{A]}{}^N \partial_M \overset{(5)}{E}{}_{NB} \quad .\end{aligned} \tag{2-33}$$

(See Appendix A) Here the spinor field $\overset{(5)}{\psi}$ has four components and $\overset{(5)}{\Gamma}{}^A$ satisfies the five dimensional Clifford algebra

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*) The bare mass for the Dirac particle in the five dimensional space-time causes CP-violation after the dimensional reduction.

$\{ \overset{(5)}{\Gamma}{}^A , \overset{(5)}{\Gamma}{}^B \} = 2\,\eta^{AB}$. We can utilize the four dimensional Dirac matrix γ^0, γ^1, γ^2, γ^3, $\gamma^5 (\equiv i\gamma^0\gamma^1\gamma^2\gamma^3)$ for $\overset{(5)}{\Gamma}{}^A$. After the dimensional reduction with the Kaluza-Klein ansatz of the fünfbein $\overset{(5)}{E}{}_M{}^A = \begin{pmatrix} E_\mu{}^\alpha & e\ell A_\mu \\ 0 & \ell \end{pmatrix}$ the spinor Lagrangian (2-31) becomes

$$\mathcal{L}_\psi = i\ell\sqrt{-g}\ \overset{(5)}{\bar{\psi}} \{ \gamma^\alpha (\nabla_\alpha - eA_\alpha \partial_5) + \frac{1}{\ell}\gamma^5 \partial_5 + \frac{e\ell}{16} F_{\alpha\beta} [\gamma^\alpha, \gamma^\beta] \gamma^5 \} \overset{(5)}{\psi} \tag{2-34}$$

When we perform the harmonic expansion of $\overset{(5)}{\psi}$:

$$\overset{(5)}{\psi}(x, x_5) = \psi^{(0)}(x) + \sum_{n=1}^{\infty} \{ e^{inx_5} \psi^{(n)}(x) + h.c. \} \tag{2-35}$$

we find that the second term in eq. (2-34) which corresponds the mass term includes the matrix γ_5 . Performing the chiral transformation,

$$\overset{(5)}{\psi} \to e^{\frac{i\pi}{4}\gamma_5} \overset{(5)}{\psi} \quad , \quad \overset{(5)}{\bar{\psi}} \to e^{-\frac{i\pi}{4}\gamma_5} \overset{(5)}{\bar{\psi}} \tag{2-36}$$

to obtain the ordinary mass term, we have

$$\mathcal{L}_\psi = i\ell\sqrt{-g}\ \overset{(5)}{\bar{\psi}} \{ \gamma^\alpha (\nabla_\alpha - eA_\alpha \partial_5) + \frac{i}{\ell} \partial_5 + \frac{ie\ell}{16} F_{\alpha\beta} [\gamma^\alpha, \gamma^\beta] \} \overset{(5)}{\psi} \tag{2-37}$$

In the equation of motion the coefficient of e^{inx_5} can be written as

$$\{\gamma^\alpha(\nabla_\alpha - ienA_\alpha) - \frac{n}{\ell} + \frac{ie\ell}{16} F_{\alpha\beta}[\gamma^\alpha, \gamma^\beta]\}\psi^{(n)} = 0 \quad . \tag{2-38}$$

The first term is the kinetic term including the minimal coupling with the gravitational field and the electromagnetic field. The second term is the mass term and the third term is the Pauli term. From eq. (2-38) we find that the charge and the mass of the spinor field $\psi^{(n)}$ is $q = en$ and $m = \frac{|n|}{\ell} = |n| \cdot \frac{\sqrt{\alpha}}{2} m_{Pl}$, respectively. This means that the mass of the spinor field $\psi^{(n)}$ is in the order of the Planck mass, hence we cannot assign the field $\psi^{(n)}$ directly to a ordinary particle (electron etc.). We face this problem again in §3-3.

§2-2 The Kaluza-Klein theory in the (4+D)-dimensional space-time

This section is devoted to explain the 4+D (D≥2) dimensional extended theory which unifies the gravitational field and the Yang-Mills field.

2-2-1 Spontaneous compactification

Let us first recall the five dimensional Kaluza-Klein theory reviewed in the previous section. The program was as follows: 1) Seek for the ground state solution of the five dimensional action with a compact extra space 2) Perform the harmonic expansion for the quantum fluctuation field with respect to the extra coordinate 3) Extract the zero mode of the expansion to obtain the Kaluza-Klein ansatz and the low energy effective Lagrangian 4) Calculate the massive fields from the nonzero modes of the expansion. For the (4+D)-dimensional theory we have only to take the same procedure as above. In this case, however, the procedure becomes more complicated owing to the fact that there are many ways to construct a theory with different shape of internal spaces. The extra space should be compact and have the magnitude as small as the Planck length. Provided the extra space is for example the D-dimensional torus T^D and has a symmetry of $[U(1)]^D$, the ground state solution can be written as

$$\overset{(4+D)}{g}_{MN}(x, y) = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \delta_{mn} \end{pmatrix} , \tag{2-39}$$

which is clearly a solution of the (4+D)-dimensional Einstein equation. Here x and y denote the four and internal D dimensional coordinates, respectively. As a more attractive case if we choose another extra space which has a larger symmetry that is for example the D-dimensional sphere S^D ($\sim SO(D+1)/SO(D)$) with a symmetry of $SO(D+1)$, then we will have to assume

$$\overset{(4+D)}{g}_{MN}(x, y) = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & g_{mn}(y) \end{pmatrix} \tag{2-40}$$

with g_{mn} being the metric tensor on S^D. However we immediately find that this is not a solution of the (4+D)-dimensional Einstein equation. Hence in order to obtain a solution which has a compact internal space with a large symmetry we must start from such a (4+D)-dimensional action that contains some matter field (for example extra gauge fields) in addition to the metric tensor field. Alternatively we need the cosmological term. This complication seems to be contrary to the motivation of the Kaluza-Klein theory that the gauge fields are described by some components of the metric tensor. But this is not necessarily nonsense, since for example the eleven dimensional N = 1 supergravity has an extra gauge field A_{MNP} in the eleven dimensional action.[9] Therefore it came to be studied actively to obtain a solution with a compact internal space by means of adding various additional actions to the Einstein action in the higher dimensional space-time. This procedure of finding a solution of compact internal space is called spontaneous compactification.[12] The solution has in general smaller symmetry than the (4+D)-dimensional action. This resembles the spontaneous symmetry breaking in the Higgs-Kibble mechanism. It may be an interesting problem to clarify in the spontaneous compactification what is the counterpart of the Goldstone boson in the Higgs-Kibble mechanism. In the next section we consider how to assign the ground state in general cases.

2-2-2 Finding the ground state

We expect the ground state to be the direct product $M^4 \times B$ of the Minkowski space M^4 and a compact manifold B. There are many possibilities in the way of taking the compact manifold B. If we expect the gauge field which comes from the higher dimensional metric tensor to have a symmetry described by a group G, we have two possibilities. In one case we take the group G itself (group manifold) for the internal space and in another case we take the coset G/H (homogeneous space) which is the manifold obtained by deviding the group G by its subgroup

H. (cf. the appendices B and C) It is known that the theory has the symmetry G $\times$ G in the groupmanifold case and it has the symmetry G $\times$(N/H) with N being the normalizer of the subgroup H in the homogeneous space case.[25] (See the appendices B,C) The necessary dimension of internal space to make the theory have a symmetry G is dim G in the case of a groupmanifold and is dim G-dim H in the case of a homogeneous space. For a given dimensionality of space-time a larger symmetry can be obtained in the latter case than in the former case. As the dimension of the space-time is considered to be less than or equal to eleven in supergravity, which we explain in §3-2, the economical type of theories with a homogeneous space are more often used than the theories with a groupmanifold. We might also consider a theory with a general manifold for the internal space. Let us see the way how to assign the ground state in both cases, G and G/H.

Consider the (4+D)-dimensional vielbein rather than the metric tensor in order to treat Fermion fields, too. With the parenthesis $\langle\ \rangle$ representing the vacuum expectation value we assume

$$\langle \overset{(4+D)}{E}{}_M{}^A(x,y)\rangle = \begin{pmatrix} \delta_\mu{}^\alpha & 0 \\ 0 & E_m{}^a(y) \end{pmatrix} \tag{2-41}$$

Here $E_m{}^a(y)$ is the vielbein which makes the metric tensor on the manifold B. In other words we assume there exists a (4+D)-dimensional theory which has the solution (2-41). Since the ground state ansatz is expected to be invariant under the extra dimensional coordinate transformation $y \to y'$,[13] the vielbein $\langle E_M{}^A \rangle$ should satisfy

$$\langle \overset{(4+D)}{E}{}'_M{}^A(x, y') \rangle = \begin{pmatrix} \delta_\mu{}^\alpha & 0 \\ 0 & E_m{}^a(y') \end{pmatrix} \quad . \tag{2-42}$$

This means that the vielbein $\langle E_m{}^a \rangle$ is form invariant:

$$\langle \overset{(4+D)}{E}{}'_m{}^a(x, y') \rangle = \langle \overset{(4+D)}{E}{}_m{}^a(x, y') \rangle \tag{2-43}$$

(cf. §2-1-3) In the case of a groupmanifold G we can assign the covariant basis $e_m{}^a(y)$ in the appendix B to $E_m{}^a(y)$,[26) namely

$$E_m{}^a(y) = e_m{}^a(y) \quad . \tag{2-44}$$

This identification is justified since under the coordinate transformation $y \to y'$ the vielbein $E_m{}^a$ transforms as

$$E'_m{}^a(y') = \frac{\partial y^n}{\partial y'^m} E_n{}^a(y) \qquad \text{(coordinate transformation)}, \tag{2-45}$$

and the covariant basis transforms as

$$e_m{}^a(y') = \frac{\partial y^n}{\partial y'^m} e_n{}^a(y) \tag{4-46}$$

under the left translation $y \to y'$ (See the appendix B). In the case of a homogeneous space G/H it is possible to take for $E_m{}^a(y)$ the covariant basis $e_m{}^a(y)$ in the appendix C, which can be justified as follows.[13) In comparison with the vielbein which transforms as (2-45), the covariant basis transforms under the left translation $y \to y'$ (See the appendix C) as

$$e_m{}^a(y') = \frac{\partial y^n}{\partial y'^m} e_n{}^b(y) D_b{}^a(h^{-1}) \tag{2-47}$$

Here $D_a{}^b(h)$ is the matrix of the adjoint representation of the group G. We notice that eq. (2-47) contradicts (2-43) and (2-45) by a factor $D_b{}^a(h^{-1})$. Hence the general coordinate transformation is not enough to relate the vielbein $E_m{}^a$ and the covariant basis $e_m{}^a$ of G/H. On the other hand the theory is invariant under the (4+D)-dimensional local Lorentz transformation that is the frame rotation SO(1, 3+D) besides the general coordinate invariance. Suppose we perform the frame rotation SO(D) in the extra space as well as the general coordinate transformation, we have

$$E'_m{}^a(y') = \frac{\partial y^n}{\partial y'^m} E_n{}^b(y) D_b{}^a$$

(coordinate transformation + local Lorentz transformation)

(2-48)

instead of (2-45). Here $D_b{}^a$ is a representation of SO(D). Comparing this with eq. (2-47) we find that $D_b{}^a(h^{-1})$ which comes from the left translation on G/H should be embeded in $D_b{}^a$ which is a representation of SO(D). Namely we have only to assign $D_b{}^a = D_b{}^a(h^{-1})$ in the case that dim G/H = D. To be concrete the generator $Q_{\bar{a}}$ of H should be written by a generator Σ^{ab} of SO(D) as

$$Q_{\bar{a}} = -\frac{1}{2} f_{\bar{a}bc} \Sigma^{bc} \tag{2-49}$$

(See the appendix C). This is because we can write the adjoint representation $D_{ab}(h)$ under an infinitesimal transformation of H as

$$D_{ab}(h) = \delta_{ab} + \delta h^{\bar{c}} \cdot f_{a\bar{c}b} = \delta_{ab} + \omega_{ab}$$

or $D(h) = 1 + \delta h^{\bar{c}} \cdot Q_{\bar{c}}$, and under the infinitesimal transformation of SO(D) D is written as $D = 1 + \frac{1}{2}\omega_{ab}\Sigma^{ab}$.

2-2-3 Harmonic expansion

First let us consider the case of a groupmanifold G.

For the sake of considering physical particles we take the (4+D)-dimensional inertial frame. The harmonic expansion[13)] of a (4+D)-dimensional general field $\Phi_{\alpha\beta\cdots ab\cdots}(x,y)$ can be written in terms of all the representations $D^{\{n\}}$ of a group G as

$$\Phi_{\alpha\beta\cdots ab\cdots}(x,y) = \sum_{\{n\}} D^{\{n\}}_{ab\cdots}(L^{-1}(y))\, \phi^{\{n\}}_{\alpha\beta\cdots}(x) \qquad , \qquad (2\text{-}50)$$

where $\{n\}$ implies the indices which distinguish the representations of G, $\alpha, \beta\cdots$ are the indices of the four dimensional inertial frame, $a, b\cdots$ are those of the extra space and $L(y)$ is defined in the appendix B. Here we have omitted detailed factors and indices (See the ref. 13). Let us rewrite (2-50) in a simpler form omitting indices as

$$\Phi(x,y) = \sum_{\{n\}} D^{\{n\}}(L^{-1}(y))\, \phi^{\{n\}}(x) \qquad . \qquad (2\text{-}51)$$

Since the field $\Phi(x,y)$ is a world scalar, it behaves under the general coordinate transformation as

$$\Phi(x, y) \rightarrow \Phi'(x, y') = \Phi(x, y) \quad . \tag{2-52}$$

Each term in the righthand side of the eq. (2-51) transforms under the left translation $y \rightarrow y'$ (B-1) as

$$\begin{aligned} D^{\{n\}}(L^{-1}(y))\,\phi^{\{n\}}(x) &\rightarrow D^{\{n\}}(L^{-1}(y')\,\phi'^{\{n\}}(x) \\ &= D^{\{n\}}(L^{-1}(y))\,D^{-1\{n\}}(g)\;\phi'^{\{n\}}(x) \quad . \end{aligned} \tag{2-53}$$

So that the expansion (2-51) is consistent with eqs. (2-52), (2-53) the field $\phi^{\{n\}}(x)$ has only to transform as

$$\phi'^{\{n\}}(x) = D^{\{n\}}(g)\;\phi^{\{n\}}(x) \quad . \tag{2-54}$$

Next let us consider the case of a homogeneous space G/H. Also in this case we expect that a (4+D)-dimensional field $\Phi(x, y)$ can be expanded as

$$\Phi(x, y) = \sum_{\{n\}} D^{\{n\}}(L^{-1}(y))\,\phi^{\{n\}}(x) \quad , \tag{2-55}$$

with representations $D^{\{n\}}$ of the group G and $L(y)$ defined in the appendix C. Since the field $\Phi(x, y)$ is a world scalar, it behaves similarly to (2-52) under the general coordinate transformation. Each term of the expansion (2-55) transforms under the left tanslation $y \rightarrow y'$ (C-1) as

$$D^{\{n\}}(L^{-1}(y))\phi^{\{n\}}(x) \to D^{\{n\}}(L^{-1}(y'))\phi'^{\{n\}}(x)$$
$$= D^{\{n\}}(h)\, D^{\{n\}}(L^{-1}(y))\, D^{-1\{n\}}(g)\, \phi'^{\{n\}}(x) \qquad (2\text{-}56)$$

Then we find that even if the field $\phi^{\{n\}}(x)$ transforms as eq. (2-54), eqs. (2-52)(2-55) and (2-56) are not consistent owing to the factor $D^{\{n\}}(h)$ in (2-56). Therefore in the homogeneous space case we cannot expand a field as (2-55). Now let us try to utilize the degrees of freedom of the frame rotation as in the section 2-2-2. We do not start with a general field but with a special field $\Phi(x, y)$ which transforms under the left translation $y \to y'$ as

$$\Phi'(x, y') = \mathbb{D}(h)\, \Phi(x, y) \qquad (2\text{-}57)$$

where $\mathbb{D}(h)$ is some particular representation of H.[13] In other words we decompose a general field into irreducible representations of H. Expand the field $\Phi(x, y)$ as

$$\Phi(x, y) = \sum_{\{n'\}} D^{\{n'\}}(L^{-1}(y))\, \phi^{\{n'\}}(x) \qquad (2\text{-}58)$$

Here $D^{\{n'\}}$ is chosen so as to contain the particular representation $\mathbb{D}$ among all the representation $D^{\{n\}}$ of the group G. If $\mathbb{D}$ is contained several times in $D^{\{n'\}}$, we must distinguish them, but we omit detailed indices (See the ref. 13.) In this case each term of the expansion (2-58) transforms as

$$D^{(n')}(L^{-1}(y))\, \phi^{(n')}(x) \rightarrow D^{(n')}(L^{-1}(y'))\, \phi'^{(n')}(x)$$
$$= \mathbb{D}(h)\, D^{(n')}(L^{-1}(y))\, D^{-1(n')}(g)\, \phi'^{(n')}(x) \qquad (2\text{-}59)$$

under the left translation. In this case we have only to assume

$$\phi'^{(n')}(x) = D^{(n')}(g)\, \phi^{(n')}(x) \qquad (2\text{-}60)$$

Now we must confirm that the theory is invariant under the left translation. Provided H is embeded into SO(D) as in the section 2-1-2, the factor $\mathbb{D}(h)$ in (2-57) can be absorbed into the degrees of freedom which come from the local Lorentz SO(1,3+D) invariance of the theory. Therfore we can perform the harmonic expansion in the case of homogeneous space G/H, too. In this case, however, all four dimensional field which have a nontrivial representation of H do not appear through the harmonic expansion and only special representations in which H contents are included in SO(D) can be obtained.[13)]

Let us look at an example of the harmonic expansion.[13)] Consider the case that G = SO(3), H = SO(2) and G/H = S^2. We must decompose a field into irreducible representations of H = SO(2) which are distinguished by "isohelicity" λ . When we write the field as $\phi_\lambda(x, \theta, \varphi)$, the harmonic expansion of ϕ_λ is

$$\phi_\lambda(x, \theta, \varphi) = \sum_{\ell(\geq|\lambda|)} \sqrt{2\ell+1} \sum_m D_{\lambda}{}^{\ell}{}_{m}(L^{-1}(\theta,\varphi))\, \phi^{\ell}_{\lambda m}(x) \quad . \qquad (2\text{-}61)$$

Here $D_{\lambda m}{}^{\ell}$ belongs to a $(2\ell+1)$-dimensional irreducible unitary representation of SO(3).

2-2-4 Zero mode ansatz and effective Lagrangian

To make the explanation simpler let us first consider the case of a homogeneous space G/H. The Kaluza-Klein ansatz is obtained from the zero mode of the harmonic expansion and can be written as[13)]

$$\overset{(4+D)}{E}{}_M{}^A(x,y) = \begin{pmatrix} E_\mu{}^\alpha(x) & e\ell\, A_\mu{}^{\hat{b}}(x)\, D_{\hat{b}}{}^a(L(y)) \\ 0 & \ell\, e_m{}^a(y) \end{pmatrix} \tag{2-62}$$

Here e is the coupling constant of the field $A_\mu{}^{\hat{b}}$, ℓ is a parameter which shows the size of the extra space, $D_{\hat{b}}{}^a$ is a matrix of the adjoint representation of G and $e_m{}^a$ is the covariant basis of G/H. To confirm that the ansatz (2-62) really corresponds to the zero mode, we have only to make sure that $E_\mu{}^\alpha$ implies the graviton and $A_\mu{}^{\hat{b}}$ is a massless field by inserting the ansatz (2-62) into the (4+D)-dimensional Einstein Lagrangian,

$$\mathcal{L} = \sqrt{-\overset{(4+D)}{g}}\;\overset{(4+D)}{R} \tag{2-63}$$

From the ansatz (2-62) we have

$$\overset{(4+D)}{E}{}_A{}^M(x,y) = \begin{pmatrix} E_\alpha{}^\mu(x) & -e\,E_\alpha{}^\nu(x)\,A_\nu{}^{\hat{b}}(x)\,D_{\hat{b}}{}^c(L(y))\,e_c{}^m(y) \\ 0 & e_a{}^m(y)/\ell \end{pmatrix} \tag{2-64}$$

and

$$\overset{(4+D)}{g}_{MN}(x,y) = \begin{pmatrix} g_{\mu\nu} + e^2\ell^2 A_\mu{}^{\hat{b}} A_\nu{}^{\hat{c}} D_{\hat{b}}{}^{a} D_{\hat{c}a} & e\ell^2 A_\mu{}^{\hat{b}} D_{\hat{b}}{}^{a} e_{na} \\ e\ell^2 A_\nu{}^{\hat{b}} D_{\hat{b}}{}^{a} e_{ma} & \ell^2 g_{mn} \end{pmatrix} \qquad (2\text{-}65)$$

.

From the definition in the appendix A the spinor connection reads

$$\overset{(4+D)}{B}_{\alpha[\beta\gamma]} = E_{[\alpha}{}^{\mu} E_{\beta]}{}^{\nu} \partial_\mu E_{\nu\gamma} - E_{[\beta}{}^{\mu} E_{\gamma]}{}^{\nu} \partial_\mu E_{\nu\alpha}$$

$$+ E_{[\gamma}{}^{\mu} E_{\alpha]}{}^{\nu} \partial_\mu E_{\nu\beta}$$

$$\equiv B_{\alpha[\beta\gamma]} \quad ,$$

$$\overset{(4+D)}{B}_{\alpha[\beta c]} = -\overset{(4+D)}{B}_{c[\alpha\beta]} = \frac{e\ell}{2} E_\alpha{}^{\mu} E_\beta{}^{\nu} F_{\mu\nu}{}^{\hat{a}} D_{\hat{a}c} \quad , \qquad (2\text{-}66)$$

$$\overset{(4+D)}{B}_{\alpha[bc]} = -e A_\alpha{}^{\hat{a}} (D_{\hat{a}}{}^{\bar{e}} - D_{\hat{a}}{}^{f} \Pi_f{}^{\bar{e}}) f_{\bar{e}bc} \quad ,$$

$$\overset{(4+D)}{B}_{a[bc]} = -\frac{1}{2\ell} f_{abc} - \frac{1}{\ell} \Pi_a{}^{\bar{d}} f_{\bar{d}bc}$$

$$\equiv \overset{(D)}{B}_{a[bc]} \quad ,$$

otherwise=0 ,

with $\Pi_a{}^{\bar{b}} \equiv e_a{}^m e_m{}^{\bar{b}}$ and $F_{\mu\nu}{}^{\hat{a}} \equiv \partial_\mu A_\nu{}^{\hat{a}} - \partial_\nu A_\mu{}^{\hat{a}} + e f_{\hat{b}\hat{c}}{}^{\hat{a}} A_\mu^{\hat{b}} A_\nu^{\hat{c}}$.

