Combinatorial and Algebraic Studies on Integral Convex Polytopes

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Contents

Chapter 1

Introduction

A *convex polytope* is a convex hull of finite points of a Euclidian space, and if these points have integer coordinates, then one is called an *integral convex polytope*. (See the books [20] and [78]). Integral convex polytopes are interesting objects related to many branches of mathematics. In this thesis, we pick up aspects of integral convex polytopes having close connections with combinatorics, algebraic geometry and commutative algebra. In particular, we focus on the following three topics in the studies of integral convex polytopes, Ehrhart polynomials, Fano polytopes and affine semigroup rings.

Three topics in the studies of integral convex polytopes

First, in the area of enumerative combinatorics, **Ehrhart polynomials** of integral convex polytopes appear as enumerative functions of several important combinatorial objects, for example, magic squares, Latin squares or domino tilings, etc. (See [6] and [69] for more detailed information.) In this thesis, we will give a combinatorial characterization of the Ehrhart polynomials of integral convex polytopes.

Secondly, many toric varieties can be constructed from integral convex polytopes. In particular, a toric Fano variety is constructed from, so-called, a **Fano polytope**, which is a full-dimensional integral convex polytope containing the origin in its interior as a unique integer point. Since a toric Fano variety is defined from a Fano polytope completely, it has a lot of information of a toric Fano variety. Thus, from a viewpoint of algebraic geometry, Fano polytopes are a useful combinatorial object to understand toric Fano varieties. In fact, many results on toric Fano varieties are obtained by using Fano polytopes ([42, 43, 52, 53]). In this thesis, we will construct some new examples of smooth Fano polytopes.

Thirdly, from an integral convex polytope, we can define an **affine semigroup ring**. By considering affine semigroup rings arising from integral convex polytopes, we obtain several interesting examples of commutative algebra. (Many results related with affine semigroup rings are described in the books [12], [49] and [72].) In this thesis, we will study some new classes of affine semigroup rings arising from integral convex polytopes.

Those topics themselves are not only interesting and crucial in each field but also closely related to each other. For example, Ehrhart polynomials of Fano polytopes express some properties on toric Fano varieties. A lot of properties on Ehrhart polynomials are proved by considering the Ehrhart ring, which is an affine semigroup ring arising from an integral convex polytope. As is well known, it often happens that we find a deep relationship between some properties on toric Fano varieties and affine semigroup rings associated with Fano polytopes.

Structure of this thesis

The organization of this thesis is as follows. We divide this thesis into three parts. Each part includes the author's results on each topic.

- Part I is devoted to the studies on Ehrhart polynomials and the author's results on Ehrhart polynomials are presented. There are three chapters in Part I. The first one is an introduction to Ehrhart polynomials. In the second one, we concentrate on the classification problems of Ehrhart polynomials. In the third one, we discuss root distributions of Ehrhart polynomials. This part contains the results of [34, 35, 36, 37, 38, 40, 41, 48].

- Part II is devoted to the studies on Fano polytopes and there are two chapters, the first one of which is an introduction. The second one is spent to establish examples of Fano polytopes via some combinatorial methods. This part contains the results of [29, 39].

- Part III is devoted to the studies on affine semigroup rings and is divided into three chapters, while the first one is an introduction. In the second one and third one, we investigate the properties on affine semigroup rings arising from graphs and cyclic polytopes, respectively. This part contains the results of [32, 33, 30, 31].

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Part I

Ehrhart polynomials

Chapter 2

Introduction to Ehrhart polynomials

In this part, as the first aspect of the studies on integral convex polytopes, we will consider the *Ehrhart polynomials* of integral convex polytopes. Ehrhart polynomials often appear in the area of enumerative combinatorics. Thus, to study the Ehrhart polynomials of integral convex polytopes are very important and interesting.

We will summarize some basic notions, notation and some results on Ehrhart polynomials.

First, let us review basic definitions and the studies on the classifications of Ehrhart polynomials. Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d and let $\partial\mathcal{P}$ denote the boundary of $\mathcal{P}$. Given a positive integer n, we define the numerical functions $i(\mathcal{P}, n)$ and $i^*(\mathcal{P}, n)$ by setting

$$i(\mathcal{P}, n) = |n\mathcal{P} \cap \mathbb{Z}^N| \ \text{ and } \ i^*(\mathcal{P}, n) = |n(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N|.$$

Here $n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}$ and $|X|$ is the cardinality of a finite set X.

The systematic study of $i(\mathcal{P}, n)$ originated in the work of Ehrhart [16], who established the following fundamental properties:

(a) $i(\mathcal{P}, n)$ is a polynomial in n of degree d; (Thus, in particular, $i(\mathcal{P}, n)$ can be defined for every integer n.)

(b) $i(\mathcal{P}, 0) = 1$;

(c) (loi de réciprocité) $i^*(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, -n)$ for every integer $n > 0$.

We say that $i(\mathcal{P}, n)$ is the *Ehrhart polynomial* of $\mathcal{P}$. We refer the reader to [6, Chapter 3] and [26, Part II] for an introduction to the theory of Ehrhart polynomials.

We define the sequence $\delta_0, \delta_1, \delta_2, \ldots$ of integers by the formula

$$(1-\lambda)^{d+1} \sum_{n=0}^{\infty} i(\mathcal{P}, n)\lambda^n = \sum_{i=0}^{\infty} \delta_i \lambda^i. \tag{2.1}$$

From the basic facts (a) and (b) on $i(\mathcal{P}, n)$ together with a fundamental result on generating function ([69, Corollary 4.3.1]), we have $\delta_i = 0$ for every $i > d$. We say that the sequence

$$\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$$

which appears in (2.1) is the δ-*vector* of $\mathcal{P}$ and the polynomial

$$\delta_{\mathcal{P}}(t) = \sum_{i=0}^{d} \delta_i t^i$$

is the δ-*polynomial* of $\mathcal{P}$. Thus $\delta_0 = 1$ and $\delta_1 = |\mathcal{P} \cap \mathbb{Z}^N| - (d+1)$. It follows from the reciprocity law (c) that

$$\sum_{n=1}^{\infty} i^*(\mathcal{P}, n)\lambda^n = \frac{\sum_{i=1}^{d+1} \delta_{d+1-i}\lambda^i}{(1-\lambda)^{d+1}}. \tag{2.2}$$

In particular,

$$\delta_d = |(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N|.$$

Hence $\delta_1 \geq \delta_d$. Remark that if $\delta_1 = \delta_d$, then $\mathcal{P}$ is always a simplex. It also follows from (2.2) that

$$\max\{j : \delta_j \neq 0\} + \min\{k : k(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N \neq \emptyset\} = d + 1. \tag{2.3}$$

Moreover, each δ_i is nonnegative ([68]). In addition, if $(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N$ is nonempty, then one has $\delta_1 \leq \delta_i$ for every $1 \leq i \leq d-1$ ([28]).

When $d = N$, the leading coefficient $\sum_{i=0}^{d} \delta_i/d!$ of $i(\mathcal{P}, n)$ is equal to the usual volume of $\mathcal{P}$ ([69, Proposition 4.6.30]). In general, the positive integer $\mathrm{vol}(\mathcal{P}) = \sum_{i=0}^{d} \delta_i$ is said to be the *normalized volume* of $\mathcal{P}$.

When $d \leq 2$, the Ehrhart polynomials are completely classified. In fact, the possible δ-vectors of integral convex polytopes of dimension 2 are known in Scott [67]. When $d \geq 3$, however, the classification is still unknown. Note that studying Ehrhart polynomials is equivalent to studying δ-vectors. The δ-vectors of integral convex polytopes have been studied intensively. For example, see [61, 73, 74].

Next, let us review the studies on roots of Ehrhart polynomials. Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d and $i(\mathcal{P}, n)$ its Ehrhart polynomial. A complex number $\alpha \in \mathbb{C}$ is called a *root* of $i(\mathcal{P}, n)$ if $i(\mathcal{P}, \alpha) = 0$.

Many papers on integral convex polytopes, including [5, 7, 8, 9, 23, 24, 62], discuss roots of Ehrhart polynomials. Root distribution of Ehrhart polynomials is one of the current topics on computational commutative algebra. It is well known that the coefficients of the Ehrhart polynomial reflect combinatorial and geometric properties such as the volume of an integral convex polytope in the leading coefficient, gathered information about its faces in the second coefficient, etc. The roots of Ehrhart polynomials should also reflect properties on integral convex polytopes that are hard to elicit just from the coefficients. Beck *et al.* [5] propose the following remarkable

Conjecture 2.0.1 ([5, Conjecture 1.4]). *All roots α of Ehrhart polynomials of d-dimensional integral convex polytopes satisfy $-d \leq \mathrm{Re}(\alpha) \leq d-1$, where $\mathrm{Re}(\alpha)$ denotes the real part of $\alpha \in \mathbb{C}$.*

This conjecture has been solved when $d \leq 5$ in [9]. It is also known ([8]) that every root is contained in

$$\left\{ z \in \mathbb{C} : \left| z + \frac{1}{2} \right| \leq d\left(d - \frac{1}{2}\right) \right\}.$$

Compared with this bound, the strip in the conjecture puts a tight restriction on the distribution of roots for any Ehrhart polynomial.

A *Fano polytope* is an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d such that the origin of $\mathbb{R}^d$ is a unique integer point belonging to the interior of $\mathcal{P}$. A Fano polytope is called *Gorenstein* if its dual polytope is integral. (Recall that the dual polytope $\mathcal{P}^\vee$ of a Fano polytope $\mathcal{P}$ is the convex polytope which consists of those $x \in \mathbb{R}^d$ such that $\langle x, y \rangle \leq 1$ for all $y \in \mathcal{P}$, where $\langle x, y \rangle$ is the usual inner product of $\mathbb{R}^d$.) Further information on Fano polytopes is written in Part II.

Let $\mathcal{P} \subset \mathbb{R}^d$ be a Fano polytope with $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$ its δ-vector. It follows from [3] and [27] that the following conditions are equivalent:

- $\mathcal{P}$ is Gorenstein;
- $\delta(\mathcal{P})$ is symmetric, i.e., $\delta_j = \delta_{d-j}$ for every $0 \leq j \leq d$;
- $i(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, -n-1)$.

A combinatorial characterization of for the δ-vectors to be symmetric is studied in [15] and [27].

When $\mathcal{P} \subset \mathbb{R}^d$ is a Gorenstein Fano polytope, since $i(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, -n-1)$, the roots of $i(\mathcal{P}, n)$ are distributed symmetrically in the complex plane with respect to the line $\mathrm{Re}(z) = -\frac{1}{2}$. Thus, in particular, if d is odd, then $-\frac{1}{2}$ is a root of $i(\mathcal{P}, n)$. In fact, since d is odd, the number of real roots of $i(\mathcal{P}, n)$ is odd. If a real root α of $i(\mathcal{P}, n)$ is not equal to $-\frac{1}{2}$, then $-\alpha - 1$ is also a real root. Hence $-\frac{1}{2}$ must be a root.

It is known in [7, Proposition 1.8] that, if all roots $\alpha \in \mathbb{C}$ of $i(\mathcal{P}, n)$ of an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d satisfy $\mathrm{Re}(\alpha) = -\frac{1}{2}$, then $\mathcal{P}$ is unimodular equivalent with a Gorenstein Fano polytope whose volume is at most 2^d. In a recent work [23], the roots of the Ehrhart polynomials of smooth Fano polytopes with small dimensions are completely determined.

If each of the roots of the Ehrhart polynomial of an integral convex polytope $\mathcal{P}$ has the real part $-\frac{1}{2}$, then $\mathcal{P}$ must be Gorenstein since the function equation $i(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, -n-1)$ must be held. On the contrary, each of all the roots of Gorenstein Fano polytopes α does not always satisfy $\mathrm{Re}(\alpha) = -\frac{1}{2}$. Hence, it is also meaningful to investigate roots of Gorenstein Fano polytopes.

The structure of the rest of this part is as follows. In Chapter 3, we will discuss the classification probelm on the Ehrhart polynomials of integral convex polytopes. Essentially, we will classify their possible δ-vectors. In particular, we will consider the δ-vectors of integral convex polytopes whose normalized volumes are small. In Chapter 4, we will discuss root distributions of the Ehrhart polynomials. Especially, we will present counterexamples of Conjecture 2.0.1. Moreover, we will also focus on roots of the Ehrhart polynomials of Gorenstein Fano polytopes.

Chapter 3

Classification problems on Ehrhart polynomials

In this chapter, we will study the classification problems on the Ehrhart polynomials of integral convex polytopes. Especially, we will consentrate on the case where they are simplices, which is a crucial case in some sence.

After reviewing the well-known technique how to compute the δ-vectors of integral simplices in Section 3.1, we will consider the classification problem on the Ehrhart polynomials of integral convex polytopes whose normalized volumes are at most 3 in Section 3.2, are 4 in Section 3.3 and at least 5 and prime in Section 3.4, respectively. Most parts of them will be devoted to discussing the Ehrhart polynomials of integral simplices. Finally, in Section 3.5, we will consider the specific class of δ-vectors and study some properties on integral convex polytopes with such δ-vectors.

3.1 Review on the computation of the δ-vector of a simplex

First of all, let us recall the well-known combinatorial technique to compute the δ-vector of an integral simplex.

Given an integral simplex $\mathcal{F} \subset \mathbb{R}^N$ of dimension d with the vertices $v_0, v_1, \ldots, v_d$, we set

$$S(\mathcal{P}) = \left\{ \sum_{i=0}^{d} r_i(v_i, 1) \in \mathbb{R}^{N+1} : 0 \leq r_i < 1 \right\} \cap \mathbb{Z}^{N+1}$$

and

$$S^*(\mathcal{P}) = \left\{ \sum_{i=0}^{d} r_i(v_i, 1) \in \mathbb{R}^{N+1} : 0 < r_i \leq 1 \right\} \cap \mathbb{Z}^{N+1}.$$

We define the degree of an integer point $(\alpha, n) \in S$ by $\deg(\alpha, n) = n$, where $\alpha \in \mathbb{Z}^N$ and $n \in \mathbb{Z}_{\geq 0}$. Let $\delta_i = |\{\alpha \in S(\mathcal{P}) : \deg \alpha = i\}|$ and $\delta_i^* = |\{\alpha \in S^*(\mathcal{P}) : \deg \alpha = i\}|$.

Then we have

$$\delta(\mathcal{F}) = (\delta_0, \delta_1, \ldots, \delta_d)$$

and

$$\delta_i^* = \delta_{d+1-i} \text{ for } i = 1, \ldots, d+1.$$

Notice that the elements of $S(\mathcal{P})$ form an abelian group with a unit $(0, \ldots, 0)$. For α and β in $S(\mathcal{P})$ with $\alpha = \sum_{i=0}^d r_i(v_i, 1)$ and $\beta = \sum_{i=0}^d s_i(v_i, 1)$, where $r_i, s_i \in \mathbb{Q}$ with $0 \leq r_i, s_i < 1$, we define the operation in $S(\mathcal{P})$ by setting $\alpha \oplus \beta := \sum_{i=0}^d \{r_i + s_i\}(v_i, 1)$, where $\{r\} = r - \lfloor r \rfloor$ denotes the fractional part of a rational number r. (Throughout this section, in order to distinguish the operation in S from the usual addition, we use the notation $\oplus$.)

3.2 The case where $\sum_{i=0}^d \delta_i \leq 3$

In this section, we classify the possible δ-vectors of integral convex polytopes with $\sum_{i=0}^d \delta_i \leq 3$.

For our classification, we present two well-known inequalities on δ-vectors. Let $s = \max\{i : \delta_i \neq 0\}$. Stanley [71] shows the inequalities

$$\delta_0 + \delta_1 + \cdots + \delta_i \leq \delta_s + \delta_{s-1} + \cdots + \delta_{s-i}, \quad 0 \leq i \leq [s/2] \tag{3.1}$$

by using the theory of Cohen–Macaulay rings. On the other hand, the inequalities

$$\delta_d + \delta_{d-1} + \cdots + \delta_{d-i} \leq \delta_1 + \delta_2 + \cdots + \delta_{i+1}, \quad 0 \leq i \leq [(d-1)/2] \tag{3.2}$$

appear in [28, Remark (1.4)].

Somewhat surprisingly, when $\sum_{i=0}^d \delta_i \leq 3$, the above inequalities (3.1) together with (3.2) give a characterization of the possible δ-vectors. In fact,

Theorem 3.2.1 ([35, Theorem 0.1])**.** *Given a finite sequence $(\delta_0, \delta_1, \ldots, \delta_d)$ of nonnegative integers, where $\delta_0 = 1$, which satisfies $\sum_{i=0}^d \delta_i \leq 3$, there exists an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d whose δ-vector coincides with $(\delta_0, \delta_1, \ldots, \delta_d)$ if and only if $(\delta_0, \delta_1, \ldots, \delta_d)$ satisfies all inequalities* (3.1) *and* (3.2)*. Moreover, all such polytopes can be chosen to be simplices.*

Note that the "Only if" part of Theorem 3.2.1 is obvious. In addition, no discussion will be required for the case where $\sum_{i=0}^d \delta_i = 1$.

On the other hand, the following example shows that Theorem 3.2.1 is no longer true for the case of $\sum_{i=0}^d \delta_i = 4$.

Example 3.2.2. We claim that the sequence $(1, 0, 1, 0, 1, 1, 0, 0)$ cannot be the δ-vector of an integral convex polytope of dimension 7. Suppose, on the contrary, there exists an integral convex polytope $\mathcal{P} \subset \mathbb{R}^N$ of dimension 7 with $\delta(\mathcal{P}) =$

$(\delta_0, \delta_1, \ldots, \delta_7) = (1, 0, 1, 0, 1, 1, 0, 0)$. Since $\delta_1 = 0$, we know that $\mathcal{P}$ is a simplex. Let $v_0, v_1, \ldots, v_7$ be the vertices of $\mathcal{P}$. By using the discussions described above, one has

$$S(\mathcal{P}) = \{(0, \ldots, 0), (\alpha, 2), (\beta, 4), (\gamma, 5)\}$$

and

$$S^*(\mathcal{P}) = \{(\alpha', 3), (\beta', 4), (\gamma', 6), (\sum_{i=0}^{7} v_i, 7)\}.$$

Write $\alpha' = \sum_{i=0}^{7} r_i v_i$ with each $0 < r_i \leq 1$. Since $(\alpha', 3) \notin S(\mathcal{P})$, there is $0 \leq j \leq 7$ with $r_j = 1$. If there are $0 \leq k < \ell \leq 7$ with $r_k = r_\ell = 1$, say, $r_0 = r_1 = 1$, then $0 < r_q < 1$ for each $2 \leq q \leq 7$ and $\sum_{i=2}^{7} r_i = 1$. Hence $(\alpha' - v_0 - v_1, 1) \in S(\mathcal{P})$, a contradiction. Thus there is a unique $0 \leq j \leq 7$ with $r_j = 1$, say, $r_0 = 1$. Then $\alpha = \sum_{i=1}^{7} r_i v_i$ and $\gamma = \sum_{i=1}^{7}(1 - r_i)v_i$. Let $\mathcal{F}$ denote the facet of $\mathcal{P}$ whose vertices are $v_1, v_2, \ldots, v_7$ with $\delta(\mathcal{F}) = (\delta'_0, \delta'_1, \ldots, \delta'_6) \in \mathbb{Z}^7$. Then $\delta'_2 = \delta'_5 = 1$. Since $\delta'_i \leq \delta_i$ for each $0 \leq i \leq 6$, it follows that $\delta(\mathcal{F}) = (1, 0, 1, 0, 0, 1, 0)$. This contradicts the inequalities (3.1).

3.2.1 A proof of Theorem 3.2.1 when $\sum_{i=0}^{d} \delta_i = 2$

The goal of this subsection is to prove the "If" part of Theorem 3.2.1 when $\sum_{i=0}^{d} \delta_i = 2$. First of all, we recall the following well-known

Lemma 3.2.3 ([6, Theorem 2.4]). *Suppose that $(\delta_0, \delta_1, \ldots, \delta_d)$ is the δ-vector of an integral convex polytope of dimension d. Then there exists an integral convex polytope of dimension $d + 1$ whose δ-vector is $(\delta_0, \delta_1, \ldots, \delta_d, 0)$.*

Note that the required δ-vector is obtained by forming the pyramid over the integral convex polytope.

We study a finite sequence $(\delta_0, \delta_1, \ldots, \delta_d)$ of nonnegative integers with $\delta_0 = 1$ which satisfies all inequalities (3.2) together with $\sum_{i=0}^{d} \delta_i = 2$. Since $\delta_0 = 1$, $\delta_1 \geq \delta_d$ and $\sum_{i=0}^{d} \delta_i = 2$, one has $\delta_d = 0$. Hence there is an integer $i \in \{1, \ldots, [(d + 1)/2]\}$ such that $(\delta_0, \delta_1, \ldots, \delta_d) = (1, 0, \ldots, 0, \underbrace{1}_{i\text{th}}, 0, \ldots, 0)$, where $\underbrace{1}_{i\text{th}}$ stands for $\delta_i = 1$. By virtue of Lemma 3.2.3, our work is to find an integral convex polytopes $\mathcal{P}$ of dimension d with $(1, 0, \ldots, 0, \underbrace{1}_{((d+1)/2)\text{th}}, 0, \ldots, 0) \in \mathbb{Z}^{d+1}$ its δ-vector.

Let $\mathcal{P} \subset \mathbb{R}^d$ be the integral simplex of dimension d whose vertices $v_0, v_1, \ldots, v_d$ are

$$v_i = \begin{cases} \mathbf{e}_i + \mathbf{e}_{i+1}, & i = 1, \ldots, d - 1, \\ \mathbf{e}_1 + \mathbf{e}_d, & i = d, \\ (0, 0, \ldots\ldots, 0), & i = 0. \end{cases}$$

When d is odd, one has $\mathrm{vol}(\mathcal{P}) = 2$ by using an elementary linear algebra. Since

$$\frac{1}{2}\{(v_0, 1) + (v_1, 1) + \cdots + (v_d, 1)\} = (1, 1, \ldots, 1, (d + 1)/2) \in \mathbb{Z}^{d+1},$$

Section 3.1 says that $\delta_{(d-1)/1} \geq 1$. Thus, since $\mathrm{vol}(\mathcal{P}) = 2$, one has

$$\delta(\mathcal{P}) = (1, 0, \ldots, 0, \underbrace{1}_{((d+1)/2)\mathrm{th}}, 0, \ldots, 0),$$

as desired.

3.2.2 A proof of Theorem 3.2.1 when $\sum_{i=0}^{d} \delta_i = 3$

The goal of this section is to prove the "If" part of Theorem 3.2.1 when $\sum_{i=0}^{d} \delta_i = 3$. Suppose that a finite sequence $(\delta_0, \delta_1, \ldots, \delta_d)$ of nonnegative integers with $\delta_0 = 1$ satisfies all inequalities (3.1) and (3.2) together with $\sum_{i=0}^{d} \delta_i = 3$.

When there is $1 \leq i \leq d$ with $\delta_i = 2$, the same discussion as in the previous subsection can be applied. In fact, instead of the vertices of the convex polytope arising in the last paragraph of the previous subsection, we may consider the convex polytope whose vertices $v_0, v_1, \ldots, v_d$ are

$$v_i := \begin{cases} \mathbf{e}_i + \mathbf{e}_{i+1}, & i = 1, \ldots, d-1, \\ 2\mathbf{e}_1 + \mathbf{e}_d, & i = d, \\ (0, 0, \ldots\ldots, 0), & i = 0. \end{cases}$$

Now, in what follows, a sequence $(\delta_0, \delta_1, \ldots, \delta_d)$ with each $\delta_i \in \{0, 1\}$, where $\delta_0 = 1$ which satisfies all inequalities (3.1) and (3.2) together with $\sum_{i=0}^{d} \delta_i = 3$ will be considered.

If $\delta_d = 1$, then $\delta_1 = 1$. Thus this happens only when $d = 2$ and $(1, 1, 1)$ is a possible δ-vector. If $\delta_1 = 1$, then $\delta_2 = 1$ by (3.1). Clearly, $(1, 1, 1, 0, \ldots, 0) \in \mathbb{Z}^{d+1}$ is also a possible δ-vector. Thus we will assume that $\delta_1 = \delta_d = 0$. Let $\delta_m = \delta_n = 1$ with $1 < m < n < d$. Let $p = m - 1$, $q = n - m - 1$, and $r = d - n$. By (3.1) one has $0 \leq q \leq p$. Moreover, by (3.2) one has $p \leq r$. Consequently,

$$0 \leq q \leq p \leq r, \quad p + q + r = d - 2. \tag{3.3}$$

Our work is to construct an integral convex polytope $\mathcal{P}$ with dimension d whose δ-vector coincides with $\delta(\mathcal{P}) = (1, \underbrace{0, \ldots, 0}_{p}, 1, \underbrace{0, \ldots, 0}_{q}, 1, \underbrace{0, \ldots, 0}_{r})$ for an arbitrary integer $1 < m < n < d$ satisfying the conditions (3.3).

Lemma 3.2.4. *Let $d = 3k + 2$. There exists an integral convex polytope $\mathcal{P}$ of dimension d whose δ-vector coincides with*

$$(1, \underbrace{0, \ldots, 0}_{k}, 1, \underbrace{0, \ldots, 0}_{k}, 1, \underbrace{0, \ldots, 0}_{k}) \in \mathbb{Z}^{d+1}.$$

Proof. When $k \geq 1$, let $\mathcal{P} \subset \mathbb{R}^d$ be the integral simplex of dimension d with the vertices $v_0, v_1, \ldots, v_d$, where

$$v_i = \begin{cases} \mathbf{e}_i + \mathbf{e}_{i+1} + \mathbf{e}_{i+2}, & i = 1, \ldots, d-2, \\ \mathbf{e}_1 + \mathbf{e}_{d-1} + \mathbf{e}_d, & i = d-1, \\ \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_d, & i = d, \\ (0, 0, \ldots, 0), & i = 0. \end{cases}$$

By using the induction on k, it follows that $\mathrm{vol}(\mathcal{P}) = 3$. Since

$$\frac{1}{3}\left\{(v_0, 1) + (v_1, 1) + \cdots + (v_d, 1)\right\} = (1, 1, \ldots, 1, k+1) \in \mathbb{Z}^{d+1},$$

Section 3.1 guarantees that $\delta_{k+1} \geq 1$ and $\delta^*_{k+1} \geq 1$. Hence $\delta_{k+1} = 1$ and $\delta_{2k+2} = 1$, as required. $\square$

Lemma 3.2.5. *Let $d = 3k+2$, $\ell > 0$ and $d' = d + 2\ell$. There exists an integral simplex $\mathcal{P} \subset \mathbb{R}^{d'}$ of dimension d' whose δ-vector coincides with*

$$(1, \underbrace{0, \ldots, 0}_{k+\ell}, 1, \underbrace{0, \ldots, 0}_{k}, 1, \underbrace{0, \ldots, 0}_{k+\ell}) \in \mathbb{Z}^{d'+1}.$$

Proof. **First Step.** Let $k = 0$. Thus $d = 2$ and $d' = 2\ell + 2$. Let $\mathcal{P} \subset \mathbb{R}^{d'}$ be an integer convex polytope of dimension d' whose vertices $v_0, v_1, \ldots, v_{2\ell+2}$ are

$$v_i = \begin{cases} 2\mathbf{e}_1 + \mathbf{e}_2, & i = 1, \\ 2\mathbf{e}_2 + \mathbf{e}_3, & i = 2, \\ \mathbf{e}_i + \mathbf{e}_{i+1}, & i = 3, \ldots, 2l+1, \\ \mathbf{e}_1 + \mathbf{e}_d, & i = 2l+2, \\ (0, \ldots, 0), & i = 0. \end{cases}$$

As usual, a routine computation says that $\mathrm{vol}(\mathcal{P}) = 3$. Let $v \in \mathbb{R}^{d'+1}$ be the point

$$\frac{1}{3}\left\{(v_0, 1) + (v_1, 1) + (v_2, 1)\right\} + \frac{1}{3}\sum_{q=2}^{\ell+1}(v_{2q}, 1) + \frac{2}{3}\sum_{q=2}^{\ell+1}(v_{2q-1}, 1)$$

belonging to $\mathbb{R}^{d'}$. Then

$$v = (1, 1, \ldots, 1, \ell+1) \in \mathbb{Z}^{d'+1}.$$

Thus Section 3.1 guarantees that $\delta_{\ell+1} \geq 1$ and $\delta^*_{\ell+1} \geq 1$. Hence $\delta_{\ell+1} = \delta_{\ell+2} = 1$, as required.

Second Step. Let $k \geq 1$. We write $\mathcal{P} \subset \mathbb{R}^{d'}$ for the integral simplex of dimension d' with the vertices $v_0, v_1, \ldots, v_{3k+2\ell+2}$ as follows:

$$v_i = \begin{cases} (0,0,\ldots,0), & i = 0, \\ \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_{3k+3} + \mathbf{e}_{3k+4} + \cdots + \mathbf{e}_{d'}, & i = 1, \\ \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_{3k+3} + \mathbf{e}_{3k+4} + \cdots + \mathbf{e}_{d'}, & i = 2, \\ \mathbf{e}_i + \mathbf{e}_{i+1} + \mathbf{e}_{i+2} + \mathbf{e}_{3k+3} + \mathbf{e}_{3k+5} + \cdots + \mathbf{e}_{d'-1}, & i = 3,4,5,\ldots,3k, \\ \mathbf{e}_1 + \mathbf{e}_{3k+1} + \mathbf{e}_{3k+2} + \mathbf{e}_{3k+3} + \mathbf{e}_{3k+5} + \cdots + \mathbf{e}_{d'-1}, & i = 3k+1, \\ \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_{3k+2} + \mathbf{e}_{3k+3} + \mathbf{e}_{3k+5} + \cdots + \mathbf{e}_{d'-1}, & i = 3k+2, \\ \mathbf{e}_i + \mathbf{e}_{i+2} + \cdots + \mathbf{e}_{d'-1}, & i = 3k+3, 3k+5, \ldots, 3k+2\ell+1, \\ \mathbf{e}_i + \mathbf{e}_{i+1} + \cdots + \mathbf{e}_{d'-1}, & i = 3k+4, 3k+6, \ldots, 3k+2\ell+2. \end{cases}$$

Let A denote the $(3k+2) \times (3k+2)$ matrix

$$\left|\mathrm{A}\right| = \underbrace{\begin{vmatrix} 1 & 1 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 1 & 1 & \ddots & & \mathbf{0} & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \mathbf{0} & & \ddots & 1 & 1 & 1 \\ 1 & & & & & 0 & 1 & 1 \\ 1 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 1 \end{vmatrix}}_{(3k+2)\times(3k+2)}.$$

Then a simple computation on determinants enables us to show that

$$\mathrm{vol}(\mathcal{P}) = \underbrace{\begin{vmatrix} \mathrm{A} & & * \\ & 1 & \\ \mathbf{0} & & \ddots \\ & & & 1 \end{vmatrix}}_{(3k+2+2\ell)\times(3k+2+2\ell)} = \left|\mathrm{A}\right| = 3.$$

One has

$$\frac{1}{3}\{(v_0,1) + (v_1,1) + \cdots + (v_{3k+4},1)\} + \frac{2}{3}\{(v_{3k+5},1) + (v_{3k+7},1) + \cdots + (v_{3k+2\ell+1},1)\}$$
$$+ \frac{1}{3}\{(v_{3k+6},1) + (v_{3k+8},1) + \cdots + (v_{3k+2\ell+2},1)\}$$

$$= (1, \ldots, 1, k+1, 1, k+2, 1, \ldots, k+\ell, 1, k+\ell+1) \in \mathbb{Z}^{d'+1}.$$

Hence $\delta_{k+\ell+1} = \delta_{2k+\ell+2} = 1$, as required. $\square$

In order to complete a proof of the "If" part of Theorem 3.2.1 when $\sum_{i=0}^{d} \delta_i = 3$, we must show the existence of an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d whose δ-vector coincides with $(1, 0, \ldots, 0, \underbrace{1}_{m\text{th}}, 0, \ldots, 0, \underbrace{1}_{n\text{th}}, 0, \ldots, 0)$, where $1 < m < n < d$ and $n - m - 1 \leq m - 1 \leq d - n$.

First, Lemma 3.2.4 says that there exists an integral convex polytope whose δ-vector coincides with

$$(1, 0, \ldots, 0, \underbrace{1}_{(n-m)\text{th}}, 0, \ldots, 0, \underbrace{1}_{(2n-2m)\text{th}}, 0, \ldots, 0) \in \mathbb{Z}^{3n-3m+3}.$$

Second, Lemma 3.2.5 guarantees that there exists an integral convex polytope whose δ-vector coincides with

$$(1, 0, \ldots, 0, \underbrace{1}_{m\text{th}}, 0, \ldots, 0, \underbrace{1}_{n\text{th}}, 0, \ldots, 0) \in \mathbb{Z}^{n+m+3}.$$

Finally, by using Lemma 3.2.3, there exists an integral convex polytope $\mathcal{P}$ of dimension d with

$$\delta(\mathcal{P}) = (1, 0, \ldots, 0, \underbrace{1}_{m\text{th}}, 0, \ldots, 0, \underbrace{1}_{n\text{th}}, 0, \ldots, 0) \in \mathbb{Z}^{d+1},$$

as desired.

3.3 The case where $\sum_{i=0}^{d} \delta_i = 4$

When $\sum_{i=0}^{d} \delta_i \leq 3$, the inequalities (3.1) and (3.2) are necessary and sufficient conditions for a sequence of nonnegative integers $(\delta_0, \delta_1, \ldots, \delta_d) \in \mathbb{Z}^{d+1}$ with $\delta_0 = 1$ to be a δ-vector of an integral convex polytope of dimension d. However, when $\sum_{i=0}^{d} \delta_i = 4$, as shown in Example 3.2.2, there exists a counterexample, namely, (3.1) and (3.2) are not sufficient. Thus, we have to impose more restrictions on $(\delta_0, \delta_1, \ldots, \delta_d)$. In this section, we will give the complete classification of the possible δ-vectors with $\sum_{i=0}^{d} \delta_i = 4$, see Theorem 3.3.6 below. Moreover, similar to the case $\sum_{i=0}^{d} \delta_i \leq 3$, it turns out that all the possible δ-vectors with $\sum_{i=0}^{d} \delta_i = 4$ can be chosen to be integral simplices. Such a result does not hold when $\sum_{i=0}^{d} \delta_i = 5$, see Remark 3.3.8.

3.3.1 An approach to a classification of integral simplices with a given δ-vector

Let $\mathbb{Z}^{d\times d}$ denote the set of $d \times d$ integral matrices. Recall that a matrix $A \in \mathbb{Z}^{d\times d}$ is *unimodular* if $\det(A) = \pm 1$. Given integral convex polytopes $\mathcal{P}$ and $\mathcal{Q}$ in $\mathbb{R}^d$ of dimension d, we say that $\mathcal{P}$ and $\mathcal{Q}$ are *unimodularly equivalent* if there exists a unimodular matrix $U \in \mathbb{Z}^{d\times d}$ and an integral vector w such that $\mathcal{Q} = f_U(\mathcal{P}) + w$, where f_U is the linear transformation in $\mathbb{R}^d$ defined by U, i.e., $f_U(\mathbf{v}) = \mathbf{v}U$ for all $\mathbf{v} \in \mathbb{R}^d$. Clearly, if $\mathcal{P}$ and $\mathcal{Q}$ are unimodularly equivalent, then $\delta(\mathcal{P}) = \delta(\mathcal{Q})$. Conversely, given a vector $v \in \mathbb{Z}^{d+1}_{\geq 0}$, it is natural to ask for a description of all the integral polytopes $\mathcal{P}$ under unimodular equivalence, such that $\delta(\mathcal{P}) = v$.

We will focus on the above problem for simplices with one vertex at the origin. In addition, we do not allow any shifts in the equivalence, i.e., integral convex polytopes $\mathcal{P}$ and $\mathcal{Q}$ of dimension d are equivalent if there exists a unimodular matrix U, such that $\mathcal{Q} = f_U(\mathcal{P})$. By considering the δ-vectors of all the integral simplices up to this equivalence, whose normalized volumes are 4, we obtain Theorem 3.3.6.

To discuss the representative under this equivalence of the integral simplices with one vertex at the origin, we consider Hermite normal forms.

Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral simplex of dimension d whose the vertices are $(0,\ldots,0), v_1,\ldots,v_d$. Define $M(\mathcal{P}) \in \mathbb{Z}^{d\times d}$ to be the matrix with the row vectors $v_1,\ldots,v_d$. Then we have the following connection between the matrix $M(\mathcal{P})$ and the δ-vector of $\mathcal{P}$: $|\det(M(\mathcal{P}))| = \sum_{i\geq 0} \delta_i = \mathrm{vol}(\mathcal{P})$. In this setting, $\mathcal{P}$ and $\mathcal{P}'$ are equivalent if and only if $M(\mathcal{P})$ and $M(\mathcal{P}')$ have the same Hermite normal form. Here, the *Hermite normal form* of a nonsingular integral square matrix B is a unique nonnegative lower triangular matrix $A = (a_{ij}) \in \mathbb{Z}^{d\times d}_{\geq 0}$ such that $A = BU$ for some unimodular matrix $U \in \mathbb{Z}^{d\times d}$ and $0 \leq a_{ij} < a_{ii}$ for all $1 \leq j < i$, see [66, Chapter 4]. In other words, we can pick the Hermite normal form as the representative in each equivalence class and study the following

Problem 3.3.1. Given a vector $v \in \mathbb{Z}^{d+1}_{\geq 0}$, classify all possible $d \times d$ matrices $A \in \mathbb{Z}^{d\times d}$ which are in Hermite normal form with $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d) = v$, where $\mathcal{P} \subset \mathbb{R}^d$ is the integral simplex whose vertices are the row vectors of A together with the origin in $\mathbb{R}^d$.

3.3.2 An algorithm for the computation of the δ-vector of a simplex

In this subsection, we introduce an algorithm for calculating the δ-vector of integral simplices arising from Hermite normal forms.

Let $M \in \mathbb{Z}^{d\times d}$. We write $\mathcal{P}(M)$ for the integral simplex whose vertices are the row vectors of M together with the origin in $\mathbb{R}^d$. We will present an algorithm to compute the δ-vector of $\mathcal{P}(M)$. To make the notation clear, we assume $d = 3$. The general case is completely analogous. Let A be the Hermite normal form of M. We have that $\{\mathcal{P}(M) \cap \mathbb{Z}^d\}$ is in bijection with $\{\mathcal{P}(A) \cap \mathbb{Z}^d\}$. By definition,

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where each a_{ij} is a nonnegative integer.

For a vector $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, consider

$$b(\lambda) := (\lambda_1, \lambda_2, \lambda_3)A = (a_{11}\lambda_1 + a_{21}\lambda_2 + a_{31}\lambda_3, a_{22}\lambda_2 + a_{32}\lambda_3, a_{33}\lambda_3).$$

Then it is clear that the set of interior points inside $\mathcal{P}(A)$ ($(\mathcal{P}(A) - \partial\mathcal{P}(A)) \cap \mathbb{Z}^3$) is in bijection with the set

$$\{(\lambda_1, \lambda_2, \lambda_3) \mid \lambda_i > 0,\ \lambda_1 + \lambda_2 + \lambda_3 < 1,\ b(\lambda) \in \mathbb{Z}^3\}.$$

We observe that for any $n \in \mathbb{N}$, $n(\mathcal{P}(A) - \partial\mathcal{P}(A)) \cap \mathbb{Z}^3$ is in bijection with

$$\{(\lambda_1, \lambda_2, \lambda_3) \mid \lambda_i > 0,\ \lambda_1 + \lambda_2 + \lambda_3 < n,\ b(\lambda) \in \mathbb{Z}^3\}.$$

We first consider all positive vectors λ satisfying $b(\lambda) \in \mathbb{Z}^3$. By the lower triangularity of the Hermite normal form, we can start from the last coefficient of $b(\lambda)$ and move forward. Then it is not hard to see that each vector λ has the following form:

$$\lambda_3 = \lambda_3^{k,k_3} := \frac{k}{a_{33}} + k_3, \qquad \lambda_2 = \lambda_2^{jk,k_2} := \frac{j - \{a_{32}\lambda_3^k\}}{a_{22}} + k_2$$

and

$$\lambda_1 = \lambda_1^{ijk,k_1} := \frac{i - \{a_{21}\lambda_2^{jk} + a_{31}\lambda_3^k\}}{a_{11}} + k_1$$

for some nonnegative integers k_3, k_2, k_1, where $k \in \{1, 2, \ldots, a_{33}\}$, $j \in \{1, 2, \ldots, a_{22}\}$, $i \in \{1, 2, \ldots, a_{11}\}$ and $\lambda_1^{ijk} = \lambda_1^{ijk,0}, \lambda_2^{jk} = \lambda_2^{jk,0}, \lambda_3^k = \lambda_3^{k,0}$. We call all the vectors λ with the same index (i, j, k) the *congruence class of* (i, j, k).

Now we consider the condition $\lambda_1 + \lambda_2 + \lambda_3 < n$ in the above bijection. As n increases, we wish to know when it is the first time that a congruence class (i, j, k) produces interior points inside $n\mathcal{P}(A)$. In other words, for a fixed (i, j, k) we want to find the smallest n such that $\lambda_1 + \lambda_2 + \lambda_3 < n$ with λ_1, λ_2, $\lambda_3 > 0$. It is clear that this happens when $k_1 = k_2 = k_3 = 0$ and

$$n = \lfloor \lambda_1^{ijk} + \lambda_2^{jk} + \lambda_3^k \rfloor + 1 =: s_{ijk}.$$

Finally, when n grows larger than s_{ijk}, we want to consider how many interior points this fixed congruence class produces. Let $n = s_{ijk} + \ell$, so each interior point corresponds to a choice of $k_1 \geq 0, k_2 \geq 0, k_3 \geq 0$ in the formula of λ_1^{ijk,k_1}, λ_2^{ij,k_2} and λ_3^{i,k_3} such that $k_1 + k_2 + k_3 \leq \ell$. There are $\binom{d+\ell}{\ell}$ choices in total.

In summary, the following two facts hold for each congruence class (i, j, k), $k \in \{1, 2, \ldots, a_{33}\}$, $j \in \{1, 2, \ldots, a_{22}\}$, $i \in \{1, 2, \ldots, a_{11}\}$:

1. s_{ijk} is the smallest n such that this congruence class contributes interior points in the n-th dilation of $\mathcal{P}(A)$;

2. In the $(s_{ijk}+\ell)$-th dilation of $\mathcal{P}(A)$, this congruence class contributes $\binom{d+\ell}{\ell}$ interior points.

The previous considerations imply the $d=3$ instance of the following theorem. The general d case follows in analogous manner.

Theorem 3.3.2 ([34, Theorem 2.1])**.** *Let $\mathcal{P}(A)$ be a simplex of dimension d corresponding to a $d\times d$ matrix $A=(a_{ij})\in\mathbb{Z}^{d\times d}$. Then the generating function for $i^*(\mathcal{P}(A),n)$ is given by*

$$\sum_{n=1}^{\infty} i^*(\mathcal{P}(A),n)t^n = (1-t)^{-(d+1)} \sum_{\substack{(i_1,\dots,i_d)\\ 1\le i_j\le a_{ij}}} t^{s_{i_1\cdots i_d}},$$

where

$$s_{i_1\cdots i_d} = \left\lfloor \sum_{k=1}^{d} \lambda_k^{i_k,i_{k+1},\dots i_d} \right\rfloor + 1,$$

with

$$\lambda_d^{i_d} = \frac{i_d}{a_{dd}},$$

and

$$\lambda_k^{i_k,i_{k+1},\dots i_d} = a_{kk}^{-1}\left(i_k - \left\{ \sum_{h=k+1}^{d} a_{hk}\lambda_h^{i_h i_{h+1}\dots i_d} \right\}\right), \quad \textit{for } 1\le k<d.$$

By the reciprocity law (2.2)*, we have*

$$\delta_{\mathcal{P}(A)}(t) = \sum_{\substack{(i_1,\dots,i_d)\\ 1\le i_j\le a_{ij}}} t^{d+1-s_{i_1\dots i_d}}.$$

Example 3.3.3. Let A be the 4×4 matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & 1 & 2 & 0\\ 1 & 0 & 1 & 3 \end{pmatrix}.$$

Then, for $1\le i\le 2$ and $1\le j\le 3$,

$$\lambda_2^{ij} = 1-\{\lambda_3^{ij}\}, \quad \lambda_1^{ij} = 1-\{\lambda_3^{ij}+\lambda_4^{j}\},$$

where

$$\lambda_4^{j} = \frac{j}{3}, \quad \lambda_3^{ij} = \frac{i-\{\lambda_4^{j}\}}{2}, \quad \lambda_2^{ij} = 1-\{\lambda_3^{ij}\}, \quad \lambda_1^{ij} = 1-\{\lambda_3^{ij}+\lambda_4^{j}\}.$$

From this we compute

$$s_{11} = 2,\ s_{21} = 3,\ s_{12} = 2,\ s_{22} = 3,\ s_{13} = 3,\ s_{23} = 5,$$

so that

$$\delta_{\mathcal{P}(A)}(t) = \sum_{i=1}^{3} \sum_{j=1}^{2} t^{d+1-s_{ij}} = 1 + 3t^2 + 2t^3,$$

and thus

$$\delta(\mathcal{P}(A)) = (1, 0, 3, 2, 0).$$

3.3.3 "One row" Hermite normal forms

In this subsection, we study the δ-vectors for some special Hermite normal forms. Results in this section are direct applications of the algorithm developed in the previous section.

Consider all $d \times d$ matrices with positive determinant D and the following Hermite normal form.

$$A_D = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ a_1 & \cdots & a_{k-1} & D & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \in \mathbb{Z}^{d \times d} \tag{3.4}$$

for some $k \in \{1, 2, \ldots, d\}$, where $a_1, \ldots, a_{k-1}$ are nonnegative integers smaller than D and all other entries are zero. Let d_j denote the number of j's among these a_ℓ's, for $j = 1, \ldots, D-1$. Then we can simplify Theorem 3.3.2 for these "one row" Hermite normal forms.

Corollary 3.3.4. *Let $M \in \mathbb{Z}^{d \times d}$ with $\det(M) = D$ and $\mathcal{P}(M)$ be the corresponding integral simplex. If its Hermite normal form is of the form as in (3.4), then we have*

$$\delta_{\mathcal{P}(M)}(t) = \sum_{i=1}^{D} t^{d+1-s_i},$$

where

$$s_i = \left\lfloor \frac{i}{D} - \sum_{j=1}^{D-1} \left\{ \frac{ij}{D} \right\} d_j \right\rfloor + d. \tag{3.5}$$

Proof. Consider

$$b(\lambda) = (\lambda_1, \ldots, \lambda_k, \ldots, \lambda_d) A_D = (\lambda_1 + a_1\lambda_k, \ldots, \lambda_{k-1} + a_{k-1}\lambda_k, D\lambda_k, \lambda_{k+1}, \ldots, \lambda_d).$$

Using the notation from the proof of Theorem 3.3.2, we have, for $i = 1, 2, \dots, D$,

$$\lambda_k^i = \frac{i}{D}, \ \lambda_\ell^i = 1 - \left\{ a_\ell \frac{i}{D} \right\}, \ \text{for } \ell = 1, \dots, k-1$$

and

$$\lambda_{k+1}^i = \cdots = \lambda_d^i = 1.$$

Therefore, $s_i = 1 + \lfloor \lambda_1^i + \cdots + \lambda_d^i \rfloor = \left\lfloor \frac{i}{D} - \sum_{j=1}^{D-1} \left\{ \frac{ij}{D} \right\} d_j \right\rfloor + d.$ □

Assume, in addition, that $d_{D-1} = d - 1$ in Corollary 3.3.4, i.e., the Hermite normal form takes the form

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ D-1 & D-1 & \cdots & D-1 & D \end{pmatrix}. \tag{3.6}$$

Then we have

Corollary 3.3.5 (All $D-1$). *For a matrix $M \in \mathbb{Z}^{d \times d}$ with Hermite normal form* (3.6)*, we have*

$$\delta_{\mathcal{P}(M)}(t) = \sum_{i=1}^{D} t^{d+1-s_i}, \ \textit{where } s_i = \left\lfloor \frac{id}{D} \right\rfloor + 1.$$

3.3.4 Classification of Hermite normal forms with a given δ-vector

In this subsection, by applying the algorithm Theorem 3.3.2, we consider Problem 3.3.1 with the assumption that the matrix $A \in \mathbb{Z}^{d \times d}$ has prime determinant, i.e., A is of the form (3.4), with only one general row. By Corollary 3.3.4, in order to classify all possible Hermite normal forms (3.4) with a given δ-vector $(\delta_0, \delta_1, \dots, \delta_d)$, we need to find all nonnegative integer solutions $(d_1, d_2, \dots, d_{D-1})$ with $d_1 + d_2 + \cdots + d_{D-1} \leq d - 1$ such that

$$|\{i : d + 1 - s_i = j, \ \text{for } i = 1, \dots, D\}| = \delta_j, \ \text{for } j = 0, \dots, d.$$

By Corollary 3.3.4, we can build equations with "floor" expressions for $(d_1, d_2, \dots, d_{D-1})$. Removing the "floor" expressions, we obtain D linear equations of $(d_1, d_2, \dots, d_{D-1})$ with different constant terms but the same $D \times D$ coefficient matrix M. Then we first find all integer solutions $(d_1, d_2, \dots, d_{D-1})$ and check every candidate using the restrictions of nonnegativity and $d_1 + d_2 + \cdots + d_{D-1} \leq d - 1$.

For $D = 2$ and 3, the coefficient matrix M is nonsingular, so we can write down the complete solutions, as presented in the first two subsections. For larger primes, the coefficient matrix becomes singular, so there are free variables in the integer solutions $(d_1, d_2, \ldots, d_{D-1})$, which make it very hard to simplify the final solutions after the test.

The idea is similar for Hermite normal forms with non prime determinant. Instead of using Corollary 3.3.4, we need to use the formulas in Theorem 3.3.2. We will also present the complete solution for $D = 4$ below.

A solution of Problem 3.3.1 when $\sum_{i=0}^{d} \delta_i = 2$.

First, we give a solution of Problem 3.3.1 when $\sum_{i=0}^{d} \delta_i = 2$, i.e., given a δ-vector $(\delta_0, \delta_1, \ldots, \delta_d)$ with $\sum_{i=0}^{d} \delta_i = 2$, we classify all the integral simplices with $(\delta_0, \delta_1, \ldots, \delta_d)$ arising from Hermite normal forms with determinant 2.

We consider all Hermite normal forms (3.4) with $D = 2$, namely,

$$A_2 = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ * & \cdots & * & 2 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \tag{3.7}$$

where there are d_1 1's among the $a_1, \ldots, a_{k-1}$. Notice that the position of the row with a 2 does not affect the δ-vector, so the only variable is d_1. By Corollary 3.3.4, we have a formula for the δ-vector of this integral simplex $\mathcal{P}(A_2)$. Denote

$$k = 1 - \left\lfloor \frac{1 - d_1}{2} \right\rfloor.$$

Then one has $\delta_0 = \delta_k = 1$.

By this formula, we can characterize all Hermite normal forms with a given δ-vector. Let $\delta_0 = \delta_i = 1$. Then by solving the equation $i = 1 - \lfloor (1 - d_1)/2 \rfloor$, we obtain $d_1 = 2i - 2$ and $d_1 = 2i - 1$, both cases will give us the desired δ-vector.

Notice that there is a constraint on d_1 given by $0 \leq d_1 \leq d - 1$. Not all δ-vectors are obtained from simplices. But we can easily get the appropriate conditions on i and the corresponding d_1 as follows (by $d_1 \geq 0$, we have $i \geq 1$):

1. If $i \leq d/2$, $d_1 = 2i - 2$ and $d_1 = 2i - 1$ both work, and these give all the matrices with this δ-vector.

2. If $i = (d + 1)/2$, only $d_1 = 2i - 2 = d - 1$ works.

3. If $i > (d + 1)/2$, there is no solution.

Now, this result has been obtained essentially in Theorem 3.2.1. In fact, the inequality $i \leq (d + 1)/2$ means that the δ-vector satisfies (3.2).

A solution of Problem 3.3.1 when $\sum_{i=0}^{d} \delta_i = 3$.
We consider all Hermite normal forms (3.4) with $D = 3$, namely,

$$A_3 = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ * & \cdots & * & 3 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \tag{3.8}$$

where there are d_1 1's and d_2 2's among the $a_1, \ldots, a_{k-1}$. The position of the row with one 3 does not affect the δ-vector, so the only variables are d_1 and d_2. Also, by Corollary 3.3.4, we have $\delta_{\mathcal{P}(A_3)}(t) = 1 + t^{k_1} + t^{k_2}$, where

$$k_1 = 1 - \left\lfloor \frac{1 - d_1 - 2d_2}{3} \right\rfloor \text{ and } k_2 = 1 - \left\lfloor \frac{2 - 2d_1 - d_2}{3} \right\rfloor.$$

Then by the formula, we can characterize all Hermite normal forms with a given δ-vector using arguments similar to $\sum_{i=0}^{d} \delta_i = 2$. Let $\delta_{\mathcal{P}(A_3)}(t) = 1 + t^i + t^j$. Set

$$i = 1 - \left\lfloor \frac{1 - d_1 - 2d_2}{3} \right\rfloor \text{ and } j = 1 - \left\lfloor \frac{2 - 2d_1 - d_2}{3} \right\rfloor.$$

(Later reverse the role of i and j if $i \neq j$, in both equations and solutions.) The solutions for (d_1, d_2) are

$$d^{(1)} = \begin{cases} d_1 = 2j - i \\ d_2 = 2i - j - 1, \end{cases} \quad d^{(2)} = \begin{cases} d_1 = 2j - i - 1 \\ d_2 = 2i - j - 1 \end{cases} \quad \text{and} \quad d^{(3)} = \begin{cases} d_1 = 2j - i \\ d_2 = 2i - j - 2. \end{cases}$$

In addition, by the restriction on (d_1, d_2) that $d_1, d_2 \geq 0$ and $d_1 + d_2 \leq d - 1$, we have the following characterizations:

Table 3.1: Characterizations for matrices of the form A_3

$2j$	$2i$	$i+j$	solutions
$\geq i$	$\geq j+1$	$\leq d$	$d^{(1)}$
$\geq i+1$	$\geq j+1$	$\leq d+1$	$d^{(2)}$
$\geq i$	$\geq j+2$	$\leq d+1$	$d^{(3)}$

1. If $2j \geq i, 2i \geq j+1$ and $i + j \leq d$, then the solution $d^{(1)}$ will work and this gives all the matrices with this δ-vector.

2. If $2j \geq i+1, 2i \geq j+1$ and $i+j \leq d+1$, then the solution $d^{(2)}$ will work and this gives all the matrices with this δ-vector.

3. If $2j \geq i, 2i \geq j+2$ and $i+j \leq d+1$, then the solution $d^{(3)}$ will work and this gives all the matrices with this δ-vector.

4. If $\{i,j\}$ in the given vector does not satisfy any of the above cases, there is no matrix with this vector as its δ-vector.

Again, this result has been obtained in Theorem 3.2.1. In fact, for example, the inequality $2j \geq i$ means that (3.1) holds and the inequality $i+j \leq d+1$ means that (3.2) holds.

Notice that only the solution

$$d^{(2)} = \begin{cases} d_1 = d-1 \\ d_2 = 0 \end{cases}$$

works when $i = (d+2)/3$ and $j = (2d+1)/3$. This happens when $d \equiv 1 \pmod 3$ and there is only one matrix with $d_1 = d-1$ and $d_2 = 0$. Similarly, only the solution

$$d^{(3)} = \begin{cases} d_1 = 0 \\ d_2 = d-1 \end{cases}$$

works when $i = (2d+2)/3$ and $j = (d+1)/3$. This happens when $d \equiv 2 \pmod 3$ and again, there is only one matrix with $d_1 = 0$ and $d_2 = d-1$.

A solution of Problem 3.3.1 when $\sum_{i=0}^{d} \delta_i = 4$.
When the determinant is 4, there are two cases of Hermite normal forms. One is the Hermite normal forms (3.4) with $D = 4$, namely,

$$A_4 = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ * & \cdots & * & 4 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \tag{3.9}$$

where there are d_1 1's, d_2 2's and d_3 3's among $*$'s. where there are d_1 1's, d_2 2's and d_3 3's among the $a_1, \dots, a_{k-1}$. The other hermit normal form takes the form

$$A_4' = \begin{pmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ * & \cdots & * & 2 & & & & & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ \dot{*} & \cdots & \dot{*} & \bar{*} & \dot{*} & \cdots & \dot{*} & 2 & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \end{pmatrix}, \tag{3.10}$$

where there are d_1 1's (resp. d_1' 1's) among $*$'s (resp. $\dot{*}$'s), there are e_1 1's (resp. e_1' 1's) among the $*$'s (resp. $\dot{*}$'s) of which the entry of the row of $\dot{*}$ (resp. $*$) in the same column is 0. Also, set $d_1'' = e_1 + e_1'$. (For example, a 6×6 Hermite normal form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 2 \end{pmatrix}$$

is a matrix (3.10) with $d_1 = 2, d_1' = 3, e_1 = 1, e_1' = 2, d_1'' = 3$ and $\bar{*} = 1$.)

First, we consider the Hermite normal forms A_4. Then, by Corollary 3.3.4, we have $\delta_{\mathcal{P}(A_4)}(t) = 1 + t^{k_1} + t^{k_2} + t^{k_3}$, where

$$k_1 = 1 - \left\lfloor \frac{1 - d_1 - 2d_2 - 3d_3}{4} \right\rfloor, k_2 = 1 - \left\lfloor \frac{1 - d_1 - d_3}{2} \right\rfloor \text{ and } k_3 = 1 - \left\lfloor \frac{3 - 3d_1 - 2d_2 - d_3}{4} \right\rfloor.$$

Let $\delta_{\mathcal{P}(A_4)}(t) = 1 + t^i + t^j + t^k$. We get three sets of equations:

$$i = 1 - \left\lfloor \frac{1 - d_1 - 2d_2 - 3d_3}{4} \right\rfloor, j = 1 - \left\lfloor \frac{1 - d_1 - d_3}{2} \right\rfloor \text{ and } k = 1 - \left\lfloor \frac{3 - 3d_1 - 2d_2 - d_3}{4} \right\rfloor.$$

(Later replace the roles of i, j and k if any of the three are distinct.) The solutions for (d_1, d_2, d_3) are

$$d^{(1)} = \begin{cases} d_1 = -i + j + k - 1 \\ d_2 = i - 2j + k \\ d_3 = i + j - k - 1, \end{cases} \qquad d^{(2)} = \begin{cases} d_1 = -i + j + k \\ d_2 = i - 2j + k \\ d_3 = i + j - k - 2, \end{cases}$$

$$d^{(3)} = \begin{cases} d_1 = -i + j + k \\ d_2 = i - 2j + k \\ d_3 = i + j - k - 1 \end{cases} \qquad d^{(4)} = \begin{cases} d_1 = -i + j + k \\ d_2 = i - 2j + k - 1 \\ d_3 = i + j - k - 1. \end{cases}$$

In addition, by the restriction on (d_1, d_2, d_3) that $d_1, d_2, d_3 \geq 0$ and $d_1+d_2+d_3 \leq d-1$, we have the following characterizations:

Table 3.2: Characterizations for matrices of the form A_4

$j+k$	$2j$	$i+j$	solutions
$\geq i+1$	$\leq i+k \leq d+1$	$\geq k+1$	$d^{(1)}$
$\geq i$	$\leq i+k \leq d+1$	$\geq k+2$	$d^{(2)}$
$\geq i$	$\leq i+k \leq d$	$\geq k+1$	$d^{(3)}$
$\geq i$	$\leq i+k-1 \leq d$	$\geq k+1$	$d^{(4)}$

1. If $j+k \geq i+1, 2j \leq i+k \leq d+1$ and $i+j \geq k+1$, then the solution $d^{(1)}$ will work and this gives all the matrices with this δ-vector.

2. If $j+k \geq i, 2j \leq i+k \leq d+1$ and $i+j \geq k+2$, then the solution $d^{(2)}$ will work and this gives all the matrices with this δ-vector.

3. If $j+k \geq i, 2j \leq i+k \leq d$ and $i+j \geq k+1$, then the solution $d^{(3)}$ will work and this gives all the matrices with this δ-vector.

4. If $j+k \geq i, 2j+1 \leq i+k \leq d+1$ and $i+j \geq k+1$, then the solution $d^{(4)}$ will work and this gives all the matrices with this δ-vector.

5. If $\{i, j, k\}$ in the given vector does not satisfy any of the above cases, there is no matrix A_4 with this vector as its δ-vector.

Notice that only the solution

$$d^{(2)} = \begin{cases} d_1 = 0 \\ d_2 = 0 \\ d_3 = d-1 \end{cases}$$

works when $i = (3d+3)/4, j = (d+1)/2$ and $k = (d+1)/4$. This happens when $d \equiv 3 \pmod 4$ and there is only one matrix with $d_3 = d-1$. Similarly, only the solution

$$d^{(1)} = \begin{cases} d_1 = d-1 \\ d_2 = 0 \\ d_3 = 0 \end{cases}$$

works when $i = (d+3)/4, j = (d+1)/2$ and $k = (3d+1)/4$. This happens when $d \equiv 1 \pmod 4$ and again, there is only one matrix with $d_1 = d-1$.

Next, we consider the Hermite normal forms (3.10). However, we need to consider two cases, which are the cases where $\bar{*} = 0$ and $\bar{*} = 1$.

First, we consider the case with $\bar{*} = 0$. Notice that the variables are d_1, d_1' and d_1''. Obviously we cannot use Corollary 3.3.4, but we apply Theorem 3.3.2 directly. Thus we have $\delta_{\mathcal{P}(A_4')}(t) = 1 + t^{k_1} + t^{k_2} + t^{k_3}$, where

$$k_1 = \left\lfloor \frac{d_1 + 2}{2} \right\rfloor,\ k_2 = \left\lfloor \frac{d_1' + 2}{2} \right\rfloor \text{ and } k_3 = \left\lfloor \frac{d_1'' + 3}{2} \right\rfloor.$$

Let $\delta_{\mathcal{P}(A_4')}(t) = 1 + t^i + t^j + t^k$. We get three sets of equations:

$$i = \left\lfloor \frac{d_1 + 2}{2} \right\rfloor,\ j = \left\lfloor \frac{d_1' + 2}{2} \right\rfloor \text{ and } k = \left\lfloor \frac{d_1'' + 3}{2} \right\rfloor.$$

or replace the role of i, j and k if i, j and k are distinct, in all equations and solutions. Since $d_1 + d_1' + d_1''$ is even, the solutions for (d_1, d_1', d_1'') are

$$d^{(1)} = \begin{cases} d_1 = 2i - 2 \\ d_1' = 2j - 1 \\ d_1'' = 2k - 3, \end{cases} \qquad d^{(2)} = \begin{cases} d_1 = 2i - 1 \\ d_1' = 2j - 2 \\ d_1'' = 2k - 3, \end{cases}$$

$$d^{(3)} = \begin{cases} d_1 = 2i - 1 \\ d_1' = 2j - 1 \\ d_1'' = 2k - 2 \end{cases} \qquad d^{(4)} = \begin{cases} d_1 = 2i - 2 \\ d_1' = 2j - 2 \\ d_1'' = 2k - 2. \end{cases}$$

In addition, by the restriction on (d_1, d_1', d_1'') that $0 \le d_1 \le d-2$, $0 \le d_1' \le d-2$, $0 \le d_1'' \le d-2$, $d_1 + d_1' + d_1'' \le 2(d-2)$, $d_1'' \le d_1 + d_1'$, $d_1' \le d_1 + d_1''$ and $d_1 \le d_1' + d_1''$, we have the following characterizations:

Table 3.3: Characterizations for matrices of the form (3.10) with $\bar{*} = 0$

i	j	k	$i+j$	$i+k$	$j+k$	$i+j+k$	solutions
$\le \lfloor \frac{d}{2} \rfloor$	$\le \lfloor \frac{d-1}{2} \rfloor$	≥ 2, $\le \lfloor \frac{d+1}{2} \rfloor$	$\ge k$	$\ge j+2$	$\ge i+1$	$\le d+1$	$d^{(1)}$
$\le \lfloor \frac{d-1}{2} \rfloor$	$\le \lfloor \frac{d}{2} \rfloor$	≥ 2, $\le \lfloor \frac{d+1}{2} \rfloor$	$\ge k$	$\ge j+1$	$\ge i+2$	$\le d+1$	$d^{(2)}$
$\le \lfloor \frac{d-1}{2} \rfloor$	$\le \lfloor \frac{d-1}{2} \rfloor$	$\le \lfloor \frac{d}{2} \rfloor$	$\ge k$	$\ge j+1$	$\ge i+1$	$\le d$	$d^{(3)}$
$\le \lfloor \frac{d}{2} \rfloor$	$\le \lfloor \frac{d}{2} \rfloor$	$\le \lfloor \frac{d}{2} \rfloor$	$\ge k+1$	$\ge j+1$	$\ge i+1$	$\le d+1$	$d^{(4)}$

1. If $i \le \lfloor d/2 \rfloor$, $j \le \lfloor (d-1)/2 \rfloor$, $2 \le k \le \lfloor (d+1)/2 \rfloor$, $i + j + k \le d + 1, k \le i + j, j + 2 \le i + k$ and $i + 1 \le j + k$, then the solution $d^{(1)}$ will work and this gives all the matrices with this δ-vector.

2. If $i \leq \lfloor (d-1)/2 \rfloor$, $j \leq \lfloor d/2 \rfloor$, $2 \leq k \leq \lfloor (d+1)/2 \rfloor$, $i+j+k \leq d+1, k \leq i+j, j+1 \leq i+k$ and $i+2 \leq j+k$, then the solution $d^{(2)}$ will work and this gives all the matrices with this δ-vector.

3. If $i, j \leq \lfloor (d-1)/2 \rfloor$, $k \leq \lfloor d/2 \rfloor$ $i+j+k \leq d, k \leq i+j, j+1 \leq i+k$ and $i+1 \leq j+k$, then the solution $d^{(3)}$ will work and this gives all the matrices with this δ-vector.

4. If $i, j, k \leq \lfloor d/2 \rfloor$, $i+j+k \leq d+1, k+1 \leq i+j, j+1 \leq i+k$ and $i+1 \leq j+k$, then the solution $d^{(4)}$ will work and this gives all the matrices with this δ-vector.

5. If $\{i, j, k\}$ in the given vector does not satisfy any of the above cases, there is no matrix (3.10), where $\bar{*} = 0$, with this vector as its δ-vector.

Next, we consider the case with $\bar{*} = 1$. By Theorem 3.3.2, we have $\delta_{\mathcal{P}(A_4')}(t) = 1 + t^{k_1} + t^{k_2} + t^{k_3}$, where

$$k_1 = 1 - \left\lfloor \frac{1 - d_1 - 2d_1''}{4} \right\rfloor, \; k_2 = 1 - \left\lfloor \frac{1 - d_1}{2} \right\rfloor \text{ and } k_3 = 2 - \left\lfloor \frac{3 - d_1 - 2d_1'}{4} \right\rfloor.$$

Let $\delta_{\mathcal{P}(A_4')}(t) = 1 + t^i + t^j + t^k$. We get three sets of equations:

$$i = 1 - \left\lfloor \frac{1 - d_1 - 2d_1''}{4} \right\rfloor, \; j = 1 - \left\lfloor \frac{1 - d_1}{2} \right\rfloor \text{ and } k = 2 - \left\lfloor \frac{3 - d_1 - 2d_1'}{4} \right\rfloor.$$

or replace the roles of i, j and k if i, j and k are distinct. Since $d_1 + d_1' + d_1''$ is even, the solutions for (d_1, d_1', d_1'') are

$$d^{(1)} = \begin{cases} d_1 = 2j - 1 \\ d_1' = 2k - j - 3 \\ d_1'' = 2i - j - 2, \end{cases} \qquad d^{(2)} = \begin{cases} d_1 = 2j - 1 \\ d_1' = 2k - j - 2 \\ d_1'' = 2i - j - 1, \end{cases}$$

$$d^{(3)} = \begin{cases} d_1 = 2j - 2 \\ d_1' = 2k - j - 3 \\ d_1'' = 2i - j - 1 \end{cases} \qquad d^{(4)} = \begin{cases} d_1 = 2j - 2 \\ d_1' = 2k - j - 2 \\ d_1'' = 2i - j - 2. \end{cases}$$

In addition, by the restriction on (d_1, d_1', d_1'') that $0 \leq d_1 \leq d-2$, $0 \leq d_1' \leq d-2$, $0 \leq d_1'' \leq d-2$, $d_1 + d_1' + d_1'' \leq 2(d-2)$, $d_1'' \leq d_1 + d_1'$, $d_1' \leq d_1 + d_1''$ and $d_1 \leq d_1' + d_1''$, we have the following characterizations:

1. If $j+3 \leq 2k \leq d+j+1$, $j+2 \leq 2i \leq d+j$, $2j \leq d-1$, $k \leq i+j$, $2j+2 \leq i+k \leq d+1$ and $i+1 \leq j+k$, then the solution $d^{(1)}$ will work and this gives all the matrices with this δ-vector.

