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EXPONENTIAL ATTRACTORS FOR BELOUSOV-ZHABOTINSKII REACTION MODEL

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Abstract. This paper is concerned with the Belousov-Zhabotinskii reaction model. We consider the reaction-diffusion model due to Keener-Tyson. After constructing a dynamical system, we will construct exponential attractors and will estimate the attractor dimension from below. In particular, it will be shown that, as the excitability $\varepsilon > 0$ tends to zero, the attractor dimension tends to infinity, although the exponential attractor can depend on the excitability continuously.

1. **Introduction.** The Belousov-Zhabotinskii reaction is known as a typical phenomenon of self-organization in the chemical reactions (cf. Nicolis and Prigogine [12, Chapter 13]). In 1986, Keener and Tyson [6] introduced a simple mathematical model

$$\begin{cases} \dfrac{\partial u}{\partial t} = \varepsilon a \Delta u + \varepsilon^{-1}\left[u(1-u) - cv\Big(\dfrac{u-q}{u+q}\Big)\right], \\ \dfrac{\partial v}{\partial t} = \varepsilon b \Delta v + u - v \end{cases}$$

for investigating the mechanics of the Belousov-Zhabotinskii reaction which is considered to consists of more than ten elementary chemical reactions. Here, u and v denote the concentrations in a vessel of $HBrO_2$ and Ce^{4+}, respectively, whereas $a > 0$ and $b > 0$ represent the diffusion rate of each species. Finally, ε, q and c are positive constants. By some chemical reason, q is such that $0 < q < 1$.

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We are concerned with the initial-boundary value problem for the Keener-Tyson model

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\Delta u + \dfrac{1}{\varepsilon^2}\left[u(1-u) - cv\Big(\dfrac{u-q}{u+q}\Big)\right] & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = b\Delta v + \dfrac{1}{\varepsilon}[u-v] & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } \Omega \end{cases} \tag{1}$$

in a three-dimensional $\mathcal{C}^2$ or convex, bounded domain Ω. Notice that we used the scaling of the time variable $\tau = \varepsilon t$ and rewrite the new variable τ to t, so that the diffusion coefficients of u and v are again a and b, respectively. As before, a, b, c and q are fixed constants. In turn, $0 < \varepsilon \le 1$ is treated as a control parameter of the system. Actually, ε is taken to represent excitability of the reaction in the vessel.

The aim of this note is to prove that Problem (1) generates a dynamical system possessing finite-dimensional exponential attractors. Besides, we estimate their dimension from below, showing that, provided that $|c-1| < 1$ and q is small enough, the exponential attractors dimension increases up to infinity, as the control parameter $\varepsilon > 0$ goes to zero. On the other hand, by their robustness, it is always proved that exponential attractors can dependent continuously on the parameter ε.

Problem (1) always has two nonnegative homogeneous stationary solutions, a trivial solution $(0,0)$ and a nontrivial solution $(\overline{u}, \overline{v})$, where $\overline{u} = \overline{v}$ is a positive number given by

$$\overline{u} = \overline{v} = \left[1 - c - q + \sqrt{(c+q-1)^2 + 4q(c+1)}\right]/2. \tag{2}$$

The trivial solution is easily seen to be always unstable. In turn, $(\overline{u}, \overline{v})$ is stable if ε is close to 1, but it becomes unstable if $|c-1| < 1$ and ε together with q is small enough. Therefore, in the latter case, every equilibrium that corresponds to a homogeneous stationary solution is unstable. This then previses that (1) must have some temporal or spatial pattern solutions. Moreover, since the dimension of attractors increases as $\varepsilon \to 0$, (1) may then exhibit solutions representing more complex patterns.

The notion of exponential attractors has been introduced in 1994 by Eden, Foias, Nicolaenko and Temam [10] in the theory of infinite-dimensional dynamical systems. The exponential attractor is a compact and positively invariant set having finite fractal dimension which contains the global attractor and attracts every trajectory at an exponential rate. It is also known that the exponential attractor enjoys stronger robustness than the global attractor. When the semigroup of a dynamical system depends continuously on a parameter, the global attractor is in general only upper-semicontinuous. In turn, under some reasonable assumptions, if an exponential attractor exists, it can depend continuously on the parameter. Such a continuous dependence was recently studied in a general framework by Efendiev and Yagi [5]. When the underlying space is a Hilbert space, it is known by the same reference [10] quoted above that the squeezing property of semigroup implies existence of exponential attractors and provides a sharp estimate of attractor dimensions. When the underlying space is a Banach space, it is known by Efendiev, Miranville and Zelik [3] that the compact smoothing property of semigroup implies existence of exponential attractors (Theorem 3.1). The methods for constructing exponential attractors for the semilinear abstract parabolic evolution equations were studied in the reference [10]. On the other hand, those for the quasilinear ones were studied by Aida, Efendiev and Yagi [1]. Finally, the methods for estimating attractor dimensions from below were presented by Aida, Tsujikawa, Efendiev, Yagi and Mimura [2].

2. **Global solutions.**

2.1. **Abstract formulation.** Let us formulate (1) as the Cauchy problem for a semilinear abstract equation in a suitable space of functions in Ω.

