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NECESSARY AND SUFFICIENT CONDITIONS FOR MAXIMAL REGULARITY IN THE STUDY OF ELLIPTIC DIFFERENTIAL EQUATIONS IN HÖLDER SPACES

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Abstract. In this paper we give new results on complete abstract second order differential equations of elliptic type in the framework of Hölder spaces, extending those given in [4] and [5]. More precisely we study $u'' + 2Bu' + Au = f$ in the case when f is Hölder continuous and under some natural assumptions on the operators A and B. We give necessary and sufficient conditions of compatibility to obtain a strict solution u and also to ensure that the strict solution has the maximal regularity property.

1. **Introduction and hypotheses.** Let us consider the second order abstract differential equation

$$u''(x) + 2Bu'(x) + Au(x) = f(x), \ x \in (0, 1), \tag{1}$$

with the boundary conditions

$$u(0) = u_0, \ u(1) = u_1. \tag{2}$$

Here u_0, u_1 are given elements of a complex Banach space X, A, B are two linear operators in X and $f \in C([0, 1]; X)$.

We recall that a strict solution u to (1)-(2), is a function u such that
$$u \in C^2([0,1];X) \cap C([0,1];D(A)), u' \in C([0,1];D(B)),$$
and which satisfies (1)-(2).

In general, the condition $f \in C([0,1];X)$ is not sufficient to provide a strict solution to (1)-(2), this is why we assume, in all the paper, that
$$f \in C^\theta([0,1];X), \, 0 < \theta < 1,$$
and, in this case, our aims are the following:

1. Furnish a unified approach in the analysis of (1)-(2) by using uniquely two closed linear operators L and M related to A and B.
2. Under reasonable assumptions on L and M, give necessary and sufficient conditions to obtain a unique strict solution u to (1)-(2).
3. Give also necessary and sufficient conditions to obtain a unique strict solution u satisfying the maximal regularity property
$$u'', Bu', Au \in C^\theta([0,1];X).$$

The results proved here extend those given in the recent papers [4], [5], [6]. Note that we have already used L and M, in the UMD case, see [8].

If P, Q are two linear operators in X such that
$$D(P) \subset D(Q) \text{ and } P = Q \text{ on } D(P),$$
we write $P \subset Q$.

Now, we assume that there exist L, M two closed linear operators in X satisfying

$$\begin{cases} D(L) = D(M) \\ D(ML) = D(LM), \end{cases} \tag{3}$$

$$\begin{cases} L - M \subset 2B \\ LM = ML \subset -A, \end{cases} \tag{4}$$

$$L, M \text{ generate a generalized analytic semigroup on } X, \tag{5}$$

(for the definition of a generalized analytic semigroup see section 2) and

$$L + M \text{ is boundedly invertible in } X. \tag{6}$$

Remark 1.

1. Assumptions (3) and (5) imply that $L + M$ generates an analytic semigroup $\left(e^{x(L+M)}\right)_{x \geq 0}$ on X.
2. Assumption (3) together with $LM = ML$ imply
$$D(M^n L^m) = D(L^m M^n) = D(L^{m+n}) = D(M^{m+n}), \tag{7}$$
   for $n, m \in \mathbb{N}$.
3. By our methods, we will solve
$$u''(x) + (L - M)\, u'(x) - LMu(x) = f(x), \; x \in (0,1),$$
   so a function u such that
$$u \in C^2([0,1];X) \cap C([0,1];D(LM)), u' \in C([0,1];D(L-M)),$$
   and which satisfies (1)-(2) will be called a (L, M)-strict solution of Problem (1)-(2). Of course such a solution will be in particular a strict solution of Problem (1)-(2) in the sense defined previously.

The main results in this work are given by the following Theorems.

Theorem 2. *Assume (3)~(6), $\theta \in]0, 1[$ and $f \in C^\theta([0, 1]; X)$.*
Then the two following assertions are equivalent.

1. *Problem (1)-(2) has a unique (L, M)-strict solution u.*
2. *$u_0, u_1 \in D(LM)$ and*

$$f(i) - Au_i \in \overline{D(LM)}, \ i = 0, 1.$$

Theorem 3. *Assume (3)~(6) and set $\theta \in]0, 1[$. Then the two following assertions are equivalent.*

1. *Problem (1)-(2) has a unique (L, M)-strict solution u satisfying the maximal regularity property*

$$u'', Bu', Au \in C^\theta([0, 1]; X),$$

2. *$f \in C^\theta([0, 1]; X)$, $u_0, u_1 \in D(LM)$ and*

$$f(i) - Au_i \in (D(LM), X)_{1 - \frac{\theta}{2}, \infty}, \quad i = 0, 1. \tag{8}$$

The plan of the paper is as follows.

In section 2, we recall some basic facts on generalized analytic semigroups, which will be applied in our proofs.

In section 3, we give the representation formula of the solution u of Problem (1)-(2).

Section 4 is devoted to the proof of technichal Lemmas which are useful to give a precise analysis of the representation of the solution u.

In section 5, we prove our main results.

Section 6 contains a comparison with the approach used in [7].

Finally, in section 7, we give some examples to which our theory applies.