2. If $j+2 \leq 2k \leq d+j$, $j+1 \leq 2i \leq d+j-1$, $2j \leq d-1$, $k \leq i+j$, $2j+1 \leq i+k \leq d$ and $i+1 \leq j+k$, then the solution $d^{(2)}$ will work and this gives all the matrices with this δ-vector.

Table 3.4: Characterizations for matrices of the form (3.10) with $\bar{*} = 1$

$2k$	$2i$	$2j$	$i+j$	$i+k$	$j+k$	solutions
$\geq j+3,$ $\leq d+j+1$	$\geq j+2,$ $\leq d+j$	$\leq d-1$	$\geq k$	$\geq 2j+2,$ $\leq d+1$	$\geq i+1$	$d^{(1)}$
$\geq j+2,$ $\leq d+j$	$\geq j+1,$ $\leq d+j-1$	$\leq d-1$	$\geq k$	$\geq 2j+1,$ $\leq d$	$\geq i+1$	$d^{(2)}$
$\geq j+3,$ $\leq d+j+1$	$\geq j+1,$ $\leq d+j-1$	$\leq d$	$\geq k$	$\geq 2j+1,$ $\leq d+1$	$\geq i+2$	$d^{(3)}$
$\geq j+2,$ $\leq d+j$	$\geq j+2,$ $\leq d+j$	$\leq d$	$\geq k+1$	$\geq 2j+1,$ $\leq d+1$	$\geq i+1$	$d^{(4)}$

3. If $j+3 \leq 2k \leq d+j+1$, $j+1 \leq 2i \leq d+j-1$, $2j \leq d$, $k \leq i+j$, $2j+1 \leq i+k \leq d+1$ and $i+2 \leq j+k$, then the solution $d^{(3)}$ will work and this gives all the matrices with this δ-vector.

4. If $j+2 \leq 2k \leq d+j$, $j+2 \leq 2i \leq d+j$, $2j \leq d$, $k+1 \leq i+j$, $2j+1 \leq i+k \leq d+1$ and $i+1 \leq j+k$, then the solution $d^{(4)}$ will work and this gives all the matrices with this δ-vector.

5. If $\{i, j, k\}$ in the given vector does not satisfy any of the above cases, there is no matrix (3.10) with this vector as its δ-vector.

Notice that only the solution

$$d^{(3)} = \begin{cases} d_1 = d-2 \\ d_1' = d-2 \\ d_1'' = 0 \end{cases}$$

works when $i = (d+2)/4, j = d/2$ and $k = (3d+2)/4$. This happens when $d \equiv 2 \pmod 4$ and there is only one matrix with $d_1 = d_1' = d-2$. Similarly, only the solution

$$d^{(4)} = \begin{cases} d_1 = d-2 \\ d_1' = 0 \\ d_1'' = d-2 \end{cases}$$

works when $i = 3d/4, j = d/2$ and $k = d/4+1$. This happens when $d \equiv 0 \pmod 4$ and again, there is only one matrix with $d_1 = d_1'' = d-2$.

3.3.5 A classification of the possible δ-vectors with $\sum_{i=0}^{d} \delta_i = 4$

Finally, we classify the possible δ-vectors with $\sum_{i=0}^{d} \delta_i = 4$ using results from the previous subsection.

As described above, we need some new constraints on δ-vectors. For explaining such required constraints, we introduce some notations. Let $(\delta_0, \delta_1, \ldots, \delta_d)$ be the δ-vector of some integral convex polytope with $\sum_{i=0}^{d} \delta_i = 4$ and let $1 + t^{i_1} + t^{i_2} + t^{i_3}$ with $1 \leq i_1 \leq i_2 \leq i_3 \leq d$ be a polynomial in t satisfying $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + t^{i_2} + t^{i_3}$. Note that $(\delta_0, \delta_1, \ldots, \delta_d)$ satisfies the inequalities (3.1) and (3.2) which are necessary conditions to be a possible δ-vector. Then (3.1) and (3.2) lead into the following inequalities on (i_1, i_2, i_3):

$$i_3 \leq i_1 + i_2, \ i_1 + i_3 \leq d + 1 \ \text{ and } \ i_2 \leq \lfloor (d+1)/2 \rfloor. \tag{3.11}$$

By using these, the classification of possible δ-vectors of integral convex polytopes with $\sum_{i=0}^{d} \delta_i = 4$ is given by the following

Theorem 3.3.6 ([34, Theorem 5.1]). *Let $1 + t^{i_1} + t^{i_2} + t^{i_3}$ be a polynomial with $1 \leq i_1 \leq i_2 \leq i_3 \leq d$. Then there exists an integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d whose δ-polynomial equals $1 + t^{i_1} + t^{i_2} + t^{i_3}$ if and only if (i_1, i_2, i_3) satisfies* (3.11) *and the additional condition*

$$2i_2 \leq i_1 + i_3 \ \textit{or} \ i_2 + i_3 \leq d + 1. \tag{3.12}$$

Moreover, all these polytopes can be chosen to be simplices.

Proof. There are four cases: (1) $i_1 = i_2 = i_3$, (2) $i_1 < i_2 = i_3$, (3) $i_1 = i_2 < i_3$, (4) $i_1 < i_2 < i_3$. We will show that in each case (3.11) together with (3.12) are the necessary and sufficient conditions for $1 + t^{i_1} + t^{i_2} + t^{i_3}$ to be the δ-polynomial of some integral convex polytope.

(1) Assume $i_1 = i_2 = i_3 = \ell$. By the inequalities (3.11), we have $1 \leq \ell \leq \lfloor (d+1)/2 \rfloor$. Set $i = j = k = \ell$. We have

$$j + k \geq i + 1, \ 2j \leq i + k \leq d + 1 \text{ and } i + j \geq k + 1. \tag{3.13}$$

Thus, by our result on the classification in the case of a matrix of the form A_4 (Table 3.2, the solution $d^{(1)}$), there exists an integral simplex whose δ-vector is of the form $(1, 0, \ldots, 0, 3, 0, \ldots, 0)$.

On the other hand, if there exists an integral convex polytope with this δ-vector, then (3.11) holds since it is a necessary condition. In this case, both inequalities in (3.12) hold.

(2) Assume $\ell = i_1 < i_2 = i_3 = \ell'$. By (3.11), we have $1 \leq \ell < \ell' \leq \lfloor (d+1)/2 \rfloor$. Let $j = \ell$ and $i = k = \ell'$. Then the inequalities (3.13) hold. Thus there exists an integral simplex whose δ-vector is $(1, 0, \ldots, 0, 1, 0, \ldots, 0, 2, 0, \ldots, 0)$.

On the other hand, if there exists an integral convex polytope with this δ-vector, then we have (3.11) and $i_2 + i_3 \leq d + 1$ follows from $i_2 \leq \lfloor (d+1)/2 \rfloor$.

(3) Assume $\ell = i_1 = i_2 < i_3 = \ell'$. Set $i = \ell'$ and $j = k = \ell$. Then it follows from (3.11) that

$$j + k \geq i, 2j + 1 \leq i + k \leq d + 1 \text{ and } i + j \geq k + 1.$$

Thus, by our result (Table 3.2, the solution $d^{(4)}$), there exists an integral simplex whose δ-vector is $(1, 0, \dots, 0, 2, 0, \dots, 0, 1, 0, \dots, 0)$.

On the other hand, if there exists an integral convex polytope with this δ-vector, then (3.11) holds. In this case, both inequalities in (3.12) hold.

(4) Assume $1 \leq i_1 < i_2 < i_3 \leq d$. Suppose $2i_2 \leq i_1 + i_3$ holds. Set $i = i_3, j = i_2$ and $k = i_1$. Then we have $j + k = i_1 + i_2 \geq i_3 = i$, $2j = 2i_2 \leq i_1 + i_3 = i + k \leq d + 1$ and $i + j = i_2 + i_3 \geq 2i_2 + 1 \geq 2i_1 + 3 > i_1 + 2 = k + 2$. Thus, by our result (Table 3.2, the solution $d^{(2)}$), there exists an integral simplex whose δ-vector is $(1, 0 \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$.

Suppose $i_2 + i_3 \leq d + 1$ holds. Set $i = i_3, j = i_1$ and $k = i_2$. Then we have $j + k = i_1 + i_2 \geq i_3 = i$, $2j = 2i_1 < i_2 + i_3 = i + k \leq d + 1$ and $i + j = i_1 + i_3 \geq i_1 + i_2 + 1 \geq i_2 + 2 = k + 2$. Thus, by our result (Table 3.2, the solution $d^{(2)}$), there exists an integral simplex whose δ-vector is $(1, 0 \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$.

On the other hand, assume the contrary of (3.12): both $2i_2 > i_1 + i_3$ and $i_2 + i_3 > d + 1$ hold. We claim that there exists no integral convex polytope $\mathcal{P}$ with this δ-vector. First we want to show that if there exists such a polytope, it must be a simplex. Note that the δ-vector satisfies (3.11). Suppose $i_1 = 1$. It then follows from (3.11) and $i_2 + i_3 > d + 1$ that $i_2 = (d + 1)/2$ and $i_3 = (d + 3)/2$. However, this contradicts (3.2). Therefore $i_1 > 1$, and thus $\delta_1 = 0$. By the explanation after equation (2.1), $\mathcal{P}$ must be a simplex. Now we can apply our characterization results for simplices.

If we set $j = i_3$, then $2j = 2i_3 > i_1 + i_2 = i + k$. If we set $j = i_2$, then $2j = 2i_2 > i_1 + i_3 = i + k$. If we set $j = i_1$, then $i + k = i_2 + i_3 > d + 1$. In any case there does not exist an Hermite normal form A_4 whose δ-polynomial coincides with $1 + t^{i_1} + t^{i_2} + t^{i_3}$.

Moreover, since $i + j + k = i_1 + i_2 + i_3 > i_2 + i_3 > d + 1$, there does not exist an Hermite normal form (3.10) with $\bar{*} = 0$ whose δ-polynomial coincides with $1 + t^{i_1} + t^{i_2} + t^{i_3}$.

In addition, if we set $j = i_3$, then $2j = 2i_3 > i_1 + i_2 = i + k$. If we set $j = i_2$, then $2j = 2i_2 > i_1 + i_3 = i + k$. If we set $j = i_1$, then $i + k = i_2 + i_3 > d + 1$. Thus there does not exist an Hermite normal form (3.10) with $\bar{*} = 1$ whose δ-polynomial coincides with $1 + t^{i_1} + t^{i_2} + t^{i_3}$. $\square$

Examples 3.3.7. (a) We consider the integer sequence $(1, 0, 1, 1, 0, 1, 0)$. Then one has $i_1 = 2, i_2 = 3, i_3 = 5$ and $d = 6$. Since (3.1) and (3.2) are satisfied and $2i_2 \leq i_1 + i_3$ holds, there is an integral convex polytope whose δ-vector coincides with $(1, 0, 1, 1, 0, 1, 0)$ by Theorem 3.3.6. In fact, let $M \in \mathbb{Z}^{6\times 6}$ be the Hermite normal form (3.9) with $(d_1, d_2, d_3) = (0, 1, 4)$ or $(0, 0, 5)$. Then we have $\delta(\mathcal{P}(M)) = (1, 0, 1, 1, 0, 1, 0)$.

(b) There is no integral convex polytope with its δ-vector $(1, 0, 1, 0, 1, 1, 0, 0)$ since we have $2i_2 > i_1 + i_3$ and $i_2 + i_3 > d + 1$, although this integer sequence satisfies (3.1) and (3.2). (See Example 3.2.2.) However, there exists an integral convex polytope with its δ-vector $(1, 0, 1, 0, 1, 1, 0, 0, 0)$ since $i_2 + i_3 = d + 1$ holds.

Remark 3.3.8. We see that when $\sum_{i=0}^{d} \delta_i = 4$, all the possible δ-vectors can be obtained by simplices. This is also true for all δ-vectors with $\sum_{i=0}^{d} \delta_i \leq 3$. However, when $\sum_{i=0}^{d} \delta_i = 5$, the δ-vector $(1,3,1)$ cannot be obtained from any simplex, while it is a possible δ-vector of an integral convex polygon. In fact, suppose that $(1,3,1)$ can be obtained from a simplex because of [40, Theorem 0.1].

3.4 Towards the case where $\sum_{i=0}^{d} \delta_i \geq 5$

As shown in Remark 3.3.8, not all the possible δ-vectors can be realized as the δ-vectors of integral simplices when $\sum_{i=0}^{d} \delta_i \geq 5$. Therefore, for the classification of the δ-vectors with $\sum_{i=0}^{d} \delta_i \geq 5$, it is natural to investigate the δ-vectors of integral simplices. In particular, the case where $\sum_{i=0}^{d} \delta_i$ is prime is of interest, which we shall explain precisely in this section.

3.4.1 New inequalities on δ-vectors of integral simplices with prime volumes

In this subsection, we present new inequalities on δ-vectors of integral simplices whose normlized prime volumes are prime. Concretely, we establish the following

Theorem 3.4.1 ([40, Theorem 0.1]). *Let $\mathcal{P}$ be an integral simplex of dimension d and $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$ its δ-vector. Suppose that $\sum_{i=0}^{d} \delta_i = p$ is an odd prime number. Let $i_1, \ldots, i_{p-1}$ be the positive integers such that $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + \cdots + t^{i_{p-1}}$ with $1 \leq i_1 \leq \cdots \leq i_{p-1} \leq d$. Then,*

(a) *one has*

$$i_1 + i_{p-1} = i_2 + i_{p-2} = \cdots = i_{(p-1)/2} + i_{(p+1)/2} \leq d+1;$$

(b) *one has*

$$i_k + i_\ell \geq i_{k+\ell} \quad \textit{for} \quad 1 \leq k \leq \ell \leq p-1 \quad \textit{with} \quad k+\ell \leq p-1.$$

Proof. Let $v_0, v_1, \ldots, v_d$ be the vertices of the integral simplex $\mathcal{P}$ and $S(\mathcal{P})$ the group appearing in Section 3.1. Then, since $\mathrm{vol}(\mathcal{P}) = p$ is prime, it follows that the order of $S(\mathcal{P})$ is also prime. In particular, $S(\mathcal{P}) \cong \mathbb{Z}/p\mathbb{Z}$.

(a) Write $g_{i_1}, \ldots, g_{i_{p-1}} \in S(\mathcal{P}) \setminus \{(0,\ldots,0)\}$ for $(p-1)$ distinct elements with $\deg(g_{i_j}) = i_j$ for $1 \leq j \leq p-1$, that is, $S(\mathcal{P}) = \{(0,\ldots,0), g_{i_1}, \ldots, g_{i_{p-1}}\}$. Then, for each g_{i_j}, there exists its inverse $-g_{i_j}$ in $S(\mathcal{P}) \setminus \{(0,\ldots,0)\}$. Let $-g_{i_j} = g_{i'_j}$. If g_{i_j} has the expression $g_{i_j} = \sum_{q=0}^{d} r_q(v_q, 1)$, where $r_q \in \mathbb{Q}$ with $0 \leq r_q < 1$, then its inverse has the expression $g_{i'_j} = \sum_{q=0}^{d} \{1 - r_q\}(v_q, 1)$. Thus, one has

$$\deg(g_{i_j}) + \deg(g_{i'_j}) = \sum_{q=0}^{d} (r_q + \{1 - r_q\}) \leq \sum_{q=0}^{d} (r_q + 1 - r_q) = d+1$$

for all $1 \le j \le p-1$.

For $j_1, j_2 \in \{1, \dots, p-1\}$ with $j_1 \ne j_2$, let $g_{i_{j_1}} = \sum_{q=0}^{d} r_q^{(1)}(v_q, 1)$ and $g_{i_{j_2}} = \sum_{q=0}^{d} r_q^{(2)}(v_q, 1)$. Since $S(\mathcal{P}) \cong \mathbb{Z}/p\mathbb{Z}$, $g_{i_{j_1}}$ generates $S(\mathcal{P})$, which implies that we can write $g_{i_{j_2}}$ and $g_{i'_{j_2}}$ as follows:

$$g_{i_{j_2}} = \underbrace{g_{i_{j_1}} \oplus \cdots \oplus g_{i_{j_1}}}_{t}, \quad g_{i'_{j_2}} = \underbrace{g_{i'_{j_1}} \oplus \cdots \oplus g_{i'_{j_1}}}_{t}$$

for some integer $t \in \{2, \dots, p-1\}$. Thus, we have

$$\sum_{q=0}^{d} (r_q^{(2)} + \{1 - r_q^{(2)}\}) = \deg(g_{i_{j_2}}) + \deg(g_{i'_{j_2}})$$

$$= \deg(\underbrace{g_{i_{j_1}} \oplus \cdots \oplus g_{i_{j_1}}}_{t}) + \deg(\underbrace{g_{i'_{j_1}} \oplus \cdots \oplus g_{i'_{j_1}}}_{t}) = \sum_{q=0}^{d} (\{tr_q^{(1)}\} + \{t(1 - r_q^{(1)})\}).$$

Moreover, $\underbrace{g_{i_{j_1}} \oplus \cdots \oplus g_{i_{j_1}}}_{p} = (0, \dots, 0)$ holds. Thus, we have $\{pr_q^{(1)}\} = 0$ for all $0 \le q \le d$. Again, since p is prime, it follows that the denominator of each rational number $r_q^{(1)}$ must be p. Hence, if $0 < r_q^{(1)} < 1$ (resp. $0 < \{1 - r_q^{(1)}\} < 1$), then $0 < \{tr_q^{(1)}\} < 1$ (resp. $0 < \{t(1 - r_q^{(1)})\} < 1$), so $r_q^{(1)} + \{1 - r_q^{(1)}\} = \{tr_q^{(1)}\} + \{t(1 - r_q^{(1)})\} = 1$. In addition, obviously, if $r_q^{(1)} = \{1 - r_q^{(1)}\} = 0$, then $\{tr_q^{(1)}\} = \{t(1 - r_q^{(1)})\} = 0$, so $r_q^{(1)} + \{1 - r_q^{(1)}\} = \{tr_q^{(1)}\} + \{t(1 - r_q^{(1)})\} = 0$. Thus, $\deg(g_{i_{j_1}}) + \deg(g_{i'_{j_1}}) = \deg(g_{i_{j_2}}) + \deg(g_{i'_{j_2}})$, i.e., $i_{j_1} + i'_{j_1} = i_{j_2} + i'_{j_2}$. Hence, we obtain

$$i_1 + i'_1 = \cdots = i_{(p-1)/2} + i'_{(p-1)/2} (= i_{(p+1)/2} + i'_{(p+1)/2} = \cdots = i_{p-1} + i'_{p-1}) \le d + 1.$$

Our work is to show that $i'_j = i_{p-j}$ for all $1 \le j \le (p-1)/2$.

First, we consider i'_1. Suppose that $i'_1 \ne i_{p-1}$. Then, there is $m \in \{1, \dots, p-2\}$ with $i'_1 = i_m < i_{p-1}$. Thus, it follows that

$$i_{p-1} + i'_{p-1} = i_1 + i'_1 = i_1 + i_m < i_1 + i_{p-1} \le i'_{p-1} + i_{p-1},$$

a contradiction. Thus, i'_1 must be i_{p-1}. Next, we consider i'_2. Since $g_{i'_2} \ne g_{i_1}$ and $g_{i'_2} \ne g_{i_{p-1}}$, we may consider i'_2 among $\{i_2, \dots, i_{p-2}\}$. Then, the same discussion can be done. Hence, $i'_2 = i_{p-2}$. Similarly, we have $i'_3 = i_{p-3}, \dots, i'_{(p-1)/2} = i_{(p+1)/2}$.

Therefore, we obtain the desired conditions

$$i_1 + i_{p-1} = i_2 + i_{p-2} = \cdots = i_{(p-1)/2} + i_{(p+1)/2} \le d + 1.$$

(b) Write $g_{i_1}, \dots, g_{i_\ell} \in S(\mathcal{P}) \setminus \{(0, \dots, 0)\}$ for ℓ distinct elements with $\deg(g_{i_j}) = i_j$ for $1 \le j \le \ell$. Let $A = \{g_{i_1}, \dots, g_{i_\ell}\}$. Then there are k distinct elements $h_{i_1}, \dots, h_{i_k}$ in A with $\deg(h_{i_j}) = i_j$ for $1 \le j \le k$ satisfying $|A| + |B| = k + \ell \le p - 1$, where $B = \{h_{i_1}, \dots, h_{i_k}\} \subset A$. Moreover, for each $g \in A \oplus B = \{a \oplus b : a \in A, b \in B\}$, g satisfies $\deg(g) \leq i_k + i_\ell$. In fact, for $g_{i_j} \in A$ and $h_{i_{j'}} \in B$, if they have the expressions

$$g_{i_j} = \sum_{q=0}^{d} r_q (v_q, 1) \quad \text{and} \quad h_{i_{j'}} = \sum_{q=0}^{d} r'_q (v_q, 1),$$

where $r_q, r'_q \in \mathbb{Q}$ with $0 \leq r_q, r'_q < 1$, then one has

$$\deg(g_{i_j} \oplus h_{i_{j'}}) = \sum_{q=0}^{d} \{r_q + r'_q\} \leq \sum_{q=0}^{d} (r_q + r'_q) = i_j + i_{j'} \leq i_k + i_\ell.$$

Now, by applying the well-known theorem, so-called *Cauchy-Davenport theorem* (cf. [50]), it follows that there exist at least k elements in $A \oplus B \setminus A \cup \{(0, \dots, 0)\}$. In addition, each g_{i_j} in A satisfies $\deg(g_{i_j}) \leq i_\ell \leq i_k + i_\ell$. Thus, we can say that there exist at least $(k + \ell)$ distinct elements in $S(\mathcal{P}) \setminus \{(0, \dots, 0)\}$ whose degrees are at most $i_k + i_\ell$. From the definition of $i_1, \dots, i_{p-1}$, this means that $i_k + i_\ell \geq i_{k+\ell}$, as desired. $\square$

Remark 3.4.2. (a) When $i_1 + i_{p-1} = \cdots = i_{(p-1)/2} + i_{(p+1)/2} = d + 1$, the δ-vector is shifted symmetric. Shifted symmetric δ-vectors are studied in [37]. Moreover, the theorem [37, Theorem 2.3] says that if $i_1 + i_{p-1} = d + 1$, then we have $i_1 + i_{p-1} = \cdots = i_{(p-1)/2} + i_{(p+1)/2} = d + 1$.

(b) The inequalities $i_1 + i_\ell \geq i_{\ell+1}$ are not new. In fact, for example, when $i_1 < \cdots < i_{p-1}$, by (3.1), one has

$$\delta_0 + \cdots + \delta_{i_1} \leq \delta_{i_{p-1}} + \cdots + \delta_{i_{p-1} - i_1}.$$

Thus, we obtain $i_{p-1} - i_1 \leq i_{p-2}$, i.e., $i_1 + i_{p-2} \geq i_{p-1}$. Similarly, one has

$$\delta_0 + \cdots + \delta_{i_2} \leq \delta_{i_{p-1}} + \cdots + \delta_{i_{p-1} - i_2}.$$

Thus, we obtain $i_{p-1} - i_2 \leq i_{p-3}$. Since $i_1 + i_{p-1} = i_2 + i_{p-2}$, this is equivalent to $i_1 + i_{p-3} \geq i_{p-2}$. In the same way, we can obtain all inequalities $i_1 + i_\ell \geq i_{\ell+1}$. On the other hand, when $k \geq 2$, there are many new inequalities.

Remark 3.4.3. We note that we cannot characterize the possible δ-vectors of integral simplices with higher prime normalized volumes only by Theorem 3.4.1. In fact, since the volume of an integral convex polytope containing a unique integer point in its interior has an upper bound, if p is a sufficiently large prime number, then the integer sequence $(1, 1, p - 3, 1)$ cannot be a δ-vector of some integral simplex of dimension 3, although $(1, 1, p - 3, 1)$ satisfies all the conditions of Theorem 3.4.1.

3.4.2 A classification of the possible δ-vectors of integral simplices with $\sum_{i=0}^{d} \delta_i = 5$

As an application of Theorem 3.4.1, we give a complete characterization of the possible δ-vectors of integral simplices when $\sum_{i=0}^{d} \delta_i = 5$.

Theorem 3.4.4 ([40, Theorem 0.2]). *Given a finite sequence $(\delta_0, \delta_1, \ldots, \delta_d)$ of nonnegative integers, where $\delta_0 = 1$ and $\sum_{i=0}^{d} \delta_i = 5$, there exists an integral simplex $\mathcal{P} \subset \mathbb{R}^d$ of dimension d whose δ-vector coincides with $(\delta_0, \delta_1, \ldots, \delta_d)$ if and only if $i_1, \ldots, i_4$ satisfy $i_1 + i_4 = i_2 + i_3 \leq d + 1$ and $i_k + i_\ell \geq i_{k+\ell}$ for $1 \leq k \leq \ell \leq 4$ with $k + \ell \leq 4$, where $i_1, \ldots, i_4$ are the positive integers such that $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + \cdots + t^{i_4}$ with $1 \leq i_1 \leq \cdots \leq i_4 \leq d$.*

By virtue of Theorem 3.4.1, the "Only if" parts of Theorem 3.4.4 are obvious. In this subsection, we give a proof of the "If" part of Theorem 3.4.4, i.e., we classify all the possible δ-vectors of integral simplices whose normalized volume is 5.

Let $(\delta_0, \delta_1, \ldots, \delta_d)$ be a nonnegative integer sequence with $\delta_0 = 1$ and $\sum_{i=0}^{d} \delta_i = 5$ which satisfies $i_1 + i_4 = i_2 + i_3 \leq d+1$, $2i_1 \geq i_2$ and $i_1 + i_2 \geq i_3$, where $i_1, \ldots, i_4$ are the positive integers such that $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + \cdots + t^{i_4}$ with $1 \leq i_1 \leq \cdots \leq i_4 \leq d$. Since $i_1 + i_4 = i_2 + i_3$, we notice that $i_1 + i_3 \geq i_4$ (resp. $2i_2 \geq i_4$) is equivalent to $2i_1 \geq i_2$ (resp. $i_1 + i_2 \geq i_3$). From the conditions $\delta_0 = 1$, $\sum_{i=0}^{d} \delta_i = 5$ and $i_1 + i_4 = i_2 + i_3$, the possible sequences are only the following forms:

(i) $(1, 0, \ldots, 0, 4, 0, \ldots, 0)$;

(ii) $(1, 0, \ldots, 0, 2, 0, \ldots, 0, 2, 0, \ldots, 0)$;

(iii) $(1, 0, \ldots, 0, 1, 0, \ldots, 0, 2, 0, \ldots, 0, 1, 0, \ldots, 0)$;

(iv) $(1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$.

Our work is to find integral simplices whose δ-vectors are of the above forms.

To construct integral simplices, we define the following integer matrix

$$A_5(d_1, \ldots, d_4) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ * & \cdots & * & 5 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \tag{3.14}$$

where there are d_j j's among the $*$'s for $j = 1, \ldots, 4$ and the rest of the entries are all 0. Note that (3.14) is nothing but the Hermite normal form (3.4) with $D = 5$. Then, clearly, it must be $d_j \geq 0$ and $d_1 + \cdots + d_4 \leq d - 1$. By determining

$d_1, \dots, d_4$, we obtain an integer matrix $A_5(d_1, \dots, d_4)$ and we define the integral simplex $\mathcal{P}_5(d_1, \dots, d_4)$ from the matrix as follows:

$$\mathcal{P}_5(d_1, \dots, d_4) = \mathrm{conv}(\{(0, \dots, 0), v_1, \dots, v_d\}) \subset \mathbb{R}^d,$$

where v_i is the ith row vector of $A_5(d_1, \dots, d_4)$.

The case (i). Let $i_1 = i_2 = i_3 = i_4 = i$. Thus, one has $i-1 \geq 0$ and $2i-2 \leq d-1$ from our conditions. Hence, we can define $\mathcal{P}_5(0, i-1, i-1, 0)$. Then, by Corollary 3.3.4, $\delta(\mathcal{P}_5(0, i-1, i-1, 0))$ coincides with (i) since $s_1 = s_2 = s_3 = s_4 = -i+1$.

The case (ii). Let $i_1 = i_2 = i$ and $i_3 = i_4 = j$. Thus, one has $2i \geq j$, $2j-2i-2 \geq 0$ and $i+j-2 \leq d-1$. Hence, we can define $\mathcal{P}_5(0, i, 2i-j, 2j-2i-2)$ and its δ-vector coincides with (ii) since $s_1 = s_2 = -j+1$ and $s_3 = s_4 = -i+1$.

The case (iii). Let $i_1 = i, i_2 = i_3 = j$ and $i_4 = k$. Thus, one has $2i \geq j$, $3j-3i-2 \geq 0$ and $2j-2 \leq d-1$. Hence, we can define $\mathcal{P}_5(0, 2i-j, i, 3j-3i-2)$ and its δ-vector coincides with (iii) since $s_1 = -2j+i+1 = -k+1$, $s_2 = s_3 = -j+1$ and $s_4 = -i+1$.

The case (iv). In this case, one has $2i_1 \geq i_2, i_1+i_2 \geq i_3, i_2+2i_3-3i_1-2 \geq 0$ and $i_2+i_3-2 \leq d-1$. Hence, we can define $\mathcal{P}_5(0, 2i_1-i_2, i_1+i_2-i_3, i_2+2i_3-3i_1-2)$ and its δ-vector coincides with (iv) since $s_1 = i_1-i_2-i_3+1 = -i_4+1, s_2 = -i_3+1, s_3 = -i_2+1$ and $s_4 = -i_1+1$.

Remark 3.4.5. The inequalities $2i_1 \geq i_2$ and $i_1+i_2 \geq i_3$ can be obtained from (3.1) as we mentioned in Remark 3.4.2 (b). Thus, the possible δ-vectors of integral simplices with normalized volume 5 can be essentially characterized only by Theorem 3.4.1 (a) and the inequalities (3.1).

3.4.3 A classification of the possible δ-vectors of integral simplices with $\sum_{i=0}^{d} \delta_i = 7$

Similar to the previous subsection, we give a complete characterization of the possible δ-vectors of integral simplices when $\sum_{i=0}^{d} \delta_i = 7$, that is,

Theorem 3.4.6 ([40, Theorem 0.3]). *Given a finite sequence $(\delta_0, \delta_1, \dots, \delta_d)$ of nonnegative integers, where $\delta_0 = 1$ and $\sum_{i=0}^{d} \delta_i = 7$, there exists an integral simplex $\mathcal{P} \subset \mathbb{R}^d$ of dimension d whose δ-vector coincides with $(\delta_0, \delta_1, \dots, \delta_d)$ if and only if $i_1, \dots, i_6$ satisfy $i_1 + i_6 = i_2 + i_5 = i_3 + i_4 \leq d+1$ and $i_k + i_\ell \geq i_{k+\ell}$ for $1 \leq k \leq \ell \leq 6$ with $k + \ell \leq 6$, where $i_1, \dots, i_6$ are the positive integers such that $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + \cdots + t^{i_6}$ with $1 \leq i_1 \leq \cdots \leq i_6 \leq d$.*

By virtue of Theorem 3.4.1, the "Only if" parts of Theorem 3.4.6 are obvious. In this subsection, similarly to the previous one, we give a proof of the "If" part of Theorem 3.4.6, i.e., we classify all the possible δ-vectors of integral simplices whose normalized volume is 7.

Let $(\delta_0, \delta_1, \dots, \delta_d)$ be a nonnegative integer sequence with $\delta_0 = 1$ and $\sum_{i=0}^{d} \delta_i = 7$ which satisfies $i_1+i_6 = i_2+i_5 = i_3+i_4 \leq d+1$, $i_1+i_l \geq i_{l+1}$ for $1 \leq l \leq 3$ and $2i_2 \geq i_4$, where $i_1, \dots, i_6$ are the positive integers such that $\sum_{i=0}^{d} \delta_i t^i = 1 + t^{i_1} + \cdots + t^{i_6}$ with $1 \le i_1 \le \cdots \le i_6 \le d$. Since $i_1 + i_6 = i_2 + i_5 = i_3 + i_4$, we need not consider the inequalities $i_1 + i_4 \ge i_5$, $i_1 + i_5 \ge i_6$, $i_2 + i_3 \ge i_5$, $i_2 + i_4 \ge i_6$ and $2i_3 \ge i_6$. From the conditions $\delta_0 = 1$, $\sum_{i=0}^{d} \delta_i = 7$ and $i_1 + i_6 = i_2 + i_5 = i_3 + i_4$, the possible sequences are only the following forms:

(i) $(1, 0, \dots, 0, 6, 0, \dots, 0)$;

(ii) $(1, 0, \dots, 0, 3, 0, \dots, 0, 3, 0, \dots, 0)$;

(iii) $(1, 0, \dots, 0, 1, 0, \dots, 0, 4, 0, \dots, 0, 1, 0, \dots, 0)$;

(iv) $(1, 0, \dots, 0, 2, 0, \dots, 0, 2, 0, \dots, 0, 2, 0, \dots, 0)$;

(v) $(1, 0, \dots, 0, 1, 0, \dots, 0, 2, 0, \dots, 0, 2, 0, \dots, 0, 1, 0, \dots, 0)$;

(vi) $(1, 0, \dots, 0, 2, 0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0, 2, 0, \dots, 0)$;

(vii) $(1, 0, \dots, 0, 1, 0, \dots, 0, 1, \dots, 0, 2, 0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$;

(viii) $(1, 0, \dots, 0, 1, 0, \dots, 0, 1, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots, 0)$.

In the same way as the previous section, we define the following integer matrix:

$$A_7(d_1, \dots, d_6) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ * & \cdots & * & 7 & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}, \tag{3.15}$$

where there are d_j j's among the $*$'s for $j = 1, \dots, 6$ and the rest of the entries are all 0. Then it must be $d_j \ge 0$ and $d_1 + \cdots + d_6 \le d - 1$. By determining $d_1, \dots, d_6$, we obtain the integral simplex

$$\mathcal{P}_7(d_1, \dots, d_6) = \mathrm{conv}(\{(0, \dots, 0), v_1, \dots, v_d\}) \subset \mathbb{R}^d,$$

where v_i is the ith row vector of $A_7(d_1, \dots, d_6)$.

The case (i). Let $i_1 = \cdots = i_6 = i$. Thus, one has $i - 1 \ge 0$ and $2i - 2 \le d - 1$ from our conditions. Hence, we can define $\mathcal{P}_7(0, 0, i-1, i-1, 0, 0)$. Then, by Corollary 3.3.4, $\delta(\mathcal{P}_7(0, 0, i-1, i-1, 0, 0))$ coincides with (i) since $s_1 = \cdots = s_6 = -i + 1$.

The case (ii). Let $i_1 = \cdots = i_3 = i$ and $i_4 = \cdots = i_6 = j$. Thus, one has $j - i \ge 0, 2i \ge j, 2j - 2i - 2 \ge 0$ and $i + j - 2 \le d - 1$. Hence, we can define $\mathcal{P}_7(0, j - i, 2i - j, 2i - j, 0, 2j - 2i - 2)$ and its δ-vector coincides with (ii) since $s_1 = s_2 = s_3 = -j + 1$ and $s_4 = s_5 = s_6 = -i + 1$.

The case (iii). Let $i_1 = i, i_2 = \cdots = i_5 = j$ and $i_6 = k$. Thus, one has $i + j \geq k, k - j \geq 0, k - i - 1 \geq 0, i - 1 \geq 0$ and $i + k - 2 \leq d - 1$. Hence, we can define $\mathcal{P}_7(i + j - k, k - j, k - i - 1, 0, 0, i - 1)$ and its δ-vector coincides with (iii) since $s_1 = \frac{-4i+j-4k}{7} + 1 = -j + 1, s_2 = \frac{-i+2j-8k}{7} + 1 = -k + 1, s_3 = \frac{-5i+3j-5k}{7} + 1 = -j + 1, s_4 = \frac{-2i-3j-2k}{7} + 1 = -j + 1, s_5 = \frac{-6i-2j+k}{7} + 1 = -i + 1$ and $s_6 = \frac{-3i-j-3k}{7} + 1 = -j + 1$.

The case (iv). Let $i_1 = i_2 = i, i_3 = i_4 = j$ and $i_5 = i_6 = k$. Thus, one has $i - 1 \geq 0, i + j \geq k, 3k - 3j - 1 \geq 0$ and $2i - 2j + 2k - 2 = i + k - 2 \leq d - 1$. Hence, we can define $\mathcal{P}_7(0, 0, i - 1, i + j - k, 0, 3k - 3j - 1)$ and its δ-vector coincides with (iv) since $s_1 = s_2 = -i + 2j - 2k + 1 = -k + 1, s_3 = s_4 = -i + j - k + 1 = -j + 1$ and $s_5 = s_6 = -i + 1$.

The case (v). Let $i_1 = k_1, i_2 = i_3 = k_2, i_4 = i_5 = k_3$ and $i_6 = k_4$. Thus, one has $2k_1 \geq k_2, k_2 - k_1 \geq 0, k_1 + k_2 \geq k_3, 2k_3 - 2k_1 - 2 \geq 0$ and $k_2 + k_3 - 2 \leq d - 1$. Hence, we can define $\mathcal{P}_7(0, 2k_1 - k_2, 0, k_2 - k_1, k_1 + k_2 - k_3, 2k_3 - 2k_1 - 2)$ and its δ-vector coincides with (v) since $s_1 = k_1 - k_2 - k_3 + 1 = -k_4 + 1, s_2 = s_3 = -k_3 + 1, s_4 = s_5 = -k_2 + 1$ and $s_6 = -k_1 + 1$.

The case (vi). Let $i_1 = i_2 = k_1, i_3 = k_2, i_4 = k_3$ and $i_5 = i_6 = k_4$. Thus, one has $k_3 - k_2 - 1 \geq 0, k_1 + k_2 \geq k_3, 2k_1 \geq k_3, k_2 + 2k_3 - 3k_1 - 1 \geq 0$ and $k_2 + k_3 - 2 \leq d - 1$. Hence, we can define $\mathcal{P}_7(0, k_3 - k_2 - 1, k_1 + k_2 - k_3, 2k_1 - k_3, 0, k_2 + 2k_3 - 3k_1 - 1)$ and its δ-vector coincides with (vi) since $s_1 = s_2 = k_1 - k_2 - k_3 + 1 = -k_4 + 1, s_3 = -k_3 + 1, s_4 = -k_2 + 1$ and $s_5 = s_6 = -k_1 + 1$.

The case (vii). Let $i_1 = k_1, i_2 = k_2, i_3 = i_4 = k_3, i_5 = k_4$ and $i_6 = k_5$. Thus, one has $2k_1 \geq k_2, k_1 + k_2 \geq k_3, k_2 - k_1 \geq 0, 3k_3 - 2k_1 - k_2 - 2 \geq 0$ and $2k_3 - 2 \leq d - 1$. Hence, we can define $\mathcal{P}_7(0, 0, 2k_1 - k_2, k_1 + k_2 - k_3, k_2 - k_1, 3k_3 - 2k_1 - k_2 - 2)$ and its δ-vector coincides with (vii) since $s_1 = k_1 - 2k_3 + 1 = -k_5 + 1, s_2 = k_2 - 2k_3 + 1 = -k_4 + 1, s_3 = s_4 = -k_3 + 1, s_5 = -k_2 + 1$ and $s_1 = -k_1 + 1$.

The case (viii). In this case, one has $i_1 + i_2 \geq i_3, 2i_2 \geq i_4, i_3 + 2i_4 - 2i_1 - i_2 - 2 \geq 0, 2i_1 \geq i_2, i_1 + i_3 \geq i_4$ and $i_3 + i_4 - 2 \leq d - 1$. Hence, we can define $\mathcal{P}_7(0, i_1 + i_2 - i_3, i_1 + i_3 - 2i_2, 0, 2i_2 - i_4, i_3 + 2i_4 - 2i_1 - i_2 - 2)$ if $i_1 + i_3 \geq 2i_2$ and $\mathcal{P}_7(0, 2i_1 - i_2, 0, 2i_2 - i_1 - i_3, i_1 + i_3 - i_4, i_3 + 2i_4 - 2i_1 - i_2 - 2)$ $i_1 + i_3 \leq 2i_2$. Moreover, each of δ-vectors of them coincides with (viii) since $s_1 = i_1 - i_3 - i_4 + 1 = -i_6 + 1$, $s_2 = i_2 - i_3 - i_4 + 1 = -i_5 + 1$, $s_3 = -i_4 + 1$, $s_4 = -i_3 + 1$, $s_5 = -i_2 + 1$ and $s_6 = -i_1 + 1$.

Remark 3.4.7. When we discuss the cases of (vi) and (viii), we need the new inequality $2i_2 \geq i_4$. In fact, for example, the sequence $(1, 0, 2, 0, 1, 1, 0, 2, 0)$ cannot be the δ-vector of an integral simplex, although this satisfies $i_1 + i_l \geq i_{l+1}, l = 1, \ldots, 3$. Similarly, the sequence $(1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 1, 0)$ also cannot be the δ-vector of an integral simplex, although this satisfies $i_1 + i_l \geq i_{l+1}, l = 1, \ldots, 3$.

3.5 Shifted symmetric δ-vectors of convex polytopes

In this section, we introduce shifted symmetric δ-vectors of integral convex polytopes and discuss some properties.

A δ-vector $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$ is called *symmetric* if the equalities hold in (3.1) for each $0 \leq i \leq [s/2]$, i.e., $\delta_i = \delta_{s-i}$ for each $0 \leq i \leq [s/2]$. The δ-vector $\delta(\mathcal{P})$ of $\mathcal{P}$ is symmetric if and only if the Ehrhart ring [26, Chapter X] of $\mathcal{P}$ is Gorenstein. A combinatorial characterization for the δ-vector to be symmetric is studied in [15] and [27].

We say that a δ-vector $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$ is *shifted symmetric* if the equalities hold in (3.2) for each $0 \leq i \leq [(d-1)/2]$, i.e., $\delta_{d-i} = \delta_{i+1}$ for each $0 \leq i \leq [(d-1)/2]$. It seems likely that an integral convex polytope with a shifted symmetric δ-vector is quite rare. Thus it is reasonable to study properties of and to find a natural family of integral convex polytopes with shifted symmetric δ-vectors. We note that since $\delta_1 = \delta_d$, integral convex polytopes with shifted symmetric δ-vectors are always a d-simplex.

Examples 3.5.1. (a) We define $v_i \in \mathbb{R}^d$ for $i = 0, 1, \ldots, d$ by setting $v_i = \mathbf{e}_i$ with $i = 1, \ldots, d$ and $v_0 = (-e, \ldots, -e)$, where e is some nonnegative integer. Let $\mathcal{P} = \text{conv}(\{v_0, v_1, \ldots, v_d\})$. Then one has $\text{vol}(\mathcal{P}) = ed + 1$ by using an elementary linear algebra. When $e = 0$, it is clear that $\delta(\mathcal{P}) = (1, 0, 0, \ldots, 0)$. When e is positive, we know that

$$\frac{j}{ed+1}\sum_{i=1}^{d}(v_i, 1) + \frac{(e-j)d+1}{ed+1}(v_0, 1) = (j-e, j-e, \ldots, j-e, 1)$$

and $0 < \frac{j}{ed+1}, \frac{(e-j)d+1}{ed+1} < 1$ for every $1 \leq j \leq e$. Then, from Section 3.1, we have $\delta_1, \delta_d \geq e$. Since $\delta_i \geq \delta_1$ for $1 \leq i \leq d-1$ and $\text{vol}(\mathcal{P}) = ed + 1$, we obtain $\delta(\mathcal{P}) = (1, e, e, \ldots, e)$.

(b) Let $d \geq 3$. We define $v_i \in \mathbb{R}^d$ for $i = 0, 1, \ldots, d$ by setting $v_i = \mathbf{e}_i$ with $i = 1, \ldots, d$ and $v_0 = (e, \ldots, e)$, where e is some positive integer. Let $\mathcal{P} = \text{conv}(\{v_0, v_1, \ldots, v_d\})$. Then one has $\text{vol}(\mathcal{P}) = ed - 1$ by using an elementary linear algebra. And we know that

$$\frac{ke+j}{ed-1}\sum_{i=1}^{d}(v_i, 1) + \frac{(e-j)d-1-k}{ed-1}(v_0, 1) = (e-j, e-j, \ldots, e-j, k+1)$$

and $0 < \frac{ke+j}{ed-1}, \frac{(e-j)d-1-k}{ed-1} < 1$ for every $0 \leq j \leq e-1$ and $0 \leq k \leq d-2$ with $(j,k) \neq (0,0)$. Hence one has $\delta(\mathcal{P}) = (1, e-1, e, e, \ldots, e, e-1)$ by Section 3.1.

3.5.1 Some characterizations of integral convex polytopes with shifted symmetric δ-vectors

In this subsection, two results on integral convex polytopes with shifted symmetric δ-vectors are given.

Theorem 3.5.2 ([37, Theorem 2.1])**.** *Let $\mathcal{P}$ be a d-simplex whose vertices are $v_0, v_1, \ldots, v_d \in \mathbb{R}^d$ and $S(\mathcal{P})$ the set which appears in Section 3.1. Then the following conditions are equivalent:*

(a) *$\delta(\mathcal{P})$ is shifted symmetric;*

(b) *the normalized volume of all facets of $\mathcal{P}$ is equal to* 1*;*

(c) *each element $(\alpha, n) \in S(\mathcal{P}) \setminus \{(0, \ldots, 0, 0)\}$ has a unique expression on the form:*

$$(\alpha, n) = \sum_{j=0}^{d} r_j (v_j, 1) \ \text{ with } \ 0 < r_j < 1 \ \text{ for } \ j = 0, 1, \ldots, d, \tag{3.16}$$

where $\alpha \in \mathbb{Z}^d$ and $n \in \mathbb{Z}$.

Proof. **((a) $\Leftrightarrow$ (c))** If each element $x \in S(\mathcal{P}) \setminus \{(0, \ldots, 0)\}$ has the form (3.16), then each inverse of x also belongs to $S(\mathcal{P})$, which means that $\delta(\mathcal{P})$ is shifted symmetric. On the other hand, suppose that there exists an element $x \in S(\mathcal{P}) \setminus \{(0, \ldots, 0)\}$ which does not have the form (3.16). Then one has $\deg(x) + \deg(-x) < d + 1$, which implies obviously that $\delta(\mathcal{P})$ is not shifted symmetric.

((b) $\Leftrightarrow$ (c)) Let $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d) \in \mathbb{Z}^{d+1}$ be the δ-vector of $\mathcal{P}$ and $\delta(\mathcal{F}) = (\delta'_0, \delta'_1, \ldots, \delta'_{d-1}) \in \mathbb{Z}^d$ the δ-vector of a facet $\mathcal{F}$ of $\mathcal{P}$. Then one has $\delta'_i \leq \delta_i$ for $0 \leq i \leq d - 1$. If there is a facet $\mathcal{F}$ with $\mathrm{vol}(\mathcal{F}) \neq 1$, say, its vertices are $v_0, v_1, \ldots, v_{d-1}$, then there exists an element $(\alpha, n) \in S(\mathcal{P})$ with $\alpha = \sum_{j=0}^{d-1} r_j v_j + 0 \cdot v_d$ and $n > 0$. This implies that there exists an element of $S(\mathcal{P}) \setminus \{(0, \ldots, 0)\}$ which does not have the form (3.16). On the other hand, suppose that there exists an element $(\alpha, n) \in S(\mathcal{P}) \setminus \{(0, \ldots, 0)\}$ which does not have the form (3.16), i.e., $(\alpha, n) = \sum_{j=0}^{d} r_j (v_j, 1)$ and there is $0 \leq j \leq d$ with $r_j = 0$, say, $r_d = 0$. Then the normalized volume of the facet whose vertices are $v_0, v_1, \ldots, v_{d-1}$ is not equal to 1. $\square$

Remark 3.5.3. In the language of [73], a δ-vector is shifted symmetric if and only if $a(t) = 1 + t + \cdots + t^d$. In fact, when $\mathcal{P}$ is shifted symmetric, $\mathrm{Box}(\sigma_{\mathcal{F}}) \cap N'$ is empty, where $\emptyset \neq \mathcal{F} \subsetneq \mathcal{P}$ is a face of $\mathcal{P}$, which means that the normalized volume of $\mathcal{F}$ is equal to 1. In addition, since $\mathcal{P}$ is a simplex, one has $h_{\mathcal{F}}(t) = \sum_{i=0}^{d-1-\dim(\mathcal{F})} t^i$. Thus it follows that

$$\sum_{\mathcal{F} \in \mathcal{T}} B_{\mathcal{F}}(t) h_{\mathcal{F}}(t) = B_{\emptyset}(t) h_{\emptyset}(t) = 1 + t + \cdots + t^d.$$

3.5.2 A family of $(0,1)$-polytopes with shifted symmetric δ-vectors

In this subsection, a family of $(0,1)$-polytopes with shifted symmetric δ-vectors is studied. We classify completely the δ-vectors of those polytopes. Moreover, we consider when those δ-vectors are both symmetric and shifted symmetric.

Let m be a positive integer with $1 \leq m < d$. We study the δ-vector of the integral convex polytope $\mathcal{P} \subset \mathbb{R}^d$ whose vertices are of the form

$$v_i = \begin{cases} \mathbf{e}_i + \mathbf{e}_{i+1} + \cdots + \mathbf{e}_{i+m-1}, & i = 1, \ldots, d, \\ (0, \ldots, 0), & i = 0, \end{cases} \tag{3.17}$$

where $\mathbf{e}_{d+i} = \mathbf{e}_i$.

The normalized volume of $\mathcal{P}$ is equal to the absolute value of the determinant of the circulant matrix

$$\begin{vmatrix} v_1 \\ \vdots \\ v_d \end{vmatrix}. \tag{3.18}$$

This determinant (3.18) can be calculated easily. In fact,

Proposition 3.5.4. *When $(m,d) = 1$, the determinant (3.18) is equal to $\pm m$. And when $(m,d) \neq 1$, the determinant (3.18) is equal to 0. Here (m,d) denotes the greatest common divisor of m and d.*

A proof of this proposition can be given by the formula of the determinant of the circulant matrix. Thus one has $\mathrm{vol}(\mathcal{P}) = m$ when $(m,d) = 1$. We assume only the case of $(m,d) = 1$.

For $j = 1, 2, \ldots, d-1$, let q_j be the quotient of jm divided by d and r_j its remainder i.e., one has the equalities

$$jm = q_j d + r_j \quad \text{for} \quad j = 1, 2, \ldots, d-1.$$

It then follows from $(m,d) = 1$ that

$$0 \leq q_j \leq m-1, 1 \leq r_j \leq d-1$$

and

$$r_j \neq r_{j'} \text{ if } j \neq j'$$

for every $1 \leq j, j' \leq d-1$. In addition, for $k = 1, 2, \ldots, m-1$, let $j_k \in \{1, 2, \ldots . d-1\}$ be the integer with $r_{j_k} = k$, i.e., one has the equalities

$$j_k m = q_{j_k} d + r_{j_k} = q_{j_k} d + k \quad \text{for} \quad k = 1, 2, \ldots, m-1.$$

Then $q_{j_k} > 0$. Thus one has

$$1 \leq q_{j_k}, r_{j_k} \leq m - 1$$

for every $1 \leq k \leq m - 1$.

For an integer a, let $\overline{a}$ denote the residue class in $\mathbb{Z}/d\mathbb{Z}$.

Theorem 3.5.5 ([37, Theorem 3.2]). *Let $\mathcal{P}$ be the integral convex polytope whose vertices are of the form (3.17) and $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$ its δ-vector. For each $1 \leq i \leq d$, one has $\overline{im} \in \{\overline{1}, \overline{2}, \ldots, \overline{m-1}\}$ if and only if one has $\delta_i = 1$. Moreover, $\delta(\mathcal{P})$ is shifted symmetric, i.e., $\delta_{i+1} = \delta_{d-i}$ for each $0 \leq i \leq [(d-1)/2]$.*

Proof. By using the above notations, we obtain

$$\frac{q_{j_k}}{m}\{(v_1, 1) + (v_2, 1) + \cdots + (v_d, 1)\} + \frac{r_{j_k}}{m}(v_0, 1) = (q_{j_k}, \ldots, q_{j_k}, j_k) \in \mathbb{Z}^{d+1}$$

and $0 < \frac{q_{j_k}}{m}, \frac{r_{j_k}}{m} < 1$ for every $1 \leq k \leq m - 1$. Then Section 3.1 guarantees that one has $\delta_{j_k} \geq 1$ for $k = 1, \ldots, m - 1$. Considering $\sum_{i=0}^{d} \delta_i = m$ by Proposition 3.5.4, it turns out that $\delta(\mathcal{P})$ coincides with

$$\delta_i = \begin{cases} 1 & i = 0, j_1, j_2, \ldots, j_{m-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Now $\overline{im} \in \{\overline{1}, \overline{2}, \ldots, \overline{m-1}\}$ is equivalent with $i \in \{j_1, \ldots, j_{m-1}\}$. Therefore one has $\delta_i = 1$ if and only if $\overline{im} \in \{\overline{1}, \overline{2}, \ldots, \overline{m-1}\}$ for each $1 \leq i \leq d$.

In addition, by virtue of Theorem 3.5.2, $\delta(\mathcal{P})$ is shifted symmetric, as required. □

Corollary 3.5.6. *Let $\mathcal{P}$ be the integral convex polytope whose vertices are of the form (3.17) and $\delta(\mathcal{P}) = (\delta_0, \delta_1, \ldots, \delta_d)$ its δ-vector. Then $\delta(\mathcal{P})$ is symmetric, i.e., $\delta_i = \delta_{s-i}$ for each $0 \leq i \leq [s/2]$ if and only if one has $d \equiv m - 1 \pmod{m}$.*

Proof. Let p be the quotient of d divided by m and r its remainder, i.e., one has $d = mp+r$. And let $j_t = \min\{j_1, j_2, \ldots, j_{m-1}\}$. On the one hand, one has $j_t m = d+t$. On the other hand, one has $(p + 1)m = d + m - r$ and $1 \leq m - r \leq m - 1$. It then follows from Theorem 3.5.5 that $p + 1 = j_t = \min\{i : \delta_i \neq 0, i > 0\}$. Hence $d - p = s = \max\{i : \delta_i \neq 0\}$ since $\delta(\mathcal{P})$ is shifted symmetric.

When $d \equiv m - 1 \pmod{m}$, i.e., $r = m - 1$, we can obtain the equalities

$$d - p = mp + r - p = mp + m - 1 - p = (m - 1)(p + 1).$$

In addition, for nonnegative integers $l(p + 1)$, $l = 1, 2, \ldots, m - 1$, the following equalities hold:

$$\overline{l(p+1)m} = \overline{l(mp+m)} = \overline{l(mp+m-1)+l} = \overline{ld+l} = \overline{l} \in \{\overline{1}, \overline{2}, \ldots, \overline{m-1}\}.$$

Thus it turns out that $\delta(\mathcal{P})$ coincides with

$$\delta_i = \begin{cases} 1 & i = 0, p+1, 2(p+1), \ldots, (m-1)(p+1), \\ 0 & otherwise, \end{cases}$$

by Theorem 3.5.5. It then follows that

$$\delta_{k(p+1)} = \delta_{(m-1-k)(p+1)} = \delta_{s-k(p+1)} = 1$$

for every $0 \leq k \leq m-1$ and

$$\delta_i = \delta_{s-i} = 0$$

for every $0 \leq i \leq s$ with $i \neq k(p+1)$, $k = 0, 1, \ldots, m-1$. These equalities imply that $\delta(\mathcal{P})$ is symmetric.

Suppose that $\delta(\mathcal{P})$ is symmetric. Our work is to show that $r = m-1$. Then one has

$$\delta_0 = \delta_s = \delta_{d-p} = \delta_{(m-1)p+r} = 1.$$

Since $\delta(\mathcal{P})$ is also shifted symmetric, one has $\delta_{(m-1)p+r} = \delta_{p+1}$. Hence one has $\delta_{p+1} = \delta_{(m-2)p+r-1} = \delta_{2(p+1)} = \cdots = \delta_{[(m-1)/2](p+1)}=1$ since $\delta(\mathcal{P})$ is both symmetric and shifted symmetric. When m is odd, one has $\frac{d-p}{2} = \frac{m-1}{2}(p+1)$ since $\delta(\mathcal{P})$ is symmetric. Thus $r = m-1$. When m is even, one has $\frac{d+1}{2} = \frac{m}{2}(p+1)$ since $\delta(\mathcal{P})$ is shifted symmetric. Thus $r = m-1$.

Therefore $\delta(\mathcal{P})$ is symmetric if and only if $d \equiv m-1 \pmod{m}$, as desired. $\square$

3.5.3 Shifted symmetric δ-vectors of Hermite normal forms

In this subsection, we consider the problem when the integral simplices arising from (3.4) have shifted symmetric δ-vectors. By using Corollary 3.3.4, we deduce a symmetry property of the δ-vectors.

Proposition 3.5.7 (Shifted symmetry for "one row")**.** *For a matrix $M \in \mathbb{Z}^{d\times d}$ with Hermite normal form (3.4), we have $s_i + s_{D-i} = d+1$, for $i = 1, \ldots, D-1$, which implies $\delta_i = \delta_{d+1-i}$ by reciprocity, if and only if the following three conditions hold:*

(a) $\sum_{j=1}^{D-1} jd_j - 1$ *is coprime with D;*

(b) $d_j = 0$ *for all j which is not coprime with D;*

(c) $\sum_{j=1}^{D-1} d_j = d-1$.

Proof. Let us consider $s_i + s_{D-i}$. For an integer a, let $\overline{a}$ denote its residue class in $\mathbb{Z}/D\mathbb{Z}$. Then we have

$$\begin{aligned} s_i + s_{D-i} &= \left\lfloor \frac{i}{D} - \sum_{j=1}^{D-1} \left\{ \frac{ij}{D} \right\} d_j \right\rfloor + \left\lfloor \frac{D-i}{D} - \sum_{j=1}^{D-1} \left\{ \frac{(D-i)j}{D} \right\} d_j \right\rfloor + 2d \\ &= \left\lfloor \frac{i - \sum_{j=1}^{D-1} \overline{ij} d_j}{D} \right\rfloor + \left\lfloor \frac{D-i-\sum_{j=1}^{D-1} \overline{(D-i)j} d_j}{D} \right\rfloor + 2d. \end{aligned}$$

Since

$$\begin{cases} i - \sum_{j=1}^{D-1} \overline{ij} d_j \equiv i\Big(1 - \sum_{j=1}^{D-1} j d_j\Big) \pmod D, \\ D - i - \sum_{j=1}^{D-1} \overline{(D-i)j} d_j \equiv (D-i)\Big(1 - \sum_{j=1}^{D-1} j d_j\Big) \pmod D, \end{cases} \tag{3.19}$$

if the condition (a) is not satisfied, then one has

$$\begin{aligned} s_i + s_{D-i} &= \frac{D - \sum_{j=1}^{D-1} \big(\overline{ij} + \overline{(D-i)j}\big) d_j}{D} + 2d \\ &= 2d + 1 - \sum_{j=1}^{D-1} \frac{\overline{ij} + \overline{(D-i)j}}{D} d_j \\ &\geq 2d + 1 - \sum_{j=1}^{D-1} d_j \geq d + 2 > d + 1 \end{aligned}$$

for some i with $1 \leq i \leq D-1$. Thus, the condition (a) is a necessary condition to have $s_i + s_{D-i} = d + 1$ for all i. On the other hand, when the condition (a) is satisfied, again from (3.19), we have

$$\begin{aligned} s_i + s_{D-i} &= \frac{D - \sum_{j=1}^{D-1} \big(\overline{ij} + \overline{(D-i)j}\big) d_j}{D} + 2d - 1 \\ &= 2d - \sum_{j=1}^{D-1} \frac{\overline{ij} + \overline{(D-i)j}}{D} d_j \\ &= 2d - \sum_{D \nmid ij} d_j. \end{aligned}$$

If the condition (b) is not satisfied, then we have

$$s_i + s_{D-i} = 2d - \sum_{D \nmid ij} d_j > d + 1$$

for some i with $1 \leq i \leq D-1$. Hence, the condition (b) is also a necessary condition. In addition, if the condition (c) is not satisfied, then we have $s_i + s_{D-i} > d + 1$. Thus, the condition (c) is also a necessary condition. On the other hand, when the conditions (a), (b) and (c) are all satisfied, we have $s_i + s_{D-i} = D + 1$ for all i. □

In particular, if we assume in addition that $d_{D-1} = d - 1$ in Corollary 3.3.4, then we have

Proposition 3.5.8 (Shifted symmetry for "all $D-1$ one row")**.** *Let $M \in \mathbb{Z}^{d\times d}$ with Hermite normal form* (3.6)*. Then*

(a) $\delta_i = \delta_{d+1-i}$ *if and only if D and d are coprime.*

(b) *When $D = kd$, for $k \in \mathbb{N}$ and $k \geq 2$, the δ-vector is*

$$(1, \underbrace{k, \ldots, k}_{d-1}, k-1),$$

which is not shifted symmetric. But for $k = 2$, we have $\delta_k = \delta_{d-k}$ (i.e., symmetric).

Chapter 4

Roots of Ehrhart polynomials

In this chapter, we will study roots of the Ehrhart polynomials of integral convex polytopes. As we described above, Conjecture 2.0.1 is an outstanding and important problem, which we will discuss. Moreover, we will also concentrate on roots of the Ehrhart polynomials of Gorenstein Fano polytopes.

In Section 4.1, in order to examine whether Conjecture 2.0.1 is affirmative, we will investigate roots of the Ehrhart polynomials of integral convex polytopes arising from graphs. We will discuss them in terms of graphs. However, in Section 4.2, counterexamples for Conjecture 2.0.1 will appear. Moreover, Section 4.3 will be devoted to studying roots of the Ehrhart polynomials of Gorenstein Fano polytopes. Finally, we will also consider roots of SSNN polynomials, which are generalized Ehrhart polynomials of Gorenstein Fano polytopes, i.e., a class of polynomials containing all their Ehrhart polynomials.

4.1 The conjecture on roots of Ehrhart polynomials

First, let us consider roots of the Ehrhart polynomials of integral convex polytopes arising from finite connected simple graphs, which we call edge polytopes. Concretely, the aim of this section is to provide evidence for Conjecture 2.0.1 for the Ehrhart polynomials of edge polytopes constructed from connected simple graphs, mainly by computational means.

4.1.1 Exhaustive computation of roots of Ehrhart polynomials arising from graphs

Let G be a graph having no multiple edges on the vertex set $V(G) = \{1, \ldots, d\}$ and the edge set $E(G) = \{e_1, \ldots, e_m\} \subset V(G)^2$. Graphs may have loops in their edge sets unless explicitly excluded; in which case the graphs are called *simple* graphs. We refer the reader to e.g., [77] for the introduction to graph theory.

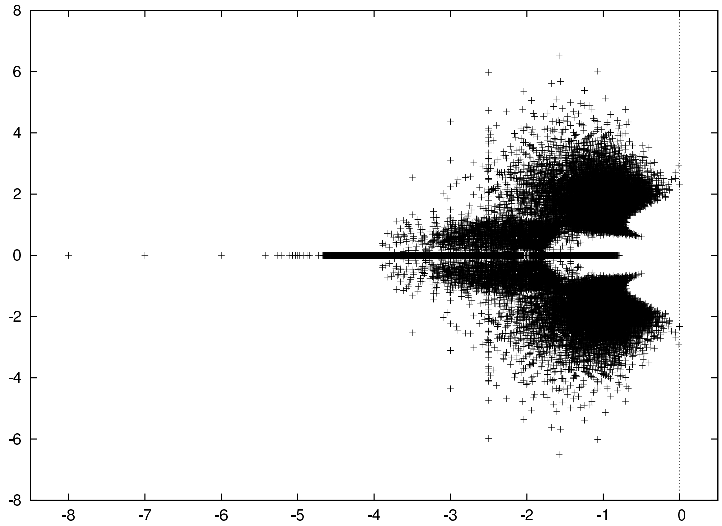


Figure 4.1: $\mathbf{V}_9^{cs}$

Definition 4.1.1. For an edge $e = \{i, j\}$ of G, we define $\rho(e) = \mathbf{e}_i + \mathbf{e}_j$. In particular, for a loop $e = \{i, i\}$ at $i \in V(G)$, one has $\rho(e) = 2\mathbf{e}_i$. The *edge polytope* of G is the convex polytope $\mathcal{P}_G \subset \mathbb{R}^d$, which is the convex hull of the finite set

$$\{\rho(e_1), \ldots, \rho(e_m)\}.$$

The dimension of $\mathcal{P}_G$ equals to $d-2$ if the graph G is a connected bipartite graph, or $d-1$, other connected graphs [54]. The edge polytopes of complete multipartite graphs are studied in [56]. Note that if the graph G is a complete graph, the edge polytope $\mathcal{P}_G$ is also called the second hypersimplex in [75, Section 9].

Let $\mathbb{C}[X]$ denote the polynomial ring in one variable over $\mathbb{C}$. Given a polynomial $f = f(X) \in \mathbb{C}[X]$, we write $\mathbf{V}(f)$ for the set of roots of f, i.e.,

$$\mathbf{V}(f) = \{a \in \mathbb{C} \mid f(a) = 0\}.$$

We computed the Ehrhart polynomial $i(\mathcal{P}_G, n)$ of each edge polytope $\mathcal{P}_G$ for connected simple graphs G of orders up to nine; there are $1, 2, \ldots, 261080$ connected simple graphs of orders $2, 3, \ldots, 9$[1]. Then, we solved each equation $i(\mathcal{P}_G, X) = 0$ in $\mathbb{C}$.

Let $\mathbf{V}_d^{cs}$ denote $\bigcup \mathbf{V}(i(\mathcal{P}_G, m))$, where the union runs over all connected simple graphs G of order d. Figure 4.1 plots points of $\mathbf{V}_9^{cs}$, as a representative of all results. For all connected simple graphs of order 2–9, Conjecture 2.0.1 holds.

Since an edge polytope is a kind of 0/1-polytope, the points in Figure 4.1 for $\mathbf{V}_9^{cs}$ are similar to those in Figure 6 of [5]. However, the former has many more points, which form three clusters: one on the real axis, and other two being complex

[1]These numbers of such graphs are known; see, e.g., [22, Chapter 4] or `A001349` of the On-Line Encyclopedia of Integer Sequences.

conjugates of each other and located nearer to the imaginary axis than the first cluster. The interesting thing is that no roots appear in the right half plane of the figure. The closest points to the imaginary axis are approximately $-0.583002 \pm 0.645775i \in \mathbf{V}_7^{cs}$, $-0.213574 \pm 2.469065i \in \mathbf{V}_8^{cs}$, and $-0.001610 \pm 2.324505i \in \mathbf{V}_9^{cs}$. A polynomial with roots only in the left half plane is called a *stable* polynomial. This observation raises an open question:

Question 4.1.2. For any d and any connected simple graph G of order d, is $i(\mathcal{P}_G, n)$ always a stable polynomial ?

For a few infinite families of graphs, rigorous proofs are known, e.g., Proposition 4.1.3 and Examples below.

Proposition 4.1.3. *A root α of the Ehrhart polynomial $i(\mathcal{P}_{K_d}, n)$ of the complete graph K_d satisfies*

1. $\alpha \in \{-1, -2\}$ *if* $d = 3$ *or*
2. $-\frac{d}{2} < \mathrm{Re}(\alpha) < 0$ *if* $d \geq 4$.

Proof. The Ehrhart polynomial $i(\mathcal{P}_{K_d}, n)$ of the complete graph K_d is given in [75, Corollary 9.6]:

$$i(\mathcal{P}_{K_d}, n) = \binom{d + 2m - 1}{d - 1} - d\binom{m + d - 2}{d - 1}.$$

In cases where $d = 2$ or 3, the Ehrhart polynomials are binomial coefficients, since the edge polytopes are simplices. Actually, they are:

$$i(\mathcal{P}_{K_2}, n) = 1 \quad \text{and} \quad i(\mathcal{P}_{K_3}, n) = \binom{m+2}{2}.$$

Thus, there are no roots for $d = 2$, whereas $\{-1, -2\}$ are the roots for $d = 3$.

