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EXPONENTIAL ATTRACTORS FOR COMPETING SPECIES MODEL WITH CROSS-DIFFUSIONS

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Abstract. This paper is concerned with the competing species model presented by Shigesada-Kawasaki-Teramoto in 1979. Under a suitable condition on self-diffusions and cross-diffusions, we construct a dynamical system determined from the model. Furthermore, under the same condition we construct exponential attractors of the dynamical system.

1. **Introduction.** We consider the initial-boundary value problem for a competing species system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta(au + \alpha_{11}u^2 + \alpha_{12}uv) + cu - \gamma_{11}u^2 - \gamma_{12}uv & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = \Delta(bv + \alpha_{21}uv + \alpha_{22}v^2) + dv - \gamma_{21}uv - \gamma_{22}v^2 & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } \Omega \end{cases} \tag{1}$$

in a two-dimensional bounded $\mathcal{C}^3$ domain $\Omega \subset \mathbb{R}^2$. Here, $a > 0$, $b > 0$, $\alpha_{ij} \geq 0$, $c > 0$, $d > 0$ and $\gamma_{ij} > 0$ are given constants.

In 1979, this system was introduced by Shigesada-Kawasaki-Teramoto [18] to describe the segregation process of a biological system consisting of two competing species, say A and B, in Ω by the cross-diffusions. The unknown functions $u = u(x,t)$ and $v = v(x,t)$ denote the densities of species A and B in Ω at time $t \geq 0$, respectively. They are subjected to homogeneous Neumann boundary conditions on $\partial\Omega$. The terms $\alpha_{ij}\Delta uv$ $(i \neq j)$ denote the cross-diffusions between A and B. On the other hand, the terms $\alpha_{11}\Delta u^2$ and $\alpha_{22}\Delta v^2$ denote the self-diffusions of A and B, respectively. The competitions of species are described by the two kinetic functions $(c - \gamma_{11}u - \gamma_{12}v)u$ and $(d - \gamma_{21}u - \gamma_{22}v)v$. The initial functions u_0 and v_0 are given in such a way that $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ is in the product space

$$\mathcal{K} = \left\{ \begin{pmatrix} u_0 \\ v_0 \end{pmatrix};\ 0 \leq u_0 \in H^{1+\varepsilon}(\Omega) \quad \text{and} \quad 0 \leq v_0 \in H^{1+\varepsilon}(\Omega) \right\}, \tag{2}$$

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where ε denotes an arbitrarily fixed exponent in such a way that $0 < \varepsilon < \frac{1}{2}$. Problem (1) is handled in the product space of $L_2(\Omega)$, i.e.,

$$X = \mathbb{L}_2(\Omega) = \left\{ \begin{pmatrix} f \\ g \end{pmatrix};\ f \in L_2(\Omega) \quad \text{and} \quad g \in L_2(\Omega) \right\}. \tag{3}$$

This system has in fact attracted interest of many mathematicians for these thirty years. For the two-dimensional problem, the global existence for all initial values in $\mathcal{K}$ is known so far under the following condition of α_{ij}:

$$0 \le \alpha_{12}\alpha_{21} \le 64\,\alpha_{11}\alpha_{22}. \tag{4}$$

Note that, if $\alpha_{12}\alpha_{21} = 0$, i.e., one of the cross-diffusions does not exists, then it is allowed that $\alpha_{11} = \alpha_{22} = 0$. A global existence result was first obtained by the author [25] (cf. also [26]) in the case when $0 < \alpha_{21} < 8\alpha_{11}$ and $0 < \alpha_{12} < 8\alpha_{22}$; afterward, this result was extended by Ichikawa-Yamada [8] to the case when $0 < \alpha_{12}\alpha_{21} < 64\alpha_{11}\alpha_{22}$. For the critical case $\alpha_{12}\alpha_{21} = 0$, the global existence result was first obtained by Masuda-Mimura (cf. [13]) for the one-dimensional problem. For the two-dimensional case, this was shown by the author [27]; afterward, the similar result was shown by Lou-Ni-Wu [12] but in a framework of the $L_p\,(2 < p < \infty)$ theory. For the other critical case $0 < \alpha_{12}\alpha_{21} = 64\alpha_{11}\alpha_{22}$, this will be seen in the present paper.

It is then very natural to ask whether Condition (4) is necessary for the global existence of solutions for all $U_0 \in \mathcal{K}$ or not. In a particular case when $a = b$, $\alpha_{12}\alpha_{21} > 0$, $\alpha_{11} = \alpha_{22} = 0$, Kim [9] proved the global existence for the one-dimensional problem. Such a result can be extended to the two-dimensional one, too (cf. [25, Remark 4.6]). But, for the moment, it is very difficult to give a satisfactory answer to the question; for example, no blowup results for (1) are known.

For the N-dimensional problem ($N \ge 3$), Deuring [5] first considered the global existence in the case when α_{12} and α_{21} are sufficiently small. Afterward, Pozio-Tesei [16] got rid of such smallness (under the Dirichlet boundary conditions) but assuming a higher order of decaying in the growth function of u or v. Yamada [24] also studied the problem in the same spirit. Wiegner [22] applied the Amann's theory [1] on abstract parabolic equations to the N-dimensional problem of (1). More recently, Choi-Lui-Yamada [3, 4] tried to extend the result [12] to the N-dimension case; but still they need smallness of $\alpha_{12} > 0$ ($\alpha_{21} = 0$) or positivity of $\alpha_{11} > 0$.

In the meantime, little is known for the asymptotic behavior of solutions. Redlinger [17] constructed the global attractor for the one-dimensional problem containing growth functions at a higher order of decaying. Shim [19, 20, 21] established uniformly bounded estimates and convergence of solutions in some suitable cases.

A number of papers on the stationary problem for (1) have already been published. We will here quote only some of them such as [10, 11, 13, 14, 23]. For the full references we refer the reader to References therein.

This paper is then concerned with constructing a dynamical system determined from (1) in the two-dimensional case and constructing exponential attractors for the dynamical system. We also consider the case when the cross-diffusion coefficients and the self-diffusion coefficients satisfy Condition (4). The notion of exponential attractors has been introduced in 1994 by Eden-Foias-Nicolaenko-Temam [29]. The exponential attractor is a compact, positively invariant set of finite fractal dimension containing the global attractor and attracting every trajectory at an exponential rate. It is also known that the exponential attractor enjoys stronger robustness with respect to the global attractor, see [29] and also, e.g., [6, 7].

As shown in [25, 27], local existence of solutions for (1) is obtained by directly applying the general results concerning abstract parabolic evolution equations. It is however necessary to verify that a realisation of the matrix differential operator

$$\begin{pmatrix} \nabla\cdot\{(a+2\alpha_{11}u+\alpha_{12}v)\nabla\cdot\} & \alpha_{12}\nabla\cdot\{u\nabla\cdot\} \\ \alpha_{21}\nabla\cdot\{v\nabla\cdot\} & \nabla\cdot\{(b+\alpha_{21}u+2\alpha_{22}v)\nabla\cdot\} \end{pmatrix}$$

in the space X is a sectorial operator with angle $< \frac{\pi}{2}$. This verification is not immediate. Especially in the case of $\alpha_{12}\alpha_{21} > 0$, namely, the matrix is a full matrix, we need to use some techniques. In showing the local existence, Condition (4) is not at all necessary. For any $U_0 \in \mathcal{K}$, a unique local solution to (1) can be constructed.

For constructing global solutions, we have to build up norm estimates (for the local solutions) which ensure that the $H^{1+\varepsilon}$ norms of solutions never blow up in finite time. For constructing the global attractor (a fortiori, the exponential attractor), we have to build up stronger norm estimates of solutions which show that the $H^{1+\varepsilon}$ norms of solutions having large norms of initial data decrease asymptotically and become smaller than a universal constant, say $\widetilde{C} > 0$, which is independent of solutions as $t \to \infty$. As a matter of fact, building up such a priori estimates will occupy the main part of the present paper. For this aim we need more careful calculations than before ([25, 27]) and need various techniques. The critical case $\alpha_{12}\alpha_{21} = 0$ of (4) may be more delicate than the other favorable case. We will employ analogous techniques utilized in [15] for the chemotaxis-growth model.

In constructing exponential attractors, we know two kinds of sufficient conditions concerning the nonlinear semigroup of the dynamical system under consideration. The first one is the squeezing property which has been introduced by Eden-Foias-Nicolaenko-Temam in the mentioned book [29]. The second one is the compact smoothing property, see (83), introduced by Efendiev-Miranville-Zelik [6]. In the sense of logic, these two properties are mutually equivalent when the universal space is a Hilbert space. But, in the viewpoint of applications, the squeezing property fits more to the semilinear diffusion equations than the quasilinear diffusion equations. So, to the present system, we will apply the compact smoothing property by verifying Condition (83). General procedure for verifying (83) was present in the paper Aida-Efendiev-Yagi [2] in which we utilized representing formulas of quasilinear abstract parabolic evolution equations provided by the semigroup methods. These methods are reviewed in Section 6 of this paper. The semigroup methods are known as powerful tools for solving parabolic equations and systems. We will just follow the general procedure to verify the compact smoothing property of our nonlinear semigroup.

The $\mathcal{C}^3$ regularity of the boundary $\partial\Omega$ is needed only in the proof of the a priori estimates for the critical case $\alpha_{12}\alpha_{21} = 0$ of (4). All other results are valid under $\mathcal{C}^2$ regularity of $\partial\Omega$ or even under convexity of Ω. The $\mathcal{C}^3$ regularity of $\partial\Omega$ ensures the shift property that $\Delta u \in H^1(\Omega)$ with $\frac{\partial u}{\partial n} = 0$ implies $u \in H^3(\Omega)$. Such a shift property is used in Step 5 of the proof of Proposition 2.