Using the definition (A-20) we obtain

$$\overset{(4+D)}{R} = R - \frac{e^2 \ell^2}{4} F_{\alpha\beta}{}^{\hat{a}} F^{\alpha\beta\hat{e}} D_{\hat{a}c} D_{\hat{e}}{}^c + \frac{1}{\ell^2} \overset{(D)}{R} , \tag{2-67}$$

$$\sqrt{-\overset{(4+D)}{g}} = \sqrt{-g} \sqrt{\overset{(D)}{g}} \ell^D = \det E(x) \det e(y) \ell^D . \tag{2-68}$$

The action is written as

$$\begin{aligned} S &\equiv \frac{1}{\ell^D \overset{(D)}{V}} \int d^{4+D} z \sqrt{-\overset{(4+D)}{g}} \overset{(4+D)}{R} \\ &= \int d^4 x \sqrt{-g} \left[R - \frac{k e^2 \ell^2}{4} F_{\alpha\beta}{}^{\hat{c}} F^{\alpha\beta}{}_{\hat{c}} + \frac{1}{\ell^2} \overset{(D)}{R} \right] , \end{aligned} \tag{2-69}$$

with $z \equiv (x, y)$ and $\overset{(D)}{V} \equiv \int d^D y \sqrt{\overset{(D)}{g}(y)}$. Here we have used the relation:

$$\frac{1}{\overset{(D)}{V}} \int d^D y \sqrt{\overset{(D)}{g}(y)} D_{\hat{a}c}(L(y)) D_{\hat{b}}{}^c(L(y)) = k \delta_{\hat{a}\hat{b}} , \tag{2-70}$$

with $k \equiv \dim(G/H)/\dim G$. In eq. (2-69) the first term is the Einstein action, the second term is the Yang-Mills action and the third term is the cosmological term. We have seen that $E_\mu{}^\alpha$ and $A_\mu{}^{\hat{b}}$ are massless as expected.

Next let us consider the case of a groupmanifold G. In this case the ansatz (2-62) is replaced by

$$\overset{(4+D)}{E}{}_M{}^A(x,y) = \begin{pmatrix} E_\mu{}^\alpha(x) & e\ell A_\mu{}^a(x) \\ 0 & \ell\, e_m{}^a(y) \end{pmatrix} \qquad (2\text{-}71)$$

Pay attention to the fact that (2-71) is not the special case of (2-62) with H = 1. Of cource we may take such an ansatz, but the factor $D_b{}^a(L(y))$ is merely unnecessary complication. From the ansatz (2-71) we get

$$\begin{aligned} S &\equiv \frac{1}{\ell^D \overset{(D)}{V}} \int d^{4+D}z \sqrt{-\overset{(4+D)}{g}}\; \overset{(4+D)}{R} \\ &= \int d^4x \sqrt{-g}\, \Big[R - \frac{e^2\ell^2}{4} F_{\alpha\beta}{}^c F^{\alpha\beta}{}_c + \frac{1}{\ell^2} \overset{(D)}{R} \Big] \end{aligned} \qquad (2\text{-}72)$$

in place of eq. (2-69).

2-2-5 Gauge transformation

Let us examine the gauge transformation. In the following explanation we use the unit, $\ell = 1$. We begin by the case of a homogeneous space G/H. We can write the infinitesimal left translation depending on x as

$$\begin{aligned} \delta x^\mu &= \epsilon^\mu(x) , \\ \delta y^m &= K^m{}_{\hat{b}}(y)\, \epsilon^{\hat{b}}(x) . \end{aligned} \qquad (2\text{-}73)$$

using the Killing vector $K^m{}_{\hat{b}}$ in the appendix C. Here we have set $\delta g^{\hat{b}} \equiv \epsilon^{\hat{b}}$ in eq. (C-13). Under the general coordinate transformation the metric tensor behaves as

$$\delta \overset{(4+D)}{g}{}^{MN}(x,y) \equiv \overset{(4+D)}{g}{}'^{MN}(x,y) - \overset{(4+D)}{g}{}^{MN}(x,y) \quad ,$$

$$\delta \overset{(4+D)}{g}{}^{MN} = \overset{(4+D)}{g}{}^{MP}\partial_P \epsilon^N + \overset{(4+D)}{g}{}^{PN}\partial_P \epsilon^M - \epsilon^P \partial_P \overset{(4+D)}{g}{}^{MN} \quad . \tag{2-74}$$

From eqs. (2-62), (2-73) and (2-74) we get

$$\delta A^{\mu\hat{a}} = \partial_\nu \epsilon^\mu \cdot A^{\nu\hat{a}} - \epsilon^\nu \partial_\nu A^{\mu\hat{a}} - \frac{1}{e} g^{\mu\nu} D_\nu \epsilon^{\hat{a}} \quad , \tag{2-75}$$

with $D_\nu \epsilon^{\hat{a}} \equiv \partial_\nu \epsilon^{\hat{a}} + e f_{\hat{b}\hat{c}}{}^{\hat{a}} A_\nu{}^{\hat{b}} \epsilon^{\hat{c}}$. The equation (2-75) implies the gauge transformation of the Yang-Mills field $A_\mu{}^{\hat{a}}$. We see that the gauge transformation corresponds to a special (4+D)-dimensional coordinate transformation which preserves the form of the D-dimensional metric tensor (isometry i.e. $\delta g^{mn} = 0$ for the internal space).

In the case of the group manifold G eq. (2-73) is replaced by

$$\delta x^\mu = \epsilon^\mu(x) \quad ,$$

$$\delta y^m = e^m{}_b(y)\, \epsilon^b(x) \quad , \tag{2-76}$$

since $e^m{}_b$ is a Killing vector on G. (cf. the appendix B) And we have

$$\delta A^{\mu a} = \partial_\nu \epsilon^\mu \cdot A^{\nu a} - \epsilon^\nu \partial_\nu A^{\mu a} - \frac{1}{e} g^{\mu\nu} D_\nu \epsilon^a \quad , \tag{2-77}$$

with $D_\nu \epsilon^a \equiv \partial_\nu \epsilon^a - e f_{bc}{}^a A_\nu{}^b \epsilon^c$ in place of eq. (2-75). Of cource $\delta g^{mn} = 0$ also in this case.

2-2-6 Scalar field and spinor field

Scalar fields can be included in the framework of the Kaluza-Klein theory in a similar way to the section 2-1-4. We can introduce a scalar field just in the same way as eq. (2-16), and we may also obtain scalar fields from the harmonic expansion of the vielbein $E_m{}^a(x,y)$. Provided there exist an extra gauge field $A_{MNP}(x,y)$ in the (4+D)-dimensional theory, it is possible for scalar fields to come from the harmonic expansion of $A_{mnp}(x,y)$. These scalar fields are considered either to determine the magnitude of the extra space or to serve as Higgs fields.

As for spinor fields we can introduce them alike in the section 2-1-7.*) The Dirac Lagrangian in the (4+D)-dimensional space-time is

$$\mathcal{L}_\psi = \frac{i}{2} \det \overset{(4+D)}{E}(x,y)\, \overset{(4+D)}{\bar{\psi}}(x,y)\, \overset{(4+D)}{\Gamma}{}^A\, \overset{(4+D)}{E}{}_A{}^M(x,y)\, \overset{(4+D)}{\nabla}_M\, \overset{(4+D)}{\psi}(x,y) + h.c. \qquad (2\text{-}78)$$

with $\overset{(4+D)}{\nabla} \equiv \partial_M + \frac{1}{8} \overset{(4+D)}{B}_{M[AB]} [\overset{(4+D)}{\Gamma}{}^A, \overset{(4+D)}{\Gamma}{}^B]$. Here $\overset{(4+D)}{\Gamma}{}^A$ is the (4+D)-dimensional γ-matrix and satisfies $\{\overset{(4+D)}{\Gamma}{}^A, \overset{(4+D)}{\Gamma}{}^B\} = 2\eta^{AB}$. Now we consider the case of the homogeneous space G/H only. Expressions in the case of the groupmanifold G can be obtained just in the same manner of this section. Assuming the Kaluza-

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*) It is not always possible to introduce the spinor field in the compact manifold. For example, CP^2 manifold does not admit the spin structure.[27)]

Klein ansatz (2-62) for the background vielbein we get

$$\mathcal{L}_\psi = \frac{i}{2} \det E(x) \det e(y) \Big\{ \overline{\overset{(4+D)}{\Psi}}(x,y) \overset{(4+D)}{\Gamma}{}^{\alpha} E_\alpha{}^\mu(x) \overset{(4)}{D}_\mu \overset{(4+D)}{\Psi}(x,y)$$

$$+ \frac{1}{\ell} \overline{\overset{(4+D)}{\Psi}}(x,y) \overset{(4+D)}{\Gamma}{}^{a} e_a{}^m(y) \overset{(D)}{\nabla}_m \overset{(4+D)}{\Psi}(x,y) \tag{2-79}$$

$$+ \frac{e\ell}{16} F_{\alpha\beta}{}^{\hat{c}}(x) D_{\hat{c}d}(L(y)) \overline{\overset{(4+D)}{\Psi}}(x,y) \overset{(4+D)}{\Gamma}{}^{d} [\overset{(4+D)}{\Gamma}{}^{\alpha}, \overset{(4+D)}{\Gamma}{}^{\beta}] \overset{(4+D)}{\Psi}(x,y) \Big\}$$

$$+ \text{h. c.} \qquad ,$$

with $\overset{(4)}{D}_\mu \equiv \overset{(4)}{\nabla}_\mu + ieA_\mu{}^{\hat{b}}(x) T_{\hat{b}}$ and $T_{\hat{b}} \equiv i K_{\hat{b}}{}^m(y) \overset{(D)}{\nabla}_m + \frac{i}{8}(\overset{(D)}{\nabla}_m K_{\hat{b}n}(y)) [\overset{(4+D)}{\Gamma}{}^m, \overset{(4+D)}{\Gamma}{}^n]$.[28] Here, we have used eq. (C-15). Let us perform the harmonic expansion of the spinor field $\overset{(4+D)}{\Psi}(x,y)$, that is

$$\overset{(4+D)}{\Psi}(x,y) = \sum_{\{n\}} D^{\{n\}}(L^{-1}(y)) \psi^{\{n\}}(x) \qquad . \tag{2-80}$$

Then we obtain the action of the four dimensional spinor field $\psi^{\{n\}}(x)$ as

$$S_\psi \equiv \frac{1}{\overset{(D)}{V}} \int d^{4+D}z \, \mathcal{L}_\psi$$

$$= \sum_{\substack{\{m\}\\ \{n\}}} \frac{i}{2} \int d^4x \det E(x) \overline{\psi}^{\{m\}}(x) \Big[\overset{(4+D)}{\Gamma}{}^{\alpha} \Big\{ E_\alpha{}^\mu(x) \overset{(4)}{\nabla}_\mu - eA_\alpha{}^{\hat{b}}(x) D^{\{n\}}(Q_{\hat{b}}) \Big\} \tag{2-81}$$

$$- M - \frac{1}{8} F_{\alpha\beta}{}^{\hat{a}}(x) \overset{(4+D)}{\Gamma}{}^{\alpha\beta} \tilde{\Gamma}_{\hat{a}} \Big] \psi^{\{n\}}(x) + \text{h. c.} \; ,$$

where

$$M \equiv \frac{1}{\ell \overset{(D)}{V}} \int d^D y \, D^{\{m\}}(L(y)) \Big\{ \overset{(4+D)}{\Gamma}{}^{a} D^{\{n\}}(Q_a) + \frac{1}{8} f_{abc} \overset{(4+D)}{\Gamma}{}^{abc} \Big\} D^{\{n\}}(L^{-1}(y)) \tag{2-82}$$
,

$$\tilde{\Gamma}_{\hat{a}} \equiv \frac{1}{\overset{(D)}{V}} \int d^D y \, D^{\{m\}}(L(y)) \overset{(4+D)}{\Gamma}{}^{c} D^{\{n\}}(L^{-1}(y)) \, D_{c\hat{a}}(L^{-1}(y)) \tag{2-83}$$
,

and $\overset{(4+D)}{\Gamma}{}^{abc}$ is defined by antisymmetrizing $\overset{(4+D)}{\Gamma}{}^{a}\overset{(4+D)}{\Gamma}{}^{b}\overset{(4+D)}{\Gamma}{}^{c}$ about a,b,c.[13] In the dimensional reduction the γ -matrix is often decomposed as

$$\overset{(4+D)}{\Gamma}{}^{\alpha} \equiv \gamma^{\alpha} \otimes \overset{(D)}{1} \qquad , \qquad \overset{(4+D)}{\Gamma}{}^{a} \equiv \gamma^{5} \otimes \overset{(D)}{\Gamma}{}^{a} \qquad (2\text{-}84)$$

We notice that in both eqs. (2-79) and (2-81) the first term is the kinetic term including the minimal coupling with the Yang-Mills field, the second term is the mass term and the last term is the Pauli term.

§3. Problems in constructing a realistic unified model
—— overview ——

As we have seen in the previous sections the Kaluza-Klein theory has a mathematical beauty and is a good candidate of a unified theory of all interactions including gravitation. However there remain many unsolved problems in constructing a realistic unified theory. Let us overview some of them in this section.

§3-1 Fermions and necessity of supergravity

We may add the Dirac Lagrangian by hand to the Einstein Lagrangian in the higher dimensional space-time in order to include spinor fields in the Kaluza-Klein theory. This seems rather ad hoc and cannot be a way of constructing a "unified" theory. On the other hand we have the theory of supergravity which naturally contains both the graviton and Fermion fields under the principle of local supersymmetry. The advantage of the supergravity is that it has the possibility to be a consistent theory of quantum gravity because the supersymmetry may give rise to cancellation of all the ultraviolet divergences.[29] Therefore it may be preferable to start from the supergravity. As the cancellation of divergence is more probable in extended supergravities with larger N and these extended supergravities are known to be easily obtained from a higher dimensional simple supergravity with $N = 1$,[9] many physicists have examined the dimensional reduction of it. If we set the vacuum expectation value of Fermions in the supergravity to be zero and consider the Bosonic sector only, the remaining theory is nothing but a Kaluza-Klein theory with some extra gauge fields in the higher dimensional space-time. There are component field supergravity and superspace supergravity, and they are known to be equivalent.[29] Suppose we prefer to start from the simplest Lagrangian in constructing a unified theory, we had better to take the superspace supergravity. Then the ultimate goal of Kaluza-Klein unified theory will be something like the one illustrated in the table 1.

table 1

§3-2 Restriction on the gauge group

When we attempt to construct a realistic unified model, the easiest way in choosing the gauge symmetry may be to take the groups such as SU(5), SO(10) which succeeded to some extent in GUTs.[6] Now people belive that supergravity exists only in less than or equal to eleven dimensions.[10] This is because the supergravity in a more than eleven dimensional space-time requires a particle with a spin $s \gtrsim 5/2$, and it is widely believed to be difficult to construct a consistent theory of such a higher spin particle.[30] Consequently, as long as we start from supergravity, we must consider it in the less than or equal to eleven dimensional space-time and therefore the symmetry group in the Kaluza-Klein theory should be severely restricted. Even if we take the most economical theory with a homogeneous space, the SU(5) gauge symmetry requires the twelve dimensional space-time owing to the fact that dim

[SU(5)/SU(4) X U(1)] = 8. In the case of SO(10) we need thirteen dimensions because dim [SO(10)/SO(9)] = 9. This forces us to abandon SU(5) and SO(10). The best we can conceive is the direct product gauge group SU(3) X SU(2) X U(1) on account of the fact dim [{SU(3)/SU(2) X U(1)} X {SU(2)/U(1)} X U(1)] = 7.[10)*)] Since the eleven dimensional N = 1 supergravity is unique and contains sufficient variety of fields to possibly describe the nature, the dimensional reduction of it has been studied actively in these days.[9)29)31)] However as we shall see in the next section the chirality cannot be defined in an odd dimensional space-time and it is difficult to obtain chiral Fermions (this is definitely needed to describe the weak interactions) after dimensional reduction.[9)] It is possible for a less than eleven dimensional theory to have the symmetry of SU(3) X SU(2) X U(1), provided the theory has an extra U(1) gauge field. Such a theory is known in a ten dimensional supergravity, in which the extra space is compactified into $Cp^2 \times S^2$ and the problem of the spin structure is solved using the U(1) gauge symmetry.[32)] As the ten dimensions are important for the string theory and the chirality can be defined in even dimensions, the ten dimensional supergravity has also been studied recently.[32)] Another possibility is to consider composite particles. Though the vielbein $E_M{}^A$ corresponds to an elementary particle, the spinor

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*) In this case it is difficult to introduce the spinor field due to the fact SU(3)/SU(2) X U(1) $\sim Cp^2$ which does not admit spin structure.[27)]

connection $B_{M[AB]}$ may be regarded as a composite operator of $E_M{}^A$.[31] There is an effort to make the field $B_{M[AB]}$ propagate and to regard it as a gauge field,[33] and some physicists have tried to utilize SO(7) in the frame rotation group SO(1, 10).[31]

Although a theory of interacting particles with $S \geq 5/2$ is not yet found, it may be constructed in future.[34] Then supergravity will be constructed in a higher than eleven dimensional space-time. Furthermore supergravity is not necessarily inevitable even in a theory with Fermions. Accordingly it is not meaningless to discuss any dimensional Kaluza-Klein theory.

Since the harmonic expansion on a homogeneous space G/H includes only restricted representations of G(cf. §2-2-3), some realistic fields with nontrivial representation may not appear from that expansion in some models. (See examples in ref. 13) To construct a realistic unified model we must make all quantum numbers of realistic particles appear in that model. If the Kaluza-Klein theory had any principle to choose a certain symmetry and some representations, and if they happened to be identical to those in nature, it would be very nice for a realistic unified theory. At any rate the assignment of the gauge group and the dimensions is one of the open problems.

§3-3 Chirality and massless spinor fields

There exist chiral Fermions like neutrinos in nature. In the four dimensional space-time we can define chirality using $\gamma_5 \; (\equiv i\gamma^0\gamma^1\gamma^2\gamma^3)$ and we can assure using the chiral invariance that the massless Fermions remain massless even with quantum corrections. However we cannot define chirality in an odd dimensional space-time, because for example $i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5 = 1$ in the five dimensions. Hence it is difficult, though not yet verified impossible, to obtain four dimensional chiral Fermions through dimensional reduction of such a space-time.[9] There are some literatures which consider the dimensional reduction of spinor fields in any dimension,[35] examine the conditions to obtain chiral Fermions[35][36] or discuss discrete symmetries such as C,P,T.[37]

Since the Kaluza-Klein theory does not have parameters with the mass dimension except for the gravitational constant G, the mass of the particles in this theory is necessarily either zero or of the order of the Planck mass. (cf. §2-1-5) On the other hand there are variety of mass in nature such as electron mass 0.5 MeV and Z Boson mass 90GeV. The X boson predicted in GUTs has the mass about 10^{15} GeV.[38] We must explain the variety of mass. This hierarchy problem is the hardest one to solve not only for the Kaluza-Klein theory but also for all the unified theories, since a unified theory is preferred to have as few parameters as possible. We have two possibilities to explain the realistic mass spectrum:
i) Obtain a light mass $(\sim m_{pl}\, e^{-\frac{1}{e}})$ by some nonperturbative effects from the Planck mass, m_{pl},[10] or ii) Create a light mass from the zero mode through the Higgs-Kibble mechanism.[39] However no way of calculation is known in the case i). In the case ii) we have the difficulty that there does not exist the zero mode of spinor fields in many Kaluza-klein theories. Let us explain this massless spinor problem below.

The mass operator of the spinor field is written as

$$\hat{M} \equiv -i \overset{(D)}{e}_a{}^m \overset{(D)}{\Gamma}{}^a \overset{(D)}{\nabla}_m \quad , \tag{3-1}$$

from eqs. (2-79,84) with $\ell = 1$. In order to examine whether this operator has a zero eigenvalue we calculate the square of it, $\hat{M}^2$. Using eq. (A-20) and the cyclicity of the Riemann tensor ($R_{abcd} + R_{acdb} + R_{adbc} = 0$) we get

$$\hat{M}^2 = - \overset{(D)}{g}{}^{mn} \overset{(D)}{\nabla}_m \overset{(D)}{\nabla}_n + \frac{1}{4} \overset{(D)}{R} \quad . \tag{3-2}$$

This equation is called the theorem of Lichnerowicz.[40)]Provided the D-dimensional space is compact, the Laplacian $\overset{(D)}{g}{}^{mn} \overset{(D)}{\nabla}_m \overset{(D)}{\nabla}_n$ is negative-semi-definite.*) Since in many cases the curvature of a compact D-dimensional space is positive, namely $\overset{(D)}{R} > 0$,**) we find that

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*) See that $\int d^D y \sqrt{\overset{(D)}{g}}\, \varphi^{pq\cdots} \overset{(D)}{g}{}^{mn} \overset{(D)}{\nabla}_m \overset{(D)}{\nabla}_n \varphi_{pq\cdots} = - \int d^D y \sqrt{\overset{(D)}{g}}\, \overset{(D)}{g}{}^{mn} (\overset{(D)}{\nabla}_m \varphi^{pq\cdots})(\overset{(D)}{\nabla}_n \varphi_{pq\cdots}) \leq 0$.