2. **Analytic semigroups.** Let Q be a linear operator in X such that

$$\begin{cases} \rho(Q) \supset S_{\omega, \delta} = \left\{ \lambda \in \mathbb{C} \backslash \{\omega\} \ / \ |\arg(\lambda - \omega)| < \frac{\pi}{2} + \delta \right\} \text{ and} \\ \sup_{\lambda \in S_{\omega, \delta}} \left\| (\lambda - \omega)(\lambda I - Q)^{-1} \right\|_{L(X)} < +\infty, \end{cases}$$

for some given $\omega \in \mathbb{R}$ and $\delta \in]0, \frac{\pi}{2}[$. This says exactly that Q is the infinitesimal generator of a generalized analytic semigroup $\left(e^{xQ} \right)_{x \geq 0}$, "generalized" in the sense that Q is not supposed to be densely defined and so $\left(e^{xQ} \right)_{x \geq 0}$ is not supposed to be a strongly continuous semi-group (see E. Sinestrari [13], A. Lunardi [12]).

Remark 4. Fix $r > 0$, $\delta_0 \in]0, \delta[$. Then $\left(e^{xQ} \right)_{x \geq 0}$ is defined by

$$e^{xQ} = \begin{cases} \frac{1}{2i\pi} \int_\gamma e^{\lambda x} (\lambda I - Q)^{-1} \, d\lambda \ \text{ if } x > 0 \\ \\ I \ \text{ if } x = 0, \end{cases}$$

where γ is the sectorial boundary curve of $S_{\omega, \delta_0} \backslash B(\omega, r)$ oriented positively.

Let us recall the following classical result.

Proposition 5.

1. *Let $\varphi \in X$. Then the two following assertions are equivalent.*
   (a) *$e^{\cdot Q} \varphi \in C([0, 1]; X)$.*
   (b) *$\varphi \in \overline{D(Q)}$.*

2. Let $\theta \in \,]0,1[\,, g \in C^\theta([0,1];X), \varphi \in X$. Set

$$v(x) = e^{xQ}\varphi + \int_0^x e^{(x-s)Q}g(s)ds, \ x \in [0,1]\,.$$

Then the two following assertions are equivalent.
(a) $v \in C^1([0,1];X) \cap C([0,1];\underline{D(Q)})$.
(b) $\varphi \in D(Q)$ and $g(0) + Q\varphi \in \overline{D(Q)}$.

Considering the well known real interpolation space

$$(D(Q),X)_{1-\theta,\infty} = (X,D(Q))_{\theta,\infty}\,,$$

(see H. Triebel [14] p. 25 and 76), we have also:

Theorem 6.

1. Let $\theta \in \,]0,1[$. Then the two following assertions are equivalent.
(a) $e^{\cdot Q}\varphi \in C^\theta([0,1];X)$.
(b) $\varphi \in (D(Q),X)_{1-\theta,\infty}$.
2. Let $\theta \in \,]0,1[$ and $g \in C^\theta([0,1];X)$. Set

$$v(x) = \int_0^x e^{(x-s)Q}\left[g(s) - g(0)\right]ds, \ x \in [0,1]\,.$$

Then

$$v \in C^{1,\theta}([0,1];X) \cap C^\theta([0,1];D(Q))\,.$$

3. Let $g \in C([0,1];X)$ and $\varphi \in X$. Set

$$w(x) = e^{xQ}\varphi + \int_0^x e^{(x-s)Q}g(s)ds, \ x \in [0,1]\,.$$

Then the two following assertions are equivalent.
(a) $w \in C^{1,\theta}([0,1];X) \cap C^\theta([0,1];D(Q))$.
(b) $g \in C^\theta([0,1];X), \ \varphi \in D(Q)$ and $g(0) + Q\varphi \in (D(Q),X)_{1-\theta,\infty}$.
4. Let $g \in C^\theta([0,1];X)$. Then

$$Q\int_0^1 e^{sQ}\left(g(s) - g(0)\right)ds \in (D(Q)),X)_{1-\theta,\infty}\,.$$

Statement 2 is obtained by applying the Da Prato-Grisvard sum theory [2]. Statement 3 which improves Statement 2 is due to E. Sinestrari [13], see also G. Da Prato [1]. We can find in D. Guidetti [9] a simple proof of these results (see Corollary 2.1. and Theorem 2.4, p.136).

Notation 7. Let g and h be two given X-valued functions defined on $[0,1]$ and $\theta \in \,]0,1[$. We write

$$g \simeq_\theta h,$$

if

$$g - h \in C^\theta([0,1];X)\,.$$

Proposition 8. Let $h \in C^\theta([0,1];X), \ \varphi \in D(Q)$ and set

$$w(x) = e^{xQ}\varphi + \int_0^x e^{(x-s)Q}h(s)ds, \ x \in [0,1]\,.$$

Then

$$Qw(\cdot) \simeq_\theta e^{\cdot Q}\left(Q\varphi + h(0)\right)\,.$$

Proof. It is an easy consequence of Theorem 6 and Proposition 1.2, statement (ii) in [13], that

$$
\begin{aligned}
Qw(x) &= Qe^{xQ}\varphi + Q\int_0^x e^{(x-s)Q}\left(h(s)-h(0)\right)ds + Q\int_0^x e^{(x-s)Q}h(0)ds \\
&= e^{xQ}Q\varphi + Q\int_0^x e^{(x-s)Q}\left(h(s)-h(0)\right)ds - \left(h(0)-e^{xQ}h(0)\right) \\
&= e^{xQ}\left(Q\varphi + h(0)\right) + Q\int_0^x e^{(x-s)Q}\left(h(s)-h(0)\right)ds - h(0).
\end{aligned}
$$

$\square$

3. **Representation formula.** In virtue of Lunardi [12] (p. 60), $I - e^{L+M}$ admits a bounded inverse T.