Hereafter, we assume $d \geq 4$. It is easy to see that $\{-1, -2, \ldots, -\left\lfloor \frac{d-1}{2} \right\rfloor\}$ are included in $\mathbf{V}(i(\mathcal{P}_{K_d}, n))$.

We shall first prove that $\mathrm{Re}(\alpha) < 0$. Let $q_d^{(1)}(n) = (2n + d - 1) \cdots (2n + 1)$ and $q_d^{(2)}(n) = d(n + d - 2) \cdots n$. Then for a complex number z, $i(\mathcal{P}_{K_d}, z) = 0$ if and only if $q_d^{(1)}(z) = q_d^{(2)}(z)$, since $q_d^{(1)}(z) - q_d^{(2)}(z)$ is $(d-1)!\, i(\mathcal{P}_{K_d}, z)$. Let us prove $|q_d^{(1)}(z)| > |q_d^{(2)}(z)|$ for any complex number z with a nonnegative real part by mathematical induction on $d \geq 4$.

If $d = 4$,

$$\begin{aligned} |q_4^{(1)}(z)| = |(2z+3)(2z+2)(2z+1)| &= |2z+3||z+1||4z+2| \\ &> |z+2||z+1||4z| = |q_4^{(2)}(z)| \end{aligned}$$

holds for any complex number z with $\mathrm{Re}(z) \geq 0$.

Assume for d that $|q_d^{(1)}(z)| > |q_d^{(2)}(z)|$ is true for any complex number z with $\mathrm{Re}(z) \geq 0$.

Then, by

$$\begin{aligned} |q_{d+1}^{(1)}(z)| &= |2z+d||q_d^{(1)}(z)| \\ |q_{d+1}^{(2)}(z)| &= \frac{d+1}{d}|z+d-1||q_d^{(2)}(z)| \end{aligned}$$

and

$$|2dz+d^2| \;>\; |(d+1)z+d^2-1|$$

from $2d > d+1$ and $d^2 > d^2-1$, one can deduce

$$\begin{aligned} d|q_{d+1}^{(1)}(z)| &= |2dz+d^2||q_d^{(1)}(z)| \\ &> |(d+1)z+d^2-1||q_d^{(2)}(z)| \\ &= (d+1)|z+d-1||q_d^{(2)}(z)| \\ &= d\frac{d+1}{d}|z+d-1||q_d^{(2)}(z)| \;=\; d|q_{d+1}^{(2)}(z)|. \end{aligned}$$

Thus, $|q_{d+1}^{(1)}(z)| > |q_{d+1}^{(2)}(z)|$ holds for any complex number z with $\mathrm{Re}(z) \geq 0$.

Therefore, for any $d \geq 4$, the inequality $|q_d^{(1)}(z)| > |q_d^{(2)}(z)|$ holds for any complex number z with a nonnegative real part. This implies that the real part of any complex root of $i(\mathcal{P}_{K_d}, n)$ is negative.

We shall also prove the other half, that $-\frac{d}{2} < \mathrm{Re}(\alpha)$. To this end, it suffices to show that all roots of $j_d(l) = i\left(\mathcal{P}_{K_d}, -l-\frac{d}{2}\right)$ have negative real parts. Let $r_d^{(1)}(l)$ and $r_d^{(2)}(l)$ be

$$\begin{aligned} r_d^{(1)}(l) &= (-1)^{d-1}q_d^{(1)}\left(-l-\frac{d}{2}\right) = (2l+1)\cdots(2l+d-1) \\ r_d^{(2)}(l) &= (-1)^{d-1}q_d^{(2)}\left(-l-\frac{d}{2}\right) = d\left(l-\frac{d-4}{2}\right)\cdots\left(l+\frac{d}{2}\right). \end{aligned}$$

Then for a complex number z, it holds that

$$j_d(z) = 0 \iff r_d^{(1)}(z) = r_d^{(2)}(z).$$

Let us prove $|r_d^{(1)}(z)| > |r_d^{(2)}(z)|$ for any complex number z with a nonnegative real part by mathematical induction on $d \geq 4$.

For $d = 4$, it immediately follows from the inequality between $q_4^{(1)}$ and $q_4^{(2)}$:

$$|r_4^{(1)}(z)| \;=\; |q_4^{(1)}(z)| \;>\; |q_4^{(2)}(z)| \;=\; |r_4^{(2)}(z)|.$$

And so we need $d = 5$ also as a base case:

$$\begin{aligned}
|r_5^{(1)}(z)| &= |2z+1||2z+2||2z+3||2z+4| \\
&> \frac{5}{4}|z+1||2z+1||2z+3||2z+4| \\
&> \frac{5}{4}|z-\tfrac{1}{2}||2z+1||2z+3||z+\tfrac{5}{2}| \\
&= 5\left|z-\frac{1}{2}\right|\left|z+\frac{1}{2}\right|\left|z+\frac{3}{2}\right|\left|z+\frac{5}{2}\right| \\
&= |r_5^{(2)}(z)|.
\end{aligned}$$

Assume for d the validity of $|r_d^{(1)}(z)| > |r_d^{(2)}(z)|$ for any complex number z with $\mathrm{Re}(z) \geq 0$.

Then, from the fact that

$$\begin{aligned}
|r_{d+2}^{(1)}(z)| &= |2z+d||2z+d+1||r_d^{(1)}(z)| \\
|r_{d+2}^{(2)}(z)| &= \frac{d+2}{d}\left|z-\frac{d}{2}+1\right|\left|z+\frac{d}{2}+1\right||r_d^{(2)}(z)|,
\end{aligned}$$

it follows that

$$\begin{aligned}
d|r_{d+2}^{(1)}(z)| &= d|2z+d||2z+d+1||r_d^{(1)}(z)| \\
&> d|2z+d|\left|z+\tfrac{d}{2}+1\right||r_d^{(2)}(z)| \\
&= |2dz+d^2|\left|z+\tfrac{d}{2}+1\right||r_d^{(2)}(z)| \\
&> |(d+2)z+d^2-4|\left|z+\tfrac{d}{2}+1\right||r_d^{(2)}(z)| \\
&> (d+2)\left|z-\frac{d-2}{2}\right|\left|z+\frac{d}{2}+1\right||r_d^{(2)}(z)| \\
&= d|r_{d+2}^{(2)}(z)|.
\end{aligned}$$

Thus, $|r_{d+2}^{(1)}(z)| > |r_{d+2}^{(2)}(z)|$ holds for any complex number z with $\mathrm{Re}(z) \geq 0$.

Therefore, for any $d \geq 4$, the inequality $|r_d^{(1)}(z)| > |r_d^{(2)}(z)|$ holds for any complex number z with a nonnegative real part. This implies that any complex root of $j_d(l)$ has a negative real part. $\square$

Next, we comput the roots of the Ehrhart polynomials $i(\mathcal{P}_G, n)$ of complete multipartite graphs G as well. A complete multipartite graph of type $(q_1, \ldots, q_t)$, denoted by $K_{q_1,\ldots,q_t}$, is constructed as follows. Let $V(K_{q_1,\ldots,q_t}) = \bigcup_{i=1}^t V_i$ be a disjoint union of vertices with $|V_i| = q_i$ for each i and the edge set $E(K_{q_1,\ldots,q_t})$ be $\{\{u,v\} \mid u \in V_i,\ v \in V_j\ (i \neq j)\}$. The graph $K_{q_1,\ldots,q_t}$ is unique up to isomorphism.

The Ehrhart polynomials for complete multipartite graphs are explicitly given in [56]:

$$i(\mathcal{P}_G, n) = \binom{d+2n-1}{d-1} - \sum_{k=1}^{t} \sum_{1\leq i\leq j\leq q_k} \binom{j-i+n-1}{j-i}\binom{d-j+n-1}{d-j} \tag{4.1}$$

where $d = \sum_{k=1}^{t} q_k$ is a partition of d and $G = K_{q_1,\ldots,q_t}$.

Another simpler formula is newly obtained.

Proposition 4.1.4. *The Ehrhart polynomial $i(\mathcal{P}_G, n)$ of the edge polytope of a complete multipartite graph $G = K_{q_1,\ldots,q_t}$ is*

$$i(\mathcal{P}_G, n) = f(n; d, d) - \sum_{k=1}^{t} f(n; d, q_k),$$

where $d = \sum_{k=1}^{t} q_k$ and

$$f(n; d, j) = \sum_{k=1}^{j} p(n; d, k)$$

with

$$p(n; d, j) = \binom{j+n-1}{j-1}\binom{d-j+n-1}{d-j}.$$

Proof. Let G denote a complete multipartite graph $K_{q_1,\ldots,q_t}$. We start from the formula (4.1).

First, it holds that

$$\binom{d+2n-1}{d-1} = f(n; d, d).$$

On the one hand, $\binom{d+2n-1}{d-1}$ is the number of combinations with repetitions choosing $2n$ elements from a set of cardinality d. On the other hand,

$$f(n; d, d) = \sum_{j=1}^{d} \binom{j+n-1}{j-1}\binom{d-j+n-1}{d-j}$$

counts the same number of combinations as the sum of the number of combinations in which the $(n+1)$th smallest number is j.

Second, it holds that

$$\sum_{k=1}^{t} \sum_{1 \leq i \leq j \leq q_k} \binom{j-i+n-1}{j-i}\binom{d-j+n-1}{d-j} = \sum_{k=1}^{t} f(n; d, q_k).$$

Since the outermost summations are the same on both sides, it suffices to show that

$$\sum_{1 \leq i \leq j \leq q_k} \binom{j-i+n-1}{j-i}\binom{d-j+n-1}{d-j} = f(n; d, q_k).$$

The summation of the left-hand side can be transformed as follows:

$$\begin{aligned}
& \sum_{1\le i\le j\le q_k} \binom{j-i+n-1}{j-i}\binom{d-j+n-1}{d-j} \\
= & \sum_{j=1}^{q_k}\sum_{i=1}^{j} \binom{j-i+n-1}{j-i}\binom{d-j+n-1}{d-j} \\
= & \sum_{j=1}^{q_k} \binom{d-j+n-1}{d-j} \sum_{i=1}^{j}\binom{j-i+n-1}{j-i} \\
= & \sum_{j=1}^{q_k} \binom{d-j+n-1}{d-j}\binom{n+j-1}{j-1} \\
= & \sum_{j=1}^{q_k} p(n;d,j) \\
= & f(n;d,q_k)
\end{aligned}$$

Finally, substituting these transformed terms into the original formula (4.1) gives the desired result. □

By the new formula above, we computed the roots of Ehrhart polynomials. Let $\mathbf{V}_d^{\mathrm{mp}}$ denote $\bigcup \mathbf{V}(i(\mathcal{P}_G, n))$, where the union runs over all complete multipartite graphs G of order d. Figure 4.2 plots the points of $\mathbf{V}_{22}^{\mathrm{mp}}$. For all complete multipartite graphs of order 10–22, Conjecture 2.0.1 holds.

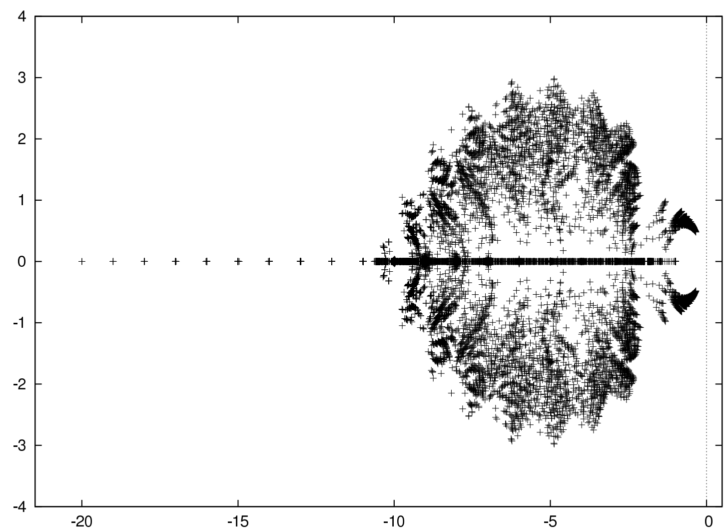


Figure 4.2: $\mathbf{V}_{22}^{\mathrm{mp}}$

Figure 4.2, for $\mathbf{V}_{22}^{\mathrm{mp}}$, shows that the noninteger roots lie in the circle $\left|z+\frac{11}{2}\right| \le \frac{11}{2}$. This fact is not exclusive to 22 alone, but similar conditions hold for all $d \le 22$. We conjecture the following

Conjecture 4.1.5. *For any* $d \geq 3$*,*

$$\mathbf{V}_d^{mp} \subset \left\{ z \in \mathbb{C} \mid \left| z + \tfrac{d}{4} \right| \leq \tfrac{d}{4} \right\} \cup \{-(d-1), \ldots, -2, -1\}.$$

Remark 4.1.6. (1) The leftmost point $-(d-1)$ can only be attained by K_3; this is shown in Proposition 4.1.10. Therefore, if we choose $d \geq 4$, the set of negative integers in the statement can be replaced with the set $\{-(d-2), \ldots, -2, -1\}$. However, $-(d-2)$ can be attained by the tree $K_{d-1,1}$ for any d; see Example 4.1.7 below.

(2) Since 0 can never be a root of an Ehrhart polynomial, Conjecture 4.1.5 answers Question 4.1.2 in the affirmative for complete multipartite graphs. Moreover, if Conjecture 4.1.5 holds, then Conjecture 2.0.1 holds for those graphs.

(3) The method of Pfeifle [62] might be useful if the δ-vector can be determined for edge polytopes of complete multipartite graphs.

Example 4.1.7. The Ehrhart polynomial for complete bipartite graph $K_{p,q}$ is given in, e.g., [56, Corollary 2.7 (b)]:

$$i(\mathcal{P}_{K_{p,q}}, n) = \binom{n+p-1}{p-1}\binom{n+q-1}{q-1},$$

and thus the roots are

$$\mathbf{V}(i(\mathcal{P}_{K_{p,q}}, n)) = \{-1, \ldots, -\max(p-1, q-1)\}$$

and all of them are negative integers satisfying the condition in Conjecture 4.1.5.

Example 4.1.8. The edge polytope of a complete 3-partite graph $\mathcal{P}_{K_{m,1,1}}$ for $m \geq 2$ can be obtained as a pyramid from $\mathcal{P}_{K_{m,2}}$ by adjoining a vertex. Therefore, its Ehrhart polynomial is the following:

$$i(\mathcal{P}_{K_{m,1,1}}, n) = \sum_{j=0}^{n} i(\mathcal{P}_{K_{m,2}}, j).$$

Each term on the right-hand side is given in Example 4.1.7 above. By some elementary algebraic manipulations of binomial coefficients, it becomes,

$$i(\mathcal{P}_{K_{m,1,1}}, n) = \binom{m+n}{m}\frac{nm+m+1}{m+1}.$$

The noninteger root $\frac{-(m+1)}{m}$ is a real number in the circle of Conjecture 4.1.5.

Now we prepare the following lemma for proving Proposition 4.1.10.

Lemma 4.1.9. *For any integer* $1 \leq j \leq \frac{d}{2}$*, the polynomial* $p(n; d, j)$ *in Proposition 4.1.4 satisfies:*

$$p(n; d, d-j) = \left(\frac{d}{j} - 1\right) p(n; d, j).$$

Proof. It is an easy transformation:

$$
\begin{aligned}
p(n; d, d-j) &= \binom{(d-j)+n-1}{(d-j)-1}\binom{d-(d-j)+n-1}{d-(d-j)} \\
&= \binom{d-j+n-1}{d-j-1}\binom{j+n-1}{j} \\
&= \frac{d-j}{j}\binom{d-j+n-1}{d-j}\binom{j+n-1}{j-1} \\
&= \left(\frac{d}{j}-1\right) p(n; d, j).
\end{aligned}
$$

□

Proposition 4.1.10. *Let $(q_1, \ldots, q_t)$ be a partition of $d \geq 3$, satisfying $q_1 \geq q_2 \geq \cdots \geq q_t$. The Ehrhart polynomial $i(\mathcal{P}_G, n)$ of the edge polytope of the complete multipartite graph $G = K_{q_1,\ldots,q_t}$ does not have a root at $-(d-1)$ except when the graph is K_3.*

Proof. From Proposition 4.1.4, the Ehrhart polynomial of the edge polytope of $G = K_{q_1,\ldots,q_t}$ is

$$
\begin{aligned}
i(\mathcal{P}_G, n) &= f(n; d, d) - \sum_{k=1}^{t} f(n; d, q_k) \\
&= p(n; d, d) + \sum_{j=1}^{d-1} p(n; d, j) - \sum_{k=1}^{t}\sum_{j=1}^{q_k} p(n; d, j)
\end{aligned}
$$

Since $p(n; d, d)$ has $-(d-1)$ as one of its roots, it suffices to show that the rest of the expression does not have $-(d-1)$ as one of its roots.

We evaluate $p(n; d, j)$ at $-(d-1)$ for j from 1 to $d-1$:

$$p(-(d-1); d, j) = \binom{j-d}{j-1}\binom{-j}{d-j}$$

by the definition of $p(n; d, j)$. If $j > 1$, its sign is $(-1)^{j-1+d-j} = (-1)^{d-1}$ since $j - d < 0$ and $-j < 0$. In case where $j = 1$, since $j - 1$ is zero,

$$p(-(d-1); d, 1) = \binom{-1}{d-1} = (-1)^{d-1}$$

gives the same sign with other values of j.

By the conjugate partition $(q'_1, \ldots, q'_{t'})$ of $(q_1, \ldots, q_t)$, which is given by $q'_j = |\{i \leq t \mid q_i \geq j\}|$, we obtain

$$\sum_{j=1}^{d-1} p(n; d, j) - \sum_{k=1}^{t}\sum_{j=1}^{q_k} p(n; d, j) \quad = \quad \sum_{j=1}^{d-1} \left(1 - q'_j\right) p(n; d, j), \tag{4.2}$$

where we set, for simplicity, $q'_j = 0$ for $j > t'$.

We show that all the coefficients of $p(n; d, j)$ are nonnegative for any j from 1 to $d-1$ and there is at least one positive coefficient among them.

(I) $q_1 \geq \frac{d}{2}$:

The coefficients of $p(n; d, j)$ are zero for $q_1 \geq j \geq d - q_1$, unless $d = q_1 + q_2$, i.e., when the graph is a complete bipartite graph; the exceptional case will be discussed later. We assume, therefore, $q_2 < d - q_1$ for a while. Though equation (4.2) gives the coefficient of $p(n; d, j)$ as 1 for $d > j > q_1$, by using Lemma 4.1.9, we are able to let them be zero and the coefficient of $p(n; d, j)$ be $\frac{d}{j} - q'_j$ for $d - q_1 > j > 0$. Then all the coefficients of $p(n; d, j)$'s are positive, since the occurrence of integers greater than or equal to j in a partition of $d - q_1$ cannot be greater than $\frac{d-q_1}{j}$.

(II) $q_1 < \frac{d}{2}$:

Each coefficient of $p(n; d, j)$ in equation (4.2) is 1 for $d > j > \frac{d}{2}$. By Lemma 4.1.9, we transfer them to lower j terms so as to make the coefficients for $\frac{d}{2} > j > 0$ be $\frac{d}{j} - q'_j$. Then all the coefficients of $p(n; d, j)$'s are nonnegative, since the occurrence of integers greater than or equal to j in a partition of d cannot be greater than $\frac{d}{j}$. Moreover, the coefficient is zero for at most one j, less than $\frac{d}{2}$. If $d = 3$ and $q_1 = q_2 = q_3 = 1$, i.e., in case of K_3, there does not remain a positive coefficient. This exceptional case will be discussed later.

For both (I) and (II), ignoring the exceptional cases, the terms on the right-hand side of equation (4.2) are all nonnegative when $d \equiv 1 \pmod 2$, or nonpositive otherwise, and there is at least one nonzero term. That is, $-(d-1)$ is not a root of

$$\sum_{j=1}^{d-1} p(n; d, j) - \sum_{k=1}^{t} \sum_{j=1}^{q_k} p(n; d, j).$$

The Ehrhart polynomial $i(\mathcal{P}_G, n)$ is a sum of a polynomial whose roots include $-(d-1)$ and another polynomial whose roots do not include $-(d-1)$. Therefore, $-(d-1)$ is not a root of $i(\mathcal{P}_G, n)$.

Finally, we discuss the exceptional cases. The complete bipartite graphs are treated in Example 4.1.7. In these cases, $-(d-1)$ is not a root of the Ehrhart polynomials. However, $-(d-1) = -2$ is actually a root of the Ehrhart polynomial of the edge polytope constructed from the complete graph K_3, as shown in Proposition 4.1.3 (1). $\square$

4.1.2 Roots of Ehrhart polynomials of edge polytopes with loops

In this subsection, we will investigate roots of the Ehrhart polynomials of edge polytopes allowing loops.

A convex polytope $\mathcal{P}$ of dimension d is *simple* if each vertex of $\mathcal{P}$ belongs to exactly d edges of $\mathcal{P}$. A simple polytope $\mathcal{P}$ is *smooth* if at each vertex of $\mathcal{P}$, the primitive edge directions form a lattice basis.

Now, if $e = \{i, j\}$ is an edge of G, then $\rho(e)$ cannot be a vertex of $\mathcal{P}_G$ if and only if $i \neq j$ and G has a loop at each of the vertices i and j. Suppose that G has a loop at $i \in V(G)$ and $j \in V(G)$ and that $\{i, j\}$ is not an edge of G. Then $\mathcal{P}_G = \mathcal{P}_{G'}$ for the graph G' defined by $E(G') = E(G) \cup \{\{i, j\}\}$. Considering this fact, throughout this section, we assume that G satisfies the following condition:

($*$) If i, $j \in V(G)$ and if G has a loop at each of i and j, then the edge $\{i, j\}$ belongs to G.

The graphs G (allowing loops) whose edge polytope $\mathcal{P}_G$ is simple are completely classified by the following

Theorem 4.1.11 ([59, Theorem 1.8])**.** *Let W denote the set of vertices $i \in V(G)$ such that G has no loop at i and let G' denote the induced subgraph of G on W. Then the following conditions are equivalent :*

(i) $\mathcal{P}_G$ is simple, but not a simplex;

(ii) $\mathcal{P}_G$ is smooth, but not a simplex;

(iii) $W \neq \emptyset$ and G is one of the following graphs:

(α) G is a complete bipartite graph with at least one cycle of length 4;

(β) G has exactly one loop, G' is a complete bipartite graph and if G has a loop at i, then $\{i, j\} \in E(G)$ for all $j \in W$;

(γ) G has at least two loops, G' has no edge and if G has a loop at i, then $\{i, j\} \in E(G)$ for all $j \in W$.

From the theory of Gröbner bases, we obtain the Ehrhart polynomial $i(\mathcal{P}_G, n)$ of the edge polytope $\mathcal{P}_G$ above. In fact,

Theorem 4.1.12 ([59, Theorem 3.1])**.** *Let G be a graph as in Theorem 4.1.11 (iii). Let W denote the set of vertices $i \in V(G)$ such that G has no loop at i and let G' denote the induced subgraph of G on W. Then the Ehrhart polynomial $i(\mathcal{P}_G, n)$ of the edge polytope $\mathcal{P}_G$ are as follows:*

(α) If G is the complete bipartite graph on the vertex set $V_1 \cup V_2$ with $|V_1| = p$ and $|V_2| = q$, then we have

$$i(\mathcal{P}_G, n) = \binom{p+n-1}{p-1}\binom{q+n-1}{q-1};$$

(β) If G' is the complete bipartite graph on the vertex set $V_1 \cup V_2$ with $|V_1| = p$ and $|V_2| = q$, then we have

$$i(\mathcal{P}_G, n) = \binom{p+n}{p}\binom{q+n}{q};$$

(γ) If G possesses p loops and $|V(G)| = d$, then we have

$$i(\mathcal{P}_G, n) = \sum_{j=1}^{p} \binom{j+n-2}{j-1}\binom{d-j+n}{d-j}.$$

The goal of this section is to discuss the roots of Ehrhart polynomials of simple edge polytopes in Theorem 4.1.11 (Theorems 4.1.15, 4.1.16, and 4.1.17). The consequences of the theorems above support Conjecture 2.0.1.

Example 4.1.13. The Ehrhart polynomial for a graph G, the induced subgraph G' of which is a complete bipartite graph $K_{p,q}$, is given in Theorem 4.1.12 (β):

$$i(\mathcal{P}_G, n) = \binom{p+n}{p}\binom{q+n}{q},$$

and thus the roots are

$$\mathbf{V}\left(\binom{p+n}{p}\binom{q+n}{q}\right) = \{-1, -2, \ldots, -\max(p,q)\}.$$

Example 4.1.14. Explicit computation of the roots of the Ehrhart polynomials obtained in Theorem 4.1.12 (γ) seems, in general, to be rather difficult.

Let $p = 2$. Then

$$\begin{aligned}
&\binom{n-1}{0}\binom{d-1+n}{d-1} + \binom{n}{1}\binom{d-2+n}{d-2} \\
= &\binom{d-1+n}{d-1} + n\binom{d-2+n}{d-2} \\
= &\left(\frac{d-1+n}{d-1} + n\right)\binom{d-2+n}{d-2} \\
= &\frac{dn+d-1}{d-1}\binom{d-2+n}{d-2}.
\end{aligned}$$

Thus,

$$\mathbf{V}\left(i(\mathcal{P}_G, n)\right) = \left\{-1, -2, \ldots, -(d-2), -\frac{d-1}{d}\right\}.$$

Let $p = 3$. Then

$$\begin{aligned}
&\binom{n-1}{0}\binom{d-1+n}{d-1} + \binom{n}{1}\binom{d-2+n}{d-2} + \binom{n+1}{2}\binom{d-3+n}{d-3} \\
= &\binom{d-1+n}{d-1} + n\binom{d-2+n}{d-2} + \frac{n(n+1)}{2}\binom{d-3+n}{d-3} \\
= &\left(\frac{(d-1+n)(d-2+n)}{(d-1)(d-2)} + n\frac{d-2+n}{d-2} + \frac{n(n+1)}{2}\right)\binom{d-3+n}{d-3}
\end{aligned}$$

and

$$\begin{aligned}
& \frac{(d-1+n)(d-2+n)}{(d-1)(d-2)} + m\frac{d-2+n}{d-2} + \frac{n(n+1)}{2} \\
= \ & \frac{2(d-1+n)(d-2+n) + 2(d-1)n(d-2+n) + (d-1)(d-2)n(n+1)}{2(d-1)(d-2)} \\
= \ & \frac{(d^2-d+2)n^2 + (3d^2-5d)n + (2d^2-6d+4)}{2(d-1)(d-2)}.
\end{aligned}$$

Let

$$f(n) = (d^2-d+2)n^2 + (3d^2-5d)n + (2d^2-6d+4).$$

Since $d > p = 3$, one has

$$\begin{aligned}
f(0) &= 2d^2-6d+4 = 2(d-1)(d-2) > 0; \\
f(-1) &= (d^2-d+2) - (3d^2-5d) + (2d^2-6d+4) = -2d+6 < 0; \\
f(-2) &= 4(d^2-d+2) - 2(3d^2-5d) + (2d^2-6d+4) = 12 > 0.
\end{aligned}$$

Hence,

$$\mathbf{V}\left(i(\mathcal{P}_G, n)\right) = \{-1, -2, \ldots, -(d-3), \alpha, \beta\}$$

where $-2 < \alpha < -1 < \beta < 0$.

We try to find information about the roots of the Ehrhart polynomials obtained in Theorem 4.1.12 (γ) with $d > p \geq 2$.

Theorem 4.1.15 ([48, Theorem 2.5])**.** *Let d and p be integers with $d > p \geq 2$ and let*

$$f_{d,p}(n) = \sum_{j=1}^{p} \binom{j+n-2}{j-1}\binom{d-j+n}{d-j}$$

be a polynomial of degree $d-1$ in the variable n. Then

$$\{-1, -2, \ldots, -(d-p)\} \quad \subset \quad \mathbf{V}(f_{d,p}) \cap \mathbb{R} \quad \subset \quad [-(d-2), 0).$$

Proof. It is easy to see that $f_{d,p}(0) = 1$ and $f_{d,p}(n) > 0$ for all $n > 0$.

From Example 4.1.14, we may assume that $4 \leq p < d$. Then

$$\begin{aligned}
& f_{d,p}(n) \\
= \ & \binom{d-1+n}{d-1} + n\binom{d-2+n}{d-2} + \sum_{j=3}^{p} \binom{j+n-2}{j-1}\binom{d-j+n}{d-j} \\
= \ & \left(\frac{d-1+n}{d-1} + n\right)\binom{d-2+n}{d-2} + \sum_{j=3}^{p} \binom{j+n-2}{j-1}\binom{d-j+n}{d-j} \\
= \ & \frac{nd+d-1}{d-1}\binom{d-2+n}{d-2} + \sum_{j=3}^{p} \binom{j+n-2}{j-1}\binom{d-j+n}{d-j}.
\end{aligned}$$

If $n < -(d-2)$, then $n+d-2 < 0$, $nd+d-1 < -(d-2)d+d-1 = -(d-3)d-1 < 0$,

$$\begin{aligned} n+d-j &\leq n+d-3 < 0 \\ n+j-2 &\leq n+p-2 \leq n+d-3 < 0 \end{aligned}$$

for each $j = 3, 4, \ldots, p$. Hence, we have $(-1)^{d-1} f_{d,p}(n) > 0$ for all $n < -(d-2)$. Thus, we have $\mathbf{V}(f_{d,p}) \cap \mathbb{R} \subset [-(d-2), 0)$.

Since

$$f_{d,p}(n) = \binom{d-p+n}{d-p} \sum_{j=1}^{p} \binom{j+n-2}{j-1} \frac{(d-j+n)\cdots(d-p+1+n)}{(d-j)\cdots(d-p+1)},$$

it follows that

$$\mathbf{V}\left(\binom{d-p+n}{d-p}\right) = \{-1, -2, \ldots, -(d-p)\} \subset \mathbf{V}(f_{d,p}).$$

□

Theorem 4.1.16 ([48, Theorem 2.6])**.** *Let d and p be integers with $d > p \geq 2$ and let $f_{d,p}(m)$ be the polynomial defined above. If $d - 2p + 2 \geq 0$, then*

$$\mathbf{V}(f_{d,p}) = \{-1, -2, \ldots, -(d-p), \alpha_1, \alpha_2, \ldots, \alpha_{p-1}\}$$

where

$$-(p-1) < \alpha_{p-1} < -(p-2) < \alpha_{p-2} < -(p-3) < \cdots < -1 < \alpha_1 < 0.$$

Proof. Let

$$g_{d,p}(n) = \frac{f_{d,p}(m)}{\binom{d-p+n}{d-p}} = \sum_{j=1}^{p} \binom{j+n-2}{j-1} \frac{(d-j+n)\cdots(d-p+1+n)}{(d-j)\cdots(d-p+1)}.$$

It is enough to show that

$$(-1)^k g_{d,p}(k) > 0$$

for $k = 0, -1, -2, \ldots, -(p-1)$.

First Step. We claim that $(-1)^{-(p-1)} g_{d,p}(-(p-1)) > 0$. A routine computation on binomial coefficients yields the equalities

$$\begin{aligned} & g_{d,p}(-(p-1)) \\ = \ & \frac{\sum_{j=1}^{p} (-1)^{j-1} \binom{p-1}{j-1} \prod_{i=1}^{j-1} (d-i) \prod_{k=j}^{p-1} (d-k-(p-1))}{(d-1)\cdots(d-p+1)} \end{aligned}$$

and

$$\begin{aligned} & \sum_{j=1}^{p} (-1)^{j-1} \binom{p-1}{j-1} \prod_{i=1}^{j-1} (d-i) \prod_{k=j}^{p-1} (d-k-(p-1)) \\ = \ & (-1)^{p-1}(p-1)p\cdots(2p-3). \end{aligned}$$

Hence,

$$(-1)^{p-1} g_{d,p}(-(p-1)) = \frac{(p-1)p \cdots (2p-3)}{(d-1) \cdots (d-p+1)} > 0.$$

Second Step. Working with induction on p, we now show that

$$(-1)^k g_{d,p}(k) > 0$$

for $k = 0, -1, -2, \ldots, -(p-2)$. Again, a routine computation on binomial coefficients yields

$$g_{d,p}(n) = \binom{p+n-2}{p-1} + \frac{d-p+1+n}{d-p+1} g_{d,p-1}(n).$$

Hence,

$$(-1)^k g_{d,p}(k) = \frac{d-p+1+k}{d-p+1} (-1)^k g_{d,p-1}(k).$$

Since $d - 2p + 2 \geq 0$, one has

$$d - p + 1 + k \geq d - p + 1 - (p-2) = d - 2p + 3 > 0.$$

By virtue of $(-1)^{-(p-1)} g_{d,p}(-(p-1)) > 0$, together with the hypothesis of induction, it follows that

$$(-1)^k g_{d,p-1}(k) > 0.$$

Thus,

$$(-1)^k g_{d,p}(k) > 0,$$

as desired. □

If $d - 2p + 2 \geq 0$, then it follows that

$$\left\lfloor \frac{d-1}{2} \right\rfloor \leq d - p.$$

In this case, around half of the elements of $\mathbf{V}(f_{d,p})$ are negative integers. This fact remains true even if $d - 2p + 2 < 0$.

Theorem 4.1.17 ([48, Theorem 2.7])**.** *Let d and p be integers with $d > p \geq 2$ and let $f_{d,p}(n)$ be the polynomial defined above. Then*

$$\left\{ -1, -2, \ldots, -\left\lfloor \frac{d-1}{2} \right\rfloor \right\} \subset \mathbf{V}(f_{d,p}).$$

Proof. If $d-2p+2 \geq 0$, then it follows from Theorem 4.1.15. (Note that if $p=2$, then $d-2p+2=d-2>0$.)

Work with induction on p. Let $d-2p+2<0$. By Theorem 4.1.15, it is enough to show that $g_{d,p}(k)=0$ for all $k=-(d-p+1),\ldots,-\left\lfloor\frac{d-1}{2}\right\rfloor$. As in the proof of Theorem 4.1.16, we have

$$g_{d,p}(n)=\binom{p+n-2}{p-1}+\frac{d-p+1+n}{d-p+1}g_{d,p-1}(n).$$

Since $d-2p+2<0$, it follows that $\left\lfloor\frac{d-1}{2}\right\rfloor \leq p-2$. Thus,

$$g_{d,p}(k)=\frac{d-p+1+k}{d-p+1}g_{d,p-1}(k).$$

By virtue of

$$g_{d,p}(-(d-p+1))=\frac{0}{d-p+1}g_{d,p-1}(-(d-p+1))=0$$

together with the hypothesis of induction, it follows that $g_{d,p}(k)=0$ for all $k=-(d-p+1),\ldots,-\left\lfloor\frac{d-1}{2}\right\rfloor$. $\square$

Example 4.1.18. Let $d=12$. Then $d-2p+2\geq 0$ if and only if $p\leq 7$. For $p=2,3,\ldots,7$, the roots of the Ehrhart polynomials are $-1,-2,\ldots,-(d-p)=p-12$, together with the real numbers listed as follows:

$p=2$	-0.92					
$p=3$	-1.92	-0.85				
$p=4$	-2.90	-1.83	-0.80			
$p=5$	-3.83	-2.77	-1.74	-0.76		
$p=6$	-4.67	-3.65	-2.65	-1.66	-0.72	
$p=7$	-5.31	-4.42	-3.47	-2.53	-1.58	-0.69

For $p=8,9,10,11$, the roots of the Ehrhart polynomials are $-1,-2,-3,-4,-5=-\left\lfloor\frac{d-1}{2}\right\rfloor$, together with the following complex numbers:

$p=8$	-5.56	-4.19	-3.31	-2.41	-1.51	-0.65
$p=9$	-5.47	-4.79	-3.16	-2.29	-1.43	-0.62
$p=10$	-5.51	$-4.16+0.18i$	$-4.16-0.18i$	-2.16	-1.34	-0.59
$p=11$	-5.50	-4.53	$-3.08+0.06i$	$-3.08-0.06i$	-1.24	-0.55

(Computed by `Maxima`.) Thus, in particular, the real parts of all roots are negative.

4.2 Counterexamples of Conjecture 2.0.1

However, we discover counterexamples of Conjecture 2.0.1.

4.2.1 A significant family of integral simplices

This section is devoted to giving some counterexamples of Conjecture 2.0.1. First, we prove

Theorem 4.2.1 ([38, Theorem 2.1]). *Let $m, d, k \in \mathbb{Z}_{>0}$ be arbitrary positive integers satisfying*

$$m \geq 1,\ d \geq 2 \ \text{ and } \ 1 \leq k \leq \lfloor (d+1)/2 \rfloor. \tag{4.3}$$

Then there exists an integral convex polytope whose Ehrhart polynomial coincides with

$$\binom{d+n}{d} + m\binom{d+n-k}{d}. \tag{4.4}$$

Proof. We may show that there exists an integral convex polytope of dimension d whose δ-vector coincides with

$$\delta_i = \begin{cases} 1, & i = 0, \\ m, & i = k, \\ 0, & \text{otherwise.} \end{cases}$$

When $k = 1$, it is obvious that $(1, m, 0, \ldots, 0)$ is a δ-vector. Thus, we assume that $k \geq 2$. By Lemma 3.2.3, it is enough to construct an integral convex polytope of dimension d with its δ-vector

$$\delta_i = \begin{cases} 1, & i = 0, \\ m, & i = (d+1)/2, \\ 0, & \text{otherwise,} \end{cases}$$

for any positive integer m and any odd number d with $d \geq 3$.

Let $d \geq 3$ be an odd number and $c = (d-1)/2$. We define the integral simplex $\mathcal{P} \subset \mathbb{R}^d$ of dimension d by setting the convex hull of the integer points $v_0, v_1, \ldots, v_d \in \mathbb{Z}^d$, which are of the form

$$v_i = \begin{cases} \mathbf{e}_i, & i = 1, \ldots, d-1, \\ \sum_{j=1}^{c} \mathbf{e}_j + \sum_{j=c+1}^{2c} m\mathbf{e}_j + (m+1)\mathbf{e}_d, & i = d, \\ (0, 0, \ldots, 0), & i = 0, \end{cases}$$

where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d$ denote the unit coordinate vectors of $\mathbb{R}^d$. In other words, for $i = 1, \ldots, d$, v_i is equal to the ith row vector of the $d \times d$ lower triangular integer matrix

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \\ 1 & \cdots & 1 & m & \cdots & m & m+1 \end{pmatrix}, \tag{4.5}$$

where there are c 1's and c m's in the dth row. Then we notice that $\mathrm{vol}(\mathcal{P}) = m+1$, which coincides with the determinant of (4.5).

For $j = 1, 2, \ldots, m$, since

$$\sum_{i=0}^{c} \frac{m+1-j}{m+1}(v_i, 1) + \sum_{i=c+1}^{d} \frac{j}{m+1}(v_i, 1) = (\underbrace{1, 1, \ldots, 1}_{c}, \underbrace{j, j, \ldots, j}_{c+1}, c+1) \in \mathbb{Z}^{d+1}$$

and

$$0 \leq \frac{m+1-j}{m+1} < 1, \quad 0 \leq \frac{j}{m+1} < 1,$$

we have $\delta_{c+1} \geq m$. Moreover, from $\mathrm{vol}(\mathcal{P}) = m+1$ together with the nonnegativity of δ-vectors, we obtain $\delta_{(d+1)/2} = m$. Therefore, we conclude that $\mathcal{P}$ has the required δ-vector. $\square$

We consider the roots of the polynomial (4.4) given in Theorem 4.2.1.

Let $f(n)$ be the polynomial (4.4) in n of degree d. Since

$$f(n) = \frac{\prod_{j=1}^{d-k}(n+j)}{d!} \left(\prod_{j=d-k+1}^{d} (n+j) + m \prod_{j=0}^{k-1} (n-j) \right),$$

negative integers $-1, -2, \ldots, -d+k$ are always the roots of $f(n)$. Let

$$g_{m,d,k}(n) = \prod_{j=d-k+1}^{d} (n+j) + m \prod_{j=0}^{k-1} (n-j)$$

be a polynomial in n of degree k. We consider the roots of $g_{m,d,k}(n)$.

Example 4.2.2. Let us consider the polynomial $g_{m,15,8}(n)$. When $1 \leq m \leq 8$, all its roots satisfy (4.7). On the other hand, when $m = 9$, its eight roots are approximately

$$14.37537447 \pm 25.02096544\sqrt{-1}, \quad -0.77681486 \pm 10.23552765\sqrt{-1},$$
$$-2.56596317 \pm 4.52757516\sqrt{-1} \text{ and } -3.03259644 \pm 1.31223697\sqrt{-1}.$$

By virtue of Theorem 4.2.1, this implies that there exists a counterexample of Conjecture 2.0.1. Moreover, it can be verified that for every $15 \leq d \leq 100$, $g_{9,d,\lfloor (d+1)/2 \rfloor}(n)$ possesses a root which violates (4.7), that is, there exists a counterexample of Conjecture 2.0.1 for each dimension $15 \leq d \leq 100$. There also seems to exist a counterexample when $d \geq 101$. In addition, we remark that when $d \geq 17$, we can verify that $g_{9,d,\lfloor (d+1)/2 \rfloor}(n)$ possesses a root whose real part is greater than d. (Those are computed by `Maple` and `Maxima`.)

These computational results are also supported theoretically. For example on the roots of $g_{9,15,8}(n)$, by applying the *Routh-Hurwitz stability criterion*, (e.g., [18, pp. 226–233],) we can check that $g_{9,15,8}(n+14.3)$ possesses a root whose real part is nonnegative but $g_{9,15,8}(n+14.4)$ possesses no root whose real part is nonnegative. Of course, this means that $g_{9,15,8}(n)$ possesses a root α with $14.3 \leq \mathrm{Re}(\alpha) < 14.4$.

Remark 4.2.3. On the order of the largest real part of the non-real roots of $g_{9,d,\lfloor (d+1)/2 \rfloor}(n)$, the order seems not to be linear on d. For example, when $d = 30, 50, 100$ and 200, the largest real parts of non-real roots of $g_{9,d,\lfloor (d+1)/2 \rfloor}(n)$ are as follows:

d	approximate real part
30	60
50	174
100	722
200	2940

Thus, it is more natural to claim that the real parts of roots of Ehrhart polynomials are bounded with $O(d^2)$, which is known as the best possible norm bound of roots of Ehrhart polynomials.

Remark 4.2.4. (a) When $m = 1$, the real parts of all the roots of $g_{1,d,k}(n)$ are $(-d+k-1)/2$, which satisfies $-d < (-d+k-1)/2 < -1/2$. In fact, since all the roots of $1+\lambda^k$ are on the unit circle in the complex plane, we can apply the theorem of [64] to the polynomial $\binom{n+d}{d} + \binom{n+d-k}{d}$. On the other hand, when $m = 2$, we can obtain other counterexample of Conjecture 2.0.1 when $d = 37$ and $k = 19$.
(b) When $k = 1$, one has $g_{m,d,1}(n) = (m+1)n + d$. Thus, its root is $-d/(m+1)$, which satisfies $-d < -d/(m+1) < 0$. When $k = 2$, one has $g_{m,d,2}(n) = (m+1)n^2 + (2d-m-1)n + d(d-1)$. If its discriminant is negative, then the real part of its roots is $-d/(m+1) + 1/2$, which satisfies $-d + 1/2 < -d/(m+1) + 1/2 < 1/2$.

Remark 4.2.5. Finally, we remark that there exists other counterexample of Conjecture 2.0.1. In [60], Ohsugi and Shibata found an integral convex polytope of dimension 124 which is a certain counterexample.

4.3 Roots of Ehrhart polynomials of Gorenstein Fano polytopes

In this section, we discuss roots of the Ehrhart polynomials of Gorenstein Fano polytopes, which have many interesting distribution.

4.3.1 Gorenstein Fano polytopes arising from graphs

Let us study roots of Ehrhart polynomials of Gorenstein Fano polytopes arising from finite connected simple graphs. Throughout this subsection, G denotes a simple graph on the vertex set $V(G) = \{1, \ldots, d\}$ with $E(G) = \{e_1, \ldots, e_m\}$ being the edge set.

Definition 4.3.1. Given an edge $e = \{i, j\} \in E(G)$, we define $\sigma(e) = \mathbf{e}_i - \mathbf{e}_j \in \mathbb{R}^d$. Moreover, we write $\mathcal{P}_G^{\pm} \subset \mathbb{R}^d$ for the convex hull of

$$\{\pm\sigma(e) : e \in E(G)\},$$

which we call an *symmetric edge polytope.*

Let $\mathcal{H} \subset \mathbb{R}^d$ denote the hyperplane defined by the equation $x_1 + x_2 + \cdots + x_d = 0$. Now, since the integral points $\pm\sigma(e_1), \ldots, \pm\sigma(e_m)$ lie on the hyperplane $\mathcal{H}$, we have $\dim(\mathcal{P}_G^{\pm}) \leq d - 1$.

Proposition 4.3.2. *One has* $\dim(\mathcal{P}_G^{\pm}) = d - 1$ *if and only if* G *is connected.*

Proof. Suppose that G is not connected. Let $G_1, \ldots, G_k$ with $k > 1$ denote the connected components of G. Let, say, $\{1, \ldots, d_1\}$ be the vertex set of G_1 and $\{d_1 + 1, \ldots, d_2\}$ the vertex set of G_2. Then $\mathcal{P}_G^{\pm}$ lies on two hyperplanes defined by the equations $x_1 + \cdots + x_{d_1} = 0$ and $x_{d_1+1} + \cdots + x_{d_2} = 0$. Thus, $\dim(\mathcal{P}_G^{\pm}) < d - 1$.

Next, we assume that G is connected. Suppose that $\mathcal{P}_G^{\pm}$ lies on the hyperplane defined by the equation $a_1x_1 + \cdots + a_dx_d = b$ with $a_1, \ldots, a_d, b \in \mathbb{Z}$. Let $e = \{i, j\}$ be an edge of G. Then because $\sigma(e)$ lies on this hyperplane together with $-\sigma(e)$, we obtain

$$a_i - a_j = -(a_i - a_j) = b.$$

Thus $a_i = a_j$ and $b = 0$. For all edges of G, since G is connected, we have $a_1 = a_2 = \cdots = a_d$ and $b = 0$. Therefore, $\mathcal{P}_G^{\pm}$ lies only on the hyperplane $x_1 + x_2 + \cdots + x_d = 0$. $\square$

For the rest of this section, we assume that G is connected.

Proposition 4.3.3. *Let* $\mathcal{P}_G^{\pm}$ *be a symmetric edge polytope of a graph* G. *Then* $\mathcal{P}_G^{\pm} \subset \mathcal{H}$ *is a Gorenstein Fano polytope of dimension* $d - 1$.

Proof. Let $\varphi : \mathbb{R}^{d-1} \to \mathcal{H}$ be the bijective homomorphism with

$$\varphi(y_1, \ldots, y_{d-1}) = (y_1, \ldots, y_{d-1}, -(y_1 + \cdots + y_{d-1})).$$

Thus, we can identify $\mathcal{H}$ with $\mathbb{R}^{d-1}$. Therefore, $\varphi^{-1}(\mathcal{P}_G^{\pm})$ is isomorphic to $\mathcal{P}_G^{\pm}$.

Since one has

$$\frac{1}{2m}\sum_{j=1}^{m}\sigma(e_j) + \frac{1}{2m}\sum_{j=1}^{m}(-\sigma(e_j)) = (0, \ldots, 0) \in \mathbb{R}^d,$$

the origin of $\mathbb{R}^d$ is contained in the relative interior of $\mathcal{P}_G^{\pm} \subset \mathcal{H}$. Moreover, since

$$\mathcal{P}_G^{\pm} \subset \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid -1 \leq x_i \leq 1, i = 1, \ldots, d\},$$

it is not possible for an integral point to exist anywhere in the interior of $\mathcal{P}_G^{\pm}$ except at the origin. Thus, $\mathcal{P}_G^{\pm} \subset \mathcal{H}$ is a Fano polytope of dimension $d-1$.

Next, we prove that $\mathcal{P}_G^{\pm}$ is Gorenstein. Let M be an integer matrix whose row vectors are $\sigma(e)$ or $-\sigma(e)$ with $e \in E(G)$. Then M is a totally unimodular matrix. From the theory of totally unimodular matrices ([66, Chapter 9]), it follows that a system of equations $yA = (1, \ldots, 1)$ has integral solutions, where A is a submatrix of M. This implies that the equation of each supporting hyperplane of $\mathcal{P}_G^{\pm}$ is of the form $a_1x_1 + \cdots + a_dx_d = 1$ with each $a_i \in \mathbb{Z}$. In other words, the dual polytope of $\mathcal{P}_G^{\pm}$ is integral. Hence, $\mathcal{P}_G^{\pm}$ is Gorenstein, as required. □

We consider the conditions under which $\mathcal{P}_G^{\pm}$ is unimodular equivalent with $\mathcal{P}_{G'}^{\pm}$ for graphs G and G'.

Recall that for a connected graph G, we call G a *2-connected* graph if the induced subgraph with the vertex set $V(G)\backslash\{i\}$ is still connected for any vertex i of G.

Let us say a Fano polytope $\mathcal{P} \subset \mathbb{R}^d$ *splits* into $\mathcal{P}_1$ and $\mathcal{P}_2$ if $\mathcal{P}$ is the convex hull of the two Fano polytopes $\mathcal{P}_1 \subset \mathbb{R}^{d_1}$ and $\mathcal{P}_2 \subset \mathbb{R}^{d_2}$ with $d = d_1 + d_2$. That is, by arranging the numbering of coordinates, we have

$$\mathcal{P} = \operatorname{conv}(\{(\alpha_1, \mathbf{0}) \in \mathbb{R}^d \mid \alpha_1 \in \mathcal{P}_1\} \cup \{(\mathbf{0}, \alpha_2) \in \mathbb{R}^d \mid \alpha_2 \in \mathcal{P}_2\}).$$

Lemma 4.3.4. *$\mathcal{P}_G^{\pm}$ cannot split if and only if G is 2-connected.*

Proof. **("Only if")** Suppose that G is not 2-connected, i.e., there is a vertex i of G such that the induced subgraph G' of G with the vertex set $V(G)\backslash\{i\}$ is not connected. For a matrix

$$\begin{pmatrix} \sigma(e_1) \\ -\sigma(e_1) \\ \vdots \\ \sigma(e_m) \\ -\sigma(e_m) \end{pmatrix} \tag{4.6}$$

whose row vectors are the vertices of $\mathcal{P}_G^{\pm}$, we add all the columns of (4.6) except the ith column to the ith column. Then the ith column vector becomes equal to the zero vector. Let, say, $\{1, \ldots, i-1\}$ and $\{i+1, \ldots, d\}$ denote the vertex set of the connected components of G'. Then, by arranging the row vectors of (4.6) if necessary, the matrix (4.6) can be transformed into

$$\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}.$$

This means that $\mathcal{P}_G^{\pm}$ splits into $\mathcal{P}_1$ and $\mathcal{P}_2$, where the vertex set of $\mathcal{P}_1$ (resp, $\mathcal{P}_2$) constitutes the row vectors of M_1 (resp. M_2).

("If") We assume that G is 2-connected. Suppose that $\mathcal{P}_G^{\pm}$ splits into $\mathcal{P}_1, \ldots, \mathcal{P}_q$ and each $\mathcal{P}_i$ cannot split, where $q > 1$. Then by arranging the row vectors if necessary, the matrix (4.6) can be transformed into

$$\begin{pmatrix} M_1 & & 0 \\ & \ddots & \\ 0 & & M_q \end{pmatrix}.$$

Now, for a row vector v of each matrix M_i, $-v$ is also a row vector of M_i. Let

$$v_{i_1}, \ldots, v_{i_{k_i}}, -v_{i_1}, \ldots, -v_{i_{k_i}}$$

denote the row vectors of M_i, where $e_{i_1}, \ldots, e_{i_{k_i}}$ are the edges of G with $v_{i_j} = \sigma(e_{i_j})$ or $v_{i_j} = -\sigma(e_{i_j})$, and G_i denote the subgraph of G with the edge set $\{e_{i_1}, \ldots, e_{i_{k_i}}\}$. Then for the subgraphs $G_1, \ldots, G_q$ of G, one has

$$|V(G_1)| + \cdots + |V(G_q)| \geq d + 2(q-1), \tag{4.7}$$

where $V(G_i)$ is the vertex set of G_i.
(In fact, the inequality (4.7) follows by induction on q. When $q = 2$, since G is 2-connected, G_1 and G_2 share at least two vertices. Thus, one has $|V(G_1)|+|V(G_2)| \geq d + 2$. When $q = k + 1$, since G is 2-connected, one has

$$\left|(\cup_{i=1}^{k} V(G_i)) \cap V(G_{k+1})\right| \geq 2.$$

Let d' be the sum of the numbers of the columns of $M_1, \ldots, M_{k-1}$ and M_k and d'' be the number of the columns of M_{k+1}, where $d' + d'' = d$. Then one has

$$\begin{aligned} |V(G_1)| + \cdots + |V(G_k)| + |V(G_{k+1})| &\geq d' + 2(k-1) + |V(G_{k+1})| \\ &\geq d' + d'' + 2(k-1) + 2 = d + 2k \end{aligned}$$

by the hypothesis of induction.)
In addition, each $\mathcal{P}_{G_i}^{\pm}$ cannot split. Thus one has $\dim(\mathcal{P}_{G_i}^{\pm}) = |V(G_i)| - 1$ since each G_i is connected by the proof of the "only if" part. It then follows from this equality and the inequality (4.7) that

$$\begin{aligned} d - 1 &= \dim(\mathcal{P}_{G_1}^{\pm}) + \cdots + \dim(\mathcal{P}_{G_q}^{\pm}) = |V(G_1)| + \cdots + |V(G_q)| - q \\ &\geq d + 2q - 2 - q = d + q - 2 \geq d \qquad (q \geq 2), \end{aligned}$$

a contradiction. Therefore, $\mathcal{P}_G^{\pm}$ cannot split. $\square$

Lemma 4.3.5. *Let G be a 2-connected graph. Then, for a graph G', $\mathcal{P}_G^{\pm}$ is unimodular equivalent with $\mathcal{P}_{G'}^{\pm}$ as an integral convex polytope if and only if G is isomorphic to G' as a graph.*

Proof. If $|V(G)| = 2$, the statement is obvious. Thus, we assume that $|V(G)| > 2$.
("Only if") Suppose that $\mathcal{P}_G^{\pm}$ is unimodular equivalent with $\mathcal{P}_{G'}^{\pm}$. Let M_G (resp. $M_{G'}$) denote the matrix whose row vectors are the vertices of $\mathcal{P}_G^{\pm}$ (resp. $\mathcal{P}_{G'}^{\pm}$). Then there is a unimodular transformation U such that one has

$$M_G U = M_{G'}. \tag{4.8}$$

Thus, each row vector of M_G, i.e., each edge of G, one-to-one corresponds to each edge of G'. Hence, G and G' have the same number of edges. Moreover, since G is 2-connected, $\mathcal{P}_G^{\pm}$ cannot split by Lemma 4.3.4. Thus, $\mathcal{P}_{G'}^{\pm}$ also cannot split; that is to say, G' is also 2-connected. In addition, if we suppose that G and G' do not have the same number of vertices, then $\dim(\mathcal{P}_G^{\pm}) \neq \dim(\mathcal{P}_{G'}^{\pm})$ since G and G' are connected, a contradiction. Thus, the number of the vertices of G is equal to that of G'.

Now an arbitrary 2-connected graph with $|V(G)| > 2$ can be obtained by the following method: start from a cycle and repeatedly append an H-path to a graph H that has been already constructed. (Consult, e.g., [77].) In other words, there is one cycle C_1 and $(q-1)$ paths $\Gamma_2, \dots, \Gamma_q$ such that

$$G = C_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_q. \tag{4.9}$$

Under the assumption that G is 2-connected and one has the equality (4.8), we show that G is isomorphic to G' by induction on q.

If $q = 1$, i.e., G is a cycle, then G has d edges. Let $a_i, i = 1, \dots, d$ denote the degree of each vertex i of G'. Then one has

$$a_1 + a_2 + \cdots + a_d = 2d.$$

If there is i with $a_i = 1$, then G' is not 2-connected. Thus, $a_i \geq 2$ for $i = 1, \dots, d$. Hence, $a_1 = \cdots = a_d = 2$. It then follows that G' is also a cycle of the same length as G, which implies that G is isomorphic to G'.

When $q = k+1$, we assume (4.9). Let $\tilde{G}$ denote the subgraph of G with

$$\tilde{G} = C_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k.$$

Then $\tilde{G}$ is a 2-connected graph. Since each edge of G has one-to-one correspondence with each edge of G', there is a subgraph $\tilde{G}'$ of G' each of whose edges corresponds to those of $\tilde{G}$. Then one has $M_{\tilde{G}} U = M_{\tilde{G}'}$, where $M_{\tilde{G}}$ (resp. $M_{\tilde{G}'}$) is a submatrix of M_G (resp. $M_{G'}$) whose row vectors are the vertices of $\mathcal{P}_{\tilde{G}}^{\pm}$ (resp. $\mathcal{P}_{\tilde{G}'}^{\pm}$). Thus, $\tilde{G}$ is isomorphic to $\tilde{G}'$ by the hypothesis of induction. Let $\Gamma_{k+1} = (i_0, i_1, \dots, i_p)$ with $i_0 < i_1 < \cdots < i_p$ and $e_{i_l} = \{i_{l-1}, i_l\}, l = 1, \dots, p$ denote the edges of Γ_{k+1}. In addition, let $e'_{i_1}, \dots, e'_{i_p}$ denote the edges of G' corresponding to the edges $e_{i_1}, \dots, e_{i_p}$ of G. Here, the edges $e'_{i_1}, \dots, e'_{i_p}$ of G' are not the edges of $\tilde{G}'$. Since i_0 and i_p are distinct vertices of $\tilde{G}$ and $\tilde{G}$ is connected, there is a path $\Gamma = (i_0, j_1, j_2, \dots, j_{q-1}, i_p)$

with $i_0 = j_0 < j_1 < j_2 < \cdots < j_{q-1} < j_q = i_p$ in $\tilde{G}$. Let $e_{j_l} = \{j_{l-1}, j_l\}, l = 1, \ldots, q$ denote the edges of Γ. Then by renumbering the vertices of $\tilde{G}'$ if necessary, there is a path $\Gamma' = (i'_0, j'_1, j'_2, \ldots, j'_{q-1}, i'_p)$ with $i'_0 = j'_0 < j'_1 < j'_2 < \cdots < j'_{q-1} < j'_q = i'_p$ in $\tilde{G}'$ since $\tilde{G}$ is isomorphic to $\tilde{G}'$. Let $e'_{j_l} = \{j'_{l-1}, j'_l\}, l = 1, \ldots, q$ denote the edges of Γ'. However, by (4.8), each edge e_{j_l} of $\tilde{G}$ has one-to-one correspondence with each edge e''_{j_l} of $\tilde{G}'$. Thus, each edge e'_{j_l} of $\tilde{G}'$ has one-to-one correspondence with each edge e''_{j_l} of $\tilde{G}'$. In other words, one has

$$\{e'_{j_l} : l = 1, \ldots, q\} = \{e''_{j_l} : l = 1, \ldots, q\}.$$

Since there are Γ_{k+1} and Γ that are paths from i_0 to i_p, one has

$$\sum_{l=1}^{p} \sigma(e_{i_l}) = \sum_{l=1}^{q} \sigma(e_{j_l}). \tag{4.10}$$

On the one hand, if we multiply the left-hand side of the equation (4.10) with U, then we have

$$\sum_{l=1}^{p} \sigma(e_{i_l})U = \sum_{l=1}^{p} \sigma(e'_{i_l}).$$

On the other hand, if we multiply the right-hand side of the equation (4.10) with U, then we have

$$\sum_{l=1}^{q} \sigma(e_{j_l})U = \sum_{l=1}^{q} \sigma(e''_{j_l}) = \sum_{l=1}^{q} \sigma(e'_{j_l}) = \mathbf{e}_{i'_0} - \mathbf{e}_{i'_p}.$$

Hence, we have $\sum_{l=1}^{p} \sigma(e'_{i_l}) = \mathbf{e}_{i'_0} - \mathbf{e}_{i'_p}$. This means that the edges $e'_{i_1}, \ldots, e'_{i_p}$ of G' construct a path from the vertex i'_0 to i'_p, which is isomorphic to Γ_{k+1}. Therefore, G is isomorphic to G'.

(" if ") Suppose that G is isomorphic to G'. Then by renumbering the vertices if necessary, it can be easily verified that $\mathcal{P}_G^{\pm}$ is unimodular equivalent with $\mathcal{P}_{G'}^{\pm}$. $\square$

Theorem 4.3.6 ([48, Theorem 3.5])**.** *For a connected simple graph G (resp. G'), let $G_1, \ldots, G_q$ (resp. $G'_1, \ldots, G'_{q'}$) denote the 2-connected components of G (resp. G'). Then $\mathcal{P}_G^{\pm}$ is unimodular equivalent with $\mathcal{P}_{G'}^{\pm}$ if and only if $q = q'$ and G_i is isomorphic to G'_i by renumbering if necessary.*

Proof. It is clear from Lemma 4.3.4 and Lemma 4.3.5. If G_i is isomorphic to G'_i for $i = 1, \ldots, q$, by virtue of Lemma 4.3.4 and Lemma 4.3.5, then $\mathcal{P}_G^{\pm}$ is unimodular equivalent with $\mathcal{P}_{G'}^{\pm}$. On the contrary, suppose that $\mathcal{P}_G^{\pm}$ is unimodular equivalent with $\mathcal{P}_{G'}^{\pm}$. If $q \neq q'$, one has a contradiction by Lemma 4.3.4. Thus, $m = m'$. Moreover, by our assumption, G_i is isomorphic to G'_i by Lemma 4.3.5. $\square$

Now, we study the Ehrhart polynomials of $\mathcal{P}_G^{\pm}$ and their roots.

Proposition 4.3.7. *If G is a tree, then $\mathcal{P}_G^{\pm}$ is unimodular equivalent with*

$$\mathrm{conv}(\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_{d-1}\}). \tag{4.11}$$

Proof. If G is a tree, then any 2-connected component of G consists of one edge and G possesses $(d-1)$ 2-connected components. Thus, by Theorem 4.3.6, for any tree G, $\mathcal{P}_G^{\pm}$ is unimodular equivalent. Hence we should prove only the case where G is a path, i.e., the edge set of G is $\{\{i, i+1\} : i = 1, \ldots, d-1\}$.

Let

$$\begin{pmatrix} \sigma(e_1) \\ -\sigma(e_1) \\ \vdots \\ \sigma(e_{d-1}) \\ -\sigma(e_{d-1}) \end{pmatrix}$$

denote the matrix whose row vectors are the vertices of $\mathcal{P}_G^{\pm}$, where $e_i = \{i, i+1\}, i = 1, \ldots, d-1$ are the edges of G. If we add the dth column to the $(d-1)$th column, the $(d-1)$th column to the $(d-2)$th column, ..., and the second column to the first column, then the above matrix is transformed into

$$\begin{pmatrix} 0 & M & & \mathbf{0} \\ \vdots & & \ddots & \\ 0 & \mathbf{0} & & M \end{pmatrix},$$

where M is the 2×1 matrix $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. This implies that $\mathcal{P}_G^{\pm}$ is unimodular equivalent with (4.11). $\square$

Let $(\delta_0, \delta_1, \ldots, \delta_{d-1}) \in \mathbb{Z}^d$ be the δ-vector of (4.11). Then it can be calculated that

$$\delta_i = \binom{d-1}{i}, i = 0, 1, \ldots, d-1.$$

It then follows from the well-known theorem [64] that if G is tree, the real parts of all the roots of $i(\mathcal{P}_G^{\pm}, n)$ are equal to $-\frac{1}{2}$. That is to say, all the roots α of $i(\mathcal{P}_G^{\pm}, n)$ lie on the vertical line $\mathrm{Re}(z) = -\frac{1}{2}$, which is the bisector of the vertical strip $-(d-1) \leq \mathrm{Re}(z) \leq d-2$.

We consider the other two classes of graphs. Let G be a complete bipartite graph of type $(2, d-2)$, i.e., the edges of G are either $\{1, j\}$ or $\{2, j\}$ with $3 \leq j \leq d$. Then the δ-polynomial of $\mathcal{P}_G^{\pm}$ coincides with

$$(1+t)^{d-3}(1 + 2(d-2)t + t^2).$$

By computational experiences, we propose the following:

Conjecture 4.3.8. *Let G be a complete bipartite graph of type $(2, d-2)$. Then the real parts of all the roots of $i(\mathcal{P}_G^{\pm}, n)$ are equal to $-\frac{1}{2}$.*

Let G be a complete graph with d vertices and $\delta(\mathcal{P}_G^{\pm}) = (\delta_0, \delta_1, \ldots, \delta_{d-1})$ be its δ-vector. In [1, Theorem 13], the $\delta(\mathcal{P}_G^{\pm})$ is calculated; that is,

$$\delta_i = \binom{d-1}{i}^2, i = 0, 1, \ldots, d-1.$$

By computational experiences, we also propose the following:

Conjecture 4.3.9. *Let G be a complete graph. Then the real parts of all the roots of $i(\mathcal{P}_G^{\pm}, n)$ are equal to $-\frac{1}{2}$.*

In addition, by computational results, we can say the following:

Proposition 4.3.10. *If $d \leq 6$, then the real parts of all the roots of $i(\mathcal{P}_G^{\pm}, n)$ are equal to $-\frac{1}{2}$ for any graph with d vertices.*

However, it is not true for $d = 7$ or $d = 8$. In fact, there are some counterexamples. The following Figures 4.3 and 4.4 illustrate how the roots are distanced from the line $\mathrm{Re}(z) = -\frac{1}{2}$. (They are computed by `CoCoA` and `Maple`.)

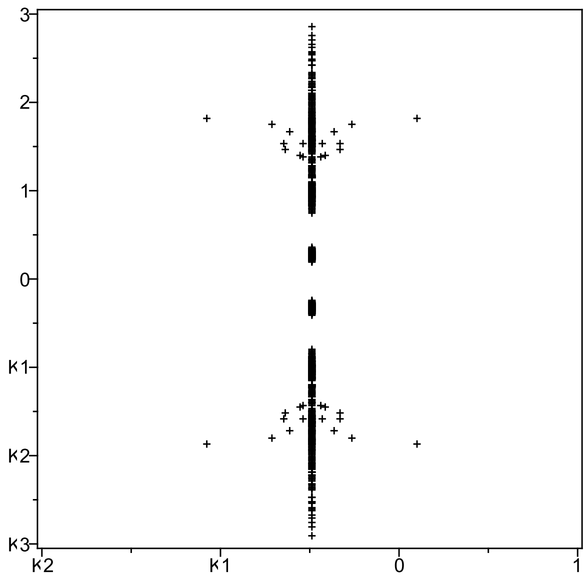


Figure 4.3: $d = 7$

Let G be a cycle of length d. When $d \leq 6$, although the real parts of all the roots of $i(\mathcal{P}_G^{\pm}, n)$ are equal to $-\frac{1}{2}$, there are also some counterexamples when $d \geq 7$. The following Figure 4.5 illustrates the behavior of the roots for $7 \leq d \leq 30$.