**) Let $\overset{(D)}{\nabla}_m$ be the covariant derivative under the D-dimensional coordinate transformation. From eq. (A-20), the cyclicity of the Riemann tensor and the Killing equation (c-20) we obtain $\overset{(D)}{\nabla}_p \overset{(D)}{\nabla}_q K_{\hat{\alpha} m} - \overset{(D)}{\nabla}_q \overset{(D)}{\nabla}_p K_{\hat{\alpha} m} - \overset{(D)}{\nabla}_m \overset{(D)}{\nabla}_q K_{\hat{\alpha} p} = 0$. Using this equation and eq. (A-20) we find that the Killing vectors $K_{\hat{\alpha} m}$ satisfy $\overset{(D)}{g}{}^{pq} \overset{(D)}{\nabla}_p \overset{(D)}{\nabla}_q K_{\hat{\alpha} m} = - \overset{(D)}{R}_m{}^n K_{\hat{\alpha} n}$. Suppose the extra space is an Einstein space i.e. $\overset{(D)}{R}_{mn} = C \overset{(D)}{g}_{mn}$ with a constant C . Then we have $\overset{(D)}{g}{}^{mn} \overset{(D)}{\nabla}_m \overset{(D)}{\nabla}_n K_{\hat{\alpha} p} = - C K_{\hat{\alpha} p}$. Therefore the negative-semi-definiteness of the Laplacian implies that $C \geq 0$ and therefore $\overset{(D)}{R} \geq 0$.[28)]

$$\hat{M}^2 > 0 \quad , \tag{3-3}$$

and there is no zero mode of spinor fields. When we stick to a particular space with a large symmetry, for example a maximally symmetric space (cf. eq. A-25), for the extra space, we will be confronted with this difficulty. But there are several ways to avoid this difficulty. First there exist compact spaces with $\overset{(D)}{R} = 0$ or $\overset{(D)}{R} < 0$ (non-Einstein space), for example $\overset{(D)}{R} = 0$ on the D-dimensional torus. Futhermore it is possible for the spinor zero mode to exist in such theories that include (4+D)-dimensional extra gauge fields or Rarita-Schwinger fields with spin 3/2.

§3-4 Puzzle of the cosmological constant

When we perform the dimensional reduction, we have a huge cosmological constant in general as can be seen in eq. (2-69). In the case $\overset{(D)}{R} > 0$ this cosmological constant is as big as $\Lambda_{KK} \sim \frac{1}{\ell^2} \sim 10^{38}\ \mathrm{GeV}^2$. On the other hand astrophysical observations show that $\Lambda_{exp} < 10^{-83}\ \mathrm{GeV}^2$.[41)42)] This puzzle is the problem of the cosmological constant.

Although this problem has not been solved, some attempts have been made to settle it. The conventional but unsatisfactory treatment of cosmological constant is to put an adjustable cosmological term by hand in higher dimensions and to fine-tune the four dimensional effective cosmological constant to be zero.[43)] On the other hand it is known that supersymmetry prohibits nonzero vacuum energy and cosmological constant.[44)] However supersymmetry must be broken at low energy, because in the real world there are no Boson-Fermion pair with the same mass predicted by supersymmetry. And in supergravities, contrary to the globally supersymmetric theory, the cosmological constant is *not* zero even with unbroken supersymmetry.[31)] Consequently it is not at all clear whether supersymmetry can explain the extraordinary small cosmological constant.

§3-5 Stability of solutions and the principle to choose a solution

When we consider spontaneous compactification, we sometimes have several different solutions and need to choose one of them. For example several solutions have been known in the eleven dimensional supergravity,[31)] after the solution with a seven torus T^7 was found.[9)] The table 2 is devoted to the summary of them.

table 2

Suppose our universe corresponds to one of these solutions, the right solution must be stable against small perturbation. For the purpose of examing stability we have to calculate the mass spectrum of fluctuations arround the solution. In all the known solutions of that supergravity the four dimensional space-time is not that of Minkowski but that of anti-de Sitter.[31)] It is also known that the existence of the tachyonic mode $(m^2 < 0)$ does not necessarily correspond to the instability in the anti-de Sitter background.[49)] For example a scalar particle with an apparently tachyonic mass $(m^2 < 0)$ does not violate the stability unless m^2 is below a certain critical value ($\Box\varphi + a^2\alpha\varphi = 0$, $R = 12a^2$, $\alpha \leq \frac{9}{4}$). Recently the round S^7 and left squashed solutions were shown to be stable under dilatation and squashing,[50] and the round S^7 solution is also known to be stable against all fluctuations.[51] Among these fluctuations there are the modes with $m^2 < 0$, but they correspond to the critical value which is just allowed in the anti-deSitter background.

Even if a solution is stable against classical fluctuations and corresponds to a local minimum point, it may be unstable under a quantum tunnel effect when there is another minimum point with lower energy. For instance the standard five dimensional Kaluza-Klein theory is known to be semiclassically unstable.[52] As is well known in the theory of gravitation, energy density cannot be defined in a covariant way and we can only define total energy in an asymptotically flat space.[41] We can hardly tell which solution is realized among the solutions with different boundary conditions. Accordingly we have a reservation to the claim that we can explain the reason why the dimensionality of our world is four on the basis of the classical solutions of supergravity. What we do not have is a principle for choosing a solution.

§3-6 Cosmology

We have said that the size of the extra space is as small as the Planck length to explain the invisibility of the extra space. Now we must explain why it is so small compared to our three dimensional space. As the vacuum expectation value of the scalar field φ in eq. (2-19) can be considered to express the size of the extra space, we can determine this size by calculating the effective potential of φ and finding its minimum point. There are some literatures which attempt to explain the smallness of the extra space along this line of thought.[53] If the minimum point of the effective potential corresponds to a compact manifold with different circumferences along the different symmetry directions, we can determine the ratios among the gauge coupling constants corresponding to the symmetry groups. For the ratios of the circumferences are just those of coupling constants as Weinberg showed.[54] (cf. §2-1-2). Provided that the ratios of couplings in a theory with SU(3) X SU(2) X U(1) are determined from the circumferences of extra space, it will not be necessary to embed this group into a larger group as in GUTs.

Another way to account the smallness of the extra space is to consider time dependence of its size.[55][56] In this cosmological approach the extra space is considered to be in the same size of our three dimensional space at the Planck time of the universe. The extra space contracts relatively to the three dimensional space as time passes. In order to illustrate this let us take the five dimensional Kasner type space-time,[55] namely we take the line element:

$$ds^2 = -dt^2 + \sum_{i=1,2,3,5} (t/t_0)^{2p_i} (dx^i)^2 \quad , \tag{3-4}$$

with $x^0 \equiv t$. We can find the following solution which satisfies the five dimensional Einstein equation $\overset{(5)}{R}_{MN} = 0$ i.e.

$$ds^2 = -dt^2 + \sum_{i=1,2,3} (t/t_0)(dx^i)^2 + (t_0/t)(dx^5)^2 \quad . \tag{3-5}$$

In this solution the three dimensional space explodes and the extra space contracts for the time duration. Such a particle has also been obtained in the eleven and ten dimensional supergravities.[56] However there is an argument that the extra space does not contract so fast relatively to the three dimensional space, because the higher dimensional anisotropic space-time may be isotropized by particle creation due to the quantum effect in a background space-time[57] in the same manner as the four dimensional anisotropic universe[58]. Since the ratio between the size of the extra space and the Planck length determins the magnitude of the coupling constant α , we may have the difficulty in a time-dependent extra space that $\dot{\alpha}$ becomes too large compared to the observed upper limit.[55]

In the recent cosmology the inflational scenario that the universe expands exponentially is considered to solve many difficulties such as flatness and horizontal problems.[42] Hence we may have to examine whether this scenario matches to the Kaluza-Klein cosmology. It has been pointed out that during the cosmological dimensional reduction entropy is pumped from the extra space into the four dimensional universe.[59] This idea is based on the analogy of the sudden smashing of a d-dimensional lattice into a (d-1)-dimensional one with the consequent increase in disorder. This entropy may be useful to solve the cosmological problems.

As we have seen above, there are several unsolved problems in constructing a realistic model based on the Kaluza-Klein theory. But we might expect they will be solved in future and it would be very fascinating if the existence of the extra space is predicted on more sound grounds.

§4. Quantum Kaluza-Klein theory

In this section we consider the quantization of the Kaluza-Klein theory. After some comments on the problem of ultraviolet divergences we discuss the BRS symmetry and the physical S-matrix unitarity.

§4-1 The Problem of ultraviolet divergence

If the extra space does exist, it is natural to consider the quantization in a higher dimensional space-time. The problem is that the higher the dimension of space-time is, the worse is the ultraviolet divergence. On the other hand the original Einstein theory is not renormalizable even in the four dimensions and a consistent theory of gravitation without uncontrollable ultraviolet divergences is not yet known.[29] In order to handle the ultraviolet divergences we might have to consider nonperturbative methods. Some physicists have tried for example the methods to expand quantum gravity in powers of $\frac{1}{N}$ or $\frac{1}{d}$, where N is the number of matter fields coupled to gravitation and d (=4+D) is the number of space-time dimensions.[60][61]*) There are also many attempts to get rid of divergence perturbatively. For instance the Lagrangian with bilinear terms of Riemann tensors can be added for the renormalizability.[62] The

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*) In the latter method even such a speculation has been made that the theory becomes finite at the limit of $d \to \infty$, namely the dimensions of the space-time are infinite.

Einstein Lagrangian may be induced by the quantum fluctuation of some fundamental fields.[63] However one of the most promising theories to solve the ultraviolet catastroph may be the supergravity.[29] Since the modern Kaluza-Klein theory is motivated by the supergravity (cf. §3-1), we think that the quantization of the Kaluza-Klein theory is meaningful and hope that the problem of ultraviolet divergences are ultimately solved by supergravity.

When we consider the quantization of this theory, we have to examine the consistency between the higher and four dimensional quantum theories. There are literatures which discuss the ultraviolet behavior of Kaluza-Klein theories in both the higher and lower dimensional space-time.[64] It has been pointed out that they are consistent with each other when all the modes in the harmonic expansion of the fields are taken into consideration.

§4-2 BRS symmetry and physical S-matrix unitarity

The Kaluza-Klein theory includes the gauge fields as well as the gravitational fields after the dimensional reduction. Both in quantum gravity and in the theory of quantum non-Abelian gauge fields, it is well known that we need the gauge fixing term together with the Faddeev-Popov ghost term[14] in the lagrangian to ensure the unitarity of the S-matrix. The key to the proof of the unitarity is the invariance of the total Lagrangian under the BRS transformation[15].[16][65] Therefore if we consider the extra space to exist and perform the quantization of the Kaluza-Klein theory in the higher dimensional space-time, we must introduce the gauge fixing term and the

Faddeev-Popov ghost term to the higher dimensional Einstein Lagrangian and make the theory have the BRS symmetry so as to ensure the unitarity. In this case it is necessary to examine the consistency between the BRS symmetries in the higher dimensional and in the four dimensional theories. When we perform the harmonic expansion of the higher dimensional metric tensor field, the zero modes correspond to the graviton and the gauge particles. The nonzero modes correspond to a series of massive tensor fields. (cf. §2-1-6, §2-2-4) In this section let us first establish the relation between the BRS symmetries in the higher dimensional Einstein theory and the four dimensional Einstein-Yang-Mills theory which corresponds to the zero modes in the harmonic expansion.[18] In the subsequent section we generalize this investigation to the extended BRS symmetry.[19)20)] Next using the dimensional reduction technique we derive the BRS symmetry of massive tensor fields which corresponds to the nonzero modes of the expansion.[21] Finally we show that the physical S-matrix is unitary in the whole quantum Kaluza-Klein theory including both massless and massive fields.[21]

4-2-1 Kaluza-Klein theory and BRS symmetry[18]

First let us consider the Kaluza-Klein theory with a group manifold G. We write down the lagrangian for the (4+D)-dimensional quantum gravity as[65]

$$\mathcal{L} = \mathcal{L}_E + \mathcal{L}_G \quad , \quad (4\text{-}1)$$

$$\mathcal{L}_E = \sqrt{-\overset{(4+D)}{g}}\ \overset{(4+D)}{R} \quad , \quad (4\text{-}2)$$

$$\mathcal{L}_G = i\,\delta\,(\partial_M \overset{(4+D)}{\bar{C}}_N \cdot \sqrt{-\overset{(4+D)}{g}}\; \overset{(4+D)}{g}{}^{MN}) \quad . \tag{4-3}$$

The BRS transformation δ in eq. (4-3) is defined by

$$\delta \overset{(4+D)}{g}{}^{MN} = \overset{(4+D)}{g}{}^{MP}\partial_P \overset{(4+D)}{C}{}^N + \overset{(4+D)}{g}{}^{PN}\partial_P \overset{(4+D)}{C}{}^M - \overset{(4+D)}{C}{}^P\partial_P \overset{(4+D)}{g}{}^{MN} \quad ,$$

$$\delta \overset{(4+D)}{b}_M = 0 \,, \quad \delta \overset{(4+D)}{C}{}^M = - \overset{(4+D)}{C}{}^P\partial_P \overset{(4+D)}{C}{}^M \,, \quad \delta \overset{(4+D)}{\bar{C}}_M = i\, \overset{(4+D)}{b}_M \quad , \tag{4-4}$$

$$\left[\ \delta\sqrt{-\overset{(4+D)}{g}} = -\partial_P\left(\sqrt{-\overset{(4+D)}{g}}\; \overset{(4+D)}{C}{}^P\right)\ \right] \quad ,$$

with the convention of the left-differentiation rule. Here $\overset{(4+D)}{b}_M$ is an auxiliary field and $\overset{(4+D)}{C}{}^M$, $\overset{(4+D)}{\bar{C}}_M$ are the Faddeev-Popov ghost fields.

The general invariance of the Eistein action (4-2) and the nilpotency property of the BRS transformation guarantee that the total action is invariant under the BRS transformation. $\mathcal{L}_G$ includes both the gauge fixing and the Faddeev-Popov ghost terms:

$$\begin{aligned} \mathcal{L}_G = {} & -\partial_M \overset{(4+D)}{b}_N \sqrt{-\overset{(4+D)}{g}}\; \overset{(4+D)}{g}{}^{MN} - i\,\partial_M \overset{(4+D)}{\bar{C}}_N \Big[\sqrt{-\overset{(4+D)}{g}}\; \overset{(4+D)}{g}{}^{MP}\partial_P \overset{(4+D)}{C}{}^N \\ & + \sqrt{-\overset{(4+D)}{g}}\; \overset{(4+D)}{g}{}^{PN}\partial_P \overset{(4+D)}{C}{}^M - \partial_P\Big(\sqrt{-\overset{(4+D)}{g}}\; \overset{(4+D)}{g}{}^{MN} \overset{(4+D)}{C}{}^P\Big)\Big] \quad . \end{aligned} \tag{4-5}$$

The variation with respect to $\overset{(4+D)}{b}_M$ gives the [(4+D)-dimensional] De Donder (harmonic) gauge condition, $\partial_M\left(\sqrt{-\overset{(4+D)}{g}}\; \overset{(4+D)}{g}{}^{MN}\right) = 0$.

We will now look at the BRS transformation in the Kaluza-Klein decomposition. To do this we must specify the y-dependence (the y's are "internal" coordinates) of the Faddeev-Popov ghost fields $\overset{(4+D)}{C}{}^{M}$, $\overset{(4+D)}{\bar{C}}{}_{M}$ and the auxiliary field $\overset{(4+D)}{b}{}_{M}$. Let us recall the gauge transformation in the Kaluza-Klein theory because the BRS transformation is the quantum counterpart of it. The gauge transformation was nothing but the special (4+D)-dimensional coordinate transformation that preserves the form of the D-dimensional metric tensor. See eqs. (2-76,27). Since the ghost field C^{M} corresponds to the parameters of the coordinate transformation ϵ^{M} , we take the ansatz:

$$\overset{(4+D)}{C}{}^{\mu}(x,y) = C^{\mu}(x) \quad , \quad \overset{(4+D)}{C}{}^{m}(x,y) = e^{m}{}_{a}(y)\, C^{a}(x) \quad . \tag{4-6}$$

About $\bar{C}_{M}$ and b_{M} we assume that they do not depend on y , namely

$$\begin{aligned} \overset{(4+D)}{\bar{C}}{}_{\mu}(x,y) &= \bar{C}_{\mu}(x) \quad , \quad \overset{(4+D)}{\bar{C}}{}_{m}(x,y) = \bar{C}^{0}{}_{m}(x) \quad , \\ \overset{(4+D)}{b}{}_{\mu}(x,y) &= b_{\mu}(x) \quad , \quad \overset{(4+D)}{b}{}_{m}(x,y) = b^{0}{}_{m}(x) \quad , \end{aligned} \tag{4-7}$$

so that we get De Donder-Landau gauge in the four dimensions. The factorized form for $\bar{C}$ and b will in general give a non-linear gauge condition. The Kaluza-Klein ansatz (2-71) together with (4-6) immediately gives the "gravitational part" of the BRS transformation correctly:

$$\begin{aligned} &\delta g^{\mu\nu} = g^{\mu\rho}\partial_{\rho} C^{\nu} + g^{\rho\nu}\partial_{\rho} C^{\mu} - C^{\rho}\partial_{\rho} g^{\mu\nu} \quad , \\ &\delta b_{\mu} = 0 \; , \quad \delta C^{\mu} = -C^{\rho}\partial_{\rho} C^{\mu} \; , \quad \delta \bar{C}_{\mu} = i b_{\mu} \quad . \end{aligned} \tag{4-8}$$

As for the "gauge part" if we redefine the auxiliary field

$b^{0}{}_{m}$ as

$$b^{1}_{m} \equiv b^{0}_{m} - i\, C^{\rho} \partial_{\rho} \bar{C}^{0}_{m} \qquad , \tag{4-9}$$

and use Lie's differential equation (B-6) and the anti-commuting property of ghost fields,

$$\delta A^{a\mu} = - g^{\mu\rho} D_{\rho} C^{a} + A^{a\rho} \partial_{\rho} C^{\mu} - C^{\rho} \partial_{\rho} A^{a\mu} \qquad ,$$

$$\delta C^{a} = - C^{\rho} \partial_{\rho} C^{a} - \tfrac{1}{2} f_{bc}{}^{a} C^{b} C^{c} \qquad , \tag{4-10}$$

$$\delta \bar{C}^{0}_{m} = i\, b^{1}_{m} - C^{\rho} \partial_{\rho} \bar{C}^{0}_{m} \qquad , \qquad \delta b^{1}_{m} = - C^{\rho} \partial_{\rho} b^{1}_{m} \ .$$

with $D_{\rho} C^{a} = \partial_{\rho} C^{a} - f_{bc}{}^{a} A^{b}_{\rho} C^{c}$.

Let us look at the gauge fixing and the Faddeev-Popov ghost terms in the action .
Substituting the ansatz (2-71) and (4-7) we have

$$\begin{aligned} S_{G} &= i \overset{(D)}{V} \int d^{4}x\, \delta \left(\sqrt{-g}\, \partial_{\mu} \bar{C}_{\nu} \cdot g^{\mu\nu} \right) \\ &\quad - i \int d^{D}y \sqrt{\overset{(D)}{g}}\, e^{m}{}_{a} \int d^{4}x\, \delta \left(\sqrt{-g}\, \partial_{\mu} \bar{C}^{0}_{m} \cdot A^{a\mu} \right) \qquad , \end{aligned} \tag{4-11}$$

with $\overset{(D)}{V} \equiv \int d^{D}y \sqrt{\overset{(D)}{g}}$ being the volume of the group space. We find that the gauge fixing and ghost terms for the De Donder gauge have been obtained in the "gravitational part". In order to get expected terms in the "gauge part" we must redefine the fields $\bar{C}$ and b as

$$\bar{C}_a(x) \equiv \bar{C}^0{}_m(x)\int d^D y \sqrt{\overset{(D)}{g}(y)}\, e^m{}_a(y) \Big/ \overset{(D)}{V} \quad ,$$

$$b_a(x) \equiv b^1{}_m(x)\int d^D y \sqrt{\overset{(D)}{g}(y)}\, e^m{}_a(y) \Big/ \overset{(D)}{V} \quad . \tag{4-12}$$

Then we find that the "gauge part" of the action S_G becomes as

$$-i \int d^D y \sqrt{\overset{(D)}{g}}\, e^m{}_a \int d^4 x\, \delta\left(\sqrt{-g}\, \partial_\mu \bar{C}^0_m \cdot A^{a\mu}\right)$$

$$= -i\, \overset{(D)}{V} \int d^4 x\, \delta\left(\sqrt{-g}\, \partial_\mu \bar{C}_a \cdot A^{a\mu}\right) \tag{4-13}$$

$$= \overset{(D)}{V} \int d^4 x \sqrt{-g}\left[\partial_\mu b_a \cdot A^{a\mu} - \partial_\mu \bar{C}_a \cdot g^{\mu\nu} D_\nu C^a\right] \quad .$$

The first and second terms represent the gauge fixing and ghost terms for the Landau gauge, respectively. With the definition (4-12) the last two terms in (4-10) are replaced by

$$\delta \bar{C}_a = i\, b_a - C^\rho \partial_\rho \bar{C}_a \quad , \quad \delta b_a = -C^\rho \partial_\rho b_a \quad . \tag{4-14}$$

We find that the "gauge part" of the BRS transformation has been derived.