Now, taking into account the representation formula used in [8], we set for $x \in (0,1)$

$$
\begin{aligned}
u(x) &= e^{xM}Tu_0 + (L+M)^{-1}\int_0^x e^{(x-s)M}f(s)ds \tag{9}\\
&\quad -T(L+M)^{-1}e^{xM}\int_0^1 e^{sL}f(s)ds \\
&\quad +e^{(1-x)L}Tu_1 + (L+M)^{-1}\int_x^1 e^{(s-x)L}f(s)ds \\
&\quad -T(L+M)^{-1}e^{(1-x)L}\int_0^1 e^{(1-s)M}f(s)ds \\
&\quad -Te^{(1-x)L}e^M f_0 - Te^{xM}e^L f_1,
\end{aligned}
$$

where

$$
\begin{cases}
f_0 = u_0 - (L+M)^{-1}\int_0^1 e^{sL}f(s)ds \\[2mm]
f_1 = u_1 - (L+M)^{-1}\int_0^1 e^{(1-s)M}f(s)ds.
\end{cases}
$$

Now, setting for $g \in C^\theta([0,1];X), Q \in \{L,M\}$, and $\phi \in D(Q)$

$$
\begin{cases}
S_1(x,\phi,g,Q) = e^{xQ}T\phi + \int_0^x e^{(x-s)Q}g(s)ds \\
S_2(x,g,Q) = -Te^{xQ}\int_0^1 e^{s(L+M-Q)}g(s)ds \\
R(x,\phi,Q) = Te^{xQ}e^{L+M-Q}\phi,
\end{cases}
$$

we see that the regularity of u is given by the one of

$$
S_1(\cdot,\phi,g,Q), S_2(\cdot,g,Q), R(\cdot,\psi,Q),
$$

since

$$
\begin{aligned}
u(x) &= (L+M)^{-1}S_1(x,(L+M)u_0,f,M) \tag{10}\\
&\quad +(L+M)^{-1}S_2(x,f,M) - R(x,f_1,M) \\
&\quad +(L+M)^{-1}S_1(1-x,(L+M)u_1,f(1-.),L) \\
&\quad +(L+M)^{-1}S_2(1-x,f(1-.),L) - R(1-x,f_0,L).
\end{aligned}
$$

4. **Technical Lemmas.**

Lemma 9. *Assume (3)∼(6). For any $g \in C^\theta([0,1];X), Q \in \{L,M\}, \psi \in X$ and $\phi \in D(Q)$, we have*

1. *$R(\cdot,\psi,Q), LMR(\cdot,\psi,Q), Q^2R(\cdot,\psi,Q) \in C^\infty([0,1];X)$.*
2. *$QS_1(\cdot,\phi,g,Q) \simeq_\theta e^{\cdot Q}(QT\phi + g(0))$.*
3. *$(L+M-Q)S_2(\cdot,g,Q) \simeq_\theta Te^{\cdot Q}g(0)$.*

Proof.

1. Since $L+M-Q$ generates a generalized analytic semigroup, we have for any $m \in \mathbb{N}$
$$e^{L+M-Q} \in L\left(X, D\left((L+M-Q)^m\right)\right),$$
and so, taking into account (7), we get for any $\psi \in X$
$$\begin{cases} R(\cdot,\psi,Q) = e^{\cdot Q}e^{L+M-Q}\psi \in C^\infty([0,1],X). \\ LMR(\cdot,\psi,Q) = e^{\cdot Q}LMe^{L+M-Q}\psi \in C^\infty([0,1],X). \\ Q^2R(\cdot,\psi,Q) = -e^{\cdot Q}Q^2e^{L+M-Q}\psi \in C^\infty([0,1],X). \end{cases}$$

2. Note that $T\phi \in D(Q)$ and apply Proposition 8.
3. We write

$$(L+M-Q)S_2(x,g,Q)$$
$$= -Te^{xQ}(L+M-Q)\int_0^1 e^{s(L+M-Q)}(g(s)-g(0))\,ds$$
$$\qquad -Te^{xQ}(L+M-Q)\int_0^1 e^{s(L+M-Q)}g(0)ds$$
$$= -Te^{xQ}(L+M-Q)\int_0^1 e^{s(L+M-Q)}(g(s)-g(0))\,ds$$
$$\qquad -Te^{xQ}\left[e^{(L+M-Q)}g(0)-g(0)\right]$$
$$= -Te^{xQ}(L+M-Q)\int_0^1 e^{s(L+M-Q)}(g(s)-g(0))\,ds$$
$$\qquad -Te^{xQ}e^{(L+M-Q)}g(0) + Te^{xQ}g(0);$$

but it is known that

$$(L+M-Q)\int_0^1 e^{s(L+M-Q)}(g(s)-g(0))\,ds \in (D(L+M-Q),X)_{1-\theta,\infty},$$

see [13], so from Theorem 6, statement 1,

$$-Te^{\cdot Q}(L+M-Q)\int_0^1 e^{s(L+M-Q)}(g(s)-g(0))\,ds \in C^\theta([0,1];X).$$

