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AN L^p-APPROACH TO SINGULAR LINEAR PARABOLIC EQUATIONS WITH LOWER ORDER TERMS

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Abstract. Singular means here that the parabolic equation is *neither* in normal form nor can it be reduced to such a form. For this class of problems we generalizes the results proved in [4] introducing first-order terms.

1. **Introduction.**

Let Ω be a bounded open set of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Let

$$\mathcal{L} = -\sum_{i,j=1}^{n} D_{x_j}(a_{i,j}(x)D_{x_i}) + \sum_{i=1}^{n} a_i(x)D_{x_i} + a_0(x) \tag{1.1}$$

be a linear second-order differential operator such that $a_{i,j}$, a_i and a_0 are real-valued functions satisfying

$$\begin{aligned} &a_{i,j} \in C(\overline{\Omega}), \quad D_{x_j}a_{i,j}, a_i, D_{x_i}a_i, a_0 \in L^\infty(\Omega), \quad i,j = 1,\dots,n, \\ &\{a_{i,j}(x)\} \text{ is a positive definite symmetric matrix for each } x \in \overline{\Omega}, \end{aligned} \tag{1.2}$$

for which there exists a positive constant c_0 such that

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geq c_0|\xi|^2, \quad \text{for all } x \in \overline{\Omega},\ \xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n. \tag{1.3}$$

As is well-known, there is a large literature concerning analytic semigroups generated by realizations of $-\mathcal{L}$ in $L^p(\Omega)$, $p \in (1,+\infty)$, when $-\mathcal{L}$ is endowed with different boundary conditions characterizing the domain of the realization (cf., e.g. the monographs [6, 8, 10]).

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This approach yields suitable regularity properties for the solution to the corresponding Cauchy problem.

In addition to this we stress that much attention has been devoted also to *singular* parabolic Cauchy problems, i.e. to problem of the form

$$D_t[m(x)u(x,t)] + \mathcal{L}u(x,t) = f(x,t), \qquad \text{for all } (x,t) \in \Omega \times [0,\tau], \tag{1.4}$$

$$\mathcal{B}u(x,t) = 0, \qquad \text{for all } (x,t) \in \partial\Omega \times [0,\tau], \tag{1.5}$$

$$m(x)u(x,t) \to m(x)u_0(x), \qquad \text{for almost every } x \in \Omega, \text{ as } t \to 0+. \tag{1.6}$$

Singular means here that m is a non-negative function in $L^\infty(\Omega)$, which *may* vanish, while u_0 and f are given functions.

If L denotes the operator with domain in $L^p(\Omega)$ realized by $(-\mathcal{L}, \mathcal{B})$ where $\mathcal{B}$ is the linear operator corresponding to Dirichlet boundary conditions and M is the multiplication operator by m in $L^p(\Omega)$, it is shown in [5] that the resolvent estimate

$$\|M(\lambda M + L)^{-1}\|_{\mathcal{L}(L^p(\Omega))} \le C(1+|\lambda|)^{-\beta}$$

holds for any λ in the region $\Sigma = \{z \in \mathbb{C} : \operatorname{Re} z \ge -c(1+|\lambda|)\}$ for some $\beta \in (0,1)$ and $c > 0$.

The previous assumption allows to develop a maximal regularity in time theory for the solution corresponding to $f \in C^\theta([0,T]; L^p(\Omega))$ (cf. [5, Theorem 3.26]). The basic point, however, is that the regularity decreases with respect to the *non-singular* case, in the sense that in the first case we can show that $u \in C^{\theta+\beta-1}([0,T]; \mathcal{D}(L))$, with $\beta \in (0,1)$, while in the latter case we have $\beta = 1$ and $u \in C^\theta([0,T]; \mathcal{D}(L))$.

In the paper [4], making use of a result by Okazawa [9], we have improved the results in [5], where the operator $-\mathcal{L}$ is symmetric and $\mathcal{B}$ corresponds to Dirichlet boundary conditions. In [4] we also showed that the index β can be improved to a larger one, if m is ρ-regular, i.e.

$$m \in C^1(\overline{\Omega}), \qquad |\nabla m(x)| \le C m(x)^\rho, \quad \text{for all } x \in \overline{\Omega},$$

for some $\rho \in (0,1)$.

The fact to have at our disposal a higher regularity for solutions plays an essential role, e.g., in recovering unknown kernels in degenerate linear integrodifferential equations.

The aim of this paper is two-fold. From one hand we want to deal with *non-symmetric* operators $\mathcal{L}$ and, from the other one, we intend to handle Robin boundary conditions, too (cf. e.g., [1, pp. 206-207]). This will be the most delicate aspect in the development of the present paper.

Concerning this aspect we note that L^2-theory for degenerate integrodifferential equations of parabolic type, with Robin boundary conditions and time-dependent multiplication operator $M(t) = m(t,\cdot)$, was developed quite recently in [3]. Such equations with Dirichlet and Neumann boundary conditions were dealt with in the space $L^2(\Omega)$ in [2], where a treatment in $L^p(\Omega)$, $p \in (1,+\infty)$, is also given for Dirichlet boundary conditions.

Finally, we will mention that inverse problems for non-autonomous degenerate integrodifferential equations with Dirichlet boundary conditions are treated in [7].

2. **Dirichlet and Robin problems in $L^p(\Omega)$, $p \in (1, +\infty)$.** In this section we make the following assumptions and suppose that all the listed functions are real-valued:

$$a_i \in W^{1,\infty}(\Omega), \quad i = 1, \dots, n, \quad a_0 - \frac{1}{p}\sum_{i=1}^n D_{x_i} a_i \geq c_1 > 0 \quad \text{in } \Omega, \tag{2.1}$$

$$b \in L^\infty(\partial\Omega). \tag{2.2}$$

The realization L of $\mathcal{L}$ in $L^p(\Omega)$, $1 < p < +\infty$, is defined by

$$D(L) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \qquad Lu = \mathcal{L}u, \quad u \in D(L), \tag{2.3}$$

in case of the Dirichlet boundary condition, and by

$$D(L) = \Big\{ u \in W^{2,p}(\Omega) : \sum_{i,j=1}^n a_{i,j}\nu_j \, D_{x_i} u + bu = 0 \ \text{on} \ \partial\Omega \Big\}, \quad Lu = \mathcal{L}u, \quad u \in D(L), \tag{2.4}$$

in case of the Robin boundary condition, where also the following assumption is needed:

$$b(x) + \frac{1}{p}\sum_{i=1}^n a_i(x)\nu_i(x) \geq 0, \quad \text{for } x \in \partial\Omega. \tag{2.5}$$

We note that, when $b = 0$, the Robin boundary condition simplifies to the Neumann one.

Finally, we observe that assumptions (2.1) and (2.1), (2.2), (2.5) guarantee that operator L admits a continuous inverse L^{-1} under both Dirichlet and Robin boundary conditions, respectively.

Let

$$D(L_0) = D(L), \quad L_0 = -\sum_{i,j=1}^n D_{x_i}(a_{i,j} D_{x_j} u), \quad u \in D(L),$$

be the principal part of L.

Consider now the identity

$$\int_\Omega |u|^{p-2}\overline{u} L u \, dx = \int_\Omega |u|^{p-2}\overline{u} L_0 u \, dx + \int_\Omega \sum_{i=1}^n a_i(x)|u|^{p-2}\overline{u} D_{x_i} u \, dx + \int_\Omega a_0(x)|u|^p \, dx. \tag{2.6}$$

Observe now that

$$\int_\Omega |u|^{p-2}\overline{u} L_0 u \, dx = -\lim_{\delta \to 0+} \int_\Omega \sum_{i,j=1}^n g_{p-2,\delta}(u)\overline{u} D_{x_j}(a_{i,j} D_{x_i} u) \, dx, \tag{2.7}$$

where

$$g_{q,\varepsilon}(u) = \begin{cases} (|u|^2 + \varepsilon)^{q/2} & \text{if } q \in (-1, 0) \\ |u|^q & \text{if } q \in [0, +\infty). \end{cases} \tag{2.8}$$

Integrating by parts, we easily obtain

$$\int_\Omega |u|^{p-2}\overline{u} L_0 u \, dx = \lim_{\delta \to 0+} I_p(u, \delta) + \eta \int_{\partial\Omega} b|u|^p \, dS, \tag{2.9}$$

where $\eta = 0$ or $\eta = 1$ according as the Dirichlet or the Robin boundary conditions hold and

$$I_p(u,\delta) = \int_\Omega g_{p-2,\delta}(u) \sum_{i,j=1}^n a_{i,j} D_{x_i} u D_{x_j} \overline{u}\, dx \\ + (p-2) \int_\Omega g_{p-4,\delta}(u)|u|\overline{u} \sum_{i,j=1}^n a_{i,j} D_{x_i} u D_{x_j} |u|\, dx. \tag{2.10}$$