2. **Local solutions.** As a matter of fact, we already know (see [25, Section 3]) that the theory of quasilinear abstract parabolic evolution equations is available for

constructing a unique local solution to (1) for any pair of u_0 and v_0 from $\mathcal{K}$. Some results of the theory are reviewed in Subsection 6.1.

Fix an exponent ε' so that $0 < \varepsilon' < \varepsilon$ (remember that ε was already taken in (2)) and let

$$Z = \mathbb{H}^{1+\varepsilon'}(\Omega) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix};\ u \in H^{1+\varepsilon'}(\Omega) \quad \text{and} \quad v \in H^{1+\varepsilon'}(\Omega) \right\}. \tag{5}$$

For any $0 < R < \infty$, let $K_R = \{U;\ \|U\|_Z < R\}$ be an open ball of Z. For each $U \in K_R$, let $A(U)$ denote a closed linear operator given by [25, (3.4)]. Let F be a nonlinear operator from K_R into X given by [25, (3.7)].

Let us take any initial value $U_0 \in \mathcal{K}$ and take an R sufficiently large so that $U_0 \in K_R$. Then (1) can be written as the Cauchy problem of the form

$$\begin{cases} \dfrac{dU}{dt} + A(U)U = F(U), \qquad 0 < t < \infty, \\ U(0) = U_0 \end{cases} \tag{6}$$

in the space X given by (3).

Then, $A(U)$, $U \in K_R$, are seen to be sectorial operators of X with angle $< \frac{\pi}{2}$ fulfilling (68) announced in Section 6. Their domains are given by

$$\mathcal{D}(A(U)) = \mathbb{H}^2_N(\Omega) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix};\ u \in H^2_N(\Omega) \quad \text{and} \quad v \in H^2_N(\Omega) \right\},$$

where $H^2_N(\Omega) = \{u \in H^2(\Omega);\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$. Moreover, according to [25, Proposition 3.2], we see for any $0 \le \theta < \frac{3}{4}$ that

$$\mathcal{D}(A(U)^\theta) = \mathbb{H}^{2\theta}(\Omega) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix};\ u \in H^{2\theta}(\Omega) \quad \text{and} \quad v \in H^{2\theta}(\Omega) \right\}. \tag{7}$$

Similarly, the operator-valued function $A(\cdot)$ fulfills the Lipschitz condition of form (69) with $Y = X$, namely, with $\alpha = 0$.

The nonlinear operator F also fulfils (70) with $Y = X$. Finally, on account of (7), the initial value U_0 fulfils the compatibility condition given by (71) with $\gamma = \frac{1+\varepsilon}{2}$ (note that $\varepsilon < \frac{1}{2}$). Therefore, Corollary 1 in Section 6 provides existence and uniqueness of a local solution to (6).

Furthermore, the truncation method deduces nonnegativity of the local solution. In this way, as stated in [25, Theorem 3.5], for any $U_0 \in \mathcal{K}$, (6) and hence (1) possesses a unique nonnegative local solution U in the function space:

$$0 \le U \in \mathcal{C}((0, T_{U_0}]; \mathbb{H}^2_N(\Omega)) \cap \mathcal{C}([0, T_{U_0}]; \mathbb{H}^{1+\varepsilon}(\Omega)) \cap \mathcal{C}^1((0, T_{U_0}]; \mathbb{L}_2(\Omega)) \tag{8}$$

with the estimates

$$t^{\frac{1-\varepsilon}{2}} \|U(t)\|_{\mathbb{H}^2} + \|U(t)\|_{\mathbb{H}^{1+\varepsilon}} \le C_{U_0}, \qquad 0 < t \le T_{U_0},$$

where $T_{U_0} > 0$ is determined by the norm $\|U_0\|_{\mathbb{H}^{1+\varepsilon}}$ and the constant $C_{U_0} > 0$ is also determined by $\|U_0\|_{\mathbb{H}^{1+\varepsilon}}$.

In addition, as shown in [25, Theorem 3.6], we can utilize the maximal regularity of abstract parabolic evolution equations to verify the following regularity

$$U \in \mathcal{C}^1((0, T_{U_0}]; \mathbb{H}^{2\theta}(\Omega)) \cap \mathcal{C}^2((0, T_{U_0}]; \mathbb{H}^1(\Omega)^*), \qquad 0 \le \theta < 1, \tag{9}$$

where $\mathbb{H}^1(\Omega)^*$ denotes the dual space of $\mathbb{H}^1(\Omega)$.

3. **A priori estimates.** We shall establish a priori estimates for local solutions to (1). Let $U_0 \in \mathcal{K}$ and let U denote any nonnegative local solution to (1) in the function space:

$$0 \le u,\, v \in \mathcal{C}((0,T_U]; H^2_N(\Omega)) \cap \mathcal{C}([0,T_U]; H^{1+\varepsilon}(\Omega)) \\ \cap\, \mathcal{C}^1((0,T_U]; H^1(\Omega)) \cap \mathcal{C}^2((0,T_U]; H^1(\Omega)^*), \quad (10)$$

where $[0,T_U]$ denotes the interval on which U is defined. As shown in the preceding section, such a local solution exists at least on some interval $[0,T_{U_0}]$.

In this section, Assumption (4) will be used. But the techniques of proof are quite different depending on the cases when $\alpha_{12}\alpha_{21} > 0$ and when $\alpha_{12}\alpha_{21} = 0$. When $0 < \alpha_{12}\alpha_{21} \le 64\alpha_{11}\alpha_{22}$, (4) is seen to be equivalent to (12). Furthermore, (12) implies nonnegativity of a quadratic function for the variables p and q (u and v being nonnegative parameters) given by (13). Such nonnegativity of the quadratic function will be used several times in the a priori estimates below. In the meantime, when $\alpha_{12}\alpha_{21} = 0$, it is allowed that $\alpha_{11} = \alpha_{22} = 0$. So the estimate of form (13) is no longer valid in general. But if $\alpha_{21} = 0$ (resp. $\alpha_{12} = 0$), we can derive an L_∞-norm estimate of the solution $v(t)$ (resp. $u(t)$) in a direct way, and we can use this estimate for deriving other norm estimates concerning partial derivatives of the solutions $u(t)$ and $v(t)$.

For simplicity, we shall use the following quadratic functions

$$P(u,v) = au + \alpha_{11}u^2 + \alpha_{12}uv, \quad Q(u,v) = bv + \alpha_{21}uv + \alpha_{22}v^2,$$
$$f(u,v) = cu - \gamma_{11}u^2 - \gamma_{12}uv, \quad g(u,v) = dv - \gamma_{21}uv - \gamma_{22}v^2.$$

3.1. **Case when $\boldsymbol{\alpha_{12}\alpha_{21} > 0}$.** Let us begin with noticing some scaling property. Let $\lambda > 0$ and $\mu > 0$ be two parameters and multiply the equations for u and v by λ and μ, respectively. Then we obtain an equivalent problem to (1):

$$\begin{cases} \dfrac{\partial u_\lambda}{\partial t} = \Delta(au_\lambda + \alpha_{11}\lambda^{-1}u_\lambda^2 + \alpha_{12}\mu^{-1}u_\lambda v_\mu) \\ \qquad\qquad + cu_\lambda - \gamma_{11}\lambda^{-1}u_\lambda^2 - \gamma_{12}\mu^{-1}u_\lambda v_\mu & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial v_\mu}{\partial t} = \Delta(bv_\mu + \alpha_{21}\lambda^{-1}u_\lambda v_\mu + \alpha_{22}\mu^{-1}v_\mu^2) \\ \qquad\qquad + dv_\mu - \gamma_{21}\lambda^{-1}u_\lambda v_\mu - \gamma_{22}\mu^{-1}v_\mu^2 & \text{in } \Omega\times(0,\infty), \\ \dfrac{\partial u_\lambda}{\partial n} = \dfrac{\partial v_\mu}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty), \\ u_\lambda(x,0) = \lambda u_0(x), \quad v_\mu(x,0) = \mu v_0(x) & \text{in } \Omega, \end{cases} \qquad (11)$$

where $u_\lambda(x,t) = \lambda u(x,t)$ and $v_\mu(x,t) = \mu v(x,t)$. It is clear that, if $0 < \alpha_{12}\alpha_{21} \le 64\alpha_{11}\alpha_{22}$, then the new self-diffusion constants and cross-diffusion constants given in (11) satisfy again the same condition.

Under (4) with $\alpha_{12}\alpha_{21} > 0$, if we choose parameters λ and μ so that the relation $\alpha_{12}\lambda = \sqrt{8\alpha_{11}\alpha_{21}}\mu$ is valid, then it is easily observed that

$$(\alpha_{12}\mu^{-1})^2 = 8(\alpha_{11}\lambda^{-1})(\alpha_{21}\lambda^{-1}) \quad \text{and} \quad (\alpha_{21}\lambda^{-1})^2 \le 8(\alpha_{22}\mu^{-1})(\alpha_{12}\mu^{-1}).$$

This means that the self-diffusion constants and cross-diffusion constants appearing in (11) fulfil a relation of the form

$$0 < \alpha_{12}^2 \le 8\alpha_{11}\alpha_{21} \qquad \text{and} \qquad 0 < \alpha_{21}^2 \le 8\alpha_{22}\alpha_{12}. \qquad (12)$$

Since (12) clearly implies (4), (12) is a stronger assumption than (4). But, since (11) is completely equivalent to (1) as the initial-boundary value problem, any a priori estimates which hold for all local solutions to (11) hold equally for all local solutions to (1).