Let us quickly find the BRS current and the ghost current. Since the BRS current $\overset{(4+D)}{J_B}{}^M$ is the BRS transform of the ghost current $\overset{(4+D)}{J_c}{}^M$, $\overset{(4+D)}{J_B}{}^M = \delta\, \overset{(4+D)}{J_c}{}^M$, it is sufficient to find $\overset{(4+D)}{J_c}{}^M$. The Kaluza-Klein decomposition of $\overset{(4+D)}{J_c}{}^M$,

$$\overset{(4+D)}{J_c}{}^M = i\sqrt{-\overset{(4+D)}{g}}\, \partial_N \overset{(4+D)}{\bar{C}_P}\left(\overset{(4+D)}{g}{}^{MN} \overset{(4+D)}{C}{}^P + \overset{(4+D)}{g}{}^{MP} \overset{(4+D)}{C}{}^N - \overset{(4+D)}{g}{}^{NP} \overset{(4+D)}{C}{}^M\right)$$

$$\qquad - i\, \overset{(4+D)}{\bar{C}_N}\, \delta\left(\sqrt{-\overset{(4+D)}{g}}\, \overset{(4+D)}{g}{}^{MN}\right) \tag{4-15}$$

gives

$$
\begin{aligned}
J_c{}^{\mu} &\equiv \int d^D y \overset{(4+D)}{J}{}^{\mu} / \overset{(D)}{V} \\
&= i\sqrt{-g}\, \partial_\nu \bar{C}_\rho (g^{\mu\nu} C^\rho + g^{\mu\rho} C^\nu - g^{\nu\rho} C^\mu) \\
&\quad -i\, \bar{C}_\nu\, \delta (\sqrt{-g}\, g^{\mu\nu}) \\
&\quad +i\sqrt{-g}\, (\partial_\nu \bar{C}_a \cdot g^{\mu\nu} C^a - \bar{C}_a\, g^{\mu\nu} D_\nu C^a) \quad .
\end{aligned}
\tag{4-16}
$$

The current $J_c{}^{\mu}$ is conserved in the usual sense, $\partial_\mu J_c{}^{\mu} = 0$. We notice the gravitational and gauge parts of the ghost current in the four dimensional space-time are derived from the gravitational ghost current in the higher dimensional space-time as expected. A brief comment may be needed for other gauge conditions than the De Donder gauge. An additional term like $\frac{\alpha}{2} \eta^{MN} \overset{(4+D)}{b_M} \overset{(4+D)}{b_N}$ gives rise to quartic terms for the coloured ghosts and complicates the Feynman rule.

Above we have treated the theory with a groupmanifold G. The extension to the homogeneous space G/H is easy. We have only to replace eqs. (4-6, 12) by

$$
\overset{(4+D)}{C}{}^{\mu}(x,y) = C^{\mu}(x) \, , \quad \overset{(4+D)}{C}{}^{m}(x,y) = C^{\hat{b}}(x) D_{\hat{b}}{}^{c}(L(y)) e_c{}^{m}(y) \, , \tag{4-17}
$$

$$
\begin{aligned}
\bar{C}_{\hat{a}}(x) &\equiv \bar{C}^{\circ}{}_{m}(x) \int d^D y \sqrt{\overset{(D)}{g}(y)}\, D_{\hat{a}}{}^{b}(L(y))\, e_b{}^{m}(y) / \overset{(D)}{V} \, , \\
b_{\hat{a}}(x) &\equiv b^{1}{}_{m}(x) \int d^D y \sqrt{\overset{(D)}{g}(y)}\, D_{\hat{a}}{}^{b}(L(y))\, e_b{}^{m}(y) / \overset{(D)}{V} \, ,
\end{aligned}
\tag{4-18}
$$

where $D_{\hat{b}}^{\ c}$ is the adjoint representation of G. (cf. the appendix C)

We conclude that the BRS symmetry in the (4+D)-dimensional Einstein theory is consistently decomposed into the sum of the BRS symmetry in the four-dimensional Einstein and Yang-Mills theory.

4-2-2 Kaluza-Klein theory and extended BRS symmetry[19]

In the preceding section the Faddeev-Popov ghost fields C and $\bar{C}$ have not been treated symmetrically and a certain redefinition of C-field has been needed. In this section treating the ghost fields C and $\bar{C}$ symmetrically we extend the previous results to the extended BRS transformation.[66)∿68)]

We write down the Lagrangian[65)68)69)] for the (4+D)-dimensional quantum gravity as

$$\mathcal{L} = \mathcal{L}_E + \mathcal{L}_G \quad , \quad (4\text{-}19)$$

$$\mathcal{L}_E = \sqrt{-\overset{(4+D)}{g}}\ \overset{(4+D)}{R} \quad , \quad (4\text{-}20)$$

$$\mathcal{L}_G = -\frac{i}{2}\delta\bar{\delta}\{\tilde{\eta}_{MN}\sqrt{-\overset{(4+D)}{g}}(\overset{(4+D)}{g}{}^{MN} + i\alpha \overset{(4+D)}{C}{}^{M}\overset{(4+D)}{\bar{C}}{}^{N}) \quad , \quad (4\text{-}21)$$

Here δ and $\bar{\delta}$ represent the BRS transformation and anti- BRS transformation[68)69)], respectively, which are defined by

$$\delta \overset{(4+D)}{g}{}^{MN} = -(\overset{(4+D)}{g}{}^{MP}\partial_P \overset{(4+D)}{C}{}^{N} + \overset{(4+D)}{g}{}^{PN}\partial_P \overset{(4+D)}{C}{}^{M} - \overset{(4+D)}{C}{}^{P}\partial_P \overset{(4+D)}{g}{}^{MN}) \quad ,$$

$$\delta \overset{(4+D)}{C}{}^{M} = \overset{(4+D)}{C}{}^{P}\partial_P \overset{(4+D)}{C}{}^{M} \quad , \qquad (4\text{-}22)$$

$$\delta \overset{(4+D)}{\bar{C}}{}^{M} = i\overset{(4+D)}{b}{}^{M} + \overset{(4+D)}{C}{}^{P}\partial_P \overset{(4+D)}{\bar{C}}{}^{M} \quad ,$$

$$\delta \overset{(4+D)}{b}{}^{M} = \overset{(4+D)}{C}{}^{P}\partial_P \overset{(4+D)}{b}{}^{M} \quad ,$$

$$\bar{\delta} \overset{(4+D)}{g}{}^{MN} = \overset{(4+D)}{g}{}^{MP}\partial_P \overset{(4+D)}{C}{}^{N} + \overset{(4+D)}{g}{}^{PN}\partial_P \overset{(4+D)}{C}{}^{M} - \overset{(4+D)}{C}{}^{P}\partial_P \overset{(4+D)}{g}{}^{MN} \quad ,$$

$$\bar{\delta} \overset{(4+D)}{C}{}^{M} = i\overset{(4+D)}{b}{}^{M} - \overset{(4+D)}{\bar{C}}{}^{P}\partial_P \overset{(4+D)}{C}{}^{M} \quad , \qquad (4\text{-}23)$$

$$\bar{\delta} \overset{(4+D)}{\bar{C}}{}^{M} = -\overset{(4+D)}{\bar{C}}{}^{P}\partial_P \overset{(4+D)}{\bar{C}}{}^{M} \quad ,$$

$$\bar{\delta} \overset{(4+D)}{b}{}^{M} = -\overset{(4+D)}{\bar{C}}{}^{P}\partial_P \overset{(4+D)}{b}{}^{M} \quad ,$$

with the convention of the left-differentiation rule. In Eq. (4-21) $\tilde{\eta}_{MN} \equiv \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \eta_{mn} \end{pmatrix}$, where μ and ν run from 0 to 3 and m and n from 5 to 4+D. We choose $\eta_{\mu\nu}$ as $\begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$ and η_{mn} as δ_{mn}. The total action is invariant under the extended BRS transformation owing to the general invariance of the Einstein action (4-20) and the nilpotency property $\delta^2 = \bar{\delta}^2 = \{\delta, \bar{\delta}\} = 0$. Substituting (4-22), (4-23) into (4-21) we find $\mathcal{L}_G$ to be the sum of the gauge fixing term and the Faddeev-Popov ghost term:

$$\mathcal{L}_G = \sqrt{-\overset{(4+D)}{g}}\,\tilde{\eta}_{MN}\left(\overset{(4+D)}{g}{}^{MP}\partial_P \overset{(4+D)}{b}{}^N - \frac{\alpha}{2}\overset{(4+D)}{b}{}^M \overset{(4+D)}{b}{}^N - i\,\overset{(4+D)}{g}{}^{PQ}\partial_P \overset{(4+D)}{\bar{C}}{}^M \partial_Q \overset{(4+D)}{C}{}^N\right) \qquad (4\text{-}24)$$

+ (total divergence).

In order to study the dimensional reduction of the extended BRS transformation first we must determine the y-dependence of the Faddeev-Popov ghost fields $\overset{(4+D)}{C}{}^M$, $\overset{(4+D)}{\bar{C}}{}^M$ and the auxiliary field $\overset{(4+D)}{b}{}^M$. From the similar consideration as we did in the section 4-2-1 the y-dependence of the Faddeev-Popov ghost fields $\overset{(4+D)}{C}{}^M$, $\overset{(4+D)}{\bar{C}}{}^M$ is naturally assumed as

$$\overset{(4+D)}{C}{}^\mu(x,y) = C^\mu(x) \quad , \quad \overset{(4+D)}{C}{}^m(x,y) = e^m{}_a(y)\, C^a(x) \quad , \qquad (4\text{-}25)$$

$$\overset{(4+D)}{\bar{C}}{}^\mu(x,y) = \bar{C}^\mu(x) \quad , \quad \overset{(4+D)}{\bar{C}}{}^m(x,y) = e^m{}_a(y)\, \bar{C}^a(x) \quad . \qquad (4\text{-}26)$$

Next we assume the y-dependence of the auxiliary field $\overset{(4+D)}{b}{}^M$ as

$$\overset{(4+D)}{b}{}^\mu(x,y) = b^\mu(x) \quad ,$$

$$\begin{aligned} \overset{(4+D)}{b}{}^m(x,y) = {} & e^m{}_a(y)\, b^a(x) \\ & + \frac{i}{2}\left(e^n{}_a \partial_n e^m{}_b + e^n{}_b \partial_n e^m{}_a\right) C^a(x)\, \bar{C}^b(x) . \end{aligned} \qquad (4\text{-}27)$$

An apparently complicated form for $\overset{(4+D)}{b}{}^m$ is justified a posteriori.

From the ansatz (2-71) and (4-25,26,27) we immediately get the gravitational part of the extended BRS transformation correctly,

$$\begin{aligned}
\delta g^{\mu\nu} &= -(g^{\mu\rho}\partial_\rho C^\nu + g^{\rho\nu}\partial_\rho C^\mu - C^\rho\partial_\rho g^{\mu\nu}) \quad ,\\
\delta C^\mu &= C^\rho\partial_\rho C^\mu \quad ,\\
\delta \bar{C}^\mu &= i b^\mu + C^\rho\partial_\rho \bar{C}^\mu \quad ,\\
\delta b^\mu &= C^\rho\partial_\rho b^\mu \quad ,
\end{aligned} \tag{4-28}$$

$$\begin{aligned}
\bar{\delta} g^{\mu\nu} &= g^{\mu\rho}\partial_\rho \bar{C}^\nu + g^{\rho\nu}\partial_\rho \bar{C}^\mu - \bar{C}^\rho\partial_\rho g^{\mu\nu} \quad ,\\
\bar{\delta} C^\mu &= i b^\mu - \bar{C}^\rho\partial_\rho C^\mu \quad ,\\
\bar{\delta} \bar{C}^\mu &= -\bar{C}^\rho\partial_\rho \bar{C}^\mu \quad ,\\
\bar{\delta} b^\mu &= -\bar{C}^\rho\partial_\rho b^\mu \quad .
\end{aligned} \tag{4-29}$$

The gauge part of the extended BRS transformation can be calculated using the anticommuting property of C and $\bar{C}$,and Lie's differential equation (B- 6) together with the Jacobi identities $(C\times C)\times\bar{C} + (C\times\bar{C})\times C + (\bar{C}\times C)\times C = 0$ etc. with $(C\times\bar{C})^a = f^{abc}C^b\bar{C}^c$ and $f_{bc}{}^a = -f^{abc}$.
We obtain

$$\begin{aligned}
\delta A^{a\mu} &= g^{\mu\rho} D_\rho C^a - A^{a\rho}\partial_\rho C^\mu + C^\rho\partial_\rho A^{a\mu} \quad ,\\
\delta C^a &= -\tfrac{1}{2}(C\times C)^a + C^\rho\partial_\rho C^a \quad ,\\
\delta \bar{C}^a &= i b^a - \tfrac{1}{2}(C\times\bar{C})^a + C^\rho\partial_\rho \bar{C}^a \quad ,\\
\delta b^a &= \tfrac{1}{2}\{(b - \tfrac{i}{2} C\times C)\times C\}^a + C^\rho\partial_\rho b^a \quad .
\end{aligned} \tag{4-30}$$

$$\bar{\delta}\, A^{a\mu} = -g^{\mu\rho} D_\rho \bar{C}^a + A^{a\rho}\partial_\rho \bar{C}^\mu - \bar{C}^\rho \partial_\rho A^{a\mu}$$

$$\bar{\delta}\, C^a = i b^a + \frac{1}{2}(C\times\bar{C})^a - \bar{C}^\rho\partial_\rho C^a$$

$$\bar{\delta}\, \bar{C}^a = \frac{1}{2}(\bar{C}\times\bar{C})^a - \bar{C}^\rho\partial_\rho \bar{C}^a \tag{4-31}$$

$$\bar{\delta}\, b^a = -\frac{1}{2}\{(b + \frac{i}{2} C\times\bar{C})\times\bar{C}\}^a - \bar{C}^\rho\partial_\rho b^a .$$

Here we notice that on the right-hand side of eqs.(4-30,31) the gauge part and the gravitational part of the extended BRS transformation are distinctly separated. Now from these results the previous assumption (4-27) can be justified as follows. Consistency of the assumptions (4-25,26) and Eqs. (4-22,23, 30,31) requires

$$\begin{aligned}\delta\, \bar{C}^a &= e^a{}_m (i \overset{(4+D)}{b}{}^m + \overset{(4+D)}{C}{}^p \partial_p \overset{(4+D)}{\bar{C}}{}^m) \\ &= e^a{}_m (i \overset{(4+D)}{b}{}^m + e^n{}_b \partial_n e^m{}_c C^b \bar{C}^c) + C^\rho\partial_\rho \bar{C}^a \\ &= i b^a - \frac{1}{2}(C\times\bar{C})^a + C^\rho\partial_\rho\bar{C}^a ,\end{aligned}$$

$$\begin{aligned}\bar{\delta}\, C^a &= e^a{}_m (i \overset{(4+D)}{b}{}^m - \overset{(4+D)}{\bar{C}}{}^p \partial_p \overset{(4+D)}{C}{}^m) \\ &= e^a{}_m (i \overset{(4+D)}{b}{}^m - e^n{}_b \partial_n e^m{}_c \bar{C}^b C^c) - \bar{C}^\rho\partial_\rho C^a \\ &= i b^a + \frac{1}{2}(C\times\bar{C})^a - \bar{C}^\rho\partial_\rho C^a .\end{aligned}$$

Adding these two equations we get

$$2 i b^a = e^a{}_m \{ 2 i \overset{(4+D)}{b}{}^m + (e^n{}_b \partial_n e^m{}_c + e^n{}_c \partial_n e^m{}_b) C^b \bar{C}^c \} ,$$

which is nothing but eq.(4-27)

Finally we are going to examine the gauge fixing and the Faddeev-Popov ghost parts of the action,

$$S_G = -\frac{i}{2}\int d^{4+D}z\, \delta\bar{\delta}\{\tilde{\eta}_{MN}\sqrt{-\overset{(4+D)}{g}}(\overset{(4+D)}{g}{}^{MN} + i\alpha \overset{(4+D)}{C}{}^{M}\overset{(4+D)}{\bar{C}}{}^{N})\} \quad .$$

$[z \equiv (x, y)]$

From the ansatz (2-71) and (4-25,26) we get

$$\begin{aligned}
&\delta\bar{\delta}\{\tilde{\eta}_{MN}\sqrt{-\overset{(4+D)}{g}}(\overset{(4+D)}{g}{}^{MN} + i\alpha \overset{(4+D)}{C}{}^{M}\overset{(4+D)}{\bar{C}}{}^{N})\} \\
&= \sqrt{\overset{(D)}{g}}\ \delta\bar{\delta}\{\eta_{\mu\nu}\sqrt{-g}\,(g^{\mu\nu} + i\alpha\, C^{\mu}\bar{C}^{\nu})\} \\
&+ \sqrt{\overset{(D)}{g}}\ \delta_{mn}\, g^{mn}\, \delta\bar{\delta}\sqrt{-g} \\
&+ \sqrt{\overset{(D)}{g}}\ \delta_{mn}\, e^{m}_{a} e^{n}_{b}\ \delta\bar{\delta}\{\sqrt{-g}\,(A^{a\rho}A^{b}_{\rho} + i\alpha\, C^{a}\bar{C}^{b})\} \quad .
\end{aligned} \tag{4-32}$$

The first term implies the correct 4-dimensional gravitational term. The second term becomes a total divergence. Using eqs.(4-28,29,30,31) we get an explicit form of the third term:

$$\begin{aligned}
&\int d^{4+D}z\,\sqrt{\overset{(D)}{g}}\,\delta_{mn}\, e^{m}_{a} e^{n}_{b}\ \delta\bar{\delta}\{\sqrt{-g}\,(A^{a\rho}A^{b}_{\rho} + i\alpha\, C^{a}\bar{C}^{b})\} \\
&= \int d^4x\,(-2iM_{ab})\sqrt{-g}\,[\,A^{a\mu}D_{\mu}b^{b} + \frac{\alpha}{2}\, b^{a}b^{b} \\
&- \frac{i\alpha}{2}\{(b\times C)^{a}\bar{C}^{b} - (b\times\bar{C})^{a}C^{b}\} \\
&+ i\, g^{\mu\nu}D_{\mu}\bar{C}^{a}D_{\nu}C^{b} - \frac{i}{2}A^{a\mu}(\bar{C}\times D_{\mu}C - C\times D_{\mu}\bar{C})^{b} \\
&- \frac{\alpha}{8}\{(C\times C)^{a}(\bar{C}\times\bar{C})^{b} - (C\times\bar{C})^{a}(C\times\bar{C})^{b}\} \\
&- \frac{\alpha}{8}\{C^{a}((\bar{C}\times\bar{C})\times C)^{b} + \bar{C}^{a}((C\times C)\times\bar{C})^{b}\}\,] \quad ,
\end{aligned} \tag{4-33}$$

where $M_{ab} = \int d^D y \sqrt{g^{(0)}}\, \delta_{mn} e^m{}_a e^n{}_b$. Since obviously δ_{mn} is not a tensor the matrix M_{ab} depends on which coordinate system we take for y. If we can take the coordinate system such that $M_{ab} \propto \delta_{ab}$ the third term on the right-hand side of eq.(4-32) becomes

$$
\begin{aligned}
&\int d^4x\, \delta_{ab}\, \delta\bar{\delta}\{\sqrt{-g}\,(A^{a\rho}A^b_\rho + i\alpha\, C^a \bar{C}^b)\} \\
&= \int d^4x(-2i\sqrt{-g})\{ A^\mu \cdot \partial_\mu b + \frac{\alpha}{2} b^2 \\
&\quad + i\, g^{\mu\nu}\partial_\mu \bar{C}\cdot\partial_\nu C + \frac{i}{2} A^\mu \cdot (\bar{C}\times\partial_\mu C - C\times\partial_\mu \bar{C}) \\
&\quad - \frac{\alpha}{8}(C\times\bar{C})^2\} \qquad (4\text{-}34)
\end{aligned}
$$

This action is identical to that of Curci and Ferrari.[66)]

Now some comments are needed about the proportionality $M_{ab} \propto \delta_{ab}$. In the case of the SU(2) group we can check that $M_{ab} \propto \delta_{ab}$ by an explicit calculation choosing $L^{-1}\partial_m L = e_m{}^a Q_a$, $L = \exp(y^a Q_a)$ and $Q_a = \frac{i}{2}\sigma^a$, where σ^a is the Pauli matrix.(cf. the appendix B) For general cases, however, the author has no proof that we can always choose the coordinate so that $M_{ab} \propto \delta_{ab}$. If the proportionality $M_{ab} \propto \delta_{ab}$ does not hold, the expression (4-33) turns out to be a little messy. In principle, however, such a gauge fixing term is nothing wrong.

We can take another way. Namely we can choose η_{mn} as $g_{mn}(y)$ instead of δ_{mn}. Also in this case the total action is invariant under the extended BRS transformation, and eqs. (4-28,29,30,31) hold without modification.

But the explicit form of $\mathcal{L}_G$ becomes

$$\mathcal{L}_G = \sqrt{-\overset{(4+D)}{g}}\,\tilde{\eta}_{MN}\left(\overset{(4+D)}{g}{}^{MP}\partial_P \overset{(4+D)}{b}{}^N - \frac{\alpha}{2}\overset{(4+D)}{b}{}^M\overset{(4+D)}{b}{}^N - i\,\overset{(4+D)}{g}{}^{PQ}\partial_P\overset{(4+D)}{\bar{C}}{}^M\partial_Q\overset{(4+D)}{C}{}^N\right)$$
$$+\frac{i}{2}\tilde{\eta}_{MN}\partial_P\Big[\sqrt{-\overset{(4+D)}{g}}\Big\{i\,\tilde{g}^{MN}\overset{(4+D)}{b}{}^P + \alpha\overset{(4+D)}{b}{}^M\left(\overset{(4+D)}{C}{}^N\overset{(4+D)}{\bar{C}}{}^P - \overset{(4+D)}{\bar{C}}{}^N\overset{(4+D)}{C}{}^P\right) \qquad (4\text{-}35)$$
$$+2\,\overset{(4+D)}{g}{}^{MQ}\left(\overset{(4+D)}{\bar{C}}{}^P\partial_Q\overset{(4+D)}{C}{}^N - \overset{(4+D)}{C}{}^P\partial_Q\overset{(4+D)}{\bar{C}}{}^N\right)\Big\} + \partial_Q\left(\sqrt{-\overset{(4+D)}{g}}\,\tilde{g}^{MN}\overset{(4+D)}{C}{}^Q\overset{(4+D)}{\bar{C}}{}^P\right)\Big].$$

where $\tilde{g}^{MN} = \overset{(4+D)}{g}{}^{MN} + i\alpha\overset{(4+D)}{C}{}^M\overset{(4+D)}{\bar{C}}{}^N$. The second term is not a total divergence on account of the y-dependence of the matrix $\tilde{\eta}_{MN}$. In the case of $\alpha=0$ the gauge fixing condition in the (4+D)-dimensional space is not the De Donder gauge but

$$\partial_N\left(2\tilde{\eta}_{MP}\sqrt{-\overset{(4+D)}{g}}\,\overset{(4+D)}{g}{}^{MN}\right) - \left(\partial_P\,\tilde{\eta}_{MN}\right)\sqrt{-\overset{(4+D)}{g}}\,\overset{(4+D)}{g}{}^{MN} = 0 .$$

Now the equation corresponding to eq. (4-32) is

$$\delta\bar{\delta}\left\{\tilde{\eta}_{MN}\sqrt{-\overset{(4+D)}{g}}\left(\overset{(4+D)}{g}{}^{MN} + i\alpha\overset{(4+D)}{C}{}^M\overset{(4+D)}{\bar{C}}{}^N\right)\right\}$$
$$= \sqrt{\overset{(D)}{g}}\,\delta\bar{\delta}\left\{\eta_{\mu\nu}\sqrt{-g}\left(g^{\mu\nu} + i\alpha C^\mu\bar{C}^\nu\right)\right\}$$
$$+ D\sqrt{\overset{(D)}{g}}\,\delta\bar{\delta}\sqrt{-g} \qquad (4\text{-}36)$$
$$+ \sqrt{\overset{(D)}{g}}\,\delta_{ab}\,\delta\bar{\delta}\left\{\sqrt{-g}\left(A^{a\rho}A^b{}_\rho + i\alpha C^a\bar{C}^b\right)\right\} ,$$

where we have used the property $g_{mn}g^{mn} = D$ and $g_{mn}e^m_a e^n_b = \delta_{ab}$. Again the first term implies the 4-dimensional gravitational term and the second term becomes a total divergence. The third term becomes the Lagrangian of Curci and Ferrari[66] no matter which coordinate system we choose for the y-space.