Then from the proof of a remarkable result by Okazawa [9], we deduce the inequalities:

$$\operatorname{Re} I_p(u,\delta) \geq \begin{cases} (p-1)c_0 \displaystyle\int_\Omega |\nabla u|^2 (|u|^2+\delta)^{(p-2)/2}\, dx & \text{if } 1<p<2, \\ c_0 \displaystyle\int_\Omega |\nabla u|^2 |u|^{p-2}\, dx & \text{if } 2 \leq p < +\infty, \end{cases} \tag{2.11}$$

$$|\operatorname{Im} I_p(u,\delta)| \leq \frac{|p-2|}{2\sqrt{p-1}} \operatorname{Re} I_p(u,\delta), \qquad \text{for all } \delta \in \mathbb{R}_+. \tag{2.12}$$

Taking the limit as $\delta \to 0+$, from (2.9)–(2.12) we deduce the following inequalities, where χ_E denotes the characteristic function of the set E:

$$\operatorname{Re} \int_\Omega |u|^{p-2}\overline{u} L_0 u\, dx = \lim_{\delta\to 0+} \operatorname{Re} I_p(u,\delta) + \eta \int_{\partial\Omega} b|u|^p\, dS \\ \geq [(p-1)\chi_{(1,2)}(p) + \chi_{[2,+\infty]}(p)] \lim_{\delta\to 0+} \int_\Omega g_{p-2,\delta} |\nabla u|^2\, dx + \eta \int_{\partial\Omega} b|u|^p\, dS, \tag{2.13}$$

$$\left|\operatorname{Im} \int_\Omega |u|^{p-2}\overline{u} L_0 u\, dx\right| = |\lim_{\delta\to 0+} \operatorname{Im} I_p(u,\delta)| \leq \frac{|p-2|}{2\sqrt{p-1}} \lim_{\delta\to 0+} \operatorname{Re} I_p(u,\delta) \\ = \frac{|p-2|}{2\sqrt{p-1}} \Big(\operatorname{Re} \int_\Omega |u|^{p-2}\overline{u} L_0 u\, dx - \eta \int_{\partial\Omega} b|u|^p\, dS \Big). \tag{2.14}$$

Let now $u \in W^{1,p}(\Omega)$, $p \in (1,+\infty]$ and $\varepsilon \geq 0$. Noting that

$$D_{x_i}(|u|^2+\varepsilon)^{p/2} = \frac{p}{2}(|u|^2+\varepsilon)^{(p-2)/2} D_{x_i}(|u|^2) \iff D_{x_i} g_{p,\varepsilon}(u) = \frac{p}{2} g_{p-2,\varepsilon} D_{x_i}(|u|^2)$$

we get

$$\begin{aligned}
&\mathrm{Re}\int_\Omega a_i|u|^{p-2}\overline{u}D_{x_i}u\,dx = \lim_{\delta\to 0+}\mathrm{Re}\int_\Omega a_i g_{p-2,\delta}(u)\overline{u}D_{x_i}u\,dx\\
&= \lim_{\delta\to 0+}\int_\Omega a_i \mathrm{Re}\,(\overline{u}D_{x_i}u) g_{p-2,\delta}(u)\,dx\\
&= \lim_{\delta\to 0+}\frac{1}{2}\int_\Omega a_i(\overline{u}D_{x_i}u + u\overline{D_{x_i}u})g_{p-2,\delta}(u)\,dx\\
&= \frac{1}{2}\lim_{\delta\to 0+}\int_\Omega a_i D_{x_i}(|u|^2)g_{p-2,\delta}(u)\,dx\\
&= \frac{1}{p}\lim_{\delta\to 0+}\int_\Omega a_i D_{x_i}g_{p,\delta}(u)\,dx\\
&= \frac{\eta}{p}\lim_{\delta\to 0+}\int_{\partial\Omega}\nu_i a_i g_{p,\delta}(u)\,dS - \frac{1}{p}\lim_{\delta\to 0+}\int_\Omega D_{x_i}a_i g_{p,\delta}(u)\,dx\\
&= \frac{\eta}{p}\int_{\partial\Omega}\nu_i a_i|u|^p\,dS - \frac{1}{p}\int_\Omega |u|^p D_{x_i}a_i\,dx. \qquad (2.15)
\end{aligned}$$

Hence we observe that, according to our assumptions (cf. (2.2) and (2.5)), the following inequalities hold for all $u \in D(L)$:

$$\begin{aligned}
&\mathrm{Re}(Lu, |u|^{p-2}u)\\
&= \mathrm{Re}\int_\Omega |u|^{p-2}\overline{u}L_0u\,dx + \mathrm{Re}\int_\Omega\sum_{i=1}^n a_i|u|^{p-2}\overline{u}D_{x_i}u\,dx + \int_\Omega a_0|u|^p\,dx\\
&\geq \eta\int_{\partial\Omega}\Big(b + p^{-1}\sum_{i=1}^n \nu_i a_i\Big)|u|^p\,dS + \int_\Omega\Big[a_0 - p^{-1}\sum_{i=1}^n D_{x_i}a_i\Big]|u|^p\,dx\\
&\geq c_1\int_\Omega |u|^p\,dx. \qquad (2.16)
\end{aligned}$$

Then, using (2.13), we deduce, for any $\varepsilon \in \mathbb{R}_+$ and $c_2 = \|\big(\sum_{i=1}^n |a_i|^2\big)^{1/2}\|_{L^\infty(\Omega)}$,

$$\begin{aligned}
&\Big|\int_\Omega\sum_{i=1}^n a_i|u|^{p-2}\overline{u}D_{x_i}u\,dx\Big| \leq c_2\int_\Omega |\nabla u||u|^{p-1}\,dx\\
&\quad= c_2\limsup_{\delta\to 0+}\int_\Omega |\nabla u| g_{(p-2)/2,\delta}g_{p/2,\delta}\,dx\\
&\quad\leq c_2\limsup_{\delta\to 0+}\Big\{\int_\Omega |\nabla u|^2 g_{p-2,\delta}\,dx\Big\}^{1/2}\Big\{\int_\Omega g_{p,\delta}\,dx\Big\}^{1/2}\\
&\quad\leq \limsup_{\delta\to 0+}\Big\{\frac{c_2}{2\varepsilon}\int_\Omega |\nabla u|^2 g_{p-2,\delta}\,dx + \frac{c_2\varepsilon}{2}\int_\Omega g_{p,\delta}\,dx\Big\}\\
&\quad= \frac{c_2}{2\varepsilon}\limsup_{\delta\to 0+}\int_\Omega |\nabla u|^2 g_{p-2,\delta}\,dx + \frac{c_2\varepsilon}{2}\int_\Omega |u|^p\,dx\\
&\quad\leq \frac{c_2}{2\varepsilon c_0}\Big[\frac{\chi_{(1,2)}(p)}{p-1} + \chi_{[2,+\infty]}(p)\Big]\Big(\mathrm{Re}\int_\Omega |u|^{p-2}\overline{u}L_0u\,dx - \eta\int_{\partial\Omega} b|u|^p\,dS\Big)\\
&\quad+\frac{c_2\varepsilon}{2}\int_\Omega |u|^p\,dx. \qquad (2.17)
\end{aligned}$$

With the aid of (2.14), (2.15), (2.16), (2.17) we obtain

$$\begin{aligned}\Big|\mathrm{Im}\int_\Omega |u|^{p-2}\overline{u}Lu\,dx\Big| &= \Big|\mathrm{Im}\int_\Omega |u|^{p-2}\overline{u}L_0u\,dx + \mathrm{Im}\int_\Omega \sum_{i=1}^n a_i|u|^{p-2}\overline{u}D_{x_i}u\,dx\Big| \\ &\le c_3\Big(\mathrm{Re}\int_\Omega |u|^{p-2}\overline{u}L_0u\,dx - \eta\int_{\partial\Omega} b|u|^p\,dS\Big) + \frac{c_2\varepsilon}{2}\int_\Omega |u|^p\,dx \\ &\le c_3\mathrm{Re}\int_\Omega |u|^{p-2}\overline{u}Lu\,dx - c_3\eta\int_{\partial\Omega}\Big[b+p^{-1}\sum_{i=1}^n \nu_i a_i\Big]|u|^p\,dS \\ &\quad - c_3\int_\Omega\Big[a_0 - p^{-1}\sum_{i=1}^n D_{x_i}a_i\Big]|u|^p\,dx + \frac{c_2\varepsilon}{2}\int_\Omega |u|^p\,dx \\ &\le c_3\mathrm{Re}(Lu,|u|^{p-2}u) - \Big(c_3c_1 - \frac{c_2\varepsilon}{2}\Big)\int_\Omega |u|^p\,dx, \quad (2.18)\end{aligned}$$

where

$$c_3 = \frac{|p-2|}{2\sqrt{p-1}} + \frac{c_2}{2\varepsilon c_0}\Big[\frac{\chi_{(1,2)}(p)}{(p-1)} + \chi_{[2,+\infty]}(p)\Big].$$

Let $\varepsilon > 0$ be so small that

$$c_4 = c_3c_1 - c_2\varepsilon/2 > 0.$$

Then (2.18) is rewritten as

$$\Big|\mathrm{Im}\int_\Omega |u|^{p-2}\overline{u}Lu\,dx\Big| \le c_3\mathrm{Re}(Lu,|u|^{p-2}u) - c_4\int_\Omega |u|^p\,dx. \quad (2.19)$$