We are thus allowed to assume Condition (12) instead of $0 < \alpha_{12}\alpha_{21} \le 64\alpha_{11}\alpha_{22}$ in establishing our a priori estimates for the local solutions of (1).

Proposition 1. *Let (12) (or, as mentioned above, $0 < \alpha_{12}\alpha_{21} \le 64\alpha_{11}\alpha_{22}$) be satisfied. There exists a continuous increasing function $p(\cdot)$ such that, for any local solution U to (1) lying in (10) with initial value $U_0 \in \mathcal{K} \cap \mathbb{H}^2(\Omega)$, it holds that*

$$\|U(t)\|_{\mathbb{H}^2} \le p(\|U_0\|_{\mathbb{H}^2}), \qquad 0 \le t \le T_U.$$

Proof. In the proof, a unified notation C will be used to denote various constants which are determined from the initial constants a, b, c, d, $\alpha_{ij}(1 \le i, j \le 2)$ and $\gamma_{ij}(1 \le i, j \le 2)$ and the domain Ω alone in a specific way. So, C may change from occurrence to occurrence. When a constant C depends on a particular parameter, say ζ, we shall denote it by C_ζ.

Similarly, a unified notation $p(\cdot)$ will be used to denote various continuous increasing functions which may change from occurrence to occurrence.

Step 1. Consider the inner product of the two evolution equations in (1) and $U(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ in $\mathbb{L}_2(\Omega)$. From the equation for u,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega \nabla P(u,v) \cdot \nabla u\, dx = \int_\Omega f(u,v)u\, dx.$$

Since

$$\nabla P(u,v) = P_u \nabla u + P_v \nabla v = (a + 2\alpha_{11}u + \alpha_{12}v)\nabla u + \alpha_{12}u\nabla v.$$

It then follows that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega \{(a + 2\alpha_{11}u + \alpha_{12}v)|\nabla u|^2 + \alpha_{12}u\nabla u \cdot \nabla v\}dx \le \int_\Omega (cu^2 - \gamma_{11}u^3)dx.$$

To estimate the integral in the right hand side we notice the inequality

$$cu^2 - \gamma_{11}u^3 \le -\tfrac{1}{2}u^2 + \tfrac{(2c+1)^3}{54\gamma_{11}^2}, \qquad 0 \le u < \infty.$$

Then,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \frac{1}{2}\int_\Omega u^2 + \int_\Omega \{(a + 2\alpha_{11}u + \alpha_{12}v)|\nabla u|^2 + \alpha_{12}u\nabla u \cdot \nabla v\}dx \le (2c+1)^3\gamma_{11}^{-2}|\Omega|/54.$$

As a similar estimate holds for the integral $\int_\Omega v^2 dx$ also, we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u^2 + v^2)dx + \frac{1}{2}\int_\Omega (u^2+v^2)dx + \int_\Omega (a|\nabla u|^2 + b|\nabla v|^2)dx + \int_\Omega \{(2\alpha_{11}u + \alpha_{12}v)|\nabla u|^2 + (\alpha_{12}u + \alpha_{21}v)\nabla u \cdot \nabla v + (\alpha_{21}u + 2\alpha_{22}v)|\nabla v|^2\}dx \le \{(2c+1)^3\gamma_{11}^{-2} + (2d+1)^3\gamma_{22}^{-2}\}|\Omega|/54.$$

Here we notice that (12) is a necessary and sufficient condition in order that the inequality

$$4(2\alpha_{11}u + \alpha_{12}v)(\alpha_{21}u + 2\alpha_{22}v) - (\alpha_{12}u + \alpha_{21}v)^2 \geq 0$$

holds for all $u,\, v \geq 0$. From this it is observed that

$$(2\alpha_{11}u + \alpha_{12}v)p^2 + (\alpha_{12}u + \alpha_{21}v)pq + (\alpha_{21}u + 2\alpha_{22}v)q^2 \geq 0 \qquad \text{for all} \quad u,\, v \geq 0;\ p,\, q \in \mathbb{R}. \tag{13}$$

Therefore we conclude that the fourth integral in the left hand side is nonnegative. Hence,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u^2 + v^2)dx + \frac{1}{2}\int_\Omega (u^2 + v^2)dx + \int_\Omega (a|\nabla u|^2 + b|\nabla v|^2)dx \leq C. \tag{14}$$

In particular,

$$\frac{d}{dt}\int_\Omega (u^2 + v^2)dx + \int_\Omega (u^2 + v^2)dx \leq 2C,$$

where C is given precisely by $C = \{(2c+1)^3\gamma_{11}^{-2} + (2d+1)^3\gamma_{22}^{-2}\}|\Omega|/54$.

We thus conclude that

$$\|u(t)\|_{L_2}^2 + \|v(t)\|_{L_2}^2 \leq e^{-t}\|U_0\|_{\mathbb{L}_2}^2 + C, \qquad 0 \leq t \leq T_U. \tag{15}$$

Integrating (14) on $(0, t)$, we as well conclude that

$$\int_0^t (\|\nabla u(s)\|_{L_2}^2 + \|\nabla v(s)\|_{L_2}^2)ds \leq C(t + \|U_0\|_{\mathbb{L}_2}^2), \qquad 0 \leq t \leq T_U. \tag{16}$$

Step 2. We next consider the inner product of the two evolution equations of (1) and $\frac{d}{dt}\begin{pmatrix} P(u,v) \\ Q(u,v) \end{pmatrix}$ in $\mathbb{L}_2(\Omega)$. From the equation for u,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla P|^2 dx + \int_\Omega (P_u u_t^2 + P_v u_t v_t)dx = \int_\Omega f(P_u u_t + P_v v_t)dx$$

because of $\frac{d}{dt}P(u,v) = P_u u_t + P_v v_t$. A similar energy equality is valid for Q, too. From these two equalities it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega (|\nabla P|^2 + |\nabla Q|^2)dx &+ \int_\Omega (a u_t^2 + b v_t^2)dx \\ &+ \int_\Omega \{(2\alpha_{11}u + \alpha_{12}v)u_t^2 + (\alpha_{12}u + \alpha_{21}v)u_t v_t + (\alpha_{21}u + 2\alpha_{22}v)v_t^2\}dx \\ &\leq C\int_\Omega (1 + u^3 + v^3)(|u_t| + |v_t|)dx.\end{aligned}$$

We can use (13) again to observe nonnegativity of the third integral in the left hand side. After obvious calculations,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (|\nabla P|^2 + |\nabla Q|^2)dx + \frac{1}{2}\int_\Omega (a u_t^2 + b v_t^2)dx \leq C\int_\Omega (1 + u^6 + v^6)dx.$$

Using the Gagliardo-Nirenberg inequality (cf. [25, (1.3)]), we notice from (15) that

$$\|u(t)\|_{L_6}^6 \leq C\|u(t)\|_{H^1}^4\|u(t)\|_{L_2}^2 \leq p(\|U_0\|_{\mathbb{L}_2})(\|\nabla u(t)\|_{L_2}^4 + 1)$$

and that

$$\begin{pmatrix} \nabla u \\ \nabla v \end{pmatrix} = \begin{pmatrix} P_u & P_v \\ Q_u & Q_v \end{pmatrix}^{-1} \begin{pmatrix} \nabla P \\ \nabla Q \end{pmatrix}.$$

Since $P_uQ_v - P_vQ_u \geq (a+2\alpha_{11}u)(b+2\alpha_{22}v)$, we observe that all the L_∞-norms of the components of $\begin{pmatrix} P_u & P_v \\ Q_u & Q_v \end{pmatrix}^{-1}$ are estimated by a constant C (remember $\alpha_{11} > 0$ and $\alpha_{22} > 0$). Hence we can verify that

$$|\nabla u| \leq C(|\nabla P(u,v)| + |\nabla Q(u,v)|), \qquad \text{a.e. } x \in \Omega. \tag{17}$$

Consequently,

$$\|u(t)\|_{L_6}^6 \leq p(\|U_0\|_{\mathbb{L}_2})\Big\{1 + \|\nabla u(t)\|_{L_2}^2 \int_\Omega (|\nabla P|^2 + |\nabla Q|^2)dx\Big\}. \tag{18}$$

Of course a similar estimate holds for $\|v(t)\|_{L_6}$, too. Therefore,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega (|\nabla P|^2 + |\nabla Q|^2)dx + \frac{1}{2}\int_\Omega (au_t^2 + bv_t^2)dx \\ &\qquad \leq p(\|U_0\|_{\mathbb{L}_2})(1 + \|\nabla u(t)\|_{L_2}^2 + \|\nabla v(t)\|_{L_2}^2)\int_\Omega (|\nabla P|^2 + |\nabla Q|^2)dx. \end{aligned} \tag{19}$$

Let us next consider the inner product of the two equations in (1) and $\begin{pmatrix} P(u,v) \\ Q(u,v) \end{pmatrix}$ in $\mathbb{L}_2(\Omega)$. After easy calculations,

$$\begin{aligned} \int_\Omega (|\nabla P|^2 + |\nabla Q|^2)dx &= -\int_\Omega (Pu_t + Qv_t)dx + \int_\Omega (fP + gQ)dx \\ &\leq \zeta \int_\Omega (u_t^2 + v_t^2)dx + C_\zeta \int_\Omega (1 + u^4 + v^4)dx \end{aligned}$$

with any number $\zeta > 0$. So, for any parameter $\xi > 0$, it is possible to observe that

$$\xi \int_\Omega (|\nabla P|^2 + |\nabla Q|^2)dx \leq \frac{1}{4}\int_\Omega \big(au_t^2 + bv_t^2\big)dx + C_\xi \int_\Omega (1 + u^4 + v^4)dx.$$

