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## STATIONARY SOLUTIONS TO FOREST KINEMATIC MODEL

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**Abstract.** We continue the study of a mathematical model for a forest ecosystem which has been presented by Y. A. Kuznetsov, M. Y. Antonovsky, V. N. Biktashev and A. Aponina (A cross-diffusion model of forest boundary dynamics, *J. Math. Biol.* **32** (1994), 219–232). In the preceding two papers (L. H. Chuan and A. Yagi, Dynamical system for forest kinematic model, *Adv. Math. Sci. Appl.* **16** (2006), 393–409; L. H. Chuan, T. Tsujikawa and A. Yagi, Asymptotic behavior of solutions for forest kinematic model, *Funkcial. Ekvac.* **49** (2006), 427–449), the present authors already constructed a dynamical system and investigated asymptotic behaviour of trajectories of the dynamical system. This paper is then devoted to studying not only the structure (including stability and instability) of homogeneous stationary solutions but also the existence of inhomogeneous stationary solutions. Especially it shall be shown that in some cases, one can construct an infinite number of discontinuous stationary solutions.

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**1. Introduction.** Conservation of forest resources is one of the main subjects in environmental issues. The fundamental problems in the theoretical studies of this subject are to know the physical principles of growth for individual trees, trees in a plot of forest and even all trees in a forest and to know mathematical structures for these growing dynamics. Many researchers have already challenged these problems. The work due to Botkin et al. [6] (cf. also [5]) may give the first and most basic model in the forest kinematic models. In their papers, considering the Individual-Based Model in a plot of forest (100–300 m<sup>2</sup>), they presented a growth equation of individual trees which describes the growth of a tree per year by

$$\Delta D^2 H = \alpha L \left( 1 - \frac{DH}{D_{\max} H_{\max}} \right).$$

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Here  $D$  denotes diameter of the tree at the breast height,  $H$  entire height of the tree (so  $D^2H$  describes a volume) and  $L$  width of foliate area. The constants  $D_{\max}$  and  $H_{\max}$  are the possible maximum diameter and height, respectively, which the tree can attain, and the coefficient  $\alpha > 0$  denotes various environmental conditions surrounding the tree including the effect of interactions with other trees in the plot. After this model, the Individual-Based Continuous Space Model was presented by Pacala et al. [15, 16]. In the meantime, macroscopic forest models concerning with the age-dependent tree relationship have been introduced by many authors, e.g., Antonovsky [2] and Antonovsky and Korzukhin [3]. Such a model is called the Age-Structured Model. In this paper, we are concerned with the Age-Structured Continuous Space Model. Among others we consider a prototype model describing the growth of a forest by age-dependent trees relationships and by regeneration processes, which was proposed by Kuznetsov et al. [11].

They considered a mono-species ecosystem with only two age classes of trees, the young age class and the old age class, and modelled the regeneration process by seed production, seed dispersion and establishment of seeds. Their system of equations reads

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} = \beta\delta w - \gamma(v)u - fu & \text{in } \Omega \times (0, \infty), \\ \frac{\partial v}{\partial t} = fu - hv & \text{in } \Omega \times (0, \infty), \\ \frac{\partial w}{\partial t} = d\Delta w - \beta w + \alpha v & \text{in } \Omega \times (0, \infty), \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x) & \text{in } \Omega. \end{array} \right. \quad (1.1)$$

Here,  $\Omega$  is a two-dimensional bounded domain. The unknown functions  $u = u(x, t)$  and  $v = v(x, t)$  denote the tree densities of young and old age classes, respectively, at a position  $x \in \Omega$  and at time  $t \in [0, \infty)$ . The third unknown function  $w = w(x, t)$  denotes the density of seeds in the air at  $x \in \Omega$  and  $t \in [0, \infty)$ . The third equation describes the kinetics of seeds;  $d > 0$  is a diffusion constant of seeds, and  $\alpha > 0$  and  $\beta > 0$  are seed production and seed deposition rates, respectively. While the first and second equations describe the growth of young and old trees, respectively;  $0 < \delta \leq 1$  is an establishment rate of seeds,  $f > 0$  is an aging rate and  $h > 0$  is a mortality of old trees. And  $\gamma(v) > 0$  is a mortality of young trees which is allowed to depend on the old-tree density  $v$  and is expected to hit a minimum at a certain optimal value of  $v$ .

In the preceding two papers [7, 8], the authors have already studied this system analytically. In [7], they constructed a dynamical system  $(S(t), K, X)$  determined from the initial-boundary value problem (1.1). As the underlying space  $X$ , we set a space of the form

$$X = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix}; u \in L^\infty(\Omega), v \in L^\infty(\Omega), w \in L^2(\Omega) \right\}. \quad (1.2)$$

It is necessary to handle the first and second ordinary differential equations in the Banach space  $L^\infty(\Omega)$ . Indeed, since  $\gamma(v)u$  contains a non-linear term like  $v^2u$  (see (1.4)), the Banach space to be chosen must enjoy a norm property  $\|v^2u\| \leq C\|v\|^2\|u\|$ ,

namely, the space must be a Banach algebra. Moreover, even if the initial functions  $u_0$ ,  $v_0$  and  $w_0$  are smooth, its solution  $(u, v, w)$  can tend to a discontinuous stationary solution as  $t \rightarrow \infty$  (see [8, Section 6] and [13]). That is, the continuous function space  $\mathcal{C}(\bar{\Omega})$  is not suitable. The phase space  $K$  consists of triplets of non-negative functions of  $X$ , i.e.,

$$K = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix}; 0 \leq u \in L^\infty(\Omega), 0 \leq v \in L^\infty(\Omega), 0 \leq w \in L^2(\Omega) \right\} \quad (1.3)$$

(see [8, Remark 2.1]). The non-linear semigroup  $S(t)$  acts on  $K$  for  $0 \leq t < \infty$ . In [8], the authors found a Lyapunov function and investigated asymptotic behaviour of trajectories  $S(t)U_0$ ,  $U_0 \in K$ . Since some  $S(t)U_0$  can converge to a discontinuous stationary solution even if the initial value  $U_0 \in K$  consists of smooth functions and since if so the trajectory  $S(t)U_0$  has an empty  $\omega$ -limit set in  $X$ , the dynamical system  $(S(t), K, X)$  never enjoys any compact attractor in general. Due to this reason we introduced three kinds of  $\omega$ -limit sets for  $U_0 \in K$ , i.e.,  $\omega(U_0) \subset L^2$ - $\omega(U_0) \subset w^*$ - $\omega(U_0) \neq \emptyset$ , here  $\omega(U_0)$  denotes the usual one (see [4, 17]),  $L^2$ - $\omega(U_0)$  is an  $\omega$ -limit set with respect to the  $L^2$  topology and  $w^*$ - $\omega(U_0)$  is that with respect to the weak\* topology of  $L^\infty(\Omega)$ . And we proved by utilizing the Lyapunov function that  $L^2$ - $\omega(U_0)$  consists of stationary solutions only. So, roughly speaking, every trajectory  $S(t)U_0$ ,  $U_0 \in K$  converges asymptotically to some stationary solution of (1.1).

In the next stage of researches, we are led to study the structure of stationary solutions of (1.1). In this paper, we first seek homogeneous stationary solutions and investigate their stability and instability. Secondly, we seek inhomogeneous stationary solutions. The structure depends on the parameter  $h$  drastically. In fact, when  $0 < h < (f\alpha\delta)/(ab^2 + c + f)$ , where  $a$ ,  $b$  and  $c$  are positive constants contained in  $\gamma(v)$  (see (1.4)), it is shown that there exist two homogeneous stationary solutions  $P_+$  (which is non-zero solution) and the zero solution  $O = (0, 0, 0)$  and that  $P_+$  is stable and  $O$  is unstable. This means that in this case any forest starting from a non-zero initial state holds alive. In the meantime, when  $(f\alpha\delta)/(c + f) < h < \infty$ , the zero solution  $O$  is a unique stationary solution and is globally stable, that is, every forest is going to vanish asymptotically. When  $\frac{f\alpha\delta}{ab^2+c+f} < h < \frac{f\alpha\delta}{c+f}$ , there exist three homogeneous stationary solutions  $P_\pm$  (which are non-zero) and the zero solution  $O$ ; here,  $P_+$  and  $O$  are stable meanwhile  $P_-$  is unstable (see Figure 1). This means that some forests can hold alive and others are going to vanish. What is more interesting is that, in this case, there exist many inhomogeneous stationary solutions. Especially when  $a$  and  $b$  are sufficiently large (see Remark 3.2), one can construct an infinite number of discontinuous stationary solutions  $(\bar{u}, \bar{v}, \bar{w})$ 's,  $\bar{u}$ ,  $\bar{v} \in L^\infty(\Omega)$  being discontinuous and  $\bar{w} \in H^2(\Omega)$  being continuous.