We first recall the known results in the theory on semilinear abstract parabolic evolution equations. Consider the Cauchy problem

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), \qquad 0 < t < \infty, \\ U(0) = U_0 \end{cases} \tag{3}$$

in a Banach space X with norm $\|\cdot\|$. Here, A is a sectorial linear operator of X the spectrum of which is contained in a sectorial domain $\Sigma = \{\lambda \in \mathbb{C};\ |\arg\lambda| < \omega\}$ with some angle $0 < \omega < \frac{\pi}{2}$, and the resolvent satisfies the estimate

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \le \frac{M}{|\lambda| + 1}, \qquad \lambda \notin \Sigma \tag{4}$$

with some constant $M > 0$. Then we denote by A^η the η-fractional power of A, whose domain $\mathcal{D}(A^\eta)$ is an intermediate space between $\mathcal{D}(A)$ and X, provided that $\eta \in [0, 1)$. It is well known that the following interpolation inequality holds true

$$\|A^\eta U\| \le C\|AU\|^\eta \|U\|^{1-\eta}, \qquad U \in \mathcal{D}(A).$$

The operator F is a nonlinear operator from $\mathcal{D}(A^\eta)$ to X, where $0 \le \eta < 1$ is some exponent. And, F is assumed to satisfy a Lipschitz condition of the form

$$\begin{aligned} &\|F(U) - F(\widetilde{U})\| \le \varphi(\|A^\alpha U\| + \|A^\alpha \widetilde{U}\|) \\ &\quad \times [\|A^\eta(U - \widetilde{U})\| + (\|A^\eta U\| + \|A^\eta \widetilde{U}\|)\|A^\alpha(U - \widetilde{U})\|], \qquad U, \widetilde{U} \in \mathcal{D}(A^\eta) \end{aligned} \tag{5}$$

with a second exponent α such that $0 \le \alpha \le \eta < 1$, where $\varphi(\cdot)$ is some continuous increasing function. The initial value U_0 is taken from $\mathcal{D}(A^\alpha)$. Let $0 < R < \infty$ be any given number. Then, U_0 is assumed to satisfy

$$\|A^\alpha U_0\| \le R. \tag{6}$$

Theorem 2.1 ([7, Theorem 3.1])**.** *Let $0 \le \alpha \le \eta < 1$ and let (4), (5) and (6) be satisfied. Then, (3) has a unique local solution in the function space:*

$$U \in \mathcal{C}([0, T_R]; \mathcal{D}(A^\alpha)) \cap \mathcal{C}^1((0, T_R]; X) \cap \mathcal{C}((0, T_R]; \mathcal{D}(A)),$$

where $T_R > 0$ is determined by R alone. Moreover, the estimate

$$t^{1-\alpha}\|AU(t)\| + t^{\eta-\alpha}\|A^\eta U(t)\| + \|A^\alpha U(t)\| \le C_R, \qquad 0 < t \le T_R \tag{7}$$

holds for the local solutions with a constant $C_R > 0$ determined by R alone.

It is in addition possible to derive the Lipschitz dependence of the local solutions with respect to the initial values satisfying (6). To see this, consider a closed ball $\mathcal{B}_R \equiv \overline{B}^{\mathcal{D}(A^\alpha)}(0;R)$ of $\mathcal{D}(A^\alpha)$ centered at 0 with the radius R. As stated above, for each $U_0 \in \mathcal{B}_R$, a unique local solution U to (3) exists on an interval $[0, T_R]$. Let U, V be local solutions of (3) for initial values U_0, $V_0 \in \mathcal{B}_R$, respectively. Then, it is valid that

$$t^\eta \|A^\eta[U(t)-V(t)]\| + \|U(t)-V(t)\| \le C_R\|U_0-V_0\|, \quad U_0,\, V_0 \in \mathcal{B}_R;\ 0 \le t \le T_R \quad (8)$$

with some constant $C_R > 0$. For the proof, refer to [7].

In the second half of this subsection, we will apply the general results to (1) by setting the underlying space

$$X = \left\{ \begin{pmatrix} f \\ g \end{pmatrix};\ f \in L_2(\Omega) \quad \text{and} \quad g \in L_2(\Omega) \right\}. \quad (9)$$

Let A be a linear operator of X given by

$$AU = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \qquad U \in \mathcal{D}(A) = \mathbb{H}^2_N(\Omega) \equiv [H^2_N(\Omega)]^2. \quad (10)$$

Here, A_1 and A_2 are realizations of $-a\Delta + 1$ and $-b\Delta + 1$ under the homogeneous Neumann boundary conditions $\frac{\partial u}{\partial n} = 0$ and $\frac{\partial v}{\partial n} = 0$ on $\partial\Omega$, respectively. Thanks to [11, Theorem 3.2.1.3], $A_i\,(i = 1, 2)$ are positive definite self-adjoint operators of $L_2(\Omega)$ with domains $H^2_N(\Omega) = \{u \in H^2(\Omega);\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$. Furthermore, according to [9], the domains of their fractional powers are characterized by

$$\begin{cases} \mathcal{D}(A_i^\theta) = H^{2\theta}(\Omega), & \text{if } 0 \le \theta < \frac{3}{4}, \\ \mathcal{D}(A_i^\theta) = H^{2\theta}_N(\Omega) \equiv \{u \in H^{2\theta}(\Omega);\ \frac{\partial u}{\partial n} = 0 \ \text{ on } \ \partial\Omega\}, & \text{if } \frac{3}{4} < \theta \le 1, \end{cases} \quad (11)$$

where $i = 1, 2$. As a consequence, A is a positive definite self-adjoint operator of X with domain $\mathbb{H}^2_N(\Omega)$ (which is the product space of $H^2_N(\Omega)$ as denote above). In view of (11), the domains of the fractional powers A^θ are also given by $\mathcal{D}(A^\theta) = \mathbb{H}^{2\theta}(\Omega)$ if $0 \le \theta < \frac{3}{4}$ and by $\mathcal{D}(A^\theta) = \mathbb{H}^{2\theta}_N(\Omega)$ if $\frac{3}{4} < \theta \le 1$, $\mathbb{H}^{2\theta}(\Omega)$ and $\mathbb{H}^{2\theta}_N(\Omega)$ being the product spaces of $H^{2\theta}(\Omega)$ and $H^{2\theta}_N(\Omega)$, respectively.