However, in the range of graphs which we computed, all the roots z of $i(\mathcal{P}_G^{\pm}, n)$ whose real parts are not equal to $-\frac{1}{2}$ satisfy $-(d-1) \leq \mathrm{Re}(z) \leq d-2$. In more

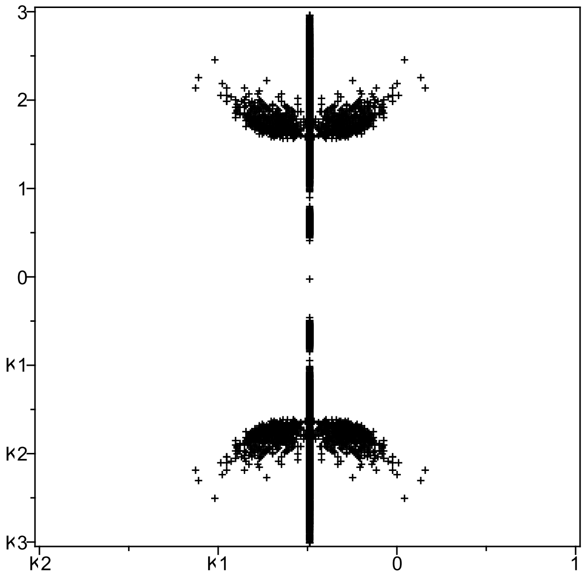


Figure 4.4: $d = 8$

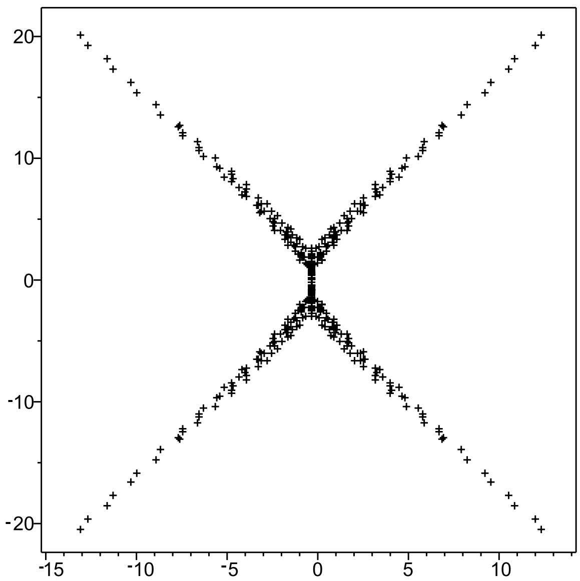


Figure 4.5: all cycles $7 \leq d \leq 30$

detail, they satisfy $-\frac{d-1}{2} \le \mathrm{Re}(z) \le \frac{d-1}{2} - 1$, though we do not know the general case. Then we propose the following:

Conjecture 4.3.11. *All roots α of the Ehrhart polynomials of Gorenstein Fano polytopes of dimension d satisfy $-\frac{d}{2} \le \mathrm{Re}(\alpha) \le \frac{d}{2} - 1$.*

In the table drawn below, in the second row, the number of connected simple graphs with $d(\le 8)$ vertices, up to isomorphism, is written. In the third row, among these, the number of graphs, up to unimodular equivalence, i.e., satisfying the condition in Theorem 4.3.6, is written. In the fourth row, among these, in turn, the number of graphs that are counterexamples, i.e., there is a root of $i(\mathcal{P}_G^{\pm}, n)$ whose real part is not equal to $-\frac{1}{2}$, is written.

	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
Connected graphs	1	2	6	21	112	853	11117
Non equivalent	1	2	5	16	75	560	7772
Counterexamples	0	0	0	0	0	12	1092

4.3.2 An interesting root distribution of Gorenstein Fano polytopes

There is an interesting result on roots of the Ehrhart polynomials of Gorenstein Fano polytopes. In fact,

Theorem 4.3.12 ([36, Theorem 0.1])**.** *Given arbitrary nonnegative integers k and d with $0 \le 2k \le d$, there exists a Gorenstein Fano polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d such that*

(i) *$i(\mathcal{P}, n)$ possesses d distinct roots;*

(ii) *$i(\mathcal{P}, n)$ possesses exactly $2k$ non-real roots and $d - 2k$ real roots;*

(iii) *the real part of each of the non-real roots is equal to $-1/2$;*

(iv) *all of the real roots belong to the open interval $(-1, 0)$.*

Proof. Let $\mathcal{Q} \subset \mathbb{R}^d$ be the convex polytope which is the convex hull of $\mathbf{e}_1, \ldots, \mathbf{e}_{2k}$ and $-(\mathbf{e}_1 + \cdots + \mathbf{e}_{2k})$. Then $\mathcal{Q}$ is an integral convex polytope of dimension $2k$ with $\delta(\mathcal{Q}) = (1, 1, \ldots, 1) \in \mathbb{Z}^{2k+1}$.

In general, when $\mathcal{F} \subset \mathbb{R}^N$ is an integral convex polytope of dimension d, if we define $\mathcal{F}' \subset \mathbb{R}^{N+1}$ by setting the convex hull of $\mathcal{F} \cup \{\mathbf{e}_{N+1}\}$, then one has

$$i(\mathcal{F}', n) = 1 + \sum_{k=1}^{n} i(\mathcal{F}, k).$$

It then follows that

$$\delta(\mathcal{F}') = (\delta(\mathcal{F}), 0) \in \mathbb{Z}^{d+2}.$$

Let $\mathcal{Q}^c \subset \mathbb{R}^d$ be the convex polytope which is the convex hull of $\mathcal{Q} \cup \{\mathbf{e}_{2k+1}, \ldots, \mathbf{e}_d\}$. Then $\delta(\mathcal{Q}^c) = (\delta(\mathcal{Q}), 0, \ldots, 0) \in \mathbb{Z}^{d+1}$. Hence, by (2.3), the convex polytope $(d-2k+1)\mathcal{Q}^c$ possesses a unique integer point $\mathbf{a}$ in its interior. Now, write $\mathcal{P} \subset \mathbb{R}^d$ for the integral convex polytope $(d-2k+1)\mathcal{Q}^c - \mathbf{a}$. Then $\mathcal{P}$ is a Fano polytope.

Since

$$\sum_{n=0}^{\infty} i(\mathcal{Q}^c, n)\lambda^n = \frac{1 + \lambda + \lambda^2 + \cdots + \lambda^{2k}}{(1-\lambda)^{d+1}},$$

one has

$$\begin{aligned}
i(\mathcal{Q}^c, n) &= \sum_{i=n-2k}^{n} \binom{d+i}{d} = \sum_{i=0}^{2k} \binom{d+(n-2k)+i}{d} \\
&= \sum_{i=0}^{2k} \binom{d+n-(2k-i)}{d} = \sum_{i=0}^{2k} \binom{n+d-i}{d} \\
&= \sum_{i=0}^{2k} \left(\binom{n+d-i+1}{d+1} - \binom{n+d-i}{d+1} \right) \\
&= \binom{n+d+1}{d+1} - \binom{n+d-2k}{d+1} \\
&= \frac{1}{(d+1)!} \prod_{i=1}^{d-2k} (n+i) \left(\prod_{i=0}^{2k} (n+d+1-i) - \prod_{i=0}^{2k} (n-i) \right).
\end{aligned}$$

Since

$$i(\mathcal{P}, n) = i((d-2k+1)\mathcal{Q}^c, n) = i(\mathcal{Q}^c, (d-2k+1)n),$$

one has

$$i(\mathcal{P}, n) = \frac{(d-2k+1)^{d+1}}{(d+1)!} \prod_{i=1}^{d-2k} \left(n + \frac{i}{d-2k+1} \right) F(n),$$

where

$$\begin{aligned}
F(n) &= \prod_{i=0}^{2k} \left(n + \frac{d+1-i}{d-2k+1} \right) - \prod_{i=0}^{2k} \left(n - \frac{i}{d-2k+1} \right) \\
&= \prod_{i=0}^{2k} \left(n + \frac{d+1-(2k-i)}{d-2k+1} \right) - \prod_{i=0}^{2k} \left(n - \frac{i}{d-2k+1} \right).
\end{aligned}$$

Thus we obtain the following equalities:

$$\begin{aligned}
\prod_{i=1}^{d-2k} \left(-n - 1 + \frac{i}{d-2k+1} \right) &= (-1)^{d-2k} \prod_{i=1}^{d-2k} \left(n + \frac{d-2k+1-i}{d-2k+1} \right) \\
&= (-1)^{d-2k} \prod_{i=1}^{d-2k} \left(n + \frac{i}{d-2k+1} \right);
\end{aligned}$$

$$
\begin{aligned}
F(-n-1) &= \prod_{i=0}^{2k}\left(-n-1+\frac{d+1-i}{d-2k+1}\right)-\prod_{i=0}^{2k}\left(-n-1-\frac{i}{d-2k+1}\right)\\
&= (-1)^{2k+1}\prod_{i=0}^{2k}\left(n+\frac{d-2k+1-d-1+i}{d-2k+1}\right)\\
&\qquad\qquad -(-1)^{2k+1}\prod_{i=0}^{2k}\left(n+\frac{d-2k+1+i}{d-2k+1}\right)\\
&= (-1)^{2k}\prod_{i=0}^{2k}\left(n+\frac{d-2k+1+i}{d-2k+1}\right)-(-1)^{2k}\prod_{i=0}^{2k}\left(n-\frac{2k-i}{d-2k+1}\right)\\
&= (-1)^{2k}\prod_{i=0}^{2k}\left(n+\frac{d+1-i}{d-2k+1}\right)-(-1)^{2k}\prod_{i=0}^{2k}\left(n-\frac{i}{d-2k+1}\right)\\
&= (-1)^{2k}F(n).
\end{aligned}
$$

It then follows that

$$(-1)^d i(\mathcal{P},-n-1)=i(\mathcal{P},n),$$

which implies that $\mathcal{P}$ is Gorenstein. Hence our work is to show that $\mathcal{P}$ enjoys the required properties (i) – (iv).

Now, since

$$-\frac{d+1-(2k-i)}{d-2k+1}<-\frac{1}{2}<\frac{i}{d-2k+1}$$

and since

$$-\frac{d+1-(2k-i)}{d-2k+1}+\frac{i}{d-2k+1}=-1,$$

Lemma 4.3.13 below guarantees that $F(n)$ possesses $2k$ distinct roots and each of them is a non-real root with $-1/2$ its real part. Finally, the real roots of $i(\mathcal{P},n)$ are

$$-\frac{i}{d-2k+1},\qquad 1\le i\le d-2k.$$

Each of those roots belongs to the open interval $(-1,0)$, as desired.

□

Lemma 4.3.13. *Let $\alpha_0,\alpha_1,\ldots,\alpha_{2k}$ and $\beta_0,\beta_1,\ldots,\beta_{2k}$ be rational numbers satisfying $\alpha_i<-1/2<\beta_i$ and $\alpha_i+\beta_i=-1$ for all i. Let*

$$f(x)=\prod_{i=0}^{2k}(x-\alpha_i)-\prod_{i=0}^{2k}(x-\beta_i)$$

be a polynomial in x of degree $2k$. Then $f(x)$ possesses $2k$ distinct roots and each of them is a non-real root with $-1/2$ its real part.

Proof. We employ a basis technique appearing in [64]. Let $a \in \mathbb{C}$ with $\mathrm{Re}(a) > -1/2$. Since $\alpha_i < \beta_i$ and $\alpha_i + \beta_i = -1$, it follows that

$$\begin{aligned} |a-\alpha_i|^2 - |a-\beta_i|^2 &= (\mathrm{Re}(a)-\alpha_i)^2 - (\mathrm{Re}(a)-\beta_i)^2 \\ &= (2\,\mathrm{Re}(a)-\alpha_i-\beta_i)(\beta_i-\alpha_i) \\ &= (2\,\mathrm{Re}(a)+1)(\beta_i-\alpha_i) \\ &> 0. \end{aligned}$$

Hence we have $|a-\alpha_i| > |a-\beta_i|$. Thus $\prod_{i=0}^{2k} |a-\alpha_i| > \prod_{i=0}^{2k} |a-\beta_i|$. Hence $f(a) \neq 0$. Similarly, if $a \in \mathbb{C}$ with $\mathrm{Re}(a) < -1/2$, then $|a-\alpha_i| < |a-\beta_i|$ for all i. Thus $\prod_{i=0}^{2k} |a-\alpha_i| < \prod_{i=0}^{2k} |a-\beta_i|$. Hence $f(a) \neq 0$. Consequently, all roots $a \in \mathbb{C}$ of $f(x)$ satisfy $\mathrm{Re}(a) = -1/2$.

Substituting $y = x + 1/2$ and $\gamma_i = \beta_i + 1/2$ in $f(x)$, it follows that each of the roots $a \in \mathbb{C}$ of the polynomial

$$g(y) = \prod_{i=0}^{2k}(\gamma_i + y) + \prod_{i=0}^{2k}(\gamma_i - y)$$

in y of degree $2k$ satisfies $\mathrm{Re}(a) = 0$. Since $\gamma_i > 0$, one has $g(0) \neq 0$. Hence $g(y)$ possesses no real root. Thus all roots of $f(x)$ are non-real roots.

What we must prove is that $g(y)$ possesses $2k$ distinct roots. Let $b \in \mathbb{R}$ and $\theta_i(b)$ the argument of $\gamma_i + b\sqrt{-1}$, where $-\pi/2 < \theta_i(b) < \pi/2$. Then $b\sqrt{-1}$ is a root of $g(y)$ if and only if

$$\prod_{i=0}^{2k} e^{\sqrt{-1}\,\theta_i(b)} = -\prod_{i=0}^{2k} e^{-\sqrt{-1}\,\theta_i(b)}.$$

In other words, $b\sqrt{-1}$ is a root of $g(y)$ if and only if

$$\prod_{i=0}^{2k} e^{2\sqrt{-1}\,\theta_i(b)} = -1,$$

which is equivalent to saying that

$$\sum_{i=0}^{2k} \theta_i(b) \equiv \frac{\pi}{2} \pmod{\pi}.$$

Now, we study the function $h(y) = \sum_{i=0}^{2k} \theta_i(y)$ with $y \in \mathbb{R}$. Since $\gamma_i > 0$, it follows that $h(y)$ is strictly increasing with

$$\lim_{y\to\infty} h(y) = k\pi + \pi/2, \quad \lim_{y\to-\infty} h(y) = -(k+1)\pi + \pi/2.$$

Hence the equation

$$h(y) \equiv \frac{\pi}{2} \pmod{\pi}$$

possesses $2k$ distinct real roots, as desired. $\square$

Here is an example of Theorem 4.3.12.

Example 4.3.14. Let $k = 1$ and $d = 4$. Then there exists a 4-dimensional Gorenstein Fano polytope $\mathcal{P} \subset \mathbb{R}^4$ such that $i(\mathcal{P}, n)$ satisfies the properties (i)–(iv) of Theorem 4.3.12. In fact, we define $\mathcal{Q}^c$ by setting the convex hull of

$$\{(1,0,0,0),(0,1,0,0),(-1,-1,0,0),(0,0,1,0),(0,0,0,1)\}.$$

Then $3\mathcal{Q}^c$ contains a unique integer point $(0,0,1,1)$ in its interior. Thus $\mathcal{P} := 3\mathcal{Q}^c - (0,0,1,1)$ is a Gorenstein Fano polytope, which is the convex hull of

$$\{(3,0,-1,-1),(0,3,-1,-1),(-3,-3,-1,-1),(0,0,2,-1),(0,0,-1,2)\}.$$

It can be computed easily that the Ehrhart polynomial of $\mathcal{P}$ is equal to

$$\frac{81}{8}n^4 + \frac{81}{4}n^3 + \frac{135}{8}n^2 + \frac{27}{4}n + 1$$

and its roots are

$$-\frac{1}{3}, -\frac{2}{3}, -\frac{1}{2} + \frac{\sqrt{-7}}{6} \text{ and } -\frac{1}{2} - \frac{\sqrt{-7}}{6}.$$

Remark 4.3.15. (a) It is disproved in [23] that all of the roots α of the Hilbert polynomial of any Fano variety satisfy $-1 < \mathrm{Re}(\alpha) < 0$, so-called *the canonical strip hypothesis*, which is stated in [19]. On Theorem 4.3.12, however, all of the roots of Ehrhart polynomials of our Gorenstein Fano polytopes satisfy this condition. In more detail, they satisfy *the narrowed canonical strip hypothesis*, which is the condition $-1 + 1/(d+1) \leq \mathrm{Re}(\alpha) \leq -1/(d+1)$. Moreover, if we set $2k = d$ when d is even or $2k = d-1$ when d is odd, then they also satisfy *the canonical line hypothesis*, which is the condition $\mathrm{Re}(\alpha) = -1/2$.
(b) It should be considered that we speculate the connections of the Ehrhart polynomials of our Gorenstein Fano polytopes with L-functions. Let $i(\mathcal{P}, s)$ be the Ehrhart polynomial of our Gorenstein Fano polytope $\mathcal{P}$ with $2k = d$ when d is even or with $2k = d - 1$ when d is odd. Then we set $z(s) = i(\mathcal{P}, -s)$. Then the function equation

$$z(1-s) = (-1)^d z(s)$$

holds and all of its roots α satisfy $\mathrm{Re}(\alpha) = 1/2$, which is, of course, the Reimann zeta function.

4.4 Roots of SSNN polynomials

On the conjecture on roots of the Ehrhart polynomials of Gorenstein Fano polytopes, which is Conjecture 4.3.11, there is a partial answer. In this section, we will show this.

On many results of the studies on roots of Ehrhart polynomials, Stanley's nonnegativity of δ-vectors [68] plays a crucial role. (For example, see [7, 8, 9].) Derived from the definition [9, Definition 1.2], we define the following polynomial.

Definition 4.4.1. Given a sequence of nonnegative real numbers $(\delta_0, \delta_1, \ldots, \delta_d) \in \mathbb{R}_{\geq 0}^{d+1}$ which satisfies these numbers are symmetric, i.e., $\delta_i = \delta_{d-i}$ for $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$, we define the polynomial

$$f(n) = \sum_{i=0}^{d} \delta_i \binom{n+d-i}{d}$$

in n of degree d. We call $f(n)$ a *symmetric Stanley's nonnegative* or *SSNN* polynomial of degree d.

We remark that this class of polynomials is mentioned in [62, Remark 2.2], although it is not pursued deeply there.

In this section, we study roots of SSNN polynomials. We consider the following question as a generalized form of Conjecture 4.3.11.

Question 4.4.2. Do all roots α of an SSNN polynomial of degree d satisfy

$$-\frac{d}{2} \leq \mathrm{Re}(\alpha) \leq \frac{d}{2} - 1 \ ?$$

This is true when the roots are real numbers or when $d \leq 5$. In fact,

Theorem 4.4.3 ([41, Theorem 0.5])**.** *Let $f(n)$ be an SSNN polynomial of degree d and $\alpha \in \mathbb{C}$ an arbitrary root of $f(n)$.*
(a) *If $\alpha \in \mathbb{R}$, then α satisfies $-\frac{d}{2} \leq \alpha \leq \frac{d}{2} - 1$, more strictly,*

$$-\left\lfloor \frac{d}{2} \right\rfloor \leq \alpha \leq \left\lfloor \frac{d}{2} \right\rfloor - 1.$$

(b) *If $d \leq 5$, then α satisfies $-\frac{d}{2} \leq \mathrm{Re}(\alpha) \leq \frac{d}{2} - 1$, more strictly,*

$$-\left\lfloor \frac{d}{2} \right\rfloor \leq \mathrm{Re}(\alpha) \leq \left\lfloor \frac{d}{2} \right\rfloor - 1.$$

4.4.1 A proof of Theorem 4.4.3

This subsection is devoted to giving a proof of Theorem 4.4.3.

Let $f(n) = \sum_{i=0}^{d} \delta_i \binom{n+d-i}{d}$ be an SSNN polynomial of degree d. First of all, we verify that $f(n)$ satisfies

$$f(n) = (-1)^d f(-n-1). \tag{4.12}$$

Let

$$N_i(n) = \prod_{j=0}^{d-1}(n+d-i-j) + \prod_{j=0}^{d-1}(n+i-j)$$

for $0 \le i \le \lfloor \frac{d}{2} \rfloor - 1$ and

$$N_{\lfloor \frac{d}{2} \rfloor}(n) = \begin{cases} \prod_{j=0}^{d-1}(n + \frac{d}{2} - j), & \text{if } d \text{ is even,} \\ \prod_{j=0}^{d-1}(n + \frac{d+1}{2} - j) + \prod_{j=1}^{d}(n + \frac{d-1}{2} - j), & \text{if } d \text{ is odd.} \end{cases}$$

It then follows that

$$f(n) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \frac{\delta_i N_i(n)}{d!}.$$

Since one has

$$\begin{aligned} (-1)^d N_i(-n-1) &= (-1)^d \prod_{j=0}^{d-1}(-n-1+d-i-j) + (-1)^d \prod_{j=0}^{d-1}(-n-1+i-j) \\ &= \prod_{j=0}^{d-1}(n+1-d+i+j) + \prod_{j=0}^{d-1}(n+1-i+j) \\ &= \prod_{j=0}^{d-1}(n+i-j) + \prod_{j=0}^{d-1}(n+d-i-j) = N_i(n) \end{aligned}$$

for $0 \le i \le \lfloor \frac{d}{2} \rfloor - 1$ and $(-1)^d N_{\lfloor \frac{d}{2} \rfloor}(-n-1) = N_{\lfloor \frac{d}{2} \rfloor}(n)$, we obtain $f(n) = (-1)^d f(-n-1)$.

We prove Theorem 4.4.3 (a) by using the above notations.

Proof of Theorem 4.4.3 (a). Let

$$g(n) = d! f\left(n - \frac{1}{2}\right) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \delta_i N_i \left(n - \frac{1}{2}\right).$$

Then, it suffices to prove that all the real roots of $g(n)$ are contained in the closed interval $[-\lfloor \frac{d}{2} \rfloor + \frac{1}{2}, \lfloor \frac{d}{2} \rfloor - \frac{1}{2}]$. It follows from (4.12) that

$$g(n) = (-1)^d g(-n). \tag{4.13}$$

For $N_i\left(n - \frac{1}{2}\right), 0 \le i \le \lfloor \frac{d}{2} \rfloor$, we have the following:

$$\begin{aligned} N_i\left(n - \frac{1}{2}\right) &= \prod_{j=0}^{d-1}\left(n + d - \frac{1}{2} - i - j\right) + \prod_{j=0}^{d-1}\left(n - \frac{1}{2} + i - j\right) \\ &= \prod_{l=0}^{2i-1}\left(n - \frac{1}{2} + i - l\right) M_i(n), \end{aligned}$$

where

$$M_i(n) = \prod_{j=0}^{d-2i-1}\left(n + \frac{1}{2} + i + j\right) + \prod_{j=0}^{d-2i-1}\left(n - \frac{1}{2} - i - j\right),$$

and

$$N_{\lfloor \frac{d}{2} \rfloor}\left(n-\frac{1}{2}\right)=\prod_{j=0}^{d-1}\left(n+\frac{d}{2}-\frac{1}{2}-j\right)$$

when d is even. Let α be a real number with $\alpha > \lfloor \frac{d}{2} \rfloor - \frac{1}{2}$. On the coefficients of $n^j, 0 \le j \le d-2i-1$, in $M_i(n)$, it is obvious that those are all nonnegative. Thus, we have $M_i(\alpha) > 0$ since $\alpha > 0$. In addition, one has $\prod_{l=0}^{2i-1}\left(\alpha - \left(\frac{1}{2} - i + l\right)\right) > 0$ since $0 \le l \le 2i-1$ and $0 \le i \le \lfloor \frac{d}{2} \rfloor$. Hence, α cannot be a root of $g(n)$ from the nonnegativity of $\delta_0, \delta_1, \dots, \delta_{\lfloor \frac{d}{2} \rfloor}$. Moreover, by virtue of (4.13), for a real number β with $\beta < -\left\lfloor \frac{d}{2} \right\rfloor + \frac{1}{2}$, β cannot be a root of $g(n)$, as desired. □

In the rest of this subsection, we prove Theorem 4.4.3 (b).

The case where $d = 2$ and 3.

- An SSNN polynomial of degree 2 has two roots. If both of them are real numbers, then the assertion holds from Theorem 4.4.3 (a). If both of them are non-real numbers, then it follows from (4.12) that each of their real parts is $-\frac{1}{2}$.
- An SSNN polynomial of degree 3 has three roots and one is $-\frac{1}{2}$. On the other roots, the same discussion as the case where $d = 2$ can be done.

The case where $d = 4$.

Let $f(n) = \frac{a}{4!}N_0(n) + \frac{b}{4!}N_1(n) + \frac{c}{4!}N_2(n)$, where $a, b, c \in \mathbb{R}_{\ge 0}$. Then $f(n)$ has four roots and the possible cases are as follows:

(i) those four roots are all real numbers;

(ii) two of them are real numbers and the others are non-real numbers;

(iii) those four roots are all non-real numbers.

We do not have to discuss the cases (i) and (ii) by virtue of Theorem 4.4.3 (a). Thus, we consider the case (iii), i.e., we assume that $f(n)$ has four non-real roots. Moreover, we may also assume that $a \neq 0$ since both 0 and -1 are their roots when $a = 0$. In addition, we may set $a = 1$ since the roots of $f(n)$ exactly coincide with those of $\frac{f(n)}{a}$.

We define

$$g(n) = 4!f\left(n-\frac{1}{2}\right) = (2+2b+c)n^4 + \left(43+7b-\frac{5}{2}c\right)n^2 + \frac{105}{8} - \frac{15}{8}b + \frac{9}{16}c.$$

Our work is to show that if the roots α of $g(n)$ are all non-real numbers, then α satisfies $-\frac{3}{2} \le \mathrm{Re}(\alpha) \le \frac{3}{2}$. Let

$$G(X) = (2+2b+c)X^2 + \left(43+7b-\frac{5}{2}c\right)X + \frac{105}{8} - \frac{15}{8}b + \frac{9}{16}c.$$

We consider the roots of $G(X)$. Let α and β (resp. $D(G(X))$) be the roots (resp. the discriminant) of $G(X)$. By our assumption, we may set $D(G(X)) < 0$. In fact, when $D(G(X)) \geq 0$, i.e., both α and β are real numbers, then the roots of $g(n)$ are $\pm\sqrt{\alpha}, \pm\sqrt{\beta}$. Even if α (resp. β) is positive or negative, $\pm\sqrt{\alpha}$ (resp. $\pm\sqrt{\beta}$) are either real numbers or pure imaginary numbers.

Let, say, $\alpha = re^{\theta\sqrt{-1}}$ with $r > 0$ and $0 < \theta < \pi$. Then $\beta = \bar{\alpha} = re^{-\theta\sqrt{-1}}$. Thus the roots of $g(n)$ are $\sqrt{r}e^{\pm\frac{\theta}{2}\sqrt{-1}}$ and $\sqrt{r}e^{\pm(\pi-\frac{\theta}{2})\sqrt{-1}}$. Hence, it is enough to show that

$$0 < \mathrm{Re}(\sqrt{r}e^{\frac{\theta}{2}\sqrt{-1}}) = \sqrt{r}\cos\frac{\theta}{2} = \sqrt{r}\sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{r+r\cos\theta}{2}} \leq \frac{3}{2}.$$

Since $G(X) = (2+2b+c)(X-\alpha)(X-\beta)$, we have

$$r = \frac{1}{4}\sqrt{\frac{210-30b+9c}{2+2b+c}} \quad \text{and} \quad r\cos\theta = -\frac{1}{4}\cdot\frac{86+14b-5c}{2+2b+c}.$$

By the way, one has

$$\begin{aligned}
D(G(X)) &= \left(43+7b-\frac{5}{2}c\right)^2 - 4(2+2b+c)\left(\frac{105}{8}-\frac{15}{8}b+\frac{9}{16}c\right) \\
&= 4(c^2-4(2b+17)c+4(4b^2+32b+109)).
\end{aligned}$$

Let $h(c) = \frac{D(G(X))}{4}$. Then one has $h(c) < 0$ and the range of c satisfying $h(c) < 0$ is

$$2(2b+17) - 12\sqrt{b+5} < c < 2(2b+17) + 12\sqrt{b+5}.$$

When b and c satisfy this, we have the following:

$$\begin{aligned}
4(r+r\cos\theta) &= \sqrt{\frac{210-30b+9c}{2+2b+c}} - \frac{86+14b-5c}{2+2b+c} \\
&= \sqrt{9-48\cdot\frac{b-4}{2+2b+c}} - 24\cdot\frac{b+4}{2+2b+c} + 5 \\
&< \sqrt{9-48\cdot\frac{b-4}{2+2b+2(2b+17)+12\sqrt{b+5}}} \\
&\qquad\qquad -24\cdot\frac{b+4}{2+2b+2(2b+17)+12\sqrt{b+5}} + 5 \\
&= \sqrt{9-8\cdot\frac{b-4}{b+6+2\sqrt{b+5}}} - 4\cdot\frac{b+4}{b+6+2\sqrt{b+5}} + 5 \ \ (=: H(b)) \\
&\leq \sqrt{9-8\cdot\frac{-4}{6+2\sqrt{5}}} - 4\cdot\frac{4}{6+2\sqrt{5}} + 5, \ \left(\text{since } \frac{dH(b)}{db} < 0 \text{ when } b \geq 0,\right) \\
&= 4\sqrt{5} - 2.
\end{aligned}$$

Therefore, one has

$$\sqrt{\frac{r+r\cos\theta}{2}} < \sqrt{\frac{4\sqrt{5}-2}{8}} = \frac{\sqrt{2\sqrt{5}-1}}{2} < \frac{3}{2}, \tag{4.14}$$

as required.

The case where $d = 5$. Finally, we consider the case where $d = 5$.

Let $f(n) = \frac{a}{5!}N_0(n) + \frac{b}{5!}N_1(n) + \frac{c}{5!}N_2(n)$. When $d = 5$, $f(n)$ has five roots and one of them is $-\frac{1}{2}$. For the other roots, the possible cases are as follows:

(i) the other four roots are all real numbers;

(ii) two of them are real numbers and the rests are non-real numbers;

(iii) those four roots are all non-real numbers.

Similarly to the case where $d = 4$, we discuss only the case (iii) and assume that $a = 1$.

We define $g(n)$ by setting

$$\begin{aligned} g(n) &= 5!f\left(n-\frac{1}{2}\right) \\ &= n\left(2(1+b+c)n^4 + 5(23+7b-c)n^2 + \frac{1689}{8} - \frac{71}{8}b + \frac{9}{8}c\right). \end{aligned}$$

Let

$$\tilde{g}(n) = 2(1+b+c)n^4 + 5(23+7b-c)n^2 + \frac{1689}{8} - \frac{71}{8}b + \frac{9}{8}c.$$

Our work is to show that if the roots α of $\tilde{g}(n)$ are all non-real numbers, then α satisfies $-\frac{3}{2} \leq \mathrm{Re}(\alpha) \leq \frac{3}{2}$. Let $G(X)$ be the polynomial replacing n^2 of $\tilde{g}(n)$ with X, that is,

$$G(X) = 2(1+b+c)X^2 + 5(23+7b-c)X + \frac{1689}{8} - \frac{71}{8}b + \frac{9}{8}c.$$

We consider the roots of $G(X)$. Similarly to the case where $d = 4$, we assume that $D(G(X)) < 0$ and prove that

$$\sqrt{\frac{r+r\cos\theta}{2}} \leq \frac{3}{2},$$

where $\alpha = re^{\theta\sqrt{-1}} (r > 0, 0 < \theta < \pi)$ is one of the roots of $G(X)$.

Since $G(X) = 2(1+b+c)(X-\alpha)(X-\beta)$, where $\beta = \bar{\alpha} = re^{-\theta\sqrt{-1}}$, we have

$$r = \sqrt{\frac{\frac{1689}{8} - \frac{71}{8}b + \frac{9}{8}c}{2(1+b+c)}} = \frac{1}{4}\sqrt{\frac{1689 - 71b + 9c}{1+b+c}}$$

and

$$r\cos\theta = \frac{-5(23+7b-c)}{4(1+b+c)} = \frac{1}{4}\cdot\frac{-115-35b+5c}{1+b+c}.$$

By the way, one has

$$\begin{aligned} D(G(X)) &= 25(23+7b-c)^2 - 8(1+b+c)\left(\frac{1689}{8} - \frac{71}{8}b + \frac{9}{8}c\right) \\ &= 16(81b^2 - 6(3c-67)b + c^2 - 178c + 721). \end{aligned}$$

Let $h(b) = \frac{D(G(X))}{16}$. Then $h(b) < 0$ by our assumption. The range of b with $h(b) < 0$ is as follows:

$$\frac{3c-67-\sqrt{D(h(b))}}{27} < b < \frac{3c-67+\sqrt{D(h(b))}}{27},$$

where $4\cdot D(h(b)) = 4\cdot 60^2(3c-5)$ is the discriminant of $h(b)$. (In particular, it must be $c \geq \frac{5}{3}$.) Moreover, since $b \geq 0$, it must be $\frac{3c-67+20\sqrt{3c-5}}{27} > 0$. Thus, $c > 89 - 60\sqrt{2}(> \frac{5}{3})$. Hence, the condition $D(G(X)) < 0$ is equivalent to the followings:

$$c > 89 - 60\sqrt{2}$$

and

$$\begin{cases} 0 \leq b < \frac{3c-67+20\sqrt{3c-5}}{27}, & \text{when } \ 89-60\sqrt{2} < c \leq 89+60\sqrt{2}, \\ \frac{3c-67-20\sqrt{3c-5}}{27} < b < \frac{3c-67+20\sqrt{3c-5}}{27}, & \text{when } \ c > 89+60\sqrt{2}. \end{cases} \tag{4.15}$$

When b and c satisfy the first condition of (4.15), we have

$$\begin{aligned} &\sqrt{\frac{1689-71b+9c}{1+b+c}} + \frac{-115-35b+5c}{1+b+c} \\ &\quad= \sqrt{\frac{-71(1+b+c)+80c+1760}{1+b+c}} + \frac{-35(1+b+c)+40c-80}{1+b+c} \\ &\quad= \sqrt{-71+80\cdot\frac{c+22}{1+b+c}} + 40\cdot\frac{c-2}{1+b+c} - 35 \\ &\quad\leq \sqrt{-71+80\cdot\frac{c+22}{c+1}} + 40\cdot\frac{c-2}{c+1} - 35 (=: H(c)) \\ &\quad\leq \sqrt{-71+80\cdot\frac{41+22}{41+1}} + 40\cdot\frac{41-2}{41+1} - 35, \\ &\left(\text{since } \frac{dH(c)}{dc} > 0 \text{ when } c < 41 \text{ and } \frac{dH(c)}{dc} < 0 \text{ when } c > 41,\right) \\ &\quad= \sqrt{-71+120} + \frac{260}{7} - 35 = \frac{64}{7} < 18. \end{aligned}$$

Thus, one has

$$\sqrt{\frac{r + r\cos\theta}{2}} < \sqrt{\frac{18}{4} \cdot \frac{1}{2}} = \frac{3}{2}.$$

On the other hand, when b and c satisfy the second condition of (4.15), we have

$$\begin{aligned}
\frac{1689 - 71b + 9c}{1 + b + c} &= -71 + 80 \cdot \frac{c + 22}{1 + b + c} \\
&< -71 + 80 \cdot \frac{c + 22}{\frac{3c - 67 - 20\sqrt{3c-5}}{27} + c + 1} \\
&= -71 + 8 \cdot \frac{27(c + 22)}{3c - 4 - 2\sqrt{3c - 5}} \\
&< -71 + 8 \cdot \frac{27(89 + 60\sqrt{2} + 22)}{3(89 + 60\sqrt{2}) - 4 - 2\sqrt{3(89 + 60\sqrt{2}) - 5}} \\
&\quad \left(\text{since } \frac{dH(c)}{dc} < 0 \text{ when } c \geq 89 + 60\sqrt{2},\right) \\
&< 81.
\end{aligned}$$

Therefore, one has

$$\sqrt{\frac{r + r\cos\theta}{2}} \leq \sqrt{r} = \frac{1}{2}\left(\frac{1689 - 71b + 9c}{1 + b + c}\right)^{\frac{1}{4}} < \frac{3}{2}, \tag{4.16}$$

as required.

4.4.2 The case where $d \geq 6$

In this subsection, in order to observe that Theorem 4.4.3 seems to be also true when $d = 6$ and 7, we make computational experiments. Moreover, we present an example which shows that Theorem 4.4.3 is no longer true when $d = 8$. In addition, we suggest a possible counterexample of Conjecture 4.3.11 with $d = 10$, while such example is already known in [60].

Our method how to make experiments, say, $d = 6$, is as follows. We produce 4 nonnegative real numbers a, b, c, d at random, construct the polynomial

$$a\left(\binom{n+6}{6} + \binom{n}{6}\right) + b\left(\binom{n+5}{6} + \binom{n+1}{6}\right) + c\left(\binom{n+4}{6} + \binom{n+2}{6}\right) + d\binom{n+3}{6},$$

compute its roots and plot them on the complex plane. Figure 4.6 drawn below shows the root distributions of a large sample (approximately 20,000) of SSNN polynomials of degree 6. Similarly, in Figure 4.7, we see the root distributions of a large sample (approximately 20,000) of SSNN polynomials

$$a\left(\binom{n+7}{7}+\binom{n}{7}\right)+b\left(\binom{n+6}{7}+\binom{n+1}{7}\right)+ c\left(\binom{n+5}{7}+\binom{n+2}{7}\right)+d\left(\binom{n+4}{7}+\binom{n+3}{7}\right)$$

with random nonnegative real numbers a, b, c, d. (Those are computed by `Maple`.)

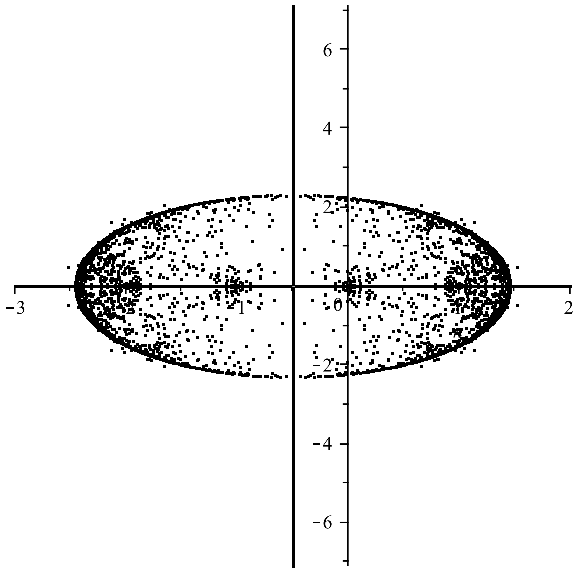


Figure 4.6: $d = 6$

Remark 4.4.4. There is an SSNN polynomial of degree 8 one of whose root α does not satisfy $-4 \leq \mathrm{Re}(\alpha) \leq 3$. In fact, if we set $(\delta_0, \delta_1, \ldots, \delta_8) = (1, 0, 0, 0, 14, 0, 0, 0, 1)$ and $f(n) = \sum_{i=0}^{8} \delta_i \binom{n+8-i}{8}$, then the roots of $f(n)$ are approximately

$$\begin{aligned}&-0.5 \pm 0.44480014\sqrt{-1}, \quad -0.5 \pm 1.78738687\sqrt{-1},\\ &3.00099518 \pm 5.29723208\sqrt{-1} \ \text{ and } \ -4.00099518 \pm 5.29723208\sqrt{-1},\end{aligned}$$

while $f(n)$ cannot be the Ehrhart polynomial of some Gorenstein Fano polytope of dimension 8 since $\delta_1 < \delta_8$. When $d = 10$, however, there are some possible candidates of counterexamples of Conjecture 4.3.11. For example, let $(\delta_0, \delta_1, \ldots, \delta_{10}) = (1, 1, 1, 1, 1, 23, 1, 1, 1, 1, 1)$ and $f(n) = \sum_{i=0}^{10} \delta_i \binom{n+10-i}{10}$. Then one of approximate roots of $f(n)$ is

$$4.02470021 + 8.22732653\sqrt{-1}.$$

On the other hand, in a recent paper [60], a certain counterexample of Conjecture 4.3.11 is provided. There exists a Gorenstein Fano polytope of dimension 34 whose Ehrhart polynomial has a root α which violates $-17 \leq \mathrm{Re}(\alpha) \leq 16$.

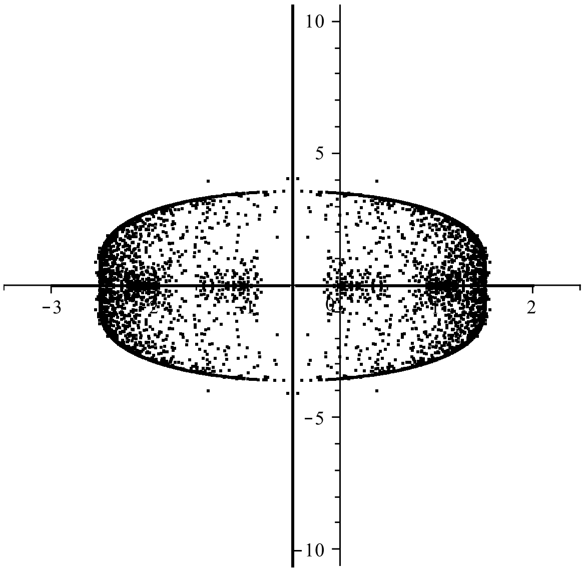


Figure 4.7: $d = 7$

4.4.3 Some comparisons of SSNN polynomials with Ehrhart polynomials of Gorenstein Fano polytopes

In this subsection, we discuss some differences of root distributions between the Ehrhart polynomials of Gorenstein Fano polytopes and SSNN polynomials when $d \leq 4$. We determine the complete range of the roots of SSNN polynomials and Ehrhart polynomials of Gorenstein Fano polytopes when $d = 2$ and 3 (Proposition 4.4.5 and 4.4.6). Moreover, we see in Theorem 4.4.8 that the real numbers in the closed interval $[-\frac{d}{2}, \frac{d}{2} - 1]$ are all the real roots of SSNN polynomials of degree d.

Proposition 4.4.5. (a) *The set of the roots of the Ehrhart polynomials of Gorenstein Fano polytopes of dimension 2 coincides with*

$$\left\{-\frac{2}{3}, -\frac{1}{2}, -\frac{1}{3}\right\} \cup \left\{-\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{6-i}{i+2}}\sqrt{-1} \in \mathbb{C} : i = 1, 2, \ldots, 5\right\}.$$

(b) *The set of the roots of SSNN polynomials of degree 2 coincides with*

$$[-1, 0] \cup \left\{\alpha \in \mathbb{C} : \mathrm{Re}(\alpha) = -\frac{1}{2},\ 0 < |\,\mathrm{Im}(\alpha)| \leq \frac{\sqrt{3}}{2}\right\}.$$

Proof. Let $f(n) = \binom{n+2}{2} + b\binom{n+1}{2} + \binom{n}{2}$. Then $2f(n) = (b+2)n^2 + (b+2)n + 2$. Thus its roots are

$$n = \frac{-(b+2) \pm \sqrt{(b+2)(b-6)}}{2(b+2)} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{b-6}{b+2}}.$$

It is well known that $(1, b, 1) \in \mathbb{Z}^3$ is the δ-vector of some Gorenstein Fano polytope of dimension 2 if and only if $b \in \{1, 2, \ldots, 7\}$. Hence we obtain the assertion (a).

On the other hand, when $b \in \mathbb{R}_{\geq 0}$, the set of the roots of $f(n)$ coincides with

$$(-1, 0) \cup \left\{ \alpha \in \mathbb{C} : \mathrm{Re}(\alpha) = -\frac{1}{2},\ 0 < |\,\mathrm{Im}(\alpha)| \leq \frac{\sqrt{3}}{2} \right\}.$$

In fact, the function $\frac{1}{2}\sqrt{\frac{b-6}{b+2}}$ is monotone increasing and $\lim_{b\to+\infty} \frac{1}{2}\sqrt{\frac{b-6}{b+2}} = \frac{1}{2}$ when $b \geq 6$, and $\frac{1}{2}\sqrt{\frac{6-b}{b+2}}$ is monotone decreasing when $0 \leq b < 6$. Moreover, -1 and 0 are the roots of $\binom{n+1}{2}$. Therefore, the assertion (b) holds, as desired. □

Proposition 4.4.6. (a) *The set of the roots of the Ehrhart polynomials of Gorenstein Fano polytopes of dimension 3 coincides with*

$$\left\{ -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{i-23}{i+1}} \in \mathbb{R} : i = 23, 24, \ldots, 32, 35 \right\} \cup \\ \left\{ -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{23-i}{i+1}}\sqrt{-1} \in \mathbb{C} : i = 1, 2, \ldots, 22 \right\}.$$

(b) *The set of the roots of SSNN polynomials of degree 3 coincides with*

$$[-1, 0] \cup \left\{ \alpha \in \mathbb{C} : \mathrm{Re}(\alpha) = -\frac{1}{2},\ 0 < |\,\mathrm{Im}(\alpha)| \leq \frac{\sqrt{23}}{2} \right\}.$$

Proof. Let $f(n) = \binom{n+3}{3} + b\binom{n+2}{3} + b\binom{n+1}{3} + \binom{n}{3}$. Then $3!f(n) = (2n+1)((b+1)n^2 + (b+1)n + 6)$. Thus its roots are $n = -\frac{1}{2}$ and

$$n = \frac{-(b+1) \pm \sqrt{(b+1)(b-23)}}{2(b+1)} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{b-23}{b+1}}.$$

By the complete classification of Kreuzer and Skarke [46], we know that $(1, b, b, 1) \in \mathbb{Z}^4$ is the δ-vector of some Gorenstein Fano polytope of dimension 3 if and only if $b \in \{1, 2, \ldots, 35\} \setminus \{33, 34\}$. (See also `http://tph16.tuwien.ac.at/kreuzer/CY/`.)

The rest parts are similar to Proposition 4.4.5. □

By (4.14) and (4.16) together with the norm bound [8], we also obtain the following

Proposition 4.4.7. (a) *The roots of SSNN polynomials of degree 4 are contained in*

$$[-2, 1] \cup \left\{ \alpha \in \mathbb{C} \setminus \mathbb{R} : \left| \alpha - \frac{1}{2} \right| \leq 14,\ \left| \mathrm{Re}(\alpha) + \frac{1}{2} \right| \leq \frac{\sqrt{2\sqrt{5}-1}}{2} \right\}.$$

(b) *The roots of SSNN polynomials of degree 5 are contained in*

$$[-2,1] \cup \left\{ \alpha \in \mathbb{C} \setminus \mathbb{R} : \left| \alpha - \frac{1}{2} \right| \le \frac{45}{2}, \ \left| \mathrm{Re}(\alpha) + \frac{1}{2} \right| \le \frac{\sqrt{8\sqrt{2}-7}}{2} \right\}.$$

The complete classification of Gorenstein Fano polytopes of dimension 4 also exists [47], but the number of them are too enormous (473,800,776). The following Figure 4.8 (resp. Figure 4.9) shows the root distribution of a large sample (approximately 20,000) of Ehrhart polynomials of Gorenstein Fano polytopes of dimension 4 (resp. SSNN polynomials of degree 4). From the following figures, we notice that the above Proposition 4.4.7 is not so sharp.

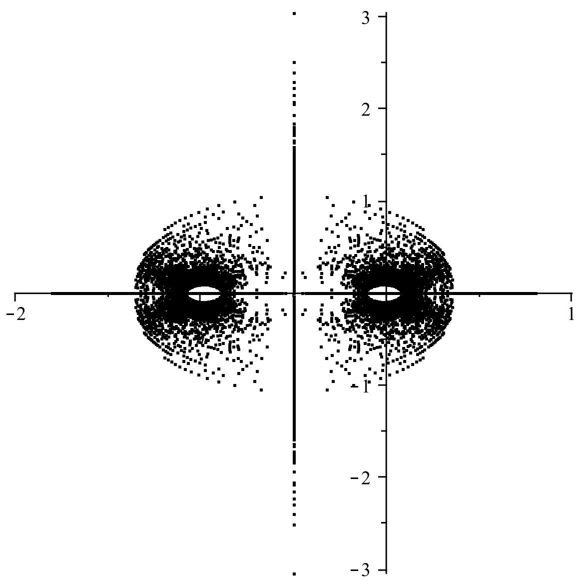


Figure 4.8: Ehrhart polynomials of Gorenstein Fano polytopes of dimension 4

Finally, we prove the following. All the real numbers in $[-\frac{d}{2}, \frac{d}{2}-1]$ can be realized as roots of SSNN polynomials of degree d.

Theorem 4.4.8 ([41, Theorem 3.4])**.** *The set of the real roots of SSNN polynomials of degree d coincides with the closed interval $[-\lfloor \frac{d}{2} \rfloor, \lfloor \frac{d}{2} \rfloor - 1]$.*

Proof. First, let us consider the case where d is even. Let $k = \frac{d}{2}$ and

$$f_0(n) = \binom{n+k+1}{d} + a\binom{n+k}{d} + \binom{n+k-1}{d},$$

where a is a real numer with $a \ge \frac{2(2k+1)}{2k-1}$. Then $f_0(n)$ is an SSNN polynomial of degree d. Let

$$g_0(n) = \frac{d!}{\prod_{j=-k+2}^{k-1}(n+j)} f_0(n) = (a+2)n^2 + (a+2)n + ak(-k+1) + 2k^2.$$

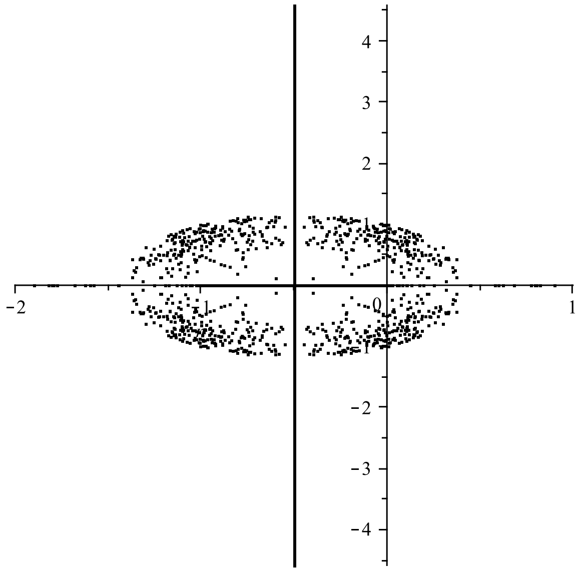


Figure 4.9: SSNN polynomials of degree 4

From $a \geq \frac{2(2k+1)}{2k-1}$ and

$$D(g_0(n)) = (2k-1)(a+2)((2k-1)a - 2(2k+1)),$$

we have $D(g_0(n)) \geq 0$. Thus the roots of $g_0(n)$ are all real numbers and those are

$$\begin{aligned} n &= \frac{-(a+2) \pm \sqrt{(2k-1)(a+2)((2k-1)a-2(2k+1))}}{2(a+2)} \\ &= -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{(2k-1)((2k-1)a-2(2k+1))}{a+2}} \\ (&=: -\frac{1}{2} \pm h_0(a) \ ). \end{aligned}$$

Now the function $h_0(a)$ on a is monotone increasing and

$$\lim_{a \to +\infty} h_0(a) = \frac{2k-1}{2} = k - \frac{1}{2}.$$

Hence, for $a \geq \frac{2(2k+1)}{2k-1}$, all the roots of each $f_0(n)$ are contained in the open interval $(-k, k-1)$. Moreover, $-k$ and $k-1$ are roots of $\binom{n+k}{d}$, which is an SSNN polynomial of degree d.

Next, let us consider the case where d is odd. Let $k = \frac{d-1}{2}$ and

$$f_1(n) = \binom{n+k+2}{d} + a\binom{n+k+1}{d} + a\binom{n+k}{d} + \binom{n+k-1}{d},$$

where a is a real number with $a \geq \frac{12k^2+12k-1}{(2k-1)^2}$. Then $f_1(n)$ is an SSNN polynomial of degree d. Let

$$g_1(n) = \frac{d!}{(2n+1)\prod_{j=-k+2}^{k-1}(n+j)} f_2(n) = (a+1)n^2+(a+1)n+ak(-k+1)+3k(k+1).$$

From $a \geq \frac{12k^2+12k-1}{(2k-1)^2}$ and

$$D(g_1(n)) = (a+1)((2k-1)^2a - (12k^2+12k-1)),$$

we have $D(g_1(n)) \geq 0$. Thus the roots of $g_1(n)$ are all real numbers and those are

$$\begin{aligned} n &= \frac{-(a+1) \pm \sqrt{(a+1)((2k-1)^2a-(12k^2+12k-1))}}{2(a+1)} \\ &= -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{(2k-1)^2a-(12k^2+12k-1)}{a+1}} \\ (&=: -\frac{1}{2} \pm h_1(a) \ ). \end{aligned}$$

Now the function $h_1(a)$ on a is monotone increasing and

$$\lim_{a\to+\infty} h_1(a) = \frac{2k-1}{2} = k - \frac{1}{2}.$$

Hence, for $a \geq \frac{12k^2+12k-1}{(2k-1)^2}$, all the roots of each $f_1(n)$ are contained in the open interval $(-k, k-1)$. Moreover, $-k$ and $k-1$ are roots of $\binom{n+k+1}{d} + \binom{n+k}{d}$, which is an SSNN polynomial of degree d. $\square$

Part II

Fano polytopes

Chapter 5

Introduction to Fano polytopes

In this part, as the second aspect of the studies on integral convex polytopes, we will consider *Fano polytopes*. Fano polytope is an integral convex polytopes arising naturally from a toric Fano variety, which is of significance in algebraic geometry.

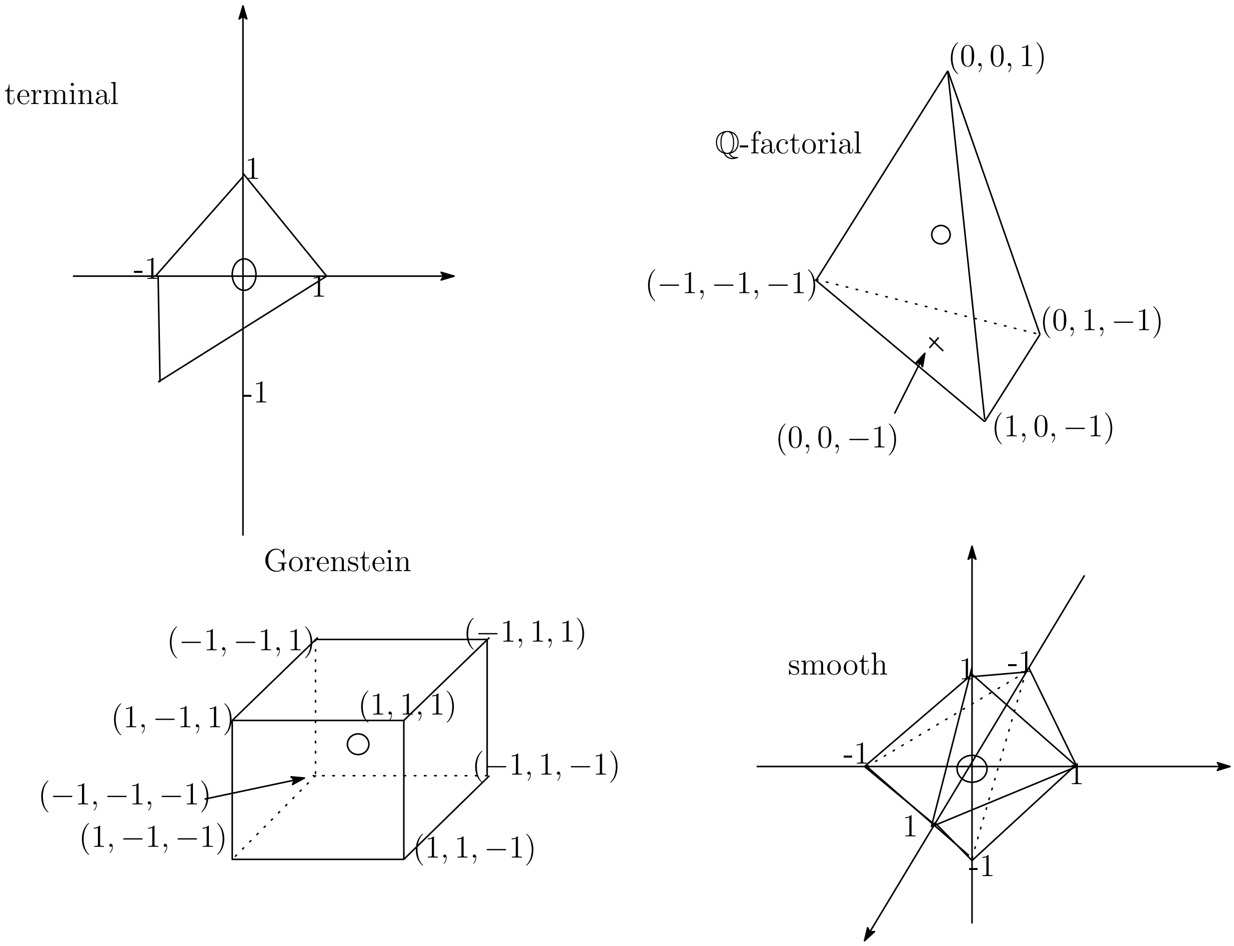
We will summarize some basic notions, definitions and some recent studies on Fano polytopes or toric Fano varieties.

Let $\mathcal{P} \subset \mathbb{R}^d$ be an integral convex polytope of dimension d.

- We say that $\mathcal{P}$ is a *Fano polytope* if the origin of $\mathbb{R}^d$ is a unique integer point belonging to the interior of $\mathcal{P}$.
- A Fano polytope $\mathcal{P}$ is called *terminal* if each integer point belonging to the boundary of $\mathcal{P}$ is a vertex of $\mathcal{P}$.
- A Fano polytope is called *Gorenstein* if its dual polytope is integral. (Recall that the dual polytope $\mathcal{P}^\vee$ of a Fano polytope $\mathcal{P}$ is the convex polytope which consists of those $x \in \mathbb{R}^d$ such that $\langle x, y \rangle \leq 1$ for all $y \in \mathcal{P}$, where $\langle x, y \rangle$ is the usual inner product of $\mathbb{R}^d$.)
- A Fano polytope is called $\mathbb{Q}$-*factorial* if it is simplicial, i.e., each of its faces is a simplex.
- A *smooth* Fano polytope is a Fano polytope such that the vertices of each facet form a $\mathbb{Z}$-basis of $\mathbb{Z}^d$.

Thus, in particular, a smooth Fano polytope is $\mathbb{Q}$-factorial, Gorenstein and terminal.

Example 5.0.9. Among four pictures drawn below, the 2-dimensional Fano polytope depicted on the upper left-hand side is terminal, the 3-dimensional Fano polytope depicted on the upper right-hand side is $\mathbb{Q}$-factorial, the 3-dimensional Fano polytope depicted on the lower left-hand side is Gorenstein and the 3-dimensional Fano polytope depicted on the lower right-hand side is smooth. The dual polytope of the lower left-hand side one coinsides with the lower right-hand side one.



M. Øbro [53] succeeded in finding an algorithm which yields the classification list of the smooth Fano polytopes for given d. It is proved in Casagrande [13] that the number of vertices of a Gorenstein $\mathbb{Q}$-factorial Fano polytope is at most $3d$ if d is even, and at most $3d-1$ if d is odd. B. Nill and M. Øbro [52] classified the Gorenstein $\mathbb{Q}$-factorial Fano polytopes of dimension d with $3d-1$ vertices. Gorenstein Fano polytopes are classified when $d \leq 4$ by Kreuzer and Skarke [46, 47] and the relevance of Gorenstein Fano polytopes to Mirror Symmetry was studied by Batyrev [3]. Gorenstein Fano polytope is often said to be a *reflexive polytope*. We refer the reader to [42, 45, 46, 47, 51] on the related works on toric Fano varieties or Gorenstein toric Fano varieties. The study of the classification of terminal or canonical Fano polytopes was done by Kasprzyk [42, 43]. The combinatorial conditions for what it implies to be terminal and canonical are explained in Reid [63].

On the rest of this part Chapter 6, we will introduce Fano polytopes arising from finite posets in Section 6.1 and study the problem of which finite posets yield smooth Fano polytopes. Similarly, in Section 6.2, we will also present Fano polytopes arising from finite directed graphs and consider the problem of which finite directed graphs yield smooth Fano polytopes. Moreover, by using them, we will construct many examples of smooth Fano polytopes.

Chapter 6

Examples of smooth Fano polytopes

In this chapter, we will establish two classes of Fano polytoes arising from combinatorial objects, finte posets (Section 6.1) and finite directed graphs (Section 6.2). To give many uselful examples of smooth Fano polytopes is very important. Hence, the descriptions how to construct smooth Fano polytopes via combinatorial methods written in this chapter are meaningful.

6.1 Smooth Fano polytopes arising from posets

In this section, we introduce Fano polytopes arising from posets and consider the problem of which poset yields smooth Fano polytopes.

6.1.1 Fano polytopes arising from posets

Let $P = \{y_1, \ldots, y_d\}$ be a finite poset and

$$\hat{P} = P \cup \{\hat{0}, \hat{1}\},$$

where $\hat{0}$ (resp. $\hat{1}$) is a unique minimal (resp. maximal) element of $\hat{P}$ with $\hat{0} \notin P$ (resp. $\hat{1} \notin P$). Let $y_0 = \hat{0}$ and $y_{d+1} = \hat{1}$. We say that $e = \{y_i, y_j\}$, where $0 \leq i, j \leq d+1$ with $i \neq j$, is an *edge* of $\hat{P}$ if e is an edge of the Hasse diagram of $\hat{P}$. (The Hasse diagram of a finite poset can be regarded as a finite nondirected graph.) In other words, $e = \{y_i, y_j\}$ is an edge of $\hat{P}$ if y_i and y_j are comparable in $\hat{P}$, say, $y_i < y_j$, and there is no $z \in P$ with $y_i < z < y_j$.

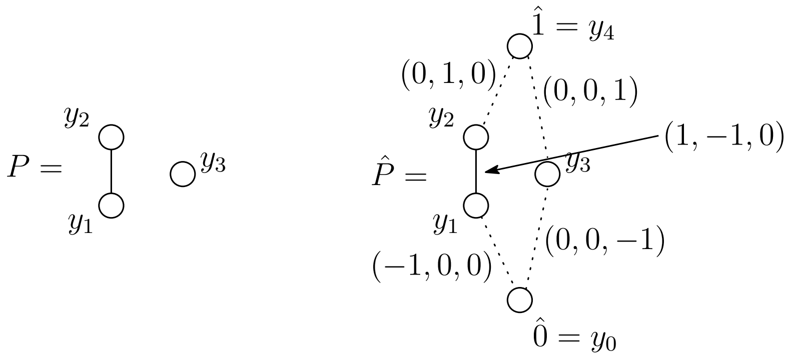
Definition 6.1.1. *Let $\hat{P} = \{y_0, y_1, \ldots, y_d, y_{d+1}\}$ be a finite poset with $y_0 = \hat{0}$ and $y_{d+1} = \hat{1}$. Let $\mathbf{e}_i$ denote the ith canonical unit coordinate vector of $\mathbb{R}^d$. Given an* *edge* $e = \{y_i, y_j\}$ *of* $\hat{P}$ *with* $y_i < y_j$*, we define* $\rho(e) \in \mathbb{R}^d$ *by setting*

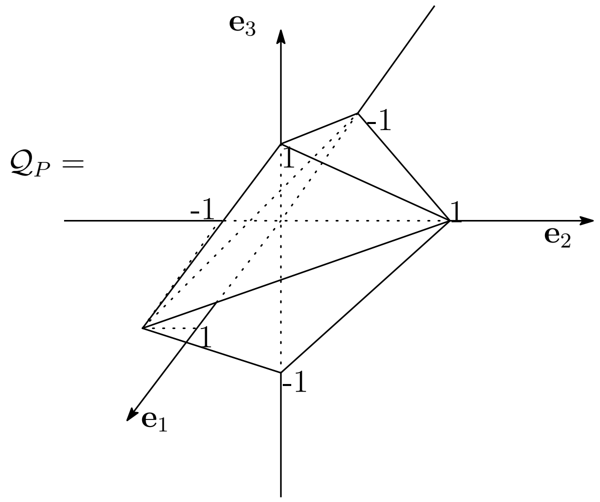
$$\rho(e) = \begin{cases} \mathbf{e}_i & \text{if} \quad j = d+1, \\ -\mathbf{e}_j & \text{if} \quad i = 0, \\ \mathbf{e}_i - \mathbf{e}_j & \text{if} \quad 1 \leq i,\, j \leq d. \end{cases}$$

Moreover, we write $\mathcal{Q}_P \subset \mathbb{R}^d$ *for the convex hull of the finite set*

$$\{\, \rho(e) : e \text{ is an edge of } \hat{P}\,\}.$$

Example 6.1.2. *Let* $P = \{y_1, y_2, y_3\}$ *be the finite poset with the partial order* $y_1 < y_2$*. Then* $\hat{P}$ *together with* $\rho(e)$*'s and* $\mathcal{Q}_P$ *are drawn below:*





Let P be a finite poset. A subset Q of P is called a *chain* of P if Q is a totally ordered subset of P. The *length* of a chain Q is $\ell(Q) = \sharp(Q) - 1$. A chain Q of P is *saturated* if $x, y \in Q$ with $x < y$, then there is no $z \in P$ with $x < z < y$. A *maximal* chain of $\hat{P}$ is a saturated chain Q of $\hat{P}$ with $\{\hat{0}, \hat{1}\} \subset Q$.

Lemma 6.1.3. *The convex polytope* $\mathcal{Q}_P$ *is a Fano polytope.*

Proof. Let $e = \{y_i, y_j\}$ be an edge of $\hat{P}$ with $y_i < y_j$. Let c_e denote the number of maximal chains Q of $\hat{P}$ with $\{y_i, y_j\} \subset Q$. If $\{y_{i_1}, y_{i_2}, \ldots, y_{i_m}\}$ is a maximal chain of $\hat{P}$ with $y_0 = y_{i_1} < y_{i_2} < \ldots < y_{i_m} = y_{d+1}$, then

$$\sum_{j=1}^{m-1} \rho(\{y_{i_j}, y_{i_{j+1}}\}) = (0, \ldots 0).$$

Hence

$$\sum_{e} c_e \rho(e) = (0, \ldots 0),$$

where e ranges all edges of $\hat{P}$. Thus the origin of $\mathbb{R}^d$ belongs to the interior of $\mathcal{Q}_P$. Since $\mathcal{Q}_P$ is a convex polytope which is contained in the convex hull of the finite set $\{\sum_{i=1}^{d} \varepsilon_i \mathbf{e}_i : \varepsilon_i \in \{0, 1, -1\}\}$ in $\mathbb{R}^d$, it follows that the origin of $\mathbb{R}^d$ is the unique integer point belonging to the interior of $\mathcal{Q}_P$. Thus $\mathcal{Q}_P$ is a Fano polytope, as desired. $\square$

Lemma 6.1.4. *The Fano polytope $\mathcal{Q}_P$ is terminal.*

Proof. Suppose that $\mathcal{Q}_P$ contains an integer point $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d$ with $\alpha \neq (0, \ldots, 0)$. Then, obviously, $\alpha_1, \ldots, \alpha_d \in \{-1, 0, 1\}$. Let, say, $\alpha_1 = 1$. Let $e_1, \ldots, e_n$ be all edges of $\hat{P}$ and $e_{i_1}, \ldots, e_{i_m}$ the edges with $y_1 \in e_{i_j}$ for $j = 1, \ldots, m$. If we set $e_{i_j} = \{y_{i_j}, y_{i_{j'}}\}$ with $y_{i_j} < y_{i_{j'}}$, since α belongs to the convex hull of $\{\rho(e_1), \ldots, \rho(e_n)\}$, then one has

$$\sum_{j=1}^{m} r_{i_j} q_{i_j} = \alpha_1 = 1,$$

where $0 \leq r_{i_1}, \ldots, r_{i_m} \leq 1$ and $q_{i_j} = 1$ (resp. $q_{i_j} = -1$) if $y_1 < y_{i_{j'}}$ (resp. $y_{i_j} < y_1$). By removing all r_{i_j} with $r_{i_j} = 0$, we may assume that

$$\sum_{j=1}^{m'} r_{i_j} q_{i_j} = 1,$$

where $0 < r_{i_1}, \ldots, r_{i_{m'}} \leq 1$. Since $\sum_{j=1}^{m'} r_{i_j} \leq 1$, there is no j with $q_{i_j} = -1$. Hence $\sum_{j=1}^{m'} r_{i_j} = 1$. If $m' > 1$, then $0 < r_{i_1}, \ldots, r_{i_{m'}} < 1$. Thus $\sum_{j=1}^{m'} r_{i_j} \rho(e_{i_j}) = \alpha \notin \mathbb{Z}^d$. Hence $m' = 1$. In other words, if $\mathcal{Q}_P$ contains an integer point $\alpha \neq (0, \ldots, 0)$, then α must be one of $\rho(e_1), \ldots, \rho(e_n)$ and $\rho(e_1), \ldots, \rho(e_n)$ are precisely the vertices of $\mathcal{Q}_P$. $\square$

Lemma 6.1.5. *The Fano polytope $\mathcal{Q}_P$ is Gorenstein.*

Proof. Via the theory of totally unimodular matrices ([66, Chapter 19]), it follows that the equation of each supporting hyperplane of $\mathcal{Q}_P$ is of the form $a_1 x_1 + \cdots + a_d x_d = 1$ with each $a_i \in \mathbb{Z}$. In other words, the dual polytope of $\mathcal{Q}_P$ is integral. Hence $\mathcal{Q}_P$ is Gorenstein, as required. $\square$

Remark 6.1.6. There is a well-known integral convex polytope arising from a finite poset P, which is called an *order polytope* $\mathcal{O}_P$. (See [69, Chapter 3] and [70].) One can verify immediately that the primitive outer normals of each facet of $\mathcal{O}_P$ one-to-one corresponds to each vertex of $\mathcal{Q}_P$. Now $\mathcal{O}_P$ is Gorenstein if and only if P is pure, i.e., all maximal chains of $\hat{P}$ have the same length. When P is pure, let l denote the length of each maximal chain of $\hat{P}$. Then the dilated polytope $l\mathcal{O}_P$ contains a unique integer point $\alpha \in \mathbb{Z}^d$, where d is the cardinality of P, belonging to the interior of $l\mathcal{O}_P$. Then the dual polytope of the Gorenstein Fano polytope $l\mathcal{O}_P - \alpha$ coincides with $\mathcal{Q}_P$. Thus, when P is pure, we can associate $\mathcal{Q}_P$ with the dual polytope of an order polytope $\mathcal{O}_P$.