Moreover, as in statement 1,

$$TQe^{\cdot Q}e^{(L+M-Q)}g(0) \in C^\infty([0,1];X),$$

from which we deduce

$$(L+M-Q)S_2(\cdot,g,Q) \simeq_\theta Te^{\cdot Q}g(0).$$

$\square$

Now we set for $g \in C^\theta([0,1];X), Q \in \{L, M\}$, and $\phi \in D(Q)$

$$\begin{aligned} S(x, \phi, g, Q) &= (L+M)^{-1} S_1(x, (L+M)\phi, g, Q) \\ &\quad + (L+M)^{-1} S_2(x, g, Q). \end{aligned} \tag{11}$$

Then

Lemma 10. *Assume (3)~(6). For any $g \in C^\theta([0,1];X), Q \in \{L, M\}$ and*

$$\phi \in D(Q^2),$$

we have

$$LMS(\cdot, \phi, g, Q) \simeq_\theta T e^{\cdot Q} g(0) + T(L+M-Q) e^{\cdot Q} Q \phi.$$

Proof. From hypothesis (3) we get

$$(L+M-Q)Q = Q(L+M-Q) = LM,$$

so

$$\begin{aligned} LMS(x, \phi, g, Q) &= (L+M-Q)(L+M)^{-1} Q S_1(x, (L+M)\phi, g, Q) \\ &\quad + Q(L+M)^{-1} (L+M-Q) S_2(x, g, Q). \end{aligned}$$

Now taking into account the fact that

$$\begin{cases} (L+M-Q)(L+M)^{-1}, Q(L+M)^{-1} \in L(X) \text{ and} \\ (L+M)\phi \in D(Q), \end{cases}$$

we can apply Lemma 9 to obtain

$$\begin{aligned} LMS(\cdot, &\phi, g, Q) \\ \simeq_\theta\ & (L+M-Q)(L+M)^{-1} e^{\cdot Q} (QT(L+M)\phi + g(0)) \\ & + Q(L+M)^{-1} T e^{\cdot Q} g(0) \\ \simeq_\theta\ & e^{\cdot Q} ((L+M-Q) + QT)(L+M)^{-1} g(0) \\ & + (L+M-Q) e^{\cdot Q} QT\phi \\ \simeq_\theta\ & T e^{\cdot Q} ((I - e^{L+M})(L+M-Q) + Q)(L+M)^{-1} g(0) \\ & + (L+M-Q) e^{\cdot Q} QT\phi \\ \simeq_\theta\ & T e^{\cdot Q} [(L+M) - e^{L+M}(L+M-Q)](L+M)^{-1} g(0) \\ & + (L+M-Q) e^{\cdot Q} QT\phi \\ \simeq_\theta\ & e^{\cdot Q} T g(0) - T e^{\cdot Q} e^{L+M}(L+M-Q)(L+M)^{-1} g(0) \\ & + (L+M-Q) e^{\cdot Q} QT\phi; \end{aligned}$$

but, as in Lemma 9, statement 1,

$$T e^{\cdot Q} e^{L+M} (L+M-Q)(L+M)^{-1} g(0) \in C^\infty([0,1];X),$$

since $L + M$ generates an analytic semigroup.

Finally

$$LMS(x, \phi, g, Q) \simeq_\theta T e^{\cdot Q} g(0) + T(L+M-Q) e^{\cdot Q} Q \phi.$$

$\square$

Lemma 11. *Assume (3)~(6). For any $g \in C^\theta([0,1];X), Q \in \{L, M\}$,*

$$\phi \in D(Q^2) \text{ and } \lambda \in \rho(L+M-Q),$$

we have

1. $QS(\cdot, \phi, g, Q) \simeq_\theta 0.$
2. $Q^2 S(\cdot, \phi, g, Q) \simeq_\theta -Q(L+M-Q-\lambda)^{-1} LMS(\cdot, \phi, g, Q).$