We introduce a nonlinear operator $F_\varepsilon : \mathcal{D}(A^\eta) \to X$ given by

$$F_\varepsilon(U) = \begin{pmatrix} u + \varepsilon^{-2}[u(1-u) - cv(u-q)(|u|+q)^{-1}] \\ \varepsilon^{-1}u + (1-\varepsilon^{-1})v \end{pmatrix}, \qquad U \in \mathcal{D}(A^\eta), \quad (12)$$

where the exponent η is fixed as $\frac{3}{4} < \eta < 1$. Since $H^{2\eta}_N(\Omega) \subset H^{2\eta}(\Omega) \subset L_\infty(\Omega)$ due to the Sobolev embedding theorem, it follows that $\mathcal{D}(A^\eta) \subset L_\infty(\Omega)$. It is immediate to verify the following Lipschitz condition on F_ε:

$$\|F_\varepsilon(U) - F_\varepsilon(\widetilde{U})\| \le C_\varepsilon(\|A^\eta U\| + \|A^\eta \widetilde{U}\| + 1)\|U - \widetilde{U}\|, \quad U, \widetilde{U} \in \mathcal{D}(A^\eta). \quad (13)$$

Here, for any fixed $0 < \varepsilon_0 < 1$, the constant $C_\varepsilon > 0$ is uniformly bounded for $\varepsilon \in [\varepsilon_0, 1]$.

This shows that (5) is fulfilled with $\alpha = 0$ and the η fixed above. In view of this fact, we are led to set the space of initial values as

$$\mathcal{K} = \left\{ \begin{pmatrix} u_0 \\ v_0 \end{pmatrix};\ 0 \le u_0 \in L_2(\Omega) \quad \text{and} \quad 0 \le v_0 \in L_2(\Omega) \right\}. \quad (14)$$

In this manner, we have seen that, for any $U_0 \in \mathcal{K}$, the abstract equation of form (3) has a unique local solution in the function space:

$$U \in \mathcal{C}((0, T_{U_0}]; \mathcal{D}(A)) \cap \mathcal{C}([0, T_{U_0}]; X) \cap \mathcal{C}^1((0, T_{U_0}]; X),$$

equivalently,

$$u,\, v \in \mathcal{H}(T_{U_0}) \equiv \mathcal{C}((0,T_{U_0}];H_N^2(\Omega)) \cap \mathcal{C}([0,T_{U_0}];L_2(\Omega)) \cap \mathcal{C}^1((0,T_{U_0}];L_2(\Omega)), \quad (15)$$

here $T_{U_0} > 0$ depends only on the norm $\|U_0\|$.

2.2. **Nonnegativity of local solutions.** For $U_0 \in \mathcal{K}$, let $U = {}^t(u,v)$ be the local solution of (3) constructed above. By the usual truncation methods, we can show that $u(t) \geq 0$ and $v(t) \geq 0$ for every $0 < t \leq T_{U_0}$.

We thus conclude that, for any $U_0 \in \mathcal{K}$, (3) possesses a unique local solution in the function space:

$$0 \leq u,\, v \in \mathcal{H}(T_{U_0}), \quad (16)$$

$T_{U_0} > 0$ being determined by the norm $\|U_0\|$ alone.

2.3. ***A priori* estimates.** Let us see next *a priori* estimates of the local solutions.

For $U_0 \in \mathcal{K}$, let $U = {}^t(u,v)$ denote any local solution of (3) on an interval $[0,T_U]$ in the function space:

$$0 \leq u,\, v \in \mathcal{H}(T_U) \qquad \text{(for the space } \mathcal{H}(T_U)\text{, see (15))}. \quad (17)$$

Consider the scalar product of the equation of (3) and U in X. After some calculations, it is possible to obtain that

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + \|U\|^2 \leq C\varepsilon^{-3}.$$

Solving this differential inequality, we conclude that

$$\|U(t)\|_{L_2}^2 \leq e^{-t}\|U_0\|_{L^2}^2 + C(1-e^{-t})\varepsilon^{-3}, \qquad 0 \leq t \leq T_U. \quad (18)$$

In particular,

$$\|U(t)\|_X \leq \sqrt{\|U_0\|_X^2 + C\varepsilon^{-3}}, \qquad 0 \leq t \leq T_U. \quad (19)$$

2.4. **Construction of global solutions.** We remember that Theorem 2.1 ensures the local existence of solution to Problem (1) on some interval $[0,T_{U_0}]$, depending only on $\|U_0\|$. Then we have obtained the *a priori* estimate (19). By the standard argument of continuation of local solutions, these two facts infer the global existence of solution. Thus, for any $U_0 \in \mathcal{K}$, (1) possesses a unique global solution in the function space:

$$0 \leq u,\, v \in \mathcal{C}((0,\infty);H_N^2(\Omega)) \cap \mathcal{C}([0,\infty);L_2(\Omega)) \cap \mathcal{C}^1((0,\infty);L_2(\Omega)). \quad (20)$$

For $U_0 \in \mathcal{K}$, let $U(t;U_0)$ denote the global solution. From (18) it is clear that

$$\|U(t;U_0)\|_X \leq e^{-\frac{t}{2}}\|U_0\|_X + C\varepsilon^{-\frac{3}{2}}, \qquad 0 \leq t < \infty;\ U_0 \in \mathcal{K}. \quad (21)$$

Moreover, on account of (7),

$$\|AU(t;U_0)\|_X \leq (1+t^{-1})p_{1,\varepsilon}(\|U_0\|_X), \qquad 0 < t < \infty;\ U_0 \in \mathcal{K} \quad (22)$$

is obtained as well, where $p_{1,\varepsilon}(\cdot)$ is a suitable continuous increasing function.