6.1.2 When is $\mathcal{Q}_P$ smooth ?

Let $P = \{y_1, \ldots, y_d\}$ be a finite poset and $\hat{P} = P \cup \{y_0, y_{d+1}\}$, where $y_0 = \hat{0}$ and $y_{d+1} = \hat{1}$. A sequence $\Gamma = (y_{i_1}, y_{i_2}, \ldots, y_{i_m})$ is called a *path* in $\hat{P}$ if Γ is a path in the Hasse diagram of $\hat{P}$. In other words, $\Gamma = (y_{i_1}, y_{i_2}, \ldots, y_{i_m})$ is a path in $\hat{P}$ if $y_{i_j} \neq y_{i_k}$ for all $1 \leq j < k \leq m$ and if $\{y_{i_j}, y_{i_{j+1}}\}$ is an edge of $\hat{P}$ for all $1 \leq j \leq m-1$. In particular, if $\{y_{i_1}, y_{i_m}\}$ is also an edge of $\hat{P}$, then Γ is called a *cycle*. The *length* of a path $\Gamma = (y_{i_1}, y_{i_2}, \ldots, y_{i_m})$ is $\ell(\Gamma) = m - 1$ or $\ell(\Gamma) = m$ if Γ is a cycle.

A path $\Gamma = (y_{i_1}, y_{i_2}, \ldots, y_{i_{m+1}})$ is called *special* if

$$\sharp\{\, j : y_{i_j} < y_{i_{j+1}}, 1 \leq j \leq m-1 \,\} = \sharp\{\, k : y_{i_k} > y_{i_{k+1}}, 1 \leq k \leq m-1 \,\}.$$

Given a special path $\Gamma = (y_{i_1}, y_{i_2}, \ldots, y_{i_m})$, there exists a unique function

$$\mu_\Gamma : \{y_{i_1}, y_{i_2}, \ldots, y_{i_m}\} \to \{0, 1, 2, \ldots\}$$

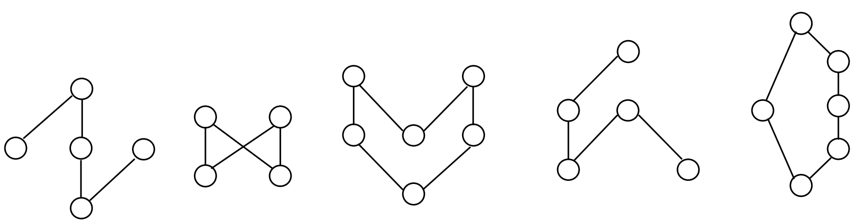
such that

- $\mu_\Gamma(y_{i_{j+1}}) = \mu_\Gamma(y_{i_j}) + 1$ (resp. $\mu_\Gamma(y_{i_j}) = \mu_\Gamma(y_{i_{j+1}}) + 1$) if $y_{i_j} < y_{i_{j+1}}$ (resp. $y_{i_j} > y_{i_{j+1}}$);
- $\min\{\mu_\Gamma(y_{i_1}), \mu_\Gamma(y_{i_2}), \ldots, \mu_\Gamma(y_{i_m})\} = 0$.

In particular, Γ is special if and only if $\mu_\Gamma(y_{i_1}) = \mu_\Gamma(y_{i_m})$.

Similary, a special cycle is defined and given a special cycle C, there exists a unique function μ_C which is defined the same way as above.

Example 6.1.7. *Among the two paths and three cycles drawn below, the three ones depicted on the left-hand side (one path and two cycles) are special; the remaining two ones (one path and one cycle) are not.*



We say that a path $\Gamma = (y_{i_1}, y_{i_2}, \ldots, y_{i_{m+1}})$ or a cycle $C = (y_{i_1}, y_{i_2}, \ldots, y_{i_m})$ of $\hat{P}$ *belongs to a facet of* $\mathcal{Q}_P$ if there is a facet $\mathcal{F}$ of $\mathcal{Q}_P$ with $\rho(\{y_{i_j}, y_{i_{j+1}}\}) \in \mathcal{F}$ for all $1 \leq j \leq m$, where $y_{i_{m+1}} = y_{i_1}$.

We say that a cycle $C = (y_{i_1}, y_{i_2}, \ldots, y_{i_m})$ is *very special* if C is special and if $\{y_0, y_{d+1}\} \not\subset \{y_{i_1}, y_{i_2}, \ldots, y_{i_m}\}$.

Lemma 6.1.8. (a) *Let $C = (y_{i_1}, y_{i_2}, \ldots, y_{i_m})$ be a cycle in $\hat{P}$. If C belongs to a facet of $\mathcal{Q}_P$, then C is a special cycle. In particular, C is a very special cycle or C contains a special path $(y_{i_1}, y_{i_2}, \ldots, y_{i_{r+1}})$ with $y_{i_1} = y_0$ and $y_{i_{r+1}} = y_{d+1}$.*

(b) *Let $\Gamma = (y_{i_1}, y_{i_2}, \ldots, y_{i_m})$ with $y_{i_1} = y_0$ and $y_{i_m} = y_{d+1}$ be a path in $\hat{P}$. If Γ belongs to a facet of $\mathcal{Q}_P$, then Γ is a special path.*

Proof. (a) Let $a_1x_1 + \cdots + a_dx_d = 1$, where each $a_i \in \mathbb{Q}$, denote the equation of the supporting hyperplane of $\mathcal{Q}_P$ which defines the facet. Since $\{y_{i_j}, y_{i_{j+1}}\}$ are edges of $\hat{P}$ for $1 \leq j \leq m$, where $y_{i_{m+1}} = y_{i_1}$, it follows that $a_{i_j} - a_{i_{j+1}} = q_j$, where $q_j = 1$ if $y_{i_j} < y_{i_{j+1}}$ and $q_j = -1$ if $y_{i_j} > y_{i_{j+1}}$. Now,

$$\sum_{i=1}^{m} q_j = \sum_{i=1}^{m} (a_{i_j} - a_{i_{j+1}}) = 0.$$

Hence C must be special.

Suppose that $\{y_0, y_{d+1}\} \subset \{y_{i_1}, y_{i_2}, \ldots, y_{i_m}\}$. Let $y_{i_1} = y_0$ and $y_{i_{r+1}} = y_{d+1}$. Since $\{y_{i_j}, y_{i_{j+1}}\}$, $1 \leq j \leq r$, are edges of $\hat{P}$, one has $-a_{i_2} = 1$, $a_{i_r} = 1$ and $a_{i_j} - a_{i_{j+1}} = q_j$ for $j = 2, 3, \ldots, r-1$. On the one hand, one has

$$-a_{i_2} + \sum_{j=2}^{r-1} (a_{i_j} - a_{i_{j+1}}) + a_{i_r} = 0.$$

On the other hand, one has

$$\begin{aligned}
-a_{i_2} + \sum_{j=2}^{r-1} (a_{i_j} - a_{i_{j+1}}) + a_{i_r} &= 1 + \sum_{j=2}^{r-1} q_j + 1 \\
&= -\mu_C(y_0) + \mu_C(y_{i_2}) + \sum_{j=2}^{r-1} (\mu_C(y_{i_{j+1}}) - \mu_C(y_{i_j})) + \mu_C(y_{d+1}) - \mu_C(y_{i_r}) \\
&= \mu_C(y_{d+1}) - \mu_C(y_0).
\end{aligned}$$

It then follows that one must be $\mu_C(y_0) = \mu_C(y_{d+1})$. Let $\Gamma = (y_{i_1}, y_{i_2}, \ldots, y_{i_{r+1}})$. Then it is clear that $\mu_\Gamma(y_0) = \mu_\Gamma(y_{d+1})$. Thus Γ is a special path. Hence C contains a special path Γ.

(b) A proof can be given by the similar way of a proof of (a). $\square$

Let P be a finite poset and $y, z \in \hat{P}$ with $y < z$. The *distance* of y and z in $\hat{P}$ is the smallest integer s for which there is a saturated chain $Q = \{z_0, z_1, \ldots, z_s\}$ with

$$y = z_0 < z_1 < \cdots < z_s = z.$$

Let $\mathrm{dist}_{\hat{P}}(y, z)$ denote the distance of y and z in $\hat{P}$.

Theorem 6.1.9 ([29, Theorem 2.3]). *Let $P = \{y_1, \ldots, y_d\}$ be a finite poset and $\hat{P} = P \cup \{y_0, y_{d+1}\}$, where $y_0 = \hat{0}$ and $y_{d+1} = \hat{1}$. Then the following conditions are equivalent:*

(i) $\mathcal{Q}_P$ *is $\mathbb{Q}$-factorial;*

(ii) $\mathcal{Q}_P$ *is smooth;*

(iii) $\hat{P}$ *possesses no very special cycle $C = (y_{i_1}, \ldots, y_{i_m})$ such that*

$$\mu_C(y_{i_a}) - \mu_C(y_{i_b}) \leq \mathrm{dist}_{\hat{P}}(y_{i_b}, y_{i_a}) \tag{6.1}$$

for all $1 \leq a, b \leq m$ with $y_{i_b} < y_{i_a}$, and

$$\mu_C(y_{i_a}) - \mu_C(y_{i_b}) \leq \mathrm{dist}_{\hat{P}}(y_0, y_{i_a}) + \mathrm{dist}_{\hat{P}}(y_{i_b}, y_{d+1}) \tag{6.2}$$

for all $1 \leq a, b \leq m$, and no special path $\Gamma = (y_{i_1}, \ldots, y_{i_m})$ with $y_{i_1} = y_0$ and $y_{i_m} = y_{d+1}$ such that

$$\mu_\Gamma(y_{i_a}) - \mu_\Gamma(y_{i_b}) \leq \mathrm{dist}_{\hat{P}}(y_{i_b}, y_{i_a}) \tag{6.3}$$

for all $1 \leq a, b \leq m$ with $y_{i_b} < y_{i_a}$.

Proof. **((i) $\Rightarrow$ (iii))** If $C = (y_{i_1}, \ldots, y_{i_m})$ is a cycle in $\hat{P}$ with $y_{i_{m+1}} = y_1$, then

$$\sum_{j=1}^{m} q_j \, \rho(\{y_{i_j}, y_{i_{j+1}}\}) = (0, \ldots, 0),$$

where $q_j = 1$ if $y_{i_j} < y_{i_{j+1}}$ and $q_j = -1$ if $y_{i_j} > y_{i_{j+1}}$. Thus in particular $\rho(\{y_{i_j}, y_{i_{j+1}}\})$, $1 \leq j \leq m$, cannot be affinely independent if C is special.

Now, suppose that $\hat{P}$ possesses a very special cycle $C = (y_{i_1}, \ldots, y_{i_m})$ which satisfies the inequalities (6.1) and (6.2). Our work is to show that $\mathcal{Q}_P$ is not simplicial. Let $v_j = \rho(\{y_{i_j}, y_{i_{j+1}}\})$, $1 \leq j \leq m$, where $y_{i_{m+1}} = y_{i_1}$. Since $v_1, \ldots, v_m$ cannot be affinely independent, to show that $\mathcal{Q}_P$ is not simplicial, what we must prove is the existence of a face of $\mathcal{Q}_P$ which contains the vertices $v_1, \ldots, v_m$.

Let $a_1, \ldots, a_d$ be integers. Write $\mathcal{H} \subset \mathbb{R}^d$ for the hyperplane defined by the equation $a_1 x_1 + \cdots + a_d x_d = 1$ and $\mathcal{H}^{(+)} \subset \mathbb{R}^d$ for the closed half-space defined by the inequality $a_1 x_1 + \cdots + a_d x_d \leq 1$. We will determine $a_1, \ldots, a_d$ such that $\mathcal{H}$ is a supporting hyperplane of a face $\mathcal{F}$ of $\mathcal{Q}_P$ with $\{v_1, \ldots, v_m\} \subset \mathcal{F}$ and with $\mathcal{Q}_P \subset \mathcal{H}^{(+)}$.

First Step. It follows from (6.2) that

$$\max_{1 \leq a \leq m} (\mu_C(y_{i_a}) - \mathrm{dist}_{\hat{P}}(y_0, y_{i_a})) \leq \min_{1 \leq b \leq m} (\mu_C(y_{i_b}) + \mathrm{dist}_{\hat{P}}(y_{i_b}, y_{d+1})). \tag{6.4}$$

By using (6.1), if $y_0 \in \{y_{i_1}, \ldots, y_{i_m}\}$, then the left-hand side of (6.4) is equal to $\mu_C(y_0)$. Similarly, if $y_{d+1} \in \{y_{i_1}, \ldots, y_{i_m}\}$, then the right-hand side of (6.4) is equal to $\mu_C(y_{d+1})$.

Now, fix an arbitrary integer a with

$$\max_{1 \le a \le m} (\mu_C(y_{i_a}) - \mathrm{dist}_{\hat{P}}(y_0, y_{i_a})) \le a \le \min_{1 \le b \le m} (\mu_C(y_{i_b}) + \mathrm{dist}_{\hat{P}}(y_{i_b}, y_{d+1})).$$

However, exceptionally, if $y_0 \in \{y_{i_1}, \dots, y_{i_m}\}$, then $a = \mu_C(y_0)$. If $y_{d+1} \in \{y_{i_1}, \dots, y_{i_m}\}$, then $a = \mu_C(y_{d+1})$. Let $a_{i_j} = a - \mu_C(y_{i_j})$ for $1 \le j \le m$. Then one has

$$-a_{i_j} \le \mathrm{dist}_{\hat{P}}(y_0, y_{i_j}), \quad a_{i_j} \le \mathrm{dist}_{\hat{P}}(y_{i_j}, y_{d+1}). \tag{6.5}$$

Moreover, it follows easily that each v_j lies on the hyperplane of $\mathbb{R}^d$ defined by the equation

$$\sum_{i_j \notin \{0, d+1\}} a_{i_j} x_{i_j} = 1.$$

Second Step. Let $A = \hat{P} \setminus (\{y_0, y_{d+1}\} \cup \{y_{i_1}, \dots, y_{i_m}\})$ and $y_i \in A$.

- Suppose that there is y_{i_j} with $y_{i_j} < y_i$ and that there is no y_{i_k} with $y_{i_k} > y_i$. Then we define a_i by setting

  $$a_i = \max(\{a_{i_j} - \mathrm{dist}_{\hat{P}}(y_{i_j}, y_i) : y_{i_j} < y_i\} \cup \{0\}).$$

- Suppose that there is no y_{i_j} with $y_{i_j} < y_i$ and that there is y_{i_k} with $y_{i_k} > y_i$. Then we define a_i by setting

  $$a_i = \min(\{a_{i_k} + \mathrm{dist}_{\hat{P}}(y_i, y_{i_k}) : y_i < y_{i_k}\} \cup \{0\}).$$

- Suppose that there is y_{i_j} with $y_{i_j} < y_i$ and that there is y_{i_k} with $y_{i_k} > y_i$. Then either

  $$b_i = \max(\{a_{i_j} - \mathrm{dist}_{\hat{P}}(y_{i_j}, y_i) : y_{i_j} < y_i\} \cup \{0\})$$

  or

  $$c_i = \min(\{a_{i_k} + \mathrm{dist}_{\hat{P}}(y_i, y_{i_k}) : y_i < y_{i_k}\} \cup \{0\})$$

  must be zero. In fact, if $b_i \neq 0$ and $c_i \neq 0$, then there are j and k with $a_{i_j} > \mathrm{dist}_{\hat{P}}(y_{i_j}, y_i)$ and $-a_{i_k} > \mathrm{dist}_{\hat{P}}(y_i, y_{i_k})$. Since $\mu_C(y_{i_k}) - \mu_C(y_{i_j}) = a_{i_j} - a_{i_k}$ and since $\mathrm{dist}_{\hat{P}}(y_{i_j}, y_i) + \mathrm{dist}_{\hat{P}}(y_i, y_{i_k}) \ge \mathrm{dist}_{\hat{P}}(y_{i_j}, y_{i_k})$, it follows that

  $$\mu_C(y_{i_k}) - \mu_C(y_{i_j}) > \mathrm{dist}_{\hat{P}}(y_{i_j}, y_{i_k}).$$

  This contradicts (6.1). Hence either $b_i = 0$ or $c_i = 0$. If $b_i \neq 0$, then we set $a_i = b_i$. If $c_i \neq 0$, then we set $a_i = c_i$. If $b_i = c_i = 0$, then we set $a_i = 0$.

- Suppose that there is no y_{i_j} with $y_{i_j} < y_i$ and that there is no y_{i_k} with $y_{i_k} > y_i$. Then we set $a_i = 0$.

Third Step. Finally, we finish determining the integers $a_1, \ldots, a_d$. Let $\mathcal{H} \subset \mathbb{R}^d$ denote the hyperplane defined by the equation $a_1x_1 + \ldots + a_dx_d = 1$ and $\mathcal{H}^{(+)} \subset \mathbb{R}^d$ the closed half-space defined by the inequality $a_1x_1 + \ldots + a_dx_d \leq 1$. Since each v_j lies on the hyperplane $\mathcal{H}$, in order for $\mathcal{F} = \mathcal{H} \cap \mathcal{Q}_P$ to be a face of $\mathcal{Q}_P$, it is required to show $\mathcal{Q}_P \subset \mathcal{H}^{(+)}$. Let $\{y_i, y_j\}$ with $y_i < y_j$ be an edge of $\hat{P}$.

- Let $y_i \in \{y_{i_1}, \ldots, y_{i_m}\}$ with $y_j \notin \{y_{i_1}, \ldots, y_{i_m}\}$. If $y_j \neq y_{d+1}$, then

  $$a_j \geq \max\{a_i - 1, 0\},$$

  where $a_0 = 0$. Thus $a_i - a_j \leq 1$. If $y_j = y_{d+1}$, then by using (6.5) one has $a_i \leq 1$, as desired.

- Let $y_j \in \{y_{i_1}, \ldots, y_{i_m}\}$ with $y_i \notin \{y_{i_1}, \ldots, y_{i_m}\}$. If $y_i \neq y_0$, then

  $$a_i \leq \min\{a_j + 1, 0\},$$

  where $a_{d+1} = 0$. Thus $a_i - a_j \leq 1$. If $y_i = y_0$, then by using (6.5) one has $-a_j \leq 1$, as desired.

Let $A' = \hat{P} \setminus \{y_{i_1}, \ldots, y_{i_m}\}$. Write B for the subset of A' consisting of those $y_i \in A'$ such that there is j with $y_{i_j} < y_i$. Write C for the subset of A' consisting of those $y_i \in A'$ such that there is k with $y_i < y_{i_k}$. Again, let $e = \{y_i, y_j\}$ with $y_i < y_j$ be an edge of $\hat{P}$. In each of the nine cases below, a routine computation easily yields that $\rho(e) \in \mathcal{H}^{(+)}$.

- $y_i \in B \setminus C$ and $y_j \in B \setminus C$;
- $y_i \in C \setminus B$ and $y_j \in C \setminus B$;
- $y_i \in C \setminus B$ and $y_j \in B \setminus C$;
- $y_i \in C \setminus B$ and $y_j \in B \cap C$;
- $y_i \in C \setminus B$ and $y_j \notin B \cup C$;
- $y_i \in B \cap C$ and $y_j \in B \cap C$;
- $y_i \in B \cap C$ and $y_j \in B \setminus C$;
- $y_i \notin B \cup C$ and $y_j \in B \setminus C$;
- $y_i \notin B \cup C$ and $y_j \notin B \cup C$.

For example, in the first case, a routine computation is as follows. Let $y_j \neq y_{d+1}$. Let $a_i = 0$. Then, since $a_j \geq 0$, one has $a_i - a_j \leq 1$. Let $a_i > 0$. Then, since $a_j \geq a_i - 1$, one has $a_i - a_j \leq 1$. Let $y_j = y_{d+1}$ and $a_i > 0$. Then there is j

with $a_i = a_{i_j} - \text{dist}_{\hat{P}}(y_{i_j}, y_i)$. By using (6.5) one has $a_{i_j} \leq \text{dist}_{\hat{P}}(y_{i_j}, y_{d+1})$. Thus $a_i \leq \text{dist}_{\hat{P}}(y_{i_j}, y_{d+1}) - \text{dist}_{\hat{P}}(y_{i_j}, y_i)$. Hence $a_i \leq 1$, as required.

Fourth step. Suppose that $\hat{P}$ possesses a special path $\Gamma = (y_{i_1}, y_{i_2}, \ldots, y_{i_m})$ with $y_{i_1} = y_0$ and $y_{i_m} = y_{d+1}$ which satisfies the inequalities (6.3). Then one has

$$\sum_{j=1}^{m-1} q_j \, \rho(\{y_{i_j}, y_{i_{j+1}}\}) = (0, \ldots, 0),$$

where $q_j = 1$ if $y_{i_j} < y_{i_{j+1}}$ and $q_j = -1$ if $y_{i_j} > y_{i_{j+1}}$. Thus $\rho(\{y_{i_j}, y_{i_{j+1}}\})$, $1 \leq j \leq m-1$, cannot be affinely independent. Our work is to show that $\mathcal{Q}_P$ is not simplicial. In this case, however, the same discussion can be given as the case which $\hat{P}$ possesses a very special cycle. (We should set $a = \mu_\Gamma(y_0)$ $(= \mu_\Gamma(y_{d+1}))$.)

((iii) $\Rightarrow$ (i)) Now, suppose that $\mathcal{Q}_P$ is not $\mathbb{Q}$-factorial. Thus $\mathcal{Q}_P$ possesses a facet $\mathcal{F}$ which is not a simplex. Let $v_1, \ldots, v_n$ denote the vertices of $\mathcal{F}$, where $n > d$, and e_j the edge of $\hat{P}$ with $v_j = \rho(e_j)$ for $1 \leq j \leq n$. Let $a_1x_1 + \cdots + a_dx_d = 1$ denote the equation of the supporting hyperplane $\mathcal{H} \subset \mathbb{R}^d$ of $\mathcal{Q}_P$ with $\mathcal{F} = \mathcal{Q}_P \cap \mathcal{H}$ and with $\mathcal{Q}_P \subset \mathcal{H}^{(+)}$, where $\mathcal{H}^{(+)} \subset \mathbb{R}^d$ is the closed-half space defined by the inequality $a_1x_1 + \cdots + a_dx_d \leq 1$. Since $v_1, \ldots, v_n$ are not affinely independent, there is $(r_1, \ldots, r_n) \in \mathbb{Z}^n$ with $(r_1, \ldots, r_n) \neq (0, \ldots, 0)$ such that $r_1v_1 + \cdots + r_nv_n = (0, \ldots, 0)$. By removing r_j with $r_j = 0$, we may assume that $r_1v_1 + \cdots + r_{n'}v_{n'} = (0, \ldots, 0)$, where $r_j \neq 0$ for $1 \leq j \leq n'$ with $r_1 + \cdots + r_{n'} = 0$. Let $e_j = \{y_{i_j}, y_{i_{j'}}\}$ with $1 \leq i_j, i_{j'} \leq d$. If either y_{i_j} or $y_{i_{j'}}$ appears only in e_j among the edges $e_1, \ldots, e_{n'}$, then $r_j = 0$. Hence both y_{i_j} and $y_{i_{j'}}$ must appear in at least two edges among $e_1, \ldots, e_{n'}$. Let G denote the subgraph of the Hasse diagram of $\hat{P}$ with the edges $e_1, \ldots, e_{n'}$. Then there is no end point of G in P. Thus G possesses a cycle of $\hat{P}$ or G is a path of $\hat{P}$ from y_0 to y_{d+1}. Since $v_1, \ldots, v_{n'}$ are contained in the facet $\mathcal{F}$, Lemma 6.1.8 says that every cycle in G is very special or else G contains a special path.

Suppose that G possesses a very special cycle $C = (y_{i_1}, y_{i_2}, \ldots, y_{i_m})$. Our goal is to show that C satisfies the inequalities (6.1) and (6.2).

Let $y_{k_0} < y_{k_1} < \cdots < y_{k_\ell}$ be a saturated chain of $\hat{P}$ with $\ell = \text{dist}_{\hat{P}}(y_{k_0}, y_{k_\ell})$ such that each of y_{k_0} and y_{k_ℓ} belongs to $\{y_{i_1}, y_{i_2}, \ldots, y_{i_m}\}$. We claim

$$\mu_C(y_{k_\ell}) - \mu_C(y_{k_0}) \leq \text{dist}_{\hat{P}}(y_{k_0}, y_{k_\ell}).$$

- Let $y_0 \neq y_{k_0}$ and $y_{d+1} \neq y_{k_\ell}$. Since $\mathbf{e}_{k_j} - \mathbf{e}_{k_{j+1}} \in \mathcal{Q}_P$, one has $a_{k_j} - a_{k_{j+1}} \leq 1$ for each $0 \leq j \leq \ell - 1$. Hence $a_{k_0} - a_{k_\ell} \leq \ell$. On the other hand, $a_{k_0} - a_{k_\ell} = \mu_C(y_{k_\ell}) - \mu_C(y_{k_0})$. Thus $\mu_C(y_{k_\ell}) - \mu_C(y_{k_0}) \leq \text{dist}_{\hat{P}}(y_{k_0}, y_{k_\ell})$.

- Let $y_0 = y_{k_0}$ and $y_{d+1} \neq y_{k_\ell}$. Since $-\mathbf{e}_{k_1} \in \mathcal{Q}_P$, one has $-a_{k_1} \leq 1$. Since $\mathbf{e}_{k_j} - \mathbf{e}_{k_{j+1}} \in \mathcal{Q}_P$, one has $a_{k_j} - a_{k_{j+1}} \leq 1$ for each $1 \leq j \leq \ell - 1$. Hence $a_{k_1} - a_{k_\ell} \leq \ell - 1$. Thus $-a_{k_\ell} \leq \ell$. On the other hand, $-a_{k_\ell} = \mu_C(y_{k_\ell}) - \mu_C(y_{k_0})$. Thus $\mu_C(y_{k_\ell}) - \mu_C(y_{k_0}) \leq \text{dist}_{\hat{P}}(y_{k_0}, y_{k_\ell})$.

- Let $y_0 \neq y_{k_0}$ and $y_{d+1} = y_{k_\ell}$. Since $\mathbf{e}_{k_j} - \mathbf{e}_{k_{j+1}} \in \mathcal{Q}_P$, one has $a_{k_j} - a_{k_{j+1}} \leq 1$ for each $0 \leq j \leq \ell - 2$. Hence $a_{k_0} - a_{k_{\ell-1}} \leq \ell - 1$. Since $\mathbf{e}_{k_{\ell-1}} \in \mathcal{Q}_P$, one has $a_{k_{\ell-1}} \leq 1$. Hence $a_{k_0} \leq \ell$. On the other hand, $a_{k_0} = \mu_C(y_{k_\ell}) - \mu_C(y_{k_0})$. Thus $\mu_C(y_{k_\ell}) - \mu_C(y_{k_0}) \leq \mathrm{dist}_{\hat{P}}(y_{k_0}, y_{k_\ell})$.

Finally, fix arbitrary y_{i_j} and y_{i_k} with $\mu_C(y_{i_j}) < \mu_C(y_{i_k})$. Then $-a_{i_k} \leq \mathrm{dist}_{\hat{P}}(y_0, y_{i_k})$ and $a_{i_j} \leq \mathrm{dist}_{\hat{P}}(y_{i_j}, y_{d+1})$. We claim

$$\mu_C(y_{i_k}) - \mu_C(y_{i_j}) \leq \mathrm{dist}_{\hat{P}}(y_0, y_{i_k}) + \mathrm{dist}_{\hat{P}}(y_{i_j}, y_{d+1}).$$

If $y_{i_j} \neq y_0$ and $y_{i_k} \neq y_{d+1}$, then $a_{i_j} - a_{i_k} = \mu_C(y_{i_k}) - \mu_C(y_{i_j})$. If $y_{i_j} = y_0$ and $y_{i_k} \neq y_{d+1}$, then $-a_{i_k} = \mu_C(y_{i_k}) - \mu_C(y_{i_j})$. If $y_{i_j} \neq y_0$ and $y_{i_k} = y_{d+1}$, then $a_{i_j} = \mu_C(y_{i_k}) - \mu_C(y_{i_j})$. Hence the required inequality follows immediately.

Suppose that G contains a special path $\Gamma = (y_{i_1}, y_{i_2}, \ldots, y_{i_m})$ with $y_{i_1} = y_0$ and $y_{i_m} = y_{d+1}$. Our goal is to show that C satisfies the inequalities (6.3). Now the same discussion can be given as above.

((i) $\Rightarrow$ (ii)) If P is a totally ordered set, then $\mathcal{Q}_P$ is a d-simplex with the vertices, say, $-\mathbf{e}_1, \mathbf{e}_1 - \mathbf{e}_2, \ldots, \mathbf{e}_{d-1} - \mathbf{e}_d, \mathbf{e}_d$. Thus in particular $\mathcal{Q}_P$ is smooth.

Now, suppose that P is not a totally ordered set. Then $\hat{P}$ possesses a cycle. Let $C = (y_{i_1}, \ldots, y_{i_m})$ be a cycle in $\hat{P}$. If C is not special, then Lemma 6.1.8 (a) says that C cannot belong to a facet of $\mathcal{Q}_P$. If C is special, then as was shown in the proof of (i) $\Rightarrow$ (iii) it follows that $\rho(\{y_{i_j}, y_{i_{j+1}}\})$, $1 \leq j \leq m$, where $y_{i_{m+1}} = y_{i_1}$, are not affinely independent. Hence there is no facet $\mathcal{F}$ of $\mathcal{Q}_P$ with $\rho(\{y_{i_j}, y_{i_{j+1}}\}) \in \mathcal{F}$ for all $1 \leq j \leq m$.

Let $\mathcal{F}$ be an arbitrary facet of $\mathcal{Q}_P$ with d vertices $v_j = \rho(e_j)$, $1 \leq j \leq d$. Let G denote the subgraph of the Hasse diagram of $\hat{P}$ with the edges $e_1, \ldots, e_d$ and $V(G)$ the vertex set of G. Since $\mathcal{F}$ is of dimension $d-1$, it follows that, for each $1 \leq i \leq d$, there is a vertex of $\mathcal{F}$ whose ith coordinate is nonzero. Hence $P \subset V(G)$. Suppose that $P = V(G)$. Since G has d edges, it follows that G possesses a cycle, a contradiction. Hence either $y_0 \in V(G)$ or $y_{d+1} \in V(G)$.

What we must prove is that the determinant

$$\begin{vmatrix} v_1 \\ \vdots \\ v_d \end{vmatrix} \tag{6.6}$$

is equal to ± 1. Let, say, $e_1 = \{y_1, y_{d+1}\}$. Thus $v_1 = (1, 0, \ldots, 0)$. Now, since G is a forest, by arranging the numbering of the elements of P if necessary, one has

$$\begin{vmatrix} v_1 \\ \vdots \\ v_d \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & \cdots & \cdots & 0 \\ a_{21} & a_{22} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ a_{d1} & a_{d2} & \cdots & \cdots & a_{dd} \end{vmatrix},$$

where each $a_{ij} \in \{1, 0, -1\}$. Since the determinant (6.6) is nonzero, it follows that the determinant (6.6) is equal to ± 1, as desired.

((ii) $\Rightarrow$ (i)) In general, every smooth Fano polytope is $\mathbb{Q}$-factorial. $\square$

Corollary 6.1.10. *Suppose that a finite poset P is pure. Then the following conditions are equivalent:*

(i) $\mathcal{Q}_P$ is $\mathbb{Q}$-factorial;

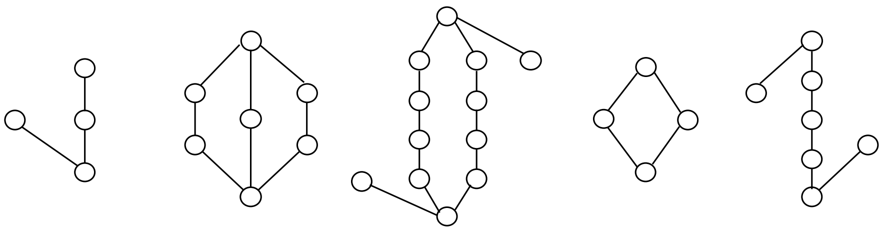
(ii) $\mathcal{Q}_P$ is smooth;

(iii) P is a disjoint union of chains;

(iv) The polytope $\mathcal{Q}_P$ is the free sum of smooth Fano simplices.

Proof. If P is pure, then every cycle of $\hat{P}$ is special and, in addition, satisfies the inequalities (6.1) and (6.2). Moreover, every path from y_0 to y_{d+1} cannot be special. Hence $\mathcal{Q}_P$ is $\mathbb{Q}$-factorial if and only if there is no very special cycle, i.e., every cycle of $\hat{P}$ possesses both $\hat{0}$ and $\hat{1}$. Now if there is a connected component of P which is not a chain, then P possesses a very special cycle. Thus $\mathcal{Q}_P$ is $\mathbb{Q}$-factorial if and only if P does not possess a connected component which is not a chain. In other words, $\mathcal{Q}_P$ is $\mathbb{Q}$-factorial if and only if P is a disjoint union of chains. Furthermore, smooth Fano simplices arising from finite posets are constructed from only totally ordered sets. That is to say, P is a disjoint union of chains if and only if the polytope $\mathcal{Q}_P$ is the free sum of smooth Fano simplices, as desired. $\square$

Example 6.1.11. *Among the five finite posets drawn below, the three finite posets depicted on the left-hand side yield a $\mathbb{Q}$-factorial Fano polytope; the remaining two finite posets do not yield a $\mathbb{Q}$-factorial Fano polytope.*



Let P and P' be finite posets. Then one can verify easily that $\mathcal{Q}_P$ is isomorphic with $\mathcal{Q}_{P'}$ as a convex polytope if and only if P is isomorphic with P' or with the dual finite poset of P' as a finite poset. (For example, a proof can be given by the induction on the number of maximal chains of $\hat{P}$.)

On the following table drawn below, the number of finite posets with $d(\leq 8)$ elements, up to isomorphic and up to isomorphic with dual finite posets, is written in the second row. Moreover, among those, the number of finite posets constructing smooth Fano polytopes is written in the third row.

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
Posets	1	2	4	12	39	184	1082	8746
Smooth	1	2	3	6	12	31	83	266

6.2 Smooth Fano polytopes arising from directed graphs

In this section, we introduce the Fano polytopes arising from directed graphs and consider the problem of which directed graphs yiled smooth Fano polytopes. Note that as written in Remark 6.2.6, Fano polytopes arising from directed graphs are a generalization of ones arising from posets in the previous section.

6.2.1 Fano polytopes associated with directed graphs

In this subsection, we construct an integral convex polytope associated with a finite directed graph and discuss the condition of which directed graph yields a Fano polytope. Most parts of this section are refered from [29, 48, 54, 57].

Let G be a finite directed graph on the vertex set $V(G) = \{1, \ldots, d\}$. An ordered pair of vertices $\vec{e} = (i, j)$ is said to be an *arrow* of G and a pair without ordering $e = \{i, j\}$ is said to be an *edge* of G. Remark that we regard (i, j) and (j, i) as two distinct arrows. Let $A(G)$ (resp. $E(G)$) denote the arrow set (resp. the edge set) of G. Throughout this paper, we allow the case where both (i, j) and (j, i) are contained in $A(G)$ and assume that G is connected.

Definition 6.2.1. Let $\mathbf{e}_i$ denote the i-th unit vector of $\mathbb{R}^d$. Given an arrow $\vec{e} = (i, j)$ in G, we define $\rho(\vec{e}) \in \mathbb{R}^d$ by setting $\rho(\vec{e}) = \mathbf{e}_i - \mathbf{e}_j$. Moreover, we write $\mathcal{P}_G \subset \mathbb{R}^d$ for the convex hull of $\{\rho(\vec{e}) : \vec{e} \in A(G)\}$.

Remark 6.2.2. In [57], $\mathcal{P}_G$ is introduced for a tournament graph G, which is called the *edge polytope* of G, and some properties on $\mathcal{P}_G$ are studied in [57, Section 1]. Similarly, in [48, Section 4], $\mathcal{P}_G$ is also defined for a symmetric graph G, which is denoted by $\mathcal{P}_G^{\pm}$ and said to be the *symmetric edge polytope* of G.

Let $\mathcal{H} \subset \mathbb{R}^d$ denote the hyperplane defined by the equation $x_1 + \cdots + x_d = 0$. Since each integer point of $\{\rho(\vec{e}) : \vec{e} \in A(G)\}$ lies on $\mathcal{H}$, one has $\mathcal{P}_G \subset \mathcal{H}$. Thus, $\dim(\mathcal{P}_G) \leq d - 1$. First of all, we discuss the dimension of $\mathcal{P}_G$.

A sequence $\Gamma = (i_1, \ldots, i_l)$ of vertices of G is called a *cycle* of length l in G with the arrows $\vec{e_1}, \ldots, \vec{e_l}$ if $e_j = \{i_j, i_{j+1}\}$ for $1 \leq j \leq l$ with $i_{l+1} = i_1$ and $i_j \neq i_{j'}$ for $1 \leq j < j' \leq l$. In other words, the edges $e_1, \ldots, e_l$ form a cycle in G. For short, we often write $\Gamma = (\vec{e_1}, \ldots, \vec{e_l})$. For a cycle $\Gamma = (\vec{e_1}, \ldots, \vec{e_l})$ in G, let $\Delta_\Gamma^{(+)} = \{\vec{e_j} \in \{\vec{e_1}, \ldots, \vec{e_l}\} : \vec{e_j} = (i_j, i_{j+1})\}$ and $\Delta_\Gamma^{(-)} = \{\vec{e_1}, \ldots, \vec{e_l}\} \setminus \Delta_\Gamma^{(+)}$. Then we may assume that $|\Delta_\Gamma^{(+)}| \geq |\Delta_\Gamma^{(-)}|$ without loss of generality. A cycle Γ is called *nonhomogeneous* if $|\Delta_\Gamma^{(+)}| > |\Delta_\Gamma^{(-)}|$ and *homogeneous* if $|\Delta_\Gamma^{(+)}| = |\Delta_\Gamma^{(-)}|$. Note that two arrows (i, j) and (j, i) form a nonhomogeneous cycle of length 2. In particular, every odd cycle is nonhomogeneous. The following result can be proved by using similar techniques appearing in the proof of [54, Proposition 1.3].

Proposition 6.2.3 (See also [57, Lemma 1.1]). *One has* $\dim(\mathcal{P}_G) = d - 1$ *if and only if* G *has a nonhomogeneous cycle.*

We assume that G has at least one nonhomogeneous cycle.

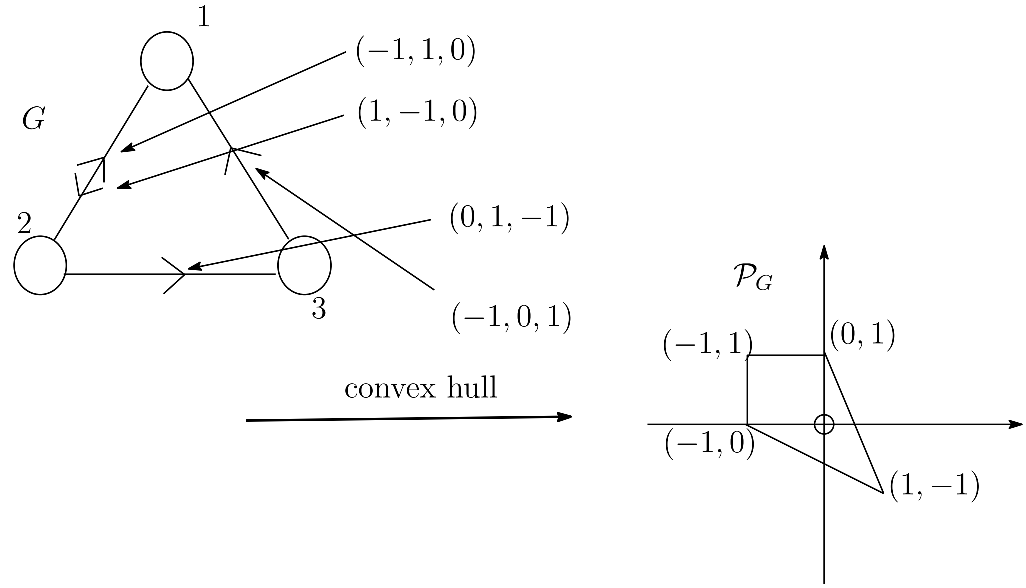
Next, we consider the problem of which directed graphs construct Fano polytopes. Once we know that G constructs a Fano polytope, one can verify that $\mathcal{P}_G$ is terminal and Gorenstein ([29, Lemma 1.4 and 1.5]). A proof of the following result can be also given by using similar techniques used in the proofs of [57, Lemma 1.2] and [48, Proposition 4.2].

Proposition 6.2.4. *$\mathcal{P}_G \subset \mathcal{H}$ is a terminal Gorenstein Fano polytope of dimension $d-1$ if and only if every arrow of G appears in a directed cycle in G.*

Here, a cycle Γ is called a *directed cycle* if $\Delta_\Gamma^{(-)}$ is empty.

Hereafter, we assume that every arrow of G appears in a directed cycle in G. Then we notice that G has a nonhomogeneous cycle since every directed cycle is nonhomogeneous.

Example 6.2.5. Let G be a directed graph on the vertex set $\{1,2,3\}$ with the arrow set $\{(1,2),(2,1),(2,3),(3,1)\}$. Then G, $\rho(\vec{e})$'s and $\mathcal{P}_G$ are drawn below:



Remark that the arrows $(1,2),(2,3),(3,1)$ and the arrows $(1,2),(2,1)$ form directed cycles. Before having the convex hull of $\rho(\vec{e})$'s, we ignore the third element of each integer point. Then the convex polytope $\mathcal{P}_G$ of this example becomes a terminal Gorenstein Fano polytope of dimension 2, in particular, smooth.

Remark 6.2.6. In [29], terminal Gorenstein Fano polytopes arising from finite partially ordered sets $\mathcal{Q}_P$ are introduced. Let $P = \{y_1,\ldots,y_d\}$ be a partially ordered set and $\hat{P} = P \cup \{y_0, y_{d+1}\}$, where $y_0 = \hat{0}$ and $y_{d+1} = \hat{1}$. Then we can regard $\hat{P}$ as the directed graph on the vertex set $\{0,1,\ldots,d+1\}$ with the arrow set

$$\{(i,j) : y_j \text{ covers } y_i\}.$$

Identifying 0 with $d+1$ as the same vertex, we construct the directed graph on the vertex set $\{1, \ldots, d+1\}$. Let G_P denote such directed graph. Then $\mathcal{Q}_P$ is nothing but $\mathcal{P}_{G_P}$. Therefore, terminal Gorenstein Fano polytopes associated with directed graphs are a natural generalization of those defined in [29] and we can consider the problem studying in section 2 in the similar way.

6.2.2 When is $\mathcal{P}_G$ smooth ?

In this section, we consider the problem of which directed graphs yield smooth Fano polytopes.

First, we prove the following

Lemma 6.2.7. (a) *Let $C = (\vec{e_1}, \ldots, \vec{e_l})$ be a cycle in G. If there exists a facet $\mathcal{F}$ of $\mathcal{P}_G$ with $\{\rho(\vec{e_1}), \ldots, \rho(\vec{e_l})\} \subset \mathcal{F}$, then C is homogeneous.*
(b) *For $(i,j) \in A(G)$, suppose that $(j,i) \in A(G)$. If $\rho((i,j))$ is contained in some facet $\mathcal{F}$ of $\mathcal{P}_G$, then $\rho((j,i))$ is never contained in $\mathcal{F}$.*

Proof. (a) Let $a_1x_1 + \cdots + a_dx_d = 1$, where each $a_i \in \mathbb{Q}$, denote the equation of the supporting hyperplane of $\mathcal{P}_G$ which defines a facet $\mathcal{F}$. Let $e_j = \{i_j, i_{j+1}\}$ for $1 \leq j \leq l$, where $i_{l+1} = i_1$. It then follows that

$$\sum_{j=1}^{l}(a_{i_j} - a_{i_{j+1}}) = \sum_{\vec{e_j} \in \Delta_C^{(+)}} (a_{i_j} - a_{i_{j+1}}) - \sum_{\vec{e_j} \in \Delta_C^{(-)}} (a_{i_{j+1}} - a_{i_j}) = |\Delta_C^{(+)}| - |\Delta_C^{(-)}| = 0.$$

Hence, C must be homogeneous.

(b) We set $a_1x_1 + \cdots + a_dx_d = 1$ as above and suppose that $\rho((i,j))$ lies on this supporting hyperplane. Then one has $a_i - a_j = 1$. Thus, $a_j - a_i = -1$. This implies that $\rho((j,i))$ cannot be contained in the same supporting hyperplane. $\square$

Next, we prepare two notions, μ_C and dist_G.

Let $C = (\vec{e_1}, \ldots, \vec{e_l})$ be a homogeneous cycle of length l, where $e_j = \{i_j, i_{j+1}\}$ for $1 \leq j \leq l$ with $i_{l+1} = i_1$. Then there exists a unique function

$$\mu_C : \{i_1, \ldots, i_l\} \to \mathbb{Z}_{\geq 0}$$

such that

- $\mu_C(i_{j+1}) = \mu_C(i_j) - 1$ (resp. $\mu_C(i_{j+1}) = \mu_C(i_j) + 1$) if $\vec{e_j} = (i_j, i_{j+1})$ (resp. $\vec{e_j} = (i_{j+1}, i_j)$) for $1 \leq j \leq l$;
- $\min(\{\mu_C(i_1), \ldots, \mu_C(i_l)\}) = 0$.

For two distinct vertices i and j of G, the *distance* from i to j, denoted by $\mathrm{dist}_G(i,j)$, is the length of the directed shortest path in G from i to j. If there exists no directed path from i to j, then we define the distance from i to j by infinity.

We now come to the position to prove the following

Theorem 6.2.8 ([39, Theorem 2.2]). *Let G be a connected directed graph on the vertex set $\{1, \dots, d\}$ satisfying that every arrow of G appears in a directed cycle of G. Then the following conditions are equivalent:*

(i) $\mathcal{P}_G$ *is $\mathbb{Q}$-factorial;*

(ii) $\mathcal{P}_G$ *is smooth;*

(iii) G *possesses no homogeneous cycle* $C = (\vec{e_1}, \dots, \vec{e_l})$ *such that*

$$\mu_C(i_a) - \mu_C(i_b) \leq \mathrm{dist}_G(i_a, i_b) \tag{6.7}$$

for all $1 \leq a, b \leq l$, *where* $e_j = \{i_j, i_{j+1}\}$ *for* $1 \leq j \leq l$ *with* $i_{l+1} = i_1$.

Proof. **((i) ⇒ (iii))** Suppose that G possesses a homogeneous cycle C in G which satisfies (6.7) and let $C = (\vec{e_1}, \dots, \vec{e_l})$ be such homogeneous cycle, where $e_j = \{i_j, i_{j+1}\}$ for $1 \leq j \leq l$ with $i_{j+1} = i_1$. Then one has

$$\sum_{j=1}^{l} q_j \rho(\vec{e_j}) = (0, \dots, 0),$$

where $q_j = 1$ (resp. $q_j = -1$) if $\vec{e_j} = (i_j, i_{j+1})$ (resp. if $\vec{e_j} = (i_{j+1}, i_j)$) for $1 \leq j \leq l$. Since C is homogeneous, one has $\sum_{j=1}^{l} q_j = 0$, which implies that the integer points $\rho(\vec{e_1}), \dots, \rho(\vec{e_l})$ are not affinely independent.

Let $v_j = \rho(\vec{e_j})$ for $1 \leq j \leq l$. In order to show that $\mathcal{P}_G$ is not simplicial, we may find a face of $\mathcal{P}_G$ containing $v_1, \dots, v_l$.

Let $a_1, \dots, a_d$ be integers. We write $\mathcal{H} \subset \mathbb{R}^d$ for the hyperplane defined by the equation $a_1x_1 + \cdots + a_dx_d = 1$ and $\mathcal{H}^{(+)} \subset \mathbb{R}^d$ for the closed half space defined by the inequality $a_1x_1 + \cdots + a_dx_d \leq 1$. By determining $a_1, \dots, a_d$, we make $\mathcal{H}$ a supporting hyperplane of a face $\mathcal{F}$ of $\mathcal{P}_G$ satisfying $\{v_1, \dots, v_l\} \subset \mathcal{F}$ and $\mathcal{P}_G \subset \mathcal{H}^{(+)}$.

First, let $a_{i_j} = \mu_C(i_j)$ for $1 \leq j \leq l$. It then follows easily that v_j lies on the hyperplane defined by the equation $\sum_{j=1}^{l} a_{i_j} x_{i_j} = 1$.

Next, we determine a_k with $k \in A$, where $A = \{1, \dots, d\} \setminus \{i_1, \dots, i_l\}$. We set

$$a_k = \max(\{a_{i_j} - \mathrm{dist}_G(i_j, k)\} \cup \{0\}).$$

In particular, we have $a_k = 0$ when there is no i_j with $\mathrm{dist}_G(i_j, k) < \infty$. Here, we notice that one has

$$a_k \leq a'_k, \tag{6.8}$$

where $a'_k = \min(\{a_{i_{j'}} + \mathrm{dist}_G(k, i_{j'})\})$. In fact, if $a_k > a'_k$, then there are i_j and $i_{j'}$ such that $\mathrm{dist}_G(i_j, k) < \infty, \mathrm{dist}_G(k, i_{j'}) < \infty$ and $a_{i_j} - \mathrm{dist}_G(i_j, k) > a_{i_{j'}} + \mathrm{dist}_G(k, i_{j'})$. Since $\mathrm{dist}_G(i_j, k) + \mathrm{dist}_G(k, i_{j'}) \geq \mathrm{dist}_G(i_j, i_{j'})$, one has

$$\mu_C(i_j) - \mu_C(i_{j'}) = a_{i_j} - a_{i_{j'}} > \mathrm{dist}_G(i_j, k) + \mathrm{dist}_G(k, i_{j'}) \geq \mathrm{dist}_G(i_j, i_{j'}).$$

This contradicts (6.7).

We finish determining the integers $a_1, \ldots, a_d$. Since each v_j lies on $\mathcal{H}$, in order for $\mathcal{F} = \mathcal{P}_G \cap \mathcal{H}$, we may prove $\mathcal{P}_G \subset \mathcal{H}^{(+)}$.

Let $(i,j) \in A(G)$. When $i \in \{i_1, \ldots, i_l\}$ and $j \in A$, then one has $a_j \geq \max(\{a_i - 1, 0\})$ by the definition of a_j. Hence, $a_i - a_j \leq 1$. When $i \in A$ and $j \in \{i_1, \ldots, i_l\}$, then one has $a_i \leq a_j + 1$ by (6.8). Hence, $a_i - a_j \leq 1$.

Let

$$B = \{k \in A : \text{there is } i_j \text{ with } \operatorname{dist}_G(i_j, k) < \infty\}$$

and

$$C = \{k \in A : \text{there is } i_{j'} \text{ with } \operatorname{dist}_G(k, i_{j'}) < \infty\}.$$

Again, let $(i,j) \in A(G)$. In each case of the nine cases below, a routine computation easily leads that $\rho((i,j)) \in \mathcal{H}^{(+)}$.

(1) $i \in B \setminus C$ and $j \in B \setminus C$;
(2) $i \in C \setminus B$ and $j \in C \setminus B$;
(3) $i \in C \setminus B$ and $j \in B \setminus C$;
(4) $i \in C \setminus B$ and $j \in B \cap C$;
(5) $i \in C \setminus B$ and $j \notin B \cup C$;
(6) $i \in B \cap C$ and $j \in B \setminus C$;
(7) $i \in B \cap C$ and $j \in B \cap C$;
(8) $i \notin B \cup C$ and $j \in B \setminus C$;
(9) $i \notin B \cup C$ and $j \notin B \cup C$.

For example, a routine computation of (1) is as follows. When $a_i = 0$, since $a_j \geq 0$, one has $a_i - a_j \leq 0 \leq 1$. When $a_i > 0$, since $a_j \geq a_i - 1$, one has $a_i - a_j \leq 1$.

Therefore, it follows that $\mathcal{H}$ is a supporting hyperplane of a face of $\mathcal{P}_G$ which is not a simplex.

((iii) $\Rightarrow$ (i)) Suppose that $\mathcal{P}_G$ is not simplicial, i.e., $\mathcal{P}_G$ possesses a facet $\mathcal{F}$ which is not a simplex. Let $v_1, \ldots, v_n$ denote the vertices of $\mathcal{F}$, where $n > d-1$, and $\vec{e_1}, \ldots, \vec{e_n}$ the arrows with $v_j = \rho(\vec{e_j})$ for $1 \leq j \leq n$. We write $\mathcal{H} \subset \mathbb{R}^d$ for the supporting hyperplane $a_1x_1 + \cdots + a_dx_d = 1$ defining $\mathcal{F}$. Since $v_1, \ldots, v_n$ are not affinely independent, there is $(r_1, \ldots, r_n) \in \mathbb{R}^n$ with $(r_1, \ldots, r_n) \neq (0, \ldots, 0)$ satisfying $\sum_{j=1}^n r_j = 0$ and $\sum_{j=1}^n r_jv_j = (0, \ldots, 0)$. By removing r_j with $r_j = 0$, we may assume that $\sum_{j=1}^{n'} r_jv_j = (0, \ldots, 0)$, where $r_j \neq 0$ for $1 \leq j \leq n'$ with $\sum_{j=1}^{n'} r_j = 0$. Let $\vec{e_j} = (i_j, i'_j)$ with $1 \leq i_j, i'_j \leq d$ and let G' denote the subgraph of G with the arrow set $\{\vec{e_1}, \ldots, \vec{e_{n'}}\}$. If $\deg_{G'}(i_j) = 1$ or $\deg_{G'}(i'_j) = 1$, then $r_j = 0$, a contradiction. Thus, $\deg_{G'}(i_j) \geq 2$ and $\deg_{G'}(i'_j) \geq 2$. By Lemma 6.2.7 (b), since $\{\rho(\vec{e_1}), \ldots, \rho(\vec{e_{n'}})\} \subset \mathcal{F}$, it cannot happen that $e_j = e_k$ with $1 \leq j \neq k \leq n'$. Moreover, since every vertex in G' is at least degree 2, there are many cycles in G'. Now, Lemma 6.2.7 (a) says that G' cannot contain any nonhomogeneous cycle. Hence, there is at least one homogeneous cycle in G.

Let $C = (\vec{e_1}, \ldots, \vec{e_l})$ be a homogeneous cycle in G, where $e_j = \{i_j, i_{j+1}\}$ for $1 \leq j \leq l$ with $i_{j+1} = i_1$. Our goal is to show that C satisfies the inequality (6.7).

Let $\Gamma = (k_0, k_1, \ldots, k_m)$ be the directed shortest path in G of length m, where k_0 and k_m belong to $\{i_1, \ldots, i_l\}$. On the one hand, since $\mathbf{e}_{k_j} - \mathbf{e}_{k_{j+1}} \in \mathcal{P}_G$, one has

$a_{k_j} - a_{k_{j+1}} \leq 1$ for $0 \leq j \leq m-1$. Hence, $a_{k_0} - a_{k_m} \leq m = \mathrm{dist}_G(k_0, k_m)$. On the other hand, we have $a_{k_0} - a_{k_m} = \mu_C(k_0) - \mu_C(k_m)$. Thus, $\mu_C(k_0) - \mu_C(k_m) \leq \mathrm{dist}_G(k_0, k_m)$. Therefore, the required inequality (6.7) holds.

((i) ⇒ (ii)) Suppose that $\mathcal{P}_G$ is simplicial. Then there are just $d-1$ vertices in each facet which are linearly independent. Let $M = \begin{pmatrix} v_1 \\ \vdots \\ v_{d-1} \end{pmatrix}$ be the matrix whose row vectors $v_1, \ldots, v_{d-1} \in \mathbb{Z}^d$ are the vertices of an arbitrary facet of $\mathcal{P}_G$ and M' the $(d-1) \times (d-1)$ submatrix of M ignoring the d-th column of M. From the theory of totally unimodular matrices [66], the determinant of M' is equal to ± 1, which means that $\mathcal{P}_G$ is smooth.

((ii) ⇒ (i)) In general, every smooth Fano polytope is $\mathbb{Q}$-factorial. □

For a directed graph G, we say that G is *symmetric* if both (i, j) and (j, i) are contained in $A(G)$, that is, $2|E(G)| = |A(G)|$. Note that when G is symmetric, every arrow of G is contained in a directed cycle of length 2, so $\mathcal{P}_G$ is always a terminal Gorenstein Fano polytope.

Recall that for a connected graph G, we say that G is *2-connected* if the induced subgraph with the vertex set $V(G) \backslash \{i\}$ is still connected for any vertex $i \in V(G)$ and a subgraph of G is a *2-connected component* of G if it is a maximal 2-connected subgraph in G.

For symmetric directed graphs, we obtain the following

Corollary 6.2.9. *Let G be a connected symmetric directed graph. Then the following conditions are equivalent:*

(i) $\mathcal{P}_G$ is $\mathbb{Q}$-factorial;

(ii) $\mathcal{P}_G$ is smooth;

(iii) G contains no even cycle;

(iv) every 2-connected component of G is either one edge or an odd cycle.

Proof. **((i) ⇔ (ii))** It is obvious from the proof of Theorem 6.2.8.

((i) ⇒ (iii)) Suppose that G possesses an even cycle C in G of length $2l$. Let $C = (e_{i_1}, \ldots, e_{i_{2l}})$ be a cycle, where $e_j = \{i_j, i_{j+1}\}$ for $1 \leq j \leq 2l$ with $i_{2l+1} = i_1$. Since G is symmetric, there are arrows of G

$$(i_2, i_1), (i_2, i_3), (i_4, i_3), (i_4, i_5), \ldots, (i_{2l}, i_{2l-1}), (i_{2l}, i_1).$$

We define $v_1, \ldots, v_{2l} \in \mathbb{R}^d$ by setting

$$v_j = \begin{cases} \rho((i_{j+1}, i_j)), & j = 1, 3, \ldots, 2l-1, \\ \rho((i_j, i_{j+1})), & j = 2, 4, \ldots, 2l. \end{cases}$$

Then one has

$$\sum_{j=1}^{l} v_{2j-1} + \sum_{j=1}^{l}(-1)v_{2j} = (0, \ldots, 0).$$

Thus, $v_1, \ldots, v_{2l}$ are not affinely independent. Hence, we may show that there is a face $\mathcal{F}$ of $\mathcal{P}_G$ with $\{v_1, \ldots, v_{2l}\} \subset \mathcal{F}$.

Now, we have $v_{2j-1} = -\mathbf{e}_{i_{2j-1}} + \mathbf{e}_{i_{2j}}$ and $v_{2j} = \mathbf{e}_{i_{2j}} - \mathbf{e}_{i_{2j+1}}$ for $1 \leq j \leq l$. Thus, $v_1, \ldots, v_{2l}$ lie on the hyperplane $\mathcal{H} \subset \mathbb{R}^d$ defined by the equation $x_{i_2}+x_{i_4}+\cdots+x_{i_{2l}} = 1$. In addition, it is clear that $\rho(\vec{e})$ is contained in $\mathcal{H}^{(+)} \subset \mathbb{R}^d$ for any arrow $\vec{e}$ of G. Hence, $\mathcal{H}$ is a supporting hyperplane defining a face $\mathcal{F}$ of $\mathcal{P}_G$ with $\{v_1, \ldots, v_{2l}\} \subset \mathcal{F}$. Therefore, $\mathcal{P}_G$ is not simplicial.

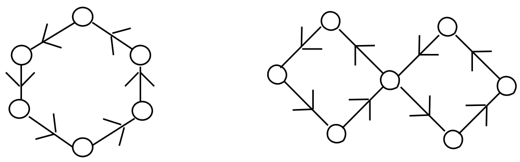
((iii) $\Rightarrow$ (iv)) We prove this implication by elementary graph theory. Suppose that there is a 2-connected component of G which is neither one edge nor an odd cycle. Let G' be such 2-connected subgraph of G. Now, an arbitrary 2-connected graph with at least 3 vertices can be obtained by the following method: starting from a cycle and repeatedly appending an H-path to a graph H that has been already constructed. (Consult, e.g., [77].) Since G' is not one edge, G' has at least 3 vertices. Thus, there is one cycle C_1 and $(m-1)$ paths $\Gamma_2, \ldots, \Gamma_m$ such that $G' = C_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_m$. Since G' is not an odd cycle, one has $G' = C_1$, where C_1 is an even cycle, or $m > 1$. Suppose that $m > 1$ and C_1 is an odd cycle. Let v and w be distinct two vertices of C_1 which are intersected with Γ_2. Then there are two paths in C_1 from v to w. Since C_1 is odd, the parities of the lengths of such two paths are different. By attaching the path Γ_2 to one or another of such two paths, we can construct an even cycle. Therefore, there exists an even cycle.

((iv) $\Rightarrow$ (i)) Suppose that each 2-connected component of G is either one edge or an odd cycle. Then there is no homogeneous cycle in G. Hence, by Theorem 6.2.8, $\mathcal{P}_G$ is simplicial. □

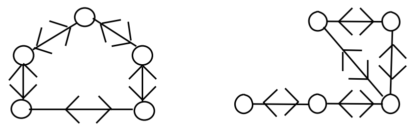
Let G and G' be connected symmetric directed graphs. The conditions under which $\mathcal{P}_G$ is unimodular equivalent to $\mathcal{P}_{G'}$ are discussed in [48, Section 4.2]. As its analogue, we obtain the following

Theorem 6.2.10 (See [48, Theorem 4.5])**.** *For a directed graph G (resp. G'), let $G_1, \ldots, G_m$ (resp. $G'_1, \ldots, G'_{m'}$) denote the 2-connected components of G (resp. G'). Then $\mathcal{P}_G$ is unimodular equivalent to $\mathcal{P}_{G'}$ if and only if $m = m'$ and G_i is isomorphic to G'_i by renumbering if necessary.*

Example 6.2.11. (a) When G is a directed cycle of length $d+1$, $\mathcal{P}_G$ is a smooth Fano polytope, whose corresponding toric Fano variety is a d-dimensional complex projective space $\mathbb{P}^d$. Moreover, each 2-connected component of a directed graph corresponds to each direct factor of a corresponding toric Fano variety. For example, the graph depicted on the left-hand side (resp. right-hand side) yields a smooth Fano polytope which corresponds to $\mathbb{P}^5$ (resp. $\mathbb{P}^3 \times \mathbb{P}^3$).



(b) When G is a symmetric directed graph without even cycle, $\mathcal{P}_G$ is a smooth Fano polytope, whose corresponding toric Fano variety is a direct product of copies of $\mathbb{P}^1$ or del Pezzo variety V^{2k}. (See Section 3.) For example, the graph depicted on the left-hand side (resp. right-hand side) yields a smooth Fano polytope which corresponds to V^4 (resp. $\mathbb{P}^1 \times \mathbb{P}^1 \times V^2$).



6.2.3 The case where G possesses no even cycle

In this subsection, we show that every pseudo symmetric smooth Fano polytope can be obtained from some directed graph with no even cycle. This fact includes the case of centrally symmetric smooth Fano polytopes.

Let $\mathcal{P} \subset \mathbb{R}^d$ be a Fano polytope.

- We call $\mathcal{P}$ *centrally symmetric* if $\mathcal{P} = -\mathcal{P} = \{-\alpha : \alpha \in \mathcal{P}\}$.
- We call $\mathcal{P}$ *pseudo symmetric* if there is a facet $\mathcal{F}$ of $\mathcal{P}$ such that $-\mathcal{F}$ is also a facet of $\mathcal{P}$. By the definition, every centrally symmetric Fano polytope is pseudo symmetric.
- A *del Pezzo polytope* of dimension $2k$ is a convex polytope

$$\mathrm{conv}(\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_{2k}, \pm(\mathbf{e}_1 + \cdots + \mathbf{e}_{2k})\}),$$

  whose corresponding variety is so-called a *del Pezzo variety* V^{2k}. In particular, del Pezzo polytopes are centrally symmetric smooth Fano polytopes.
- A *pseudo del Pezzo polytope* of dimension $2k$ is a convex polytope

$$\mathrm{conv}(\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_{2k}, \mathbf{e}_1 + \cdots + \mathbf{e}_{2k}\}),$$

  whose corresponding variety is so-called a *pseudo del Pezzo variety* $\tilde{V}^{2k}$. In particular, pseudo del Pezzo polytopes are pseudo symmetric smooth Fano polytopes.

There is a well-known fact on the characterization of centrally symmetric or pseudo symmetric smooth Fano polytopes.

Theorem 6.2.12 ([76])**.** *Any centrally symmetric smooth Fano polytope splits into copies of the closed interval* $[-1, 1]$ *or a del Pezzo polytope.*

Theorem 6.2.13 ([17, 76])**.** *Any pseudo symmetric smooth Fano polytope splits into copies of the closed interval* $[-1, 1]$ *or a del Pezzo polytope or a pseudo del Pezzo polytope.*

Somewhat surprisingly, we also give the complete characterization of centrally symmetric or pseudo symmetric smooth Fano polytopes by means of directed graphs. In fact,

Theorem 6.2.14 ([39, Theorem 3.3])**.** *(i) Any centrally symmetric smooth Fano polytope can be obtained from a symmetric directed graph with no even cycle.*
(ii) Any pseudo symmetric smooth Fano polytope can be obtained from a directed graph with no even cycle.

Proof. First, we prove (ii). Let $\mathcal{P}$ be an arbitrary pseudo symmetric smooth Fano polytope of dimension d. By Theorem 6.2.13, $\mathcal{P}$ splits into $\mathcal{P}_1, \dots, \mathcal{P}_m$ which are copies of $[-1, 1]$ or a del Pezzo polytope or a pseudo del Pezzo polytope. Let $\mathcal{P}_1, \dots, \mathcal{P}_{m'}$ be del Pezzo polytopes, $\mathcal{P}_{m'+1}, \dots, \mathcal{P}_{m''}$ pseudo del Pezzo polytopes and $\mathcal{P}_{m''+1}, \dots, \mathcal{P}_m$ the closed intervals $[-1, 1]$. Then the following arguments easily follow.

- Let, say, $\mathcal{P}_1$ be a del Pezzo polytope of dimension $2k_1$ and G_1 a symmetric directed graph with its arrow set

  $$A(G_1) = \{(1, 2), (2, 1), \dots, (2k_1, 2k_1{+}1), (2k_1{+}1, 2k_1), (2k_1{+}1, 1), (1, 2k_1{+}1)\}.$$

  Then G_1 is an odd cycle, i.e., there is no even cycle, so $\mathcal{P}_{G_1}$ is smooth by Corollary 6.2.9 and we can check that $\mathcal{P}_{G_1}$ is unimodular equivalent to $\mathcal{P}_1$.

- Let, say, $\mathcal{P}_{m'+1}$ be a pseudo del Pezzo polytope of dimension $2k_1$ and G_1' a directed graph with its arrow set

  $$A(G_1') = A(G_1) \setminus \{(2, 1)\},$$

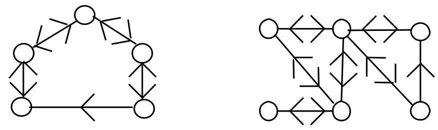
  i.e., we miss one arrow from G_1. Then we can also check that $\mathcal{P}_{G_1'}$ is unimodular equivalent to $\mathcal{P}_{m'+1}$.