Such a discontinuous stationary solution is very important in the view point of forestry also (see [11]). The interface of discontinuity of a stationary solution is considered as an internal and proper forest boundary, which is called an ecotone. So we can re-create the ecotone of forest by using the prototype model (1.1). Many interesting problems concerning discontinuous stationary solutions, however, remain to be solved. For example, we have no rigorous argument on their stability or instability, and do not know how many discontinuous solutions exist. It seems very hard to say how the interface is determined from the parameters in the parabolic-ordinary system (1.1). Only some numerical evidences suggest that the structure of discontinuous stationary

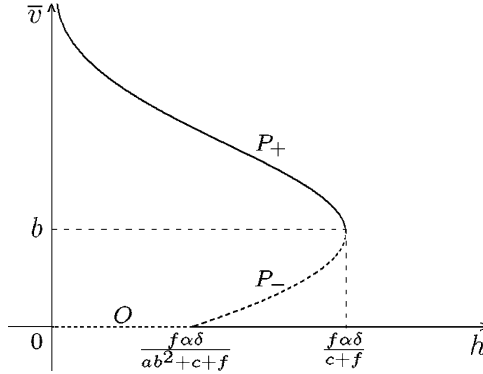


Figure 1. Homogeneous stationary solutions.

solutions might be immensely complicated. In [8, Section 6] we found an example having a symmetric and smooth interface of discontinuity. On the other hand, we shall find in this paper an example which has an irregular and non smooth interface .

Throughout the paper,  $\Omega$  is a  $C^2$  or convex, bounded domain in  $\mathbb{R}^2$ . But, in Section 3,  $\Omega$  will be a rectangular domain. We assume as in the paper [11] that the function  $\gamma(v)$  is given by a quadratic function

$$\gamma(v) = a(v - b)^2 + c, \quad (1.4)$$

where  $a, b, c > 0$  are all positive constants.

## 2. Homogeneous stationary solutions.

**2.1. Structure of homogeneous stationary solutions.** Let  $(\bar{u}, \bar{v}, \bar{w})$  be a non-negative homogeneous stationary solution of system (1.1). Then  $\bar{u} \geq 0$ ,  $\bar{v} \geq 0$  and  $\bar{w} \geq 0$  satisfy the system of equations

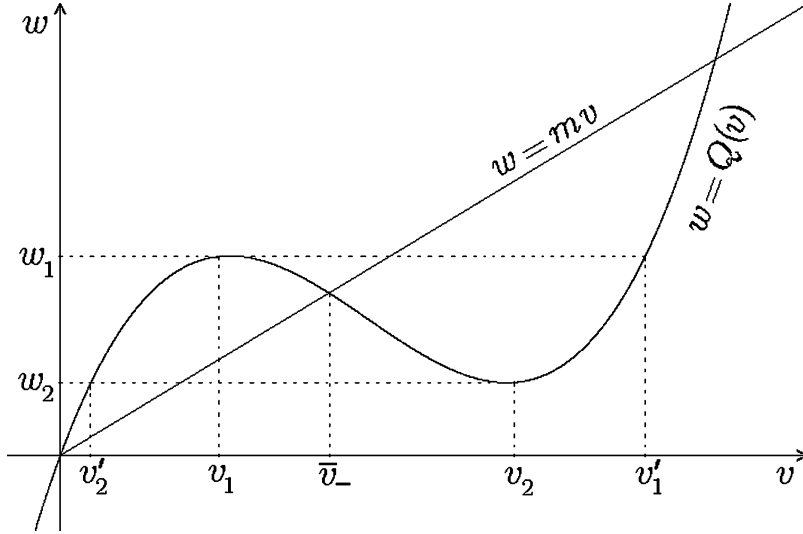
$$\begin{cases} \beta\delta\bar{w} - \gamma(\bar{v})\bar{u} - f\bar{u} = 0, \\ f\bar{u} - h\bar{v} = 0, \\ -\beta\bar{w} + \alpha\bar{v} = 0. \end{cases} \quad (2.1)$$

Clearly, this system is reduced to

$$\begin{cases} \bar{w} = Q(\bar{v}) \equiv \frac{h}{f\beta\delta} \{\gamma(\bar{v}) + f\}\bar{v}, \\ \bar{w} = m\bar{v} \equiv \frac{\alpha}{\beta}\bar{v}, \end{cases} \quad (2.2)$$

where  $Q$  denotes a cubic function and  $m = \frac{\alpha}{\beta}$  is a gradient of linear curve (see Figure 2). So, the structure of non-negative solutions to (2.2), and hence (2.1), is described by

- (1) When  $0 < h \leq \frac{f\alpha\delta}{ab^2+c+f}$ , (2.1) has two solutions  $O = (0, 0, 0)$  and  $P_+ = (\frac{h}{f}(b + \sqrt{D}), b + \sqrt{D}, \frac{\alpha}{\beta}(b + \sqrt{D}))$ , where  $D = \frac{f\alpha\delta - (c+f)h}{ah}$ ;
- (2) When  $\frac{f\alpha\delta}{ab^2+c+f} < h < \frac{f\alpha\delta}{c+f}$ , (2.1) has three solutions  $O = (0, 0, 0)$  and  $P_{\pm} = (\frac{h}{f}(b \pm \sqrt{D}), b \pm \sqrt{D}, \frac{\alpha}{\beta}(b \pm \sqrt{D}))$ , where  $D$  is as in (1);


 Figure 2. Graphs of  $w = Q(v)$  and  $w = mv$ .

- (3) When  $h = \frac{f\alpha\delta}{c+f}$ , (2.1) has two solutions  $O = (0, 0, 0)$  and  $P = P_{\pm} = (\frac{bh}{f}, b, \frac{\alpha b}{\beta})$ ;  
 (4) When  $\frac{f\alpha\delta}{c+f} < h < \infty$ , (2.1) has a unique solution  $O = (0, 0, 0)$ .

**2.2. Stability and instability of homogeneous stationary solutions.** Let  $\bar{P} = (\bar{u}, \bar{v}, \bar{w})$  be one of the three homogeneous stationary solutions  $O$ ,  $P_+$  and  $P_-$ . We now study its stability or instability. For this purpose, we will localize problem (1.1) in a neighbourhood of  $\bar{P}$  and extend the phase space  $K$  of the dynamical system  $(S(t), K, X)$  determined from (1.1) to a suitable one containing complex-valued functions in the neighbourhood. And we will apply the general strategy announced in the Appendix for the complexified dynamical system to construct the stable and unstable manifolds in the neighbourhood.

We introduce three cutoff functions  $\chi_{\bar{u}}(\lambda)$ ,  $\chi_{\bar{v}}(\lambda)$  and  $\chi_{\bar{w}}(\lambda)$  defined on the complex plane  $\mathbb{C}$  as follows:  $\chi_{\bar{u}}(\lambda) = \lambda$  for  $\lambda : |\lambda - \bar{u}| < 1$ ,  $\chi_{\bar{u}}(\lambda)$  vanishes for  $\lambda : |\lambda - \bar{u}| > 2$ , and  $\chi_{\bar{u}}(\lambda)$  is a smooth function in the real variables  $\lambda'$  and  $\lambda''$  such that  $\lambda = \lambda' + i\lambda''$ . It is similar for the definitions of  $\chi_{\bar{v}}(\lambda)$  and  $\chi_{\bar{w}}(\lambda)$ .