A comment on the gauge fixing and the Faddeev-Popov ghost terms (4-21,34). By the usage of $\delta, \bar{\delta}$ we have written the gauge fixing and the Faddeev-Popov ghost parts of the Lagrangians (4-21,34) in a very compact way. This form is suggested by the superspace formulation of the extended BRS transformation.[67)69)] In the superspace formulation the extended BRS symmetry can be elegantly expressed. In this formulation the superfields of the Yang-Mills theory are

$$\begin{aligned} \phi^a_\mu &= A^a_\mu + \theta \bar{\delta} A^a_\mu + \bar{\theta} \delta A^a_\mu + \theta \bar{\theta} \delta \bar{\delta} A^a_\mu , \\ \phi^a &= C^a + \theta \bar{\delta} C^a + \bar{\theta} \delta C^a + \theta \bar{\theta} \delta \bar{\delta} C^a \\ \bar{\phi}^a &= \bar{C}^a + \theta \bar{\delta} \bar{C}^a + \bar{\theta} \delta \bar{C}^a + \theta \bar{\theta} \delta \bar{\delta} \bar{C}^a , \end{aligned} \tag{4-37}$$

where θ and $\bar{\theta}$ are the Grassmann coordinates of the superspace, and δ and $\bar{\delta}$ represent the BRS transformation and another BRS transformation. Then the gauge fixing and the Faddeev-Popov ghost parts of the Lagrangian are written as

$$\begin{aligned} \mathcal{L}_G &= \int d\theta d\bar{\theta} \left(\phi^a_\mu \phi^{a\mu} + i \alpha \phi^a \bar{\phi}^a \right) \\ &= - \delta \bar{\delta} \left(A^a_\mu A^{a\mu} + i \alpha C^a \bar{C}^a \right) . \end{aligned} \tag{4-38}$$

This equation suggests the left-hand side of eq. (4-34).

In conclusion, the extended BRS transformation in the (4+D)-dimensional Einstein theory is consistently decomposed into the sum of the extended BRS transformation in the

4-dimensional Einstein and Yang-Mills theory.

4-2-3 BRS symmetry of massive tensor fields and physical S-matrix unitarity[21)]

The BRS symmetry of massive tensor fields was first derived in ref.21 by the present author. Following this work let us examine the BRS symmetry of massive tensor fields by means of the dimensional reduction technique and establish the total physical S-matrix unitarity in the Kaluza-Klein theory.

We consider the five dimensional Kaluza-Klein theory to illustrate the procedure in the simplest example. The five dimensional quantum Einstein action[65)] is written as

$$S = S_E + S_G \quad , \tag{4-39}$$

$$S_E = \int_0^{2\pi} dx_5 \int d^4x \sqrt{-\overset{(5)}{g}}\, \overset{(5)}{R} \quad , \tag{4-40}$$

$$S_G = \int_0^{2\pi} dx_5 \int d^4x\, (-i)\, \delta\left(\partial_M \overset{(5)}{\bar{C}}_N \cdot \sqrt{-\overset{(5)}{g}}\, \overset{(5)}{g}{}^{MN} \right) \quad , \tag{4-41}$$

where $R_{MN} = \partial_P \Gamma^P_{MN} - \cdots$ and

$$\begin{aligned} -i\delta\left(\partial_M \overset{(5)}{\bar{C}}_N \cdot \sqrt{-\overset{(5)}{g}}\, \overset{(5)}{g}{}^{MN}\right) = & -\sqrt{-\overset{(5)}{g}}\, \overset{(5)}{g}{}^{MN} \partial_M \overset{(5)}{b}_N \\ & + i\, \partial_M \overset{(5)}{\bar{C}}_N \Big\{ \sqrt{-\overset{(5)}{g}} \big(\overset{(5)}{g}{}^{MP} \partial_P \overset{(5)}{C}{}^N + \overset{(5)}{g}{}^{PN} \partial_P \overset{(5)}{C}{}^M \big) \\ & - \partial_P \big(\sqrt{-\overset{(5)}{g}}\, \overset{(5)}{g}{}^{MN} \overset{(5)}{C}{}^P \big) \Big\} \quad . \end{aligned} \tag{4-42}$$

Hereafter we employ the convention that capital latin letters M, N run 0,1,2,3,5 and greek letters μ, ν run 0,1,2,3. In the action (4-41) we do not take the unitary gauge condition ($\partial_5 \overset{(5)}{g}_{5M}=0$) but take the De Donder gauge condition($\partial_M(\sqrt{-\overset{(5)}{g}}\,\overset{(5)}{g}{}^{MN})=0$) , since we want to investigate the BRS symmetry. The total action (4-39) is invariant under the BRS transformation[65]:

$$
\begin{aligned}
\delta \overset{(5)}{g}{}^{MN} &= \partial_P \overset{(5)}{C}{}^M \cdot \overset{(5)}{g}{}^{PN} + \partial_P \overset{(5)}{C}{}^N \cdot \overset{(5)}{g}{}^{PM} - \overset{(5)}{C}{}^P \partial_P \overset{(5)}{g}{}^{MN} &, \\
\delta \overset{(5)}{b}_M &= 0 &, \\
\delta \overset{(5)}{C}{}^M &= -\overset{(5)}{C}{}^P \partial_P \overset{(5)}{C}{}^M &, \\
\delta \overset{(5)}{\bar{C}}_M &= -i \overset{(5)}{b}_M &,
\end{aligned}
\qquad (4\text{-}43)
$$

where we use the left-differentiation convention.

When we expand the field in the five dimensional space-time in the harmonic functions of the compact internal manifold, we have a series of massive fields in addition to massless fields in the physical four dimensional space-time[13]. Then the full action and the full BRS transformation become considerably complicated. However in order to establish the physical S-matrix unitarity in the standard Kugo-Ojima formalism we have only to consider asymptotic fields provided we assume asymptotic completeness, since the total action is obviously invariant under the full BRS transformation[16]. Let us consider the harmonic expansion of the asymptotic fields:

$$
\begin{aligned}
\overset{(5)}{g}_{MN}(x, x_5) &\longrightarrow \eta_{MN} + \sum_{n=-\infty}^{\infty} h^{(n)}_{MN}(x)\, e^{inx_5} \,, \\
\overset{(5)}{b}_{M}(x, x_5) &\longrightarrow \sum_{n=-\infty}^{\infty} \beta^{(n)}_{M}(x)\, e^{inx_5} \,, \\
\overset{(5)}{C}{}^{M}(x, x_5) &\longrightarrow \sum_{n=-\infty}^{\infty} \gamma^{M}_{(n)}(x)\, e^{inx_5} \,, \\
\overset{(5)}{\bar{C}}_{M}(x, x_5) &\longrightarrow \sum_{n=-\infty}^{\infty} \bar{\gamma}^{(n)}_{M}(x)\, e^{inx_5} \,,
\end{aligned}
\qquad (4\text{-}44)
$$

where $\eta_{MN} = diag(-++++)$. In terms of these asymptotic fields the action becomes

$$
S \longrightarrow 2\pi \sum_{n=0}^{\infty} S^{(n)}_{as} \,,
$$

$$
S^{(n)}_{as} = \int d^4x\, \mathcal{L}^{(n)}_{as} \qquad (4\text{-}45)
$$

$$
\begin{aligned}
\mathcal{L}^{(n)}_{as} = \sum_{m=\pm n} \Big[& \tfrac{1}{2} \partial_\mu h^{(-m)\nu}{}_{P} \cdot \partial_\nu h^{(m)P\mu} - \tfrac{1}{2} \partial_\mu h^{(-m)P}{}_{P} \cdot \partial_\nu h^{(m)\mu\nu} \\
& + \tfrac{1}{4} \partial^\rho h^{(-m)M}{}_{M} \cdot \partial_\rho h^{(m)N}{}_{N} - \tfrac{1}{4} \partial_\rho h^{(-m)MN} \cdot \partial^\rho h^{(m)}_{MN} \\
& + im\, \partial_\mu h^{(-m)5}{}_{P} \cdot h^{(m)P\mu} - im\, \partial_\mu h^{(-m)P}{}_{P} \cdot h^{(m)\mu 5} \\
& + \tfrac{m^2}{2} h^{(-m)5}{}_{P}\, h^{P5}_{(m)} - \tfrac{m^2}{2} h^{(-m)P}{}_{P}\, h^{55}_{(m)} + \tfrac{m^2}{4} h^{(-m)M}{}_{M}\, h^{(m)N}{}_{N} - \tfrac{m^2}{4} h^{(-m)}_{MN}\, h^{MN}_{(m)} \\
& - \partial_\mu h^{\mu N}_{(-m)} \cdot \beta^{(m)}_{N} + \tfrac{1}{2} \partial^\mu h^{(-m)P}{}_{P} \cdot \beta^{(m)}_{\mu} + im\, h^{(-m)}_{5N}\, \beta^{N}_{(m)} - \tfrac{im}{2} h^{(-m)P}{}_{P}\, \beta^{(m)}_{5} \\
& + i\, \partial_\mu \bar{\gamma}^{(-m)}_{N} \cdot \partial^\mu \gamma^{N}_{(m)} + i m^2 \bar{\gamma}^{(-m)}_{N}\, \gamma^{N}_{(m)} \Big] \,,
\end{aligned}
\qquad (4\text{-}46)
$$

and the BRS transformation reads

$$\delta h^{(n)}_{\mu\nu} = -(\partial_\mu \gamma_\nu^{(n)} + \partial_\nu \gamma_\mu^{(n)}) \quad ,$$

$$\delta h^{(n)}_{\mu 5} = -(\partial_\mu \gamma_5^{(n)} + i n \gamma_\mu^{(n)}) \quad ,$$

$$\delta h^{(n)}_{55} = -2 i n \gamma_5^{(n)} \quad , \qquad (4\text{-}47)$$

$$\delta \beta_M^{(n)} = \delta \gamma_M^{(n)} = 0 \quad ,$$

$$\delta \bar{\gamma}_M^{(n)} = -i \beta_M^{(n)} \quad .$$

The asymptotic action $S^{(n)}_{as}$ is invariant under the asymptotic BRS transformation as it should be. From the asymptotic action (4-46) we get equations of motion as

$$\left.\begin{aligned} (\Box - n^2) h^{(n)}_{\mu\nu} + \partial_\mu \beta_\nu^{(n)} + \partial_\nu \beta_\mu^{(n)} &= 0 \quad , \\ (\Box - n^2) h^{(n)}_{\mu 5} + \partial_\mu \beta_5^{(n)} + i n \beta_\mu^{(n)} &= 0 \quad , \\ (\Box - n^2) h^{(n)}_{55} + 2 i n \beta_5^{(n)} &= 0 \quad , \end{aligned}\right\} \qquad (4\text{-}48)$$

$$\partial^\rho h^{(n)}_{\mu\rho} - \frac{1}{2} \partial_\mu h^{(n)\rho}{}_{\rho} + i n h^{(n)}_{55} = 0 \quad , \qquad (4\text{-}49)$$

$$\partial^\rho h^{(n)}_{5\rho} - \frac{in}{2} h^{(n)\rho}{}_{\rho} + \frac{in}{2} h^{(n)}_{55} = 0 \quad , \qquad (4\text{-}50)$$

$$(\Box - n^2)\beta_M^{(n)} = 0 \quad , \quad (\Box - n^2)\gamma_M^{(n)} = 0 \quad , \quad (\Box - n^2)\bar{\gamma}_M^{(n)} = 0 \quad , \qquad (4\text{-}51)$$

which can be written in a compact form

$$(\Box - n^2) h^{(n)}_{MN} + \partial_M \beta^{(n)}_N + \partial_N \beta^{(n)}_M = 0 \quad , \tag{4-52}$$

$$\partial^N h^{(n)}_{MN} - \frac{1}{2} \partial_M h^{(n)N}_{\ \ N} = 0 \quad , \tag{4-53}$$

$$\left[\ (\Box - n^2)^2 h^{(n)}_{MN} = 0 \ \right] \quad , \tag{4-54}$$

with the convention that $\partial_5 \equiv i n$. The equations (4-48) imply that the fields $h^{(n)}_{MN}$ ($n \neq 0$) are massive tensor fields with the mass $|n|$, and the equations (4-49,50) correspond to the divergenceless condition ($\partial_\mu h^{\mu\nu} \sim 0$) and the traceless condition ($h_\mu^{\ \mu} \sim 0$) for massive tensor fields, respectively.

As the action (4-46) has no derivative terms of $\beta^{(n)}_M$, we regard the $\beta^{(n)}_M$ fields as dependent variables and we require the following canonical (anti-)commutation relations:

$$\left.\begin{aligned}
&[\, h^{(n)}_{KL}(x) \, , \, \Pi^{(n)}_{MN}(y)]_{x^0=y^0} = \frac{i}{2} (\eta_{KM}\eta_{LN} + \eta_{KN}\eta_{LM}) \delta(\vec{x}-\vec{y}) \quad , \\
&\{\, \gamma^{(n)}_M(x) \, , \, \Pi^{(n)}_{\gamma_N}(y) \}_{x^0=y^0} = -i \eta_{MN} \delta(\vec{x}-\vec{y}) \quad , \\
&\{\, \bar{\gamma}^{(n)}_M(x) \, , \, \Pi^{(n)}_{\bar{\gamma}_N}(y) \}_{x^0=y^0} = -i \eta_{MN} \delta(\vec{x}-\vec{y}) \quad ,
\end{aligned}\right. \tag{4-55}$$

otherwise=0, where

$$\left.\begin{aligned}
\Pi^{(n)}_{\mu\nu} = {} & \partial_{(\mu} h^{(-n)0}_{\nu)} - \frac{1}{2}\eta_{\mu\nu}\partial_\rho h^{0\rho}_{(-n)} - \frac{1}{2}\delta^0_{(\mu}\partial_{\nu)} h^{(-n)\rho}_{\ \ \ \rho} + \frac{1}{2}\eta_{\mu\nu}\partial^0 h^{(-n)\rho}_{\ \ \ \rho} \\
& - \frac{1}{2}\partial^0 h^{(-n)}_{\mu\nu} + i n \eta_{\mu\nu} h^{05}_{(-n)} - \delta^0_{(\mu}\beta^{(-n)}_{\nu)} + \frac{1}{2}\eta_{\mu\nu}\beta^0_{(-n)} \quad , \\
\Pi^{(n)}_{\mu 5} = {} & \frac{1}{2}\partial_\mu h^{(-n)0}_{\ \ 5} - \frac{1}{2}\partial^0 h^{(-n)}_{\mu 5} - \frac{in}{2} h^{(-n)0}_{\mu} - \frac{1}{2}\delta^0_\mu \beta^{(-n)}_5 \quad ,
\end{aligned}\right\} \tag{4-56}$$

$$\Pi^{(n)}_{55} = -\frac{1}{2}\partial_\rho h^{0\rho}_{(-n)} + \frac{1}{2}\partial^0 h^{(-n)\rho}_{\rho} + \frac{1}{2}\beta^{c}_{(-n)} ,$$

$$\Pi^{(n)}_{\gamma_N} = -i\,\partial^0\bar{\gamma}^{(-n)}_N , \quad \Pi^{(n)}_{\bar{\gamma}_N} = i\,\partial^0\gamma^{(-n)}_N ,$$

$$\left[\ T_{(\mu\nu)} \equiv \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})\ \right] .$$

Now it is easy to see that the fields h , β , γ and $\bar{\gamma}$ can be written as

$$h^{(n)}_{MN}(x) = \int d^3z \left[\Delta(x-z;n^2)\overleftrightarrow{\partial}{}^{z}_{0}\, h^{(n)}_{MN}(z) + E^{(n)}(x-z)\,\overleftrightarrow{\partial}{}^{z}_{0}(\Box_z - n^2)\,h^{(n)}_{MN}(z)\right] ,$$

$$\beta^{(n)}_M(x) = \int d^3z\ \Delta(x-z;n^2)\overleftrightarrow{\partial}{}^{z}_{0}\,\beta^{(n)}_M(z) ,$$

$$\gamma^{(n)}_M(x) = \int d^3z\ \Delta(x-z;n^2)\overleftrightarrow{\partial}{}^{z}_{0}\,\gamma^{(n)}_M(z) , \tag{4-57}$$

$$\bar{\gamma}^{(n)}_M(x) = \int d^3z\ \Delta(x-z;n^2)\overleftrightarrow{\partial}{}^{z}_{0}\,\bar{\gamma}^{(n)}_M(z) ,$$

$$\left(A\overleftrightarrow{\partial}{}^{z}_{0} B \equiv A\,\partial^z_0 B - \partial^z_0 A\cdot B \right) .$$

and that the right-hand side of these equations is independent of z^0 . Here the functions Δ and $E^{(n)}$ satisfy the following relations:

$$(\Box - n^2)\Delta(x;n^2) = 0 , \quad \partial^x_0\Delta\,|_{x^0=0} = \delta(\vec{x}) ,$$

$$(\Box - n^2)^2 E^{(n)}(x) = 0 , \quad (\Box - n^2)E^{(n)}(x) = \Delta(x;n^2) , \tag{4-58}$$

$$\Delta(x;n^2) = \frac{i}{(2\pi)^3}\int d^4p\ \epsilon(p^0)\,\delta(p^2+n^2)\,e^{ip\cdot x} ,$$

$$E^{(n)}(x) = \frac{i}{(2\pi)^3}\int d^4p\ \epsilon(p^0)\,\delta'(p^2+n^2)\,e^{ip\cdot x} .$$

Employing the equations (4-48~58) we get the four-dimensional (anti-)commutation relations*):

$$
\begin{aligned}
[h^{(n)}_{KL}(x), h^{(-n)}_{MN}(y)] &= i\left(\tfrac{2}{3}\eta_{KL}\eta_{MN}-\eta_{KM}\eta_{LN}-\eta_{KN}\eta_{LM}\right)\Delta(x-y;n^2) \\
&\quad + i\left(\eta_{KN}\partial^x_M\partial^x_L+\eta_{LN}\partial^x_M\partial^x_K+\eta_{KM}\partial^x_N\partial^x_L+\eta_{LM}\partial^x_N\partial^x_K\right)E^{(n)}(x-y) \quad , \\
[h^{(n)}_{KL}(x), \beta^{(-n)}_{M}(y)] &= i\left(\eta_{KM}\partial^x_L+\eta_{LM}\partial^x_K\right)\Delta(x-y;n^2) \quad , \\
[\beta^{(n)}_{M}(x), \beta^{(-n)}_{N}(y)] &= 0 \quad , \\
\{\gamma^{(n)}_{M}(x), \bar{\gamma}^{(-n)}_{N}(y)\} &= \eta_{MN}\Delta(x-y;n^2) \quad ,
\end{aligned}
\tag{4-59}
$$

with $\partial_5 \equiv i n$. If we formally write the fields as

---

*) The factor 2/3 in the equation (4-59) may be understood if we notice that the (anti-)commutation relations in the 4+N-dimensional quantum Einstein theory:

$$
\begin{aligned}
[h_{KL}(x), h_{MN}(y)] &= i\left(\tfrac{2}{2+N}\eta_{KL}\eta_{MN}-\eta_{KM}\eta_{LN}-\eta_{KN}\eta_{LM}\right)D(x-y) \\
&\quad + i\left(\eta_{KN}\partial^x_M\partial^x_L+\eta_{LN}\partial^x_M\partial^x_K+\eta_{KM}\partial^x_N\partial^x_L+\eta_{LN}\partial^x_N\partial^x_K\right)E(x-y) \quad , \\
[h_{KL}(x), \beta_M(y)] &= i\left(\eta_{KM}\partial^x_L+\eta_{LM}\partial^x_K\right)D(x-y) \quad , \\
[\beta_M(x), \beta_N(y)] &= 0 \quad , \quad \{\gamma_M(x), \bar{\gamma}_N(y)\} = \eta_{MN}D(x-y) \quad ,
\end{aligned}
$$

where

$$
D(x-y) \equiv \frac{i}{(2\pi)^{3+N}}\int d^{4+N}p\; \epsilon(p^0)\,\delta(p^2)\,e^{ip\cdot(x-y)} \quad ,
$$

$$
E(x-y) \equiv \frac{i}{(2\pi)^{3+N}}\int d^{4+N}p\; \epsilon(p^0)\,\delta'(p^2)\,e^{ip\cdot(x-y)} \quad .
$$

$$h^{(n)}_{MN}(x) = \int \frac{d^4p}{(2\pi)^{\frac{3}{2}}}\, \theta(p^0)\left[h^{(n)}_{MN}(p)\, e^{ip\cdot x} + h^{(n)\dagger}_{MN}(p)\, e^{-ip\cdot x} \right] ,$$

$$\beta^{(n)}_{M}(x) = \int \frac{d^4p}{(2\pi)^{\frac{3}{2}}}\, \theta(p^0)\left[\beta^{(n)}_{M}(p)\, e^{ip\cdot x} + \beta^{(n)\dagger}_{M}(p)\, e^{-ip\cdot x} \right] ,$$

$$\gamma^{(n)}_{M}(x) = \int \frac{d^4p}{(2\pi)^{\frac{3}{2}}}\, \theta(p^0)\left[\gamma^{(n)}_{M}(p)\, e^{ip\cdot x} + \gamma^{(n)\dagger}_{M}(p)\, e^{-ip\cdot x} \right] , \tag{4-60}$$

$$\bar{\gamma}^{(n)}_{M}(x) = \int \frac{d^4p}{(2\pi)^{\frac{3}{2}}}\, \theta(p^0)\left[\bar{\gamma}^{(n)}_{M}(p)\, e^{ip\cdot x} + \bar{\gamma}^{(n)\dagger}_{M}(p)\, e^{-ip\cdot x} \right] ,$$

we get

$$\begin{aligned}\left[h^{(n)}_{KL}(p), h^{(n)\dagger}_{MN}(q) \right] = {}& \left(\eta_{KM}\eta_{LN} + \eta_{KN}\eta_{LM} - \tfrac{2}{3}\eta_{KL}\eta_{MN} \right)\theta(p^0)\,\delta(p^2+n^2)\,\delta^4(p-q) \\ & + \left(\eta_{KN}P_M P_L + \eta_{LN}P_M P_K + \eta_{KM}P_N P_L + \eta_{LM}P_N P_K \right)\times \\ & \times \theta(p^0)\,\delta'(p^2+n^2)\,\delta^4(p-q) ,\end{aligned}$$

$$\left[h^{(n)}_{KL}(p), \beta^{(n)\dagger}_{M}(q) \right] = -i\left(\eta_{KM}P_L + \eta_{LM}P_K \right)\theta(p^0)\,\delta(p^2+n^2)\,\delta^4(p-q) , \tag{4-61}$$

$$\left[\beta^{(n)}_{M}(p), \beta^{(n)\dagger}_{N}(q) \right] = 0 ,$$

$$\left\{ \gamma^{(n)}_{M}(p), \bar{\gamma}^{(n)\dagger}_{N}(q) \right\} = i\,\eta_{MN}\,\theta(p^0)\,\delta(p^2+n^2)\,\delta^4(p-q) ,$$

with $P_5 \equiv n$.