- A directed graph consisting of only one symmetric edge yields the smooth Fano polytope of dimension 1, which is nothing but the closed interval $[-1, 1]$.

By connecting the above graphs with one vertex, we obtain the directed graph with no even cycle which yields the required smooth Fano polytope $\mathcal{P}$.

Moreover, del Pezzo polytopes and the closed interval $[-1, 1]$ are constructed by symmetric directed graphs. Therefore, in the similar way to the above construction, by Theorem 6.2.12, we can also find the symmetric directed graph G with no even cycle such that $\mathcal{P}_G$ is unimodular equivalent to $\mathcal{P}$ for any centrally symmetric smooth Fano polytope $\mathcal{P}$. □

Example 6.2.15. The graph depicted on the left-hand side (resp. right-hand side) yields a smooth Fano polytope which corresponds to $\tilde{V}^4$ (resp. $\mathbb{P}^1 \times V^2 \times \tilde{V}^2$).



Example 6.2.16. In [4], a *symmetric* (not centrally symmetric) smooth toric Fano variety is given, which is important from the viewpoint whether smooth toric Fano variety admits an Einstein-Kähler metric, and some examples of symmetric smooth Fano varieties are provided in [4, Example 4.2 – 4.4]. Note that smooth toric Fano varieties corresponding to centrally symmetric smooth Fano polytopes and direct products of copies of complex projective spaces are always symmetric.

Let m be a positive integer and G_1 a directed graph with its arrow set

$$A(G_1) = \{(1,2),(2,3),\dots,(2m+1,2m+2),(2m+2,1),(1,m+2),(m+2,1)\}.$$

Then $\mathcal{P}_{G_1}$ is a smooth Fano polytope of dimension $2m+1$ which corresponds to the example of the case where $k=1$ described in [4, Example 4.2].

Let G_2 be a directed graph with its arrow set

$$A(G_2) = A(G_1) \cup \{(1,2m+3),(2m+3,1),(m+2,2m+3),(2m+3,m+2)\}.$$

Then $\mathcal{P}_{G_2}$ is a smooth Fano polytope of dimension $2m+2$ which is the example of the case where $k=1$ described in [4, Example 4.3].

6.2.4 Primitive collections of $\mathcal{P}_G$

In this section, we describe the primitive collections of $\mathcal{P}_G$ in terms of directed graphs, where we assume that $\mathcal{P}_G$ is smooth.

Primitive collections, introduced by Batyrev [2], are very important and convenient for investigating smooth toric Fano varieties. We refer the reader to, e.g., [65], for some aspects on algebraic geometry of smooth toric Fano varieties using primitive collections.

Let $\mathcal{P}$ be a smooth Fano polytope and $V(\mathcal{P})$ the set of its vertices. A nonempty subset $P \subset V(\mathcal{P})$ is called a *primitive collection* of $\mathcal{P}$ if $\mathrm{conv}(P)$ is not a face of $\mathcal{P}$ but $\mathrm{conv}(P \setminus \{v\})$ is a face of $\mathcal{P}$ for every $v \in P$.

Theorem 6.2.17 ([39, Theorem 4.1])**.** *Let G be a connected directed graph on the vertex set $\{1,\dots,d\}$ such that $\mathcal{P}_G$ is a smooth Fano polytope of dimension $d-1$. Let $\mathcal{A}_G \subset 2^{A(G)}$ be the set consisting of $A \subset A(G)$ which satisfies that there exists some nonhomogeneous cycle C in G with $\Delta_C^{(+)} \subset A$. Then there is a one-to-one correspondence between the primitive collections of $\mathcal{P}_G$ and the minimal elements in $\mathcal{A}_G$ by inclusion.*

Proof. For $A \subset A(G)$, let $(A) = \{\rho(\vec{e}) : \vec{e} \in A\}$. Since $\mathcal{P}_G$ is terminal, there is a one-to-one correspondence between the vertices of $\mathcal{P}_G$ and the arrows of G. Thus, it suffices to show that for every $A \in \mathcal{A}_G$, (A) is not contained in any face of $\mathcal{P}_G$, and that for every $A' \in 2^{A(G)} \setminus \mathcal{A}_G$, there exists a face of $\mathcal{P}_G$ containing (A').

For $A \in \mathcal{A}_G$, suppose that there exists a face containing (A) and let $\mathcal{H}$ be a supporting hyperplane of such face defined by $a_1x_1+\cdots+a_dx_d = 1$ with $\mathcal{P}_G \subset \mathcal{H}^{(+)}$, where $a_i \in \mathbb{Q}$. Since $A \in \mathcal{A}_G$, there exists a nonhomogeneous cycle $C = (\vec{e_1}, \ldots, \vec{e_l})$ in G, where $e_j = \{i_j, i_{j+1}\}$ for $1 \leq j \leq l$ with $i_{l+1} = i_1$, such that $\Delta_C^{(+)} \subset A$. Then $(\Delta_C^{(+)}) \subset (A) \subset \mathcal{H}$ and $(\Delta_C^{(-)}) \subset \mathcal{H}^{(+)}$. Thus, one has

$$0 = \sum_{j=1}^{l}(a_{i_j} - a_{i_{j+1}}) = \sum_{\vec{e_j} \in \Delta_C^{(+)}} (a_{i_j} - a_{i_{j+1}}) - \sum_{\vec{e_j} \in \Delta_C^{(-)}} (a_{i_{j+1}} - a_{i_j}) \geq |\Delta_C^{(+)}| - |\Delta_C^{(-)}| > 0,$$

a contradiction. Hence, (A) is not contained in any face of $\mathcal{P}_G$. In particular, this assertion holds for every minimal element in $\mathcal{A}_G$.

Moreover, by Lemma 9.1.11 below, for every $A' \in 2^{A(G)} \setminus \mathcal{A}_G$, there exists a face of $\mathcal{P}_G$ containing (A'). In particular, for every minimal element A in $\mathcal{A}_G$, there exists a face containing $(A \setminus \{\vec{e}\})$, where $\vec{e}$ is an arbitrary arrow in A. This implies that if A is minimal in $\mathcal{A}_G$, then (A) is a primitive collection of $\mathcal{P}_G$. On the other hand, we know that if $A' \subset A(G)$ is not a minimal element in $\mathcal{A}_G$, then A' cannot be a primitive collection of $\mathcal{P}$.

Therefore, we conclude that there is a one-to-one correspondence between the primitive collections of $\mathcal{P}_G$ and the minimal elements in $\mathcal{A}_G$. □

Lemma 6.2.18. *Work with the same notations as in Theorem 6.2.17. For every $A' \in 2^{A(G)} \setminus \mathcal{A}_G$, there exists a face of $\mathcal{P}_G$ containing (A').*

Proof. For $A' \in 2^{A(G)} \setminus \mathcal{A}_G$, let G' denote the subgraph in G with $A(G') = A'$ and $G'_1, \ldots, G'_m$ connected components of G'. Then there is no cycle in each G'_i, i.e., each G'_i is a tree. In fact, there is no nonhomogeneous cycle since $A' \notin \mathcal{A}_G$ and no homogeneous cycle since $\mathcal{P}_G$ is simplicial. (See the proof of ((i) $\Rightarrow$ (iii)) of Theorem 6.2.8.)

Let $a_1, \ldots, a_d$ be integers and let $\mathcal{H} \subset \mathbb{R}^d$ and $\mathcal{H}^{(+)} \subset \mathbb{R}^d$ denote as in the proof of ((i) $\Rightarrow$ (iii)) of Theorem 6.2.8. In order to find a face of $\mathcal{P}_G$ containing (A'), we determine $a_1, \ldots, a_d$ such that $\mathcal{H}$ becomes a supporting hyperplane of a face $\mathcal{F}$ of $\mathcal{P}_G$ with $(A') \subset \mathcal{F}$ and $\mathcal{P}_G \subset \mathcal{H}^{(+)}$.

The first step. In this step and the next step, we determine a_j for all $j \in V(G')$.

Let $V(G'_i) = \{q_1^{(i)}, \ldots, q_{k_i}^{(i)}\}$ for $1 \leq i \leq m$ and $c_1, \ldots, c_m$ some integers. We choose one vertex from each G'_i, say, $q_1^{(1)}, \ldots, q_1^{(m)}$, and set $a_{q_1^{(i)}} = c_i$. For $1 \leq i \leq m$ and $2 \leq j \leq k_i$, we define $a_{q_j^{(i)}}$ by setting

$$a_{q_j^{(i)}} = \begin{cases} a_{q_{j'}^{(i)}} - 1, & \text{if } (q_{j'}^{(i)}, q_j^{(i)}) \in A(G'_i), \\ a_{q_{j'}^{(i)}} + 1, & \text{if } (q_j^{(i)}, q_{j'}^{(i)}) \in A(G'_i). \end{cases}$$

Notice that since G'_i is a tree, each $a_{q_j^{(i)}}$ is uniquely determined. It then follows that (A') is contained in the hyperplane defined by $\sum_{1\leq j\leq k_i, 1\leq i\leq m} a_{q_j^{(i)}} x_{q_j^{(i)}} = 1$.

The second setp. Next, we give the exact values of $a_{q_j^{(i)}}$'s by determining integers $c_1, \ldots, c_m$. For this, we define a directed graph. Let $\tilde{H}$ be (not necessary connected) a directed graph on the vertex set $\{1, \ldots, m\}$ with the arrow set $A(\tilde{H})$ consisting of (i, j), where $1 \leq i, j \leq m$, such that there exists a directed path in G from some vertex of G'_i to some vertex of G'_j. Remark that $\tilde{H}$ may have some loops. And we give a weight b_{ij} on every arrow $(i, j) \in A(\tilde{H})$ defined by

$$b_{ij} = \max(\{a_\alpha - a_\beta - \mathrm{dist}_G(\alpha, \beta) + 1 - (c_i - c_j)\}),$$

where $\alpha \in V(G'_i)$ and $\beta \in V(G'_j)$. Then we have

$$l \geq \sum_{j=1}^{l} b_{i_j i_{j+1}} \tag{6.9}$$

for every directed cycle $\tilde{C} = (i_1, \ldots, i_l)$ in $\tilde{H}$ of length $l \geq 1$. In fact, for $1 \leq j \leq l$, let $b_{i_j i_{j+1}} = a_{\alpha_{i_j}} - a_{\beta_{i_{j+1}}} - \mathrm{dist}_G(\alpha_{i_j}, \beta_{i_{j+1}}) + 1 - (c_{i_j} - c_{i_{j+1}})$, where $\alpha_{i_j} \in V(G'_{i_j})$ and $\beta_{i_{j+1}} \in V(G'_{i_{j+1}})$. Since each G'_{i_j} is a tree, there is a unique path in G'_{i_j} from β_{i_j} to α_{i_j}, say, $\Gamma'_{i_j} = (e_{k_1}, \ldots, e_{k_{l_j - 1}})$, where $e_{k_a} = \{\gamma_{k_a}, \gamma_{k_{a+1}}\} \in E(G'_{i_j})$ for $1 \leq a \leq l_j - 1$ with $\gamma_{k_1} = \beta_{i_j}$ and $\gamma_{k_{l_j}} = \alpha_{i_j}$. Then we notice that there is a cycle

$$C = (\Gamma'_{i_1}, \Gamma_{i_1}, \Gamma'_{i_2}, \Gamma_{i_2}, \ldots, \Gamma'_{i_l}, \Gamma_{i_l})$$

in G, where Γ_{i_j} is a directed path from α_{i_j} to $\beta_{i_{j+1}}$ of length $\mathrm{dist}_G(\alpha_{i_j}, \beta_{i_{j+1}})$. Let $\delta_{i_j}^{(+)}$ (resp. $\delta_{i_j}^{(-)}$) denote the number of arrows such that $\vec{e_{k_a}} = (\gamma_{k_a}, \gamma_{k_{a+1}})$ (resp. $\vec{e_{k_a}} = (\gamma_{k_{a+1}}, \gamma_{k_a})$). By the definitions of $a_{\beta_{i_j}}$ and $a_{\alpha_{i_j}}$, we have $a_{\beta_{i_j}} - a_{\alpha_{i_j}} = \delta_{i_j}^{(+)} - \delta_{i_j}^{(-)}$. Moreover, by our assumption, we have

$$\sum_{j=1}^{l} \delta_{i_j}^{(+)} + \sum_{j=1}^{l} \mathrm{dist}_G(\alpha_{i_j}, \beta_{i_{j+1}}) \geq \sum_{j=1}^{l} \delta_{i_j}^{(-)},$$

otherwise C becomes a nonhomogeneous cycle satisfying $\Delta_C^{(+)} \subset \cup_{j=1}^{l} A(G'_{i_j}) = A'$. Hence, we have

$$\begin{aligned} \sum_{j=1}^{l} (\delta_{i_j}^{(+)} - \delta_{i_j}^{(-)} + \mathrm{dist}_G(\alpha_{i_j}, \beta_{i_{j+1}})) &= \sum_{j=1}^{l} (a_{\beta_{i_{j+1}}} - a_{\alpha_{i_j}} + \mathrm{dist}_G(\alpha_{i_j}, \beta_{i_{j+1}})) \\ &= \sum_{j=1}^{l} (-b_{i_j i_{j+1}} + 1 - (c_{i_j} - c_{i_{j+1}})) = l - \sum_{j=1}^{l} b_{i_j i_{j+1}} \geq 0. \end{aligned}$$

By considering the directed graph $\tilde{H}$, we give the exact values of $c_1, \ldots, c_m$. (Here, even if $\tilde{H}$ is not connected, we may do the same operations as the following to each connected component, so we assume the connectedness of $\tilde{H}$.) In $\tilde{H}$, if there are some directed cycles, then we pick up one $\tilde{C} = (i_1, \ldots, i_l)$ satisfying that a nonnegative integer $l - \sum_{j=1}^{l} b_{i_j i_{j+1}}$ is the smallest. And, for $1 \leq j \leq l-1$, we set $c_{i_j} - c_{i_{j+1}} = 1 - b_{i_j i_{j+1}}$. Next, we do this to other directed cycle $\tilde{C}' = (i'_1, \ldots, i'_{l'})$ in $\tilde{H}$ such that $l' - \sum_{j=1}^{l'} b_{i'_j i'_{j+1}}$ is the second smallest. If $\tilde{C}$ and $\tilde{C}'$ are distinct in $\tilde{H}$, then we find a path in $\tilde{H}$ from some vertex of $\tilde{C}$ to some vertex of $\tilde{C}'$, say, $(i''_1, \ldots, i''_{l''})$ with $i''_1 = i_1$ and $i''_{l''} = i'_1$, and we also define $c_{i''_j} - c_{i''_{j+1}} = 1 - b_{i''_j i''_{j+1}}$ in the similar way. After this, similarly, we define $c_{i'_j} - c_{i'_{j+1}} = 1 - b_{i'_j i'_{j+1}}$. In this way, we define $c_a - c_b$ for all $1 \leq a, b \leq m$ with $(a, b) \in A(\tilde{H})$.

After all, thanks to (6.9), it is easy to see that we have

$$c_i - c_j \leq \min\left(\left\{l - \sum_{j=1}^{l} b_{k_j k_{j+1}}\right\}\right) \tag{6.10}$$

for every $(i, j) \in A(\tilde{H})$, where l is the length of some directed path $(k_1, \ldots, k_{l+1})$ in $\tilde{H}$ from $i = k_1$ to $j = k_{l+1}$.

Finally, we set

$$\min(\{a_{q_j^{(i)}} : 1 \leq j \leq k_i, 1 \leq i \leq m\}) = 0.$$

Then we obtain the exact values of all $a_{q_j^{(i)}}$'s.

The third step. In this step, we determine a_k for all $k \in V(G) \setminus V(G')$. For $k \in V(G) \setminus V(G')$, let

$$a_k = \max(\{a_\alpha - \mathrm{dist}_G(\alpha, k)\} \cup \{0\}),$$

where $\alpha \in V(G'_i)$ for some $1 \leq i \leq m$. In particular, we have $a_k = 0$ when there is no α with $\mathrm{dist}_G(\alpha, k) < \infty$. Then one has

$$a_k \leq a'_k, \tag{6.11}$$

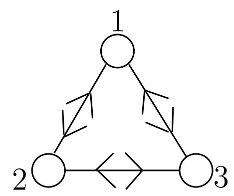
where $a'_k = \min(\{a_\beta + \mathrm{dist}_G(k, \beta)\})$ with $\beta \in V(G'_j)$ for some $1 \leq j \leq m$. In fact, if $a_k > a'_k$, then there exist α and β such that $\mathrm{dist}_G(\alpha, k) < \infty$ and $\mathrm{dist}_G(k, \beta) < \infty$. Since $\mathrm{dist}_G(\alpha, k) + \mathrm{dist}_G(k, \beta) \geq \mathrm{dist}_G(\alpha, \beta)$, one has

$$\begin{aligned}
0 > a'_k - a_k &= a_\beta + \mathrm{dist}_G(k, \beta) - a_\alpha + \mathrm{dist}_G(\alpha, k) \\
&\geq a_\beta - a_\alpha + \mathrm{dist}_G(\alpha, \beta) \\
&\geq -b_{ij} + 1 - (c_i - c_j) \quad \text{(by the definition of } b_{ij}) \\
&\geq -b_{ij} + 1 - (1 - b_{ij}) \quad \text{(by (9.7))} \\
&= 0,
\end{aligned}$$

a contradiction.

The fourth step. By the previous three steps, we finish determining the integers $a_1, \ldots, a_d$. Thus, what we need is to prove $\mathcal{P} \subset \mathcal{H}^{(+)}$. However, since the definition of a_k for $k \in V(G) \setminus V(G')$ is the same as the proof of ((i) $\Rightarrow$ (iii)) and we also have (6.11), the rest parts are also the same, proving the assertion. $\square$

Example 6.2.19. Let us consider the primitive collections of the del Pezzo polytope of dimension 2, whose corresponding directed graph is the following:



Then there are 9 primitive collections, which correspond to

$$\begin{gathered}\{(1,2),(2,1)\},\{(2,3),(3,2)\},\{(1,3),(3,1)\},\\ \{(1,2),(2,3)\},\{(2,3),(3,1)\},\{(3,1),(1,2)\},\\ \{(1,3),(3,2)\},\{(3,2),(2,1)\},\{(2,1),(1,3)\}.\end{gathered}$$

Part III

Affine semigroup rings

Chapter 7

Introduction to affine semigroup rings

In this part, as the third aspect of the studies on integral convex polytopes, we consider *affine semigroup rings* associated with integral convex polytopes. Affine semigroup rings often appear and play several important roles in the area of not only commutative algebra but also combinatorics and other fields.

We will summarize some basic notions, definitions and well-known results on affine semigroup rings. Most parts are refered from [12, Chapter 6].

An *affine semigroup* C is a finitely generated semigroup which for some n is isomorphic to a subsemigroup of $\mathbb{Z}^n$ containing 0. Let K be a field. We write $K[C]$ for the vector space whose basis consists of all the elemnts of C, which is denoted by X^c for $c \in C$. Then $K[C]$ carries also a natural multiplication whose table is given by $X^c X^{c'} = X^{c+c'}$. Thus, in particular, $K[C]$ is a K-algebra, which we call an *affine semigroup ring*.

An affine semigroup C is called *normal* if it satisfies the following condition: if $mz \in C$ for some $z \in \mathbb{Z}C$ and $m \in \mathbb{Z}_{>0}$, then $z \in C$. One sees immediately that C must be normal if $K[C]$ is a normal domain. Then it is well known (e.g. [12, Theorem 6.1.4]) that $K[C]$ is normal if and only if C is a normal semigroup. Moreover, it is also well known as Hochster's Theorem (e.g. [12, Theorem 6.3.5]) that when C is normal, then $K[C]$ is Cohen–Macaulay.

Let $\mathcal{P} \subset \mathbb{R}^N$ be an integral convex polytope. A typical example of normal semigroup rings is the Ehrhart ring of $\mathcal{P}$, which is constructed as follows. We define $\mathcal{P}^* \subset \mathbb{R}^{N+1}$ to be the convex hull of all points $(1, \alpha) \in \mathbb{R}^{N+1}$ with $\alpha \in \mathcal{P}$ and let $\mathcal{A}_\mathcal{P} = \mathcal{P}^* \cap \mathbb{Z}^{N+1}$ denote the set of integer points in $\mathcal{P}^*$. Then $\mathbb{R}_{\geq 0}\mathcal{A}_\mathcal{P} \cap \mathbb{Z}^{N+1}$ is a normal semigroup and so $K[\mathbb{R}_{\geq 0}\mathcal{A}_\mathcal{P} \cap \mathbb{Z}^{N+1}]$ is a normal semigroup ring, which is called the *Ehrhart ring* of $\mathcal{P}$.

On the rest of this part, we will discuss the affine semigroup rings associated graphs, which we call edge rings, in Chapter 8. We will study depth of edge rings.

In Chapter 9, we will consider the affine semigroup rings arising from cyclic polytopes. We will study their normality, non-very ampleness, Cohen–Macaualyness and Gorensteinness. And we will also introduce the other affine semigroup rings arising from cyclic polytopes, which are generated only by their vertices, and discuss their properties.

Chapter 8

Affine semigroup rings arising from graphs

In this chapter, we will study the depth of edge rings. In Section 8.1, we will consider the depth of edge rings (toric ideals) of non-normal graphs. Note that when graphs are normal, then their edge rings are always Cohen–Macaulay, which means that the depth is equal to its Krull dimension. In Section 8.2, we will discuss the depth of initial ideals of toric ideals of normal graphs.

8.1 Depth of non-normal edge ring

First, we consider the depth of non-normal edge rings.

Let G be a finite simple graph on the vertex set $[d] = \{1, \ldots, d\}$ and $E(G) = \{e_1, \ldots, e_m\}$ its edge set. Let $K[\mathbf{t}] = K[t_1, \ldots, t_d]$ be the polynomial ring in d variables over a field K and write $K[G]$ for the subring of $K[\mathbf{t}]$ generated by those squarefree quadratic monomials $\mathbf{t}^e = t_i t_j$ with $e = \{i, j\} \in E(G)$. The affine semigroup ring $K[G]$ is called the *edge ring* of G. Let Krull-dim $K[G]$ denote the Krull dimension of $K[G]$ and depth $K[G]$ the depth of $K[G]$. Let $K[\mathbf{x}] = K[x_1, \ldots, x_m]$ be the polynomial ring in m variables over a field K. The kernel I_G of the surjective homomorphism $\pi : K[\mathbf{x}] \to K[G]$ defined by setting $\pi(x_i) = \mathbf{t}^{e_i}$ for $i = 1, \ldots, m$ is called the *toric ideal* of G. One has $K[G] \cong K[\mathbf{x}]/I_G$. If G is connected and is non-bipartite (resp. bipartite), then Krull-dim $K[G] = d$ (resp. Krull-dim $K[G] = d-1$).

The criterion of normality [54, Corollary 2.3] of edge rings guarantees that $K[G]$ is normal if either G is bipartite or $d \leq 6$. If $d = 7$, then there exists a finite graph G for which $K[G]$ is nonnormal. However, it follows easily that $K[G]$ is Cohen–Macaulay whenever $d \leq 7$. Computing depth of edge rings of all connected nonbipartite graphs G with 7 vertices shows that the depth of $K[G]$ is at least 7. Moreover, our computational experiment would naturally lead the authors into the temptation to give the following

Conjecture 8.1.1. *Let G be a finite graph on $[d]$ with $d \geq 7$. Then* depth $K[G] \geq 7$.

Now, even though Conjecture 8.1.1 is completely open, by taking Conjecture 8.1.1 into consideration, this section will be devoted to proving the following

Theorem 8.1.2 ([32, Theorem 0.2])**.** *Given integers f and d with $7 \leq f \leq d$, there exists a finite graph G on $[d]$ with* $\operatorname{depth} K[G] = f$ *and with* $\text{Krull-dim}\, K[G] = d$.

Let $k \geq 1$ be an arbitrary integer and G_{k+6} the finite graph on $[k+6]$ of Figure 8.1. The essential part of a proof of Theorem 8.1.2 is to show that

$$\operatorname{depth} K[G_{k+6}] = \operatorname{depth} K[\mathbf{x}]/I_{G_{k+6}} = 7. \tag{8.1}$$

In Subsection 8.1.1, by virtue of the formula given in [10, Theorem 2.1], the inequality $\operatorname{depth} K[G_{k+6}] \leq 7$ will be proved. In Subection 8.1.2, we compute a Gröbner basis of $I_{G_{k+6}}$ and an initial ideal $\operatorname{in}(I_{G_{k+6}})$ of $I_{G_{k+6}}$, and show the inequality $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}(I_{G_{k+6}}) \geq 7$. In general, one has $\operatorname{depth} K[\mathbf{x}]/I_{G_{k+6}} \geq \operatorname{depth} K[\mathbf{x}]/\operatorname{in}(I_{G_{k+6}})$ (e.g., [25, Theorem 3.3.4 (d)]). Thus the desired equality (8.1) follows.

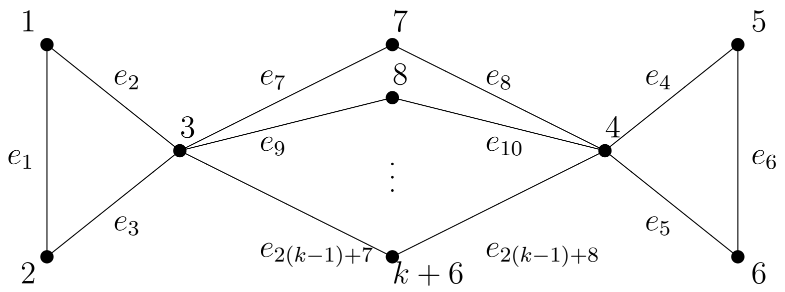


Figure 8.1 (finite graph G_{k+6})

Once we know that $\operatorname{depth} K[G_{k+6}] = 7$, to prove Theorem 8.1.2 is straightforward. In fact, given integers f and d with $7 \leq f \leq d$, let Γ denote the finite graph G_{d-f+7} on $[d-f+7]$ and write G for the finite graph on $[d]$ obtained from Γ by adding $f-7$ edges

$$\{1, d-f+8\}, \{1, d-f+9\}, \ldots, \{1, d\}$$

to Γ. It then follows that $\operatorname{depth} K[G] = \operatorname{depth} K[\Gamma] + f - 7$. Since $\operatorname{depth} K[\Gamma] = 7$, one has $\operatorname{depth} K[G] = f$, as required.

8.1.1 Proof of $\operatorname{depth} K[G_{k+6}] \leq 7$

Let $G = G_{k+6}$ of Figure 8.1. In this section, we prove that $\operatorname{depth} K[G] \leq 7$. Since the number of edges of G is $m = 2(k-1)+8$, Auslander–Buchsbaum formula implies that we may prove $\operatorname{pd} K[G] \geq m - 7 = 2k-1$, where $\operatorname{pd} K[\mathbf{x}]/I$ stands for the projective dimension of $K[\mathbf{x}]/I$.

Let S_G be the semigroup arising from G. Let $\mathcal{A}_G = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ be the set of columns of the incidence matrix of G, where $\mathbf{a}_l$ corresponds to the edge e_l which corresponds to the variable x_l. Actually, $S_G = \mathbb{Z}_{\geq 0}\mathcal{A}_G$.

To prove $\operatorname{pd} K[G] \geq 2k-1$, we use the following theorem due to Briales, Campillo, Marijuán, and Pisón [10]. For $s \in S_G$, we define the simplicial complex

$$\Delta_s = \{F \subset [r] \ : \ s - n_F \in S_G\},$$

where $n_F = \sum_{l \in F} \mathbf{a}_l$. We denote by $\beta_{i,s}(K[G])$, the ith multi-graded Betti number of $K[G]$ in multi-degree s.

Lemma 8.1.3 ([10, Theorem 2.1])**.** *Let G be a finite simple graph. Then*

$$\beta_{j+1,s}(K[G]) = \dim_K \tilde{H}_j(\Delta_s; K).$$

We consider the case where

$$s = (1, 1, k+1, k+1, 1, 1, 1, 2, 2, \dots, 2).$$

By Lemma 8.1.3, it is sufficient to prove the following

Lemma 8.1.4. *Set $s = (1, 1, k+1, k+1, 1, 1, 1, 2, 2, \dots, 2)$. Then*

$$\dim_K \tilde{H}_{2k-2}(\Delta_s; K) \neq 0.$$

Let $\Delta = \Delta_s$. Before proving Lemma 8.1.4, we find all the facets of Δ.

Lemma 8.1.5. *A subset $F \subset [r]$ is a facet of Δ_s if and onlyl if F is one of the following ones :*

$$F_{1,i} = \{1, 4, 5, 7, 8, \dots, 2(k-1)+8\} \setminus \{2(i-1)+8\}, \qquad i = 1, \dots, k;$$
$$F_{2,j} = \{2, 3, 6, 7, 8, \dots, 2(k-1)+8\} \setminus \{2(j-1)+7\}, \qquad j = 1, \dots, k.$$

Proof. Since $s - n_{F_{1,i}} = \mathbf{a}_{2(i-1)+7} \in S_G$, we have $F_{1,i} \in \Delta_s = \Delta$. (It follows that $s \in S_G$.) Similarly, we have $F_{2,j} \in \Delta$.

To prove that there are no facet other than $F_{1,i}, F_{2,j}$, it is enough to show that

- $\{1,2\}, \{1,3\}, \{4,6\}, \{5,6\} \notin \Delta$;
- $\{1,6\} \notin \Delta$;
- $\{2,4\}, \{2,5\}, \{3,4\}, \{3,5\} \notin \Delta$;

- $F_0 = \{7, 8, \dots, 2(k-1)+8\} \notin \Delta$.

Since the first entry of $s - n_{\{1,2\}}$ is $-1 < 0$, it follows that $s - n_{\{1,2\}} \notin S_G$. Therefore $\{1,2\} \notin \Delta$. By the symmetry, we also have $\{1,3\}, \{4,6\}, \{5,6\} \notin \Delta$.

Second we show that $\{1,6\} \notin \Delta$. Suppose, on the contrary, that $\{1,6\} \in \Delta$, i.e.,

$$s - n_{\{1,6\}} = (0, 0, k+1, k+1, 0, 0, 2, 2, \dots, 2) \in S_G.$$

Then we can write $s - n_{\{1,6\}} = \sum_{l=1}^{r} c_l \mathbf{a}_l$ where $c_l \in \mathbb{Z}_{\geq 0}$. Since $(s - n_{\{1,6\}})_1 = (s - n_{\{1,6\}})_2 = 0$ and $(s - n_{\{1,6\}})_3 = k+1$, where $(\mathbf{a})_i$ means the ith entry of $\mathbf{a} \in \mathbb{Z}^n$, we have $c_1 = c_2 = c_3 = 0$ and $\sum_{i=1}^{k} c_{2(i-1)+7} = k+1$. Similarly, we have $c_4 = c_5 = c_6 = 0$ and $\sum_{j=1}^{k} c_{2(j-1)+8} = k+1$. Then $\sum_{i=1}^{k} c_{2(i-1)+7} + \sum_{j=1}^{k} c_{2(j-1)+8} = 2(k+1)$, but it must be $2k$. This is a contradiction.

Next we show that $\{2,4\}, \{2,5\}, \{3,4\}, \{3,5\} \notin \Delta$. Suppose that $\{2,4\} \in \Delta$, i.e.,

$$s - n_{\{2,4\}} = (0, 1, k, k, 0, 1, 2, 2, \dots, 2) \in S_G.$$

Then we can write $s - n_{\{2,4\}} = \sum_{l=1}^{r} c_l \mathbf{a}_l$ where $c_l \in \mathbb{Z}_{\geq 0}$. Since $(s - n_{\{2,4\}})_1 = 0$ and $(s - n_{\{2,4\}})_2 = 1$, we have $c_3 = 1$. Similarly, we have $c_5 = 1$. Thus

$$(0, 0, k-1, k-1, 0, 0, 2, 2, \dots, 2) \in S_G.$$

Then the similar argument on the proof of $\{1,6\} \notin \Delta$ yields a contradiction. Therefore $\{2,4\} \notin \Delta$. By the symmetry, we also have $\{2,5\}, \{3,4\}, \{3,5\} \notin \Delta$.

Last, we show $F_0 \notin \Delta$. It follows from

$$s - n_{F_0} = (1, 1, 1, 1, 1, 1, 0, 0, \dots, 0) \notin S_G.$$

□

Now we prove Lemma 8.1.4.

Proof of Lemma 8.1.4. Let Δ_1 (resp. Δ_2) be the subcomplex of Δ whose facets are $F_{1,i}$, $i = 1, \dots, k$, (resp. $F_{2,j}$, $j = 1, \dots, k$). Then $\Delta = \Delta_1 \cup \Delta_2$. Also facets of the simplicial complex $\Delta_1 \cap \Delta_2$ are

$$\{7, 8, \dots, 2(k-1)+8\} \setminus \{2(j-1)+7, 2(i-1)+8\}, \qquad i, j = 1, \dots, k.$$

In particular, $\dim(\Delta_1 \cap \Delta_2) = 2k-3$. Note that both of Δ_1 and Δ_2 are cone of some simplicial complexes and reduced homologies of these all vanish (cf. [12, Exercise 5.3.10]). Therefore, Mayer–Vietoris exact sequence

$$\begin{aligned} \cdots \longrightarrow \tilde{H}_{i-1}(\Delta_1 \cap \Delta_2; K) \longrightarrow \tilde{H}_i(\Delta_1; K) \oplus \tilde{H}_i(\Delta_2; K) \longrightarrow \tilde{H}_i(\Delta; K) \\ \longrightarrow \tilde{H}_{i-1}(\Delta_1 \cap \Delta_2; K) \longrightarrow \tilde{H}_{i-1}(\Delta_1; K) \oplus \tilde{H}_{i-1}(\Delta_2; K) \longrightarrow \cdots \end{aligned}$$

yields

$$\tilde{H}_i(\Delta; K) \cong \tilde{H}_{i-1}(\Delta_1 \cap \Delta_2; K) \qquad \text{for all } i.$$

One can see that $\tilde{H}_{2k-3}(\Delta_1 \cap \Delta_2; K) \neq 0$ by considering the alternating sum of all facets of $\Delta_1 \cap \Delta_2$, which is

$$\sum_{1 \leq i,j \leq k} (-1)^{i+j}\{7, 8, \ldots, 2(k-1)+8\} \setminus \{2(j-1)+7, 2(i-1)+8\}.$$

This implies that $\tilde{H}_{2k-2}(\Delta; K) \neq 0$, as desired. $\square$

8.1.2 Proof of $\operatorname{depth} K[G_{k+6}] \geq 7$

Let, as before, $G = G_{k+6}$ as in Figure 8.1. In this subsection, we prove another inequality $\operatorname{depth} K[G] \geq 7$.

We set $C_1 = (e_2, e_1, e_3)$ and $C_2 = (e_4, e_6, e_5)$, both of which are 3-cycles of G. By [55, Lemma 3.2], there are three kinds of primitive even closed walks Γ of G up to the way:

(I) a 4-cycle: $\Gamma = (e_{2(i-1)+7}, e_{2(i-1)+8}, e_{2(j-1)+8}, e_{2(j-1)+7})$, where $i < j$;

(II) a walk on two 3-cycles C_1, C_2 and the same path combining C_1 and C_2: $\Gamma = (C_1, e_{2(i-1)+7}, e_{2(i-1)+8}, C_2, e_{2(i-1)+8}, e_{2(i-1)+7})$, where $i = 1, \ldots, k$;

(III) a walk on two 3-cycles C_1, C_2 and the different paths combining C_1 and C_2: $\Gamma = (C_1, e_{2(i-1)+7}, e_{2(i-1)+8}, C_2, e_{2(j-1)+8}, e_{2(j-1)+7})$, where $i < j$.

It was proved in [55, Lemma 3.1] that binomials corresponding to these primitive even closed walks generate the toric ideal I_G. Let us consider the lexicographic order $<=<_{\text{lex}}$ induced with $x_1 > x_2 > x_3 > \cdots > x_{2(k-1)+8}$.

Lemma 8.1.6. *The set of binomials corresponding to primitive even closed walks* (I), (II), (III) *is a Gröbner basis of* I_G *with respect to* $<_{\text{lex}}$.

Proof. The result follows from a straightforward application of Buchberger's algorithm to the set of generators of I_G corresponding to the primitive even closed walks listed above. Let f and g be two such generators. We will prove that the S-polynomial, $S(f,g)$, yielding from Buchberger's algorithm will reduce to 0 by generators of type (I), (II) and (III). For convenience of notation, we will assume that i, j, p, and q are all odd integers such that $7 \leq i < j, 7 \leq p < q$.

Case 1. Let $f = x_i x_{j+1} - x_{i+1} x_j$ and $g = x_p x_{q+1} - x_{p+1} x_q$ be generators of type (I). If $i \neq p$ and $j \neq q$, then the leading terms of f and g are relatively prime and thus the S-polynomial $S(f,g)$ will reduce to 0 (e.g., [25, Lemma 2.3.1]). Suppose $i = p$, then

$$\begin{aligned} S(f,g) &= \frac{\operatorname{lcm}(f,g)}{LT_{<_{\text{lex}}}(f)} f - \frac{\operatorname{lcm}(f,g)}{LT_{<_{\text{lex}}}(g)} g \\ &= x_{q+1}(x_i x_{j+1} - x_{i+1} x_j) - x_{j+1}(x_i x_{q+1} - x_{i+1} x_q) \\ &= x_{i+1} x_{j+1} x_q - x_{i+1} x_j x_{q+1} \\ &= x_{i+1}(x_{j+1} x_q - x_j x_{q+1}). \end{aligned}$$

Note that, up to sign, $x_{j+1}x_q - x_j x_{q+1}$ is a generator of I_G of type (I) and therefore $S(f,g)$ will reduce to 0. The case of $j = q$ is similar.

Case 2. Let f be the same as above and $g = x_1x_4x_5x_p^2 - x_2x_3x_6x_{p+1}^2$ a generator of type (II). If $i \neq p$, then the leading terms of f and g are relatively prime and therefore negligible. If $i = p$, then

$$\begin{aligned} S(f,g) &= x_1x_4x_5x_i(x_ix_{j+1} - x_{i+1}x_j) - x_{j+1}(x_1x_4x_5x_i^2 - x_2x_3x_6x_{i+1}^2) \\ &= x_2x_3x_6x_{i+1}^2x_{j+1} - x_1x_4x_5x_ix_{i+1}x_j \\ &= -x_{i+1}(x_1x_4x_5x_ix_j - x_2x_3x_6x_{i+1}x_{j+1}), \end{aligned}$$

where $x_1x_4x_5x_ix_j - x_2x_3x_6x_{i+1}x_{j+1}$ is a generator of type (III).

Case 3. Again, we assume that f is the same as above. Now assume g is of type (III), $g = x_1x_4x_5x_px_q - x_2x_3x_6x_{p+1}x_{q+1}$. If $i \neq p, q$, then the leading terms of f and g will be relatively prime. Suppose $i = p$, then

$$\begin{aligned} S(f,g) &= x_1x_4x_5x_q(x_ix_{j+1} - x_{i+1}x_j) - x_{j+1}(x_1x_4x_5x_ix_q - x_2x_3x_6x_{i+1}x_{q+1}) \\ &= -x_{i+1}(x_1x_4x_5x_qx_j - x_2x_3x_6x_{q+1}x_{j+1}) \end{aligned}$$

and again we have that $x_1x_4x_5x_qx_j - x_2x_3x_6x_{q+1}x_{j+1}$ is a type either (II) or (III) generator of I_G. The case of $i = q$ is similar.

Case 4. Now let f and g both be generators of type (II), $f = x_1x_4x_5x_i^2 - x_2x_3x_6x_{i+1}^2, g = x_1x_4x_5x_j^2 - x_2x_3x_6x_{j+1}^2$. Then the S-polynomial

$$\begin{aligned} S(f,g) &= x_j^2(x_1x_4x_5x_i^2 - x_2x_3x_6x_{i+1}^2) - x_i^2(x_1x_4x_5x_j^2 - x_2x_3x_6x_{j+1}^2) \\ &= x_2x_3x_6(x_i^2x_{j+1}^2 - x_{i+1}^2x_j^2) \\ &= x_2x_3x_6(x_ix_{j+1} + x_{i+1}x_j)(x_ix_{j+1} - x_{i+1}x_j) \end{aligned}$$

is a multiple of a type (I) generator.

Case 5. Let f be the same as in Case 4 and $g = x_1x_4x_5x_px_q - x_2x_3x_6x_{p+1}x_{q+1}$ of type (III). First suppose that $i \neq p, q$. Let us consider the case of $i < p$. Then

$$\begin{aligned} S(f,g) &= x_px_q(x_1x_4x_5x_i^2 - x_2x_3x_6x_{i+1}^2) - x_i^2(x_1x_4x_5x_px_q - x_2x_3x_6x_{p+1}x_{q+1}) \\ &= x_2x_3x_6(x_i^2x_{p+1}x_{q+1} - x_{i+1}^2x_px_q) \\ &= x_2x_3x_6(x_ix_{q+1}(x_ix_{p+1} - x_{i+1}x_p) + x_ix_{i+1}x_px_{q+1} - x_{i+1}^2x_px_q) \\ &= x_2x_3x_6(x_ix_{q+1}(x_ix_{p+1} - x_{i+1}x_p) + x_{i+1}x_p(x_ix_{q+1} - x_{i+1}x_q)) \end{aligned}$$

and so $S(f,g)$ reduce to 0 by two type (I) generators. The cases of $p < i < q$ and $q < i$ are similar.

Now suppose $i = p$, then the S-polynomial

$$\begin{aligned} S(f,g) &= x_q(x_1x_4x_5x_i^2 - x_2x_3x_6x_{i+1}^2) - x_i(x_1x_4x_5x_ix_q - x_2x_3x_6x_{i+1}x_{q+1}) \\ &= x_2x_3x_6x_{i+1}(x_ix_{q+1} - x_{i+1}x_q). \end{aligned}$$

is a multiple of a type (I) generator. The case of $i = q$ is similar.

Case 6. Finally, we consider the case that both f and g are of type (III): $f = x_1x_4x_5x_ix_j - x_2x_3x_6x_{i+1}x_{j+1}$, $g = x_1x_4x_5x_px_q - x_2x_3x_6x_{p+1}x_{q+1}$. We may assume that $i \leq p$. Let us first suppose that $i, j \neq p, q$. Then

$$\begin{aligned} S(f,g) &= x_px_q(x_1x_4x_5x_ix_j - x_2x_3x_6x_{i+1}x_{j+1}) - x_ix_j(x_1x_4x_5x_px_q - x_2x_3x_6x_{p+1}x_{q+1}) \\ &= x_2x_3x_6(x_ix_jx_{p+1}x_{q+1} - x_{i+1}x_{j+1}x_px_q) \\ &= x_2x_3x_6(x_jx_{q+1}(x_ix_{p+1} - x_{i+1}x_p) + x_{i+1}x_p(x_jx_{q+1} - x_{j+1}x_q)). \end{aligned}$$

Now let $i = p$. We then have

$$\begin{aligned} S(f,g) &= x_qf - x_jg = -x_qx_2x_3x_6x_{i+1}x_{j+1} + x_jx_2x_3x_6x_{i+1}x_{q+1} \\ &= x_2x_3x_6x_{i+1}(x_jx_{q+1} - x_{j+1}x_q). \end{aligned}$$

The cases of $j = p$ and $j = q$ are similar. $\square$

Now we prove that $\operatorname{depth} K[G] \geq 7$. We denote by $\operatorname{in}(I_G)$, the initial ideal of I_G with respect to $<_{\text{lex}}$. Since

$$\operatorname{depth} K[G] = \operatorname{depth} K[\mathbf{x}]/I_G \geq \operatorname{depth} K[\mathbf{x}]/\operatorname{in}(I_G),$$

it is sufficient to prove the following

Lemma 8.1.7. $\operatorname{depth}_{K[\mathbf{x}]} K[\mathbf{x}]/\operatorname{in}(I_G) \geq 7$.

Proof. First, we compute $\operatorname{in}(I_G)$.

The binomials corresponding to type (I) are

$$x_{2(i-1)+7}x_{2(j-1)+8} - x_{2(i-1)+8}x_{2(j-1)+7}, \qquad \text{where } i < j.$$

The initial term of this binomial is $x_{2(i-1)+7}x_{2(j-1)+8}$ $(i < j)$. We denote by I', the ideal generated by these monomials. Note that x_8 and $x_{2(k-1)+7}$ do not appear in the minimal system of monomial generators of I'.

The binomials corresponding to types (II), (III) are

$$x_2x_3x_6x_{2(i-1)+8}x_{2(j-1)+8} - x_1x_4x_5x_{2(i-1)+7}x_{2(j-1)+7}, \qquad \text{where } i \leq j.$$

The initial term of this binomial is $-x_1x_4x_5x_{2(i-1)+7}x_{2(j-1)+7}$ $(i \leq j)$.

Therefore,

$$\begin{aligned} \operatorname{in}(I_G) &= x_1x_4x_5(x_7, x_9, \ldots, x_{2(k-1)+7})^2 + I' \\ &= ((x_7, x_9, \ldots, x_{2(k-1)+7})^2 + I') \cap ((x_1x_4x_5) + I'). \end{aligned}$$

We set

$$I_1 = (x_7, x_9, \ldots, x_{2(k-1)+7})^2 + I', \quad I_2 = (x_1x_4x_5) + I'.$$

By the short exact sequence $0 \to K[\mathbf{x}]/I_1 \cap I_2 \to K[\mathbf{x}]/I_1 \oplus K[\mathbf{x}]/I_2 \to K[\mathbf{x}]/(I_1 + I_2) \to 0$, we have

$$\operatorname{depth} K[\mathbf{x}]/\operatorname{in}(I_G) \geq \max\{\operatorname{depth} K[\mathbf{x}]/I_1, \operatorname{depth} K[\mathbf{x}]/I_2, \operatorname{depth} K[\mathbf{x}]/(I_1 + I_2) + 1\}. \tag{8.2}$$

We investigate each of $\operatorname{depth} K[\mathbf{x}]/I_1, \operatorname{depth} K[\mathbf{x}]/I_2, \operatorname{depth} K[\mathbf{x}]/(I_1 + I_2)$.

First, it is easy to see that $\operatorname{depth} K[\mathbf{x}]/I_1 \geq 7$ because $x_1, x_2, x_3, x_4, x_5, x_6$ and x_8 do not appear in $G(I_1)$, where $G(I)$ stands for a minimal system of generators of an ideal $I \subset K[\mathbf{x}]$. Moreover, since $x_1x_4x_5$ is a $K[\mathbf{x}]/I_1$-regular element, we have $\operatorname{depth} K[\mathbf{x}]/I_2 = \operatorname{depth} K[\mathbf{x}]/I' - 1$. Then $x_1, \ldots, x_6, x_8, x_{2(k-1)+7}$ do not appear in $G(I')$. Thus, $\operatorname{depth} K[\mathbf{x}]/I_2 \geq 7$. Similarly, we also have $\operatorname{depth} K[\mathbf{x}]/(I_1 + I_2) \geq 6$, proving the assertion. □

8.2 Depth of initial ideals of normal edge rings

Following the previous section, the topic of this section is the depth of initial ideals of normal edge rings.

We refer the reader to [25, Chapter 2] for fundamental materials on Gröbner bases. Let $<$ be a monomial order on $K[\mathbf{x}]$ and $\operatorname{in}_<(I_G)$ the initial ideal of I_G with respect to $<$. The topic of this section is $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_<(I_G)$, the depth of $K[\mathbf{x}]/\operatorname{in}_<(I_G)$, when $K[G]$ is normal. Computational experience yields the following

Conjecture 8.2.1. *Let G be a finite connected nonbipartite graph on $[d]$ with $d \geq 6$ and suppose that its edge ring $K[G]$ is normal. Then* $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_<(I_G) \geq 6$ *for any monomial order $<$ on $K[\mathbf{x}]$.*

Now, even though Conjecture 8.2.1 is completely open, by taking Conjecture 8.2.1 into consideration, we prove the following

Theorem 8.2.2 ([33, Theorem 0.2])**.** *Given integers f and d with $6 \leq f \leq d$, there exists a finite connected nonbipartite graph G on $[d]$ together with a reverse lexicographic order $<_{\text{rev}}$ on $K[\mathbf{x}]$ and a lexicographic order $<_{\text{lex}}$ on $K[\mathbf{x}]$ such that*

(i) *$K[G]$ is normal with* $\operatorname{Krull-dim} K[G] = d$*;*

(ii) $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_{<_{\text{rev}}}(I_G) = f$*;*

(iii) *$K[\mathbf{x}]/\operatorname{in}_{<_{\text{lex}}}(I_G)$ is Cohen–Macaulay.*

Let $k \geq 1$ be an arbitrary integer. We introduce the finite connected nonbipartite graph H_{k+5} on $[k+5]$ which is drawn in Figure 8.2. Clearly, the edge ring $K[H_{k+5}]$ is normal. It will turn out that H_{k+5} plays an important role in our proof of Theorem 8.2.2.

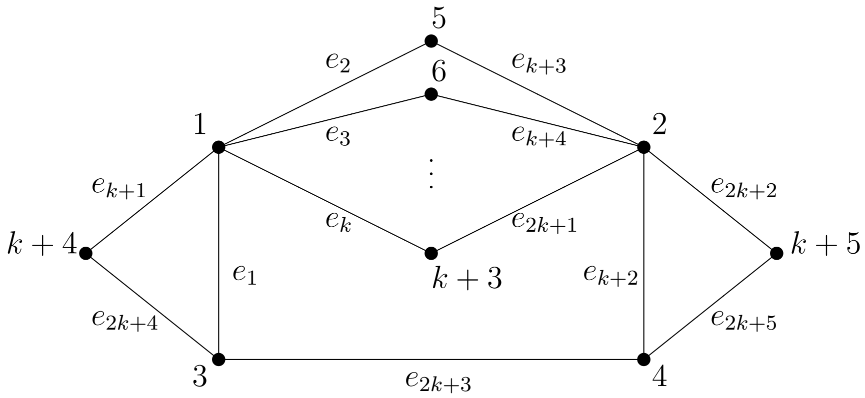


Figure 8.2 (finite graph H_{k+5})

The essential step in order to prove Theorem 8.2.2 is to show the following

Lemma 8.2.3. *Let $<_{\text{rev}}$ (resp. $<_{\text{lex}}$) denote the reverse lexicographic order (resp. the lexicographic order) on $K[\mathbf{x}] = K[x_1, \ldots, x_{2k+5}]$ induced by the ordering $x_1 > \cdots > x_{2k+5}$ of the variables. Then*

(i) $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_{<_{\text{rev}}}(I_{H_{k+5}}) = 6$*;*

(ii) $K[\mathbf{x}]/\operatorname{in}_{<_{\text{lex}}}(I_{H_{k+5}})$ *is Cohen–Macaulay.*

Once we establish Lemma 8.2.3, to prove Theorem 8.2.2 is straightforward. In fact, given integers f and d with $6 \leq f \leq d$, we define the finite graph Γ on $[d-f+6]$ to be H_{d-f+6} with the edges $e_1, e_2, \ldots, e_{2(d-f)+7}$ and then introduce the finite connected nonbipartite graph G on $[d]$ which is obtained from Γ by adding $f-6$ edges

$$e_{2(d-f)+7+i} = \{1, d-f+6+i\}, \quad i = 1, \ldots, f-6$$

to Γ. Clearly, both edge rings $K[\Gamma]$ and $K[G]$ are normal, and

$$I_G = I_\Gamma(K[x_1, \ldots, x_{2d-f+1}]).$$

Thus, in particular,

$$\operatorname{in}_<(I_G) = \operatorname{in}_<(I_\Gamma)(K[x_1, \ldots, x_{2d-f+1}]),$$

where $<$ is any monomial order on $K[x_1, \ldots, x_{2d-f+1}]$. Thus Lemma 8.2.3 guarantees that

$$\operatorname{depth} K[x_1, \ldots, x_{2d-f+1}]/\operatorname{in}_{<_{\text{rev}}}(I_G) = f$$

and $K[x_1, \ldots, x_{2d-f+1}]/\operatorname{in}_{<_{\text{lex}}}(I_G)$ is Cohen–Macaulay, as desired.

8.2.1 Preliminaries

Let $H = H_{k+5}$. In this subsection, we will find a Gröbner basis of I_G and a set of generators of the initial ideal of I_G with respect to the reverse lexicographic order.

Let $K[\mathbf{x}] = K[x_1, \ldots, x_{2k+5}]$ be the polynomial ring in $2k+5$ variables over a field K. There are 4 kinds of primitive even closed walks of H:

(I) a 4-cycle: $(e_i, e_{k+1+i}, e_{k+1+j}, e_j)$, where $2 \leq i < j \leq k$;

(II) a walk on two 3-cycles and the same edge e_{2k+3} combining two cycles: $(e_1, e_{k+1}, e_{2k+4}, e_{2k+3}, e_{k+2}, e_{2k+2}, e_{2k+5}, e_{2k+3})$;

(III) a 6-cycle: $(e_i, e_{k+1+i}, e_{k+2}, e_{2k+3}, e_{2k+4}, e_{k+1})$ or $(e_i, e_{k+1+i}, e_{2k+2}, e_{2k+5}, e_{2k+3}, e_1)$, where $2 \leq i \leq k$;

(VI) a walk on two 3-cycles and the length 2 paths combining two cycles: $(e_{k+2}, e_{2k+5}, e_{2k+2}, e_{k+1+i}, e_i, e_1, e_{2k+4}, e_{k+1}, e_j, e_{k+1+j})$, where $2 \leq i \leq j \leq k$.

It was proved in [55, Lemma 3.1] that the binomials corresponding to these primitive even closed walks generate the toric ideal I_H. Let $<_{\text{rev}}$ be the reverse lexicographic order with $x_1 > x_2 > \cdots > x_{2k+5}$.

Lemma 8.2.4. *The set of binomials corresponding to primitive even closed walks* (I), (II), (II), (VI) *is a Gröbner basis of* I_H *with respect to* $<_{\text{rev}}$.

Proof. Similar to Lemma 8.1.6, the result follows from a direct application of Buchberger's criterion. Let f and g be two such generators. We can prove that the S-polynomial $S(f,g)$ will reduce to 0 by generators of type (I), (II), (III) and (VI). Let i, j, p, q be integers with $2 \leq i, j, p, q \leq k$. On the following proof, we will underline the leading monomial of a binomial with respect to $<_{\text{rev}}$.

Case 1. Let $f = x_i x_{k+1+j} - \underline{x_j x_{k+1+i}}$ and $g = x_p x_{k+1+q} - \underline{x_q x_{k+1+p}}$ be generators of type (I), where $i < j$ and $p < q$. If $i \neq p$ and $j \neq q$, then the leading monomials of f and g are coprime. Thus $S(f,g)$ will reduce to 0. We assume that $i = p$. Then

$$\begin{aligned} S(f,g) &= -x_q(x_i x_{k+1+j} - x_j x_{k+1+i}) - (-x_j)(x_i x_{k+1+q} - x_q x_{k+1+i}) \\ &= -x_i(x_q x_{k+1+j} - x_j x_{k+1+q}). \end{aligned}$$

Note that, up to sign, $x_q x_{k+1+j} - x_j x_{k+1+q}$ is a generator of I_G of type (I). Therefore $S(f,g)$ will reduce to 0. The case of $j = q$ is similar.

Case 2. Let f be the same as above and $g = x_1 x_{k+2} x_{2k+4} x_{2k+5} - \underline{x_{k+1} x_{2k+2} x_{2k+3}^2}$ a generator of type (II). Since $2 \leq i < j \leq k$, the leading monomials of f and g are always coprime.

Case 3. Again, we set that f is the same as above. Let g be of type (III). First, let $g = x_p x_{k+2} x_{2k+4} - \underline{x_{k+1} x_{k+1+p} x_{2k+3}}$. If $i \neq p$, then the leading monomials of f

and g are coprime. We assume that $i = p$. Then

$$\begin{aligned} S(f,g) &= -x_{k+1}x_{2k+3}f - (-x_j)g \\ &= -x_i x_{k+1} x_{k+1+j} x_{2k+3} + x_i x_j x_{k+2} x_{2k+4} \\ &= x_i(x_j x_{k+2} x_{2k+4} - x_{k+1} x_{k+1+j} x_{2k+3}), \end{aligned}$$

where $x_j x_{k+2} x_{2k+4} - x_{k+1} x_{k+1+j} x_{2k+3}$ is of type (3). Next, let $g = \underline{x_p x_{2k+2} x_{2k+3}} - x_1 x_{k+1+p} x_{2k+5}$. If $j \neq p$, then the leading monomials of f and g are coprime. We assume that $j = p$. Then

$$\begin{aligned} S(f,g) &= -x_{2k+2}x_{2k+3}f - x_{k+1+i}g \\ &= -x_{k+1+j}(x_i x_{2k+2} x_{2k+3} - x_1 x_{k+1+i} x_{2k+5}) \end{aligned}$$

and again we have that $x_i x_{2k+2} x_{2k+3} - x_1 x_{k+1+i} x_{2k+5}$ is of type (III).

Case 4. Again, we assume that f is the same as above. Let $g = \underline{x_p x_q x_{k+2} x_{2k+2} x_{2k+4}} - x_1 x_{k+1} x_{k+1+p} x_{k+1+q} x_{2k+5}$ be of type (VI), where $p \leq q$. If $j \neq p$ and $j \neq q$, then the leading monomials of f and g are coprime. If $j = p$, then

$$\begin{aligned} S(f,g) &= -x_q x_{k+2} x_{2k+2} x_{2k+4} f - x_{k+1+i} g \\ &= -x_{k+1+j}(x_i x_q x_{k+2} x_{2k+2} x_{2k+4} - x_1 x_{k+1} x_{k+1+i} x_{k+1+q} x_{2k+5}), \end{aligned}$$

which is a multiple of type (VI) generator. The case of $j = q$ is similar.

Case 5. Let $f = x_1 x_{k+2} x_{2k+4} x_{2k+5} - \underline{x_{k+1} x_{2k+2} x_{2k+3}^2}$ be a generator of type (II), and g a generator of type (III). First we consider the case where $g = x_p x_{k+2} x_{2k+4} - \underline{x_{k+1} x_{k+1+p} x_{2k+3}}$. Then

$$\begin{aligned} S(f,g) &= -x_{k+1+p}f - (-x_{2k+2}x_{2k+3})g \\ &= x_{k+2}x_{2k+4}(x_p x_{2k+2} x_{2k+3} - x_1 x_{k+1+p} x_{2k+5}), \end{aligned}$$

where $x_p x_{2k+2} x_{2k+3} - x_1 x_{k+1+p} x_{2k+5}$ is of type (3). Next, let $g = \underline{x_p x_{2k+2} x_{2k+3}} - x_1 x_{k+1+p} x_{2k+5}$. Then

$$\begin{aligned} S(f,g) &= -x_p f - x_{k+1} x_{2k+3} g \\ &= -x_1 x_{2k+5}(x_p x_{k+2} x_{2k+4} - x_{k+1} x_{k+1+p} x_{2k+3}) \end{aligned}$$

and we have that $x_p x_{k+2} x_{2k+4} - x_{k+1} x_{k+1+p} x_{2k+3}$ is of type (III).

Case 6. Let f be the same as in Case 5 and $g = \underline{x_p x_q x_{k+2} x_{2k+2} x_{2k+4}} - x_1 x_{k+1} x_{k+1+p} x_{k+1+q} x_{2k+5}$ be of type (VI) generator, where $p \leq q$. Then

$$\begin{aligned} S(f,g) &= -x_p x_q x_{k+2} x_{2k+4} f - x_{k+1} x_{2k+3}^2 g \\ &= -x_1 x_{2k+5}(x_p x_q x_{k+2}^2 x_{2k+4}^2 - \underline{x_{k+1}^2 x_{k+1+p} x_{k+1+q} x_{2k+3}^2}) \\ &= -x_1 x_{2k+5}\{x_{k+1} x_{k+1+q} x_{2k+3}(x_p x_{k+2} x_{2k+4} - x_{k+1} x_{k+1+p} x_{2k+3}) \\ &\qquad + x_p x_{k+2} x_{2k+4}(x_q x_{k+2} x_{2k+4} - x_{k+1} x_{k+1+q} x_{2k+3})\}. \end{aligned}$$

Thus $S(f,g)$ reduce to 0 by generators of type (III).

Case 7. We assume that both f and g are of type (III). First, we consider the case where $f = x_i x_{k+2} x_{2k+4} - \underline{x_{k+1} x_{k+1+i} x_{2k+3}}$ and $g = x_p x_{k+2} x_{2k+4} - \underline{x_{k+1} x_{k+1+p} x_{2k+3}}$, where $i \neq p$. Then

$$\begin{aligned} S(f,g) &= -x_{k+1+p} f - (-x_{k+1+i}) g \\ &= -x_{k+2} x_{2k+4} (x_i x_{k+1+p} - x_p x_{k+1+i}), \end{aligned}$$

which is a multiple of type (I) generator. Next, let f be the same one and $g = \underline{x_p x_{2k+2} x_{2k+3}} - x_1 x_{k+1+p} x_{2k+5}$. Then

$$\begin{aligned} S(f,g) &= -x_p x_{2k+2} f - x_{k+1} x_{k+1+i} g \\ &= -(x_i x_p x_{k+2} x_{2k+2} x_{2k+4} - x_1 x_{k+1} x_{k+1+i} x_{k+1+p} x_{2k+5}), \end{aligned}$$

which is a generator of type (VI) up to sign. Finally, let $f = \underline{x_i x_{2k+2} x_{2k+3}} - x_1 x_{k+1+i} x_{2k+5}$ and $g = \underline{x_p x_{2k+2} x_{2k+3}} - x_1 x_{k+1+p} x_{2k+5}$, where $i \neq p$. Then

$$\begin{aligned} S(f,g) &= x_p f - x_i g \\ &= x_1 x_{2k+5} (x_i x_{k+1+p} - x_p x_{k+1+i}). \end{aligned}$$

Case 8. Let f be of type (III) and $g = \underline{x_p x_q x_{k+2} x_{2k+2} x_{2k+4}} - x_1 x_{k+1} x_{k+1+p} x_{k+1+q} x_{2k+5}$ be of type (VI) with $p \leq q$. First, we set that $f = x_i x_{k+2} x_{2k+4} - \underline{x_{k+1} x_{k+1+i} x_{2k+3}}$. Then the leading monomials of f and g are coprime. Next, we set that $f = \underline{x_i x_{2k+2} x_{2k+3}} - x_1 x_{k+1+i} x_{2k+5}$. If $i \neq p$ and $i \neq q$, then

$$\begin{aligned} S(f,g) &= x_p x_q x_{k+2} x_{2k+4} f - x_i x_{2k+3} g \\ &= x_1 x_{2k+5} (\underline{x_i x_{k+1} x_{k+1+p} x_{k+1+q} x_{2k+3}} - x_p x_q x_{k+2} x_{k+1+i} x_{2k+4}) \\ &= x_1 x_{2k+5} \{ -x_i x_{k+1+q} (x_p x_{k+2} x_{2k+4} - x_{k+1} x_{k+1+p} x_{2k+3}) \\ &\qquad + x_p x_{k+2} x_{2k+4} (x_i x_{k+1+q} - x_q x_{k+1+i}) \}. \end{aligned}$$

Thus $S(f,g)$ reduce to 0 by generators of type (I) and (III). If $i = p$, then

$$\begin{aligned} S(f,g) &= x_q x_{k+2} x_{2k+4} f - x_{2k+3} g \\ &= -x_1 x_{k+1+i} x_{2k+5} (x_q x_{k+2} x_{2k+4} - x_{k+1} x_{k+1+q} x_{2k+3}). \end{aligned}$$

The case of $i = q$ is similar.

Case 9. Finally, we consider the case that both f and g are of type (VI). Let $f = \underline{x_i x_j x_{k+2} x_{2k+2} x_{2k+4}} - x_1 x_{k+1} x_{k+1+i} x_{k+1+j} x_{2k+5}$ and $g = \underline{x_p x_q x_{k+2} x_{2k+2} x_{2k+4}} - x_1 x_{k+1} x_{k+1+p} x_{k+1+q} x_{2k+5}$, where $i \leq j$ and $p \leq q$. Without loss of generality, we may assume that $j \geq q$. First, we assume that $j > q (\geq p)$. If $i \neq p$ and $i \neq q$, then

$$\begin{aligned} S(f,g) &= x_p x_q f - x_i x_j g \\ &= x_1 x_{k+1} x_{2k+5} (\underline{x_i x_j x_{k+1+p} x_{k+1+q}} - x_p x_q x_{k+1+i} x_{k+1+j}) \\ &= x_1 x_{k+1} x_{2k+5} \{ -x_i x_{k+1+q} (x_p x_{k+1+j} - x_j x_{k+1+p}) + x_p x_{k+1+j} (x_i x_{k+1+q} - x_q x_{k+1+i}) \}. \end{aligned}$$

Thus we have that $S(f,g)$ reduce to 0 by generators of type (I). If $i = p$, then

$$\begin{aligned} S(f,g) &= x_q f - x_j g \\ &= x_1 x_{k+1} x_{k+1+i} x_{2k+5} (x_j x_{k+1+q} - x_q x_{k+1+j}). \end{aligned}$$

The case of $i = q$ is similar. Next, we consider the case where $j = q$. Then $i \neq p$ and

$$\begin{aligned} S(f,g) &= x_p f - x_i g \\ &= x_1 x_{k+1} x_{k+1+j} x_{2k+5} (x_i x_{k+1+p} - x_p x_{k+1+i}), \end{aligned}$$

which is a multiple of type (I) generator. □

Corollary 8.2.5. *The initial ideal of I_G with respect to $<_{\text{rev}}$ is generated by the following monomials:*

$$\begin{aligned} &x_j x_{k+1+i}, \quad 2 \leq i < j \leq k, \\ &x_{k+1} x_{2k+2} x_{2k+3}^2, \\ &x_{k+1} x_{k+1+r} x_{2k+3},\ x_r x_{2k+2} x_{2k+3}, \quad 2 \leq r \leq k, \\ &x_p x_q x_{k+2} x_{2k+2} x_{2k+4}, \quad 2 \leq p \leq q \leq k. \end{aligned}$$

For the rest part of this section, we will denote by I, the initial ideal of I_H with respect to $<_{\text{rev}}$.

8.2.2 Proof of $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_{<_{\text{rev}}}(I_G) \leq 6$

In this subsection, we will prove that $\operatorname{depth} K[\mathbf{x}]/I \leq 6$. Since the number of edges of G, which coincides with $2k+5$, is equal to the number of variables of $K[\mathbf{x}]$, we may prove that $\operatorname{pd} K[\mathbf{x}]/I \geq 2k-1$.

First, we recall from [49] the fundamental technique to compute the Betti numbers of (non-squarefree) monomial ideals.

For a monomial ideal J and a multi degree $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$, define

$$\mathbf{K}^{\mathbf{a}}(J) = \{\text{squarefree vectors } \alpha : \mathbf{x}^{\mathbf{a}-\alpha} \in J\}$$

to be the *Koszul simplicial complex* of J in degree $\mathbf{a}$, where a squarefree vector α means that each entry of α is 0 or 1.