3. Dynamical system.

3.1. Construction of dynamical system.

Let us first recall some basic matters on the dynamical system. Let X be a Banach space and let $\mathcal{K}$ be a subset of X, $\mathcal{K}$ being a metric space equipped with the distance induced from the norm of X. Let $S(t)$, $0 \le t < \infty$ be a family of mappings from $\mathcal{K}$ into itself having the following properties: i) $S(0) = 1$ (the identity mapping); ii) $S(t)S(s) = S(t+s)$, $0 \le t, s < \infty$ (the semigroup property); and iii) the mapping $G : [0, \infty) \times \mathcal{K} \to \mathcal{K}$, $(t, U_0) \mapsto S(t)U_0$, is continuous. Such a family is called a continuous (nonlinear) semigroup acting on $\mathcal{K}$. The image of $S(\cdot)U_0$ drawn in $\mathcal{K}$ is called the trajectory starting from $\mathcal{K}$. The whole of such trajectories is the dynamical system $(S(t), \mathcal{K}, X)$, where $\mathcal{K}$ and X are called the phase-space and the universal space, respectively.

A subset $\mathcal{A}$ of the phase-space $\mathcal{K}$ is the global attractor of $(S(t), \mathcal{K}, X)$ if the following conditions are satisfied: (1) $\mathcal{A}$ is a compact subset of X; (2) $\mathcal{A}$ is an invariant set, i.e., $S(t)\mathcal{A} = \mathcal{A}$ for every $0 < t < \infty$; and (3) $\mathcal{A}$ attracts every bounded subset of $\mathcal{K}$, namely, for any bounded subset $B \subset \mathcal{K}$, it holds that $\lim_{t\to\infty} h(S(t)B, \mathcal{A}) = 0$, where $h(B_1, B_2) = \sup_{U\in B_1} \inf_{V\in B_2} \|U - V\|$ denotes the Hausdorff pseudodistance between two sets B_1 and B_2. In turn, $\mathcal{M} \subset \mathcal{K}$ is called an exponential attractor of $(S(t), \mathcal{K}, X)$ if the following conditions are satisfied: (1) $\mathcal{M}$ is a compact subset of X with finite fractal dimension $d_F(\mathcal{M})$; (2) $\mathcal{M}$ is a positively invariant set, i.e., $S(t)\mathcal{M} \subset \mathcal{M}$ for every $0 < t < \infty$; and (3) $\mathcal{M}$ contains the global attractor $\mathcal{A}$ and attracts every bounded subset of $\mathcal{K}$ at an exponential rate, namely, there exists a positive exponent $k > 0$ such that, for any bounded subset $B \subset \mathcal{K}$, it holds that $h(S(t)B, \mathcal{M}) \le C_B e^{-kt}$ with some constant $C_B > 0$ depending on B.

Let us now verify that our problem (1) defines a dynamical system with the phase-space $\mathcal{K}$ and the universal space X given by (14) and (9), respectively. Then, for $0 \le t < \infty$, we put $S(t)U_0 = U(t; U_0)$, where $U(t; U_0)$ is the global solution of (1) with initial value $U_0 \in \mathcal{K}$. Then, $S(t)$ is a semigroup acting on $\mathcal{K}$. In addition, thanks to (8), for any $0 < R < \infty$, there exist $0 < T_R < \infty$ and a constant $L_R > 0$ such that

$$\|S(t)U_0 - S(t)V_0\| \le L_R\|U_0 - V_0\|, \qquad 0 \le t \le T_R;\ U_0,\ V_0 \in \mathcal{K}_R,$$

where $\mathcal{K}_R \equiv \mathcal{K} \cap \overline{B}^X(0; R)$. From this, $S(t)$ is shown to be continuous on $\mathcal{K}$ for $0 \le t < \infty$. Consequently, a dynamical system $(S(t), \mathcal{K}, X)$ is determined.

3.2. Compact smoothing of $S(t)$.

Consider a dynamical system $(S(t), \mathcal{K}, X)$ in a Banach space X. We recall the conditions on $S(t)$ which ensure existence of exponential attractors.

We make the following assumptions:

1. There exists an absorbing and positively invariant set $\mathcal{X} \subset \mathcal{K}$ of $S(t)$ which is a compact subset of X;
2. There exist a Banach space Z which is compactly embedded in X and a time $t^* > 0$ such that

$$\|S(t^*)U_0 - S(t^*)V_0\|_Z \le D\|U_0 - V_0\|_X, \qquad U_0,\ V_0 \in \mathcal{X} \tag{23}$$

with some constant $D > 0$;

3. $G(t, U_0) = S(t)U_0$ satisfies the Lipschitz condition

$$\|G(t, U_0) - G(s, V_0)\|_X \le L(|t-s| + \|U_0 - V_0\|_X), \qquad t,\, s \in [0, t^*];\; U_0,\, V_0 \in \mathcal{X} \tag{24}$$

with some constant $L > 0$.