Now we consider the massive modes($n \neq 0$) first. Let us take the rest frame, $P_1 = P_2 = P_3 = 0$. Then the equations (4-49, 50) imply that

$$n\, h^{(n)}_{5k} = -p^0 h^{(n)}_{0k} \quad (k = 1, 2, 3) ,$$

$$n\, p^0 \left(h^{(n)}_{00} + h^{(n)}_{55} \right) = -\left(p_0^2 + n^2 \right) h^{(n)}_{05} , \tag{4-62}$$

$$n\, p^0 \left(h^{(n)}_{11} + h^{(n)}_{22} + h^{(n)}_{33} \right) = \left(p_0^2 - n^2 \right) h^{(n)}_{05} ,$$

which show that there remain ten independent components of fields for each n ($\neq 0$). We set

$$\varphi_1^{(n)} \equiv h_{12}^{(n)}, \quad \varphi_2^{(n)} \equiv h_{23}^{(n)}, \quad \varphi_3^{(n)} \equiv h_{31}^{(n)}, \quad \varphi_4^{(n)} \equiv \frac{\sqrt{3}}{2} h_{11}^{(n)}, \quad \varphi_5^{(n)} \equiv \frac{h_{22}^{(n)} - h_{33}^{(n)}}{2},$$
$$\chi_0^{(n)} \equiv \frac{1}{i}\left\{ \frac{h_{05}^{(n)}}{n} - \frac{P_0}{2n^2}\left(h_{55}^{(n)} + \frac{h_{11}^{(n)}}{2}\right)\right\}, \quad \chi_k^{(n)} \equiv \frac{h_{0k}^{(n)}}{i P_0} \ (k=1,2,3), \quad \chi_5^{(n)} \equiv \frac{h_{55}^{(n)} + \frac{h_{11}^{(n)}}{2}}{2in}. \tag{4-63}$$

After a suitable rescaling of fields the (anti-)commutation relations of these fields are found to be

$$\mathcal{M}_{ab} \equiv \begin{array}{c|ccccc} & \varphi_N^\dagger & \chi_N^\dagger & \beta_N^\dagger & \gamma_N^\dagger & \bar{\gamma}_N^\dagger \\ \hline \varphi_M & \delta_{MN} & 0 & 0 & 0 & 0 \\ \chi_M & 0 & \omega_{MN} & -\eta_{MN} & 0 & 0 \\ \beta_M & 0 & -\eta_{MN} & 0 & 0 & 0 \\ \gamma_M & 0 & 0 & 0 & 0 & i\eta_{MN} \\ \bar{\gamma}_M & 0 & 0 & 0 & -i\eta_{MN} & 0 \end{array} \tag{4-64}$$

This table is written in the symbolic notation and should be read as e.g. $[\varphi_M, \varphi_N^\dagger] = \delta_{MN}$. The BRS transformation (4-47) of these fields can be rewritten as

$$\begin{aligned} \delta \varphi_M^{(n)} &= [iQ_B, \varphi_M^{(n)}] = 0, \\ \delta \chi_M^{(n)} &= [iQ_B, \chi_M^{(n)}] = -\gamma_M^{(n)}, \\ \delta \beta_M^{(n)} &= [iQ_B, \beta_M^{(n)}] = 0, \\ \delta \gamma_M^{(n)} &= \{iQ_B, \gamma_M^{(n)}\} = 0, \\ \delta \bar{\gamma}_M^{(n)} &= \{iQ_B, \bar{\gamma}_M^{(n)}\} = -i\beta_M^{(n)}, \end{aligned} \tag{4-65}$$

where Q_B is the generator of the asymptotic BRS transformation (4-47,65):

$$Q_B = 2\pi \sum_{n=0}^{\infty} Q_B^{(n)} \quad , \tag{4-66}$$
$$Q_B^{(n)} = \sum_{m=\pm n} \int d^3x \left(\partial_0 \beta_P^{(-m)} \cdot \gamma_{(m)}^P - \beta_P^{(-m)} \partial_0 \gamma_{(m)}^P \right) \quad .$$

The equations (4-65) show that φ is a BRS singlet and that(χ, $\beta, \gamma, \bar{\gamma}$) makes a BRS quartet. Hence we regard $\varphi_M^{(n)}$ as physical particles and $\chi_M^{(n)}$ as Higgs-like particles. We impose the subsidary condition to the physical state ($|phys\rangle$) as

$$Q_B \quad |phys\rangle = 0 \quad . \tag{4-67}$$

Then the physical S-matrix unitarity of the massive tensor sector can be established as shown below for each n ($\neq 0$) in the same way as in the Kugo-Ojima formalism[16]. The projection operator $P_{(N)}^{(n)}$ onto the N-unphysical-particle sector can be written as

$$P_{(N)}^{(n)} \equiv \frac{1}{N} \sum_{a,b} \mathcal{M}^{-1}{}_{ab}\, a^+ P_{(N-1)}^{(n)} b \quad ,$$
$$\doteq \frac{1}{N} \Big(-\chi_M^{(n)+} P_{(N-1)}^{(n)} \beta_{(n)}^M - \beta_M^{(n)+} P_{(N-1)}^{(n)} \chi_{(n)}^M - \omega^{MN} \beta_M^{(n)+} P_{(N-1)}^{(n)} \beta_N^{(n)}$$
$$+ i\, \gamma_M^{(n)+} P_{(N-1)}^{(n)} \bar{\gamma}_{(n)}^M - i\, \bar{\gamma}_M^{(n)+} P_{(N-1)}^{(n)} \gamma_{(n)}^M \Big) \quad , \tag{4-68}$$

$$= \left\{ Q_B \,,\, R_{(N)}^{(n)} \right\} \qquad (N \geq 1) \quad , \tag{4-69}$$

with $R_{(N)}^{(n)} \equiv \frac{1}{N} \left(\bar{\gamma}_M^{(n)+} P_{(N-1)}^{(n)} \chi_{(n)}^M + \chi_M^{(n)+} P_{(N-1)}^{(n)} \bar{\gamma}_{(n)}^M + \omega^{MN} \beta_M^{(n)+} P_{(N-1)}^{(n)} \bar{\gamma}_N^{(n)} \right)$.

We see for any physical states $|f\rangle$, $|g\rangle$ that

$$\langle f | P_{(N)}^{(n)} | g \rangle = 0 \qquad (N \geq 1) \quad ,$$
$$\langle f | f \rangle = \langle f | \sum_{N=0}^{\infty} P_{(N)}^{(n)} | f \rangle = \langle f | P_{(0)}^{(n)} | f \rangle = \langle P_{(0)}^{(n)} f, P_{(0)}^{(n)} f \rangle \geq 0 \quad . \tag{4-70}$$

Since the hermiticity of the Hamiltonian is obvious and the time-independence of the physical subspace which is the total vector space of the physical state $|phys\rangle$ is guaranteed, eq. (4-70) shows the physical S-matrix unitarity for each massive mode ($n \neq 0$).

As for the zero modes we can proceed in the same way as in the case of massive modes stated above. In the massless case ($n=0$) we can choose the frame as $p_1 = p_2 = 0$ and write

$$\varphi_1^t \equiv \frac{h_{11}^{(0)} - h_{22}^{(0)}}{2}, \quad \varphi_2^t \equiv h_{12}^{(0)}, \quad \varphi_1^v \equiv h_{51}^{(0)}, \quad \varphi_2^v \equiv h_{52}^{(0)}, \quad \varphi^s \equiv \frac{\sqrt{3}}{2} h_{55}^{(0)},$$

$$\chi_0^t \equiv \frac{h_{00}^{(0)} - \frac{h_{55}^{(0)}}{2}}{2i p_0}, \quad \chi_k^t \equiv \frac{h_{0k}}{i p_0} \quad (k=1,2), \quad \chi_3^t \equiv \frac{h_{33}^{(0)} + \frac{h_{55}^{(0)}}{2}}{2i p_3}, \tag{4-71}$$

$$\chi^v \equiv \frac{h_{50}^{(0)}}{i p_0} .$$

The (anti-)commutation relations of these fields are

$$\mathcal{M}_{ab}^{(0)} = \begin{pmatrix} \mathcal{M}^t & 0 & 0 \\ 0 & \mathcal{M}^v & 0 \\ 0 & 0 & \mathcal{M}^s \end{pmatrix} , \tag{4-72}$$

$$\mathcal{M}^t \equiv \begin{array}{c|ccccc} & \varphi_j^{t\,\dagger} & \chi_\nu^{t\,\dagger} & \beta_\nu^\dagger & \gamma_\nu^\dagger & \bar{\gamma}_\nu^\dagger \\ \hline \varphi_i^t & \delta_{ij} & 0 & 0 & 0 & 0 \\ \chi_\mu^t & 0 & \omega_{\mu\nu} & -\eta_{\mu\nu} & 0 & 0 \\ \beta_\mu & 0 & -\eta_{\mu\nu} & 0 & 0 & 0 \\ \gamma_\mu & 0 & 0 & 0 & 0 & i\eta_{\mu\nu} \\ \bar{\gamma}_\mu & 0 & 0 & 0 & -i\eta_{\mu\nu} & 0 \end{array} ,$$

$$\mathcal{M}^v \equiv \begin{array}{c|ccccc} & \varphi_j^{v\dagger} & \chi^{v\dagger} & \beta_5^\dagger & \gamma_5^\dagger & \bar{\gamma}_5^\dagger \\ \hline \varphi_i^v & \delta_{ij} & 0 & 0 & 0 & 0 \\ \chi^v & 0 & \omega & -1 & 0 & 0 \\ \beta_5 & 0 & -1 & 0 & 0 & 0 \\ \gamma_5 & 0 & 0 & 0 & 0 & i \\ \bar{\gamma}_5 & 0 & 0 & 0 & -i & 0 \end{array} ,$$

$$\mathcal{M}^s \equiv \begin{array}{c|c} & \varphi^{s\dagger} \\ \hline \varphi^s & 1 \end{array} .$$

And the BRS transformations become

$$\begin{cases} \delta\varphi_i^t = 0 \\ \delta\chi_\mu^t = -\gamma_\mu \\ \delta\beta_\mu = \delta\gamma_\mu = 0 \\ \delta\bar{\gamma}_\mu = -i\beta_\mu \end{cases} , \quad \begin{cases} \delta\varphi_i^v = 0 \\ \delta\chi^v = -\gamma_5 \\ \delta\beta_5 = \delta\gamma_5 = 0 \\ \delta\bar{\gamma}_5 = -i\beta_5 \end{cases} , \tag{4-73}$$

$$\delta\varphi^s = 0 .$$

We notice that each φ is a BRS singlet and that each (χ , β , γ , $\bar{\gamma}$) makes a BRS quartet. As the (anti-)commutation relations and the BRS transformation are similar to those of the massive modes, we can prove the physical S-matrix unitarity in the massless sector in the same way as the massive modes. We will not repeat the procedure here.

We conclude that a unitary model of massive spin 2 fields can be constructed through the dimensional reduction technique and that we can prove the total physical S-matrix unitarity in the Kaluza-Klein theory combining the results in the massive and massless sectors. In the full five dimensional Kaluza-Klein theory the physical modes are a series of massive tensor fields, a massless graviton, a massless photon and a massless scalar field.[13] In other words we need these infinite numbers of field to construct a unitary model of massive tensor fields with interactions. We summarize the BRS symmetry of the five dimensional Kaluza-Klein theory in the table 3.

table 3

We cannot find any obstructions to conjecture that as for the Kaluza-Klein theory in the higher than five dimensional space-time the total physical S-matrix unitarity can be proved in the same way as demonstrated above in the simplest example.[*)]

§4-3 Comments on the quantum Kaluza-Klein theory

Let us comment on the relation between the higher and lower dimensional quantizations. The path-integral formalism

*) A report on a unitary quantum Kaluza-Klein theory in the six dimensional space-time has been done at meeting in Kyoto University.[70]

of the five dimensional quantum theory is formulated in the functional integration[71]:

$$W = \int\!\!\!\int d\Phi(x,x_5)\, e^{i\int_0^{2\pi} dx_5 \int d^4x\, \mathcal{L}(\Phi(x,x_5))} \qquad , \qquad (4\text{-}74)$$

where $\Phi(x,x_5)$ is a five dimensional field. When we perform the harmonic expansion (Fourier expansion) of the field as

$$\Phi(x,x_5) = \sum_{n=-\infty}^{\infty} e^{inx_5}\, \Phi^{(n)}(x) \qquad , \qquad (4\text{-}75)$$

and change the variable $\Phi(x,x_5)$ into $\Phi^{(n)}(x)$ in eq. (4-75), the eq. (4-74) will be replaced by

$$W = J \int\!\!\!\int \prod_n d\Phi^{(n)}(x)\, e^{i\sum_n \int d^4x\, \mathcal{L}^{(n)}(\Phi^{(n)}(x))} \qquad , \qquad (4\text{-}76)$$

where Jacobian J is only a constant. Though this discussion is considerably naive,*) it suggests that the higher and lower dimensional quantum theory are equivalent only when the massive modes are taken into consideration.

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*) In the path-integral formulation of the gravitational field we must take account of the problem of the measure of the integral.[72] We will not discuss this problem here.

§5. Possible test of the Kaluza-Klein theory and massive Particles

In this section we consider the possible way to test the Kaluza-Klein theory and in particular we examine one of the special features of its heavy particles, namely *the gyromagnetic ratio*.

§5-1 Test of the Kaluza-Klein theory

Let us consider the way how we can test the Kaluza-Klein theory as a matter of principle. The most direct test would be to find out the existence of the extra space. However since its size is considered to be as small as the Planck length, the best we can do is to look at the massive particles of the order of the Planck mass, which are the consequences of the existence of the extra space. (cf. §2-1-5) At such a high energy the gravitational quantum effect will be important and one might think that we cannot say anything definite about such heavy particles. However, as we have seen before the first excited mode in the expansion (2-22) has the mass $(\sim \frac{\sqrt{\alpha}}{2} m_{Pl})$ that is a little smaller than the Planck mass m_{Pl} .*) Hence it is possible to test the existence of the extra space in principle. Then it will be needed to find out the special

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*) The size of the extra space can be considerably larger than the Planck length according to Weinberg.[54)]

features of the heavy particles in the Kaluza-Klein theory,*) which cannot be seen for the more conventional heavy particles.

§5-2 Gyromagnetic ratio of heavy particles[22)]

In this section we study a particular aspect of electro-mangetic interactions of the heavy spinning particles, namely the gyromagnetic ratio, because there is a possibility that

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*) One of the effects of the extra space may be the correction on the four dimensional causality or Lorentz invariance. The (4+D)-dimensional microscopic causality can be represented as

$$[\varphi(z), \varphi(z')] = 0 \qquad \text{for} \quad (z-z')^2 > 0$$
$$\text{i.e.} \quad (x-x')^2 + (y-y')^2 > 0 ,$$

which is a little different from the four dimensional one[73)]

$$[\varphi(x), \varphi(x')] = 0 \qquad \text{for} \quad (x-x')^2 > 0 .$$

Though this correction does not tend to violate the four dimensional causality, it will change the dispersion relation of the forward scattering amplitude, though it will be tiny. If the size of the extra space is much larger than the Planck length,[54)] this correction would be used one of the tests of the Kaluza-Klein therory.

the gyromagnetic g-factor of them differs from the ordinary value[74] g=2 due to their nonminimal interaction with the electromagnetic field. (cf. eqs. (2-27, 38) To be specific let us consider the original five dimensional Kaluza-Klein theory.

5-2-1 Massive tensor field

In order to evaluate the g-factor of the massive tensor field we only have to examine the linearized field equation of h_{AB} , where h_{AB} is the fluctuation of the five dimensional metric tensor arround the Kaluza-Klein background. (See eq.(2-21)) This field equation has been already derived as (2-27). Going into the rest frame of our particle and assuming that only the magnetic field $\vec{B}$ is non-zero we can derive the change of the energy due to the magnetic interaction as[74]

$$\Delta E = -\frac{q}{2M}\,\vec{B}\cdot\vec{S} \quad , \qquad (5\text{-}1)$$

$$E \equiv M + \Delta E \quad , \qquad (5\text{-}2)$$

where $\vec{S}$ represents the spin vector of the massive tensor field and its charge q and mass M have been determined in §2-1-5. With the reduced magnetic moment $\vec{\mu}$ in the unit of the Bohr magneton $\frac{q}{2M}$ the g-factor is defined as

$$\Delta E = -\frac{q}{2M}\,\vec{B}\cdot\vec{\mu} \quad , \qquad (5\text{-}3)$$

$$g \equiv \frac{|\vec{\mu}|}{|\vec{S}|} \quad . \qquad (5\text{-}4)$$

The equations (5-1∿4) shows the magnetic g factor of the massive spin two particle is unity, g = 1. Incidentally the last term of eq. (2-27) represents the Thomas precession.

5-2-2 Massive vector field

Let us consider an extra Abelian gauge field $\overset{(5)}{V}_M$ in the background of the five dimensional Kaluza-Klein ansatz (2-2, 15), $\bar{\bar{E}}_M{}^A$. Take Lagrangian, which is added to the five dimensional Einstein Lagrangian, as

$$\mathcal{L}_V = -\frac{1}{4}\sqrt{-\overset{(5)}{g}}\,\overset{(5)}{W}_{MN}\overset{(5)}{W}{}^{MN} \qquad , \qquad (5\text{-}5)$$

with $\overset{(5)}{W}_{MN} \equiv \partial_M \overset{(5)}{V}_N - \partial_N \overset{(5)}{V}_M$. If we do the harmonic expansion for the extra gauge field

i.e.

$$\overset{(5)}{V}_A(x, x_5) = v_A{}^{(0)}(x) + \sum_{n=1}^{\infty}\left\{ e^{inx_5} v_A{}^{(n)}(x) + h.c.\right\} \qquad , \qquad (5\text{-}6)$$

the massive modes $v_A{}^{(n)}$ $(n \neq 0)$ interacts with the gauge field A_μ which comes from the background (2-2, 15) in a particular way that we shall see below. In order to extract the physical modes we impose the gauge condition $v_5^{(n)} = 0$ $(n \neq 0)$. Using the weak field approximation ($\Gamma \sim 0$, $A^2 \sim 0$, $\partial F \sim 0$) for simplicity we derive the equation of motion for the massive spin 1 particle as

$$D_\beta D^\beta v_\alpha{}^{(n)} - \frac{n^2}{\ell^2} v_\alpha{}^{(n)} - i e n F_\alpha{}^\beta v_\beta{}^{(n)} - \frac{ie\ell^2}{n} F^{\beta\gamma} D_\alpha D_\beta v_\gamma{}^{(n)} = 0 \qquad , \qquad (5\text{-}7)$$

$$D^{\alpha} \mathcal{V}_{\alpha}^{(n)} - \frac{ie\ell^2}{n} F^{\alpha\beta} D_{\alpha} \mathcal{V}_{\beta}^{(n)} = 0 \qquad , \tag{5-8}$$

with $D_{\rho} \equiv \partial_{\rho} - ienA_{\rho}$. The equation (5-7) indicates that the mass M and charge q of the massive vector field $\mathcal{V}_A^{(n)}$ are $|n| \cdot \frac{\sqrt{\alpha}}{2} m_{pl}$ and en, respectively. In the same way as the massive tensor field it is easy to observe that the gyromagnetic g-factor of the massive vector field is unity: g = 1. The last term of eq. (5-7) implies the Thomas precession.