Proof.   1. We get

$$
\begin{aligned}
QS(x,\phi,g,Q) &= Q(L+M)^{-1}S_1(x,(L+M)\phi,g,Q) \\
&\quad +Q(L+M)^{-1}S_2(x,g,Q) \\
&= Q\left[e^{xQ}T\phi + \int_0^x e^{(x-s)Q}(L+M)^{-1}g(s)ds\right] \\
&\quad -Q(L+M)^{-1}\left[Te^{xQ}\int_0^1 e^{s(L+M-Q)}g(s)ds\right] \\
&= Q\left[w_1(x)\right] - Q(L+M)^{-1}\left[w_2(x)\right],
\end{aligned}
$$

but $L+M-Q$ generates a generalized analytic semigroup, so

$$
\int_0^1 e^{s(L+M-Q)}g(s)ds \in D(L+M-Q) = D(Q),
$$

and w_2 is differentiable on $[0,1]$. Moreover, from Proposition 8

$$
Qw_1(\cdot) \simeq_\theta e^{\cdot Q}\left(QT\phi + (L+M)^{-1}g(0)\right),
$$

and since

$$
\left(QT\phi + (L+M)^{-1}g(0)\right) \in D(L),
$$

we deduce

$$
Qw_1(\cdot) \simeq_\theta 0.
$$

Finally

$$
QS(\cdot,\phi,g,Q) \simeq_\theta 0.
$$

2. We write

$$
\begin{aligned}
Q^2 S(\cdot,\phi,g,Q) &= Q\left(L+M-Q-\lambda\right)^{-1}\left(L+M-Q-\lambda\right)QS(\cdot,\phi,g,Q) \\
&= Q\left(L+M-Q-\lambda\right)^{-1}\left(L+M-Q\right)QS(\cdot,\phi,g,Q) \\
&\quad -Q\left(L+M-Q-\lambda\right)^{-1}\lambda QS(\cdot,\phi,g,Q) \\
&= Q\left(L+M-Q-\lambda\right)^{-1}LMS(\cdot,\phi,g,Q) \\
&\quad -\lambda Q\left(L+M-Q-\lambda\right)^{-1}QS(\cdot,\phi,g,Q),
\end{aligned}
$$

and conclude using statement 1.   □

5. **The main abstract results.** First we show that the function u given by (9) satisfies formally (1). Setting $\widetilde{f} = f(1-.)$, one has, due to (10) and (11)

$$
\begin{aligned}
u(x) &= S(x,u_0,f,M) + S(1-x,u_1,\widetilde{f},L) \\
&\quad -R(1-x,f_0,L) - R(x,f_1,M).
\end{aligned}
$$

So

$$\begin{cases} u''(x) = & M^2 S(x, u_0, f, M) - M^2 R(x, f_1, M) \\ & + L^2 S(1-x, u_1, \widetilde{f}, L) - L^2 R(1-x, f_0, L) \\ & + (L+M)(L+M)^{-1} f(x) \\ \\ (L-M)u'(x) = & (L-M)MS(x, u_0, f, M) - (L-M)MR(x, f_1, M) \\ & -(L-M)LS(1-x, u_1, \widetilde{f}, L) + (L-M)LR(1-x, f_0, L) \\ \\ -LMu(x) = & -LMS(x, u_0, f, M) + LMR(x, f_1, M) \\ & -LMS(1-x, u_1, \widetilde{f}, L) + LMR(1-x, f_0, L), \end{cases}$$

Then, from (4)

$$\begin{aligned} u''(x) + 2Bu'(x) + Au(x) &= u''(x) + (L-M)u'(x) - LMu(x) \\ &= f(x). \end{aligned}$$

Moreover

$$u(0) = S(0, u_0, f, M) - R(0, f_1, M) + S(1, u_1, \widetilde{f}, L) - R(1, f_0, L) = u_0,$$

and

$$u(1) = S(1, u_0, f, M) - R(1, f_1, M) + S(0, u_1, \widetilde{f}, L) - R(0, f_0, L) = u_1.$$

We show the unicity part of our theorems in the following manner. Let u be a (L, M)-strict solution of (1)-(2); then u is necessarily given by the representation (9) and this gives unicity. In fact u can be broken down into the sum

$$u = v + w,$$

where

$$\begin{cases} v = L(L+M)^{-1}u + (L+M)^{-1}u' \\ w = M(L+M)^{-1}u - (L+M)^{-1}u'. \end{cases}$$

After computation we get

$$\begin{cases} v' = Mv + (L+M)^{-1}f \\ w' = -Lw - (L+M)^{-1}f, \end{cases} \tag{12}$$

and

$$\widetilde{w}' = L\widetilde{w} + (L+M)^{-1}f(1-.),$$

where $\widetilde{w} = w(1-.)$. In the case when $L = M$ and thus

$$L^2 \subset -A,$$

it is the Krein method using square roots of operators, see [11].

Now, from (12) and (5) we deduce

$$\begin{cases} v = e^{xM}\xi_0 + (L+M)^{-1} \int_0^x e^{(x-s)M} f(s)ds \\ \widetilde{w} = e^{xL}\xi_1 + (L+M)^{-1} \int_0^x e^{(x-s)L} f(1-s)ds. \end{cases}$$

Finally

$$\begin{aligned} u &= v + \widetilde{w}(1-.) \\ &= e^{xM}\xi_0 + (L+M)^{-1} \int_0^x e^{(x-s)M} f(s)ds \\ &\quad + e^{(1-x)L}\xi_1 + (L+M)^{-1} \int_x^1 e^{(\sigma-x)L} f(\sigma)d\sigma. \end{aligned}$$

Now, since $u_0 = u(0), u_1 = u(1)$ we have

$$\xi_0 = T(u_0 - e^L u_1) - T(L+M)^{-1} \int_0^1 e^{sL} f(s) ds + T(L+M)^{-1} e^L \int_0^1 e^{(1-s)M} f(s) ds,$$

$$\xi_1 = T(u_1 - e^M u_0) - T(L+M)^{-1} \int_0^1 e^{(1-s)M} f(s) ds + T(L+M)^{-1} e^M \int_0^1 e^{sL} f(s) ds,$$

and thus u is given by (9).