Theorem 3.1 ([3, 8])**.** *Let the conditions 1∼3 in Subsection 3.2 be satisfied. Then $(S(t), \mathcal{K}, X)$ possesses a family of exponential attractors $\mathcal{M}_\theta$, where $0 < \theta < \frac{1}{2D^2}$, with dimension*

$$d_F(\mathcal{M}_\theta) \le \log K_\theta / \log \tfrac{1}{a_\theta} \tag{25}$$

attracting all bounded sets of $\mathcal{K}$ at an exponential rate with $k = (t^*)^{-1} \log a_\theta^{-1}$, where $a_\theta = 2D^2\theta$ and K_θ is the minimal number of balls with radii θ in X which cover the closed ball $\overline{B}^Z(0; 1)$.

Let us apply this theorem to our dynamical system $(S(t), \mathcal{K}, X)$. The dissipative estimate (21) together with (22) immediately yields existence of an absorbing set $\mathcal{B} \subset \mathcal{K}$ of $S(t)$ which is a compact set of X and is a bounded set of $\mathcal{D}(A)$. Then, we put

$$\mathcal{X} = \overline{\bigcup_{t_{\mathcal{B}} \le t < \infty} S(t)\mathcal{B}} \subset \mathcal{B} \qquad \text{(closure in the space } X\text{)},$$

where $t_{\mathcal{B}} > 0$ is a time when $\mathcal{B}$ is absorbed by itself. It is easy to see that this set is still positively invariant and absorbing in $\mathcal{K}$, compact in X and bounded in $\mathcal{D}(A)$. In particular, we can fix $R > 0$ such that $\mathcal{X} \subset B^X(0; R)$ and $Z = \mathcal{D}(A^\eta)$; then, (23) holds true thanks to (8), provided that $t^* > 0$ is suitably chosen. Finally, (24) is verified directly by using boundedness of $\mathcal{X}$ in $\mathcal{D}(A)$. Theorem 3.1 then concludes that $(S(t), \mathcal{K}, X)$ possesses a family of exponential attractors $\mathcal{M}_\theta$ whose fractal dimensions are estimated by (25).

3.3. **Dynamical system in $\mathcal{D}_\beta$.** For any exponent $0 < \beta < 1$, we can repeat a similar argument for constructing a dynamical system, taking as universal space and phase space $\mathcal{D}_\beta = \mathcal{D}(A^\beta)$ and $\mathcal{K}_\beta = K \cap \mathcal{D}_\beta$, respectively, so getting $(S(t), \mathcal{K}_\beta, \mathcal{D}_\beta)$. Then it is immediate to check that the inertial sets $\mathcal{M}_\theta$ given by Theorem 3.1 are exponential attractors for the dynamical system $(S(t), \mathcal{K}_\beta, \mathcal{D}_\beta)$, as well.

4. **Exponential attractors for (1).** In this section, we want to study the dependence of the exponential attractors on the control parameter ε. For this reason, we denote the dynamical system corresponding to Problem (1) for a fixed ε by $(S_\varepsilon(t), \mathcal{K}, X)$.

4.1. **Continuous dependence on the parameter ε.** In the paper Efendiev-Yagi [5] (cf. also [4]), continuous dependence of exponential attractors on a parameter was studied in a general framework. Let us review their results. Consider a family of dynamical systems $(S_\xi(t), \mathcal{X}_\xi, X)$ in a Banach space X which are parameterized by $0 \le \xi \le 1$, the phase space $\mathcal{X}_\xi$ being a compact subset of X for each ξ. Let us assume the following conditions:

1. There exist a Banach space Z which is compactly embedded in X and a time $t^* > 0$ such that
$$\|S_\xi(t^*)U_0 - S_\xi(t^*)V_0\|_Z \le D\|U - V\|_X, \quad U_0, V_0 \in \mathcal{X}_\xi$$
with some uniform constant $D > 0$ independent of $0 \le \xi \le 1$;
2. Each $G_\xi(t, U_0) = S_\xi(t)U_0$ satisfies
$$\|S_\xi(t)U_0 - S_\xi(s)V_0\|_X \le L(|t - s| + \|U_0 - V_0\|_X), \quad 0 \le s, t \le t^*, U_0, V_0 \in \mathcal{X}_\xi$$
with some uniform constant $L > 0$ independent of $0 \le \xi \le 1$;
3. There exists an absorbing set B which is uniform in ξ, namely, $B \subset \cap_{0\le\xi\le1}\mathcal{X}_\xi$ and
$$S_\xi(t)\mathcal{X}_\xi \subset B \qquad \text{for every} \quad t \ge t^*$$
for all $0 \le \xi \le 1$;
4. As $\xi \to 0$, $S_\xi(t)$ converges to $S_0(t)$ on B at the rate
$$\sup_{U\in B} \sup_{0\le t\le t^*} \|S_\xi(t)U - S_0(t)U\|_X \le K\xi$$
with some constant $K \ge 1$ for all $0 \le \xi \le 1$.

As proved by [5, Theorem 3.1], the conditions 1∼4 imply the following result.

Theorem 4.1. *Let the conditions 1∼4 in Subsection 4.1 be satisfied. Then one can construct a family of exponential attractors $\mathcal{M}_\xi$ of $(S_\xi(t), \mathcal{X}_\xi, X)$ for $0 \le \xi \le 1$, such that*
$$d(\mathcal{M}_\xi, \mathcal{M}_0) \le C\xi^\kappa$$
for some $\kappa \in (0, 1)$ and some positive constant C. Here, $d(B_1, B_2)$ denotes the distance of two sets B_1, B_2, i.e., $d(B_1, B_2) = \max\{h(B_1, B_2), h(B_2, B_1)\}$.