Furthermore, following the same arguments as for (18), it is shown that

$$C_\xi \|u(t)\|_{L_4}^4 \leq p(\|U_0\|_{\mathbb{L}_2})\Big\{1 + \|\nabla u(t)\|_{L_2}^2 \int_\Omega (|\nabla P|^2 + |\nabla Q|^2)dx\Big\} + \widetilde{C}_\xi p(\|U_0\|_{\mathbb{L}_2}),$$

$\widetilde{C}_\xi > 0$ being another constant depending on the parameter ξ. Hence,

$$\begin{aligned} \xi \int_\Omega (|\nabla P|^2 + |\nabla Q|^2)dx &\leq \frac{1}{4}\int_\Omega \big(au_t^2 + bv_t^2\big)dx \\ &\quad + p(\|U_0\|_{\mathbb{L}_2})\Big\{1 + \|\nabla u(t)\|_{L_2}^2 \int_\Omega (|\nabla P|^2 + |\nabla Q|^2)dx\Big\} + \widetilde{C}_\xi p(\|U_0\|_{\mathbb{L}_2}). \end{aligned}$$

Combining this estimate with (19), we obtain the following estimate

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega (|\nabla P|^2 + |\nabla Q|^2)dx + \xi \int_\Omega (|\nabla P|^2 + |\nabla Q|^2)dx + \frac{1}{4}\int_\Omega (au_t^2 + bv_t^2)dx \\ &\qquad \leq p(\|U_0\|_{\mathbb{L}_2})(1 + \|\nabla u(t)\|_{L_2}^2 + \|\nabla v(t)\|_{L_2}^2) \\ &\qquad\qquad \times \int_\Omega (|\nabla P|^2 + |\nabla Q|^2)dx + \widetilde{C}_\xi p(\|U_0\|_{\mathbb{L}_2}). \end{aligned} \tag{20}$$

Regarding this as a differential inequality on $\int_\Omega(|\nabla P|^2+|\nabla Q|^2)dx$ and solving it, we have

$$
\begin{aligned}
&\|\nabla P(U(t))\|_{L_2}^2+\|\nabla Q(U(t))\|_{L_2}^2\\
&\qquad \le Ce^{\int_0^t\{p(\|U_0\|_{\mathbb{L}_2})(1+\|\nabla U(s)\|_{\mathbb{L}_2}^2)-2\xi\}ds}(\|P(U_0)\|_{H^1}^2+\|Q(U_0)\|_{H^1}^2)\\
&\qquad\qquad +\widetilde{C}_\xi p(\|U_0\|_{\mathbb{L}_2})\int_0^t e^{\int_s^t\{p(\|U_0\|_{\mathbb{L}_2})(1+\|\nabla U(\tau)\|_{\mathbb{L}_2}^2)-2\xi\}d\tau}ds.
\end{aligned}
$$

Furthermore, we introduce the corresponding version of (16) obtained by integrating (14) on (s,t). In fact, thanks to (15),

$$
\begin{aligned}
\int_s^t \|\nabla U(\tau)\|_{\mathbb{L}_2}^2 d\tau &\le C\{(t-s)+\|U(s)\|_{\mathbb{L}_2}^2\}\\
&\le C\{(t-s)+\|U_0\|_{\mathbb{L}_2}^2+1\}. \qquad 0\le s\le t\le T_U.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\|\nabla P(U(t))\|_{L_2}^2+\|\nabla Q(U(t))\|_{L_2}^2\\
&\qquad \le p(\|U_0\|_{\mathbb{L}_2})e^{\{p(\|U_0\|_{\mathbb{L}_2})-2\xi\}t}(\|P(U_0)\|_{H^1}^2+\|Q(U_0)\|_{H^1}^2)\\
&\qquad\qquad +\widetilde{C}_\xi p(\|U_0\|_{\mathbb{L}_2})\int_0^t e^{\{p(\|U_0\|_{\mathbb{L}_2})-2\xi\}(t-s)}ds.
\end{aligned}
$$

Fix now the parameter ξ as $\xi=\{p(\|U_0\|_{\mathbb{L}_2})+1\}/2$. Then, thanks to (15) again, we conclude that

$$
\begin{aligned}
&\|\nabla P(U(t))\|_{L_2}^2+\|\nabla Q(U(t))\|_{L_2}^2\\
&\qquad \le e^{-t}p\big(\|P(U_0)\|_{H^1}+\|Q(U_0)\|_{H^1}\big)+p(\|U_0\|_{\mathbb{L}_2}), \qquad 0\le t\le T_U.
\end{aligned}
$$

Moreover, in view of (17),

$$
\|U(t)\|_{\mathbb{H}^1}^2\le e^{-t}p\big(\|P(U_0)\|_{H^1}+\|Q(U_0)\|_{H^1}\big)+p(\|U_0\|_{\mathbb{L}_2}), \qquad 0\le t\le T_U, \tag{21}
$$

and

$$
\begin{aligned}
&\|P(U(t))\|_{H^1}^2+\|Q(U(t))\|_{H^1}^2\\
&\qquad \le e^{-t}p\big(\|P(U_0)\|_{H^1}+\|Q(U_0)\|_{H^1}\big)+p(\|U_0\|_{\mathbb{L}_2}), \qquad 0\le t\le T_U.
\end{aligned}\tag{22}
$$

At the same time, integrating (20) on $(0,t)$ in view of (21) and (22), we conclude that

$$
\int_0^t \|U_s(s)\|_{\mathbb{L}_2}^2 ds\le p\big(\|P(U_0)\|_{H^1}+\|Q(U_0)\|_{H^1}\big)(t+1), \qquad 0\le t\le T_U. \tag{23}
$$

In view of these estimates, we may introduce a notation

$$
N_{\mathbb{H}^1}(U)=\|P(u,v)\|_{H^1}+\|Q(u,v)\|_{H^1}, \qquad U=\begin{pmatrix}u\\ v\end{pmatrix}\in\mathbb{H}^2(\Omega).
$$

Step 3. We now use the equation

$$
\frac{\partial u_t}{\partial t}=\Delta\{P_u(u,v)u_t+P_v(u,v)v_t\}+f_u(u,v)u_t+f_v(u,v)v_t
$$

in $H^1(\Omega)^*$ satisfied by the derivative $u_t = \frac{\partial u}{\partial t}$. Consider the duality product of this equation and u_t in $H^1(\Omega)^* \times H^1(\Omega)$. Then,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_t^2 dx + \int_\Omega (P_u|\nabla u_t|^2 + P_v \nabla u_t \cdot \nabla v_t)dx$$
$$= -\int_\Omega (u_t \nabla P_u + v_t \nabla P_v)\cdot \nabla u_t dx + \int_\Omega (f_u u_t + f_v v_t)u_t dx.$$

A similar energy equality holds for v_t, too. We combine these. After some calculations,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u_t^2+v_t^2)dx + \int_\Omega (a|\nabla u_t|^2 + b|\nabla v_t|^2)dx$$
$$+\int_\Omega \{(2\alpha_{11}u+\alpha_{12}v)|\nabla u_t|^2 + (\alpha_{12}u+\alpha_{21}v)\nabla u_t\cdot\nabla v_t + (\alpha_{21}u+2\alpha_{22}v)|\nabla v_t|^2\}dx$$
$$\le \frac{1}{2}\int_\Omega (a|\nabla u_t|^2+b|\nabla v_t|^2)dx + C\int_\Omega (1+|\nabla u|^2+|\nabla v|^2)(u_t^2+v_t^2)dx$$
$$+C\int_\Omega (1+u+v)(u_t^2+v_t^2)dx.$$

As before, (13) shows that the third integral in the left hand side is nonnegative. Therefore,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u_t^2+v_t^2)dx + \frac{1}{2}\int_\Omega (a|\nabla u_t|^2 + b|\nabla v_t|^2)dx$$
$$\le C\int_\Omega (1+|\nabla u|^2+|\nabla v|^2)(u_t^2+v_t^2)dx + C\int_\Omega (1+u+v)(u_t^2+v_t^2)dx.$$

By the Gagliardo-Nirenberg inequality, we notice that

$$\int_\Omega |\nabla u|^2 u_t^2 dx \le C\|\nabla u\|_{L_4}^2 \|u_t\|_{L_4}^2 \le C\|u\|_{H^2}\|u\|_{H^1}\|u_t\|_{H^1}\|u_t\|_{L_2}.$$

In view of Lemma 3.1 announced below and (22), we have

$$\begin{aligned}\|u\|_{H^2} &\le p\big(N_{\mathbb{H}^1}(U_0)\big)(\|u_t - f\|_{L_2} + \|v_t - g\|_{L_2} + 1)\\ &\le p\big(N_{\mathbb{H}^1}(U_0)\big)(\|u_t\|_{L_2} + \|v_t\|_{L_2} + 1).\end{aligned}$$

From (21), $\|u\|_{H^1} \le p(N_{\mathbb{H}^1}(U_0))$. Of course, $\|u_t\|_{H^1} = \|\nabla u_t\|_{L_2} + \|u_t\|_{L_2}$. So it follows that

$$\int_\Omega |\nabla u|^2 u_t^2 dx \le \zeta\|\nabla u_t\|_{L_2}^2 + C_\zeta p\big(N_{\mathbb{H}^1}(U_0)\big)(\|u_t\|_{L_2}^2 + \|v_t\|_{L_2}^2 + 1)\|u_t\|_{L_2}^2$$

with any $\zeta > 0$. Handling in a similar way for the other integrals, we verify the estimate

$$\int_\Omega (|\nabla u|^2+|\nabla v|^2)(u_t^2+v_t^2)dx \le \zeta(\|\nabla u_t\|_{L_2}^2 + \|\nabla v_t\|_{L_2}^2)$$
$$+ C_\zeta p\big(N_{\mathbb{H}^1}(U_0)\big)(\|u_t\|_{L_2}^2 + \|v_t\|_{L_2}^2 + 1)(\|u_t\|_{L_2}^2 + \|v_t\|_{L_2}^2).$$