Consider now the spectral problem

$$u \in \mathcal{D}(L), \qquad \lambda mu + Lu = f \in L^p(\Omega). \quad (2.20)$$

Taking the real and imaginary parts of the scalar product of both sides in (2.20) with $u|u|^{p-2}$, we get

$$\mathrm{Re}\,\lambda\int_\Omega m|u|^p\,\mathrm{d}x + \mathrm{Re}\,(Lu,u|u|^{p-2}) = \mathrm{Re}\int_\Omega f\overline{u}|u|^{p-2}\mathrm{d}x, \quad (2.21)$$

$$\mathrm{Im}\,\lambda\int_\Omega m|u|^p\,\mathrm{d}x + \mathrm{Im}\,(Lu,u|u|^{p-2}) = \mathrm{Im}\int_\Omega f\overline{u}|u|^{p-2}\mathrm{d}x. \quad (2.22)$$

From (2.22) we deduce the inequalities

$$|\mathrm{Im}\,\lambda|\int_\Omega m|u|^p\,\mathrm{d}x \le |\mathrm{Im}\,(Lu,u|u|^{p-2})| + \Big|\mathrm{Im}\int_\Omega f\overline{u}|u|^{p-2}\mathrm{d}x\Big|. \quad (2.23)$$

Multiply then both sides in (2.23) by a positive constant k and add the obtained inequality to equation (2.22). From (2.19) we get

$$\begin{aligned}&(\mathrm{Re}\,\lambda + k|\mathrm{Im}\,\lambda|)\int_\Omega m|u|^p\,\mathrm{d}x + (1-kc_3)\mathrm{Re}\,(Lu,u|u|^{p-2}) + kc_4\int_\Omega |u|^p\,dx \\ \le\;& \mathrm{Re}\int_\Omega f\overline{u}|u|^{p-2}\mathrm{d}x + k\Big|\mathrm{Im}\int_\Omega f\overline{u}|u|^{p-2}\mathrm{d}x\Big| \le (1+k)\|f\|_p\|u\|_p^{p-1}. \quad (2.24)\end{aligned}$$

Choose now $k = k_1(p)$ so small as to satisfy

$$h_1(p) := 1 - k_1(p)c_3 > 0, \qquad \text{for all } p \in (1,+\infty). \quad (2.25)$$

Therefore, (2.24) and (2.25) imply

$$\Big(\mathrm{Re}\,\lambda + k_1(p)|\mathrm{Im}\,\lambda| + \frac{k_1(p)c_4}{\|m\|_\infty}\Big)\int_\Omega m|u|^p\,\mathrm{d}x$$
$$+h_1(p)\mathrm{Re}\,(Lu,u|u|^{p-2}) \le [k_1(p)+1]\|f\|_p\|u\|_p^{p-1}, \tag{2.26}$$

since

$$m(x) \le \|m\|_\infty, \qquad \text{for all } x\in\overline{\Omega}.$$

Introduce now the sector

$$\Sigma_1 = \Big\{\lambda\in\mathbb{C} : \mathrm{Re}\,\lambda + \frac{k_1(p)}{2}|\mathrm{Im}\,\lambda| + \frac{k_1(p)c_4}{2\|m\|_\infty} \ge 0\Big\}. \tag{2.27}$$

Then, for any $\lambda\in\Sigma_1$, from (2.16) and (2.26) we deduce the estimates

$$c_1\|u\|_p^p \le \mathrm{Re}\,(Lu,u|u|^{p-2}) \le \frac{k_1(p)+1}{h_1(p)}\|f\|_p\|u\|_p^{p-1}, \tag{2.28}$$

implying

$$\|u\|_p \le \frac{(k_1(p)+1)}{c_1h_1(p)}\|f\|_p. \tag{2.29}$$

Consequently,

$$\Big(\mathrm{Re}\,\lambda + k_1(p)|\mathrm{Im}\,\lambda| + \frac{k_1(p)c_4}{\|m\|_\infty}\Big)\int_\Omega m|u|^p\,dx$$
$$+h_1(p)\mathrm{Re}\,(Lu,u|u|^{p-2}) \le c_5(p)\|f\|_p^p. \tag{2.30}$$

Then, recalling that $\mathrm{Re}\,(Lu,u|u|^{p-2})$ is non-negative (cf. (2.16)) and observing that

$$|\lambda|+1 \le \Big(1+\frac{2c_4+2\|m\|_\infty}{c_4k_1(p)}\Big)\Big(\mathrm{Re}\,\lambda + k_1(p)|\mathrm{Im}\,\lambda| + \frac{c_4k_1(p)}{\|m\|_\infty}\Big), \qquad \lambda\in\Sigma_1, \tag{2.31}$$

(cf. Proposition 2.1 in [4]), we obtain

$$(|\lambda|+1)\int_\Omega m|u|^p\,dx + \mathrm{Re}\,(Lu,u|u|^{p-2}) \le c_6(p)\|f\|_p^p, \qquad \lambda\in\Sigma_1, \tag{2.32}$$

for some positive constant $c_6(p)$.

From Proposition 2.2 in [4] we deduce that $\lambda M + L$ is surjective on $L^p(\Omega)$.

Finally, from (2.32) we deduce the desired estimate

$$\|M(\lambda M+L)^{-1}f\|_{L^p(\Omega)} \le \frac{C}{(|\lambda|+1)^{1/p}}\|f\|_{L^p(\Omega)}, \qquad f\in L^p(\Omega),\ \lambda\in\Sigma_1. \tag{2.33}$$

We can now summarize the results proved in this section in Theorem 2.1.

Theorem 2.1. *Let L and M be the linear operators defined by (1.7) and (1.8), the coefficients $a_{i,j}$, a_i, a_0, $i,j=1,\dots,n$, enjoying properties (1.2), (1.3), (2.1), (2.2), (2.5) and m being a non-negative function in $L^\infty(\Omega)$. Then the spectral equation $\lambda Mu + Lu = f$, with $f\in L^p(\Omega)$, admits, for any $\lambda\in\Sigma_1 = \{\mu\in\mathbb{C} : \mathrm{Re}\,\mu + k_1(p)|\mathrm{Im}\,\mu|/2 + k_1(p)c_4/(2\|m\|_\infty)\ge 0\}$ and $p\in(1,+\infty)$, a unique solution $u\in D(L)$ satisfying the estimates*

$$\|u\|_p \le C_1(p)\|f\|_p, \qquad \|Mu\|_p \le C_2(p)|\lambda|^{-1/p}\|f\|_p, \qquad \lambda\in\Sigma_1,$$
$$\|Lu\|_p \le C_3(p)(1+|\lambda|)^{1/p'}\|f\|_p, \qquad \lambda\in\Sigma_1.$$

3. **The case when m is ρ-regular and $p \in [2, +\infty)$.** In this section we will assume that the multiplier m is more regular, i.e. it satisfies

$$m \in C^1(\overline{\Omega}), \qquad |\nabla m(x)| \le c_7 m(x)^\rho, \quad x \in \overline{\Omega}, \text{ for some } \rho \in (0,1). \tag{3.1}$$

We will show that our β can be chosen larger than $1/p$. We recall that the previous estimate (2.32) hold for any $p \in (1, +\infty)$.

First of all we state here Lemma 3.1 in [4] concerning the computation of the gradient of the function $\overline{u}|u|^{p-2}$ when $p \in [2, +\infty)$.

Lemma 3.1. *Let $u \in W_0^{1,p}(\Omega)$ ($u \in W^{1,p}(\Omega)$) with $p \in [2, +\infty)$. Then the function $\overline{u}|u|^{p-2}$ belongs to $W_0^{1,p}(\Omega)$ ($u \in W^{1,p}(\Omega)$) and the following formulae hold a.e. in Ω:*

$$D_{x_j}\overline{u}|u|^{p-2} = |u|^{p-2}D_{x_j}\overline{u} + (p-2)g_p(u)\mathrm{Re}\,(g_p(u)D_{x_j}u), \qquad j = 1, \dots, n, \tag{3.2}$$

where

$$g_p(u)(x) = \begin{cases} \overline{u(x)}|u(x)|^{(p-4)/2}, & \text{if } u(x) \ne 0, \\ 0, & \text{if } u(x) = 0. \end{cases} \tag{3.3}$$

Remark 1. From formula (3.3) we easily deduce the identity

$$|g_p(u)(x)| = |u(x)|^{(p-2)/2}. \tag{3.4}$$

We need also the following generalization of Lemma 3.2 in [4].