5-2-3 Massive spinor field

The dimensional reduction of the five dimensional spinor field in the Kaluza-Klein background has been previously obtained in §2-1-7. Again appealing to the weak field approximation ($\Gamma \sim 0$, $A^2 \sim 0$, $\partial F \sim 0$) we write down the field equation of the massive spinor field $\psi^{(n)}$ as

$$\left(\gamma^{\alpha} D_{\alpha} - \frac{n}{\ell} + \frac{ie\ell}{16} F_{\alpha\beta} [\gamma^{\alpha}, \gamma^{\beta}] \right) \psi^{(n)} = 0 \qquad , \tag{5-9}$$

with $D_{\rho} \equiv \partial_{\rho} - ienA_{\rho}$. Multiplying eq. (5-9) by the operata: $\gamma^{\alpha} D_{\alpha} + \frac{n}{\ell} - \frac{ie\ell}{16} F_{\alpha\beta} [\gamma^{\alpha}, \gamma^{\beta}]$, we get

$$\Big(D_{\alpha} D^{\alpha} - \frac{n^2}{\ell^2} - \frac{ien}{8} F_{\alpha\beta} [\gamma^{\alpha}, \gamma^{\beta}] + \frac{ie\ell}{16} [\gamma^{\alpha}, [\gamma^{\beta}, \gamma^{\gamma}]] F_{\beta\gamma} D_{\alpha} \Big) \psi^{(n)} = 0 \ . \tag{5-10}$$

From this equation we can easily derive the magnetic g-facor of the massive spinor field as g = 1. Since the g-factor of a massive spinor field interacting minimally with the electromagnetic field is two, our result g = 1 has an origin in the Pauli term in eq. (2-38). Again the last term of the equation (5-10) represents the Thomas precession.

5-2-4 Classical spinning particle

We are now going to show that the magnetic g factor is universally g = 1 for an arbitrary charged spinning particle which emerges from the harmonic expansion in the Kaluza-Klein theory. For that purpose let us consider the geodesic equations[75] for the velocity U^M and the spin S_M of the classical charged particle in the five dimensional space-time. It reads

$$\frac{DU^M}{D\overset{(5)}{\tau}} = \frac{dU^M}{d\overset{(5)}{\tau}} + \overset{(5)}{\Gamma}{}^M{}_{PQ} U^P U^Q = 0 \quad , \qquad (5\text{-}11)$$

$$\frac{DS_M}{D\overset{(5)}{\tau}} = \frac{dS_M}{d\overset{(5)}{\tau}} - \overset{(5)}{\Gamma}{}^P{}_{MQ} U^Q S_P = 0 \quad ,$$

with the masslessness condition, $U^M U_M = 0$. Let us go over to the local inertial frame. The Kaluza-Klein ansatz gives the equations of motion as

$$\frac{dU^\alpha}{d\overset{(5)}{\tau}} - e\ell F^\alpha{}_\beta U^\beta U^5 = 0 \quad ,$$

$$\frac{dU^5}{d\overset{(5)}{\tau}} = 0 \quad , \qquad (5\text{-}12)$$

$$\frac{dS_\alpha}{d\overset{(5)}{\tau}} - \frac{e\ell}{2} F_{\alpha\beta} U^\beta S_5 - \frac{e\ell}{2} F_\alpha{}^\beta S_\beta U^5 = 0 \quad ,$$

$$\frac{dS_5}{d\overset{(5)}{\tau}} - \frac{e\ell}{2} F_\beta{}^\gamma U^\beta S_\gamma = 0 \quad .$$

we write the conserved quantity $U^{A=5}$ as $U^{A=5} = n$ which can be identified with a charge. The first equation in (5-12) implies the equation of motion of a charged particle under the Lorentz force. We redefine the velocity and the spin as

$$u^A \equiv \frac{1}{|n|} U^A \quad , \quad s_A \equiv \mathcal{S}_A - \frac{n}{|n|^2} U_A \mathcal{S}_{A=5} \quad , \tag{5-13}$$

to get $u^\alpha u_\alpha = -1$ and $s_{A=5} = 0$. If we require $\mathcal{S}_A U^A = 0$, we get the transversality $s_\alpha u^\alpha = 0$. It is easy to check that $s_\alpha s^\alpha$ is constant. (Multiply eq. (5-14) by s^α) Hence u^α and s_α can be interpreted as the physical velocity and spin of our point particle. We finally obtain

$$\frac{d s_\alpha}{d\tau} - \frac{e\ell n}{2|n|} F_\alpha{}^\beta s_\beta + \frac{e\ell n}{2|n|} F_\beta{}^\gamma u_\alpha u^\beta s_\gamma = 0 \quad , \tag{5-14}$$

where $d\tau = |n| \overset{(5)}{\tau}$ is the proper time interval in the physical four dimensional world. The equation (5-14) indicates that the gyromagnetic g factor of our classical spinning particle is unity. The last term of the equation (5-14) represents the Thomas precession.[74]

5-2-5 Discussion on g = 1

As we have seen above the magnetic g factor is unity for any massive spinning particle in the Kaluza-Klein theory of the five dimensional space-time.

Since it is known that the classical orbital motion of a charged particle causes a magnetic moment with g = 1,[76] perhaps our result $g = 1$ may be related to the fact that the electromagnetic interaction in our case has its origin in the orbital motion in the direction of the extra coordinate. The result $g = 1$ reminds us of the old story on the magnetic moment of the electron.[77)] We comment that the high energy behavior of the Compton scattering amlitudes of such particles with $g = 1$ will be worse than with $g = 2$,[78)] and this difficulty may come from the fact that the original five dimensional Einstein action is not renormalizable.

Though it is known that the gyromagnetic g-factor of the real light particles like leptons, quarks and W, Z bosons is nearly two, our particles with $g = 1$ are considered to be as heavy as the Planck mass and obviously cannot be identified with these real light particles. Even if we consider a theory where the size of the extra space is not so small, it is clear from the result $g = 1$ that we should not regard the real particles to be the massive modes in the five dimensional Kaluza-Klein theory. Since our result is based on the five dimensional theory, g-factor may differ from unity in a certain higher dimensional theory. We will have to construct a reallistic unified theory in which we can really treat these light particles in the nature. Anyhow, in principle, if super high energy experiments or cosmological observations are possible, the characteristic value of g-factor for the heavy spinning particles can be used to test the Kaluza-Klein theory.

§6. Summary

The Kaluza-Klein theory has a mathematical beauty, therefore it is now one of the most promising unified theories of all interactions in the nature. Since the Kaluza-Klein theory is closely related to supergravity, we may expect to obtain a consistent quantum theory of gravitation based on it in future.

If the extra space predicted by the Kaluza-Klein theory really exists, it is natural to perform quantization in the higher dimensional space-time. We have shown that the four dimensional BRS symmetry can be obtained correctly through the dimensional reduction from the higher dimensional BRS symmetry. This result can be extended to the extended BRS symmetry. We have also seen that the physical S-matrix unitarity can be established in the whole quantum Kaluza-Klein theory. As a biproduct of this procedure we have obtained a unitary model of interacting massive spin 2 particles.

On the other hand to test the Kaluza-Klein theory, it is necessary to examine the characteristic properties of the massive particles predicted by it. As a special aspect of these particles we have shown that the gyromagnetic g-factor of them is universally unity in the five dimensional theory.

Although we have several unsolved problems in constructing a realistic Kaluza-Klein theory, we may expect they will be solved in future. Since the Kaluza-Klein theory is a beautiful system, we may hope that at least it has some sort of truth.

Acknowledgements

The author is much obliged to Prof. A. Hosoya for helpful suggegtions and instructions as well as careful reading of the manuscript. He is also grateful to Dr. K. Yamagishi for valuable discussions, and he would like to thank Prof. K. Kikkawa, Mrs. Tsuji and other members of Kikkawa seminor for encouragement and kind help.

Appendix A. General relativity in the higher dimensional space-time

Consider (4+D)-dimensional vielbein $\overset{(4+D)}{E}{}_M{}^A(z)$.[13] Here $z=(x,y)$ and z, x, y are the [(4+D), 4, D]-dimensional coordinates, respectively. Letters M and A represent world indices and frame labels, respectively. The (4+D)-dimensional metric tensor $\overset{(4+D)}{g}{}_{MN}$ is defined as

$$\overset{(4+D)}{g}{}_{MN}(z) \equiv \overset{(4+D)}{E}{}_M{}^A(z)\,\overset{(4+D)}{E}{}_N{}^B(z)\,\eta_{AB} \tag{A-1}$$

with $\eta_{AB} \equiv diag(-1, 1, 1 \cdots 1)$. The vielbein transforms like

$$\overset{(4+D)}{E}{}_M{}^A(z) \rightarrow \overset{(4+D)}{E}{}'_M{}^A(z') = \frac{\partial z^N}{\partial z'^M}\,\overset{(4+D)}{E}{}_N{}^A(z) \tag{A-2}$$

under the general coordinate transformation, and it transforms like

$$\overset{(4+D)}{E}{}_M{}^A(z) \rightarrow \overset{(4+D)}{E}{}'_M{}^A(z) = \overset{(4+D)}{E}{}_M{}^A(z)\,a_B{}^A(z) \tag{A-3}$$

with

$$a_C{}^A(z)\,a_D{}^B(z)\,\eta_{AB} = \eta_{CD} \tag{A-4}$$

under the local Lorentz transformation. The local Lorentz transformation (frame rotation) belongs to SO(1, 3 + D) . The (4+D)-dimensional spinor $\overset{(4+D)}{\psi}(z)$ has $2^{[\frac{4+D}{2}]}$ components. It is a scalar:

$$\overset{(4+D)}{\Psi}(z) \longrightarrow \overset{(4+D)}{\Psi}{}'(z') = \overset{(4+D)}{\Psi}(z) \tag{A-5}$$

under the coordinate transformation but transforms as SO(1,3+D) spinor:

$$\overset{(4+D)}{\Psi}(z) \longrightarrow \overset{(4+D)}{\Psi}{}'(z) = S(a^{-1})\overset{(4+D)}{\Psi}(z) \tag{A-6}$$

under the frame rotation. The covariant derivative for the spinor field $\overset{(4+D)}{\Psi}(z)$ is defined as

$$\overset{(4+D)}{\nabla}_M \overset{(4+D)}{\Psi} = (\partial_M + \overset{(4+D)}{B}_M)\overset{(4+D)}{\Psi} \tag{A-7}$$

In order to obtain the covariance:
$\overset{(4+D)}{\nabla}_M \overset{(4+D)}{\Psi} \rightarrow \overset{(4+D)}{\nabla}{}'_M \overset{(4+D)}{\Psi}{}' = (\partial_M + \overset{(4+D)}{B}{}'_M)\overset{(4+D)}{\Psi}{}' = S\overset{(4+D)}{\nabla}_M \overset{(4+D)}{\Psi}$ under the frame rotation, we should assume that $\overset{(4+D)}{B}{}'_M = S\overset{(4+D)}{B}_M S^{-1} + S\partial_M S^{-1}$.
The (4+D)-dimensional Dirac-matrix $\overset{(4+D)}{\Gamma}{}^A$ satisfies the Clifford algebra $\{\overset{(4+D)}{\Gamma}{}^A, \overset{(4+D)}{\Gamma}{}^B\} = 2\eta^{AB}$. Suppose the field $\overset{(4+D)}{\overline{\Psi}} (\equiv \overset{(4+D)}{\Psi}{}^{+}\overset{(4+D)}{\Gamma}{}^0)$ transforms like $\overset{(4+D)}{\overline{\Psi}}(z) \rightarrow \overset{(4+D)}{\overline{\Psi}}{}'(z) = \overset{(4+D)}{\overline{\Psi}}(z)\, S^{-1}(a^{-1})$ under the frame rotation. The Dirac Lagrangian in the (4+D)-dimensional space-time is written as

$$\mathcal{L}_\psi = i\sqrt{-\overset{(4+D)}{g}}\;\overset{(4+D)}{\overline{\Psi}}\;\overset{(4+D)}{E}_A{}^M\;\overset{(4+D)}{\Gamma}{}^A\;\overset{(4+D)}{\nabla}_M\;\overset{(4+D)}{\Psi} \tag{A-8}$$

In order this Lagrangian to be invariant under frame rotation, S and a should satisfy $a^A{}_B\overset{(4+D)}{\Gamma}{}^B = S(a^{-1})\overset{(4+D)}{\Gamma}{}^A S^{-1}(a^{-1})$.
Such an S can be expanded by $[\overset{(4+D)}{\Gamma}{}^A, \overset{(4+D)}{\Gamma}{}^B]$,
therefore the spinor connection $\overset{(4+D)}{B}_M$ can be also expanded by $[\overset{(4+D)}{\Gamma}{}^A, \overset{(4+D)}{\Gamma}{}^B]$, namely

$$\overset{(4+D)}{B}_M \equiv \frac{1}{8} \overset{(4+D)}{B}_{M[AB]} [\overset{(4+D)}{\Gamma}{}^A, \overset{(4+D)}{\Gamma}{}^B] \quad . \tag{A-9}$$

From the vielbein $\overset{(4+D)}{E}_M{}^A$ and the spinor connection $\overset{(4+D)}{B}_{M[AB]}$ let us define the 1-form:

$$\overset{(4+D)}{E}{}^A \equiv \overset{(4+D)}{E}{}^A{}_M dZ^M \quad , \quad \overset{(4+D)}{B}_{AB} \equiv \overset{(4+D)}{B}_{M[AB]} dZ^M \quad . \tag{A-10}$$

The torsion 2-form is defined by

$$\overset{(4+D)}{T}{}^A \equiv d\overset{(4+D)}{E}{}^A + \overset{(4+D)}{E}{}^B \wedge \overset{(4+D)}{B}_B{}^A \tag{A-11}$$

and the curvature 2-form by

$$\overset{(4+D)}{R}_A{}^B \equiv d\overset{(4+D)}{B}_A{}^B + \overset{(4+D)}{B}_A{}^C \wedge \overset{(4+D)}{B}_C{}^B \quad . \tag{A-12}$$

Raising or lowering the indices M and A is performed by $\overset{(4+D)}{g}_{MN}$ and η_{AB} , respectively. Expanding these 2-forms like

$$\overset{(4+D)}{T}{}^A \equiv \frac{1}{2} \overset{(4+D)}{T}_{MN}{}^A dZ^M \wedge dZ^N \quad , \quad \overset{(4+D)}{R}_A{}^B \equiv \frac{1}{2} \overset{(4+D)}{R}_{MNA}{}^B dZ^M \wedge dZ^N , \tag{A-13}$$

we obtain

$$\overset{(4+D)}{T}_{MN}{}^A = \partial_M \overset{(4+D)}{E}_N{}^A - \partial_N \overset{(4+D)}{E}_M{}^A + \overset{(4+D)}{E}_M{}^B \overset{(4+D)}{B}_{N[B}{}^{A]} - \overset{(4+D)}{E}_N{}^B \overset{(4+D)}{B}_{M[B}{}^{A]} ,$$

$$\overset{(4+D)}{R}_{MNAB} = \partial_M \overset{(4+D)}{B}_{N[AB]} - \partial_N \overset{(4+D)}{B}_{M[AB]} + \overset{(4+D)}{B}_{M[A}{}^{C]} \overset{(4+D)}{B}_{N[CB]} - \overset{(4+D)}{B}_{N[A}{}^{C]} \overset{(4+D)}{B}_{M[CB]} \quad . \tag{A-14}$$

If we impose the torsionless condition $\overset{(4+D)}{T}{}^A = 0$, we find the relation between the spinor connection $\overset{(4+D)}{B}_{M[AB]}$ and the vielbein $\overset{(4+D)}{E}_M{}^A$ as

$$\overset{(4+D)}{B}_{M[BC]} = \frac{1}{2}\overset{(4+D)}{E}_M{}^A\left(\overset{(4+D)}{\Omega}_{[AB]C} - \overset{(4+D)}{\Omega}_{[BC]A} + \overset{(4+D)}{\Omega}_{[CA]B}\right) \tag{A-15}$$

with $\overset{(4+D)}{\Omega}_{[AB]C} \equiv \overset{(4+D)}{E}_A{}^M \overset{(4+D)}{E}_B{}^N (\partial_M \overset{(4+D)}{E}_{NC} - \partial_N \overset{(4+D)}{E}_{MC})$. In this case the Riemann tensor,

$$\overset{(4+D)}{R}{}^M{}_{NPQ} \equiv \partial_P \overset{(4+D)}{\Gamma}{}^M{}_{NQ} - \partial_Q \overset{(4+D)}{\Gamma}{}^M{}_{NP} + \overset{(4+D)}{\Gamma}{}^M{}_{PR}\overset{(4+D)}{\Gamma}{}^R{}_{NQ} - \overset{(4+D)}{\Gamma}{}^M{}_{QR}\overset{(4+D)}{\Gamma}{}^R{}_{NP} \tag{A-16}$$

with $\overset{(4+D)}{\Gamma}{}^M{}_{PQ} = \frac{1}{2}\overset{(4+D)}{g}{}^{MN}(\partial_P \overset{(4+D)}{g}_{NQ} + \partial_Q \overset{(4+D)}{g}_{NP} - \partial_N \overset{(4+D)}{g}_{PQ})$ and the field $\overset{(4+D)}{R}_{MNAB}$ in eq. (A-14) are related as [4]

$$\overset{(4+D)}{R}_{MNPQ} = \overset{(4+D)}{E}_P{}^A \overset{(4+D)}{E}_Q{}^B \overset{(4+D)}{R}_{MNAB} \tag{A-17}$$

Next we summarize useful formulae as follows. Denoting world indices M, N, frame labels A, B and spinor sufices Φ, Ψ ($\overset{(4+D)}{\bar\Psi}\overset{(4+D)}{\Psi} \equiv \overset{(4+D)}{\bar\Psi}{}^{\Phi}\overset{(4+D)}{\Psi}_{\Phi}$) we can write the covariant derivative of the most general field $T_M{}^N{}_A{}^B{}_\Phi{}^\Psi$ which is covariant under both coordinate transformation and frame rotation as

$$\begin{aligned}
&\overset{(4+D)}{\nabla}_P T_M{}^N{}_A{}^B{}_\Phi{}^\Psi \\
&= \partial_P T_M{}^N{}_A{}^B{}_\Phi{}^\Psi - \overset{(4+D)}{\Gamma}{}^Q{}_{PM} T_Q{}^N{}_A{}^B{}_\Phi{}^\Psi + \overset{(4+D)}{\Gamma}{}^N{}_{PQ} T_M{}^Q{}_A{}^B{}_\Phi{}^\Psi \\
&\quad + \overset{(4+D)}{B}_{P[A}{}^{C]} T_M{}^N{}_C{}^B{}_\Phi{}^\Psi + \overset{(4+D)}{B}_P{}^{[B}{}_{C]} T_M{}^N{}_A{}^C{}_\Phi{}^\Psi \\
&\quad + \frac{1}{8}\overset{(4+D)}{B}_{M[CD]}[\overset{(4+D)}{\Gamma}{}^C, \overset{(4+D)}{\Gamma}{}^D]_\Phi{}^\Sigma T_M{}^N{}_A{}^B{}_\Sigma{}^\Psi - \frac{1}{8}\overset{(4+D)}{B}_{M[CD]}[\overset{(4+D)}{\Gamma}{}^C, \overset{(4+D)}{\Gamma}{}^D]_\Sigma{}^\Psi T_M{}^N{}_A{}^B{}_\Phi{}^\Sigma .
\end{aligned} \tag{A-18}$$

For example,

$$\overset{(4+D)}{\nabla}_M \overset{(4+D)}{E}_N{}^A = 0 \quad , \quad \overset{(4+D)}{\nabla}_M (\overset{(4+D)}{\Gamma}{}^A)_{\Phi}{}^{\Psi} = 0 \tag{A-19}$$

and they correspond to the fact that $\overset{(4+D)}{E}_M{}^A$ is the vielbein from which the metric $\overset{(4+D)}{g}_{MN}$ is constructed and $\overset{(4+D)}{\Gamma}{}^A$ is a constant matrix, respectively. Commutation relations of the covariant derivatives satisfy

$$\begin{aligned}
[\overset{(4+D)}{\nabla}_M , \overset{(4+D)}{\nabla}_N] \Psi &= \frac{1}{8} \overset{(4+D)}{R}_{MNAB} [\overset{(4+D)}{\Gamma}{}^A , \overset{(4+D)}{\Gamma}{}^B] \Psi \quad , \\
[\overset{(4+D)}{\nabla}_M , \overset{(4+D)}{\nabla}_N] V^P &= \overset{(4+D)}{R}{}^P{}_{QMN} V^Q \quad , \\
[\overset{(4+D)}{\nabla}_M , \overset{(4+D)}{\nabla}_N] T_{PQ} &= -(\overset{(4+D)}{R}{}^R{}_{PMN} T_{RQ} + \overset{(4+D)}{R}{}^R{}_{QMN} T_{PR}) \quad .
\end{aligned} \tag{A-20}$$

Under the weak field approximation[79],

$$\overset{(4+D)}{g}_{MN} = \bar{g}_{MN} + h_{MN} \quad , \tag{A-21}$$

the curvature density can be written as

$$\begin{aligned}
\sqrt{-\overset{(4+D)}{g}} \overset{(4+D)}{R} = \sqrt{-\bar{g}} \Big[\bar{R} &+ \frac{1}{2} h_M{}^M \bar{R} - h^{MN} \bar{R}_{MN} \\
&+ \left(\frac{1}{8} h_M{}^M h_N{}^N - \frac{1}{4} h_{MN} h^{MN} \right) \bar{R} \\
&- \frac{1}{2} h_P{}^P h^{MN} \bar{R}_{MN} + h^M{}_P h^{NP} \bar{R}_{MN} \\
&+ \frac{1}{2} \bar{\nabla}_P h_M{}^Q \bar{\nabla}_Q h^{MP} - \frac{1}{2} \bar{\nabla}_P h_M{}^M \bar{\nabla}_Q h^{PQ} \\
&+ \frac{1}{4} \bar{\nabla}^P h_M{}^M \bar{\nabla}_P h_N{}^N - \frac{1}{4} \bar{\nabla}_P h^{MN} \bar{\nabla}^P h_{MN} \Big] + \text{tot. div.} \; .
\end{aligned} \tag{A-22}$$