Lemma 8.2.6 ([49, Theorem 1.34])**.** *Let S be a polynomial ring, J a monomial ideal of S and $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$ a vector. Then the Betti numbers of J and S/J in degree $\mathbf{a}$ can be expressed as*

$$\beta_{i,\mathbf{a}}(J) = \beta_{i+1,\mathbf{a}}(S/J) = \dim_K \tilde{H}_{i-1}(\mathbf{K}^{\mathbf{a}}(J); K).$$

By virtue of Lemma 8.2.6, in order to prove that $\operatorname{pd} K[\mathbf{x}]/I \geq 2k-1$, we may show the following

Lemma 8.2.7. *Let* $\mathbf{a} = \sum_{j=2}^{k}(\mathbf{e}_j + \mathbf{e}_{k+1+j}) + \mathbf{e}_{k+1} + \mathbf{e}_{2k+2} + 2\mathbf{e}_{2k+3} \in \mathbb{Z}_{\geq 0}^{2k+5}$*, where* $\mathbf{e}_i \in \mathbb{R}^{2k+5}$ *is the* i*th unit vector of* $\mathbb{R}^{2k+5}$*. Then*

$$\dim_K \tilde{H}_{2k-3}(\mathbf{K}^{\mathbf{a}}(I); K) \neq 0.$$

Proof. Let Δ be the simplicial complex on the vertex set $[2k+5]$ which is obtained by identifying a squarefree vector $\alpha \in \mathbf{K}^{\mathbf{a}}(I)$ with the set of coordinates where the entries of α are 1. To prove the assertion, we may show that $\dim_K \tilde{H}_{2k-3}(\Delta; K) \neq 0$. Let I_1 (resp. I_2) be the monomial ideal generated by the monomials

$$\begin{array}{l} x_j x_{k+1+i}, \quad 2 \leq i < j \leq k, \\ x_{k+1}x_{k+1+r}x_{2k+3},\ x_r x_{2k+2} x_{2k+3}, \quad 2 \leq r \leq k \end{array}$$

(resp. by the monomial $x_{k+1}x_{2k+2}x_{2k+3}^2$). We denote by Δ_1, Δ_2, the subcomplexes of Δ corresponding to $\mathbf{K}^{\mathbf{a}}(I_1), \mathbf{K}^{\mathbf{a}}(I_2)$, respectively. Since $(\mathbf{a})_{k+2} = 0$, one has $\Delta = \Delta_1 \cup \Delta_2$. Moreover, one can verify that all the facets of Δ_1 contain a common vertex $2k+3$. In other words, Δ_1 is a cone over some simplicial complex. In addition, Δ_2 has only one facet

$$\{2, 3, \dots, k, k+3, k+4, \dots, 2k+1\},$$

which is a $(2k-3)$-dimensional simplex. Thus the reduced homologies of both of Δ_1 and Δ_2 all vanish. Hence Mayer–Vietoris exact sequence

$$\begin{array}{l} \cdots \longrightarrow \tilde{H}_i(\Delta_1 \cap \Delta_2; K) \longrightarrow \tilde{H}_i(\Delta_1; K) \oplus \tilde{H}_i(\Delta_2; K) \longrightarrow \tilde{H}_i(\Delta; K) \\ \quad \longrightarrow \tilde{H}_{i-1}(\Delta_1 \cap \Delta_2; K) \longrightarrow \tilde{H}_{i-1}(\Delta_1; K) \oplus \tilde{H}_{i-1}(\Delta_2; K) \longrightarrow \cdots \end{array}$$

yields

$$\tilde{H}_i(\Delta; K) \cong \tilde{H}_{i-1}(\Delta_1 \cap \Delta_2; K) \qquad \text{for all } i.$$

Now we note that subsets

$$\begin{array}{ll} \{2, 3, \dots, k, k+3, k+4, \dots, 2k+1\} \setminus \{i\}, & i = 2, \dots, k, \\ \{2, 3, \dots, k, k+3, k+4, \dots, 2k+1\} \setminus \{k+1+j\}, & j = 2, \dots, k \end{array}$$

are faces of Δ_1 and $\{2, 3, \dots, k, k+3, k+4, \dots, 2k+1\}$ is not a face of Δ_1. Thus the above subsets are the facets of $\Delta_1 \cap \Delta_2$. In particular, one has $\dim(\Delta_1 \cap \Delta_2) = 2k-4$. Since $\Delta_1 \cap \Delta_2$ contains all facets of the $(2k-3)$-dimensional simplex Δ_2, the geometric realization of $\Delta_1 \cap \Delta_2$ is homeomorphic to the boundary complex of the simplex Δ_2, i.e., $\Delta_1 \cap \Delta_2$ is a simplicial $(2k-4)$-sphere.

Therefore one has $\dim_K \tilde{H}_{2k-3}(\Delta; K) = \dim_K \tilde{H}_{2k-4}(\Delta_1 \cap \Delta_2; K) \neq 0$, as desired. □

8.2.3 Proof of $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_{<_{\mathrm{rev}}}(I_G) \geq 6$

In this subsection, we will prove the following

Lemma 8.2.8. $\operatorname{depth} K[\mathbf{x}]/I \geq 6$.

Before proving Lemma 8.2.8, we prepare the following two lemmata.

Lemma 8.2.9. *Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables and $J \subset S$ a monomial ideal of S.*

(i) *If only $m(\leq n)$ variables appear in the elements of $G(J)$, then $\operatorname{depth} S/J \geq n - m$.*

(ii) *If only m variables appear in the elements of $G(J)$ and the variables $x_{i_1}, \ldots, x_{i_r}$ do not appear in there, then $\operatorname{depth} S/J' \geq n - m$, where $J' = x_{i_1} \cdots x_{i_r} J$.*

Proof. Without loss of generality, we may assume that only the variables $x_1, \ldots, x_m$ appear in the elements of $G(J)$.

(i) Since the variables $x_{m+1}, \ldots, x_n$ do not appear in the elements of $G(J)$, the sequence $x_{m+1}, \ldots, x_n$ is an S/J-regular sequence. Thus one has $\operatorname{depth} S/J \geq n-m$.

(ii) Set $x_{i_\ell} = x_{m+\ell}$ for $\ell = 1, \ldots, r$ and $J'' = (x_{m+1} \cdots x_{m+r}) \subset S$. Then, by the short exact sequence $0 \to S/J \cap J'' \to S/J \oplus S/J'' \to S/(J + J'') \to 0$, we have

$$\operatorname{depth} S/J' = \operatorname{depth} S/J \cap J'' \geq \min\{\operatorname{depth} S/J, \operatorname{depth} S/J'', \operatorname{depth} S/(J + J'') + 1\}.$$

Now we have $\operatorname{depth} S/J \geq n - m$ by (i) and $\operatorname{depth} S/J'' = n - 1$. In addition, since $x_{m+1}, \ldots, x_{m+r}$ do not appear in the elements of $G(J)$, the monomial $x_{m+1} \cdots x_{m+r}$ is an S/J-regular element. Hence one has $\operatorname{depth} S/(J + J'') = \operatorname{depth} S/J - 1 \geq n - m - 1$. $\square$

Let

$$\begin{aligned} I_1 &= (x_j x_{k+1+i} : 2 \leq i < j \leq k), \\ I_2 &= (x_{k+1} x_{2k+2} x_{2k+3}^2), \\ I_3 &= x_{2k+2} x_{2k+3} (x_2, x_3, \ldots, x_k), \\ I_4 &= x_{k+1} x_{2k+3} (x_{k+3}, x_{k+4}, \ldots, x_{2k+1}), \\ I_5 &= x_{k+2} x_{2k+2} x_{2k+4} (x_2, x_3, \ldots, x_k)^2. \end{aligned}$$

Then $I = I_1 + I_2 + \cdots + I_5$.

The following lemma can be obtained by elementary computations.

Lemma 8.2.10. *Let $J_1 = I_3 + I_4$, $J_2 = J_1 + I_1$ and $J_3 = J_2 + I_5$. Then*
(i) $I_3 \cap I_4 = x_{k+1} x_{2k+2} x_{2k+3} (x_2, \ldots, x_k)(x_{k+3}, \ldots, x_{2k+1})$.
(ii) $J_1 \cap I_1 = x_{2k+3} (x_{k+1}, x_{2k+2}) I_1$.
(iii) $J_2 \cap I_5 = x_{k+2} x_{2k+2} x_{2k+4} (x_2, \ldots, x_k)(x_{2k+3}(x_2, \ldots, x_k) + I_1)$.
(vi) $J_3 \cap I_2 = x_{k+1} x_{2k+2} x_{2k+3}^2 (x_2, \ldots, x_k, x_{k+3}, \ldots, x_{2k+1})$.

Now we will prove Lemma 8.2.8.

Proof of Lemma 8.2.8. Work with the same notations as in Lemma 8.2.10. By the short exact sequence

$$0 \to K[\mathbf{x}]/J_3 \cap I_2 \to K[\mathbf{x}]/J_3 \oplus K[\mathbf{x}]/I_2 \to K[\mathbf{x}]/(J_3 + I_2) \to 0,$$

one has

$$\begin{aligned}\operatorname{depth} K[\mathbf{x}]/I &= \operatorname{depth} K[\mathbf{x}]/(J_3 + I_2) \\ &\geq \min\{\operatorname{depth} K[\mathbf{x}]/J_3, \operatorname{depth} K[\mathbf{x}]/I_2, \operatorname{depth} K[\mathbf{x}]/J_3 \cap I_2 - 1\}.\end{aligned}$$

Thus what we must prove is that the inequalities $\operatorname{depth} K[\mathbf{x}]/J_3 \geq 6$, $\operatorname{depth} K[\mathbf{x}]/I_2 \geq 6$ and $\operatorname{depth} K[\mathbf{x}]/J_3 \cap I_2 \geq 7$. Obviously, $\operatorname{depth} K[\mathbf{x}]/I_2 = 2k + 4 \geq 6$. Moreover, by Lemmata 8.2.10 (vi) and 8.2.9 (ii), we can easily see that $\operatorname{depth} K[\mathbf{x}]/J_3 \cap I_2 \geq (2k + 5) - 2(k - 1) = 7$. Thus we investigate $\operatorname{depth} K[\mathbf{x}]/J_3$.

First step. By the short exact sequence

$$0 \to K[\mathbf{x}]/I_3 \cap I_4 \to K[\mathbf{x}]/I_3 \oplus K[\mathbf{x}]/I_4 \to K[\mathbf{x}]/(I_3 + I_4) \to 0,$$

one has

$$\begin{aligned}\operatorname{depth} K[\mathbf{x}]/J_1 &= \operatorname{depth} K[\mathbf{x}]/(I_3 + I_4) \\ &\geq \min\{\operatorname{depth} K[\mathbf{x}]/I_3, \operatorname{depth} K[\mathbf{x}]/I_4, \operatorname{depth} K[\mathbf{x}]/I_3 \cap I_4 - 1\}.\end{aligned}$$

By Lemma 8.2.9 (ii), one has $\operatorname{depth} K[\mathbf{x}]/I_3 \geq k+6 \geq 6$ and $\operatorname{depth} K[\mathbf{x}]/I_4 \geq k+6 \geq 6$. Since $I_3 \cap I_4 = x_{k+1}x_{2k+2}x_{2k+3}(x_2, \ldots, x_k)(x_{k+3}, \ldots, x_{2k+1})$ by Lemma 8.2.10 (i) and $x_{k+1}, x_{2k+2}, x_{2k+3}$ do not appear in the elements of $G((x_2, \ldots, x_k)(x_{k+3}, \ldots, x_{2k+1}))$, one has $\operatorname{depth} K[\mathbf{x}]/I_3 \cap I_4 \geq (2k + 5) - 2(k - 1) = 7$ by Lemma 8.2.9 (ii). Hence one has $\operatorname{depth} K[\mathbf{x}]/J_1 \geq 6$.

Second step. Again, by the short exact sequence

$$0 \to K[\mathbf{x}]/J_1 \cap I_1 \to K[\mathbf{x}]/J_1 \oplus K[\mathbf{x}]/I_1 \to K[\mathbf{x}]/(J_1 + I_1) \to 0,$$

one has

$$\begin{aligned}\operatorname{depth} K[\mathbf{x}]/J_2 &= \operatorname{depth} K[\mathbf{x}]/(J_1 + I_1) \\ &\geq \min\{\operatorname{depth} K[\mathbf{x}]/J_1, \operatorname{depth} K[\mathbf{x}]/I_1, \operatorname{depth} K[\mathbf{x}]/J_1 \cap I_1 - 1\}.\end{aligned}$$

By Lemma 8.2.9 (i), $\operatorname{depth} K[\mathbf{x}]/I_1 \geq (2k + 5) - 2(k - 2) \geq 6$. Also by Lemma 8.2.10 (ii), one has $J_1 \cap I_1 = x_{2k+3}(x_{k+1}, x_{2k+2})I_1$. Since only $2k-2$ variables appear in the elements of $G((x_{k+1}, x_{2k+2})I_1)$, and x_{2k+3} does not appear in there, one has $\operatorname{depth} K[\mathbf{x}]/J_1 \cap I_1 \geq 7$ by Lemma 8.2.9 (ii). In addition, one has $\operatorname{depth} K[\mathbf{x}]/J_1 \geq 6$ by the first step. Hence one has $\operatorname{depth} K[\mathbf{x}]/J_2 \geq 6$.

Third step. Similarly, by the short exact sequences

$$0 \to K[\mathbf{x}]/J_2 \cap I_5 \to K[\mathbf{x}]/J_2 \oplus K[\mathbf{x}]/I_5 \to K[\mathbf{x}]/(J_2 + I_5) \to 0,$$

one has

$$\begin{aligned}\operatorname{depth} K[\mathbf{x}]/J_3 &= \operatorname{depth} K[\mathbf{x}]/(J_2 + I_5)\\ &\geq \min\{\operatorname{depth} K[\mathbf{x}]/J_2, \operatorname{depth} K[\mathbf{x}]/I_5, \operatorname{depth} K[\mathbf{x}]/J_2 \cap I_5 - 1\}.\end{aligned}$$

By Lemma 8.2.9 (ii), one has $\operatorname{depth} K[\mathbf{x}]/I_5 \geq k+6 \geq 6$. For $\operatorname{depth} K[\mathbf{x}]/J_2 \cap I_5$, by Lemma 8.2.10 (iii), one has $J_2 \cap I_5 = x_{k+2}x_{2k+2}x_{2k+4}(x_2, \ldots, x_k)(x_{2k+3}(x_2, \ldots, x_k) + I_1)$. Notice that only $2k-2$ variables appear and $x_{k+2}, x_{2k+2}, x_{2k+4}$ do not appear in the elements of $G((x_2, \ldots, x_k)(x_{2k+3}(x_2, \ldots, x_k) + I_1))$. Thus, again by Lemma 8.2.9 (ii), one has $\operatorname{depth} K[\mathbf{x}]/J_2 \cap I_5 \geq 7$. Combining these results with the second step, one has $\operatorname{depth} K[\mathbf{x}]/J_3 \geq 6$.

Therefore, one has $\operatorname{depth} K[\mathbf{x}]/I \geq 6$, as required. □

8.2.4 Cohen–Macaulayness of $K[\mathbf{x}]/\operatorname{in}_{<_{\text{lex}}}(I_G)$

In this subsection, we will prove the following

Lemma 8.2.11. *Let $<_{\text{lex}}$ denote the lexicographic order on $K[\mathbf{x}]$ induced by the ordering $x_1 > \cdots > x_{2k+5}$ of the variables. Then $K[\mathbf{x}]/\operatorname{in}_{<_{\text{lex}}}(I_H)$ is Cohen–Macaulay.*

First of all, we need to know the generators of $\operatorname{in}_{<_{\text{lex}}}(I_H)$. As an analogue of Lemmata 8.1.6 and 8.2.4, we can prove the following

Lemma 8.2.12. *The set of binomials corresponding to primitive even closed walks* (I), (II), (III), (VI) *appeared in the previous subsection is a Gröbner basis of I_H with respect to $<_{\text{lex}}$.*

Corollary 8.2.13. *The initial ideal of I_G with respect to $<_{\text{lex}}$ is generated by the following monomials:*

$$(\flat)\qquad \begin{aligned}&x_i x_{k+1+j}, \quad 2 \leq i < j \leq k,\\ &x_1 x_{k+2} x_{2k+4} x_{2k+5},\\ &x_r x_{k+2} x_{2k+4}, \ x_1 x_{k+1+r} x_{2k+5}, \quad 2 \leq r \leq k.\end{aligned}$$

In particular, $\operatorname{in}_{<_{\text{lex}}} I_H$ is a squarefree monomial ideal.

Note that we can exclude the initial term of the binomial corresponding to the even closed walk of type (VI).

Let I' be the initial ideal of I_H with respect to $<_{\text{lex}}$. Since I' is squarefree, we can define a simplicial complex Δ' on $[2k+5]$ whose Stanley–Reisner ideal coincides with I'. In order to prove that $K[\mathbf{x}]/I'$ is Cohen–Macaulay, we will show that Δ' is shellable.

We recall the definition of the shellable simplicial complex. Let Δ be a simplicial complex. We call Δ is *pure* if every facets (maximal faces) of Δ have the same dimension. A pure simplicial complex Δ of dimension $d-1$ is called *shellable* if all its facets can be listed

$$F_1, F_2, \ldots, F_s$$

in such a way that

$$(\bigcup_{j=1}^{i-1} \langle F_j \rangle) \cap \langle F_i \rangle \quad (= \bigcup_{j=1}^{i-1} \langle F_j \cap F_i \rangle)$$

is pure of dimension $d-2$ for every $1 < i \le s$. Here $\langle F_i \rangle := \{\sigma \in \Delta : \sigma \subset F_i\}$. It is known that if Δ is shellable, then $K[\Delta]$ is Cohen–Macaulay for any field K.

To show that Δ' is shellable, we investigate the facets of Δ'. Let $F(\Delta')$ be the set of facets of Δ'. Then the standard primary decomposition of $I' = I_{\Delta'}$ is

$$I_{\Delta'} = \bigcap_{F \in F(\Delta')} P_{\bar{F}},$$

where $\bar{F}$ is the complement of F in $[2k+5]$ and $P_{\bar{F}} = (x_i : i \in \bar{F})$; see [25, Lemma 1.5.4]. Hence we can obtain $F(\Delta')$ from the standard primary decomposition of I'.

Lemma 8.2.14. *The standard primary decomposition of I' is the intersection of the following prime ideals:*

$$(\sharp) \qquad \begin{aligned} &(x_1) + (x_2, x_3, \ldots, x_k),\ (x_{2k+5}) + (x_2, x_3, \ldots, x_k), \\ &(x_{k+2}) + (x_{k+3}, x_{k+4}, \ldots, x_{2k+1}),\ (x_{2k+4}) + (x_{k+3}, x_{k+4}, \ldots, x_{2k+1}), \\ &(x_1, x_{k+2}) + I'_\ell,\ \ 2 \le \ell \le k, \\ &(x_1, x_{2k+4}) + I'_\ell,\ \ 2 \le \ell \le k, \\ &(x_{k+2}, x_{2k+5}) + I'_\ell,\ \ 2 \le \ell \le k, \\ &(x_{2k+4}, x_{2k+5}) + I'_\ell,\ \ 2 \le \ell \le k, \end{aligned}$$

where $I'_\ell = (x_2, \ldots, x_{\ell-1}, x_{k+2+\ell}, \ldots, x_{2k+1})$ *for* $\ell = 2, \ldots, k$.

Proof. Since there is no relation of inclusion among the prime ideals on $(\sharp)$, it is enough to prove that the intersection of these prime ideals coincides with I'.

First, we consider the case where $k = 1$. Then $G(I') = \{x_1x_3x_6x_7\}$ and $(\sharp)$ consist of only the first 2 rows: (x_1), (x_7), (x_3), and (x_6). Thus the assertion trivially holds.

Next, we consider the case where $k = 2$. Note that $I'_2 = 0$. Then the ideal I' is

$$\begin{aligned} I' &= (x_1x_4x_8x_9, x_1x_5x_9, x_2x_4x_8) \\ &= (x_1, x_2) \cap (x_1, x_4) \cap (x_1, x_8) \cap (x_4, x_5) \cap (x_4, x_9) \cap (x_8, x_5) \cap (x_8, x_9) \cap (x_9, x_2) \\ &= (x_1, x_2) \cap (x_9, x_2) \cap (x_4, x_5) \cap (x_8, x_5) \cap (x_1, x_4) \cap (x_1, x_8) \cap (x_4, x_9) \cap (x_8, x_9), \end{aligned}$$

as desired.

Hence we may assume that $k \ge 3$. Then the intersection of the prime ideals on the first row of $(\sharp)$ is

$$(x_1x_{2k+5}, x_2, x_3, \ldots, x_k)$$

and that on the second row of ($\sharp$) is

$$(x_{k+2}x_{2k+4}, x_{k+3}, x_{k+4}, \dots, x_{2k+1}).$$

For $\ell = 2, \dots, k$, the intersection of the prime ideals on the last 4 rows of ($\sharp$) is

$$\begin{aligned}&((x_1, x_{k+2}) + I'_\ell) \cap ((x_1, x_{2k+4}) + I'_\ell) \cap ((x_{k+2}, x_{2k+5}) + I'_\ell) \cap ((x_{2k+4}, x_{2k+5}) + I'_\ell)\\ =&((x_1, x_{k+2}x_{2k+4}) + I'_\ell) \cap ((x_{k+2}x_{2k+4}, x_{2k+5}) + I'_\ell)\\ =&(x_1x_{2k+5}, x_{k+2}x_{2k+4}) + I'_\ell.\end{aligned}$$

Hence, the intersection of the prime ideals on the last 4 rows of ($\sharp$) for all ℓ is

$$(x_1x_{2k+5}, x_{k+2}x_{2k+4}) + \bigcap_{\ell=2}^{k} I'_\ell.$$

Therefore the intersection of all prime ideals of ($\sharp$) is

$$\begin{aligned}&x_1x_{2k+5}(x_{k+2}x_{2k+4}, x_{k+3}, x_{k+4}, \dots, x_{2k+1}) + x_{k+2}x_{2k+4}(x_1x_{2k+5}, x_2, x_3, \dots, x_k)\\ &+ (\bigcap_{\ell=2}^{k} I'_\ell) \cap (x_1x_{2k+5}, x_2, x_3, \dots, x_k) \cap (x_{k+2}x_{2k+4}, x_{k+3}, x_{k+4}, \dots, x_{2k+1}).\end{aligned} \tag{8.3}$$

The ideal on the first row of (8.3) coincides with the one generated by monomials on the last 2 rows of ($\flat$). Since $I'_2 = (x_{k+4}, x_{k+5}, \dots, x_{2k+1})$ and $I'_k = (x_2, x_3, \dots, x_{k-1})$, the ideal on the second row of (8.3) coincides with $\bigcap_{\ell=2}^{k} I'_\ell$. Hence, we may prove that

$$\bigcap_{\ell=2}^{k} I'_\ell = (x_ix_{k+1+j} \ : \ 2 \le i < j \le k).$$

To show this equality, we prove

$$\bigcap_{\ell=2}^{k'} I'_\ell = (x_ix_{k+1+j} \ : \ 2 \le i < j \le k') + (x_{k+2+k'}, \dots, x_{2k+1}) \tag{8.4}$$

for $k' = 2, \dots, k$. When $k' = k$, we obtain the desired equality. We use induction on $k' \ge 2$. The case of $k' = 2$ is trivial. When (8.4) holds for k', we have

$$\begin{aligned}\bigcap_{\ell=2}^{k'+1} I'_\ell &= (\bigcap_{\ell=2}^{k'} I'_\ell) \cap I'_{k'+1}\\ &= ((x_ix_{k+1+j} \ : \ 2 \le i < j \le k') + (x_{k+2+k'}, \dots, x_{2k+1})) \cap (x_2, \dots, x_{k'}, x_{k+3+k'}, \dots, x_{2k+1})\\ &= (x_ix_{k+1+j} \ : \ 2 \le i < j \le k') + x_{k+2+k'}(x_2, \dots, x_{k'}) + (x_{k+3+k'}, \dots, x_{2k+1})\\ &= (x_ix_{k+1+j} \ : \ 2 \le i < j \le k' + 1) + (x_{k+3+k'}, \dots, x_{2k+1}),\end{aligned}$$

as desired. $\square$

Now we are in the position to prove Lemma 8.2.11.

Proof of Lemma 8.2.11. By Lemma 8.2.14, $F(\Delta')$ consists of the following subsets of $[2k+5]$:

$$
\begin{aligned}
&F_1 = \overline{\{1\} \cup \{2,3,\ldots,k\}},\ F_2 = \overline{\{2k+5\} \cup \{2,3,\ldots,k\}},\\
&F_3 = \overline{\{k+2\} \cup \{k+3,k+4,\ldots,2k+1\}},\\
&F_4 = \overline{\{2k+4\} \cup \{k+3,k+4,\ldots,2k+1\}},\\
&G_{1,\ell} = \overline{A_1 \cup G'_\ell},\ \ 2 \le \ell \le k,\\
&G_{2,\ell} = \overline{A_2 \cup G'_\ell},\ \ 2 \le \ell \le k,\\
&G_{3,\ell} = \overline{A_3 \cup G'_\ell},\ \ 2 \le \ell \le k,\\
&G_{4,\ell} = \overline{A_4 \cup G'_\ell},\ \ 2 \le \ell \le k,
\end{aligned}
$$

where $G'_\ell = \{2,\ldots,\ell-1,k+2+\ell,\ldots,2k+1\}$ for $2 \le \ell \le k$, $A_1 = \{1,k+2\}$, $A_2 = \{1,2k+4\}$, $A_3 = \{k+2,2k+5\}$, $A_4 = \{2k+4,2k+5\}$ and $\overline{F} = [2k+5] \setminus F$. Note that $G_{m,\ell} \cap A_j = \emptyset$ and $\#(G_{m,\ell}) = k-2$. In particular, Δ' is pure of dimension $k+4$.

Now we define the ordering on $F(\Delta')$ as follows:

$$G_{1,2},\ldots,G_{1,k},G_{2,2},\ldots,G_{2,k},G_{3,2},\ldots,G_{3,k},G_{4,2},\ldots,G_{4,k},F_1,F_2,F_3,F_4. \tag{8.5}$$

We will prove Δ' satisfies the condition of shellability with this ordering. For $F,G \in F(\Delta)$, we write $G \prec F$ if G lies in previous to F on (8.5).

First, we investigate $\Delta_{m,\ell} := (\bigcup_{G' \prec G_{m,\ell}} \langle G' \rangle) \cap \langle G_{m,\ell} \rangle = \bigcup_{G' \prec G_{m,\ell}} \langle G' \cap G_{m,\ell} \rangle$ for $m = 1,2,3,4$. For $\ell' < \ell$, one has

$$
\begin{aligned}
G_{m,\ell'} \cap G_{m,\ell} &= \overline{A_m \cup G'_{\ell'}} \cap \overline{A_m \cup G'_\ell}\\
&= \overline{(A_m \cup G'_{\ell'}) \cup (A_m \cup G'_\ell)}\\
&= \overline{A_m \cup \{2,\ldots,\ell-2,\ell-1,k+2+\ell',k+3+\ell',\ldots,2k+1\}}\\
&\subset \overline{A_m \cup \{2,\ldots,\ell-2,\ell-1,k+1+\ell,k+2+\ell,\ldots,2k+1\}}\\
&= G_{m,\ell-1} \cap G_{m,\ell}
\end{aligned}
$$

and $G_{m,\ell-1} \cap G_{m,\ell}$ is a $(k+3)$-dimensional face. Then we can conclude that $\Delta_{1,\ell}$ is pure of dimension $k+3$. Assume that $m = 2,3,4$. For $m' < m$, one has

$$
\begin{aligned}
G_{m',\ell'} \cap G_{m,\ell} &= \overline{A_{m'} \cup G'_{\ell'}} \cap \overline{A_m \cup G'_\ell}\\
&= \overline{(A_{m'} \cup G'_{\ell'}) \cup (A_m \cup G'_\ell)}\\
&\subset \overline{(A_{m'} \cup A_m) \cup G'_\ell}.
\end{aligned}
$$

When $m = 2$, then $m' = 1$ and

$$\overline{(A_1 \cup A_2) \cup G'_\ell} = \overline{\{1,k+2,2k+4\} \cup G'_\ell} = G_{1,\ell} \cap G_{2,\ell},$$

which is $(k+3)$-dimensional. Therefore, we can conclude that $\Delta_{2,\ell}$ is a pure simplicial complex of dimension $k+3$. Similarly, we can see that $\Delta_{m,\ell}$ is pure of dimension $k+3$ for $m=3,4$ since e.g., $A_2 \cup A_3 \supset A_1 \cup A_3 = \{1, k+2, 2k+5\}$.

Next, we investigate $\Delta_s := \bigcup_{G \prec F_s} \langle G \cap F_s \rangle$ for $s = 1,2,3,4$. It is easy to see that $G_{1,k} \cap F_1$ (resp. $G_{2,k} \cap F_1$) contains $G_{1,\ell} \cap F_1$ and $G_{3,\ell} \cap F_1$ (resp. $G_{2,\ell} \cap F_1$ and $G_{4,\ell} \cap F_1$). Thus facets of Δ_1 are $G_{1,k} \cap F_1$ and $G_{2,k} \cap F_1$, those are $(k+3)$-dimensional.

Similarly, we can see that the facets of Δ_2 are $G_{3,k} \cap F_1$, $G_{4,k} \cap F_1$, and $F_1 \cap F_2$, those are also $(k+3)$-dimensional.

For Δ_3, we can verify that $G_{1,2} \cap F_3$ (resp. $G_{3,2} \cap F_3$) is a $(k+3)$-dimensional face containing $G_{1,\ell} \cap F_3$, $G_{2,\ell} \cap F_3$ and $F_1 \cap F_3$ (resp. $G_{3,\ell} \cap F_3$, $G_{4,\ell} \cap F_3$ and $F_2 \cap F_3$). Therefore, Δ_3 is pure of dimension $k+3$.

Similarly, we can see that Δ_4 is also a pure simplicial complex of dimension $k+3$ whose facets are $G_{2,2} \cap F_4$, $G_{4,2} \cap F_4$, and $F_3 \cap F_4$. $\square$

Chapter 9

Affine semigroup rings arising from cyclic polytopes

Following the previous chapter, in this chapter, we will study some properties on the affine semigroup ring arising from cyclic polytopes. In Section 9.1, we will consider the normality and non-very ampleness of affine semigroup rings arising from cyclic polytopes. In Section 9.2, we will investigate their Cohen–Macaualyness and Gorensteinness. Finally, in Section 9.3, we will study the other semigroup rings arising from cyclic polytopes, which are generated only by the vertices of cyclic polytopes.

9.1 Normality and non-very ampleness of cyclic polytopes

The cyclic polytope is one of the most distinguished polytopes and played the essential role in the classical theory of convex polytopes ([20]). Let d and n be positive integers with $n \geq d+1$ and $\tau_1, \ldots, \tau_n$ real numbers with $\tau_1 < \cdots < \tau_n$. The convex polytope $C_d(\tau_1, \ldots, \tau_n)$ which is the convex hull of the finite set

$$\{(\tau_1, \tau_1^2, \ldots, \tau_1^d), \ldots, (\tau_n, \tau_n^2, \ldots, \tau_n^d)\} \subset \mathbb{R}^d$$

is called a *cyclic polytope*. It is known that $C_d(\tau_1, \ldots, \tau_n)$ is a simplicial polytope of dimension d with n vertices. The combinatorial type of $C_d(\tau_1, \ldots, \tau_n)$ is independent of the particular choice of real numbers $\tau_1, \ldots, \tau_n$.

The present section is devoted to the study on *integral* cyclic polytopes. A convex polytope is called integral if all of its vertices have integer coordinate. The integral convex polytope has established an active area lying between combinatorics and commutative algebra ([26, 72]).

We say that $\mathcal{P}$ is *normal* if one has

$$\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}} = \mathbb{Z}\mathcal{A}_{\mathcal{P}} \cap \mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P}}.$$

Moreover, $\mathcal{P}$ is called *very ample* if the set

$$(\mathbb{Z}\mathcal{A}_{\mathcal{P}} \cap \mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P}}) \setminus \mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}$$

is finite. One of the most fundamental questions on integral convex polytopes is to determine if given an integral convex polytope is normal ([54]).

On the other hand, we say that an integral convex polytope $\mathcal{P} \subset \mathbb{R}^N$ has the *integer decomposition property* if, for each $m = 1, 2, \ldots$ and for each $\alpha \in m\mathcal{P} \cap \mathbb{Z}^N$, there exist $\alpha_1, \ldots, \alpha_m$ belonging to $\mathcal{P} \cap \mathbb{Z}^N$ such that $\alpha = \alpha_1 + \cdots + \alpha_m$. Here $m\mathcal{P} = \{ m\alpha : \alpha \in \mathcal{P} \}$. If $\mathcal{P}$ has the integer decomposition property, then $\mathcal{P}$ is normal. However, the converse is false. For example, the tetrahedron $\mathcal{T}_3 \subset \mathbb{R}^3$ with the vertices $(0,0,0), (1,1,0), (1,0,1)$ and $(0,1,1)$ is normal, but cannot have the integer decomposition property because $(1,1,1) \in 2\mathcal{T}_3$. If $\mathcal{P} \subset \mathbb{R}^d$ is an integral convex polytope of dimension d with $\mathbb{Z}(\mathcal{P}^* \cap \mathbb{Z}^{d+1}) = \mathbb{Z}^{d+1}$, then $\mathcal{P}$ has the integer decomposition property if and only if $\mathcal{P}$ is normal. Lemma 9.1.7 says that every integral cyclic polytope $\mathcal{P} \subset \mathbb{R}^d$ satisfies $\mathbb{Z}(\mathcal{P}^* \cap \mathbb{Z}^{d+1}) = \mathbb{Z}^{d+1}$. In particular it follows that an integral cyclic polytope is normal if and only if it has the integer decomposition property.

Let, as before, d and n be positive integers with $n \geq d+1$. Given integers $\tau_1, \ldots, \tau_n$ with $\tau_1 < \cdots < \tau_n$, we wish to examine whether $C_d(\tau_1, \ldots, \tau_n)$ is normal or not. Thus our final goal is to classify the integers $\tau_1, \ldots, \tau_n$ with $\tau_1 < \cdots < \tau_n$ for which $C_d(\tau_1, \ldots, \tau_n)$ is normal. Even though to find a complete classification seems to be rather difficult, many fascinating problems arise in the natural way. As a first step toward our goal, we are interested in finding the smallest integer γ_d such that if $\tau_{i+1} - \tau_i \geq \gamma_d$ for $1 \leq i < n$, then $C_d(\tau_1, \ldots, \tau_n)$ is normal. It follows immediately from [21, Theorem 1.3 (b)] that one has $\gamma_d \leq d(d+1)$. In the present section, a new inequality $\gamma_d \leq d^2 - 1$ is proved (Theorem 9.1.9). Moreover, it is shown that if $d \geq 4$ with $\tau_3 - \tau_2 = 1$, then $C_d(\tau_1, \ldots, \tau_n)$ is non-very ample (Theorem 9.1.14).

9.1.1 Preliminaries

In this subsection, we prepare notation and lemmata for our theorems, Theorem 9.1.9 and Theorem 9.1.14.

First of all, we will review some fundamental facts on cyclic polytopes. Let d and n be positive integers with $n \geq d+1$. It is convenient to work with a homogeneous version of the cyclic polytopes, hence, throughout the present paper, we consider $C_d^*(\tau_1, \ldots, \tau_n)$ instead of $C_d(\tau_1, \ldots, \tau_n)$. For n real numbers $\tau_1, \ldots, \tau_n$ with $\tau_1 < \cdots < \tau_n$, we set

$$v_i := (1, \tau_i, \tau_i^2, \ldots, \tau_i^d) \in \mathbb{R}^{d+1} \quad \text{for} \quad 1 \leq i \leq n.$$

In other words, $C_d^*(\tau_1, \ldots, \tau_n) = \mathrm{conv}(\{v_i : 1 \leq i \leq n\}) \subset \mathbb{R}^{d+1}$. Unless stated otherwise, we will always assume the indices are ordered like $\tau_1 < \ldots < \tau_n$. See [78, Chapter 0] for some basic properties of cyclic polytopes. We will use a well-known characterization of their facets. (See, e.g., [78, Theorem 0.7]).

Let $[n] := \{1, \dots, n\}$ and let us say that a set $S \subset [n]$ *forms a facet* of $C_d^*(\tau_1, \dots, \tau_n)$ if $\mathrm{conv}(\{v_i : i \in S\})$ is its facet.

Proposition 9.1.1 (Gale's evenness condition)**.** *A set $S \subset [n]$ with d elements forms a facet of $C_d^*(\tau_1, \dots, \tau_n)$ if and only if S satisfies the following condition: If i and j with $i < j$ are not in S, then the number of elements of S between i and j is even. In other words,*

$$2 \,|\, \#\{k \in S \mid i < k < j\},$$

where $\#X$ stands for the number of elements contained in a finite set X.

Moreover, we also know the precise information on other faces of cyclic polytopes. Recall that the cyclic polytope $C_d^*(\tau_1, \dots, \tau_n)$ is simplicial. Hence its boundary complex is just a $(d-1)$-dimensional simplicial complex on $\{v_1, \dots, v_n\}$. Assigning i to v_i for each i, we can regard the simplicial complex as the one on $[n]$. Let $\Gamma_d(\tau_1, \dots, \tau_n)$ denote this simplicial complex on $[n]$. The faces of $\Gamma_d(\tau_1, \dots, \tau_n)$ are completely characterized in terms of their *type*. A non-empty subset $W \subset [n]$ is said to be *contiguous* if $W = \{i, i+1, \dots, j\}$ for some positive integers i and j with $1 < i \le j < n$, and to be an *end set* if either $W = \{1, \dots, i\}$ or $W = \{i, \dots, n\}$ for some i with $1 \le i \le n$. We set $\max \emptyset := 1$ and $\min \emptyset := n$. Any subset $W \subset [n]$ has unique decomposition

$$W := Y_1 \sqcup X_1 \sqcup X_2 \sqcup \cdots \sqcup X_t \sqcup Y_2, \tag{9.1}$$

such that

1. Y_1, Y_2 are empty or end sets, and each X_i is contiguous;
2. $\max X_i < \min X_{i+1}$ for all i with $0 \le i \le t$, where we set $X_0 := Y_1$ and $X_{t+1} := Y_2$.

The subset W is said to be of *type* (r, s) where $r = \#W$ and $s = \#\{i : \#X_i \text{ is odd}\}$.

Proposition 9.1.2 (cf. [12, pp. 226–227])**.** *Let W be a subset of $[n]$. The following statements hold.*

(i) Any $d+1$ elements of $v_1, \dots, v_n$ are linearly independent over $\mathbb{R}$.

(ii) If $\#W \le \lfloor d/2 \rfloor$, then W is a face of $\Gamma_d(\tau_1, \dots, \tau_n)$.

(iii) The subset W is a face of $\Gamma_d(\tau_1, \dots, \tau_n)$ of dimension $\#W - 1$ if and only if $0 \le \#W \le d$ and W is a type $(\#W, s)$ for some integer s with $0 \le s \le d - \#W$.

Hereafter, we will assume that $\tau_1, \dots, \tau_n$ are integers.

Let $\Delta_{ij} := \tau_j - \tau_i$ for $i, j \in [n]$. The proof of Proposition 9.1.1 yields a description of the inequality of the supporting hyperplane defining each facet. Let $S = \{k_1, \dots, k_d\} \subset [n]$ and consider the polynomial

$$\sum_{i=0}^{d} c_{S,i} t^i := \prod_{i \in S} (t - \tau_i).$$

Then all d vectors $v_{k_1}, \ldots, v_{k_d}$ vanish by the linear form

$$\sigma_S : \mathbb{R}^{d+1} \ni (w_0, w_1, \ldots, w_d) \mapsto \sum_{i=0}^{d} c_{S,i} w_i \in \mathbb{R}\,,$$

thus it defines the hyperplane spanned by them. Note that we index the first coordinate by 0. Hence, if the set S forms a facet $\mathcal{F}$ of $\mathcal{P}^* = C_d^*(\tau_1, \ldots, \tau_n)$, then σ_S is the linear form defining $\mathcal{F}$, which means that $\sigma_S(x) \geq 0$ if x is in $\mathcal{P}^*$ and $\sigma_S(x) = 0$ if x is in $\mathcal{F}$. For every $j \in [n] \setminus S$, it holds $\sigma_S(v_j) = \prod_{i \in S} \Delta_{ij}$. This has a useful implication, that is, if we write a vector $x \in \mathbb{Z}^{d+1}$ as $x = \sum_{i \in S} \lambda_i v_i + \lambda_j v_j$ with rational coefficients λ_i, then the denominator of λ_j is a divisor of $\prod_{i \in S} \Delta_{ij}$, because $\sigma_S(x) = \lambda_j \prod_{i \in S} \Delta_{ij}$ is an integer.

We introduce a special representation of cyclic polytopes which is sometimes helpful. Write the vectors $v_1, \ldots, v_n$ as row vectors into a matrix, namely,

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 1 & \tau_1 & \tau_1^2 & \ldots & \tau_1^d \\ 1 & \tau_2 & \tau_2^2 & \ldots & \tau_2^d \\ \vdots & \vdots & & & \vdots \\ 1 & \tau_n & \tau_n^2 & \ldots & \tau_n^d \end{pmatrix}. \tag{9.2}$$

Lemma 9.1.3. *The aforementioned matrix can be transformed to the following matrix by using a unimodular transformation:*

$$\begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 1 & \Delta_{12} & 0 & \ddots & \vdots \\ 1 & \Delta_{13} & \Delta_{13}\Delta_{23} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 1 & \Delta_{1,d+1} & \Delta_{1,d+1}\Delta_{2,d+1} & \cdots & \prod_{k=1}^{d} \Delta_{k,d+1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \Delta_{1,n} & \Delta_{1,n}\Delta_{2,n} & \cdots & \prod_{k=1}^{d} \Delta_{k,n} \end{pmatrix}. \tag{9.3}$$

In particular, the convex hull of the row vectors of this matrix is unimodularly equivalent to $C_d^*(\tau_1, \ldots, \tau_n)$.

A proof of the above lemma is essentially the same as a proof of the well-known Vandermonde determinant. Note that Lemma 9.1.3 is valid for any ordering of the parameters $\tau_1, \ldots, \tau_n$, i.e., any ordering of $v_1, \ldots, v_n$.

Let us identify a special case where the polytopes are indeed unimodularly equivalent.

Lemma 9.1.4. *An integral cyclic polytope $C_d^*(\tau_1, \ldots, \tau_d)$ is unimodularly equivalent to $C_d^*(-\tau_n, \ldots, -\tau_1)$. Moreover, for any integer m, $C_d^*(\tau_1, \ldots, \tau_d)$ is unimodularly equivalent to $C_d^*(\tau_1 + m, \ldots, \tau_n + m)$.*

Proof. The replacement $\tau_i \mapsto -\tau_i$ corresponds to a multiplication with -1 in every column of (9.2) with an odd exponent. This is a unimodular transformation. The second statement is immediate from Lemma 9.1.3, because the matrix (9.3) depends only on the differences $\Delta_{ij} = \tau_j - \tau_i$. $\square$

We define a certain class of vectors which we will use in the sequel. Let $S = \{i_1, \ldots, i_q\} \subset [n]$ be a non-empty set, where $i_1 < \cdots < i_q$. Then we define

$$b_S := \sum_{i \in S} \frac{1}{\prod_{j \in S \setminus \{i\}} \Delta_{ij}} v_i = \sum_{k=1}^{q} \frac{(-1)^{k+1}}{\prod_{j \in S \setminus \{i_k\}} |\Delta_{i_k j}|} v_{i_k},$$

where $b_S = v_{i_1}$ when $q = 1$, i.e., $\#S = 1$. If S is small, we will sometimes omit the brackets around the elements, thus we write, for example, $b_{ij} = b_{\{i,j\}}$. However, the vector does not depend on the order of the indices.

Example 9.1.5. *Let us write down b_S's for small sets S. Assume $1 \leq i < j < k < l \leq n$. Then*

$$\begin{aligned}
b_i &= v_i, \\
b_{ij} &= \frac{1}{\Delta_{ij}} v_i - \frac{1}{\Delta_{ij}} v_j, \\
b_{ijk} &= \frac{1}{\Delta_{ij}\Delta_{ik}} v_i - \frac{1}{\Delta_{ij}\Delta_{jk}} v_j + \frac{1}{\Delta_{ik}\Delta_{jk}} v_k, \\
b_{ijkl} &= \frac{1}{\Delta_{ij}\Delta_{ik}\Delta_{il}} v_i - \frac{1}{\Delta_{ij}\Delta_{jk}\Delta_{jl}} v_j + \frac{1}{\Delta_{ik}\Delta_{jk}\Delta_{kl}} v_k - \frac{1}{\Delta_{il}\Delta_{jl}\Delta_{kl}} v_l.
\end{aligned}$$

The sign changes are due to a reordering of the indices since $\Delta_{ij} = -\Delta_{ji}$. If v_i, v_j, v_k, v_l are given in the form (9.3), i.e., if

$$\begin{pmatrix} v_i \\ v_j \\ v_k \\ v_l \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & \Delta_{ij} & 0 & \ddots & \cdots & \cdots & \vdots \\ 1 & \Delta_{ik} & \Delta_{ik}\Delta_{jk} & \ddots & \cdots & \cdots & \vdots \\ 1 & \Delta_{il} & \Delta_{il}\Delta_{jl} & \Delta_{il}\Delta_{jl}\Delta_{kl} & 0 & \cdots & 0 \end{pmatrix},$$

then $b_i = (1, 0, \ldots, 0)$, $b_{ij} = (0, -1, 0, \ldots, 0)$, $b_{ijk} = (0, 0, 1, 0, \ldots, 0)$ and $b_{ijkl} = (0, 0, 0, -1, 0, \ldots, 0)$. In general, $b_1, b_{12}, \ldots, b_{12\cdots d+1}$ look like $(0, \ldots, 0, \pm 1, 0, \ldots, 0)$ when $v_1, \ldots, v_{d+1}$ are of the form (9.3).

The following proposition collects the basic properties on these vectors.

Proposition 9.1.6. *(i) For any non-empty set $S \subset [n]$, one has $b_S \in \mathbb{Z}^{d+1}$.*

(ii) Let $S \subset [n]$ and $a, b \in S$ with $a \neq b$. Then we have a recursion formula

$$b_S = \frac{1}{\Delta_{ba}} b_{S \setminus \{a\}} + \frac{1}{\Delta_{ab}} b_{S \setminus \{b\}}.$$

(iii) For any distinct $d+1$ indices $i_1, \dots, i_{d+1} \in [n]$ (not necessarily ordered), the vectors

$$b_{i_1}, b_{i_1 i_2}, b_{i_1 i_2 i_3}, \dots, b_{i_1 \cdots i_{d+1}}$$

form a $\mathbb{Z}$-basis for $\mathbb{Z}^{d+1}$.

(iv) If $\#S \geq d+2$, then $b_S = 0$.

Proof. The second statement can be verified by elementary computations, using $\Delta_{ij} + \Delta_{jk} = \Delta_{ik}$ for $i, j, k \in [n]$. To prove the first statement, we consider the components of b_S as rational functions in $\tau_i, i \in S$. By induction on $\#S$, we prove the following statement. The components of b_S are symmetric polynomials in $\tau_i, i \in S$, and their coefficients depend only on $\#S$.

If $\#S = 1$, then $b_S = b_i = v_i = (1, \tau_i, \tau_i^2, \dots, \tau_i^d)$, thus the claim holds. Now consider a set S with at least two distinct elements a, b. Let

$$f_j(\tau_a, \tau_i, i \in S),\ f_j(\tau_b, \tau_i, i \in S)$$

be the j-th components of $b_{S \setminus b}$, $b_{S \setminus a}$, respectively. Then the difference between these polynomials is zero if we set $\tau_a = \tau_b$, hence the quotient

$$\frac{f_j(\tau_a, \tau_i, i \in S) - f_j(\tau_b, \tau_i, i \in S)}{\tau_a - \tau_b}$$

is a polynomial as claimed. It is obviously symmetric in a and b. Since we are free to choose any two elements of S, it is symmetric in all variables. The coefficients of the polynomial depend only on $\#S$, so the claim is proven. Note that the degree of the polynomial decreases by one by taking the quotient. Since the degree of the components of v_i is at most $d+1$, we conclude that $b_S = 0$ for $\#S \geq d+2$.

To prove the third statement, we first note that the vertices $v_{i_1}, \dots, v_{i_{d+1}}$ are linearly independent. Take an element $x \in \mathbb{Z}^{d+1}$ and write it as $x = \sum \lambda_j v_{i_j}$. By considering $\sigma_{\{i_1, \dots, i_d\}}(x)$, we can say that the coefficient $\lambda_{i_{d+1}}$ is of the form

$$\lambda_{i_{d+1}} = \frac{k}{\prod_{j=1}^{d} \Delta_{i_j i_{d+1}}}$$

for an integer k. Thus, $x + (-1)^d k b_{i_1 \dots i_{d+1}} \in \mathbb{Z}^{d+1}$ is a vector in the subspace spanned by $v_{i_1}, \dots, v_{i_d}$. These vectors define a $(d-1)$-dimensional cyclic polytope again, so we can proceed by induction and obtain a representation of x as a $\mathbb{Z}$-linear combination of the $b_{i_1}, b_{i_1 i_2}, \dots, b_{i_1 \dots i_{d+1}}$. $\square$

We apply this construction to prove another useful fact on cyclic polytopes.

Lemma 9.1.7. *For an integral cyclic polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension d, one has*

$$\mathbb{Z}\mathcal{A}_{\mathcal{P}} = \mathbb{Z}^{d+1}.$$

Proof. First, we notice that $\mathbb{Z}\mathcal{A}_{\mathcal{P}} \subset \mathbb{Z}^{d+1}$ is obvious. To prove another inclusion, we construct a basis of $\mathbb{Z}^{d+1}$ from $d+1$ points in $\mathcal{A}_{\mathcal{P}}$. We choose $d+1$ vertices $v_1, \ldots, v_{d+1}$ of $\mathcal{P}^*$ and consider the vectors

$$b_{i_{d+1}}, b_{i_{d+1}} + b_{i_d i_{d+1}}, b_{i_{d+1}} + b_{i_d i_{d+1}} + b_{i_{d-1} i_d i_{d+1}}, \ldots, \sum_{l=1}^{d+1} b_{i_l \ldots i_{d+1}}.$$

Let us denote them by $c_j := \sum_{l=j}^{d+1} b_{i_l \ldots i_{d+1}}$ for $j = 1, \ldots, d+1$. By Proposition 9.1.6 (iii), they constitute a $\mathbb{Z}$-basis of $\mathbb{Z}^{d+1}$. Hence, if each c_j is contained in $\mathcal{P}^*$, then our claim follows. For this, let us consider the coefficient of a vertex v_{i_k} in the sequence of

$$b_{i_d}, b_{i_d i_{d+1}}, b_{i_{d-1} i_d i_{d+1}}, \ldots, b_{i_1 \ldots i_{d+1}}.$$

The coefficient of v_{i_k} appears first in $b_{i_k \ldots i_{d+1}}$, where it has a positive sign. After that, its sign is alternating and the absolute value is non-increasing since the denominators increase. Hence, the sum of those coefficients and thus the coefficient in c_j is nonnegative. So, c_j is a convex combination of the vertices of $\mathcal{P}^*$. □

Finally, we discuss the normality of integral cyclic polytopes.

Lemma 9.1.8. *Let $\mathcal{P}$ be an integral cyclic polytope of dimension d. If any simplex of dimension d whose vertices are chosen from those of $\mathcal{P}$ is normal, then $\mathcal{P}$ itself is also normal.*

Proof. Let $v_1, \ldots, v_n$ be the vertices of $\mathcal{P}^*$. A proof is a direct application of Carathéodory's Theorem (see, e.g., [66, Section 7]). Let $x \in \mathbb{Z}\mathcal{A}_{\mathcal{P}} \cap \mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P}}$. Now, Carathéodory's Theorem guarantees that there exist $d+1$ vertices $v_{i_1}, \ldots, v_{i_{d+1}}$ of $\mathcal{P}^*$ such that $x \in \mathbb{Z}\mathcal{A}_{\mathcal{Q}} \cap \mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{Q}}$, where $\mathcal{Q} = \mathrm{conv}(\{v_{i_1}, \ldots, v_{i_{d+1}}\})$. Here we use that $\mathbb{Z}\mathcal{A}_{\mathcal{P}} = \mathbb{Z}^{d+1} = \mathbb{Z}\mathcal{A}_{\mathcal{Q}}$ by Lemma 9.1.7. If $\mathcal{Q}$ is normal, then we have $x \in \mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{Q}}$, in particular, $x \in \mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}$. This implies that $\mathcal{P}$ is normal. □

9.1.2 Normal cyclic polytopes

Our goal of this subsection is to prove

Theorem 9.1.9 ([30, Theorem 2.1]). *Work with the same notations as in the previous section 1. If $\Delta_{i,i+1} \geq d^2 - 1$ for $1 \leq i \leq n-1$, then $\mathcal{P} = C_d(\tau_1, \ldots, \tau_n)$ is normal. In particular, $\gamma_d \leq d^2 - 1$.*

Most parts of this section are devoted to proving the simplex case. In fact, once we know that $\mathcal{P}$ is always normal when $n = d+1$ and $\Delta_{i,i+1} \geq d^2 - 1$ for $1 \leq i \leq d$, Theorem 9.1.9 follows immediately from Lemma 9.1.8.

Before giving a proof, we prepare two lemmata, Lemma 9.1.11 and Lemma 9.1.12. First, for Lemma 9.1.11, we start from proving

Proposition 9.1.10. *Let $(r_1, r_2, \dots, r_{d+1}) \in \mathbb{Q}^{d+1}$ satisfying*

$$0 \le r_1 \le r_2 \le \cdots \le r_{d+1} \le 1 \quad \text{and} \quad \sum_{i=1}^{d+1} r_i = m.$$

Then one has

$$\text{(a) } \textstyle\sum_{i=1}^{j} r_i \le \frac{jm}{d+1} \quad \text{and} \quad \text{(b) } \sum_{i=1}^{j} r_{d+2-i} \ge \frac{jm}{d+1}$$

for any integer j with $1 \le j \le d+1$.

Proof. We prove by induction on j.

First, we show $r_1 \le \frac{m}{d+1}$. Suppose that $r_1 > \frac{m}{d+1}$. Then one has $r_i > \frac{m}{d+1}$ for all $1 \le i \le d+1$ by $r_1 \le r_2 \le \cdots \le r_{d+1}$. Thus, $m = \sum_{i=1}^{d+1} r_i > (d+1) \cdot \frac{m}{d+1} = m$, a contradiction. Similarly, we also have $r_{d+1} \ge \frac{m}{d+1}$.

Now, we assume that the assertions (a) and (b) hold for any integer j' with $1 \le j' < j$, where j is some integer with $2 \le j \le d+1$. Let $d+1 = kj+q$, where k is a positive integer and $0 \le q \le j-1$, i.e., k (resp. q) is a quotient (resp. a remainder) of $d+1$ divided by j. Suppose that $\sum_{i=1}^{j} r_i > \frac{jm}{d+1}$. Then one has

$$\sum_{i=1}^{j} r_{(k-1)j+i} \ge \sum_{i=1}^{j} r_{(k-2)j+i} \ge \cdots \ge \sum_{i=1}^{j} r_i > \frac{jm}{d+1}.$$

Moreover, by the hypothesis of induction, one also has $\sum_{i=kj+1}^{d+1} r_i = \sum_{i=1}^{q} r_{d+2-i} \ge \frac{mq}{d+1}$ when $q \ne 0$. Hence, we obtain

$$m = \sum_{i=1}^{d+1} r_i > k \cdot \frac{jm}{d+1} + \frac{mq}{d+1} = m \cdot \frac{kj+q}{d+1} = m,$$

a contradiction. Therefore, the assertion (a) also holds for j. Similarly, we also have the assertion (b) for j, as required. $\square$

Lemma 9.1.11. *Let d be a positive integer and $(r_1, r_2, \dots, r_{d+1}) \in \mathbb{Q}^{d+1}$ satisfying that $0 \le r_1 \le r_2 \le \cdots \le r_{d+1} \le 1$ and that $\sum_{i=1}^{d+1} r_i$ is an integer which is greater than 1. Then one has*

$$\max_{\substack{1 \le i_1 < i_2 < \cdots < i_l \le d+1, \\ 2 \le l \le d}} \left\{ \sum_{j=1}^{l} r_{i_j} : \sum_{j=1}^{l-1} r_{i_j} \le 1 \right\} \ge 1 + \frac{1}{d+1}. \tag{9.4}$$

Proof. Let $m = \sum_{i=1}^{d+1} r_i$. When $m > d$, it must be satisfied that $r_i = 1$ for $1 \le i \le d+1$ and $m = d+1$ by our assumption. Thus, we may assume that $2 \le m \le d$. Let M denote the value of the left-hand side of (9.4).

The first step. Assume that $m-1 > \lfloor \frac{d+1}{2} \rfloor$. Then, by Proposition 9.1.10, one has $r_d + r_{d+1} \geq \frac{2m}{d+1}$, while $r_d \leq 1$. Hence,

$$M \geq r_d + r_{d+1} \geq \frac{2m}{d+1} > \frac{2}{d+1}\left(\left\lfloor \frac{d+1}{2} \right\rfloor + 1\right) \geq \frac{2}{d+1}\left(\frac{d}{2}+1\right) = 1 + \frac{1}{d+1}.$$

The second step. Assume that $m-1 \leq \lfloor \frac{d+1}{2} \rfloor$ and let $d+1 = km+q$, where k is a positive integer and $0 \leq q \leq m-1$, i.e., k (resp. q) is a quotient (resp. a remainder) of $d+1$ divided by m.

If we suppose that $\sum_{j=0}^{k-1} r_{jm+q+1} > 1$, then one has

$$1 < \sum_{j=0}^{k-1} r_{jm+q+1} \leq \sum_{j=0}^{k-1} r_{jm+q+2} \leq \cdots \leq \sum_{j=0}^{k-1} r_{jm+q+m}.$$

Thus, $m = \sum_{i=1}^{d+1} r_i \geq \sum_{i=q+1}^{d+1} r_i > m$, a contradiction. Hence, we have

$$\sum_{j=0}^{k-1} r_{jm+q+1} \leq 1.$$

The third step. If we assume that $q \neq m-1$, that is, $0 \leq q \leq m-2$, then one has $\sum_{j=0}^{k-2} r_{jm+q+2} \leq \frac{d-q-m+1}{d-q}$. In fact, on the contrary, suppose that $\sum_{j=0}^{k-2} r_{jm+q+2} > \frac{d-q-m+1}{d-q}$. Then,

$$\frac{d-q-m+1}{d-q} < \sum_{j=0}^{k-2} r_{jm+q+2} \leq \sum_{j=0}^{k-2} r_{jm+q+3} \leq \cdots \leq \sum_{j=0}^{k-2} r_{jm+q+m+1}.$$

Thus, $\sum_{i=q+2}^{(k-1)m+q+1} r_i > \frac{m(d-q-m+1)}{d-q}$. Moreover, since $\sum_{i=q+2}^{d+1} r_i = m - \sum_{i=1}^{q+1} r_i$, we also have $\sum_{i=(k-1)m+q+2}^{d+1} r_i \geq \frac{(m-1)(m-\sum_{i=1}^{q+1} r_i)}{d-q}$ by Proposition 9.1.10. Hence,

$$\begin{aligned} m - \sum_{i=1}^{q+1} r_i = \sum_{i=q+2}^{d+1} r_i &> \frac{m(d-q-m+1)}{d-q} + \frac{(m-1)(m-\sum_{i=1}^{q+1} r_i)}{d-q} \\ &= \frac{m(d-q)}{d-q} - \frac{(m-1)\sum_{i=1}^{q+1} r_i}{d-q} \geq m - \sum_{i=1}^{q+1} r_i, \end{aligned}$$

a contradiction. Here, since $m-1 \leq \lfloor \frac{d+1}{2} \rfloor \leq \frac{d+1}{2}$ and $0 \leq q \leq m-2 < d$, we have $m+q \leq 2m-2 \leq d+1$, which means that $\frac{m-1}{d-q} \leq 1$. Thus, one has

$$\sum_{j=0}^{k-2} r_{jm+q+2} \leq \frac{d-q-m+1}{d-q}.$$

Similarly, if we assume that $q = m - 1$, then one has

$$\sum_{j=0}^{k-1} r_{jm+1} \le \frac{d-m+2}{d+1}.$$

The fourth step. In this step, we prove that

$$\sum_{j=0}^{k-1} r_{jm+q+1} + r_{d+1} \ge 1 + \frac{1}{d+1}.$$

We assume that $0 \le q \le m-2$. Suppose, on the contrary, $\sum_{j=0}^{k-1} r_{jm+q+1} + r_{d+1} < 1 + \frac{1}{d+1}$. Then $\sum_{j=1}^{k-1} r_{jm+q+1} + r_{d+1} < 1 + \frac{1}{d+1} - r_{q+1} < 1 + \frac{1}{d-q} - r_{q+1}$. Thus,

$$\begin{aligned} 1 + \frac{1}{d-q} - r_{q+1} &> \sum_{j=1}^{k-1} r_{jm+q+1} + r_{km+q} \ge \sum_{j=1}^{k-1} r_{jm+q} + r_{km+q-1} \ge \cdots \\ &\ge \sum_{j=1}^{k-1} r_{jm+q+1-(m-2)} + r_{km+q-(m-2)} = \sum_{j=0}^{k-2} r_{jm+q+3} + r_{(k-1)m+q+2}. \end{aligned}$$

Moreover, by the third step, we also have $\sum_{j=0}^{k-2} r_{jm+q+2} \le \frac{d-q-m+1}{d-q}$. Hence,

$$\begin{aligned} m - \sum_{i=1}^{q+1} r_i = \sum_{i=q+2}^{d+1} r_i &< m - 1 + \frac{m-1}{d-q} - (m-1) r_{q+1} + \frac{d-q-m+1}{d-q} \\ &= m - (m-1) r_{q+1} \le m - (q+1) r_{q+1} \le m - \sum_{i=1}^{q+1} r_i, \end{aligned}$$

a contradiction. Similarly, when $q = m-1$, if we suppose that $\sum_{j=1}^{k} r_{jm} + r_{km+m-1} < 1 + \frac{1}{d+1}$, then

$$1 + \frac{1}{d+1} > \sum_{j=1}^{k} r_{jm} + r_{km+m-1} \ge \sum_{j=1}^{k} r_{jm-1} + r_{km+m-2} \ge \cdots \ge \sum_{j=0}^{k-1} r_{jm+2} + r_{km+1}$$

and $\sum_{j=0}^{k-1} r_{jm+1} \le \frac{d-m+2}{d+1}$ by the third step, so we obtain $m = \sum_{i=1}^{d+1} r_i < m - 1 + \frac{m-1}{d+1} + \frac{d-m+2}{d+1} = m$, a contradiction.

The fifth step. Thanks to the second and fourth steps, we have

$$M \ge \sum_{j=0}^{k-1} r_{jm+q+1} + r_{d+1} \ge 1 + \frac{1}{d+1},$$

as desired. $\square$

We also prepare another

Lemma 9.1.12. *Let l be an integer with $l \geq 2$ and $i_1, \ldots, i_l$ distinct integers. We set*

$$Z_l(j) = \frac{\prod_{k=1}^{j-1} \Delta_{i_k i_j}}{\prod_{1 \leq k \leq l, k \neq j} |\Delta_{i_k i_j}|} p_j + \frac{\prod_{k=1}^{j-1} \Delta_{i_k i_{j+1}}}{\prod_{1 \leq k \leq l, k \neq j+1} |\Delta_{i_k i_{j+1}}|} p_{j+1} + \cdots + \frac{\prod_{k=1}^{j-1} \Delta_{i_k i_l}}{\prod_{1 \leq k \leq l, k \neq l} |\Delta_{i_k i_l}|} p_l$$

for $2 \leq j \leq l$. Then, for any $2 \leq j \leq l-1$, we have

$$Z_l(j) = \frac{\prod_{k=1}^{j-1} \Delta_{i_k i_j}}{\prod_{1 \leq k \leq l, k \neq j} |\Delta_{i_k i_j}|} p_j + \frac{1}{\Delta_{i_j i_{j+1}}} Z_l(j+1) - \frac{1}{\Delta_{i_j i_{j+1}} \Delta_{i_j i_{j+2}}} Z_l(j+2) + \cdots + (-1)^{l-j+1} \frac{1}{\prod_{k=j+1}^{l} \Delta_{i_j i_k}} Z_l(l).$$

A proof is given by elementary computations.

Now, Lemma 9.1.12 says that if $Z_l(j+1), \ldots, Z_l(l)$ are integers, then there exists an integer p_j such that $Z_l(j)$ becomes an integer. In fact, since

$$\frac{1}{\Delta_{i_j i_{j+1}}} Z_l(j+1) - \cdots + (-1)^{l-j+1} \frac{1}{\prod_{k=j+1}^{l} \Delta_{i_j i_k}} Z_l(l) = \frac{P}{C},$$

where P is some integer and $C = \prod_{k=j+1}^{l} |\Delta_{i_j i_k}|$, and the numerator (resp. the denominator) of $\frac{\prod_{k=1}^{j-1} \Delta_{i_k i_j}}{\prod_{1 \leq k \leq l, k \neq j} |\Delta_{i_k i_j}|}$ is either 1 or -1 (resp. C), it is obvious that there exists an integer p_j such that $Z_l(j)$ becomes an integer.

Let $\mathcal{Q} \subset \mathbb{R}^N$ be an integral convex polytope of dimension d. In general, when $\mathbb{Z}\mathcal{A}_{\mathcal{Q}} = \mathbb{Z}^{N+1}$, in order to prove that $\mathcal{Q}$ is normal, it suffices to show that for any $\alpha = (m, \alpha_1, \ldots, \alpha_N) \in \mathbb{Z}\mathcal{A}_{\mathcal{Q}} \cap \mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{Q}} = \mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{Q}} \cap \mathbb{Z}^{N+1}$ with $m \geq 2$, we find $\alpha' \in \mathcal{Q}^* \cap \mathbb{Z}^{N+1}$ and $\alpha'' \in \mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{Q}} \cap \mathbb{Z}^{N+1}$ with $\alpha = \alpha' + \alpha''$. (This is equivalent to prove that $\mathcal{Q}$ satisfies the integer decomposition property.) In particular, when $\mathcal{Q}$ is a simplex, since there exists a unique $(r_1, \ldots, r_{d+1}) \in \mathbb{Q}^{d+1}$ such that $\alpha = \sum_{i=1}^{d+1} r_i u_i$ and $\sum_{i=1}^{d+1} r_i = m$, where $u_1, \ldots, u_{d+1}$ are the vertices of $\mathcal{Q}^*$, we may find $(r'_1, \ldots, r'_{d+1}) \in \mathbb{Q}^{d+1}$ with $\sum_{i=1}^{d+1} r'_i u_i \in \mathcal{Q}^* \cap \mathbb{Z}^{N+1}$ and $(r''_1, \ldots, r''_{d+1}) \in \mathbb{Q}^{d+1}$ with $\sum_{i=1}^{d+1} r''_i u_i \in \mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{Q}} \cap \mathbb{Z}^{N+1}$ satisying $r'_i + r''_i = r_i$ for $1 \leq i \leq d+1$.

Hence, it is enough to show that for any $\alpha = \sum_{i=1}^{d+1} r_i u_i \in \mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{Q}} \cap \mathbb{Z}^{N+1}$ with $\sum_{i=1}^{d+1} r_i \geq 2$, there exists $(r'_1, \ldots, r'_{d+1}) \in \mathbb{Q}^{d+1}$ such that

$$\sum_{i=1}^{d+1} r'_i = 1, \quad 0 \leq r'_i \leq r_i \ \text{ for } 1 \leq i \leq d+1 \quad \text{and} \quad \sum_{i=1}^{d+1} r'_i u_i \in \mathbb{Z}^{N+1}.$$

Now, we come to the position to verify the normality of integral cyclic polytopes in the case where $n = d+1$ and $\Delta_{i,i+1} \geq d^2 - 1$ for $1 \leq i \leq d$. Let $\mathcal{P}$ be such cyclic polytope. Let m be an integer with $m \geq 2$ and α an element in $\mathbb{Z}\mathcal{A}_{\mathcal{P}} \cap \mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P}} = \mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathbb{Z}^{d+1}$ with the first coordinate m. Since $\mathcal{P}^*$ is a simplex of dimension d, there exists a unique $(r_1, \ldots, r_{d+1}) \in \mathbb{Q}^{d+1}$, where $\sum_{i=1}^{d+1} r_i = m$, such that $\alpha = \sum_{i=1}^{d+1} r_i v_i$. Then what we must do is to show that there exists $(r'_1, \ldots, r'_{d+1}) \in \mathbb{Q}^{d+1}$ such that

$$\sum_{i=1}^{d+1} r'_i = 1, \quad 0 \leq r'_i \leq r_i \quad \text{for } 1 \leq i \leq d+1 \quad \text{and} \quad \sum_{i=1}^{d+1} r'_i v_i \in \mathbb{Z}^{d+1}. \tag{9.5}$$

The first step. If there exists r_i with $r_i \geq 1$, say, r_1, then we may set $r'_1 = 1$ and $r'_2 = \cdots = r'_{d+1} = 0$. Moreover, when $m \geq d+1$, since $\sum_{i=1}^{d+1} r_i = m$ and $r_i \geq 0$, there is at least one r_i with $r_i \geq 1$. Thus, we may assume that

$$2 \leq m \leq d \quad \text{and} \quad 0 \leq r_i \leq 1 \quad \text{for} \quad 1 \leq i \leq d+1.$$

The second step. By Lemma 9.1.11, there exist $r_{i_1}, \ldots, r_{i_l}$ among $(r_1, \ldots, r_{d+1})$ such that $\sum_{j=1}^{l} r_{i_j} \geq 1 + \frac{1}{d+1}$ and $\sum_{j=1}^{l-1} r_{i_j} \leq 1$, where $0 \leq r_{i_1} \leq \cdots \leq r_{i_l} \leq 1$ and $2 \leq l \leq d$, although we do not know whether $1 \leq i_1 < \cdots < i_l \leq d+1$. Let $r_{i_1}, \ldots, r_{i_l}$ be such ones. However, we assume that $0 \leq r_{i_l} \leq r_{i_{l-1}} \leq \cdots \leq r_{i_1} \leq 1$, i.e., we have

$$\sum_{j=2}^{l} r_{i_j} \leq 1 \quad \text{and} \quad \sum_{j=1}^{l} r_{i_j} \geq 1 + \frac{1}{d+1}.$$

Let $D = d^2 - 1$. Thus, $|\Delta_{ij}| \geq D$ for any $1 \leq i \neq j \leq d+1$. Now, we set $\epsilon(l) = \frac{l-1}{D}$ for $2 \leq l \leq d$. Then it is easy to see that $\epsilon(l)$ enjoys the following properties:

$$\begin{aligned} &\epsilon(l) \geq \sum_{a=2}^{l} \frac{1}{D^{a-1}}, \qquad \frac{1}{d+1} = \epsilon(d) > \epsilon(d-1) > \cdots > \epsilon(2), \\ &\epsilon(l) - \frac{l-j+1}{D^{j-1}} > \epsilon(j-1) \quad \text{for} \quad 3 \leq j \leq l. \end{aligned} \tag{9.6}$$

In the following two steps, by induction on l, we prove that if $\sum_{j=1}^{l} r_{i_j} \geq 1 + \epsilon(l)$ and $\sum_{j=2}^{l} r_{i_j} \leq 1$, then there is $(r'_1, \ldots, r'_{d+1}) \in \mathbb{Q}^{d+1}$ which satisfies (9.5). Once we know this, we obtain the required assertion from $2 \leq l \leq d$ and $\frac{1}{d+1} = \epsilon(d) \geq \epsilon(l)$.