By the similar procedure (actually it may be easier), we verify also the estimate

$$\int_\Omega (u+v)(u_t^2+v_t^2)dx \le \zeta(\|\nabla u_t\|_{L_2}^2 + \|\nabla v_t\|_{L_2}^2)$$
$$+ C_\zeta p\big(N_{\mathbb{H}^1}(U_0)\big)(\|u_t\|_{L_2}^2 + \|v_t\|_{L_2}^2 + 1)(\|u_t\|_{L_2}^2 + \|v_t\|_{L_2}^2).$$

Thus it has been shown that the derivatives u_t and v_t satisfy

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u_t^2+v_t^2)dx+\frac{1}{4}\int_\Omega (a|\nabla u_t|^2+b|\nabla v_t|^2)dx \\ \le p\big(N_{\mathbb{H}^1}(U_0)\big)(\|u_t\|_{L_2}^2+\|v_t\|_{L_2}^2+1)(\|u_t\|_{L_2}^2+\|v_t\|_{L_2}^2).$$

Meanwhile, on account of (19), (21) and (22), observe that the following differential inequality

$$\frac{d}{dt}\int_\Omega \eta(|\nabla P|^2+|\nabla Q|^2)dx+\int_\Omega \eta\delta(u_t^2+v_t^2)dx \le \eta p\big(N_{\mathbb{H}^1}(U_0)\big)$$

holds, where $\delta>0$ is a positive exponent given by $\delta=\min\{a,b\}$ and $\eta>0$ is a parameter.

The previous two inequalities then yield the following one

$$\frac{d\psi_1}{dt}+\eta\delta\psi_1+\frac{1}{4}\int_\Omega (a|\nabla u_t|^2+b|\nabla v_t|^2)dx \\ \le \eta^2 p\big(N_{\mathbb{H}^1}(U_0)\big)+p\big(N_{\mathbb{H}^1}(U_0)\big)(\|u_t\|_{L_2}^2+\|v_t\|_{L_2}^2+1)\psi_1 \quad (24)$$

for the function $\psi_1(t)=\int_\Omega\{\frac{1}{2}(u_t^2+v_t^2)+\eta(|\nabla P|^2+|\nabla Q|^2)\}dx$. Solving this, we have

$$\psi_1(t)\le e^{\int_0^t\{p(N_{\mathbb{H}^1}(U_0))(\|u_s\|_{L_2}^2+\|v_s\|_{L_2}^2+1)-\eta\delta\}ds}\psi_1(0) \\ +\eta^2 p\big(N_{\mathbb{H}^1}(U_0)\big)\int_0^t e^{\int_s^t\{p(N_{\mathbb{H}^1}(U_0))(\|u_\tau\|_{L_2}^2+\|v_\tau\|_{L_2}^2+1)-\eta\delta\}d\tau}ds.$$

Use the corresponding version of (23) obtained by integrating (20) on (s,t). In fact, thanks to (22),

$$\int_s^t(\|u_\tau\|_{L_2}^2+\|v_\tau\|_{L_2}^2)d\tau\le p\big(N_{\mathbb{H}^1}(U(s))\big)\{(t-s)+1\} \\ \le p\big(N_{\mathbb{H}^1}(U_0)\big)\{(t-s)+1\},\qquad 0\le s\le t\le T_U.$$

Hence,

$$\psi_1(t)\le p\big(N_{\mathbb{H}^1}(U_0)\big)e^{\{p(N_{\mathbb{H}^1}(U_0))-\eta\delta\}t}\psi_1(0) \\ +\eta^2 p\big(N_{\mathbb{H}^1}(U_0)\big)\int_0^t e^{\{p(N_{\mathbb{H}^1}(U_0))-\eta\delta\}(t-s)}ds.$$

This means that, if we fix the parameter η as $\eta=\{p(N_{\mathbb{H}^1}(U_0))+1\}/\delta$, then $\psi_1(t)$ is estimated by

$$\psi_1(t)\le p\big(N_{\mathbb{H}^1}(U_0)\big)\{e^{-t}\psi_1(0)+1\}.$$

Since $\psi_1(0)\le p(\|U_0\|_{\mathbb{H}^2})$, it follows that

$$\|u_t(t)\|_{L_2}^2+\|v_t(t)\|_{L_2}^2\le e^{-t}p(\|U_0\|_{\mathbb{H}^2})+p\big(N_{\mathbb{H}^1}(U_0)\big). \quad (25)$$

In view of Lemma 3.1 below, we therefore arrive at the estimate

$$\|u(t)\|_{H^2}+\|v(t)\|_{H^2}\le e^{-t}p(\|U_0\|_{\mathbb{H}^2})+p\big(N_{\mathbb{H}^1}(U_0)\big),\qquad 0\le t\le T_U. \quad (26)$$

We have thus accomplished the proof of proposition. ☐

Lemma 3.1. *For $0\le u,\, v\in H^2(\Omega)$ with $\frac{\partial u}{\partial n}\equiv\frac{\partial v}{\partial n}\equiv 0$ on $\partial\Omega$, it holds that*

$$\|u\|_{H^2}+\|v\|_{H^2} \\ \le C(\|P\|_{H^1}+\|Q\|_{H^1}+1)(\|\Delta P\|_{L_2}+\|\Delta Q\|_{L_2}+\|P\|_{L_2}+\|Q\|_{L_2}).$$

Proof of lemma. We know that

$$\|u\|_{H^2}+\|v\|_{H^2}\le C(\|\Delta u\|_{L_2}+\|\Delta v\|_{L_2}+\|u\|_{L_2}+\|v\|_{L_2}),\\ u,\,v\in H^2(\Omega);\ \tfrac{\partial u}{\partial n}\equiv\tfrac{\partial v}{\partial n}\equiv 0 \text{ on } \partial\Omega. \quad (27)$$

While by direct calculations it is seen that

$$\begin{pmatrix}\Delta P\\ \Delta Q\end{pmatrix}=\begin{pmatrix}P_u & P_v\\ Q_u & Q_v\end{pmatrix}\begin{pmatrix}\Delta u\\ \Delta v\end{pmatrix}+2\begin{pmatrix}\alpha_{11}|\nabla u|^2+\alpha_{21}\nabla u\cdot\nabla v\\ \alpha_{21}\nabla u\cdot\nabla v+\alpha_{22}|\nabla v|^2\end{pmatrix},$$

$$\begin{pmatrix}\Delta u\\ \Delta v\end{pmatrix}=\begin{pmatrix}P_u & P_v\\ Q_u & Q_v\end{pmatrix}^{-1}\left\{\begin{pmatrix}\Delta P\\ \Delta Q\end{pmatrix}-2\begin{pmatrix}\alpha_{11}|\nabla u|^2+\alpha_{21}\nabla u\cdot\nabla v\\ \alpha_{21}\nabla u\cdot\nabla v+\alpha_{22}|\nabla v|^2\end{pmatrix}\right\}.$$

Furthermore, $\||\nabla u|^2\|_{L_2}\le C(\|\nabla P\|_{L_4}^2+\|\nabla Q\|_{L_4}^2)$ (due to (17)) and

$$\|\nabla P\|_{L_4}^2\le C\|P\|_{H^2}\|P\|_{H^1}\le C(\|\Delta P\|_{L_2}+\|P\|_{L_2})\|P\|_{H^1},$$
$$\|\nabla Q\|_{L_4}^2\le C\|Q\|_{H^2}\|Q\|_{H^1}\le C(\|\Delta Q\|_{L_2}+\|Q\|_{L_2})\|Q\|_{H^1}.$$

Remember that $\frac{\partial u}{\partial n}\equiv\frac{\partial v}{\partial n}\equiv 0$ implies that $\frac{\partial P}{\partial n}\equiv\frac{\partial Q}{\partial n}\equiv 0$. It is the same for $\||\nabla v|^2\|_{L_2}$. ☐

As an immediate consequence of the series of estimates established above, we obtain the important dissipative estimate for U at an exponential rate. Let us apply the estimates (15), (22) and (26) in the interval $[0,\frac{t}{3}]$, $[\frac{t}{3},\frac{2t}{3}]$ and $[\frac{2t}{3},t]$, respectively. Then, it follows that

$$\begin{aligned}\|U(t)\|_{\mathbb{H}^2}&\le e^{-\frac{t}{3}}p(\|U(\tfrac{2t}{3})\|_{\mathbb{H}^2})+p(N_{\mathbb{H}^1}(U(\tfrac{2t}{3})))\\ &\le e^{-\frac{t}{3}}p(\|U_0\|_{\mathbb{H}^2})+p\big(e^{-\frac{t}{3}}p(N_{\mathbb{H}^1}(U(\tfrac{t}{3})))+p(\|U(\tfrac{t}{3})\|_{\mathbb{L}_2})\big)\\ &\le p\big(e^{-\frac{t}{3}}p(\|U_0\|_{\mathbb{H}^2})+1\big),\qquad 0\le t\le T_U,\end{aligned}\quad (28)$$

where $p(\cdot)$'s are continuous increasing functions determined in a suitable way.