The sufices M , N are raised or lowered by $\bar{g}_{MN}$. Under the conformal transformation,

$$\overset{(4+D)}{g}{}'_{MN} = K \overset{(4+D)}{g}_{MN} \quad , \tag{A-23}$$

the curvature tensors are transformed like

$$\begin{aligned}
\overset{(4+D)}{R}{}'_{MNPQ} = K \overset{(4+D)}{R}_{MNPQ} \\
&+ \frac{1}{2}\{(\overset{(4+D)}{g}_{MQ}\overset{(4+D)}{\nabla}_P - \overset{(4+D)}{g}_{MP}\overset{(4+D)}{\nabla}_Q)\partial_N K + (\overset{(4+D)}{g}_{PN}\overset{(4+D)}{\nabla}_Q - \overset{(4+D)}{g}_{NQ}\overset{(4+D)}{\nabla}_P)\partial_M K\} \\
&+ \frac{1}{4K}\{3\partial_M K(\overset{(4+D)}{g}_{NQ}\partial_P K - \overset{(4+D)}{g}_{NP}\partial_Q K) - 3\partial_N K(\overset{(4+D)}{g}_{MQ}\partial_P K - \overset{(4+D)}{g}_{MP}\partial_Q K) \\
&\quad + (\overset{(4+D)}{g}_{MQ}\overset{(4+D)}{g}_{NP} - \overset{(4+D)}{g}_{MP}\overset{(4+D)}{g}_{NQ})\overset{(4+D)}{g}{}^{RS}\partial_R K\partial_S K\} \quad ,
\end{aligned} \tag{A-24}$$

$$\begin{aligned}
\overset{(4+D)}{R}{}'_{MN} &= \overset{(4+D)}{R}_{MN} - \frac{1}{2K}\{(2+D)\overset{(4+D)}{\nabla}_M\partial_N K + \overset{(4+D)}{g}_{MN}\overset{(4+D)}{g}{}^{PQ}\overset{(4+D)}{\nabla}_P\partial_Q K\} \\
&\quad - \frac{1}{4K^2}\{-(6+3D)\partial_M K\partial_N K + D\overset{(4+D)}{g}_{MN}\overset{(4+D)}{g}{}^{PQ}\partial_P K\partial_Q K\} \quad ,
\end{aligned}$$

$$\overset{(4+D)}{R}{}' = \frac{1}{K}\overset{(4+D)}{R} - \frac{(3+D)}{K^2}\overset{(4+D)}{g}{}^{MN}\overset{(4+D)}{\nabla}_M\partial_N K - \frac{(3+D)(-2+D)}{4K^3}\overset{(4+D)}{g}{}^{MN}\partial_M K\partial_N K \quad .$$

Finally, if a d-dimensional space-time is the maximally symmetric space[75] with $\frac{d(d+1)}{2}$ Killing vectors, the curvature tensors can be written as

$$\overset{(d)}{R}_{MNPQ} = \frac{\overset{(d)}{R}}{d(d-1)}(\overset{(d)}{g}_{MP}\overset{(d)}{g}_{NQ} - \overset{(d)}{g}_{MQ}\overset{(d)}{g}_{NP}) \quad . \tag{A-25}$$

Appendix B. Group manifold (G)

Suppose the coordinates y^m ($m = 1 \sim D, D = \dim G$) on the group G, and write each element of the group G as $L(y)$. The left translation[13] $y \to y'$ is defined by

$$g \cdot L(y) = L(y') \tag{B-1}$$

with $g \in G$. Let us introduce a covariant basis[13] $e_m{}^a(y)$ from the 1-form $e(y)$:

$$\begin{aligned} e(y) &\equiv L^{-1}(y)\, dL(y) \\ &= e^a(y)\, Q_a = dy^m\, e_m{}^a(y)\, Q_a \end{aligned} \tag{B-2}$$

Here Q_a ($a = 1 \sim D$) are generators of the group and satisfy the Lie algebra:

$$[Q_a, Q_b] = f_{ab}{}^c Q_c \tag{B-3}$$

with structure constants $f_{ab}{}^c$ of the group G. Differential of the 1-form $e(y)$ reads

$$\begin{aligned} de(y) &= dL^{-1}(y) \wedge dL(y) = -L^{-1}(y)\, dL(y) L^{-1}(y) \wedge dL(y) \\ &= -e(y) \wedge e(y) \end{aligned} \tag{B-4}$$

and this can be written in components as

$$de^a(y) = -\frac{1}{2} e^b(y) \wedge e^c(y)\, f_{bc}{}^a \tag{B-5}$$

The equations (B-4,5) are called the Cartan-Maurer formula[80] or the equation of structure. From eq. (B-5) we derive the Lie's differential equation as

$$\partial_m e_n{}^a - \partial_n e_m{}^a = - e_m{}^b e_n{}^c f_{bc}{}^a \quad , \qquad \text{(B-6)}$$

$$e^m{}_a \partial_m e^n{}_b - e^m{}_b \partial_m e^n{}_a = f_{ab}{}^c e^n{}_c \quad , \qquad \text{(B-7)}$$

where $e^m{}_a$ is the inverse matrix of $e_m{}^a$. Defining the generators $\tilde{L}_a$ as $\tilde{L}_a \equiv e_a{}^m(y)\frac{\partial}{\partial y^m}$, we obtain the Lie algebra of the group G:

$$[\tilde{L}_a , \tilde{L}_b] = f_{ab}{}^c \tilde{L}_c \quad . \qquad \text{(B-8)}$$

The behavior of $e(y)$ under the left translation $y \to y'$ can be deduced from eqs. (B-1,2) as

$$e(y) \to e(y') = L^{-1}(y) g^{-1} d(gL(y)) = e(y) + L^{-1}(y) g^{-1} dg \cdot L(y) \quad . \qquad \text{(B-9)}$$

In components this can be rewritten like

$$e^a(y') = e^a(y) + (g^{-1}dg)^b D_b{}^a(L(y)) \quad , \qquad \text{(B-10)}$$

where $D_b{}^a$ is the adjoint representation of G defined by

$$g^{-1} Q_a g = D_b{}^a(g) Q_b \qquad \text{(B-11)}$$

The coefficients of dy'^m in eq.(B-10) give the behavior of the covariant basis $e_m{}^a$ under the left translation $y \to y'$ as

$$e_m{}^a(y') = \frac{\partial y^n}{\partial y'^m} e_n{}^b(y) \quad , \qquad \text{(B-12)}$$

since g does not depend on y'^m .

Let us introduce the metric tensor $\overset{(0)}{g}_{mn}$ on the group manifold G by

$$\overset{(0)}{g}_{mn}(y) \equiv e_m{}^a(y)\, e_n{}^b(y)\, \delta_{ab} \quad . \qquad \text{(B-13)}$$

The infinitesimal left translation is written like

$$L(y+\delta y) = (1 + \delta g^a Q_a) L(y) = L(y) + \delta y^m \partial_m L(y) \quad . \qquad \text{(B-14)}$$

From the definition of Killing vectors $K_a{}^m(y)$:

$$L(y+\delta y) = L(y) + \delta g^a K_a{}^m(y)\, \partial_m L(y) \qquad \text{(B-15)}$$

and eqs. (B-2, 11, 14) we can easily derive

$$\delta y^m = \delta g^a K_a{}^m(y) \quad , \quad K_a{}^m(y) = D_a{}^b(L(y))\, e_b{}^m(y) \quad . \qquad \text{(B-16)}$$

In the case that the structure constant f_{abc} is fully antisymetric it is straight forward to derive Killing's equation

$$\overset{(0)}{\nabla}_m K_{an} + \overset{(0)}{\nabla}_n K_{am} = 0 \tag{B-17}$$

with $\overset{(0)}{\nabla}_m K_{an} = \partial_m K_{an} - \overset{(0)}{\Gamma}{}^p_{mn} K_{ap}$. Here we have used the property:

$$\partial_m D_a{}^b(L(y)) = -e_m{}^c(y) D_a{}^d(L(y)) f_{cd}{}^b \tag{B-18}$$

If we define the generators L_a as $L_a \equiv K_a{}^m(y)\frac{\partial}{\partial y^m}$, we find they satisfy the Lie algebra,

$$[L_a, L_b] = -f_{ab}{}^c L_c \tag{B-19}$$

We also notice from eqs. (B-6, 13) that the covariant basis $e_m{}^a$ are another set of Killing vectors, since

$$\overset{(0)}{\nabla}_m \tilde{K}_n{}^a + \overset{(0)}{\nabla}_n \tilde{K}_m{}^a = 0 \tag{B-20}$$

with the definition:

$$\tilde{K}_m{}^a(y) \equiv e_m{}^a(y) \tag{B-21}$$

It is easy to see from eqs. (B-7, 18) that the generators L_a , $\tilde{L}_a$ commute $[L_a, \tilde{L}_b] = 0$.

As we have seen above the group manifold G admits the symmetry G x G, we shall see in app. C that the group manifold G is a special case of a homogeneous space G/H with H = 1 and that $K_m{}^a$ and $\tilde{K}_m{}^a$ correspond to the Killing vectors for the left and right translations, respectively.[25)81)]

Appendix C. Homogeneous space (G/H)

A homogeneous space (cost space) is a manifold obtained by dividing a group G by its subgroup H i.e. G/H. That is $G/H \equiv \{ g \in G;\ g \sim g' \text{ if } g' = g^{-1}h,\ h \in H \}$. For example if $G = SO(3)$ and $H = SO(2)$, then $G/H = S^2$ This means that if we freeze the rotation arround a certain coordinate axis in the space of the Euler angles, we are left with only two degrees of freedom which correspond to the freedom of the direction of the axis. Let us introduce the coordinates y^m $(m = 1 \sim D,\ D = \dim G/H)$ on the homogeneous space G/H, namely a represensative element of each coset is written as $L(y)$. Multiplication from the left by an arbitrary element $g \in G$ will generally carry into another coset. Writing its representative element as we have the left translation[13)] $y \to y'$ as

$$g \cdot L(y) = L(y')\, h \tag{C-1}$$

with $h \in H$. Compare this with eq. (B-1). Let us introduce a covariant basis[13)] $e_m{}^{\hat{a}}(y)$ from the 1-form $e(y)$:

$$\begin{aligned} e(y) &\equiv L^{-1}(y)\, dL(y) \\ &= e^{\hat{a}}(y)\, Q_{\hat{a}} = dy^m e_m{}^{\hat{a}}(y)\, Q_{\hat{a}} \end{aligned} \tag{C-2}$$

Here the indices $\hat{a}$ run from 1 to $\dim G$ and the generators $Q_{\hat{a}}$ ($\hat{a}=1\sim\dim G$) satisfy the Lie algebra:

$$[Q_{\hat{a}}, Q_{\hat{b}}] = f_{\hat{a}\hat{b}}{}^{\hat{c}} Q_{\hat{c}} \tag{C-3}$$

with the structure constants $f_{\hat{a}\hat{b}}{}^{\hat{c}}$ of the group G. The equations (B-5,6) are replaced by

$$d e^{\hat{a}}(y) = -\frac{1}{2} e^{\hat{b}}(y) \wedge e^{\hat{c}}(y) f_{\hat{b}\hat{c}}{}^{\hat{a}} \quad , \tag{C-4}$$

$$\partial_m e_n{}^{\hat{a}} - \partial_n e_m{}^{\hat{a}} = - e_m{}^{\hat{b}} e_n{}^{\hat{c}} f_{\hat{b}\hat{c}}{}^{\hat{a}} \quad . \tag{C-5}$$

The behavior of $e(y)$ under the left translation $y \to y'$ can be deduced from eqs. (C-1,2) as

$$\begin{aligned} e(y) \to e(y') &= h L^{-1}(y) g^{-1} d(g L(y) h^{-1}) \\ &= h e(y) h^{-1} + h d h^{-1} + h L^{-1}(y) g^{-1} dg \cdot L(y) h^{-1} \ . \end{aligned} \tag{C-6}$$

Writing the adjoint representation of G as $D_{\hat{a}}{}^{\hat{b}}$;

$$g^{-1} Q_{\hat{a}} g = D_{\hat{a}}{}^{\hat{b}}(g) Q_{\hat{b}} \tag{C-7}$$

we have the component form of eq. (C-6):

$$e^{\hat{a}}(y') = e^{\hat{b}}(y) D_{\hat{b}}{}^{\hat{a}}(h^{-1}) + (h d h^{-1})^{\hat{a}} + (g^{-1} d g)^{\hat{b}} D_{\hat{b}}{}^{\hat{a}}(L(y) h^{-1}) \ . \tag{C-8}$$

Let us assume that the algebra is fully reducible, i.e.

$D_a{}^{\bar{b}}(h) = D_{\bar{a}}{}^{b}(h) = 0$ where the indices a and $\bar{a}$ are those of G/H and H, respectively. Then we obtain

$$e^a(y') = e^b(y) D_b{}^a(h^{-1}) + (g^{-1}dg)^{\hat{b}} D_{\hat{b}}{}^a(L(y)h^{-1}) \quad . \tag{C-9}$$

The coefficient of dy'^m in eq. (C-9) reads.

$$e_m{}^a(y') = \frac{\partial y^n}{\partial y'^m} e_n{}^b(y) D_b{}^a(h^{-1}) \quad . \tag{C-10}$$

The infinitesimal left translation:

$$L(y+\delta y) = (1+\delta g^{\hat{a}} \cdot Q_{\hat{a}}) L(y)(1-\delta h^{\bar{a}} \cdot Q_{\bar{a}}) = L(y) + \delta y^m \partial_m L(y) \tag{C-11}$$

and the definition of Killing vectors $K_{\hat{a}}{}^m(y)$:

$$L(y+\delta y) = L(y) + \delta g^{\hat{a}} \cdot K_{\hat{a}}{}^m(y) \partial_m L(y) \tag{C-12}$$

as well as eqs. (C-2, 7) show that

$$\delta y^m = \delta g^{\hat{a}} \cdot K_{\hat{a}}{}^m(y) \quad , \quad K_{\hat{a}}{}^m(y) = D_{\hat{a}}{}^b(L(y)) e_b{}^m(y) \quad , \tag{C-13}$$

where $e_a{}^m$ is the inverse matrix of $e_m{}^a$. If we define the generator $L_{\hat{a}}$ as $L_{\hat{a}} \equiv K_{\hat{a}}{}^m(y) \frac{\partial}{\partial y^m}$, we find

$$[L_{\hat{a}}, L_{\hat{b}}] = -f_{\hat{a}\hat{b}}{}^{\hat{c}} L_{\hat{c}} \quad . \tag{C-14}$$

Here we have used the property $f_{\bar{a}\bar{b}}{}^c = 0$ which shows that H is a subgroup of G and the equation,

$$\partial_m D_{\hat{a}}{}^{\hat{b}}(L(y)) = - e_m{}^{\hat{c}}(y) D_{\hat{a}}{}^{\hat{d}}(L(y)) f_{\hat{c}\hat{d}}{}^{\hat{b}} \quad . \tag{C-15}$$

Now let us turn to the right action[81] of $g \in G$ on $L(y)$:

$$L(y) \cdot g = L(y') h \tag{C-16}$$

For the expression $L(y) \cdot g$ to make sense, it should not depend on the choice of coset representatives. This happens if and only if g belongs to the normalizer $N(H)$ of H in G which is defined as $N(H) \equiv \{ g \in G \, ; \, g H g^{-1} = H \}$. In this case if we write another representative of the coset of $L(y)$ as $L'(y)$, we find that $L(y) \cdot g$ and $L'(y) \cdot g$ are in the same coset. Since the right action of $g \in H$ on $L(y)$ is trivial, we need only to consider elements of N(H)/H. The infinitesimal right action can be written as

$$L(y + \delta y) = L(y)(1 + \delta g^{\hat{a}} \cdot Q_{\hat{a}})(1 - \delta h^{\bar{a}} \cdot Q_{\bar{a}}) \quad . \tag{C-17}$$

In the same way as the Killing vectors $K_{\hat{a}}{}^m$ we obtain another set of Killing vectors:

$$\tilde{K}_a{}^m(y) \equiv e_a{}^m(y) \quad . \tag{C-18}$$

We have to be careful that only in the case of $f_{a\bar{b}}{}^c = 0$ we can obtain $[\tilde{L}_a , \tilde{L}_b] = f_{ab}{}^c \tilde{L}_c$ with $\tilde{L}_a \equiv e_a{}^m(y) \frac{\partial}{\partial y^m}$. (cf. eq.(B-8)) Since we derive the

commutation relation of $L_{\hat{a}}$, $\tilde{L}_a$ as
$[L_{\hat{a}}, \tilde{L}_b] = (D_{\hat{a}}{}^{\bar{d}} - D_{\hat{a}}{}^{c}\pi_c{}^{\bar{d}}) f_{b\bar{d}}{}^{e} e_e{}^{n}\partial_n$ $\quad (\pi_a{}^{\bar{b}} \equiv e_a{}^{m} e_m{}^{\bar{b}})$, we find $L_{\hat{a}}$ and $\tilde{L}_a$ commute only if $f_{a\bar{b}}{}^{c} = 0$.

Provided we introduce the metric $\overset{(0)}{g}_{mn}$ on the homogeneous space G/H by

$$\overset{(0)}{g}_{mn}(y) \equiv e_m{}^{a}(y)\, e_n{}^{b}(y)\, \delta_{ab} \quad , \qquad \text{(C-19)}$$

the Killing equations hold,

$$\overset{(0)}{\nabla}_m K_{\hat{a}n} + \overset{(0)}{\nabla}_n K_{\hat{b}m} = 0 \quad , \qquad \overset{(0)}{\nabla}_m \tilde{K}_{an} + \overset{(0)}{\nabla}_n \tilde{K}_{am} = 0 \qquad \text{(C-20)}$$

in the case that $f_{\hat{a}\hat{b}\hat{c}}$ is fully antisymmetric.

Therefore we conclude that the homogeneous space G/H has the symmetry $G \times (N/H)$.[25)81)] In the case H=1 the normalizer N is equal to G, and consequently the group manifold has the symmetry $G \times G$.

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Table captions

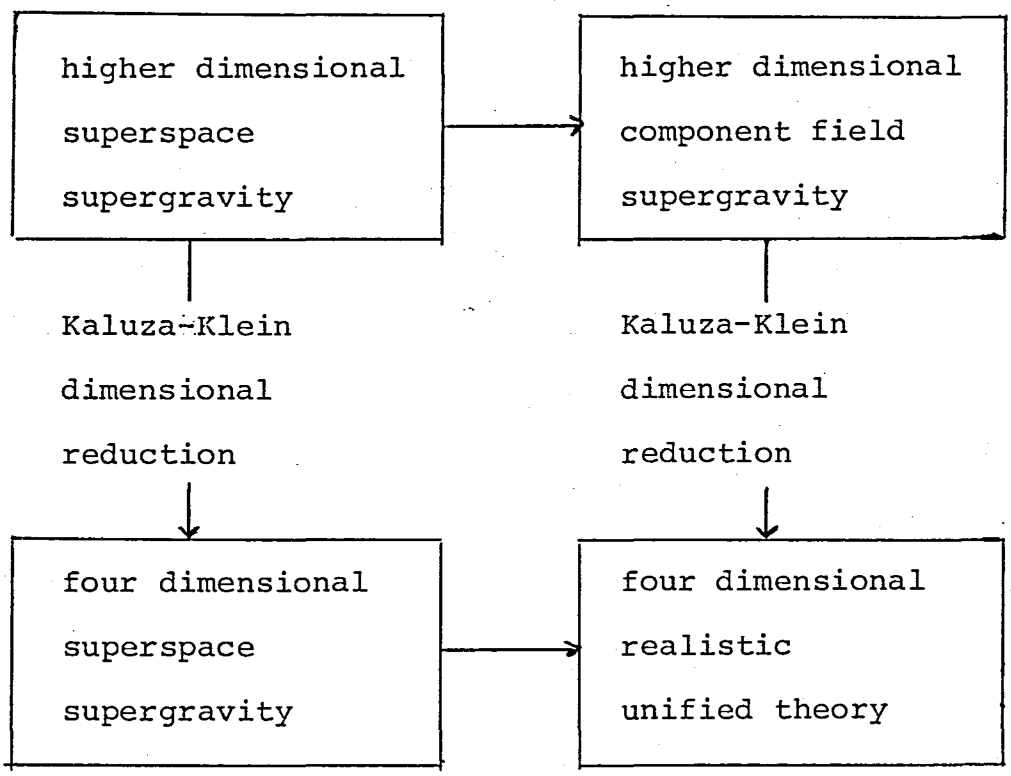

Table 1.

solution	supersymmetry	gauge group	ref.
1. T^7	N=8	$\underbrace{U(1)\times\cdots\times U(1)}_{7}$	9
2. round S^7	N=8	SO(8)	45
3. left-squashed S^7	N=1	SO(5)xSU(2)	46
4. round S^7 + torsion	N=0	SO(7)	47
5. right-squashed S^7	N=0	SO(5)xSU(2)	48
6. right-squashed S^7 + torsion	N=0	SO(5)xSU(2)	48

Table 2.

5-dimensions

	metric tensor field		auxiliary field	ghost field	anti-ghost field
	physical mode	unphys. mode			
	$\varphi^{(5)}_M(x_5)$	$\Big(\ \chi^{(5)}_M(x_5)$	$\beta^{(5)}_M(x_5)$	$\gamma^{(5)}_M(x_5)$	$\bar{\gamma}^{(5)}_M(x_5)\ \Big)$
	(5)	(5)	(5)	(5)	(5)

$\Downarrow$ dimensional reduction

4-dimensions

massive modes (n≠0)	$\varphi_M^{(n)}$ (spin 2)	$\Big(\ \chi_M^{(n)}$	$\beta_M^{(n)}$	$\gamma_M^{(n)}$	$\bar{\gamma}_M^{(n)}\ \Big)$
	(5)	(5)	(5)	(5)	(5)
massless modes (n=0)					
tensor	φ_i^t	$\Big(\ \chi^t_\mu$	β^t_μ	γ^t_μ	$\bar{\gamma}^t_\mu\ \Big)$
	(2)	(4)	(4)	(4)	(4)
vector	φ_i^v	$\Big(\ \chi^v$	β^v	γ^v	$\bar{\gamma}^v\ \Big)$
	(2)	(1)	(1)	(1)	(1)
scalar	φ^s				
	(1)				

Table 3.