Lemma 3.2. *Let $(b_{i,j})_{i,j=1,\dots,n}$ be a matrix of functions in $C^1(\overline{\Omega}; \mathbb{R})$ and let $(b_i)_{i=1,\dots,n}$ a vector in $C(\overline{\Omega}; \mathbb{R})$ such that*

$$b_{i,j} = b_{j,i} \quad i,j = 1, \dots, n, \tag{3.5}$$

$$c_8|\xi|^2\mu(x) \le \sum_{i,j=1}^n b_{i,j}(x)\xi_i\xi_j \le c_9|\xi|^2\mu(x),$$
$$\text{for all } x \in \overline{\Omega}, \text{ for all } \xi \in \mathbb{R}^n, \tag{3.6}$$

$$\Big(\sum_{i=1}^n |b_i(x)|^2\Big)^{1/2} \le c_{10}\mu(x), \quad c_{11}\mu(x) \le b_0(x) - \frac{1}{p}\sum_{i=1}^n D_{x_i}b_i(x),$$
$$\text{for all } x \in \overline{\Omega}, \; i = 0, \dots, n, \tag{3.7}$$

$$0 \le b(x) + \frac{1}{p}\sum_{i=1}^n b_i(x)\nu_i(x), \qquad \text{for all } x \in \partial\Omega. \tag{3.8}$$

where $\mu \in C(\overline{\Omega})$ is a non-negative function and c_8, c_9, c_{10}, c_{11} are four positive constants. Then for any $p \in [2, +\infty)$, the linear operator $K = -\sum_{i,j=1}^n D_{x_i}[b_{i,j}(x)D_{x_j}] + \sum_{i=1}^n b_i(x)D_{x_i} + b_0(x)$ with $\mathcal{D}(K) = \mathcal{D}(L)$ (cf. (2.3) and (2.4)) satisfies the following relations with two positive constants c_{12} and

c_{13}:

$$c_8\Big(\int_\Omega \mu|u|^{p-2}|Du|^2\,dx+(p-2)\int_\Omega \mu\sum_{j=1}^n[\mathrm{Re}\,(g_p(u)D_{x_j}u)]^2\,dx\Big)$$

$$\le \mathrm{Re}\,(Ku,\overline{u}|u|^{p-2})-\int_\Omega\Big[b_0-\frac{1}{p}\sum_{i=1}^n D_{x_i}b_i\Big]|u|^p\,dx-\int_{\partial\Omega}\Big[b+\frac{1}{p}\sum_{i=1}^n b_i\nu_i\Big]|u|^p\,dS$$

$$\le c_9\Big(\int_\Omega \mu|u|^{p-2}|Du|^2\mathrm{d}x+(p-2)\int_\Omega \mu\sum_{j=1}^n[\mathrm{Re}\,(g_p(u)D_{x_j}u)]^2\,dx\Big), \tag{3.9}$$

$$|\mathrm{Im}\,(Ku,\overline{u}|u|^{p-2})|\le c_{12}\mathrm{Re}\,(Ku,\overline{u}|u|^{p-2})-c_{13}\int_\Omega \mu|u|^p\,dx. \tag{3.10}$$

Proof. First we deal with the case $p\in(2,+\infty)$. For any $\varepsilon>0$ define $a_{i,j,\varepsilon}=b_{i,j}+\varepsilon\delta_{i,j}$, $i,j=1,\ldots,n$, and set $K_\varepsilon=-\varepsilon\Delta+K$. Since the matrix $(a_{i,j,\varepsilon})_{i,j=1,\ldots,n}$ is uniformly positive definite, from Lemma 3.1 and an integration by parts we easily deduce the identity

$$\begin{aligned}(K_\varepsilon u,\overline{u}|u|^{p-2}) &= -\int_{\partial\Omega}\overline{u}|u|^{p-2}\sum_{i,j=1}^n a_{i,j,\varepsilon}\nu_j D_{x_i}u\,dS\\ &\quad+\int_\Omega\sum_{i,j=1}^n a_{i,j,\varepsilon}D_{x_i}uD_{x_j}(\overline{u}|u|^{p-2})\,dx+\int_\Omega\sum_{i=1}^n b_i\overline{u}|u|^{p-2}D_{x_i}u\,dx+\int_\Omega b_0|u|^p\,dx\\ &=\Big\{\int_\Omega\sum_{i,j=1}^n|u|^{p-2}a_{i,j,\varepsilon}D_{x_j}uD_{x_i}\overline{u}\,dx\\ &\qquad+(p-2)\int_\Omega\sum_{i,j=1}^n a_{i,j,\varepsilon}g_p(u)D_{x_j}u\mathrm{Re}\,(g_p(u)D_{x_i}u)\,dx\Big\}\\ &\quad+\Big\{\int_{\partial\Omega}\Big[b+\frac{1}{p}\sum_{i=1}^n b_i\nu_i\Big]|u|^p\,dS+\int_\Omega\Big[b_0-\frac{1}{p}\sum_{i=1}^n D_{x_i}b_i\Big]|u|^p\,dx\Big\}\\ &\quad+i\mathrm{Im}\int_\Omega\overline{u}|u|^{p-2}\sum_{i=1}^n b_iD_{x_i}u\,dx-\varepsilon\int_{\partial\Omega}\overline{u}|u|^{p-2}\sum_{i=1}^n\nu_iD_{x_i}u\,dS\\ &=: I_1(u,\varepsilon)+I_2(u)+iI_3(u)-\varepsilon\int_{\partial\Omega}\overline{u}|u|^{p-2}\sum_{i=1}^n\nu_iD_{x_i}u\,dS.\end{aligned} \tag{3.11}$$

Set now

$$I_0(u)=\int_\Omega\Big[|u|^{p-2}|\nabla u|^2+(p-2)\sum_{i=1}^n g_p(u)D_{x_i}u\mathrm{Re}\,(g_p(u)D_{x_i}u)\Big]\,dx \tag{3.12}$$

and observe that

$$I_1(u,\varepsilon)=I_1(u,0)+\varepsilon I_0(u). \tag{3.13}$$

Then from Lemma 3.1 in [9] we easily deduce

$$c_8 \int_\Omega \mu \Big[|u|^{p-2} |\nabla u|^2 + (p-2) \sum_{j=1}^n [\mathrm{Re}\,(g_p(u) D_{x_j} u)]^2 \Big]\, dx \tag{3.14}$$

$$\leq \mathrm{Re}\, I_1(u,\varepsilon) \leq \mathrm{Re}\, I_1(u,0) + \varepsilon \mathrm{Re}\, I_0(u), \tag{3.15}$$

$$|\mathrm{Im}\, I_1(u,\varepsilon)| = |\mathrm{Im}\, [I_1(u,0) + \varepsilon I_0(u)]| \leq \frac{|p-2|}{2\sqrt{p-1}} \mathrm{Re}\, [I_1(u,0) + \varepsilon I_0(u)]. \tag{3.16}$$

Taking the limit as $\varepsilon \to 0+$ in (3.14) and (3.16), we easily deduce the inequalities

$$c_8 \int_\Omega \mu \Big[|u|^{p-2} |\nabla u|^2 + (p-2) \sum_{j=1}^n [\mathrm{Re}\,(g_p(u) D_{x_j} u)]^2 \Big] dx \leq \mathrm{Re}\, I_1(u,0)$$

$$\leq c_9 \int_\Omega \mu \Big[|u|^{p-2} |\nabla u|^2 + (p-2) \sum_{j=1}^n [\mathrm{Re}\,(g_p(u) D_{x_j} u)]^2 \Big] dx, \tag{3.17}$$

$$|\mathrm{Im}\, I_1(u,0)| \leq \frac{|p-2|}{2\sqrt{p-1}} \mathrm{Re}\, I_1(u,0), \tag{3.18}$$

(3.17) being a consequence of the definition of $I_1(u,0)$ and (3.6).

Then, taking the limit as $\varepsilon \to 0+$ in (3.11), we get

$$(Ku, \overline{u}|u|^{p-2}) = I_1(u,0) + I_2(u) + i I_3(u). \tag{3.19}$$

To prove relations (3.9) and (3.10) we observe that

$$\mathrm{Re}\,(Ku, \overline{u}|u|^{p-2}) = \mathrm{Re}\, I_1(u,0) + I_2(u), \tag{3.20}$$

and thus (3.9) follows. Further we need the estimates

$$\begin{aligned} |I_3(u)| &\leq c_{14} \int_\Omega \mu |u|^{p/2} |u|^{(p-2)/2} |\nabla u|\, dx \\ &\leq c_{14} \Big(\int_\Omega \mu |u|^p\, dx \Big)^{1/2} \Big(\int_\Omega \mu |u|^{p-2} |\nabla u|^2\, dx \Big)^{1/2} \\ &\leq \frac{1}{2} c_{14} \varepsilon \int_\Omega \mu |u|^p\, dx + \frac{1}{2} c_{14} \varepsilon^{-1} \int_\Omega \mu |u|^{p-2} |\nabla u|^2\, dx. \end{aligned} \tag{3.21}$$

Since $I_2(u) \geq 0$, according to assumptions (3.7) and (3.8), from (3.21) we deduce

$$|\mathrm{Im}\,(Ku, \overline{u}|u|^{p-2})| \leq |\mathrm{Im}\, I_1(u,0)| + |I_3(u)|$$

$$\leq \frac{|p-2|}{2\sqrt{p-1}} \mathrm{Re}\, I_1(u,0) + \frac{1}{2} c_{14} \varepsilon \int_\Omega \mu |u|^p\, dx + \frac{1}{2} c_{14} \varepsilon^{-1} \int_\Omega \mu |u|^{p-2} |\nabla u|^2\, dx$$

$$\leq \Big(\frac{|p-2|}{2\sqrt{p-1}} + \frac{c_{14}}{2c_8} \varepsilon^{-1} \Big) \mathrm{Re}\, I_1(u,0) + \frac{1}{2} c_{14} \varepsilon \int_\Omega \mu |u|^p\, dx$$

$$\leq c_{12} \mathrm{Re}\,(Ku, \overline{u}|u|^{p-2}) - \Big(\frac{|p-2|}{2\sqrt{p-1}} c_{11} - \frac{1}{2} c_{14} \varepsilon \Big) \int_\Omega \mu |u|^p\, dx, \tag{3.22}$$

where

$$c_{12} := \frac{|p-2|}{2\sqrt{p-1}} + \frac{c_{14}}{2c_8} \varepsilon^{-1}. \tag{3.23}$$

Assume $p \in (2, +\infty)$ and choose now $\varepsilon > 0$ so small that

$$c_{13} := \frac{|p-2|}{2\sqrt{p-1}} c_{11} - \frac{1}{2} c_{14}\varepsilon > 0. \tag{3.24}$$

This implies estimate (3.10).