3.2. Case when $\boldsymbol{\alpha_{12}\alpha_{21}=0}$.

Let us assume that one of α_{12} and α_{21} vanishes, say $\alpha_{21}=0$, and so $Q(u,v)=Q(v)=bv+\alpha_{22}v^2$. In this case, we have only $\alpha_{ii}\ge 0$ for $i=1,\,2$.

Proposition 2. *Let $\alpha_{21}=0$. There exists a continuous increasing function $p(\cdot)$ such that, for any local solution U to (1) lying in (10) with initial value $U_0\in\mathcal{K}\cap\mathbb{H}^2(\Omega)$, it holds that*

$$\|U(t)\|_{\mathbb{H}^2}\le p(\|U_0\|_{\mathbb{H}^2}),\qquad 0\le t\le T_U. \quad (29)$$

Proof. As before, a unified notation C will be used to denote various constants which are determined from the initial constants a, b, c, d, $\alpha_{ij}(1\le i,j\le 2)$ and $\gamma_{ij}(1\le i,j\le 2)$ and the domain Ω alone in a specific way. So, C may change from occurrence to occurrence. When a constant C depends on a particular parameter, say ζ, we shall denote it by C_ζ.

Similarly, a unified notation $p(\cdot)$ will be used to denote various continuous increasing functions which may change from occurrence to occurrence.

Step 1. Integrate the equation for u of (1) in Ω. Then,

$$\frac{d}{dt}\int_\Omega u\,dx=\int_\Omega f(u,v)dx\le\int_\Omega(cu-\gamma_{11}u^2)dx. \quad (30)$$

Since

$$cu-\gamma_{11}u^2\le -u+\tfrac{(c+1)^2}{4\gamma_{11}}\qquad \text{for } u\ge 0,$$

we have

$$\frac{d}{dt}\int_\Omega u\,dx + \int_\Omega u\,dx \le \frac{(c+1)^2}{4\gamma_{11}}|\Omega|.$$

Therefore,

$$\|u(t)\|_{L_1} \le e^{-t}\|u_0\|_{L_1} + \tfrac{(c+1)^2}{4\gamma_{11}}|\Omega|, \qquad 0 \le t \le T_U. \tag{31}$$

In addition, from

$$\|u(t)\|_{L_1} - \|u_0\|_{L_1} = \int_0^t \int_\Omega f(u,v)dxds,$$

it follows that

$$\begin{aligned}\left|\int_0^t \int_\Omega f(u,v)dxds\right| &\le \max\{\|u(t)\|_{L_1}, \|u_0\|_{L_1}\} \\ &\le \|u_0\|_{L_1} + \tfrac{(c+1)^2}{4\gamma_{11}}|\Omega|, \qquad 0 \le t \le T_U. \end{aligned}\tag{32}$$

As well, since

$$u^2 = \gamma_{11}^{-1}\{cu - \gamma_{12}uv - f(u,v)\}, \tag{33}$$

it is seen that

$$\begin{aligned}\int_0^t \|u(s)\|_{L_2}^2 ds &\le c\gamma_{11}^{-1}\int_0^t \|u\|_{L_1}ds + \gamma_{11}^{-1}\left|\int_0^t\int_\Omega f(u,v)dxds\right| \\ &\le C(\|u_0\|_{L_1}+1)(t+1), \qquad 0 \le t \le T_U. \end{aligned}\tag{34}$$

Step 2. Let q be an exponent varying in the range $2 < q < \infty$. Multiply the second equation in (1) by qv^{q-1} and integrate the product in Ω. After some calculations,

$$\frac{d}{dt}\int_\Omega v^q dx = -q(q-1)\int_\Omega Q_v v^{q-2}|\nabla v|^2 dx + q\int_\Omega g v^{q-1}dx \le q\int_\Omega (d-\gamma_{22}v)v^q dx.$$

Here we notice, for example, that

$$(d-\gamma_{22}v)v^q \le -dv^q + \gamma_{22}(2d/\gamma_{22})^{q+1} \qquad \text{for } v \ge 0$$

(indeed to see this, argue dividing the range of v into $0 \le v \le 2d/\gamma_{22}$ and $2d/\gamma_{22} < v < \infty$). By this estimate it follows that

$$\frac{d}{dt}\int_\Omega v^q dx \le -dq\int_\Omega v^q dx + \gamma_{22}q(2d/\gamma_{22})^{q+1}|\Omega|.$$

Solving this differential inequality for $\int_\Omega v^q dx$, we obtain that

$$\|v(t)\|_{L_q}^q \le e^{-dqt}\|v_0\|_{L_q}^q + 2(2d/\gamma_{22})^q|\Omega|,$$

and hence

$$\|v(t)\|_{L_q} \le \{e^{-dqt}\|v_0\|_{L_q}^q + 2(2d/\gamma_{22})^q|\Omega|\}^{\frac{1}{q}} \le 2^{\frac{1}{q}}\{e^{-dt}\|v_0\|_{L_q} + (2d/\gamma_{22})(2|\Omega|)^{\frac{1}{q}}\}.$$

Now, noting that $\lim_{q\to\infty}\|v\|_{L_q} = \|v\|_{L_\infty}$, we deduce that

$$\|v(t)\|_{L_\infty} \le e^{-dt}\|v_0\|_{L_\infty} + \tfrac{2d}{\gamma_{22}}, \qquad 0 \le t \le T_U. \tag{35}$$

Step 3. Multiply the second equation in (1) by $\frac{d}{dt}Q(v) = Q_v v_t$ and integrate the product in Ω. By obvious calculations,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla Q(v)|^2 dx + \int_\Omega Q_v(v) v_t^2 dx = \int_\Omega g Q_v(v) v_t dx \\
&\quad \le p(\|v_0\|_{L_\infty}) \int_\Omega (u+v+1)|v_t| dx \le \frac{b}{2}\int_\Omega v_t^2 dx + p(\|v_0\|_{L_\infty}) \int_\Omega (u+v+1)^2 dx
\end{aligned}$$

due to (35). Therefore,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla Q|^2 dx + \frac{b}{2}\int_\Omega v_t^2 dx \le p(\|v_0\|_{L_\infty}) \int_\Omega (u^2+1) dx. \tag{36}$$

In the meantime, multiply the second equation in (1) by $Q(v)$ and integrate the product in Ω. After some calculations,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \Xi(v(t)) dx + \int_\Omega |\nabla Q|^2 dx = \int_\Omega g Q dx &\le p(\|v_0\|_{L_\infty}) \int_\Omega (u+1) dx \\
&\le p(\|u_0\|_{L_1} + \|v_0\|_{L_\infty}),
\end{aligned}$$

where $\Xi(v) = \int_0^v Q(v) dv = \frac{b}{2}v^2 + \frac{\alpha_{22}}{3}v^3$. Here we used (31). This differential inequality together with (36) then yields that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (\Xi + \tfrac{1}{2}|\nabla Q|^2) dx + \int_\Omega (\Xi + \tfrac{1}{2}|\nabla Q|^2) dx + \frac{b}{2}\int_\Omega v_t^2 dx \\
&\qquad\qquad \le p(\|u_0\|_{L_1} + \|v_0\|_{L_\infty}) \int_\Omega (u^2+1) dx.
\end{aligned} \tag{37}$$

Therefore,

$$\begin{aligned}
&\|\Xi(v(t))\|_{L_1} + \tfrac{1}{2}\|\nabla Q(v(t))\|_{L_2}^2 \le e^{-t}(\|\Xi(v_0)\|_{L_1} + \tfrac{1}{2}\|\nabla Q(v_0)\|_{H^1}^2) \\
&\qquad\qquad + p(\|u_0\|_{L_1} + \|v_0\|_{L_\infty}) \int_0^t e^{-(t-s)}\{\|u(s)\|_{L_2}^2 + 1\} ds.
\end{aligned}$$

Using (32), we can here prove the following lemma.

Lemma 3.2. *There exists a constant C independent of the local solution such that*

$$\int_0^t e^{-(t-s)} \|u(s)\|_{L_2}^2 ds \le C(\|u_0\|_{L_1} + 1)(\|v_0\|_{L_\infty} + 1), \qquad 0 \le t \le T_U. \tag{38}$$

Proof of lemma. We verify from (33) that

$$\|u(t)\|_{L_2}^2 \le N_2(U(t)) + C\|u(t)\|_{L_1}(\|v(t)\|_{L_\infty} + 1), \qquad 0 \le t \le T_U,$$

where $N_2(U)$ denotes $N_2(U) = -\frac{1}{\gamma_{11}}\int_\Omega f(u,v) dx$. Therefore,

$$\int_0^t e^{-(t-s)} \|u(s)\|_{L_2}^2 ds \le \int_0^t e^{-(t-s)} \{N_2(U(s)) + C\|u(s)\|_{L_1}(\|v(s)\|_{L_\infty} + 1)\} ds.$$

By (31) and (35) it is clear that

$$\int_0^t e^{-(t-s)} \|u(s)\|_{L_1}(\|v(s)\|_{L_\infty} + 1) ds \le C(\|u_0\|_{L_1} + 1)(\|v_0\|_{L_\infty} + 1).$$

Meanwhile, to estimate the integral of the function $e^{-(t-s)}N_2(U(s))$, we apply the second mean value theorem. Then there exists some $\tau \in [0,t]$ for which the formula

$$\int_0^t e^{-(t-s)}N_2(U(s))ds = e^{-t}\int_0^\tau N_2(U(s))ds + \int_\tau^t N_2(U(s))ds$$
$$= \int_0^t N_2(U(s))ds + (e^{-t}-1)\int_0^\tau N_2(U(s))ds$$

is valid. Then we deduce from (32) that

$$\int_0^t e^{-(t-s)}N_2(U(s))ds \le C(\|u_0\|_{L_1}+1).$$

Hence (38) is proved. □

We have in this way concluded that

$$\|Q(v(t))\|_{H^1}^2 \le Ce^{-t}\|Q(v_0)\|_{H^1}^2 + p(\|u_0\|_{L_1}+\|v_0\|_{L_\infty}), \qquad 0\le t\le T_U. \tag{39}$$