The third step. Assume that $l = 2$, i.e., we have $r_{i_1} + r_{i_2} \geq 1 + \frac{1}{D}$, where $0 \leq r_{i_2} \leq r_{i_1} \leq 1$.

Let p be a nonnegative integer satisfying

$$\frac{p}{|\Delta_{i_1 i_2}|} \leq r_{i_2} < \frac{p+1}{|\Delta_{i_1 i_2}|}.$$

Then it is clear that there exists such a unique nonnegative integer p. Let $r'_{i_2} = \frac{p}{|\Delta_{i_1 i_2}|}$, $r'_{i_1} = 1 - r'_{i_2}$ and $r'_j = 0$ for any j with $j \in [d+1] \setminus \{i_1, i_2\}$. Thus, $\sum_{i=1}^{d+1} r'_i = 1$

and $0 \le r'_{i_2} \le r_{i_2}$. Moreover, since $r_{i_2} \le 1$, we have $r'_{i_1} = 1 - r'_{i_2} \ge 1 - r_{i_2} \ge 0$. In addition, by $r_{i_1} + r_{i_2} \ge 1 + \frac{1}{D}$ and $|\Delta_{i_1 i_2}| \ge D$, we also have

$$r_{i_1} - r'_{i_1} = r_{i_1} - 1 + \frac{p}{|\Delta_{i_1 i_2}|} \ge \frac{1}{D} - r_{i_2} + \frac{p}{|\Delta_{i_1 i_2}|} \ge \frac{p+1}{|\Delta_{i_1 i_2}|} - r_{i_2} > 0.$$

On the other hand, by Proposition 9.1.3, we may consider v_{i_1} and v_{i_2} as $v_{i_1} = (1, 0, \ldots, 0)$ and $v_{i_2} = (1, \Delta_{i_1 i_2}, 0, \ldots, 0)$. Obviously, $\sum_{i=1}^{d+1} r'_i v_i \in \mathbb{Z}^{d+1}$.

The fourth step. Assume that $l \ge 3$. For each j with $2 \le j \le l$, we define each nonnegative integer p_j as follows. Let p_l be a nonnegative integer which satisfies

$$\frac{p_l}{\prod_{k=1}^{l-1} |\Delta_{i_k i_l}|} \le r_{i_l} < \frac{p_l + 1}{\prod_{k=1}^{l-1} |\Delta_{i_k i_l}|},$$

and for $2 \le j \le l-1$, let p_j be an integer which satisfies $Z_l(j) \in \mathbb{Z}$ and

$$\frac{p_j}{\prod_{1 \le k \le l, k \ne j} |\Delta_{i_k i_j}|} \le r_{i_j} < \frac{p_j + \prod_{k=j+1}^{l} |\Delta_{i_j i_k}|}{\prod_{1 \le k \le l, k \ne j} |\Delta_{i_k i_j}|},$$

where $Z_l(j)$ is as in Lemma 9.1.12. Thanks to Lemma 9.1.12, if $Z_l(j+1), \ldots, Z_l(l) \in \mathbb{Z}$, then there exists an integer p_j with $Z_l(j) \in \mathbb{Z}$ and each p_j is uniquely determined by the above inequalities. Remark that we do not know whether p_j is nonnegative except for p_l. However, in our case, we may assume that $p_2, \ldots, p_{l-1}$ are all nonnegative because of the following discussions. In fact, on the contrary, suppose that there is j' with $p_{j'} < 0$. Let $q_{j'} \in \mathbb{Z}_{\ge 0}$ be a minimal nonnegative integer satisfying

$$\frac{\prod_{k=1}^{j'-1} \Delta_{i_k i_{j'}}}{\prod_{1 \le k \le l, k \ne j'} |\Delta_{i_k i_{j'}}|} q_{j'} + \frac{1}{\Delta_{i_{j'} i_{j'+1}}} Z_l(j'+1) - \frac{1}{\Delta_{i_{j'} i_{j'+1}} \Delta_{i_{j'} i_{j'+2}}} Z_l(j'+2) + \cdots + (-1)^{l-j'+1} \frac{1}{\prod_{k=j'+1}^{l} \Delta_{i_{j'} i_k}} Z_l(l) \in \mathbb{Z}.$$

In particular, it follows from the minimality of $q_{j'}$ that $0 \le q_{j'} < \prod_{k=j'+1}^{l} |\Delta_{i_{j'} i_k}|$. By our assumption, one has $\frac{q_{j'}}{\prod_{1 \le k \le l, k \ne j'} |\Delta_{i_k i_{j'}}|} > r_{i_{j'}}$. Thus,

$$r_{i_l} \le \cdots \le r_{i_{j'}} < \frac{q_{j'}}{\prod_{1 \le k \le l, k \ne j'} |\Delta_{i_{j'} i_k}|} < \frac{\prod_{k=j'+1}^{l} |\Delta_{i_{j'} i_k}|}{\prod_{1 \le k \le l, k \ne j'} |\Delta_{i_k i_{j'}}|} = \frac{1}{\prod_{k=1}^{j'-1} |\Delta_{i_k i_{j'}}|} \le \frac{1}{D^{j'-1}},$$

so one has $\sum_{j=j'}^{l} r_{i_j} < \frac{l-j'+1}{D^{j'-1}}$. From $\sum_{j=1}^{l} r_{i_j} \ge 1 + \epsilon(l)$ and (9.6), we have

$$\sum_{j=1}^{j'-1} r_{i_j} > 1 + \epsilon(l) - \frac{l-j'+1}{D^{j'-1}} > 1 + \epsilon(j'-1)$$

when $j' \ge 3$. Hence, we may skip such case by the hypothesis of induction. When $j' = 2$, one has $r_{i_1} > 1 + \epsilon(l) - \frac{l-1}{D} = 1$, a contradiction.

By using the above p_j's, we define $r'_1, \ldots, r'_{d+1}$ by setting

$$r'_a = \begin{cases} \dfrac{p_j}{\prod_{1 \le k \le l, k \ne j} |\Delta_{i_k i_j}|}, & \text{if } a = i_j \in \{i_2, \ldots, i_l\}, \\ 1 - \displaystyle\sum_{j=2}^{l} r'_{i_j}, & \text{if } a = i_1, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $\sum_{a=1}^{d+1} r'_a = 1$. By definition of $r'_{i_2}, \ldots, r'_{i_l}$, we have $0 \le r'_{i_j} \le r_{i_j}$ for $2 \le j \le l$. Moreover, from $\sum_{j=2}^{l} r_{i_j} \le 1$, we also have $r'_{i_1} = 1 - \sum_{j=2}^{l} r'_{i_j} \ge 1 - \sum_{j=2}^{l} r_{i_j} \ge 0$. In addition, from $\sum_{j=1}^{l} r_{i_j} \ge 1 + \epsilon(l)$ and (9.6), we also have

$$\begin{aligned} r_{i_1} - r'_{i_1} &= r_{i_1} - 1 + \sum_{j=2}^{l} \frac{p_j}{\prod_{1 \le k \le l, k \ne j} |\Delta_{i_k i_j}|} \ge \epsilon(l) - \sum_{j=2}^{l} r_{i_j} + \sum_{j=2}^{l} \frac{p_j}{\prod_{1 \le k \le l, k \ne j} |\Delta_{i_k i_j}|} \\ &\ge \sum_{j=2}^{l} \left(\frac{p_j}{\prod_{1 \le k \le l, k \ne j} |\Delta_{i_k i_j}|} + \frac{1}{D^{j-1}} - r_{i_j} \right) \ge \sum_{j=2}^{l} \left(\frac{p_j + \prod_{k=j+1}^{l} |\Delta_{i_j i_k}|}{\prod_{1 \le k \le l, k \ne j} |\Delta_{i_k i_j}|} - r_{i_j} \right) \\ &> 0. \end{aligned}$$

Finally, we verify that $\sum_{i=1}^{d+1} r'_i v_i \in \mathbb{Z}^{d+1}$. Again, by Proposition 9.1.3, we may consider $v_{i_1}, \ldots, v_{i_l}$ as follows:

$$\begin{pmatrix} v_{i_1} \\ v_{i_2} \\ \vdots \\ v_{i_l} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\ 1 & \Delta_{i_1 i_2} & 0 & \ddots & \ddots & \vdots & & \vdots \\ 1 & \Delta_{i_1 i_3} & \Delta_{i_1 i_3}\Delta_{i_2 i_3} & \ddots & \ddots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & & \vdots \\ 1 & \Delta_{i_1 i_l} & \Delta_{i_1 i_l}\Delta_{i_2 i_l} & \cdots & \prod_{k=1}^{l-1} \Delta_{i_k i_l} & 0 & \cdots & 0 \end{pmatrix}.$$

Hence, it is easy to check that

$$\sum_{i=1}^{d+1} r'_i v_i = \sum_{j=1}^{l} r'_{i_j} v_{i_j} = (1, Z_l(2), Z_l(3), \ldots, Z_l(l), 0, \ldots, 0) \in \mathbb{Z}^{d+1},$$

proving the assertion.

Remark 9.1.13. Since each lattice length of an edge $\mathrm{conv}(\{v_i, v_j\})$ of $\mathcal{P}^*$ coincides with Δ_{ij}, where $i < j$, it follows immediately from [21, Theorem 1.3 (b)] that $\mathcal{P}$ is normal if $\Delta_{i,i+1} \ge d(d+1)$ for $1 \le i \le n-1$. (We are grateful to Gábor Hegedüs for informing us the result [21, Theorem 1.3 (b)].) Thus, our constraint $\Delta_{i,i+1} \ge d^2 - 1$ on integral cyclic polytopes is better than a general case, but this bound is still very rough. For example, $C_3(0,1,2,3)$ is normal, while we have $\Delta_{12} = \Delta_{23} = \Delta_{34} = 1 < 8$. Similarly, $C_4(0,1,3,5,6)$ is also normal, although one has $\Delta_{12} = \Delta_{45} = 1$ and $\Delta_{23} = \Delta_{34} = 2$.

9.1.3 Non-very ample cyclic polytopes

Our goal of this subsection is to prove

Theorem 9.1.14 ([30, Theorem 3.1]). *Let d and n be positive integers satisfying $n \geq d+1$ and $d \geq 4$. If $\Delta_{12} = 1$ or $\Delta_{n-2,n-1} = 1$, then $C_d(\tau_1, \ldots, \tau_n)$ is not very ample.*

We obtain Theorem 9.1.14 as a conclusion of Proposition 9.1.15 and Corollary 9.1.16 below.

Proposition 9.1.15. *Let $\mathcal{P} = C_4(\tau_1, \ldots, \tau_n)$. If $\Delta_{23} = 1$ or $\Delta_{n-2,n-1} = 1$, then $\mathcal{P}$ is not very ample.*

Proof. Thanks to Lemma 9.1.4, by symmetry, we assume $\Delta_{23} = 1$. Consider the set

$$\mathcal{A}_{\mathcal{P},3} := \left\{ x - v_3 : x \in \mathcal{P}^* \cap \mathbb{Z}^5 \right\}.$$

We will prove that the monoid $\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P},3}$ is not normal, thus there exists a vector $p \in \mathbb{Z}\mathcal{A}_{\mathcal{P},3} \cap \mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P},3} = \mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P},3} \cap \mathbb{Z}^5$ such that $p \notin \mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P},3}$. Then, for every integer $k \geq 1$, it holds that $kv_3 + p \in (\mathbb{Z}\mathcal{A}_{\mathcal{P}} \cap \mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P}}) \setminus \mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}$, see [11, Excercise 2.23]. Hence, $\mathcal{P}$ is not very ample.

In the sequel, we denote the facet of $\mathcal{P}^*$ spanned by the vertices v_i, v_j, v_k and v_l with $\mathcal{F}_{ijkl}$. Moreover, we denote the corresponding linear form with σ_{ijkl}. Note that every facet of $\mathcal{P}^*$ containing v_3 defines also a facet of $\mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P},3}$.

The following vector has the required properties:

$$\begin{aligned}
p &:= b_{23} + b_{134} + b_{12345} \\
&= \frac{\Delta_{12}\Delta_{15} + 1}{\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{15}} v_1 + \frac{1}{\Delta_{23}} \left(1 - \frac{1}{\Delta_{12}\Delta_{24}\Delta_{25}} \right) v_2 - \frac{1}{\Delta_{23}} \left(1 + \frac{\Delta_{23}\Delta_{35} - 1}{\Delta_{13}\Delta_{34}\Delta_{35}} \right) v_3 \\
&+ \frac{\Delta_{24}\Delta_{45} - 1}{\Delta_{14}\Delta_{24}\Delta_{34}\Delta_{45}} v_4 + \frac{1}{\Delta_{15}\Delta_{25}\Delta_{35}\Delta_{45}} v_5.
\end{aligned}$$

First, one has $p \in \mathbb{Z}^5$ from Proposition 9.1.6 (i). Then, by the second representation of p, it is a positive linear combination of the vectors $v_1 - v_3, v_2 - v_3, v_4 - v_3$ and $v_5 - v_3$. Thus, $p \in \mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P},3}$. Moreover, since we assume $\Delta_{23} = 1$, the coefficient of v_3 is less than -1. Hence, p lies beyond the facet $\mathcal{F}_{1245}$ which is a facet of $\mathcal{P}^*$ by Gale's evenness condition (Proposition 9.1.1). Thus, we have $p \notin \mathcal{A}_{\mathcal{P},3}$.

It remains to show that p cannot be written as a sum $\sum w_j$ with $w_j \in \mathcal{A}_{\mathcal{P},3}$. Suppose that we have such a representation. Then we remark that p has at least two summands. Consider a facet $\mathcal{F}_{1234}$. Then $\sigma_{1234}(p) = \frac{1}{\Delta_{15}\Delta_{25}\Delta_{35}\Delta_{45}} \sigma_{1234}(v_5) = 1$. Since $\sigma_{1234}(w_j) \geq 0$, $\sigma_{1234}(w_j) = 0$ for every summand w_j except one. Choose one $w_j \neq 0$ with $\sigma_{1234}(w_j) = 0$ and denote it by w. Further, we set $w' := p - w \in \mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P},3}$ the remaining sum. By Carathéodory's Theorem, there exist vertices $v_{i_1}, \ldots, v_{i_4}$ of $\mathcal{P}^*$ and nonnegative numbers $\lambda_j \geq 0$, such that $w' = \sum_{j=1}^{4} \lambda_j (v_{i_j} - v_3)$. Let i_4 be the greatest of those indices. Since $\sigma_{1234}(w') = 1$ and $\sigma_{1234}(v_{i_4}) = \Delta_{1i_4}\Delta_{2i_4}\Delta_{3i_4}\Delta_{4i_4}$, we conclude that

$$\lambda_4 \leq \frac{1}{\Delta_{1i_4}\Delta_{2i_4}\Delta_{3i_4}\Delta_{4i_4}}.$$

But the vertices $v_{i_1}, \ldots, v_{i_4}$ and v_3 define an integral cyclic polytope, thus the denominator of the coefficient of v_{i_4} has to be a divisor of $\Delta_{i_1 i_4}\Delta_{i_2 i_4}\Delta_{i_3 i_4}\Delta_{3i_4}$. This is only possible if $\{i_1, i_2, i_3\} = \{1, 2, 4\}$. Thus, w' lies in the cone generated by $v_1 - v_3, v_2 - v_3, v_4 - v_3$ and $v_{i_4} - v_3$. Note that $\sigma_{1234}(w) = 0$ implies that w lies in the cone generated by $v_1 - v_3, v_2 - v_3$ and $v_4 - v_3$. Thus we can replace the polytope $\mathcal{P}^*$ by the polytope $\mathcal{Q}^*$ whose vertices are $v_1, \ldots, v_5$ and v_{i_4}. The reason for doing this is that we know the facets of $\mathcal{Q}^*$. Here, $i_4 = 5$ is possible.

We consider the representation

$$w = a_1 b_3 + a_2 b_{23} + a_3 b_{123} + a_4 b_{1234}$$

with integer coefficients a_1, a_2, a_3, a_4. This is possible from Proposition 9.1.6 (iii). Since w is in the cone generated by $v_1 - v_3, v_2 - v_3$ and $v_4 - v_3$, we have $a_1 = 0$. Now consider a facet $\mathcal{F}_{123i_4}$ of $\mathcal{Q}^*$. We compute

$$\sigma_{123i_4}(p) = \frac{1}{\Delta_{45}}(\Delta_{24}\Delta_{45} - 1)\Delta_{4i_4} + \frac{1}{\Delta_{45}}\Delta_{5i_4} = \Delta_{24}\Delta_{4i_4} - 1.$$

Moreover, $\sigma_{123i_4}(w) = -a_4\Delta_{4i_4}$. From $0 \leq \sigma_{123i_4}(w) \leq \sigma_{123i_4}(p)$, we conclude $0 \leq -a_4 \leq \Delta_{24} - 1$. Here we used that a_4 is an integer. Next, consider a facet $\mathcal{F}_{2345}$. We compute $\sigma_{2345}(w) = a_3\Delta_{14}\Delta_{15} + a_4\Delta_{15}$ and $\sigma_{2345}(p) = \Delta_{12}\Delta_{15} + 1$. As before, we conclude that $0 \leq a_3\Delta_{14} + a_4 \leq \Delta_{12}$. However, these two constraints can only be satisfied by $a_3 = a_4 = 0$, because $\Delta_{14} = \Delta_{12} + \Delta_{24}$ and $\Delta_{15} > 1$. Finally, we consider a facet $\mathcal{F}_{134i_4}$. By computing $\sigma_{134i_4}(w) = a_2\Delta_{12}\Delta_{24}\Delta_{2i_4}$ and $\sigma_{134i_4}(p) = \Delta_{12}\Delta_{24}\Delta_{2i_4} - 1$, we conclude that $a_2 = 0$. But this means $w = 0$, a contradiction to $w \neq 0$. □

By using this proposition, we also obtain

Corollary 9.1.16. *Let $\mathcal{P} = C_d(\tau_1, \ldots, \tau_n)$, where $d \geq 5$. If there is some i with $2 \leq i \leq n-2$ such that $\Delta_{i,i+1} = 1$, then $\mathcal{P}$ is not very ample.*

Proof. We prove this by induction on d.

When $d = 5$, let $\mathcal{F}_i = \mathrm{conv}(\{v_1, v_i, v_{i+1}, v_{i+2}, v_{i+3}\})$ for $2 \leq i \leq n-3$ and $\mathcal{F}_{n-2} = \mathrm{conv}(\{v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}, v_n\})$. By Gale's evenness condition, each $\mathcal{F}_i$ is a facet of $\mathcal{P}^*$. When $\Delta_{i,i+1} = 1$ for some i with $2 \leq i \leq n-2$, it then follows from Proposition 9.1.15 that $\mathcal{F}_i$ is not very ample. Thus, $\mathcal{P}$ itself is non-very ample, either. (See [54, Lemma 1].)

Now, let $d \geq 6$. For $2 \leq i \leq n-d+2$, we set

$$\mathcal{F}_i = \begin{cases} \mathrm{conv}(\{v_1, v_i, \ldots, v_{i+d-2}\}), & \text{when } d \text{ is odd,} \\ \mathrm{conv}(\{v_{i-1}, v_i, \ldots, v_{i+d-2}\}), & \text{when } d \text{ is even.} \end{cases}$$

Again, Gale's evenness condition guarantees that each $\mathcal{F}_i$ is a facet of $\mathcal{P}^*$. When $\Delta_{i,i+1} = 1$ for some i with $2 \le i \le n-2$, since each facet is also an integral cyclic polytope of dimension $d-1$, either $\mathcal{F}_i$ or $\mathcal{F}_{d-n+2}$ is not very ample by the hypothesis of induction. Therefore, $\mathcal{P}$ is non-very ample. $\square$

On the case where $d = 2$, it is well known that there exists a unimodular triangulation for every integral convex polytope of dimension 2. Therefore, integral convex polytopes of dimension 2 are always normal.

On the case where $d = 3$, exhaustive computational experiences lead us to give the following

Conjecture 9.1.17. *All cyclic polytopes of dimension* 3 *are normal.*

Moreover, by computational experiences together with Proposition 9.1.15, we also conjecture a complete characterization of normal cyclic polytopes of dimension 4.

Conjecture 9.1.18. *A cyclic polytope of dimension* 4 *is normal if and only if we have*

$$\Delta_{23} \ge 2 \quad \textit{and} \quad \Delta_{n-2,n-1} \ge 2.$$

By considering the foregoing two conjectures and Theorem 9.1.9, the following statement seems to be natural for us.

Conjecture 9.1.19. *If* $\mathcal{P} = C_d(\tau_1, \ldots, \tau_n)$ *is normal and* $\mathcal{P}' = C_d(\tau'_1, \ldots, \tau'_n)$ *satisfies* $\tau'_j - \tau'_i \ge \Delta_{ij}$ *for all* $1 \le i < j \le n$*, then* $\mathcal{P}'$ *is also normal.*

Finally, we also state

Conjecture 9.1.20. *If an integral cyclic polytope is very ample, then it is also normal.*

Actually, it often happens that a very ample integral convex polytope is also normal, that is to say, the normality of an integral convex polytope is equivalent to what it is very ample. Hence, the above conjecture occurs in the natural way. On the other hand, it is also known that there exists an integral convex polytope which is not normal but very ample. See [11, Exercise 2.24].

9.2 Cohen–Macaulayness and Gorensteinness of toric rings arising from cyclic polytopes

In the previous section, we discussed the normality of (toric rings of) cyclic polytopes $\mathcal{P}$ and gave a sufficient condition (Theorem 9.1.9) and a necessary one (Theorem 9.1.14) for $\mathcal{P}$ to be normal. This section is devoted to the continuation of the study of $\mathcal{P}$.

Let K be a field and $\mathcal{P} \subset \mathbb{R}^d$ an integral cyclic polytope of dimension d. Let $\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}$ be as above. Then $\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}$ is an affine semigroup contained in $\mathbb{Z}^{d+1}$, which is generated by the set of integer points in $\mathcal{P}^*$. For simplicity, set $Q := Q_d(\tau_1, \cdots, \tau_n)$. Following usual convention, let $K[\mathcal{P}]$ denote the affine semigroup K-algebra of $\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}$. The K-algebra $K[\mathcal{P}]$ is just the K-subalgebra of the polynomial ring $K[t_0, t_1, \ldots, t_d]$ such that

$$K[\mathcal{P}] = \bigoplus_{\mathbf{a} \in \mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}} K \cdot t^{\mathbf{a}},$$

where we set $t^{\mathbf{a}} = t_0^{a_0} t_1^{a_1} \cdots t_d^{a_d}$ for $\mathbf{a} = (a_0, a_1, \ldots, a_d) \in \mathbb{Z}_{\geq 0}^{d+1}$. Note that $K[\mathcal{P}]$ is nothing other than the *toric ring* of $\mathcal{P}$, and as is well known, $\mathcal{P}$ is normal if and only if so is $K[\mathcal{P}]$.

In this section, we will consider the Cohen-Macaulayness and Gorensteinness of $K[\mathcal{P}]$ (Theorem 9.2.3 and Theorem 9.2.5, respectively). We prove that $K[\mathcal{P}]$ always satisfies Serre's condition (R_1), which implies that $K[\mathcal{P}]$ is Cohen-Macaulay if and only if it is normal. This means that the characterization of the normality of integral cyclic polytopes is also that of its Cohen-Macaulayness. Moreover, it will turn out that $K[\mathcal{P}]$ is Gorenstein if and only if one has $d = 2$, $n = 3$ and $(\tau_2 - \tau_1, \tau_3 - \tau_2) = (2, 1)$ or $(1, 2)$, which says that there is essentially only one Gorenstein integral cyclic polytope, see Lemma 9.1.4.

9.2.1 Serre's (R_1) property

In this subsection, we prove that $K[\mathcal{P}]$ always satisfies Serre's Condition (R_1). Moreover, this fact enables us to claim that the Cohen–Macaulayness of $K[\mathcal{P}]$ is equivalent to its normality.

Recall that a Noetherian ring R is said to satisfy (S_n) if

$$\operatorname{depth} R_{\mathfrak{p}} \geq \min\{n, \dim R_{\mathfrak{p}}\}$$

for all $\mathfrak{p} \in \operatorname{Spec}(R)$, and satisfy (R_n) if $R_{\mathfrak{p}}$ is a regular local ring for all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R_{\mathfrak{p}} \leq n$. The conditions (S_n) and (R_n) are called Serre's conditions.

The well-known criterion for normality of a Noetherian ring, Serre's Criterion (cf. [12, Theorem 2.2.22]), says that a Noetherian ring is normal if and only if it satisfies (R_1) and (S_2).

We use the following combinatorial criterion of (R_1), which can be found in [11, Exercises 4.15 and 4.16].

Proposition 9.2.1 ([11]). *Let M be an affine monoid, K a field and $K[M]$ its semigroup K-algebra. Then $K[M]$ satisfies (R_1) if and only if every facet $\mathcal{F}$ of M satisfies the following two conditions:*

1. *$\mathbb{Z}(M \cap \mathcal{F}) = \mathbb{Z}M \cap \mathcal{H}$, where $\mathcal{H}$ is the supporting hyperplane of $\mathcal{F}$;*

2. *there exists $x \in M$ such that $\sigma_{\mathcal{F}}(x) = 1$, where $\sigma_{\mathcal{F}}$ is a support form of $\mathcal{F}$ with integer coefficients.*

Using this, we can prove

Proposition 9.2.2. *Let $\mathcal{P}$ be an integral cyclic polytope. Then $K[\mathcal{P}]$ always satisfies the condition (R_1).*

Proof. First, note that the facets of $\mathcal{P}^*$ are in bijection with the facets of the monoid $\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}$. Let $\mathcal{F}$ be a facet of $\mathcal{P}^*$ with vertices $v_{i_1}, \ldots, v_{i_d}$, where $i_1 < \ldots < i_d$. Using the same construction as in the proof of Lemma 9.1.7, we get a family $c_j := \sum_{l=j}^{d} b_{i_l \ldots i_d}$ of integer points in $\mathcal{F}$ that is part of a basis of $\mathbb{Z}^{d+1}$. This implies that every element $x \in \mathbb{Z}^{d+1} \cap \mathcal{H}$ can be written as a $\mathbb{Z}$-linear combination of them. Therefore, the first condition of Proposition 9.2.1 follows.

For the second condition, pick any vertex v_k of $\mathcal{P}^*$ that is not in $\mathcal{F}$. Consider the set $S := \{k, i_1, \ldots, i_d\} \subset [n]$ with its natural ordering. If the position of k in S is even (i.e., if there is an odd number of j such that $i_j < k$), then let $F \subset S$ be the set of elements of odd position. Otherwise (i.e., if the position of k in S is odd), let F be the set of elements of even position. In any case, $k \notin F$. We write $F = \{j_1, \ldots, j_r\}$. We want to do a similar construction to the one above, but this time we need to analyse it more closely. Consider the vector

$$x' := \sum_{l=1}^{r} b_{j_l \ldots j_r} \,.$$

By the reasoning above, we know that this is an integer point in $\mathcal{F}$, but we claim that it has the additional property that the coefficient of each v_{j_s} is strictly positive. Indeed, if s is an odd number, then the coefficient of v_{j_s} is an alternating sum of non-increasing values, starting and ending with a positive value. Thus it is positive and we only need to consider the case that s is even. For this, we compute the coefficient of v_{j_s} in x':

$$\sum_{l=1}^{s} \frac{1}{\prod_{\substack{m=l \\ m \neq s}}^{r} \Delta_{j_s j_m}} = \sum_{l=1}^{s} \frac{(-1)^{l+1}}{|\prod_{\substack{m=l \\ m \neq s}}^{r} \Delta_{j_s j_m}|} = \sum_{\substack{l=1 \\ l \text{ even}}}^{s} \frac{1}{|\prod_{\substack{m=l \\ m \neq s}}^{r} \Delta_{j_s j_m}|} \left(1 - \frac{1}{|\Delta_{j_s j_{l-1}}|}\right).$$

By our choice of F, for every two indices in $s_1 < s_2$ in F, there is an index in $s_3 \in S$ between them $s_1 < s_3 < s_2$. Thus every $\Delta_{j_q j_{q'}}$ in above formula is at least 2. Hence the coefficient of v_{j_s} cannot be zero. Now we define

$$x := x' \pm b_S,$$

where the sign is "+" if the position of k in S is odd and "−" if it is even. This ensures that $\sigma_{\mathcal{F}}(x) = 1$. It remains to show that x is contained in $\mathcal{P}^*$, that is that the coefficients of all $v_i, i \in S$ are nonnegative. Now for $i \in S \setminus F$, the coefficient of v_i is positive by construction. For $i \in F$, the coefficient in x' is positive and thus at least $|\prod_{j \in F \setminus \{i\}} \Delta_{ij}|^{-1}$. But the coefficient in b_S is $-|\prod_{j \in S \setminus \{i\}} \Delta_{ij}|^{-1}$, so their sum (i.e., the coefficient in x) is nonegative, because $F \subset S$. $\square$

As a consequence of this proposition, we obtain

Theorem 9.2.3 ([31, Theorem 2.3]). *Let $\mathcal{P}$ be an integral cyclic polytope and $K[\mathcal{P}]$ its associated semigroup K-algebra. Then the following conditions are equivalent:*

(i) $K[\mathcal{P}]$ *is normal;*

(ii) $K[\mathcal{P}]$ *is Cohen–Macaulay;*

(iii) $K[\mathcal{P}]$ *satisfies* (S_2).

Proof. By Hochster's Theorem (see, e.g., [11, Theorem 6.10]), normality implies Cohen–Macaulayness. Moreover, Serre's Criterion states that normality is equivalent to Serre's conditions (R_1) and (S_2). On the other hand, Cohen–Macaulayness implies (S_2), see [12, p. 62], and thus the claim follows. □

Remark 9.2.4. Using the same methods as employed above, one can also prove that an integral cyclic polytope is normal if and only if it is seminormal. See [11, p. 66] for the definition and basic properties of seminormality. We use the notation from that book. Now, assume that $\mathcal{P}$ is not normal. Then there exists a point m in $\mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}$ which is not contained in $\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}$. This point m lies in the interior of a unique face $\mathcal{F}$ of $\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}$. But using the same construction as above, we can show that $\mathbb{Z}(\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathcal{F}) = \mathbb{Z}^{d+1} \cap \mathcal{H}$, where $\mathcal{H}$ is the linear subspace spanned by $\mathcal{F}$. Thus $m \in \mathbb{Z}(\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathcal{F})$ is an exceptional point, and therefore $(\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathcal{F})_*$ is not normal. Hence, $\mathcal{P}$ is not seminormal.

9.2.2 When is $K[\mathcal{P}]$ Gorenstein ?

The goal of this subsection is to characterize completely when $K[\mathcal{P}]$ is Gorenstein, that is, this section is devoted to proving

Theorem 9.2.5 ([31, Theorem 3.1]). *Let $\mathcal{P} = C_d(\tau_1, \dots, \tau_d)$ be an integral cyclic polytope and $K[\mathcal{P}]$ its associated semigroup K-algebra. Then $K[\mathcal{P}]$ is Gorenstein if and only if $d = 2$, $n = 3$ and*

$$(\Delta_{12}, \Delta_{23}) = (1, 2) \text{ or } (2, 1).$$

Thus, by Proposition 9.1.4, there is essentially only one case where $K[\mathcal{P}]$ is Gorenstein.

Before giving a proof, we prepare the following:

- Let

$$(v_1, \dots, v_{d+1}) = \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 0 & \Delta_{12} & \Delta_{13} & \cdots & \Delta_{1,d+1} \\ \vdots & \ddots & \Delta_{13}\Delta_{23} & \cdots & \Delta_{1,d+1}\Delta_{2,d+1} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \prod_{k=1}^{d} \Delta_{k,d+1} \end{pmatrix} \in \mathbb{Z}^{(d+1)\times(d+1)}$$

and $\mathcal{P}^* = \mathrm{conv}(\{v_1, \ldots, v_{d+1}\})$.

- Let

$$\mathbf{a}_1 = \left(0, \prod_{j=3}^{d+1} \Delta_{1,j}, -\prod_{j=4}^{d+1} \Delta_{1,j}, \ldots, (-1)^d \Delta_{1,d+1}, (-1)^{d+1}\right) \in \mathbb{Z}^{d+1},$$

$$\mathbf{a}_i = \left(\underbrace{0, \ldots, 0}_{i-1}, \prod_{j=i+1}^{d+1} \Delta_{i,j}, -\prod_{j=i+2}^{d+1} \Delta_{i,j}, \ldots, (-1)^{d+i-2} \Delta_{i,d+1}, (-1)^{d+i-1}\right) \in \mathbb{Z}^{d+1}$$

for $i = 2, \ldots, d+1$.

- Let $\mathcal{H}_i$ be the closed half space in $\mathbb{R}^{d+1}$ defined by the inequality

$$\langle \mathbf{a}_1, \mathbf{x} \rangle \leq \prod_{j=2}^{d+1} \Delta_{1,j}, \qquad \text{for } i = 1,$$
$$\langle \mathbf{a}_i, \mathbf{x} \rangle \geq 0, \qquad \text{for } i = 2, \ldots, d+1,$$

where $\mathbf{x} = (x_0, x_1, \ldots, x_d) \in \mathbb{R}^{d+1}$ and $\langle \mathbf{a}_i, \mathbf{x} \rangle$ stands for the usual inner product in $\mathbb{R}^{d+1}$.

- By using the above, we have

$$\mathcal{P}^* = \bigcap_{i=1}^{d+1} \mathcal{H}_i \cap \{\mathbf{x} \in \mathbb{R}^{d+1} : x_0 = 1\}. \tag{9.7}$$

A proof of (9.7) is given by elemtary computations. This establishes an explicit description of the supporting hyperplanes of an integral cyclic polytope with $n = d+1$, i.e., a simplex case.

Proof of Theorem 9.2.5. First, we can check easily that $K[\mathcal{P}]$ is Gorenstein when $\mathcal{P} = C_2(\tau_1, \tau_2, \tau_3)$ with $(\Delta_{12}, \Delta_{23}) = (1,2)$ or $(\Delta_{12}, \Delta_{23}) = (2,1)$.

Thus, what we must do is to show that $K[\mathcal{P}]$ is never Gorenstein in other cases. Mostly, we concentrate on the case where $\mathcal{P}$ is a simplex.

The first step. Suppose that $K[\mathcal{P}]$ is not normal. Then, from Theorem 9.2.3, $K[\mathcal{P}]$ is not Cohen–Macaulay. In particular, $K[\mathcal{P}]$ cannot be Gorenstein.

Hence, in the remaining parts, we assume that $K[\mathcal{P}]$ is normal. Since $\mathbb{Z}\mathcal{A}_{\mathcal{P}} = \mathbb{Z}^{d+1}$ by Lemma 9.1.7, we notice that $K[\mathcal{P}]$ is nothing but the Ehrhart ring of $\mathcal{P}$ when $K[\mathcal{P}]$ is normal (cf. [12, pp. 275–278]). In addition, it is neccesary for the Ehrhart ring $K[\mathcal{P}]$ to be Gorenstein that $\mathcal{P}$ contains only one integer point in its relative interior when $\mathcal{P} \setminus \partial\mathcal{P} \neq \emptyset$. (See, e.g., [15].) In the following, we verify that there is no such $(\tau_1, \ldots, \tau_n)$.

The second step. Assume that $d = 2$ and let us consider when $n = 3$. Suppose that $(\Delta_{12}, \Delta_{23})$ is neither $(2,1)$ nor $(1,2)$. From Proposition 9.1.4, we may assume that $\Delta_{12} \geq \Delta_{23}$. When $(\Delta_{12}, \Delta_{23}) = (1, 1)$, we can check that $\mathcal{P}$ is not Gorenstein. Hence, we assume that either $\Delta_{12} \geq \Delta_{23} \geq 2$ or $\Delta_{12} \geq 3$ and $\Delta_{23} = 1$ is satisfied. Recall from the above statements that

$$\mathcal{H}_1 : \Delta_{13}x_1 - x_2 \leq \Delta_{12}\Delta_{13}, \quad \mathcal{H}_2 : \Delta_{23}x_1 - x_2 \geq 0, \quad \mathcal{H}_3 : x_2 \geq 0.$$

Then it is enough to that there exist at least two integer points $p_1, p_2 \in \mathbb{Z}^2$ such that

$$\langle(\Delta_{13}, -1), p_i\rangle < \Delta_{12}\Delta_{13}, \ \langle(\Delta_{23}, -1), p_i\rangle > 0 \text{ and } \langle(0, 1), p_i\rangle < 0 \quad \text{for} \ \ i = 1, 2.$$

- When $\Delta_{12} \geq \Delta_{23} \geq 2$, the integer points $(1, 1, 1)$ and $(1, 2, 2)$ are contained in $\mathcal{P}^* \setminus \partial\mathcal{P}^*$. In fact,

$$\begin{aligned} &\Delta_{13} - 1 < \Delta_{13} < \Delta_{12}\Delta_{13}, \quad \Delta_{23} - 1 > 0, \quad 1 > 0, \\ &\Delta_{13} - 2 < \Delta_{13} < \Delta_{12}\Delta_{13}, \quad 2\Delta_{23} - 2 > 0, \quad 2 > 0. \end{aligned}$$

- When $\Delta_{12} \geq 3$ and $\Delta_{23} = 1$, the integer points $(1, 2, 1)$ and $(1, 3, 1)$ are contained in the interior. In fact,

$$\begin{aligned} &2\Delta_{13} - 1 < 2\Delta_{13} < \Delta_{12}\Delta_{13}, \quad 2\Delta_{23} - 1 > 0, \quad 1 > 0, \\ &3\Delta_{13} - 1 < 3\Delta_{13} \leq \Delta_{12}\Delta_{13}, \quad 3\Delta_{23} - 1 > 0, \quad 1 > 0. \end{aligned}$$

Thus, $\mathcal{P}$ is not Gorenstein when $n = 3$ except the case where $(\Delta_{12}, \Delta_{23}) = (2, 1)$ or $(1, 2)$.

When $n = 4$ and $(\Delta_{12}, \Delta_{23}, \Delta_{34}) = (1, 1, 1)$, then we can also check that $\mathcal{P}$ is not Gorenstein. Moreover, when $n = 4$ and there is at least one $1 \leq i \leq 3$ with $\Delta_{i,i+1} \geq 2$, since either $\tau_3 - \tau_1 \geq 2$ and $\tau_4 - \tau_3 \geq 2$ or $\tau_3 - \tau_1 \geq 3$ and $\tau_4 - \tau_3 = 1$ are satisfied, $\mathcal{P}' = C_2(\tau_1, \tau_3, \tau_4)$ has at least two integer points in $\mathcal{P}' \setminus \partial\mathcal{P}' \subset \mathcal{P} \setminus \partial\mathcal{P}$ as discussed above, which implies that $\mathcal{P}$ is not Gorenstein. Similarly, when $n \geq 5$, since $\tau_4 - \tau_1 \geq 3$ and $\tau_5 - \tau_4 \geq 1$, $\mathcal{P}$ is not Gorenstein.

The third step. Assume that $d = 3$ and let us consider the case where $n = 4$. When $(\Delta_{12}, \Delta_{23}, \Delta_{34}) = (1, 1, 1)$, we can check $\mathcal{P}$ is not Gorenstein. Thus, we assume that there is at least one $1 \leq i \leq 3$ with $\Delta_{i,i+1} \geq 2$. Recall that

$$\begin{aligned} \mathcal{H}_1 : \Delta_{13}\Delta_{14}x_1 - \Delta_{14}x_2 + x_3 \leq \Delta_{12}\Delta_{13}\Delta_{14}, \quad &\mathcal{H}_2 : \Delta_{23}\Delta_{24}x_1 - \Delta_{24}x_2 + x_3 \geq 0, \\ \mathcal{H}_3 : \Delta_{34}x_2 - x_3 \geq 0, \quad &\mathcal{H}_4 : x_3 \geq 0. \end{aligned}$$

- When $\Delta_{23} \geq 2$, the integer points $(1, \Delta_{12} + 1, \Delta_{13} + 1, 1)$ and $(1, \Delta_{12} + 1, \Delta_{13} + 1, 2)$ are contained in $\mathcal{P}^* \setminus \partial\mathcal{P}^*$. In fact,

$$\begin{aligned} &\Delta_{13}\Delta_{14}(\Delta_{12} + 1) - \Delta_{14}(\Delta_{13} + 1) + q = \Delta_{12}\Delta_{13}\Delta_{14} - \Delta_{14} + q < \Delta_{12}\Delta_{13}\Delta_{14}, \\ &\Delta_{23}\Delta_{24}(\Delta_{12} + 1) - \Delta_{24}(\Delta_{13} + 1) + q \geq \Delta_{12}\Delta_{24} - \Delta_{24} + q > 0, \\ &\Delta_{34}(\Delta_{13} + 1) - q > 0, \quad q > 0, \end{aligned}$$

  where q is 1 or 2.

- When $\Delta_{23} = 1$ and $\Delta_{12} \geq 2$ and $\Delta_{34} \geq 2$, the integer points $(1,2,2,1)$ and $(1,2,2,2)$ are contained in the interior. In fact,

$$\begin{aligned}&2\Delta_{13}\Delta_{14} - 2\Delta_{14} + q = 2\Delta_{12}\Delta_{14} + q < \Delta_{12}\Delta_{13}\Delta_{14},\\ &2\Delta_{23}\Delta_{24} - 2\Delta_{24} + q = q > 0, \quad 2\Delta_{34} - q > 0, \quad q > 0,\end{aligned}$$

where q is 1 or 2.

- When $\Delta_{12} \geq 2$ and $\Delta_{23} = \Delta_{34} = 1$, the integer points $(\Delta_{12}, \Delta_{12}, 1)$ and $(\Delta_{12}+1, \Delta_{12}+2, 3)$ are contained in the interior. In fact,

$$\begin{aligned}&\Delta_{12}\Delta_{13}\Delta_{14} - \Delta_{12}\Delta_{14} + 1 < \Delta_{12}\Delta_{13}\Delta_{14},\\ &\Delta_{12}\Delta_{23}\Delta_{24} - \Delta_{12}\Delta_{24} + 1 = 1 > 0, \quad \Delta_{12}\Delta_{34} - 1 = \Delta_{12} - 1 > 0, \quad 1 > 0\end{aligned}$$

and

$$\begin{aligned}&\Delta_{13}\Delta_{14}(\Delta_{12}+1) - \Delta_{14}(\Delta_{12}+2) + 3 = \Delta_{12}\Delta_{13}\Delta_{14} - \Delta_{14} + 3 < \Delta_{12}\Delta_{13}\Delta_{14},\\ &\Delta_{23}\Delta_{24}(\Delta_{12}+1) - \Delta_{24}(\Delta_{12}+2) + 3 = -2\Delta_{24} + 3 > 0,\\ &\Delta_{34}(\Delta_{12}+2) - 3 = \Delta_{12} - 1 > 0, \quad 3 > 0.\end{aligned}$$

Thus, $\mathcal{P}$ is not Gorenstein when $n = 4$. Remark that we need not consider the case where $\Delta_{34} \geq 2$ and $\Delta_{12} = \Delta_{23} = 1$ because of Proposition 9.1.4 again.

On the other hand, when $n \geq 5$, let $\mathcal{P}' = C_3(\tau_1, \tau_3, \tau_4, \tau_5)$. Since $\tau_3 - \tau_1 \geq 2$, there exist at least two integer points in $\mathcal{P}' \setminus \partial\mathcal{P}' \subset \mathcal{P} \setminus \partial\mathcal{P}$, which means that $\mathcal{P}$ is not Gorestein.

The fourth step. Assume that $d \geq 4$ and d is even. Let us consider

$$\alpha_q = (1, \Delta_{12}+1, \Delta_{13}+1, \ldots, \Delta_{1,d-1}+1, \Delta_{1,d}, q) \in \mathbb{Z}^{d+1}$$

for $q = 1$ and 2. We show that α_1 and α_2 are contained in $\mathcal{P}^* \setminus \partial\mathcal{P}^*$.

Now, we have

$$\begin{aligned}\langle \mathbf{a}_1, \alpha_q \rangle &= \prod_{j=2}^{d+1}\Delta_{1,j} + \prod_{j=3}^{d+1}\Delta_{1,j} - \left(\prod_{j=3}^{d+1}\Delta_{1,j} + \prod_{j=4}^{d+1}\Delta_{1,j}\right) + \cdots\\ &\quad + (-1)^{d-1}\left(\prod_{j=d-1}^{d+1}\Delta_{1,j} + \prod_{j=d}^{d+1}\Delta_{1,j}\right) + (-1)^d\prod_{j=d}^{d+1}\Delta_{1,j} + (-1)^{d+1}q\\ &= \prod_{j=2}^{d+1}\Delta_{1,j} + (-1)^{d+1}q = \prod_{j=2}^{d+1}\Delta_{1,j} - q < \prod_{j=2}^{d+1}\Delta_{1,j},\end{aligned}$$

$$
\begin{aligned}
\langle \mathbf{a}_i, \alpha_q \rangle = \Delta_{1,i} \prod_{j=i+1}^{d+1} \Delta_{i,j} + \prod_{j=i+1}^{d+1} \Delta_{i,j} - \left(\Delta_{1,i+1} \prod_{j=i+2}^{d+1} \Delta_{i,j} + \prod_{j=i+2}^{d+1} \Delta_{i,j} \right) + \cdots + \\
(-1)^{d+i-3} \left(\Delta_{1,d-1} \prod_{j=d}^{d+1} \Delta_{i,j} + \prod_{j=d}^{d+1} \Delta_{i,j} \right) + (-1)^{d+i-2} \Delta_{1,d} \Delta_{i,d+1} + (-1)^{d+i-1} q \\
= \Delta_{1,i} \prod_{j=i+1}^{d+1} \Delta_{i,j} + \sum_{k=i}^{d-1} (-1)^{i+k-2} \left(\prod_{j=k+1}^{d+1} \Delta_{i,j} - \Delta_{1,k+1} \prod_{j=k+2}^{d+1} \Delta_{i,j} \right) + (-1)^{d+i-1} \\
= \Delta_{1,i} \prod_{j=i+1}^{d+1} \Delta_{i,j} + \sum_{k=i}^{d-1} (-1)^{i+k-1} \left(\Delta_{1,i} \prod_{j=k+2}^{d+1} \Delta_{i,j} \right) + (-1)^{d+i-1} \\
= \Delta_{1,i} \left(\prod_{j=i+1}^{d+1} \Delta_{i,j} - \prod_{j=i+2}^{d+1} \Delta_{i,j} \right) + \Delta_{1,i} \left(\prod_{j=i+3}^{d+1} \Delta_{i,j} - \prod_{j=i+4}^{d+1} \Delta_{i,j} \right) + \cdots + \\
\Delta_{1,i} \left(\prod_{j=d-1}^{d+1} \Delta_{i,j} - \prod_{j=d}^{d+1} \Delta_{i,j} \right) + \Delta_{1,i} \Delta_{i,d+1} - q > 0
\end{aligned}
$$

when i is even and

$$
\langle \mathbf{a}_i, \alpha_q \rangle = \Delta_{1,i} \left(\prod_{j=i+1}^{d+1} \Delta_{i,j} - \prod_{j=i+2}^{d+1} \Delta_{i,j} \right) + \cdots + \Delta_{1,i} \left(\prod_{j=d}^{d+1} \Delta_{i,j} - \prod_{j=d+1}^{d+1} \Delta_{i,j} \right) + q > 0
$$

when i is odd.

The fifth step. Assume that $d \geq 5$ and d is odd. Let us consider

$$
\beta_q = (1, \Delta_{12} + 1, \Delta_{13} + 1, \ldots, \Delta_{1,d} + 1, \Delta_{1,d+1} - q) \in \mathbb{Z}^{d+1}
$$

for $q = 1$ and 2. Similar to the fourth step, it is easy to see that

$$
\langle \mathbf{a}_1, \beta_q \rangle < \prod_{j=2}^{d+1} \Delta_{1,j} \quad \text{and} \quad \langle \mathbf{a}_i, \beta_q \rangle > 0 \quad \text{for} \quad i = 2, \ldots, d+1.
$$

In other word, both β_1 and β_2 are contained in the interior, as desired. $\square$

9.3 The semigroup ring associated only with vertices of a cyclic polytope

In this section, we will study the semigroup K-algebra generated only by the vertices of integral cyclic polytopes. That is to say, we will consider the affine semigroup K-algebra arising from

$$
Q_d(\tau_1, \ldots, \tau_n) = \mathbb{Z}_{\geq 0} \left\{ (1, \tau_i, \tau_i^2, \ldots, \tau_i^d) \in \mathbb{Z}^{d+1} : i = 1, \ldots, n \right\}.
$$

(Throughout this section, Q denotes the affine semigroup $Q_d(\tau_1, \ldots, \tau_n)$.) Let $K[Q]$ be the K-subalgebra of $K[t_0, t_1, \ldots, t_d]$ with $K[Q] = \bigoplus_{\mathbf{a} \in Q} K \cdot t^{\mathbf{a}}$. It is just the toric ring associated with the *configuration* (9.2). In this section, we study the normality of $K[Q]$ (Theorem 9.3.3).

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring over a field K. Let I_Q be the kernel of the surjective ring homomorphism $S \to K[Q]$ sending each x_i to t^{v_i}. The ideal I_Q is just the toric ideal associated with the matrix (9.2). In particular, it is homogeneous with respect to the usual $\mathbb{Z}$-grading on S. Recall that the matrix (9.2) can be transformed into the form (9.3).

By Proposition 9.1.2 (i), $K[Q]$ is regular when $n = d + 1$ and in particular is normal. When $d = 1$, the matrix (9.2) transformed as is stated above is of the following form:

$$\begin{pmatrix} 1 & 0 \\ 1 & \Delta_{1,2} \\ \vdots & \vdots \\ 1 & \Delta_{1,n} \end{pmatrix}. \tag{9.8}$$

Since I_Q is preserved even if we divide a common divisor of $\Delta_{1,2}, \ldots, \Delta_{1,n}$ out of the second row, we may assume the greatest common divisor of $\Delta_{1,2}, \cdots, \Delta_{1,n}$ is equal to 1. The ideal I_Q is a defining ideal of a projective monomial curve in $\mathbb{P}^{n-1}$, and it is well known (cf. [14]) that the corresponding curve is normal if and only if it is a rational normal curve of degree $n-1$, that is, $\Delta_{1,i} = i-1$ for all $i-1$ with $2 \leq i \leq n$ (after the above transformation and re-setting each $\Delta_{1,i}$). Consequently, in the case $d = 1$, the ring $K[Q]$ is normal if and only if $\tau_2 - \tau_1 = \tau_3 - \tau_2 = \cdots = \tau_n - \tau_{n-1}$.

We will show that $K[Q]$ is never normal if $d \geq 2$ and $n = d + 2$. Our strategy is to make use of the following criterion.

Lemma 9.3.1 ([58, Lemma 6.1])**.** *Let R be a toric ring such that the corresponding toric ideal I is homogeneous. Suppose I has a minimal system of binomial generators that contains a binomial consisting of non-squarefree monomials. Then R is not normal.*

Set $\Gamma := \Gamma_d(\tau_1, \ldots, \tau_n)$. Note that there is a one-to-one correspondence between the faces of Γ and the proper faces of $\mathbb{R}_{\geq 0}Q$; a subset $W \subseteq [n]$ is a $(d-1)$-dimensional face of Γ if and only if $\mathbb{R}_{\geq 0} \cdot \{v_i : i \in W\}$ is a d-dimensional face of $\mathbb{R}_{\geq 0}Q$. In the sequel, we tacitly use this correspondence.

If $n = d + 2$, then I_Q is principal, and we can determine the supports of both monomials appearing in the binomial generator of I_Q. Following the usual convention, we set $\operatorname{supp}(u) := \{i \in [n] : x_i \mid u\}$.

Lemma 9.3.2. *Assume $n = d + 2$. Then $K[Q] \cong S/(u - v)$ for some monomials $u, v \in S$ such that* $\operatorname{supp}(u) = \{i \in [n] : i \text{ is odd}\}$ *and* $\operatorname{supp}(v) = \{i \in [n] : i \text{ is even}\}$.

Proof. Since the rank of $\mathbb{Z}Q$ is equal to $\dim \mathbb{R}_{\geq 0}C_d(\tau_1, \ldots, \tau_n) = d+1$, the kernel of the $\mathbb{Q}$-linear map defined by (9.2) is of dimension 1. It is then clear that I_Q is principal. Choose a generator $u - v$ of I_Q. Obviously $\operatorname{supp}(u) \cap \operatorname{supp}(v) = 0$. Moreover neither $\operatorname{supp}(u)$ nor $\operatorname{supp}(v)$ is a face of Γ. Indeed, by the choice of $u - v$,

$$\sum_{i \in \operatorname{supp}(u)} a_i v_i = \sum_{j \in \operatorname{supp}(v)} b_j v_j \qquad (*)$$

for some positive integers a_i, b_j, and hence if one of $\operatorname{supp}(u)$ and $\operatorname{supp}(v)$ is a face of Γ, say W, then the corresponding cone $\mathbb{R}_{\geq 0}W$ of $\mathbb{R}_{\geq 0}Q$ contains all the v_i and v_j appearing in $(*)$. This implies $(*)$ is just a relation among vertices in $\mathbb{R}_{\geq 0}W$, which contradicts (i) of Proposition 9.1.2. Since $n = d+2$, applying (i) of Proposition 9.1.2 again, it follows from $(*)$ that $\operatorname{supp}(u) \cup \operatorname{supp}(v) = [n]$. Thus $\operatorname{supp}(u)$ and $\operatorname{supp}(v)$ give a partition of $[n]$ by *non-faces* of Γ, i.e., subsets of $[n]$ which are not in Γ.

Without loss of generality, we may assume that $1 \in \operatorname{supp}(u)$. Set

$$\Lambda := \{(F, G) \in (2^{[n]} \setminus \Gamma) \times (2^{[n]} \setminus \Gamma) : 1 \in F, F \cap G = \emptyset, F \cup G = [n]\}.$$

Then $(\operatorname{supp}(u), \operatorname{supp}(v)) \in \Lambda$. On the other hand, the pair (U, V), where $U := \{i \in [n] : i \text{ is odd}\}$ and $V := \{i \in [n] : i \text{ is even}\}$, also belongs to Λ; indeed, U and V does not satisfy the condition in (iii) of Proposition 9.1.2. Thus what we have only to show to complete the proof is $\#\Lambda = 1$. Note that by [44, Proposition 5.1], Γ is combinatorially equivalent to the join of the boundary complexes of two simplexes Γ_1, Γ_2. Hence we may identify Γ with $\partial\Gamma_1 * \partial\Gamma_2$ to prove $\#\Lambda = 1$, and may assume $1 \in F_1$. It is straightforward to verify that $\Lambda = \{(F_1, F_2)\}$. $\square$

Now we will prove the following.

Theorem 9.3.3 ([31, Theorem 4.3])**.** *Assume $d \geq 2$ and $n = d+2$. Then $K[Q]$ is never normal.*

Proof. Set $U := \{i \in [n] : i \text{ is odd}\}$ and $V := \{i \in [n] : i \text{ is even}\}$. By Lemma 9.3.2,

$$K[Q] \cong S/\left(\prod_{i \in U} x_i^{a_i} - \prod_{j \in V} x_j^{b_j}\right).$$

Set $u = \prod_{i \in U} x_i^{a_i}$ and $v = \prod_{j \in V} x_j^{b_j}$. By Lemma 9.3.1, it suffices to show that neither u nor v is squarefree. Note that the following equality holds.

$$(a_1, -b_2, a_3, -b_4, \ldots,) \begin{pmatrix} 1 & \tau_1 & \tau_1^2 & \cdots & \tau_1^d \\ 1 & \tau_2 & \tau_2^2 & \cdots & \tau_2^d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \tau_n & \tau_n^2 & \cdots & \tau_n^d \end{pmatrix} = (0, 0, \ldots, 0) \in \mathbb{Z}^{d+1}.$$

By Lemma 9.1.3,

$$(a_1, -b_2, a_3, -b_4, \dots,) \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & \Delta_{12} & 0 & \cdots & 0 \\ 1 & \Delta_{13} & \Delta_{13}\Delta_{23} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & \Delta_{1,d+1} & \Delta_{1,d}\Delta_{2,d} & \cdots & \prod_{k=1}^{d} \Delta_{k,d+1} \\ 1 & \Delta_{1,n} & \Delta_{1,n}\Delta_{2,n} & \cdots & \prod_{k=1}^{d} \Delta_{k,n} \end{pmatrix} = (0, 0, \dots, 0) \tag{9.9}$$

For a proof by contradiction, suppose either u or v is squarefree. This is equivalent to say that $\sum_{i\in U} a_i = \#U$ or $\sum_{j\in V} b_j = \#V$. By the equation (9.9), it follows that

$$\sum_{i\in U} a_i = \sum_{j\in V} b_j. \tag{9.10}$$

The case d is even. Then $d = 2l$ for some positive integer l, $n = 2l + 2$, $\#U = \#V = l + 1$, which implies both of u and v are squarefree. By the equation (9.9), we have $\prod_{k=1}^{d} \Delta_{k,d+1} = \prod_{k=1}^{d} \Delta_{k,n} = 0$, while clearly $\prod_{k=1}^{d} \Delta_{k,n} > \prod_{k=1}^{d} \Delta_{k,d+1}$ holds, a contradiction.

The case d is odd. Then $d = 2l - 1$ for some integer l with $l > 1$, $n = 2l + 1$ and $\#U = \#V + 1 = l + 1$, which implies that v cannot be squarefree since $\sum_{i\in U} a_i \geq \#U$. Thus u is squarefree, that is, $a_i = 1$ for all $i \in U$. Moreover one of the b_j is 2 and the others are 1. On the other hand, it follows from (9.9) that $-\prod_{k=1}^{d} \Delta_{k,d+1} b_{2l} + \prod_{k=1}^{d} \Delta_{k,n} a_{2l+1} = 0$. Since $\prod_{k=1}^{d} \Delta_{k,d+1} < \prod_{k=1}^{d} \Delta_{k,n}$, we conclude that $b_{2l} = 2$ and hence

$$\prod_{k=1}^{d} \Delta_{k,n} = 2 \prod_{k=1}^{d} \Delta_{k,d+1}.$$

For simplicity, we set $c_i = a_i$ for odd i and $c_i = -b_i$ for even i. Hence $c_1 = c_3 = \cdots = c_n = 1$, $c_2 = c_4 = \cdots = c_{n-3} = -1$, and $c_{n-1} = -2$. By the equation (9.9) again,

$$0 = \sum_{i=2}^{n} \Delta_{1,i} c_i = \sum_{i=2}^{n} \left(\sum_{j=1}^{i-1} \Delta_{j,j+1} \right) c_i = \sum_{j=1}^{n-1} \Delta_{j,j+1} \left(\sum_{i=j+1}^{n} c_i \right)$$

Since $n \geq 5$ by the hypothesis that n is odd and $d \geq 2$, we may divide the last summation in the above equality as follows. Set

$$s_1 := \sum_{j=1}^{n-3} \left(\sum_{i=j+1}^{n} c_i \right) \Delta_{j,j+1},$$

and

$$s_2 = \Delta_{n-2,n-1}(c_{n-1} + c_n) + \Delta_{n-1,n} c_n = -\Delta_{d,d+1} + \Delta_{d+1,n}$$

Then $s_1 + s_2 = \sum_{j=1}^{n-1} \Delta_{j,j+1} \left(\sum_{i=j+1}^{n} c_i \right) = 0$. An easy observation shows that each coefficient $\sum_{i=j+1}^{n} c_i$ of $\Delta_{j,j+1}$ in s_1 is 0 if j is even and otherwise negative. Hence the inequality $s_1 < 0$ follows since $n - 3 \geq 2$. We will show that $s_2 \leq 0$. If this is the case, then $s_1 + s_2 < 0$ holds on the contrary to the fact $s_1 + s_2 = 0$, which completes the proof.

Suppose $s_2 > 0$. Then

$$\tau_n - \tau_{d+1} = \Delta_{d+1,n} > \Delta_{d,d+1} = \tau_{d+1} - \tau_d,$$

and hence $\tau_n - \tau_d > 2(\tau_{n-1} - \tau_d)$. It follows that

$$\begin{aligned}\prod_{k=1}^{d} \Delta_{k,n} &= (\tau_n - \tau_d)(\tau_n - \tau_{d-1}) \cdots (\tau_n - \tau_1) \\ &> 2(\tau_{n-1} - \tau_d)(\tau_{n-1} - \tau_{d-1}) \cdots (\tau_{n-1} - \tau_1) = 2 \prod_{k=1}^{d} \Delta_{k,n-1},\end{aligned}$$

which is absurd. $\square$

As is stated above Lemma 9.3.1, the K-algebra $K[Q]$ is normal if and only if $\tau_2 - \tau_1 = \tau_3 - \tau_2 = \cdots = \tau_n - \tau_{n-1}$, when $d = 1$. Though we do not have a complete answer on the normality of $K[Q]$ when $n > d + 2$, we strongly believe the following holds.

Conjecture 9.3.4. *The K-algebra $K[Q]$ is normal only in one of the following cases:*

(i) $n = d + 1$;

(ii) $d = 1$ *and* $\tau_2 - \tau_1 = \tau_3 - \tau_2 = \cdots = \tau_n - \tau_{n-1}$.

The following proposition tells us that there are a lot of $K[Q]$ which are not normal when $n \geq d + 3$.

Proposition 9.3.5. *Assume $n \geq d + 3$. If $\prod_{k=1}^{d} \Delta_{k,d+1} \nmid \prod_{k=1}^{d} \Delta_{k,s}$ for some s with $d + 2 \leq s \leq n$, then $K[Q]$ is not normal.*

Proof. Suppose Q is normal. Since the subset $\{1, \ldots, d\}$ of $[n]$ satisfies the condition in (iii) of Proposition 9.1.2, the cone generated by $v_1, \ldots, v_d$ forms a facet of $\mathbb{R}_{\geq 0}Q$. Let $\mathcal{F}$ denote this facet. Then Q together with $\mathcal{F}$ satisfies the condition in Proposition 9.2.1, and in particular, there exists an element $x \in Q$ such that $\sigma_{\mathcal{F}}(x) = 1$, where $\sigma_{\mathcal{F}}$ is a support form of $\mathcal{F}$ with integer coefficients.

We will describe $\sigma_{\mathcal{F}}$ explicitly. Let $\mathcal{H}$ be the supporting hyperplane of $\mathcal{F}$. Note that we can freely identify Q with the affine semigroup associated with the matrix in

Lemma 9.1.3. After this identification, the vector $\mathbf{a}_d = (0, \dots, 0, 1) \in \mathbb{Z}^{d+1}$ defines $\mathcal{H}$ as is stated below of Theorem 9.2.5. Thus

$$\mathcal{H} = \{x \in \mathbb{R}^{d+1} : \langle \mathbf{a}_d, x \rangle = 0\},$$

and $\langle \mathbf{a}_d, x \rangle \in \mathbb{Z}_{>0}$ for all $x \in Q \setminus \mathcal{F}$. We set $\mathbb{Z}Q_{\mathcal{H}} := \mathbb{Z}Q/\mathbb{Z}Q \cap \mathcal{H}$. Note that $\mathbb{Z}Q_{\mathcal{H}} \cong \mathbb{Z}$. Let $v_0 \in \mathbb{Z}Q$ be an element whose image in $\mathbb{Z}Q_{\mathcal{H}}$ is a free basis of $\mathbb{Z}Q_{\mathcal{H}}$. Then the support form $\sigma_{\mathcal{F}}$ of Q and $\mathcal{F}$ is defined as

$$\sigma_{\mathcal{F}}(x) = \frac{\langle \mathbf{a}_d, x \rangle}{\langle \mathbf{a}_d, v_0 \rangle}$$

for all $x \in \mathbb{R}^{d+1}$, and $\sigma_{\mathcal{F}}(x) = 0$ for $x \in Q \cap \mathcal{F}$ and $\sigma_{\mathcal{F}}(x) \in \mathbb{Z}_{>0}$ for $x \in Q \setminus \mathcal{F}$ (see [11, Remark 1.72 and p.55] for the construction and the property of a support form). Recall that there exists an element $x \in Q$ such that $\sigma_{\mathcal{F}}(x) = 1$. Since $\mathcal{F}$ is generated by $v_1, \dots, v_d$, the element x can be written as $x = y + \sum_{i=d+1}^{n} \lambda_i v_i$ for some $\lambda_i \in \mathbb{Z}_{\geq 0}$ and $y \in Q \cap \mathcal{F}$. By definition, $\langle \mathbf{a}_d, v_i \rangle = \prod_{k=1}^{d} \Delta_{k,i}$ for $i = d+1, \dots, n$, and $\langle \mathbf{a}_d, y \rangle = 0$. Since $\prod_{k=1}^{d} \Delta_{k,d+1} < \prod_{k=1}^{d} \Delta_{k,d+2} < \cdots < \prod_{k=1}^{d} \Delta_{k,n}$, it follows that $0 < \langle \mathbf{a}_d, v_{d+1} \rangle < \cdots < \langle \mathbf{a}_d, v_n \rangle$, and hence

$$1 = \sigma_{\mathcal{F}}(x) \geq (\sum_{i=d+1}^{n} \lambda_i) \sigma_{\mathcal{F}}(v_{d+1}) > 0.$$

Therefore $\sigma_{\mathcal{F}}(v_{d+1}) = 1$ and $x = y + v_{d+1}$. Thus we can replace v_0 by v_{d+1}. However it follows from the fact $v_s \in Q$ that

$$\frac{\prod_{k=1}^{d} \Delta_{k,s}}{\prod_{k=1}^{d} \Delta_{k,d+1}} = \frac{\langle \mathbf{a}_d, v_s \rangle}{\prod_{k=1}^{d} \Delta_{k,d+1}} = \sigma_{\mathcal{F}}(v_s) \in \mathbb{Z},$$

contrary to the hypothesis $\prod_{k=1}^{d} \Delta_{k,d+1} \nmid \prod_{k=1}^{d} \Delta_{k,s}$. □

Since there exists a lot of non-normal $K[Q]$, it is natural to ask when $K[Q]$ is Cohen-Macaulay. Clearly if $n = d + 2$, then $K[Q]$ is a complete intersection, and hence in particular, Cohen-Macaulay. So far, we have never found an example of $K[Q]$ which is Cohen-Macaulay, in the case $d \geq 2$ and $n > d + 2$. Thus we expect the following

Conjecture 9.3.6. *The K-algebra $K[Q]$ is never Cohen-Macaulay if $d \geq 2$ and $n > d + 2$.*

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