We intend to apply this theorem to our dynamical system. But we realize that as $\varepsilon \to 0$, the estimates (21) and (22) lose uniformity. So, letting $0 < \varepsilon_0 < 1$ be arbitrarily fixed, we assume that the control parameter ε varies in the range $\varepsilon \in [\varepsilon_0, 1]$ only. Then, it is possible by similar techniques as before to verify that the structural conditions 1∼4 in Subsection 4.1 are fulfilled by $(S_\varepsilon(t), \mathcal{K}, X)$ for $\varepsilon \in [\varepsilon_0, 1]$.

As a result, the following statement is true. Take any ε such that $\varepsilon_0 \le \varepsilon \le 1$. Then, we can construct an exponential attractor $\mathcal{M}_{\widetilde{\varepsilon}}$ of $(S_{\widetilde{\varepsilon}}(t), \mathcal{K}, X)$ for each $\widetilde{\varepsilon} \in [\varepsilon_0, 1]$ that depends continuously on $\widetilde{\varepsilon}$ at ε in such a way that
$$d(\mathcal{M}_{\widetilde{\varepsilon}}, \mathcal{M}_\varepsilon) \le C|\widetilde{\varepsilon} - \varepsilon|^\kappa, \qquad \varepsilon_0 \le \widetilde{\varepsilon} \le 1$$
with some exponent $0 < \kappa < 1$.

4.2. **Lower estimate of $d_F(\mathcal{M}_\varepsilon)$.** We begin with recalling the general results. In a Banach space X, consider the Cauchy problem for a semilinear evolution equation of form (3). Let the structural conditions (4) and (5) be satisfied, and let the *a priori* estimate
$$\|A^\alpha U(t; U_0)\| \le p(\|A^\alpha U_0\|), \qquad 0 \le t \le T_U$$
hold for all local solutions with some continuous increasing function $p(\cdot)$. Then, by the same argument as in Subsection 2.4, the global existence of solution for $U_0 \in \mathcal{D}(A^\alpha)$ is established. Furthermore, a dynamical system $(S(t), \mathcal{D}_\alpha, \mathcal{D}_\alpha)$ is defined, where $\mathcal{D}_\alpha = \mathcal{D}(A^\alpha)$ is a Banach space endowed with the graph norm $\|A^\alpha \cdot\|$. Let $\overline{U} \in \mathcal{D}(A)$ be a stationary solution to (3). Clearly, $\overline{U}$ is an equilibrium

of $(S(t), \mathcal{D}_\alpha, \mathcal{D}_\alpha)$. We assume that $F : \mathcal{D}(A^\eta) \to X$ is $\mathcal{C}^{1,1}$ in $B^{\mathcal{D}(A^\eta)}(\overline{U}; r)$ for some $r > 0$. More precisely, we make the following assumptions:

$$\|F'(U)V\| \le \psi(\|A^\alpha U\|)\|A^\eta U\|\|A^\alpha V\|, \qquad U \in B^{\mathcal{D}(A^\eta)}(\overline{U}; r),\ V \in \mathcal{D}(A^\eta), \tag{26}$$

$$\begin{aligned}\|[F'(U) - F'(\widetilde{U})]V\| &\le \psi(\|A^\alpha U\| + \|A^\alpha \widetilde{U}\|) \\ &\quad \times \|A^\eta(U - \widetilde{U})\|\|A^\alpha V\|, \qquad U, \widetilde{U} \in B^{\mathcal{D}(A^\eta)}(\overline{U}; r);\ V \in \mathcal{D}(A^\eta)\end{aligned} \tag{27}$$

with some continuous increasing function $\psi(\cdot)$.

By [2, Sections 5 and 6], the following results are known.

Theorem 4.2. *Let $\overline{U}$ be an equilibrium of $(S(t), \mathcal{D}_\alpha, \mathcal{D}_\alpha)$ and let (26) and (27) be satisfied with some $r > 0$. If the spectrum of $A - F'(\overline{U})$ is separated by the imaginary axis, i.e., $\sigma(A - F'(\overline{U})) \cap i\mathbb{R} = \emptyset$ and if $\sigma(A - F'(\overline{U})) \cap \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda < 0\} \ne \emptyset$, then $\overline{U}$ is unstable and has a smooth local unstable manifold $\mathcal{M}(\overline{U}, \mathcal{O})$.*

In order to apply this theorem, let us find first homogeneous stationary solutions of (1). Obviously, for $u \ge 0$ and $v \ge 0$, the system of equations

$$\begin{cases} u(1-u) - cv\Big(\dfrac{u-q}{u+q}\Big) = 0, \\ u - v = 0 \end{cases}$$

has two solutions $(0, 0)$ and $(\overline{u}, \overline{v})$, where $\overline{u} = \overline{v} \ne 0$. Here, $\overline{u}$ is a unique positive solution given by (2) to the quadratic equation

$$(u + q)(1 - u) = c(u - q).$$

Then, $\overline{U} = {}^t(\overline{u}, \overline{v})$ is an equilibrium of $(S_\varepsilon(t), \mathcal{K}_\beta, \mathcal{D}_\beta)$. Let $\mathcal{M}_\varepsilon(\overline{U})$ be the unstable manifold of $\overline{U}$ and $\mathcal{M}_\varepsilon(\overline{U}, \mathcal{O})$ be a local unstable manifold of $\overline{U}$ in $(S_\varepsilon(t), \mathcal{K}_\beta, \mathcal{D}_\beta)$, where $\mathcal{O}$ is a neighborhood of $\overline{U}$ in $\mathcal{K}_\beta$. By definition, we have

$$\mathcal{M}_\varepsilon(\overline{U}, \mathcal{O}) \subset \mathcal{M}_\varepsilon(\overline{U}) \subset \mathcal{M}_\varepsilon.$$