5.1. **Proof of Theorem 2.** Let $f \in C^\theta([0,1]; X)$.

Assume that there exists a (L, M)-strict solution u of Problem (1)-(2). Then

$$u_0 \in D(LM),\ u_1 \in D(LM),$$

and from Lemma 9 and Lemma 10 one has

$$LMu(\cdot) \simeq_\theta LMS(\cdot, u_0, f, M) + LMS(1 - \cdot, u_1, \widetilde{f}, L),$$

with

$$
\begin{aligned}
LMS(\cdot, u_0, f, M) \quad &\simeq_\theta \quad Te^{\cdot M} f(0) + TLe^{\cdot M} Mu_0 \\
&\simeq_\theta \quad T\left[e^{\cdot M} f(0) + Le^{\cdot M} Mu_0\right] \\
&\simeq_\theta \quad T\left[e^{\cdot M} (f(0) - Au_0)\right],
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
LMS(1 - \cdot, u_1, \widetilde{f}, L) \quad &\simeq_\theta \quad Te^{(1-\cdot)L} \widetilde{f}(0) + TMe^{(1-\cdot)L} Lu_1 \\
&\simeq_\theta \quad T\left[e^{(1-\cdot)L} f(1) + Me^{(1-\cdot)L} Lu_1\right] \\
&\simeq_\theta \quad T\left[e^{(1-\cdot)L} (f(1) - Au_1)\right].
\end{aligned}
\tag{14}
$$

Taking into account the fact that

$$LMS(\cdot, u_0, f, M) \in C^1(]0,1]; X), LMS(1 - \cdot, u_1, \widetilde{f}, L) \in C^1([0,1[; X), \tag{15}$$

we get

$$
LMu \in C([0,1]; X) \iff \begin{cases} e^{\cdot M} (f(0) - Au_0) \in C([0,1]; X) \\ e^{(1-\cdot)L} (f(1) - Au_1) \in C([0,1]; X) \end{cases} \\
\iff \begin{cases} f(0) - Au_0 \in \overline{D(LM)} \\ f(1) - Au_1 \in \overline{D(LM)}. \end{cases}
\tag{16}
$$

Conversely if $u_0, u_1 \in D(LM)$ and

$$f(i) - Au_i \in \overline{D(LM)},\ i = 0, 1,$$

then from (16)

$$LMS(\cdot, u_0, f, M), LMS(1 - \cdot, u_1, \widetilde{f}, L) \in C([0,1]; X),$$

and due to Lemma 11

$$M^2 S(\cdot, u_0, f, M), L^2 S(1 - \cdot, u_1, \widetilde{f}, L) \in C([0,1]; X).$$

But

$$(L - M)u'(\cdot) \simeq_\theta (L - M) MS(\cdot, u_0, f, M) - (L - M) LS(1 - \cdot, u_1, \widetilde{f}, L), \tag{17}$$

and thus

$$(L - M)u' \in C([0,1]; X) \text{ and } u'' = f - (L - M)u' + LMu \in C([0,1]; X),$$

from which we deduce that u is a (L, M)-strict solution of Problem (1)-(2).

5.2. **Proof of Theorem 3.** Assume that there exists a (L, M)-strict solution u of Problem (1)-(2) having the maximal regularity property. Then necessarily

$$\begin{cases} u_0, u_1 \in D(LM) \text{ and} \\ f = u'' + 2Bu' + Au \in C^\theta([0, 1]; X). \end{cases}$$

Moreover

$$Au = -LMu \in C^\theta([0, 1]; X),$$

and, using (13),(14) and (15) as in the previous proof we get

$$LMu \in C^\theta([0, 1]; X) \iff \begin{cases} e^{\cdot M} (f(0) - Au_0) \in C^\theta([0, 1]; X) \\ e^{(1-\cdot)L} (f(1) - Au_1)) \in C^\theta([0, 1]; X) \end{cases}$$
$$\iff \begin{cases} f(0) - Au_0 \in (D(M), X)_{1-\theta,\infty} \\ f(1) - Au_1 \in (D(L), X)_{1-\theta,\infty}. \end{cases} \quad (18)$$

We conclude by noting that

$$\begin{aligned} (D(M), X)_{1-\theta,\infty} &= (D(L), X)_{1-\theta,\infty} \\ &= \left(D(L^2), X\right)_{1-\theta/2,\infty} \\ &= \left(D(M^2), X\right)_{1-\theta/2,\infty} \\ &= (D(LM), X)_{1-\theta/2,\infty}. \end{aligned}$$