Finally, note that relations (3.9) and (3.10), with $p = 2$, easily follow from the identity

$$\begin{aligned}(Ku, \overline{u}) = \quad & \int_\Omega \sum_{i,j=1}^n b_{i,j} D_{x_i} u D_{x_j} \overline{u} + \int_\Omega \Big[b_0 - \frac{1}{2} \sum_{i=1}^n D_{x_i} b_i \Big] |u|^2 \, dx \\ & + \int_{\partial\Omega} \Big[b + \frac{1}{2} \sum_{i=1}^n b_i \nu_i \Big] |u|^2 \, dS,\end{aligned}$$

and our assumptions on the coefficients. In this case, since $\operatorname{Im}(Ku, \overline{u}) = 0$, we can choose, e.g., $c_{12} = 1$ and $c_{13} = c_{11}$. Indeed, since

$$\operatorname{Re}(Ku, \overline{u}) \geq c_8 \int_\Omega \mu |\nabla u|^2 \, dx + c_{11} \int_\Omega \mu |u|^2 \, dx$$

we obtain

$$\operatorname{Re}(Ku, \overline{u}) - c_{11} \int_\Omega \mu |u|^2 \, dx \geq c_8 \int_\Omega \mu |\nabla u|^2 \, dx \geq 0 = |\operatorname{Im}(Ku, \overline{u})|.$$

□

To apply the previous result to our case we shall use also the following identity

$$\begin{aligned}&(Lu, m^{p-1} u |u|^{p-2}) = (m^{p-1} Lu, u|u|^{p-2}) \\ &= (Ku, u|u|^{p-2}) + (p-1)\Big(m^{p-2} \sum_{i,j=1}^n a_{i,j} D_{x_i} m D_{x_j} u, u|u|^{p-2} \Big), \quad u \in \mathcal{D}(L),\end{aligned} \tag{3.25}$$

where

$$\begin{aligned}K = & - \sum_{i,j=1}^n D_{x_i} [m(x)^{p-1} a_{i,j}(x) D_{x_j}] \\ & + \Big[\sum_{i=1}^n m(x)^{p-1} a_i(x) \Big] D_{x_i} + m(x)^{p-1} a_0(x).\end{aligned} \tag{3.26}$$

We now set

$$\mu(x) = m(x)^{p-1}, \quad b_0(x) = \mu(x) a_0(x), \quad b_{i,j}(x) = \mu(x) a_{i,j}(x), \quad i, j = 1, \ldots, n, \tag{3.27}$$

$$b_i(x) = \mu(x) a_i(x), \quad i = 1, \ldots, n. \tag{3.28}$$

and we assume that the following inequalities hold for all $x \in \overline{\Omega}$ and all $x \in \partial\Omega$, respectively:

$$m(x)^{p-1}\Big(a_0(x) - \frac{1}{p}\sum_{i=1}^{n} D_{x_i}a_i(x)\Big) - \frac{p-1}{p}m^{p-2}(x)\sum_{i=1}^{n} a_i(x)D_{x_i}m(x)$$

$$\geq c_{15}m^{p-1}(x), \tag{3.29}$$

$$b(x) + \frac{1}{p}m(x)^{p-1}\sum_{i=1}^{n} a_i(x)\nu_i(x) \geq 0. \tag{3.30}$$

Then all conditions (3.5)–(3.10) are satisfied.

Remark 2. Condition (3.29) is surely satisfied if we assume

$$a_0(x) - \frac{1}{p}\sum_{i=1}^{n} D_{x_i}a_i(x) \geq c_1, \quad \sum_{i=1}^{n} a_i(x)(x)D_{x_i}m(x) \leq 0, \quad \text{for all } x \in \overline{\Omega}. \tag{3.31}$$

Let now u be a solution to equation (2.20). Taking the scalar product of both sides in (2.6) with $m^{p-1}u|u|^{p-2}$ and using (3.25), we easily get the equalities

$$(f, m^{p-1}u|u|^{p-2}) = (\lambda mu + Lu, m^{p-1}u|u|^{p-2})$$

$$= \lambda\|Mu\|_p^p + (Ku, u|u|^{p-2}) + (p-1)\Big(m^{p-2}\sum_{i,j=1}^{n} a_{i,j}D_{x_i}mD_{x_j}u, u|u|^{p-2}\Big). \tag{3.32}$$

Taking the real and imaginary parts in (3.32) and using (3.10), we easily deduce the inequalities

$$\operatorname{Re}\lambda\|Mu\|_p^p + \operatorname{Re}(Ku, u|u|^{p-2})$$

$$\leq |(f, m^{p-1}u|u|^{p-2})| + (p-1)\Big|\Big(m^{p-2}\sum_{i,j=1}^{n} a_{i,j}D_{x_i}mD_{x_j}u, u|u|^{p-2}\Big)\Big|, \tag{3.33}$$

$$|\operatorname{Im}\lambda|\|Mu\|_p^p \leq |\operatorname{Im}(Ku, u|u|^{p-2})|$$

$$+|(f, m^{p-1}u|u|^{p-2})| + (p-1)\Big|\Big(m^{p-2}\sum_{i,j=1}^{n} a_{i,j}D_{x_i}mD_{x_j}u, u|u|^{p-2}\Big)\Big|$$

$$\leq c_{12}\operatorname{Re}(Ku, u|u|^{p-2}) - c_{13}\int_\Omega m^{p-1}|u|^p\,dx + |(f, m^{p-1}u|u|^{p-2})|$$

$$+(p-1)\Big|\Big(m^{p-2}\sum_{i,j=1}^{n} a_{i,j}D_{x_i}mD_{x_j}u, u|u|^{p-2}\Big)\Big|. \tag{3.34}$$

Multiply now by a (fixed) positive constant $k_2(p) \in (0, c_{12}^{-1})$ the first and last sides in (3.34) and add to the first and last sides in (3.33). We get the estimate

$$\begin{aligned}
&[\mathrm{Re}\,\lambda + k_2(p)|\mathrm{Im}\,\lambda| + c_{13}k_2(p)\|m\|_\infty^{-1}]\|Mu\|_p^p \\
&+\big(1-k_2(p)c_{12}\big)\mathrm{Re}\,(Ku, u|u|^{p-2}) \\
\leq\quad &[1+k_2(p)]\Big\{|(f, m^{p-1}u|u|^{p-2})| \\
&\qquad +(p-1)\Big|\Big(m^{p-2}\sum_{i,j=1}^n a_{i,j}D_{x_i}mD_{x_j}u, u|u|^{p-2}\Big)\Big|\Big\},
\end{aligned}\tag{3.35}$$

where we have made use of the elementary inequality

$$m(x)^p \leq \|m\|_\infty m(x)^{p-1}, \qquad \text{for all } x \in \overline{\Omega}.$$

We now estimate the last term in (3.35) with the aid of (1.9). Using twice Hölder's inequality, we get

$$\Big|\Big(m^{p-2}\sum_{i,j=1}^n a_{i,j}D_{x_i}mD_{x_j}u, u|u|^{p-2}\Big)\Big| \leq \int_\Omega m^{p-2}|u|^{p-1}\Big|\sum_{i,j=1}^n a_{i,j}D_{x_i}mD_{x_j}u\Big|\,dx$$

$$\leq c_{16}\int_\Omega m^{p-2+\rho}|u|^{p-1}|\nabla u|\,dx = c_{16}\int_\Omega m^{p\rho/2}|u|^{p/2}m^{(p-2)(2-\rho)/2}|u|^{-1+p/2}|\nabla u|\,dx$$

$$\leq c_{16}\Big(\int_\Omega m^{p\rho}|u|^{p\rho}|u|^{p(1-\rho)}\,dx\Big)^{1/2}\Big(\int_\Omega m^{(p-2)(2-\rho)}|u|^{p-2}|\nabla u|^2\,dx\Big)^{1/2}$$

$$\leq c_{16}\|Mu\|_p^{p\rho/2}\|u\|_p^{(1-\rho)p/2}\|m\|_\infty^{(p-2)(2-\rho)/2}\Big(\int_\Omega |u|^{p-2}|\nabla u|^2\,dx\Big)^{1/2}. \tag{3.36}$$