Since $\nabla v = Q_v^{-1}\nabla Q$, it immediately follows that

$$\|v(t)\|_{H^1}^2 \le Ce^{-t}\|Q(v_0)\|_{H^1}^2 + p(\|u_0\|_{L_1}+\|v_0\|_{L_\infty}), \qquad 0\le t\le T_U. \tag{40}$$

As well, thanks to (34), integration of (37) on $(0,t)$ yields that

$$\int_0^t \|v_t\|_{L_2}^2 ds \le p(\|u_0\|_{L_1}+\|v_0\|_{L_\infty}+\|Q(v_0)\|_{H^1})(t+1). \qquad 0\le t\le T_U. \tag{41}$$

Step 4. In this step, we shall use the following abbreviated notation

$$p_1(U_0) = p\big(\|u_0\|_{L_1}+\|v_0\|_{L_\infty}+\|Q(v_0)\|_{H^1}\big),$$

where $p(\cdot)$ is a continuous increasing function which varies in each occurrence. Introducing the quantity

$$N_{1,\log}(u) = \int_\Omega u\log(u+1)dx, \qquad 0\le u\in L_2(\Omega),$$

we intend to estimate $N_{1,\log}(u(t))$ for the local solution.

Multiply the first equation in (1) by $\log(u+1)$ and integrate the product in Ω. Then we can verify that

$$\frac{d}{dt}\int_\Omega\{(u+1)\log(u+1)-u\}dx + 4\int_\Omega P_u(u,v)|\nabla\sqrt{u+1}|^2dx$$
$$= -\alpha_{12}\int_\Omega\{\log(u+1)-u\}\Delta v\,dx + \int_\Omega f(u,v)\log(u+1)dx$$

by direct calculations utilizing the following formulae

$$\frac{\partial u}{\partial t}\cdot\log(u+1) = \frac{\partial}{\partial t}\{(u+1)\log(u+1)-u\},$$
$$\frac{1}{u+1}|\nabla u|^2 = 4|\nabla\sqrt{u+1}|^2,$$
$$\frac{u}{u+1}\nabla u = -\nabla\{\log(u+1)-u\}.$$

Here it is clear that

$$-\alpha_{12}\int_\Omega\{\log(u+1)-u\}\Delta v\,dx \le \zeta\|\Delta v\|_{L_2}^2 + C_\zeta\|u\|_{L_2}^2$$

with any $\zeta > 0$. Since

$$\Delta v = \frac{\Delta Q - 2\alpha_{22}|\nabla v|^2}{b + 2\alpha_{22}v} = \frac{\Delta Q - 2\alpha_{22}Q_v^{-2}|\nabla Q|^2}{b + 2\alpha_{22}v}, \qquad 0 \le v \in H^2(\Omega),$$

it follows by (27) that

$$\begin{aligned}\|\Delta v\|_{L_2}^2 &\le C(\|\Delta Q\|_{L_2}^2 + \|Q\|_{H_4^1}^4) \le C(\|\Delta Q\|_{L_2}^2 + \|Q\|_{H^2}^2\|Q\|_{H^1}^2) \\ &\le C\{\|\Delta Q\|_{L_2}^2(1 + \|Q\|_{H^1}^2) + \|Q\|_{H^1}^4\}, \qquad 0 \le v \in H^2(\Omega). \end{aligned} \tag{42}$$

Meanwhile, we have

$$C_\zeta u^2 \le -\tfrac{1}{2}(cu - \gamma_{11}u^2)\log(u+1) + \widetilde{C}_\zeta u \qquad \text{for all } u \ge 0,$$

whatever the parameter ζ is, where $\widetilde{C}_\zeta$ denotes some constant determined from C_ζ (and hence from ζ). Hence we obtain that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \{(u+1)\log(u+1) - u\}dx + \int_\Omega \{(u+1)\log(u+1) - u\}dx \\ &\qquad \le \zeta C\{\|\Delta Q\|_{L_2}^2(1 + \|Q\|_{H^1}^2) + \|Q\|_{H^1}^4\} + \widetilde{C}_\zeta\|u\|_{L_1}.\end{aligned}$$

Moreover, in view of (31) and (39),

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \{(u+1)\log(u+1) - u\}dx + \int_\Omega \{(u+1)\log(u+1) - u\}dx \\ &\qquad \le p_1(U_0)(\zeta\|v_t - g\|_{L_2}^2 + \widetilde{C}_\zeta).\end{aligned}$$

Let us add this differential equation to (37) and take ζ sufficiently small. Then it follows that

$$\frac{d\psi_1}{dt} + \psi_1(t) + \frac{b}{4}\int_\Omega v_t^2 dx \le p_1(U_0)(\|u\|_{L_2}^2 + 1),$$

where $\psi_1(t) = \int_\Omega \{(u+1)\log(u+1) - u + \Xi(v) + \frac{1}{2}|\nabla Q(v)|^2\}dx$. Noting (38), we conclude that

$$\psi_1(t) \le e^{-t}\psi_1(0) + p_1(U_0), \qquad 0 \le t \le T_U.$$

In particular,

$$N_{1,\log}(u(t)) \le e^{-t}N_{1,\log}(u_0) + p_1(U_0), \qquad 0 \le t \le T_U. \tag{43}$$

Step 5. In this step, we shall use the following abbreviated notation

$$p_2(U_0) = p\big(N_{1,\log}(u_0) + \|v_0\|_{L_\infty} + \|Q(v_0)\|_{H^1}\big),$$

where $p(\cdot)$ is a continuous increasing function which varies in each occurrence. The goal is to estimate $\|u(t)\|_{L_2}$ for the local solution.

Multiply the first equation in (1) by u and integrate the product in Ω. Then,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega P_u(u,v)|\nabla u|^2 dx &= -\int_\Omega P_v(u,v)\nabla u\cdot\nabla v\,dx + \int_\Omega f(u,v)u\,dx \\ &= -\alpha_{12}\int_\Omega u\nabla u\cdot\nabla v\,dx + \int_\Omega f(u,v)u\,dx.\end{aligned}$$

Here,

$$\begin{aligned}-\int_\Omega u\nabla u\cdot\nabla v\,dx &= -\frac{1}{2}\int_\Omega \nabla u^2\cdot\nabla v\,dx = \frac{1}{2}\int_\Omega u^2\Delta v\,dx \\ &\le C\|u\|_{L_3}^2\|\Delta v\|_{L_3} \le \zeta_1\|\Delta v\|_{L_3}^3 + C_{\zeta_1}\|u\|_{L_3}^3\end{aligned}$$

with any $\zeta_1 > 0$. Furthermore, by (40),

$$\|\Delta v\|_{L_3}^3 \le C\|\Delta v\|_{H^1}\|\Delta v\|_{L_2}^2 \le C\|v\|_{H^3}^2\|v\|_{H^1} \le p_2(U_0)\|v\|_{H^3}^2.$$

Here we used the fact that $v(t) \in H^3(\Omega)$ which is verified by Lemma 3.3 below. In addition, to estimate the norm $\|u\|_{L_3}$, we utilize the modified Gagliardo-Nirenberg inequality

$$\|u\|_{L_3}^3 \le \zeta_2\|u\|_{H^1}^2 N_{1,\log}(u) + C_{\zeta_2}\|u\|_{L_1}, \qquad 0 \le u \in H^1(\Omega),$$

where $\zeta_2 > 0$ is any positive number (cf. [15, Lemma 4.3]). This together with (31) and (43) yields that

$$\|u\|_{L_3}^3 \le p_2(U_0)\{\zeta_2\|u\|_{H^1}^2 + C_{\zeta_2}\}.$$

Therefore it follows that

$$\zeta_1\|\Delta v\|_{L_3}^3 + C_{\zeta_1}\|u\|_{L_3}^3 \le p_2(U_0)\{\zeta_1\|v\|_{H^3}^2 + C_{\zeta_1}\zeta_2\|u\|_{H^1}^2 + C_{\zeta_1}C_{\zeta_2}\}.$$

Hence we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega u^2dx + \gamma_{11}\int_\Omega u^3dx + \int_\Omega P_u|\nabla u|^2dx& \\ \le p_2(U_0)\{\zeta_1\|v\|_{H^3}^2 + C_{\zeta_1}\zeta_2\|u\|_{H^1}^2 + C_{\zeta_1}C_{\zeta_2}\}.& \qquad (44)\end{aligned}$$

Lemma 3.3. *We have $v(t)$, $Q(v(t)) \in H^3(\Omega)$ with the estimates*

$$\|v(t)\|_{H^3} \le C\{\|\nabla\Delta v(t)\|_{L_2} + \|v(t)\|_{H^1}\}, \qquad 0 < t \le T_U, \qquad (45)$$

$$\|Q(v(t))\|_{H^3} \le C\{\|\nabla\Delta Q(v(t))\|_{L_2} + \|Q(v(t))\|_{H^1}\}, \qquad 0 < t \le T_U. \qquad (46)$$

Proof of lemma. From (10) we see that $\Delta Q(t) \in H^1(\Omega)$. Since Ω is of class $\mathcal{C}^3$, $\Delta Q(t) \in H^1(\Omega)$ with $\frac{\partial Q}{\partial n}(t) = 0$ on $\partial\Omega$ implies $Q(t) \in H^3(\Omega)$ with (46).