Conversely if $f \in C^\theta([0, 1]; X)$, $u_0, u_1 \in D(LM)$ and

$$f(i) - Au_i \in (D(LM), X)_{1-\frac{\theta}{2},\infty}, \quad i = 0, 1,$$

then from (18)

$$LMS(\cdot, u_0, f, M), LMS(1 - \cdot, u_1, \widetilde{f}, L) \in C^\theta([0, 1]; X),$$

and due to Lemma 11

$$M^2 S(\cdot, u_0, f, M), L^2 S(1 - \cdot, u_1, \widetilde{f}, L) \in C^\theta([0, 1]; X);$$

but, from (17) we deduce

$$(L - M)u' \in C^\theta([0, 1]; X) \text{ and } u'' = f - (L - M)u' + LMu \in C^\theta([0, 1]; X).$$

6. **Comparison with the approach in [5].** In this section, we illustrate our abstract theory by building a typical model of a pair of operators (L, M) satisfying assumptions (3)~(6) as it has been done in [8].

Assume that operators A, B are such that

$$\begin{cases} B^2 - A \text{ is closed with } \mathbb{R}_- \subset \rho \left(B^2 - A\right) \text{ and} \\ \exists c > 0 : \forall \lambda \geq 0, \left\|(\lambda I + B^2 - A)^{-1}\right\|_{L(X)} \leq c/(1 + \lambda), \end{cases} \quad (19)$$

(then it is well known that $-(B^2 - A)^{1/2}$ is the infinitesimal generator of a generalized analytic semigroup)

$$\forall y \in D(B), \ B(B^2 - A)^{-1}y = (B^2 - A)^{-1}By, \quad (20)$$

$$D((B^2 - A)^{1/2}) \subseteq D(B), \quad (21)$$

$$\begin{cases} L = B - (B^2 - A)^{1/2} \text{ and } M = -B - (B^2 - A)^{1/2} \\ \text{generate a generalized analytic semigroup on } X. \end{cases} \quad (22)$$

The following lemma has been proved in [5] (see Lemma 4, p. 426, noting that the additional condition $D(A) \subset D(B^2)$ in this lemma can be dropped).

Lemma 12. *Under the hypothesis (19) we have*

1. *Assumption (20) is equivalent to*

$$\begin{cases} D(B(B^2 - A)) \subset D((B^2 - A)B) \ and \\ \forall z \in D(B(B^2 - A)), \ B(B^2 - A)z = (B^2 - A)Bz. \end{cases} \tag{23}$$

2. *Assumption (20) is equivalent to*

$$\begin{cases} \forall y \in D(B), \ (B^2 - A)^{-1/2}y \in D(B) \ and \\ B(B^2 - A)^{-1/2}y = (B^2 - A)^{-1/2}By. \end{cases} \tag{24}$$

We can find, in [8], the proof of the next Lemma which precises the domains of L and M.

Lemma 13. *Suppose that (19)∼(22) hold. Then, one has*

$$\begin{cases} D(M) = D(L) = D((B^2 - A)^{1/2}) \\ D(ML) = D(LM) = D(B^2 - A) \\ ML = LM \subset -A, \end{cases}$$

and

$$(L + M)^{-1} = -\frac{1}{2}(B^2 - A)^{-1/2} \in L(X).$$

Note that $ML = LM = -A$ if and only if $D(A) \subset D(B^2)$.

Hence, (19)∼(22) imply (3)∼(6) and we can apply Theorem 2 and Theorem 3.

7. **Applications.**

7.1. **First example.** Consider C which generates a generalized bounded analytic semigroup on X and assume that $0 \in \rho(C)$.

Set

$$B = bC, \ A = -aC^2,$$

where

$$a, b \in \mathbb{R}, \ a \geq 0, \ \sqrt{b^2 + a} > 0.$$

Then (3)∼(6) are verified for

$$\begin{cases} L = \left(b + \sqrt{b^2 + a}\right) C \\ M = \left(-b + \sqrt{b^2 + a}\right) C, \end{cases}$$

and so we can solve the following abstract problem

$$\begin{cases} u''(x) + 2bBu'(x) - aB^2u(x) = f(x), \ x \in (0, 1), \\ u(0) = u_0, \ u(1) = u_1. \end{cases}$$

As an example, take C defined in $X = C([0, 1])$ by

$$\begin{cases} D(C) = \left\{v \in C^2([0, 1]) : v(0) = v(1) = 0\right\} \\ Cv = v'', \ v \in D(C). \end{cases} \tag{25}$$

Then

$$\begin{cases} D(A) = \left\{v \in C^4([0, 1]) : v(0) = v(1) = v''(0) = v''(1) = 0\right\} \\ Av = -av^{(4)}, \ v \in D(A). \end{cases}$$

So we can deal with the problem

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial x^2}(x,y) + 2b\dfrac{\partial^3 u}{\partial y^2 \partial x}(x,y) - a\dfrac{\partial^4 u}{\partial y^4}(x,y) \\
= f(x,y), \quad (x,y) \in (0,1) \times (0,1) \\[2mm]
u(x,0) = u(x,1) = \dfrac{\partial^2 u}{\partial y^2}(x,0) = \dfrac{\partial^2 u}{\partial y^2}(x,1) = 0, \quad x \in (0,1) \\[2mm]
u(0,y) = u_0(y),\ u(1,y) = u_1(y), \quad y \in (0,1).
\end{cases}
\tag{26}
$$