On account of (2.13), with $p \in [2, +\infty)$, we easily observe the estimate

$$\int_\Omega |u|^{p-2}|\nabla u|^2\,dx \leq c_9(p)\|f\|_p^p. \tag{3.37}$$

From (2.23), (3.36) and (3.37) we finally deduce the estimates

$$\Big|\Big(m^{p-2}\sum_{i,j=1}^n a_{i,j}D_{x_i}mD_{x_j}u, u|u|^{p-2}\Big)\Big| \leq c_{17}(p)\|f\|_p^{p(2-\rho)/2}\|Mu\|_p^{p\rho/2}. \tag{3.38}$$

Moreover, we have

$$|(f, m^{p-1}\overline{u}|u|^{p-2})| \leq \|f\|_p\|Mu\|_p^{p-1}. \tag{3.39}$$

Finally, from (3.35), (3.38), (3.39) we deduce the inequality

$$[\mathrm{Re}\,\lambda + k_2(p)|\mathrm{Im}\,\lambda| + c_{13}k_2(p)\|m\|_\infty^{-1}]\|Mu\|_p^p + \big(1-k_2(p)c_{12}\big)\mathrm{Re}\,(Ku, u|u|^{p-2})$$

$$\leq c_{18}(p)[\|f\|_p\|Mu\|_p^{p-1} + \|f\|_p^{p(2-\rho)/2}\|Mu\|_p^{p\rho/2}], \qquad \text{for all } \lambda \in \Sigma_1. \tag{3.40}$$

We now introduce the sector

$$\Sigma_2 = \{\lambda \in \mathbb{C} :\ \mathrm{Re}\,\lambda + \frac{k_2(p)}{2}|\mathrm{Im}\,\lambda| + \frac{c_{13}k_2(p)}{2\|m\|_\infty} \geq 0\}.$$

Choose now $k = k_2(p)$ so small as to satisfy

$$0 < k_2(p) \leq \min\{1/c_{10}, k_1(p), c_4k_1(p)/c_{11}\}, \qquad \text{for all } p \in (1, +\infty). \tag{3.41}$$

Due to this choice we immediately deduce the inclusion $\Sigma_2 \subset \Sigma_1$ (cf. (2.30)).

Then, recalling that $\mathrm{Re}\,(Ku, u|u|^{p-2})$ is non-negative (cf. Lemma 3.2) and observing that

$$|\lambda| + 1 \le \Big(1 + \frac{2(c_{13} + \|m\|_\infty)}{c_{13}k_2(p)}\Big)\Big(\mathrm{Re}\,\lambda + k_2(p)|\mathrm{Im}\,\lambda| + \frac{c_{13}k_2(p)}{\|m\|_\infty}\Big) \tag{3.42}$$

(cf. Proposition 2.1 in [4]), we obtain

$$\begin{aligned} &(|\lambda| + 1)\|Mu\|_p^p + \mathrm{Re}\,(Ku, u|u|^{p-2}) \\ \le\ & c_{19}(p)[\|f\|_p\|Mu\|_p^{p-1} + \|f\|_p^{p(2-\rho)/2}\|Mu\|_p^{p\rho/2}], \qquad \text{for all } \lambda \in \Sigma_2. \end{aligned} \tag{3.43}$$

Consequently, since $\|u\|_p \le C_1(p)\|f\|_p$ (cf. Theorem 2.1), (3.34) and (3.42) imply

$$\begin{aligned} (|\lambda| + 1)\|Mu\|_p^{p(2-\rho)/2} \le\ & c_{20}(p)[\|f\|_p\|Mu\|_p^{p-1-p\rho/2} + \|f\|_p^{p(2-\rho)/2}], \\ & \text{for all } \lambda \in \Sigma_2. \end{aligned} \tag{3.44}$$

Since $\lambda M + L$ is surjective on $L^p(\Omega)$, estimate (2.24) holds with $\alpha = 1$ and $\beta = 2[p(2-\rho)]^{-1}$.

We can summarize the results in this section in Theorem 3.3.

Theorem 3.3. *Let L and M be the linear operators defined by (2.3) or (2.4) and by $Mu = mu$, the coefficients $a_{i,j}$, a_i, a_0, $i, j = 1, \dots, n$, enjoying properties (1.2), (1.3), (2.1), (2.2), (2.5), (3.29), (3.30) for some non-negative function $m \in C^1(\overline{\Omega})$ satisfying (3.1). Then the spectral equation $\lambda Mu + Lu = f$, with $f \in L^p(\Omega)$, admits, for any $\lambda \in \Sigma_2 = \{\mu \in \mathbb{C} : \mathrm{Re}\,\mu + \frac{1}{2}k_2(p)|\mathrm{Im}\,\mu| + \frac{1}{2}k_2(p)c_3\|m\|_\infty^{-1} \ge 0\}$ and $p \in [2, +\infty)$, a unique solution $u \in D(L)$ satisfying the estimates*

$$\|u\|_p \le C_1(p)\|f\|_p, \qquad \|Mu\|_p \le C_4(p)|\lambda|^{-2/[p(2-\rho)]}\|f\|_p, \qquad \lambda \in \Sigma_2,$$

$$\|Lu\|_p \le C_5(p)(1 + |\lambda|^{[p(2-\rho)-2]/[p(2-\rho)]})\|f\|_p, \qquad \lambda \in \Sigma_2,$$

for some positive constants $C_4(p)$ and $C_5(p)$.

Example 1. Let Ω be a bounded domain and let x_0 be a fixed point in $\partial\Omega$. Define then $r = \max_{x\in\overline{\Omega}} |x - x_0|$ and choose

$$m(x) = [(|x - x_0|(r - |x - x_0|)]^q, \qquad q \in (1, +\infty).$$

An elementary computation shows that

$$|\nabla m(x)| = q[|x - x_0|(r - |x - x_0|)]^{q-1}|2|x - x_0| - r| \le qrm(x)^{(q-1)/q}, \quad x \in \Omega.$$

Consequently, function m satisfies condition (3.1).
We notice that for *any* open interval $\Omega \subset \mathbb{R}$ we have $r = \mathrm{length}(\Omega)$.

4. **The case when $p \in (1, 2)$.** In this section we focus our attention to the case when $m \in W^{1,\infty}(\Omega)$ satisfies inequality (3.1) with

$$\rho \in (2 - p, 1]. \tag{4.1}$$

Multiplying both sides in (2.20) by $m^{p-1}\overline{u}|u|^{p-2}$ and integrating over Ω, we easily get

$$\lambda\|Mu\|_p^p - \lim_{\delta\to 0+}\int_\Omega m^{p-1}\overline{u}(|u|^2+\delta)^{(p-2)/2}\sum_{j,k=1}^n D_{x_j}[a_{j,k}D_{x_k}u]\,dx$$

$$+\int_\Omega m^{p-1}\overline{u}|u|^{p-2}\sum_{i=1}^n a_i D_{x_i}u\,dx + \int_\Omega a_0 m^{p-1}|u|^p\,dx = \int_\Omega f m^{p-1}\overline{u}|u|^{p-2}\,dx, \quad (4.2)$$

where $\overline{u}|u|^{p-2}$ stands for the function vanishing where u does.

We need now Proposition 4.1 in [4] that we restate here for the the convenience of the reader.

Proposition 1. *Let m satisfy property (3.1). Then for any $\beta\in(1-\rho,1)$, the function $m(\cdot)^\beta$ belongs to $C^1(\overline{\Omega})$ and $\nabla[m(\cdot)^\beta]=m_1$ for any $x\in\overline{\Omega}$, where*

$$m_1 = \begin{cases} 0, & x\in Z(m), \\ \beta m^{\beta-1}\nabla m, & x\notin Z(m), \end{cases} \quad (4.3)$$

and $Z(m)$ denotes the zero-set of m. Moreover,

$$|\nabla[m(\cdot)^\beta]| \le c_{21} m(\cdot)^{\beta-1+\rho}, \qquad \text{for all } x\in\overline{\Omega}.$$

An integration by parts in the first integral, which takes into account (4.1), (4.2) and the Robin condition (2.4), easily yields

$$-\lim_{\delta\to 0+}\int_\Omega m^{p-1}\overline{u}(|u|^2+\delta)^{(p-2)/2}\sum_{j,k=1}^n D_{x_j}[a_{j,k}D_{x_k}u]\,dx$$

$$=\lim_{\delta\to 0+}\int_{\partial\Omega} bm^{p-1}|u|^2(|u|^2+\delta)^{(p-2)/2}\,dS$$

$$+\lim_{\delta\to 0+}\Big\{\int_\Omega (|u|^2+\delta)^{(p-2)/2}m^{p-1}\sum_{j,k=1}^n a_{j,k}D_{x_j}\overline{u}D_{x_k}u\,dx$$

$$+(p-1)\int_\Omega \overline{u}(|u|^2+\delta)^{(p-2)/2}\sum_{j,k=1}^n m^{p-2}D_{x_j}m a_{j,k}D_{x_k}u\,dx$$

$$+(p-2)\int_\Omega m^{p-1}(|u|^2+\delta)^{(p-4)/2}\sum_{j,k=1}^n a_{j,k}\mathrm{Re}\left(\overline{u}D_{x_j}u\right)\overline{u}D_{x_k}u\,dx\Big\}$$

$$=:\int_{\partial\Omega} bm^{p-1}|u|^p\,dS + I_1(u,\delta)+(p-1)I_2(u,\delta)-(2-p)I_3(u,\delta). \quad (4.4)$$