The localized problem is then written in the form

$$\begin{cases} \frac{\partial u}{\partial t} = \beta\delta\chi_{\bar{w}}(w) - \gamma(\chi_{\bar{v}}(v))\chi_{\bar{u}}(u) - fu & \text{in } \Omega \times (0, \infty), \\ \frac{\partial v}{\partial t} = f\chi_{\bar{u}}(u) - hv & \text{in } \Omega \times (0, \infty), \\ \frac{\partial w}{\partial t} = d\Delta w - \beta w + \alpha\chi_{\bar{v}}(v) & \text{in } \Omega \times (0, \infty), \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x) & \text{in } \Omega. \end{cases} \quad (2.3)$$

We can handle this localized problem in a quite analogous way as for the original one. In fact, as before, the problem (2.3) is formulated as the Cauchy problem for an

abstract evolution equation

$$\begin{cases} \frac{dU}{dt} + AU = \tilde{F}(U), & 0 < t \leq \infty, \\ U(0) = U_0 \end{cases} \quad (2.4)$$

in the function space  $X$  (see (1.2)). Here, the linear operator  $A$  is defined by

$$A = \begin{pmatrix} f & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & \Lambda \end{pmatrix} \quad \text{with} \quad \mathcal{D}(A) = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix}; u, v \in L^\infty(\Omega) \text{ and } w \in H_N^2(\Omega) \right\},$$

where  $\Lambda$  is realization of Laplace operator  $-d\Delta + \beta$  in  $L^2(\Omega)$  under Neumann boundary conditions on the boundary  $\partial\Omega$  (see [9, Chapter VI]). It is known that  $\Lambda$  is a positive definite self-adjoint operator of  $L^2(\Omega)$  with (see [10, 12, 18])

$$\mathcal{D}(\Lambda^\theta) = \begin{cases} H^{2\theta}(\Omega) & \text{when } 0 \leq \theta < \frac{3}{4}, \\ H_N^{2\theta}(\Omega) = \{u \in H^{2\theta}(\Omega); \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\} & \text{when } \frac{3}{4} < \theta \leq 1. \end{cases}$$

It is clear that  $A$  is sectorial operator with angle less than  $\frac{\pi}{2}$ . Moreover, for  $0 \leq \theta \leq 1$ ,  $\theta \neq \frac{3}{4}$ ,

$$A^\theta = \begin{pmatrix} f^\theta & 0 & 0 \\ 0 & h^\theta & 0 \\ 0 & 0 & \Lambda^\theta \end{pmatrix} \quad \text{with} \quad \mathcal{D}(A^\theta) = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix}; u, v \in L^\infty(\Omega) \text{ and } w \in \mathcal{D}(\Lambda^\theta) \right\}.$$

The non-linear operator  $\tilde{F}$  is given by

$$\tilde{F}(U) = \begin{pmatrix} \beta\delta\chi_{\bar{w}}(w) - \gamma(\chi_{\bar{v}}(v))\chi_{\bar{u}}(u) \\ f\chi_{\bar{u}}(u) \\ \alpha\chi_{\bar{v}}(v) \end{pmatrix}, \quad U = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \in \mathcal{D}(A^\eta),$$

where  $\eta$  is an arbitrarily fixed exponent such that  $\frac{1}{2} < \eta < 1$ . Initial value  $U_0$  is taken from  $\mathcal{D}(A^\mu)$  with  $\frac{1}{2} < \mu < \eta$ .

Since  $\chi_{\bar{u}}(u)$ ,  $\chi_{\bar{v}}(v)$  and  $\chi_{\bar{w}}(w)$  are uniformly bounded, we can repeat the same arguments as in [7] (cf. also [14]) to construct local and global solutions for every initial value  $U_0$  from  $\mathcal{D}(A^\mu)$  in the function space

$$U \in \mathcal{C}([0, \infty); \mathcal{D}(A^\mu)) \cap \mathcal{C}^1((0, \infty); X) \cap \mathcal{C}((0, \infty); \mathcal{D}(A)).$$

Therefore, the localized problem (2.3) defines a semigroup  $\tilde{S}(t)$  acting on

$$\mathcal{D}_\mu = \mathcal{D}(A^\mu), \quad \frac{1}{2} < \mu < \eta < 1.$$

Since the similar Lipschitz continuity of solutions as in [7, Proposition 5.3] is also valid, the problem (2.3) defines a dynamical system  $(\tilde{S}(t), \mathcal{D}_\mu, \mathcal{D}_\mu)$  in the universal space  $\mathcal{D}_\mu$  with a whole phase space  $\mathcal{D}_\mu$ .

As  $\frac{1}{2} < \mu < 1$ , we have  $\mathcal{D}_\mu \subset \mathbb{L}^\infty(\Omega) \equiv L^\infty(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega)$ . Then, in a suitable neighbourhood of  $\bar{P}$  in  $\mathcal{D}_\mu$ , any solution of the original problem (1.1) is

a solution of (2.3). In other words, in such a neighbourhood, any trajectory of  $(S(t), \mathcal{K}_\mu, \mathcal{D}_\mu)$ , where  $\mathcal{K}_\mu = K \cap \mathcal{D}_\mu$ , is that of  $(\tilde{S}(t), \mathcal{D}_\mu, \mathcal{D}_\mu)$ ; conversely, any non-negative trajectory of  $(\tilde{S}(t), \mathcal{D}_\mu, \mathcal{D}_\mu)$  in the neighbourhood is that of  $(S(t), \mathcal{K}_\mu, \mathcal{D}_\mu)$ . Clearly,  $\bar{P}$  is an equilibrium of  $(\tilde{S}(t), \mathcal{D}_\mu, \mathcal{D}_\mu)$  also. Furthermore, we notice that, if  $\bar{P}$  is stable as an equilibrium of  $(\tilde{S}(t), \mathcal{D}_\mu, \mathcal{D}_\mu)$ , then it is the same as that of  $(S(t), \mathcal{K}_\mu, \mathcal{D}_\mu)$ . However, we cannot say that, even if  $\bar{P}$  is unstable in  $(\tilde{S}(t), \mathcal{D}_\mu, \mathcal{D}_\mu)$ , it is the same as that of  $(S(t), \mathcal{K}_\mu, \mathcal{D}_\mu)$ . Nevertheless, instability of  $\bar{P}$  in  $(\tilde{S}(t), \mathcal{D}_\mu, \mathcal{D}_\mu)$  provides crucial information concerning the behaviour of trajectories of the original system  $(S(t), \mathcal{K}_\mu, \mathcal{D}_\mu)$  in the neighbourhood of  $\bar{P}$ , for, as the Theorem A.1 shows, the unstable manifold of  $\bar{P}$  in  $(\tilde{S}(t), \mathcal{D}_\mu, \mathcal{D}_\mu)$  is tangential at  $\bar{P}$  to a subspace of the form  $\bar{P} + X_+$ , where  $X_+$  is a linear subspace of  $X$  having a basis consisting of real functions. For the details, see the proofs of Theorems 2.2 and 2.4.

Our goal is therefore to apply Theorem A.2 to the localized problem (2.4). Let us first verify Fréchet differentiability of  $\tilde{S}(t)$  in a neighbourhood of  $\bar{P}$ . In a neighbourhood of  $\bar{P}$  in  $\mathcal{D}(A^\eta) \subset \mathcal{D}(A^\mu) \subset \mathbb{L}^\infty(\Omega)$ ,  $\tilde{F}$  is Fréchet differentiable with the derivative

$$\tilde{F}'(U) = \begin{pmatrix} -\gamma(v) & -\gamma'(v)u & \beta\delta \\ f & 0 & 0 \\ 0 & \alpha & 0 \end{pmatrix}, \quad U \in B^{\mathcal{D}(A^\eta)}(\bar{P}, r).$$

By a direct calculation the derivative  $F'(U)$  is seen to fulfill the assumptions (A.4) and (A.5) of Theorem A.2. Hence, by Theorem A.2, the semigroup  $\tilde{S}(t)$  is also Fréchet differentiable in a neighbourhood. In particular, the Fréchet derivative of  $\tilde{S}(t)$  at  $\bar{P}$  is given by  $\tilde{S}'(t)\bar{P} = e^{-t\bar{A}}$ , where  $e^{-t\bar{A}}$  is an analytic semigroup on  $X$  generated by

$$\bar{A} = A - \tilde{F}'(\bar{P}) = \begin{pmatrix} M & N & -\beta\delta \\ -f & h & 0 \\ 0 & -\alpha & \Lambda \end{pmatrix}, \quad (2.5)$$

where  $M = \gamma(\bar{v}) + f$  and  $N = 2a\bar{u}(\bar{v} - b)$ .