Remember that $\frac{3}{4} < \beta < 1$ and that $\mathcal{D}_\beta \subset \mathcal{C}(\overline{\Omega})$. On account of this fact, we can consider a complexified version of Problem (1) localized at the equilibrium point $\overline{U}$, by introducing the nonlinear operator

$$\widetilde{F}_\varepsilon(U) = \begin{pmatrix} \chi(u) + \varepsilon^{-2}\Big[\chi(u)(1-\chi(u)) - c\chi(v)\Big(\frac{\chi(u)-q}{\chi(u)+q}\Big)\Big] \\ \varepsilon^{-1}\chi(u) + (1 - \varepsilon^{-1})\chi(v) \end{pmatrix}, \qquad U = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(A^\eta).$$

Here, $\chi(u)$ is a cutoff function of u in a complex neighborhood of $\overline{u}$ such that $\chi(u) \equiv u$ for $|u - \overline{u}| < \overline{u}$ and $\chi(u) = \frac{u-\overline{u}}{|u-\overline{u}|}\overline{u} + \overline{u}$ for $|u - \overline{u}| \ge \overline{u}$. It is clear that $|\chi(u) + q| \ge q$ for $u \in \mathbb{C}$, $|\chi(u)| \le 2\overline{u}$ for $u \in \mathbb{C}$ and $|\chi(u_1) - \chi(u_2)| \le |u_1 - u_2|$ for $u_1, u_2 \in \mathbb{C}$. Since $\overline{u} = \overline{v}$, $\chi(v)$ plays also a cutoff function of v in the same neighborhood of $\overline{v}$. Then the complexified version of (1) is written by

$$\begin{cases} \dfrac{dU}{dt} + AU = \widetilde{F}_\varepsilon(U), \qquad 0 < t < \infty, \\ U(0) = U_0, \end{cases} \tag{28}$$

where A is the same sectorial operator of X defined by (10).

Problem (28) also generates a dynamical system with the phase space $\mathcal{D}_\beta$ and the universal space $\mathcal{D}_\beta$, respectively, which will be denoted by $(\widetilde{S}_\varepsilon(t), \mathcal{D}_\beta, \mathcal{D}_\beta)$. We can construct exponential attractors $\widetilde{\mathcal{M}}_\varepsilon$ for $(\widetilde{S}_\varepsilon(t), \mathcal{D}_\beta, \mathcal{D}_\beta)$, as well. Of course, $\overline{U}$ is an equilibrium of $(\widetilde{S}_\varepsilon(t), \mathcal{D}_\beta, \mathcal{D}_\beta)$. In a suitable neighborhood of $\overline{U}$, any trajectory of $(S_\varepsilon(t), \mathcal{K}_\beta, \mathcal{D}_\beta)$ is that of $(\widetilde{S}_\varepsilon(t), \mathcal{D}_\beta, \mathcal{D}_\beta)$. We will then apply Theorem 4.2 to $(\widetilde{S}_\varepsilon(t), \mathcal{D}_\beta, \mathcal{D}_\beta)$ with $\alpha = \beta$. It is not difficult to verify that $\widetilde{F}_\varepsilon$ is $\mathcal{C}^{1,1}$ in a neighborhood of $\overline{U}$ and that the derivative $\widetilde{F}_\varepsilon{}'$ satisfies the conditions (26) and (27).

Put

$$\begin{cases} f(u,v) = u(1-u) - cv\Big(\dfrac{u-q}{u+q}\Big), \\ \qquad g(u,v) = u - v. \end{cases}$$

Then it is easy to see that

$$\begin{cases} \overline{f}_u = f_u(\overline{u},\overline{v}) = 1 - 2\overline{u} - \dfrac{2cq\overline{u}}{(\overline{u}+q)^2}, & \overline{f}_v = f_v(\overline{u},\overline{v}) = \overline{u} - 1, \\ \qquad \overline{g}_u = g_u(\overline{u},\overline{v}) = 1, & \overline{g}_v = g_v(\overline{u},\overline{v}) = -1. \end{cases} \tag{29}$$

Furthermore, we observe that

$$\widetilde{F}_\varepsilon{}'(\overline{U}) = \begin{pmatrix} 1 + \varepsilon^{-2} f_u(\overline{u},\overline{v}) & \varepsilon^{-2} f_v(\overline{u},\overline{v}) \\ \varepsilon^{-1} g_u(\overline{u},\overline{v}) & 1 + \varepsilon^{-1} g_v(\overline{u},\overline{v}) \end{pmatrix}.$$