Using again proposition 1 and assumption (4.1), by an integration by parts we get

$$\int_\Omega m^{p-1}\sum_{i=1}^n a_i\overline{u}|u|^{p-2}D_{x_i}u\,dx+\int_\Omega m^{p-1}a_0|u|^p\,dx=\frac{1}{p}\int_{\partial\Omega}m^{p-1}|u|^p\sum_{i=1}^n a_i\nu_i\,dS$$

$$+\int_\Omega\Big[m^{p-1}\Big(a_0-\frac{1}{p}\sum_{i=1}^n D_{x_i}a_i\Big)-\frac{p-1}{p}\sum_{i=1}^n a_i m^{p-2}D_{x_i}m\Big]|u|^p\,dx$$

$$+i\mathrm{Im}\int_\Omega m^{p-1}\overline{u}|u|^{p-2}\sum_{i=1}^n a_iD_{x_i}u\,dx. \tag{4.5}$$

Consequently, equation (4.2) can be rewritten in the form

$$\lambda\|Mu\|_p^p+\lim_{\delta\to0+}\big[I_1(u,\delta)+(p-1)I_2(u,\delta)-(2-p)I_3(u,\delta)\big]$$

$$+\int_{\partial\Omega}m^{p-1}\Big[b+\frac{1}{p}\sum_{i=1}^n a_i\nu_i\Big]|u|^p\,dS$$

$$+\int_\Omega\Big[m^{p-1}\Big(a_0-\frac{1}{p}\sum_{i=1}^n D_{x_i}a_i\Big)-\frac{p-1}{p}\sum_{i=1}^n a_im^{p-2}D_{x_i}m\Big]|u|^p\,dx$$

$$+i\mathrm{Im}\int_\Omega m^{p-1}\overline{u}|u|^{p-2}\sum_{i=1}^n a_iD_{x_i}u\,dx=\int_\Omega fm^{p-1}\overline{u}|u|^{p-2}\,dx. \tag{4.6}$$

Since the matrix $(a_{j,k})_{j,k=1,\ldots,n}$ is real-valued and positive definite, from (4.4) we immediately deduce that

$$I_1(u,\delta)\ \text{ and }\ \mathrm{Re}\,I_3(u,\delta)\ \text{ are positive for any }\ \delta\in\mathbb{R}_+. \tag{4.7}$$

Then we observe that $I_2(u,\delta)$ has a limit as $\delta\to0+$ and

$$\lim_{\delta\to0+}I_2(u,\delta)=\int_\Omega\overline{u}|u|^{p-2}\sum_{j,k=1}^n m^{p-2}D_{x_j}ma_{j,k}D_{x_k}u\,dx. \tag{4.8}$$

Note that the integral in the right-hand side is well-defined on the whole of $W^{1,p}(\Omega)$ since $\overline{u}|u|^{p-2}\in L^{p'}(\Omega)$, $D_{x_j}u\in L^p(\Omega)$ and $m^{p-2}D_{x_j}m\in L^\infty(\Omega)$, due to Proposition 1, with $\beta=p-1$, and assumption (4.1).

Further, (4.6) and (4.8) imply that there exists also $\lim_{\delta\to0+}[I_1(u,\delta)-(2-p)I_3(u,\delta)]$. Whence we deduce that there exist the limits

$$\lim_{\delta\to0+}\mathrm{Im}\,I_3(u,\delta)\quad\text{and}\quad\lim_{\delta\to0+}[I_1(u,\delta)-(2-p)\mathrm{Re}\,I_3(u,\delta)],$$

We now restate here Lemma 4.1 in [4].

Lemma 4.1. *The following estimates hold for any $\delta \in \mathbb{R}_+$, $p \in (1,2)$ and $\sigma \in (0, 2(p-1)(2-p)^{-1})$:*

$$I_1(u,\delta) - (2-p)\mathrm{Re}\, I_3(u,\delta) - \sigma(2-p)|\mathrm{Im}\, I_3(u,\delta)| \geq 0, \tag{4.9}$$

$$\begin{aligned} &I_1(u,\delta) + (p-1)\mathrm{Re}\, I_2(u,\delta) - (2-p)\mathrm{Re}\, I_3(u,\delta) \\ &-\sigma|(p-1)\mathrm{Im}\, I_2(u,\delta) - (2-p)\mathrm{Im}\, I_3(u,\delta)| \\ &\geq -(p-1)(1+\sigma^2)^{1/2}|I_2(u,\delta)|, \end{aligned} \tag{4.10}$$

$$\begin{aligned} &\lim_{\delta\to 0+} \big[I_1(u,\delta) + (p-1)\mathrm{Re}\, I_2(u,\delta) - (2-p)\mathrm{Re}\, I_3(u,\delta)\big] \\ &-\sigma \lim_{\delta\to 0+} |(p-1)\mathrm{Im}\, I_2(u,\delta) - (2-p)\mathrm{Im}\, I_3(u,\delta)| \\ &\geq -c_{22}\|f\|_p^{p/2}\|Mu\|_p^{p-2+\rho}\|u\|_p^{2-\rho-p/2}, \end{aligned} \tag{4.11}$$

c_{22} being a suitable positive constant.

Taking now the real part and the modulus of the imaginary part in (4.2) and using (4.4), we easily derive the relations

$$\begin{aligned} &\mathrm{Re}\,\lambda\|Mu\|_p^p + \lim_{\delta\to 0+}[I_1(u,\delta) + (p-1)\mathrm{Re}\, I_2(u,\delta) - (2-p)\mathrm{Re}\, I_3(u,\delta)] \\ &\quad + \int_{\partial\Omega} m^{p-1}\Big[b + \frac{1}{p}\sum_{i=1}^n a_i\nu_i\Big]|u|^p\, dS \\ &\quad + \int_\Omega \Big[m^{p-1}\Big(a_0 - \frac{1}{p}\sum_{i=1}^n D_{x_i}a_i\Big) - \frac{p-1}{p}\sum_{i=1}^n a_i m^{p-2}D_{x_i}m\Big]|u|^p\, dx \\ &= \mathrm{Re}\int_\Omega m^{p-1}f\overline{u}|u|^{p-2}\, dx, \end{aligned} \tag{4.12}$$

$$\begin{aligned} &|\mathrm{Im}\,\lambda|\|Mu\|_p^p \leq \lim_{\delta\to 0+} |[(p-1)\mathrm{Im}\, I_2(u,\delta) - (2-p)\mathrm{Im}\, I_3(u,\delta)]| \\ &+\Big|\mathrm{Im}\int_\Omega m^{p-1}\overline{u}|u|^{p-2}\sum_{i=1}^n a_iD_{x_i}u\, dx\Big| + \Big|\mathrm{Im}\int_\Omega m^{p-1}f\overline{u}|u|^{p-2}\, dx\Big|, \qquad \text{for all } \lambda \in \mathbb{C}. \end{aligned} \tag{4.13}$$

Assume now that inequalities (3.29) and (3.30) hold. Add then member by member (4.12) and (4.13) multiplied by $k_3(p) \in (0, 2\sqrt{p-1}(2-p)^{-1})$ and use (4.11) and (2.2). Then from Lemma 4.1 we easily deduce the following estimate for any $\lambda \in \Sigma =: \{\mu \in \mathbb{C} : \operatorname{Re}\mu + k_3(p)|\operatorname{Im}\mu| \geq 0\}$:

$$\Big[\operatorname{Re}\lambda + k_3(p)|\operatorname{Im}\lambda| + \frac{c_{15}}{\|m\|_\infty}\Big]\|Mu\|_p^p$$

$$\leq -\lim_{\delta\to 0+}\,[I_1(u,\delta) + (p-1)\operatorname{Re} I_2(u,\delta) - (2-p)\operatorname{Re} I_3(u,\delta)]$$

$$- k_3(p)\lim_{\delta\to 0+}|[(p-1)\operatorname{Im} I_2(u,\delta) - (2-p)\operatorname{Im} I_3(u,\delta)]|\Big]$$

$$+ \operatorname{Re}\int_\Omega f m^{p-1}\overline{u}|u|^{p-2}\,dx + k_3(p)\Big|\operatorname{Im}\int_\Omega f m^{p-1}\overline{u}|u|^{p-2}\,dx\Big|$$

$$\leq c_{22}\|f\|_p^{p/2}\|Mu\|_p^{p-2+\rho}\|u\|_p^{2-\rho-p/2} + \Big|\operatorname{Re}\int_\Omega f m^{p-1}\overline{u}|u|^{p-2}\,dx\Big|$$

$$+ k_3(p)\Big|\operatorname{Im}\int_\Omega f m^{p-1}\overline{u}|u|^{p-2}\,dx\Big| + k_3(p)\Big|\operatorname{Im}\int_\Omega m^{p-1}\overline{u}|u|^{p-2}\sum_{i=1}^n a_i D_{x_i}u\,dx\Big|$$

$$\leq c_{23}\|f\|_p^{p/2}\|Mu\|_p^{p-2+\rho}\|u\|_p^{2-\rho-p/2} + (1+k_3(p)^2)^{1/2}\|f\|_p\|Mu\|_p^{p-1}. \tag{4.14}$$