Let us next verify the hyperbolicity of  $\bar{P}$ , namely, let us verify the condition (A.2). As Theorem A.2 shows again, it is sufficient to verify that

$$\sigma(\bar{A}) \cap \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda = 0\} = \emptyset.$$

To this end, let us consider a proper value problem

$$(\lambda - \bar{A}) \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix},$$

or equivalently

$$\begin{cases} (\lambda - M)u - Nv + \beta\delta w = p, \\ fu + (\lambda - h)v = q, \\ \alpha v + (\lambda - \Lambda)w = r \end{cases}$$

for  $(u, v, w) \in \mathcal{D}(\bar{A}) = \mathcal{D}(A)$ ,  $(p, q, r) \in X$  and  $\lambda \in \mathbb{C}$ . It then follows that

$$\begin{aligned} & \{[(\lambda - M)(\lambda - h) + Nf](\lambda - \Lambda) + f\alpha\beta\delta\}w \\ & = f\alpha p - \alpha(\lambda - M)q + [(\lambda - M)(\lambda - h) + Nf]r. \end{aligned}$$



If  $\lambda$  is a solution to the quadratic equation

$$(\lambda - M)(\lambda - h) + Nf = 0, \quad (2.6)$$

then  $w$  cannot belong to  $H^2(\Omega)$  in general, i.e.,  $\lambda \in \sigma(\bar{A})$ . Now, let  $\lambda$  do not satisfy (2.6), then  $\lambda \in \sigma(\bar{A})$  if and only if  $\lambda + \frac{f\alpha\beta\delta}{(\lambda-M)(\lambda-h)+Nf} \in \sigma(\Lambda)$ . In other words,  $\lambda \in \sigma(\bar{A})$  if and only if  $\lambda$  is a solution to one of the following cubic equations:

$$[(\lambda - M)(\lambda - h) + Nf](\lambda - d\mu_n - \beta) + f\alpha\beta\delta = 0, \quad (2.7)$$

where

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \dots \rightarrow \infty$$

are the infinite number of eigenvalues of the Laplace operator  $-\Delta$  in  $L^2(\Omega)$  equipped with the Neumann boundary conditions.

Thus, we arrive at the following general result.

**THEOREM 2.1.** *The homogeneous stationary solutions  $\bar{P}$  is a hyperbolic equilibrium if and only if  $Mh + Nf \neq 0$  and  $(Mh + Nf)(d\mu_n + \beta) - f\alpha\beta\delta \neq 0$  for  $n = 0, 1, 2, \dots$*

*Proof.* Necessity is trivial because if  $Mh + Nf = 0$  or  $(Mh + Nf)(d\mu_n + \beta) - f\alpha\beta\delta = 0$  with some  $\mu_n$  then  $\lambda = 0$  is an eigenvalue of  $\bar{A}$ .

For sufficiency, let  $Mh + Nf \neq 0$  and  $(Mh + Nf)(d\mu_n + \beta) - f\alpha\beta\delta \neq 0$  for  $n = 0, 1, 2, \dots$ . It is easy to see that equation (2.6) has no imaginary solution. Assume that  $\lambda = iy$ ,  $y \in \mathbb{R}$  is a solution of (2.7) with  $\mu_n = \mu_{n_0}$ . Then, by direct calculation, we get  $y \neq 0$  and

$$\begin{cases} y^2 = (M + h)(d\mu_{n_0} + \beta) + Mh + Nf, \\ y^2(M + h + d\mu_{n_0} + \beta) = (Mh + Nf)(d\mu_{n_0} + \beta) - f\alpha\beta\delta. \end{cases}$$

But this system has no solution for every  $\mu_{n_0} \geq 0$ , which is a contradiction to the assumption.  $\square$

From now let us consider the particular cases.

*Case 1.*  $\bar{P} = O$ . In this the case, we have  $M = ab^2 + c + f$  and  $N = 0$ .

**THEOREM 2.2.** (i) *Let  $0 < h < \frac{f\alpha\delta}{ab^2+c+f}$  and let the conditions*

$$\mu_n \neq \frac{\beta\{f\alpha\delta - (ab^2 + c + f)h\}}{(ab^2 + c + f)hd}, \quad n = 0, 1, 2, \dots$$

*be satisfied. Then,  $O$  is an unstable equilibrium of  $(\tilde{S}(t), \mathcal{D}_\mu, \mathcal{D}_\mu)$ .*

(ii) *Let  $\frac{f\alpha\delta}{ab^2+c+f} < h < \infty$ . Then  $O$  is an exponentially stable equilibrium of  $(\tilde{S}(t), \mathcal{D}_\mu, \mathcal{D}_\mu)$ .*

*Proof.* (i) By using Theorem 2.1, we obtain that  $O$  is a hyperbolic equilibrium. It is now suffices to verify that  $\tilde{S}(t)O$  has spectra in the region  $\{\lambda \in \mathbb{C}; |\lambda| > 1\}$  or equivalently  $\sigma(\bar{A}) \cap \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda < 0\} \neq \emptyset$ . By virtue of Routh–Hurwitz theorem, we verify that, for  $\mu_n$  satisfying  $\mu_n > \frac{\beta\{f\alpha\delta - (ab^2 + c + f)h\}}{(ab^2 + c + f)hd}$ , equation (2.7) has all solutions in the region  $\{\lambda \in \mathbb{C}; \operatorname{Re}\lambda > 0\}$ . In addition, for  $0 \leq \mu_n < \frac{\beta\{f\alpha\delta - (ab^2 + c + f)h\}}{(ab^2 + c + f)hd}$ , equation (2.7)

has a negative real solution  $\lambda_n$ , namely,

$$\lambda_n \in \sigma_-(\bar{A}) = \sigma(\bar{A}) \cap \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda < 0\}.$$

Let  $X_-$  denotes the subspace of  $\mathcal{D}_\mu$  corresponding to the spectral set  $\sigma_-(\bar{A})$ . Then, there exists a smooth unstable manifold  $\mathcal{M}_+(O)$  with dimension  $\dim X_-$  which is tangential to the subspace  $X_-$  at  $O$ . More precisely,  $\sigma_-(\bar{A})$  consists of finite number of eigenvalues and the space  $X_-$  corresponding to  $\sigma_-(\bar{A})$  is a finite-dimensional subspace spanned by vectors of the form

$$\begin{pmatrix} \beta\delta(h - \lambda_n) \\ f\beta\delta \\ (ab^2 + c + f - \lambda_n)(h - \lambda_n) \end{pmatrix} \phi_n, \quad 0 \leq \mu_n < \frac{\beta\{f\alpha\delta - (ab^2 + c + f)h\}}{(ab^2 + c + f)hd},$$

where  $\phi_n$  denotes a real eigenfunction of  $-\Delta$  corresponding to the eigenvalue  $\mu_n$ .

(ii) In this case, we verify by Routh–Hurwitz theorem that all equations in (2.6) and (2.7) have all their solutions in the region  $\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda > 0\}$ . Therefore,  $O$  is exponentially stable equilibrium of  $(\tilde{S}(t), \mathcal{D}_\mu, \mathcal{D}_\mu)$ .  $\square$

*Case 2.  $\bar{P} = P_+$ .* In this the case, we have  $M = \frac{f\alpha\delta}{h}$  and  $N = \frac{2ah}{f}(D + b\sqrt{D})$ . Then, by Routh–Hurwitz theorem, we observe that all equations in (2.6) and (2.7) have all their solutions in the region  $\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda > 0\}$ , which implies that  $P_+$  is exponentially stable equilibrium of  $(\tilde{S}(t), \mathcal{D}_\mu, \mathcal{D}_\mu)$ .

**THEOREM 2.3.** *Let  $0 < h < \frac{f\alpha\delta}{c+f}$ . Then,  $P_+$  is an exponentially stable equilibrium of  $(\tilde{S}(t), \mathcal{D}_\mu, \mathcal{D}_\mu)$ .*

*Case 3.  $\bar{P} = P_-$ .* In this the case, we have  $M = \frac{f\alpha\delta}{h}$  and  $N = \frac{2ah}{f}(D - b\sqrt{D}) < 0$ .