Taking into account the fact that $D(C)$ is not dense since

$$
\overline{D(C)} = \{v \in C([0,1]) : v(0) = v(1) = 0\},
$$

and applying Theorem 2, we get:

Theorem 14. *Let, $\theta \in\ ]0,1[$ and $f \in C^\theta([0,1]; X)$. Then the two following asser-
tions are equivalent.*

1. *Problem (26) has a unique strict solution u.*
2. *For $i \in \{0,1\}$*

$$
\begin{cases}
u_i \in C^4([0,1]),\ u_i(0) = u_i(1) = u_i''(0) = u_i''(1) = 0 \ \text{and} \\
f(i,.) - u_i^{(4)}(.) \in C([0,1]), \\
f(i,0) - u_i^{(4)}(0) = f(i,1) - u_i^{(4)}(1) = 0.
\end{cases}
$$

Now, let $\theta \in\ ]0,1[$, then

$$
(D(A),X)_{1-\frac{\theta}{2},\infty} = (D(M),X)_{1-\theta,\infty} = (D(C),X)_{1-\theta,\infty},
$$

and it is well known that this last space coincides with

$$
\left\{v \in Z^\theta([0,1]) : v(0) = v(1) = 0\right\},
$$

where

$$
Z^\theta([0,1]) =
\begin{cases}
C^{2\theta}([0,1]) & \text{if } 2\theta < 1 \\
C^{1,*}([0,1]) & \text{if } 2\theta = 1 \\
C^{1,2\theta-1}([0,1]) & \text{if } 2\theta > 1.
\end{cases}
$$

Here $C^{1,*}([0,1])$ is the Zigmund space

$$
\left\{v \in C([0,1]) \ / \ \sup_{y_1 \neq y_2} \frac{|v(y_1) - 2v((y_1 + y_2)/2) + v(y_2)|}{|y_1 - y_2|} < \infty \right\}.
$$

We can apply Theorem 3, to obtain

Theorem 15. *The two following assertions are equivalent.*

1. *Problem (26) has unique strict solution u satisfying the maximal regularity
   property*
   $$
   u'', Bu', Au \in C^\theta([0,1]; X).
   $$
2. *$f \in C^\theta([0,1]; X)$ and for $i \in \{0,1\}$*

$$
\begin{cases}
u_i \in C^4([0,1]),\ u_i(0) = u_i(1) = u_i''(0) = u_i''(1) = 0 \ \text{and} \\
f(i,.) - u_i^{(4)}(.) \in Z^\theta([0,1]), \\
f(i,0) - u_i^{(4)}(0) = f(i,1) - u_i^{(4)}(1) = 0.
\end{cases}
$$

7.2. **Second example.** Let $\theta \in]0, \pi[$ and set

$$\Sigma_\theta = \{\lambda \in \mathbb{C} \backslash \{0\} \ / \ |\arg \lambda| < \theta\}.$$

We say that an operator Q on X belongs to $\mathrm{Sect}(\theta)$ if

$$\begin{cases} \sigma(Q) \subset \overline{\Sigma_\theta} \text{ and for } \theta < \theta' < \pi \\ \sup_{\lambda \in \mathbb{C} \ \backslash \ \overline{\Sigma_{\theta'}}} \left\| \lambda(\lambda I - Q)^{-1} \right\|_{L(X)} < +\infty. \end{cases}$$

Let us recall the following Scaling Property

$$(Q \in \mathrm{Sect}(\theta) \text{ and } \alpha \in]0, \pi/\theta[) \implies Q^\alpha \in \mathrm{Sect}(\alpha\theta), \tag{27}$$

see Haase [10], Proposition 2.2. p. 58.

Note also that $Q \in \mathrm{Sect}(\theta)$ for some $\theta \in]0, \pi/2[$ if and only if $-Q$ generates a generalized bounded analytic semigroup on X.

Consider C an operator on X such that

$$\begin{cases} C \text{ and } -C^2 \text{ generate a generalized} \\ \text{bounded analytic semigroup on } X, \end{cases} \tag{28}$$

and

$$0 \in \rho(C).$$

Note that if $-C \in \mathrm{Sect}(\theta)$ with $\theta \in]0, \pi/4[$, then from (27)

$$C^2 \in \mathrm{Sect}(2\theta), \ 2\theta \in]0, \pi/2[,$$

and we get (28).

Set

$$B = b\left(C + C^2\right), \ A = 4b^2 C^3,$$

where $b > 0$.

Then (3)$\sim$(6) are verified for

$$L = 2bC, \ M = -2bC^2,$$

and so we can solve the following abstract problem

$$\begin{cases} u''(x) + 2b\left(C + C^2\right)u'(x) + 4b^2 C^3 u(x) = f(x), \ x \in (0, 1), \\ u(0) = u_0, \ u(1) = u_1. \end{cases}$$

Then C given by (25) can be used again to furnish a concrete example.

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