Indeed, the last term in the penultimate line can be estimated as follows:

$$\Big|\operatorname{Im}\int_\Omega m^{p-1}\overline{u}|u|^{p-2}\sum_{i=1}^n a_i D_{x_i}u\,dx\Big| \leq c_{24}\int_\Omega m^{p-1}|u|^{p-1}|\nabla u|\,dx$$

$$= c_{24}\lim_{\delta\to 0}\int_\Omega m^{1-\rho}m^{p-2+\rho}|u|^{p-2+\rho}|u|^{2-\rho-p/2}(|u|^2+\delta)^{(p-2)/4}|\nabla u|\,dx$$

$$\leq c_{24}\|m\|_\infty^{1-\rho}\lim_{\delta\to 0}\int_\Omega (m|u|)^{p-2+\rho}|u|^{2-\rho-p/2}(|u|^2+\delta)^{(p-2)/4}|\nabla u|\,dx$$

$$\leq c_{24}\|m\|_\infty^{1-\rho}\|Mu\|_p^{p-2+\rho}\|u\|_p^{2-\rho-p/2}\lim_{\delta\to 0}\Big(\int_\Omega (|u|^2+\delta)^{(p-2)/2}|\nabla u|^2\,dx\Big)^{1/2}.$$

By virtue of the proof of (2.32) we obtain

$$(p-1)c_0\lim_{\delta\to 0+}\int_\Omega (|u|^2+\delta)^{(p-2)/2}|\nabla u|^2\,dx \leq c_6^{1-p}\left(\frac{(k_1(p)+1)}{h_1(p)}\right)^p\|f\|_p^p.$$

Thus

$$\Big|\operatorname{Im}\int_\Omega m^{p-1}\overline{u}|u|^{p-2}\sum_{i=1}^n a_i D_{x_i}u\,dx\Big| \leq c_{25}\|Mu\|_p^{p-2+\rho}\|u\|_p^{2-\rho-p/2}\|f\|_p^{p/2}.$$

Take now λ in the sector

$$\Sigma_3 = \Big\{\mu \in \mathbb{C} : \operatorname{Re}\mu + \frac{k_3(p)}{2}|\operatorname{Im}\mu| + \frac{c_{15}}{2\|m\|_\infty} \geq 0\Big\}. \tag{4.15}$$

Then, since $\|u\|_p \leq C_1\|f\|_p$ (cf. (2.11), (2.12) and our definition of $k_3(p)$) and $2-\rho-p/2 > 0$ (cf. (4.1)), we immediately derive the inequality

$$(|\lambda|+1)\|Mu\|_p^{2-\rho} \leq c_{26}\big[\|f\|_p^{2-\rho} + \|f\|_p\|Mu\|_p^{1-\rho}\big], \quad \text{if} \quad \lambda \in \Sigma_3. \tag{4.16}$$

Finally, $\|Mu\|_p \leq \|m\|_\infty\|u\|_p \leq c_{27}\|m\|_\infty\|f\|_p$ implies

$$(|\lambda|+1)\|Mu\|_p^{2-\rho} \leq c_{28}\|f\|_p^{2-\rho}, \quad \text{if} \quad \lambda \in \Sigma_3. \tag{4.17}$$

We can now collect the result in this section in the following Theorem 4.1.

Theorem 4.2. *Let L and M be the linear operators defined by (2.3) or (2.4) and by $Mu = mu$, the coefficients $a_{i,j}$, a_i, a_0, $i, j = 1, \dots, n$, enjoying properties (1.2), (1.3), (2.1), (2.2), (2.3), (3.29), (3.30) for some non-negative function $m \in C^1(\overline{\Omega})$ satisfying (3.1)). Then the spectral equation $\lambda Mu + Lu = f$, with $f \in L^p(\Omega)$, admits, for any $\lambda \in \Sigma_3 = \{\mu \in \mathbb{C} : \operatorname{Re}\mu + k_3(p)|\operatorname{Im}\mu|/2 + c_{15}(2\|m\|_\infty)^{-1} \geq 0\}$ and $p \in (1, 2)$, $\rho \in (2-p, 1)$, a unique solution $u \in D(L)$ satisfying the estimates*

$$\|u\|_p \leq C_1\|f\|_p, \qquad \|Mu\|_p \leq C_6(p)|\lambda|^{-(2-\rho)^{-1}}\|f\|_p, \qquad \text{for all } \lambda \in \Sigma_3,$$

$$\|Lu\|_p \leq C_7(1 + |\lambda|^{(1-\rho)(2-\rho)^{-1}})\|f\|_p, \qquad \text{for all } \lambda \in \Sigma_3. \tag{4.18}$$

Example 2. Let $n = 1$, $m(x) = x^q(1-x)^q$, $q \in (1, +\infty)$, $\Omega = (0, 1)$. Then

$$m'(x) = q(1-2x)m^{(q-1)/q}, \qquad \text{for all } x \in (0, 1).$$

Hence (3.1) holds true for any $q \in (1, +\infty)$. If we have to deal with $L^p(0, 1)$ with $p \in (1, 2)$, to satisfy (4.1) we are forced to assume $q > (p-1)^{-1}$.

5. **Solving singular parabolic problems.** Taking the spectral Theorems 2.1, 3.1, 4.1 into account, from Theorem 3.26 in [5] we can easily derive our existence and uniqueness result. For this purpose we need to introduce the following Banach space contained into $(X; D(LM^{-1}))_{\theta,\infty}$:

$$L^p_{\theta,\infty} = \Big\{ g \in L^p(\Omega) : \sup_{t \geq 1} t^\theta \|L(tM + L)^{-1}g\|_{L^p(\Omega)} < +\infty \Big\}. \tag{5.1}$$

In particular, any $g = mh$ belongs to $L^p_{\theta,\infty}$, whenever $m \in L^\infty(\Omega)$ and $h \in D(L)$, if $1 - \beta < \theta < \beta$ with $1/2 < \beta \leq 1$, while when $\beta \leq \theta < 1$ function $g = mh$ belongs to $L^p_{\theta,\infty}$ if $Lh = Mk$ for some $k \in D(L)$.

Consider now the initial and boundary value problem

$$\text{(P)} \quad \begin{cases} D_t[m(x)u(x,t)] + \mathcal{L}u(x,t) = f(x,t), & \text{for all } (x,t) \in \Omega \times [0,\tau], \\ \eta \sum_{i,j=1}^n a_{i,j}(x)\nu_j(x)\, D_{x_i}u(x,t) + [\eta(b(x)-1)+1]u(x,t) = 0, \\ \quad \text{for all } (x,t) \in \partial\Omega \times [0,\tau], \\ m(x)u(x,t) \to m(x)u_0(x), & \text{for almost every } x \in \Omega, \text{ as } t \to 0+, \end{cases}$$

where $\eta \in \{0, 1\}$.

Note that the choice $\eta = 0$ corresponds to Dirichlet boundary conditions, while the choice $\eta = 1$ does to Robin boundary conditions.

Theorem 5.1. *Let $p \in (1, +\infty)$, let $m \in L^\infty(\Omega)$ be a non-negative function and let the coefficients $a_{i,j}$ $i, j = 1, \dots, n$, a_0 enjoy properties (2.1). Further, when $\eta = 1$, let coefficient b satisfy conditions (2.2) and (2.5). Then for any*

$$u_0 \in D(L), \quad f \in C^\theta([0,T]; L^p(\Omega)), \quad \theta \in (1-\beta, 1), \tag{5.2}$$

with $\beta = 1/p$ and

$$-\mathcal{L}u_0 + f(0, \cdot) = g_0, \qquad g_0 \in L^p_{\theta,\infty}, \tag{5.3}$$

problem (P)*, with $\eta \in \{0, 1\}$, admits a unique solution*

$$mu \in C^{\theta+\beta}([0,T]; L^p(\Omega)), \qquad u \in C^{\theta+\beta-1}([0,T]; D(L)). \tag{5.4}$$

Moreover, if m is a non-negative function satisfying (3.1) and

$$\beta = \begin{cases} (2-\rho)^{-1}, & \text{if } p \in (1,2),\ \rho \in (2-p,1], \\ 2[p(2-\rho)]^{-1}, & \text{if } p \in [2,+\infty),\ \rho \in (0,1], \end{cases} \tag{5.5}$$

the same result holds under assumptions (5.1)–(5.3) on (u_0, f).

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