**THEOREM 2.4.** (i) *Let  $Mh + Nf > 0$  and let the conditions*

$$\mu_n \neq \frac{\beta}{d} \left( \frac{f\alpha\delta}{Mh + Nf} - 1 \right), \quad n = 0, 1, 2, \dots$$

*be satisfied. Then,  $P_-$  is an unstable equilibrium of  $(\tilde{S}(t), \mathcal{D}_\mu, \mathcal{D}_\mu)$ .*

(ii) *Let  $Mh + Nf < 0$ , then  $P_-$  is an unstable equilibrium of  $(\tilde{S}(t), \mathcal{D}_\mu, \mathcal{D}_\mu)$ .*

*Proof.* (i) In view of Theorem 2.1, we have  $P_-$  as a hyperbolic equilibrium. By virtue of Routh–Hurwitz theorem, for  $\mu_n$  satisfying  $\mu_n > \frac{\beta}{d} \left( \frac{f\alpha\delta}{Mh + Nf} - 1 \right)$ , all solutions of equation (2.7) lie in the region  $\{\lambda \in \mathbb{C}; \operatorname{Re} \lambda > 0\}$ . In addition, for  $0 \leq \mu_n < \frac{\beta}{d} \left( \frac{f\alpha\delta}{Mh + Nf} - 1 \right)$ , equation (2.7) has a negative real solution  $\lambda_n$ , namely,

$$\lambda_n \in \sigma_-(\bar{A}) = \sigma(\bar{A}) \cap \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda < 0\}.$$

Let  $X_-$  denote the subspace of  $\mathcal{D}_\mu$  corresponding to the spectral subset  $\sigma_-(\bar{A})$ . Then, there exists a smooth unstable manifold  $\mathcal{M}_+(P_-)$  with dimension  $\dim X_-$  which is tangential to  $P_- + X_-$  at  $P_-$ . The space  $X_-$  contains at least vectors of the form

$$\begin{pmatrix} \beta\delta(h - \lambda_n) \\ f\beta\delta \\ (M - \lambda_n)(h - \lambda_n) + Nf \end{pmatrix} \phi_n, \quad 0 \leq \mu_n < \frac{\beta}{d} \left( \frac{f\alpha\delta}{Mh + Nf} - 1 \right),$$

where  $\phi_n$  denotes a real eigenfunction of  $-\Delta$  corresponding to the eigenvalue  $\mu_n$ .

(ii) In view of Theorem 2.1, we see that  $P_-$  is a hyperbolic equilibrium. In addition, we verify that every cubic equation in (2.7) has a negative real solution  $\lambda_n$ , namely,  $\lambda_n \in \sigma(\bar{A}) \cap \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda < 0\}$  for all  $n$ . Therefore, in this case, there exists a smooth unstable manifold  $\mathcal{M}_+(P_-)$  with dimension  $\dim X_- = \infty$  which is tangential to  $P_- + X_-$  at  $P_-$ . The space  $X_-$  contains an infinite number of vectors of the form

$$\begin{pmatrix} \beta\delta(h - \lambda_n) \\ f\beta\delta \\ (M - \lambda_n)(h - \lambda_n) + Nf \end{pmatrix} \phi_n, \quad n = 0, 1, 2, \dots$$

□

REMARK 2.5. The condition  $Mh + Nf < 0$  is equivalent to  $ab^2 > 3(c + f)$  and

$$\begin{aligned} & \frac{f\alpha\delta\{ab^2 + 3(c + f) - \sqrt{ab^2[ab^2 - 3(c + f)]}\}}{2(c + f)(ab^2 + c + f)} < h \\ & < \frac{f\alpha\delta\{ab^2 + 3(c + f) + \sqrt{ab^2[ab^2 - 3(c + f)]}\}}{2(c + f)(ab^2 + c + f)}. \end{aligned}$$

**2.3. Non-existence of inhomogeneous stationary solutions.** In this subsection, we will show that in some cases, there is no non-negative stationary solution other than homogeneous ones.

THEOREM 2.6. *Let  $\frac{f\alpha\delta}{c+f} < h < \infty$ . If  $\bar{U} \in \mathcal{D}(A)$  is a non-negative stationary solution of (1.1), then  $\bar{U}$  necessarily coincides with the zero solution  $O$ .*

*Proof.* Let  $\bar{U} = (\bar{u}, \bar{v}, \bar{w})$  be any non-negative stationary solution. Then,  $\bar{v}$  and  $\bar{w}$  satisfy the following elliptic-algebraic system

$$\begin{cases} \bar{w} = Q(\bar{v}) & \text{in } \Omega, \\ d\Delta\bar{w} - \beta\bar{w} = -\alpha\bar{v} & \text{in } \Omega, \\ \frac{\partial\bar{w}}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.8)$$

Since the linear curve in (2.2) lies under the graph of the cubic curve for  $\bar{v}, \bar{w} \geq 0$ , the first equation of (2.8) implies that  $\bar{w}(x) \geq m\bar{v}(x)$  for almost all  $x \in \Omega$ , that is,  $\beta\bar{w} - \alpha\bar{v} \geq 0$  for almost all  $x \in \Omega$ . On the other hand,  $\int_{\Omega} \{\beta\bar{w} - \alpha\bar{v}\} dx = d \int_{\Omega} \Delta\bar{w} dx = 0$ . Therefore,  $\beta\bar{w} - \alpha\bar{v} = 0$  and hence  $\Delta\bar{w} = 0$  for almost all  $x \in \Omega$ . Furthermore, it follows that  $\int_{\Omega} |\nabla\bar{w}|^2 dx = - \int_{\Omega} \Delta\bar{w} \bar{w} dx = 0$  and that  $\bar{w}$  is a constant. □

In the one-dimensional case, we can show the similar result for the case when  $0 < h < \frac{f\alpha\delta}{ab^2+c+f}$ .

THEOREM 2.7. *Let  $\Omega = (0, \ell)$  and let  $0 < h < \frac{f\alpha\delta}{ab^2+c+f}$ . If  $\bar{U} \in \mathcal{D}(A)$  is a non-negative stationary solution of (1.1), then  $\bar{U}$  necessarily coincides with one of the homogeneous stationary solutions  $O$  and  $P_+$ .*

*Proof.* Let  $\bar{U} = (\bar{u}, \bar{v}, \bar{w})$  be any non-negative stationary solution. Put

$$\bar{w}(x_1) = \max_{x \in [0, \ell]} \bar{w}(x) \equiv \bar{w}_1.$$

Then, we have  $\bar{w}'(x_1) = 0$ . Indeed, it is clear if  $x_1 \in (0, \ell)$ . In the case when  $x_1 = 0$  or  $x_1 = \ell$ , this follows from the Neumann boundary conditions.

Furthermore, we can deduce that the value  $\bar{w}_1$  satisfies  $\bar{w}_1 \leq m(b + \sqrt{D})$ . In fact, assume that  $\bar{w}_1 > m(b + \sqrt{D})$ . Then, since  $\bar{w}(x)$  is a continuous function of  $x \in [0, \ell]$  and since the values  $(\bar{v}(x), \bar{w}(x))$  lie on the cubic curve in (2.2), there exists a number  $\epsilon > 0$  and a neighbourhood of  $x_1$  in  $[0, \ell]$  in which  $\bar{w}(x) \geq m\bar{v}(x) + \epsilon$  is valid. Consequently,  $\bar{w}''(x) = \frac{\beta}{d}\{\bar{w}(x) - m\bar{v}(x)\} \geq \frac{\beta\epsilon}{d}$  in the neighbourhood of  $x_1$ . Furthermore, since

$$\bar{w}(x) = \bar{w}(x_1) + \bar{w}'(x_1)(x - x_1) + \int_{x_1}^x (x - y)\bar{w}''(y)dy,$$

it follows that  $\bar{w}(x) \geq \bar{w}_1 + \frac{\beta\epsilon}{2d}(x - x_1)^2$  for all  $x$  in the neighbourhood. This is obviously a contradiction.