By the similar argument as in [2], we are able to characterize the spectrum of $A - \widetilde{F}_\varepsilon{}'(\overline{U})$. In fact, it is proved that $\lambda \in \sigma(A - \widetilde{F}_\varepsilon{}'(\overline{U}))$ if and only if $\lambda \in \sigma(\overline{A}_k)$ at least for some $k = 0, 1, 2, \ldots$. Here, $\overline{A}_k$ is a part of $A - \widetilde{F}_\varepsilon{}'(\overline{U})$ in the two-dimensional subspace X_k of X defined by

$$X_k = \left\{ U = \xi_k \begin{pmatrix} \phi_k \\ 0 \end{pmatrix} + \eta_k \begin{pmatrix} 0 \\ \phi_k \end{pmatrix};\ \xi_k,\ \eta_k \in \mathbb{C} \right\}, \qquad k = 0, 1, 2, \ldots,$$

where $\{\phi_k\}_{k=0,1,2,\ldots}$ is an orthonormal basis of $L_2(\Omega)$ consisting of real eigenfunctions of $-\varDelta$ in Ω under the homogeneous Neumann boundary conditions. Moreover, the transformation matrix of $\overline{A}_k$ is given by

$$\begin{pmatrix} a\mu_k - \varepsilon^{-2}\overline{f}_u & -\varepsilon^{-1}\overline{g}_u \\ -\varepsilon^{-2}\overline{f}_v & b\mu_k - \varepsilon^{-1}\overline{g}_v \end{pmatrix}, \qquad k = 0, 1, 2, \ldots,$$

where $\mu_k \geq 0$ denotes the eigenvalue corresponding to ϕ_k. Therefore, we have $\sigma(\overline{A}_k) = \{\lambda_k',\ \lambda_k''\}$, where λ_k' and λ_k'' are solutions of a quadratic equation

$$\begin{aligned} \lambda^2 - [(a+b)\mu_k - \varepsilon^{-1}(\varepsilon^{-1}\overline{f}_u + \overline{g}_v)]\lambda \\ + ab\mu_k^2 - \varepsilon^{-1}(a\overline{g}_v + \varepsilon^{-1} b\overline{f}_u)\mu_k + \varepsilon^{-3}(\overline{f}_u\overline{g}_v - \overline{f}_v\overline{g}_u) = 0. \end{aligned}$$

Hence, $\sigma(A - \widetilde{F}_\varepsilon'(\overline{U})) = \cup_{k=0,1,2,\ldots}\{\lambda_k',\ \lambda_k''\}$.

As seen from (29), it always holds that $f_u(\overline{u},\overline{v}) < 1$. When $f_u(\overline{u},\overline{v}) \leq 0$, we can observe that $\sigma(\overline{A}_k) \subset \{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda > 0\}$ for every k; therefore, $\sigma(A - \widetilde{F}_\varepsilon{}'(\overline{U})) \subset \{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda > 0\}$ whatever $\varepsilon > 0$ is. This means in this case that $\overline{U}$ is stable. Even when $0 < f_u(\overline{u},\overline{v}) < 1$ (indeed this can take place if $|c-1| < 1$ and q is small enough), if ε is sufficiently large in such a way that

$$\varepsilon > \max\left\{ \frac{b^2\overline{f}_u^2}{\left(\sqrt{ab(\overline{f}_u\overline{g}_v - \overline{f}_v\overline{g}_u)} + \sqrt{-ab\overline{f}_v\overline{g}_u}\right)^2},\ \overline{f}_u \right\},$$

then $\sigma(A-\widetilde{F}_\varepsilon{}'(\overline{U})) \subset \{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda > 0\}$ and $\overline{U}$ is still stable. But, if $0 < f_u(\overline{u},\overline{v}) < 1$ (as mentioned, this is the case when $|c-1| < 1$ and q is small enough) and if $\varepsilon > 0$ is sufficiently small together with the supplement conditions that

$$\begin{cases} ab\mu_k^2 - \varepsilon^{-2}(\varepsilon a\overline{g}_v + b\overline{f}_u)\mu_k + \varepsilon^{-3}(\overline{f}_u\overline{g}_v - \overline{f}_v\overline{g}_u) \neq 0 \quad \text{for all } k; \text{ and,} \\ \qquad\qquad -\varepsilon a\overline{g}_v + b\overline{f}_u > (a+b)\sqrt{-\varepsilon\overline{f}_v\overline{g}_u}, \end{cases}$$

then it is true that $\sigma(A-\widetilde{F}_\varepsilon{}'(\overline{U})) \cap i\mathbb{R} = \emptyset$ and $\sigma(A-\widetilde{F}_\varepsilon{}'(\overline{U})) \cap \{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda < 0\} \neq \emptyset$. Hence, in this case, $\overline{U}$ is unstable. Furthermore, counting the number of eigenvalues of $A-\widetilde{F}_\varepsilon{}'(\overline{U})$ lying in $\{\lambda \in \mathbb{C};\ \mathrm{Re}\,\lambda < 0\}$, the dimension of a local unstable manifold $\widetilde{\mathcal{M}}_\varepsilon(\overline{U},\mathcal{O})$ of $\overline{U}$ is estimated by $\dim\, \widetilde{\mathcal{M}}_\varepsilon(\overline{U},\mathcal{O}) \geq C\varepsilon^{-3}$. In turn, since any exponential attractor $\widetilde{\mathcal{M}}_\varepsilon$ of $(\widetilde{S}_\varepsilon(t),\mathcal{D}_\beta,\mathcal{D}_\beta)$ contains $\widetilde{\mathcal{M}}_\varepsilon(\overline{U},\mathcal{O})$, it is deduced that

$$\dim\, \widetilde{\mathcal{M}}_\varepsilon \geq \dim\, \widetilde{\mathcal{M}}_\varepsilon(\overline{U},\mathcal{O}) \geq C\varepsilon^{-3}.$$

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