Since  $0 \leq \bar{w}(x) \leq \bar{w}_1 \leq m(b + \sqrt{D})$ , and since the values  $(\bar{v}(x), \bar{w}(x))$  lie on the cubic curve, we observe that  $\bar{w}(x) \leq m\bar{v}(x)$ , that is,  $\beta\bar{w} - \alpha\bar{v} \leq 0$  for almost all  $x \in (0, \ell)$ . Then, by the same argument as in proof of Theorem 2.6, we conclude that  $\bar{w}$  is constant.  $\square$

In the two-dimensional case, if we add assumptions that  $\Omega$  is a  $\mathcal{C}^2$  domain and  $\bar{w} \in \mathcal{C}^2(\Omega)$ , then we can repeat the same argument as in the proof of Theorem 2.7 to prove the following result.

**THEOREM 2.8.** *Let  $\Omega$  be  $\mathcal{C}^2$  domain in  $\mathbb{R}^2$  and let  $0 < h < \frac{f\alpha\delta}{ab^2+c+f}$ . Let  $\bar{U} \in \mathcal{D}(A)$  is a non-negative stationary solution of (1.1) with  $\bar{w} \in \mathcal{C}^2(\Omega)$ . Then,  $\bar{U}$  necessarily coincides with one of the homogeneous stationary solutions  $O$  and  $P_+$ .*

**3. Discontinuous stationary solutions.** In this section, we intend to construct discontinuous stationary solutions of (1.1) which is obviously the solutions of (2.8).

We assume that  $\Omega$  is a rectangular domain in  $\mathbb{R}^2$  and that the coefficients satisfy the relations  $\frac{f\alpha\delta}{ab^2+c+f} < h < \frac{f\alpha\delta}{c+f}$  and  $ab^2 > 3(c+f)$ . We already know that, when  $0 < h < \frac{f\alpha\delta}{ab^2+c+f}$  or  $\frac{f\alpha\delta}{c+f} < h < \infty$ , one cannot expect existence of any inhomogeneous stationary solution (cf. Theorems 2.6–2.8). In addition, when  $ab^2 \leq 3(c+f)$ , we have  $Q'(v) \geq 0$  and therefore  $Q(v)$  is monotone increasing. Hence  $v = Q^{-1}(w)$  is a single-valued continuous function for  $-\infty < \bar{w} < \infty$  and (2.8) has no discontinuous solutions.

So, let the two relations  $\frac{f\alpha\delta}{ab^2+c+f} < h < \frac{f\alpha\delta}{c+f}$  and  $ab^2 > 3(c+f)$  be satisfied. Then, there exists the homogeneous stationary solution  $P_- = (\bar{u}_-, \bar{v}_-, \bar{w}_-)$  as defined in Section 2. In addition, the equation  $Q'(v) = 0$  has two positive solutions  $0 < v_1 < v_2 < \infty$ . We here set the two more points  $v'_1 > v_1$  and  $v'_2 < v_2$  in such a way that  $Q(v'_1) = Q(v_1)$  and  $Q(v'_2) = Q(v_2)$ , respectively. It is then clear that  $0 < v'_2 < v_1 < v_2 < v'_1 < \infty$  (see Figure 2). We now make a basic assumption

$$v'_2 < \bar{v}_- < v'_1. \quad (3.1)$$

Under this assumption, we take two points  $(V_+, W_+)$  and  $(V_-, W_-)$  on the cubic curve  $w = Q(v)$  in the  $(v, w)$ -plane which satisfy the following conditions: (1)  $Q(v_2) < W_+ = W_- < Q(v_1)$ ; (2)  $W_+ > mV_+$  and  $W_- < mV_-$ , respectively and (3)  $v'_2 < V_+ < v'_1 < V_- < v_1$ . Remember that  $\bar{v}_-$  was obtained as the intersection point of the cubic curve and the linear curve  $w = mv$ .

Let  $W_0 = W_{\pm}$  and let  $[W_0 - \omega, W_0 + \omega]$  be a sufficiently small neighbourhood of  $W_0$  in the  $w$ -axis such that one can define single-valued smooth branches  $v = Q_{\pm}^{-1}(w)$  for  $w \in [W_0 - \omega, W_0 + \omega]$  of the multi-valued function  $v = Q^{-1}(w)$  such that  $V_+ = Q_+^{-1}(W_0)$  and  $V_- = Q_-^{-1}(W_0)$ . In addition, let  $w - mQ_+^{-1}(w) \geq \varepsilon$  and  $w - mQ_-^{-1}(w) \leq -\varepsilon$  for  $w \in [W_0 - \omega, W_0 + \omega]$ , respectively, with a suitable constant  $\varepsilon > 0$ .

We now introduce the Cauchy problem for some ordinary differential equation

$$\begin{cases} dw'' = \beta\{w - mQ_-^{-1}(w)\}, & 0 \leq x < \infty, \\ w(0) = W_0, \quad w'(0) = v > 0. \end{cases} \quad (3.2)$$

For each number  $v_0 > 0$ , there exists an interval  $[0, \ell]$  ( $\ell > 0$ ) of  $x$  such that, for any initial differential quotient  $v \in (0, v_0]$ , (3.2) has a unique solution  $w_v$  at least on the fixed interval  $[0, \ell]$ . For such a solution  $w_v$ , we see that

$$w'_v(x) = w'_v(0) + \int_0^x w''_v(\tau) d\tau < v - \frac{\beta\varepsilon x}{d}, \quad 0 \leq x \leq \ell.$$

Therefore, if  $0 < v \leq \frac{\beta\varepsilon\ell}{d}$ , then each  $w_v$  has a unique point  $x = \ell_v \leq \ell$  such that  $w'_v(\ell_v) = 0$  and  $w'_v(x) > 0$  for  $0 \leq x < \ell_v$ .

LEMMA 3.1. *The point  $\ell_v$  is continuous for  $0 < v \leq \frac{\beta\varepsilon\ell}{d}$  and  $\lim_{v \rightarrow 0} \ell_v = 0$ .*

*Proof.* Let  $z = w'_v$ . Then,  $x \in [0, \ell_v]$  and  $z = w'_v(x) \in [0, v]$  have one to one correspondence. In addition,  $\ell_v$  is given by the formula

$$\ell_v - 0 = \int_v^0 \frac{dx}{dz} dz = - \int_0^v \frac{1}{w''_v} dz = - \int_0^v \frac{d}{\beta\{w_v - mQ_-^{-1}(w_v)\}} dz.$$

Meanwhile, since

$$\frac{dw_v}{dz} = \frac{dw_v}{dx} / \frac{dz}{dx} = \frac{zd}{\beta\{w_v - mQ_-^{-1}(w_v)\}},$$

it follows that

$$z \frac{dz}{dw_v} = \frac{\beta}{d} \{w_v - mQ_-^{-1}(w_v)\}.$$

By integration,

$$\Xi(w_v) = \frac{1}{2}(z^2 - v^2), \quad \text{where} \quad \Xi(w_v) = \int_{W_0}^{w_v} \frac{\beta}{d} \{w_v - mQ_-^{-1}(w_v)\} dw_v.$$

Therefore, we obtain the formula

$$\ell_v = - \int_0^v \frac{d}{\beta\{\Xi^{-1}[\frac{1}{2}(z^2 - v^2)] - mQ_-^{-1}(\Xi^{-1}[\frac{1}{2}(z^2 - v^2)])\}} dz.$$

This shows that  $\ell_v$  depends on  $v$  continuously. □

Similarly, we consider the backward Cauchy problem

$$\begin{cases} dw'' = \beta\{w - mQ_+^{-1}(w)\}, & -\infty < x \leq 0, \\ w(0) = W_0, \quad w'(0) = v > 0. \end{cases} \quad (3.3)$$

Then, for each number  $\nu_0 > 0$ , there exists an interval  $[-\ell', 0]$  ( $\ell' > 0$ ) of  $x$  such that, for any  $\nu \in (0, \nu_0]$ , (3.3) has a unique solution  $w_\nu$  at least on the fixed interval  $[-\ell', 0]$ . We can similarly verify that if  $0 < \nu \leq \frac{\beta\epsilon\ell'}{d}$  then each  $w_\nu$  has a unique point  $x = -\ell'_\nu \geq -\ell'$  such that  $w'_\nu(-\ell'_\nu) = 0$  and  $w'_\nu(x) > 0$  for  $-\ell'_\nu < x \leq 0$ . Furthermore, by the same proof as for Lemma 3.1, we see that  $\ell'_\nu$  is continuous for  $0 < \nu \leq \frac{\beta\epsilon\ell'}{d}$  and  $\lim_{\nu \rightarrow 0} \ell'_\nu = 0$ .

Joining these two solutions, we accomplish construction of the discontinuous stationary solution. Indeed, take a  $\nu$  so that  $0 < \nu \leq \min\{\frac{\beta\epsilon\ell}{d}, \frac{\beta\epsilon\ell'}{d}\}$  and consider a rectangular domain  $\Omega = (-\ell'_\nu, \ell_\nu) \times I_y$ , where  $I_y$  is any bounded open interval for the variable  $y$ . Let  $\bar{w}(x, y) = w_\nu(x)$  for  $(x, y) \in \Omega$  and  $\bar{v}(x, y) = Q_+^{-1}(w_\nu(x))$  for  $(x, y) \in (-\ell'_\nu, 0] \times I_y$  and  $\bar{v}(x, y) = Q_-^{-1}(w_\nu(x))$  for  $(x, y) \in (0, \ell_\nu) \times I_y$ . It is then easily verified that the pair of functions  $\bar{v}(x, y)$  and  $\bar{w}(x, y)$  is certainly a solution of (2.8).

Let now  $I_x$  be any bounded open interval for the variable  $x$  and let  $\Omega = I_x \times I_y$ . Since  $\ell_\nu + \ell'_\nu$  is continuous for  $0 < \nu \leq \min\{\frac{\beta\epsilon\ell}{d}, \frac{\beta\epsilon\ell'}{d}\}$  and  $\lim_{\nu \rightarrow 0} (\ell_\nu + \ell'_\nu) = 0$ , there exists an integer  $n$  and a suitable  $\nu$  such that  $|I_x| = n(\ell_\nu + \ell'_\nu)$ . We already know existence of discontinuous solution to (2.8) in the domain  $(-\ell'_\nu, \ell_\nu) \times I_y$ . Then, by reflexion, we can construct a discontinuous solution in the domain  $I_x \times I_y$  also.

In this way, when  $\frac{f\alpha\delta}{ab^2+c+f} < h < \frac{f\alpha\delta}{c+f}$  and  $ab^2 > 3(c+f)$  and (3.1) is satisfied, we have shown that, in any rectangular domain  $\Omega$ , there exists an infinite number of discontinuous stationary solutions to (1.1).

REMARK 3.2. (i) In the case when  $3(c+f) < ab^2 \leq 4(c+f)$ , condition (3.1) is equivalent to

$$\frac{9f\alpha\delta}{5ab^2 - 3(c+f) + 4\sqrt{ab^2[ab^2 - 3(c+f)]}} < h < \frac{9f\alpha\delta}{5ab^2 - 3(c+f) - 4\sqrt{ab^2[ab^2 - 3(c+f)]}}.$$

(ii) In the case when  $ab^2 > 4(c+f)$ , condition (3.1) is equivalent to

$$\frac{9f\alpha\delta}{5ab^2 - 3(c+f) + 4\sqrt{ab^2[ab^2 - 3(c+f)]}} < h < \frac{f\alpha\delta}{c+f}.$$

**4. Numerical results.** We shall present some numerical examples. The coefficients are taken as  $\alpha = \beta = 1.0$ ,  $\delta = 0.1$ ,  $f = 1.0$ ,  $h = 0.04$ ,  $a = 1.0$ ,  $b = 3.0$ ,  $c = 0.2$  and  $d = 0.05$ . Initial functions  $u_0$  and  $w_0$  are given by  $u_0 \equiv w_0 \equiv 0$ , on the other hand,  $v_0$  is constructed randomly as in Figure 3(a) in the square domain  $\Omega = [0, 5] \times [0, 5]$ . We performed numerical computations for sufficiently large time until the graph of solution and the values of Lyapunov function are stabilized numerically. The graph of  $v$  at  $t = 200,000$  in Figure 3(b) has a clear interface of discontinuity. However, the interface seems to be described by an irregular and non-smooth curve.

**Appendix** We shall review some known results for the stable and unstable manifolds of the dynamical system.

Let  $(S(t), X, X)$  be a continuous dynamical system in a complex Banach space  $X$  and let  $\bar{U} \in X$  be an equilibrium of  $(S(t), X, X)$ . Then the stable and unstable manifolds

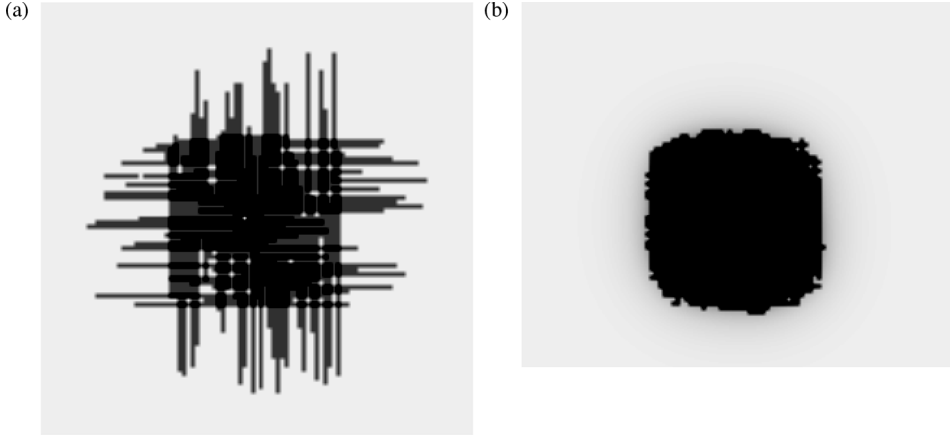


Figure 3. Interface of discontinuity.

at  $\bar{U}$  are defined by

$$\begin{aligned}\mathcal{M}_-(\bar{U}) &= \{U_0 \in X; \lim_{t \rightarrow \infty} S(t)U_0 = \bar{U}\}, \\ \mathcal{M}_+(\bar{U}) &= \{U_0 \in X; \exists U : (-\infty, 0] \rightarrow X, S(t)U(-\tau) = U(t - \tau) \text{ for } 0 \leq t \leq \tau, \\ &U(0) = U_0 \text{ and } \lim_{t \rightarrow \infty} U(-t) = \bar{U}\},\end{aligned}$$

respectively. From these definitions, it is easily verified that  $\mathcal{M}_-(\bar{U})$  and  $\mathcal{M}_+(\bar{U})$  are invariant sets of  $S(t)$  for any  $t > 0$ ; in particular,  $S(t)$  maps  $\mathcal{M}_+(\bar{U})$  onto itself, i.e.,

$$S(t)(\mathcal{M}_-(\bar{U})) \subset \mathcal{M}_-(\bar{U}) \quad \text{and} \quad S(t)(\mathcal{M}_+(\bar{U})) = \mathcal{M}_+(\bar{U}).$$

Fix any finite time  $0 < t^* < \infty$ . We obviously have a discrete dynamical system  $(S^n, X, X)$ , where  $S = S(t^*)$ . In an analogous way, the stable and unstable manifolds at  $\bar{U}$  are defined by

$$\begin{aligned}\mathcal{W}_-(\bar{U}) &= \{U_0 \in X; \lim_{n \rightarrow \infty} S^n U_0 = \bar{U}\}, \\ \mathcal{W}_+(\bar{U}) &= \{U_0 \in X; \exists \{U_{-n}\}_{n=1,2,\dots} \subset X, \\ &SU_{-n} = U_{-n+1} \text{ for } n \geq 1 \text{ and } \lim_{n \rightarrow \infty} U_{-n} = \bar{U}\}.\end{aligned}$$

Let  $\mathcal{O}$  be any neighbourhood of  $\bar{U}$ . We also consider the localized stable and unstable manifolds in  $\mathcal{O}$

$$\begin{aligned}\mathcal{W}_-(\bar{U}; \mathcal{O}) &= \{U_0 \in \mathcal{O}; S^n U_0 \in \mathcal{O} \text{ for } n \geq 1 \text{ and } \lim_{n \rightarrow \infty} S^n U_0 = \bar{U}\}, \\ \mathcal{W}_+(\bar{U}; \mathcal{O}) &= \{U_0 \in \mathcal{O}; \exists \{U_{-n}\}_{n=1,2,\dots} \subset \mathcal{O}, SU_{-n} = U_{-n+1} \text{ for } n \geq 1 \\ &\text{and } \lim_{n \rightarrow \infty} U_{-n} = \bar{U}\}.\end{aligned}$$

We can then verify the following coincidence

$$\mathcal{M}_-(\bar{U}) = \mathcal{W}_-(\bar{U}) = \bigcup_{n=0}^{\infty} S^{-n}(\mathcal{W}_-(\bar{U}; \mathcal{O})), \quad \mathcal{M}_+(\bar{U}) = \mathcal{W}_+(\bar{U}) = \bigcup_{n=0}^{\infty} S^n(\mathcal{W}_+(\bar{U}; \mathcal{O})).$$

This means that  $\mathcal{M}_-(\bar{U})$  and  $\mathcal{M}_+(\bar{U})$  could be characterized by  $\mathcal{W}_-(\bar{U}; \mathcal{O})$  and  $\mathcal{W}_+(\bar{U}; \mathcal{O})$  in some sense. We now proceed to the problem of representing  $\mathcal{W}_\pm(\bar{U}; \mathcal{O})$  as smooth manifolds under the assumption that the operator  $S$  is Fréchet differentiable in a neighbourhood of  $\bar{U}$ .

Consider a dynamical system  $(S(t), X, X)$ . Let, for some fixed time  $0 < t^* < \infty$ ,  $S(t^*)$  be Fréchet differentiable in a neighbourhood  $\tilde{\mathcal{O}}$  of  $\bar{U}$  and let a Hölder condition

$$\|S(t^*)'U - S(t^*)'V\|_{\mathcal{L}(X)} \leq D\|U - V\|^\alpha, \quad U, V \in \tilde{\mathcal{O}} \quad (\text{A.1})$$

be satisfied with some exponent  $0 < \alpha \leq 1$  and some constant  $D > 0$ . Moreover, let  $\bar{U}$  be a hyperbolic equilibrium of  $(S(t^*)^n, X, X)$ , i.e.,

$$\sigma(S(t^*)'\bar{U}) \cap \{\lambda \in \mathbb{C}; |\lambda| = 1\} = \emptyset. \quad (\text{A.2})$$

Let  $X_i = X_i(\bar{U})$  and  $X_e = X_e(\bar{U})$  be the invariant subspaces of  $S(t^*)'\bar{U}$  such that  $X = X_i + X_e$  in which each of the parts  $S(t^*)'\bar{U}|_{X_i}$  and  $S(t^*)'\bar{U}|_{X_e}$  has its spectrum in  $\{\lambda \in \mathbb{C}; |\lambda| < 1\}$  and in  $\{\lambda \in \mathbb{C}; |\lambda| > 1\}$ , respectively.

Then the following theorem is known.

**THEOREM A.1** [[17, Chapter VII, Theorem 3.1] and [19]] *Let  $\bar{U}$  be an equilibrium of a dynamical system  $(S(t), X, X)$ . Let (A.1) and (A.2) be satisfied for some fixed time  $0 < t^* < \infty$  and an open neighbourhood  $\tilde{\mathcal{O}}$  of  $\bar{U}$ . Let  $X_i$  and  $X_e$  be the invariant subspaces as above. Then, in a sufficiently small open neighbourhood  $\mathcal{O} \subset \tilde{\mathcal{O}}$  of  $\bar{U}$ ,  $\mathcal{W}_-(\bar{U}; \mathcal{O})$  and  $\mathcal{W}_+(\bar{U}; \mathcal{O})$  are  $\mathcal{C}^{1,\alpha}$  manifolds with dimensions  $\dim X_i$  and  $\dim X_e$ , respectively. Moreover, the manifolds  $\mathcal{W}_-(\bar{U}; \mathcal{O})$  and  $\mathcal{W}_+(\bar{U}; \mathcal{O})$  are tangential at  $\bar{U}$  to  $\bar{U} + X_i$  and  $\bar{U} + X_e$ , respectively.*

Let us next apply these results to a dynamical system determined from a semilinear abstract evolution equation. We consider the Cauchy problem for a semilinear abstract evolution equation

$$\begin{cases} \frac{dU}{dt} + AU = F(U), & 0 < t \leq \infty, \\ U(0) = U_0 \end{cases} \quad (\text{A.3})$$

in a Banach space  $X$ . Here,  $A$  is a sectorial operator of  $X$  with angle  $\omega_A < \frac{\pi}{2}$ ; consequently,  $-A$  is the generator of an analytic semigroup  $e^{-tA}$  on  $X$ . The operator  $F$  is a non-linear operator from  $\mathcal{D}(A^\eta)$  into  $X$ , where  $\eta$  is some exponent such that  $0 < \eta < 1$ , and is assumed to satisfy a Lipschitz condition of the form

$$\|F(U) - F(V)\| \leq \varphi(\|A^\mu U\| + \|A^\mu V\|) \times \{\|A^\eta(U - V)\| + (\|A^\eta U\| + \|A^\eta V\|)\|A^\mu(U - V)\|\}, \quad U, V \in \mathcal{D}(A^\eta),$$

where  $\mu$  is some exponent such that  $0 \leq \mu \leq \eta < 1$  and  $\varphi(\cdot)$  is some increasing continuous function. The initial value  $U_0$  is taken from  $\mathcal{D}(A^\mu)$ .

We consider the case when a dynamical system  $(S(t), \mathcal{D}(A^\mu), \mathcal{D}(A^\mu))$  is determined from the problem (A.3). Let  $\bar{U}$  be a stationary solution to (A.3), namely,  $\bar{U}$  is an equilibrium of  $(S(t), \mathcal{D}(A^\mu), \mathcal{D}(A^\mu))$ . We will investigate the Fréchet differentiability of  $S(t)$  in an open neighbourhood of  $\bar{U}$ .



Let us assume that  $F : \mathcal{D}(A^\eta) \rightarrow X$  is of class  $\mathcal{C}^{1,1}$  in  $B^{\mathcal{D}(A^\eta)}(\bar{U}; r)$ , with some  $r > 0$ . Moreover, the derivative satisfies the following conditions:

$$\|F'(U)V\| \leq \psi(\|A^\mu U\|)\|A^\eta U\|\|A^\mu V\|, \quad U, V \in B^{\mathcal{D}(A^\eta)}(\bar{U}; r), \quad (\text{A.4})$$

$$\begin{aligned} \|[F'(U_1) - F'(U_2)]V\| &\leq \psi(\|A^\mu U_1\| + \|A^\mu U_2\|) \\ &\quad \times \|A^\eta(U_1 - U_2)\|\|A^\mu V\|, \quad U_1, U_2, V \in B^{\mathcal{D}(A^\eta)}(\bar{U}; r) \end{aligned} \quad (\text{A.5})$$

with some continuous increasing function  $\psi(\cdot)$ .

Then the following theorem can be proved.

**THEOREM A.2** [1, Sections 5 and 6] *Let  $\bar{U}$  be an equilibrium of  $(S(t), \mathcal{D}(A^\mu), \mathcal{D}(A^\mu))$  and let (A.4) and (A.4) be satisfied with some  $r > 0$ . Let  $0 < t^* < \infty$  be arbitrarily fixed. Then, for a sufficiently small number  $r' > 0$ , the semigroup  $S(t)$ , where  $0 \leq t \leq t^*$ , is Fréchet differentiable in the ball  $B^{\mathcal{D}(A^\mu)}(\bar{U}; r')$ . And its derivative satisfies a Lipschitz condition*

$$\|S(t)'U - S(t)'V\|_{\mathcal{L}(\mathcal{D}(A^\mu))} \leq C\|U - V\|_{\mathcal{D}(A^\mu)}, \quad 0 \leq t \leq t^*, \quad U, V \in B^{\mathcal{D}(A^\mu)}(\bar{U}; r').$$

Furthermore, if  $\sigma(A - F'(\bar{U})) \cap \{\lambda \in \mathbb{C}; \operatorname{Re}\lambda = 0\} = \emptyset$  then  $\bar{U}$  is a hyperbolic equilibrium of  $(S(t)^n, \mathcal{D}(A^\mu), \mathcal{D}(A^\mu))$ .

This theorem has essentially been proved in the arguments of [1, Sections 5 and 6].

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