In addition, the quadratic function $w = Q(v) \equiv bv + \alpha_{22}v^2$ is monotonically increasing for $v \ge 0$ and hence has a smooth inverse $v = Q^{-1}(w)$ for $w \ge 0$. By this fact, $Q(v(t)) \in H^3(\Omega)$ implies that $v(t) \in H^3(\Omega)$. Since $\frac{\partial v}{\partial n}(t) = 0$ on $\partial\Omega$, the estimate of form (45) holds for $v(t)$, too. □

In the meantime, we have to prepare another differential inequality. Consider the duality product of the second equation of (1) and $\Delta^2 v(t)$ in $H^1(\Omega) \times H^1(\Omega)^*$. From (10) the equation for v has a meaning in the space $H^1(\Omega)$ at each $0 < t \le T_U$. Meanwhile, since $v(t) \in H^3(\Omega)$, $\Delta^2 v(t)$ is an element of $H^1(\Omega)^*$ because of the fact that Δ is a bounded operator from $H^1(\Omega)$ into $H^1(\Omega)^*$. After some calculations,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta v|^2dx + \int_\Omega Q_v|\nabla\Delta v|^2dx&\\ = -\alpha_{22}\int_\Omega(\Delta v\nabla v + \nabla|\nabla v|^2)\cdot\nabla\Delta v\,dx - \int_\Omega \nabla g\cdot\nabla\Delta v\,dx&\\ \le p_2(U_0)\|v\|_{H^3}\{\|v\|_{H^2_3}\|v\|_{H^1_6} + \|\nabla u\|_{L_2} + 1\}.& \qquad (47)\end{aligned}$$

By the Gagliardo-Nirenberg inequality,

$$\|v\|_{H^3}\|v\|_{H^2_3}\|v\|_{H^1_6} \le C\|v\|_{H^3}^{\frac{4}{3}}\|v\|_{H^2}^{\frac{4}{3}}\|v\|_{H^1}^{\frac{1}{3}} \le \zeta_3\|v\|_{H^3}^2 + C_{\zeta_3}p_2(U_0)\|v\|_{H^2}^4$$

with any $\zeta_3 > 0$. Meanwhile,

$$\|v\|_{H^3}\|\nabla u\|_{L_2} \le \zeta_4\|v\|_{H^3}^2 + C_{\zeta_4}\|\nabla u\|_{L_2}^2$$

with any $\zeta_4 > 0$. So, taking ζ_3 and ζ_4 sufficiently small and using (45), we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta v|^2 dx + \frac{1}{2}\int_\Omega Q_v|\nabla\Delta v|^2 dx \le p_2(U_0)\{\|\nabla u\|_{L_2}^2 + \|v\|_{H^2}^4 + 1\}. \tag{48}$$

We now multiply a parameter $\eta > 0$ to (44) and add the multiplied inequality to (48). Then,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega(\eta u^2 + |\Delta v|^2)dx + \gamma_{11}\eta\int_\Omega u^3 dx\\ + \int_\Omega[\{a\eta - p_2(U_0)\}|\nabla u|^2 + b|\nabla\Delta v|^2]dx\\ \le \eta p_2(U_0)\{\zeta_1\|v\|_{H^3}^2 + C_{\zeta_1}\zeta_2\|u\|_{H^1}^2 + C_{\zeta_1}C_{\zeta_2}\} + p_2(U_0)\|v\|_{H^2}^4.\end{aligned}$$

If η is fixed in such a way that $\frac{a\eta}{2} \ge p_2(U_0)$ and if ζ_1 and ζ_2 are taken sufficiently small, then it is seen that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega(\eta u^2 + |\Delta v|^2)dx + \gamma_{11}\eta\int_\Omega u^3 dx\\ + \frac{1}{4}\int_\Omega(a\eta|\nabla u|^2 + b|\nabla\Delta v|^2)dx \le p_2(U_0)(\|u\|_{L_2}^2 + \|v\|_{H^2}^4 + 1).\end{aligned}$$

Moreover, thanks to (39) and (42),

$$\begin{aligned}\|v\|_{H^2}^4 \le C(\|\Delta v\|_{L_2}^4 + \|v\|_{L_2}^4) &\le p_2(U_0)(\|\Delta v\|_{L_2}^2\|\Delta Q\|_{L_2}^2 + 1)\\ &\le p_2(U_0)\{\|\Delta v\|_{L_2}^2(\|v_t\|_{L_2}^2 + \|u\|_{L_2}^2 + 1) + 1\}.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega(\eta u^2 + |\Delta v|^2)dx + \gamma_{11}\eta\int_\Omega u^3 dx + \frac{1}{4}\int_\Omega(a\eta|\nabla u|^2 + b|\nabla\Delta v|^2)dx\\ \le p_2(U_0)\{\|\Delta v\|_{L_2}^2(\|u\|_{L_2}^2 + \|v_t\|_{L_2}^2 + 1) + 1\}.\end{aligned}$$

We add this inequality to the following one

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \xi|\nabla Q|^2 dx + \int_\Omega \xi p_2(U_0)^{-1}|\Delta v|^2 dx \le p_2(U_0)\xi(\|u\|_{L_2}^2 + 1)$$

which is obtained from (36) (due to (42)) after some calculations and multiplication of a parameter $\xi > 0$. Then we arrive at the main differential inequality of this step

$$\begin{aligned}\frac{d}{dt}\psi_2(t) + \xi p_2(U_0)^{-1}\psi_2(t) + \frac{1}{4}\int_\Omega(a\eta|\nabla u|^2 + b|\nabla\Delta v|^2)dx\\ \le p_2(U_0)[\{\|u\|_{L_2}^2 + \|v_t\|_{L_2}^2 + 1\}\psi_2(t) + C_\xi]\end{aligned} \tag{49}$$

for the function $\psi_2(t) = \frac{1}{2}\{\eta\|u(t)\|_{L_2}^2 + \xi\|\nabla Q(v(t))\|_{L_2}^2 + \|\Delta v(t)\|_{L_2}^2\}$. Here we used also a fact that the estimate

$$\{p_2(U_0) + \eta p_2(U_0)^{-1}\}\xi u^2 \le \gamma_{11}\eta u^3 + C_\xi p_2(U_0)$$

holds for all $u \ge 0$, whatever the parameter $\xi > 0$ is, with some constant $C_\xi > 0$.

Solving (49), we conclude that

$$\begin{aligned}\psi_2(t) \le e^{\int_0^t[p_2(U_0)\{\|u(s)\|_{L_2}^2 + \|v_s(s)\|_{L_2}^2 + 1\} - p_2(U_0)^{-1}\xi]ds}\psi_2(0)\\ + C_\xi p_2(U_0)\int_0^t e^{\int_s^t[p_2(U_0)\{\|u(\tau)\|_{L_2}^2 + \|v_\tau(\tau)\|_{L_2}^2 + 1 - p_2(U_0)^{-1}\xi]d\tau}ds.\end{aligned}$$

Moreover, by the corresponding versions of (34) and (41) obtained by integrating (30) and (41) on (s,t), respectively, we can verify that

$$\psi_2(t) \le p_2(U_0)e^{\{p_2(U_0)-p_2(U_0)^{-1}\xi\}t}\psi_2(0) + C_\xi p_2(U_0)\int_0^t e^{\{p_2(U_0)-p_2(U_0)^{-1}\xi\}(t-s)}ds.$$

Here it is possible to fix the parameter ξ in such a way that $p_2(U_0)-p_2(U_0)^{-1}\xi \le -1$. Then it follows that $\psi_2(t) \le p_2(U_0)\{e^{-t}\psi_2(0)+1\}$. In particular,

$$\|u(t)\|_{L_2}^2 + \|\Delta v(t)\|_{L_2}^2 \le p_2(U_0)\{e^{-t}(\|u_0\|_{L_2}^2 + \|v_0\|_{H^2}^2)+1\}, \qquad 0 \le t \le T_U.$$

In view of (40),

$$\|u(t)\|_{L_2}^2 + \|v(t)\|_{H^2}^2 \le p_2(U_0)\{e^{-t}(\|u_0\|_{L_2}^2 + \|v_0\|_{H^2}^2)+1\}, \qquad 0 \le t \le T_U. \quad (50)$$

As well, thanks to (34), (41), (45) and (50), integration of (49) on $(0,t)$ yields that

$$\int_0^t \{\|u(s)\|_{H^1}^2 + \|v(s)\|_{H^3}^2\}ds \le p\big(\|u_0\|_{L_2} + \|v_0\|_{H^2}\big)(t+1), \qquad 0 \le t \le T_U. \quad (51)$$

Step 6. In this step, we shall use the following notation

$$p_3(U_0) = p\big(\|u_0\|_{L_2} + \|v_0\|_{H^2}\big),$$

where $p(\cdot)$ is a similar continuous increasing function as before. The goal is to estimate $\|u(t)\|_{L_4}$ for the local solution.

Multiply the first equation in (1) by u^3 and integrate the product in Ω. Then,

$$\frac{1}{4}\frac{d}{dt}\int_\Omega u^4dx + \int_\Omega 3P_u u^2|\nabla u|^2dx = -\int_\Omega 3u^2P_v\nabla u\cdot\nabla v dx + \int_\Omega fu^3dx.$$

Here,

$$\begin{aligned}\int_\Omega u^3|\nabla u|\cdot|\nabla v|dx &\le \int_\Omega u^2|u\nabla u|[\|\nabla v\|_{L_\infty}]dx\\ &\le \int_\Omega (\zeta u^2|\nabla u|^2 + C_\zeta\zeta'[\|\nabla v\|_{L_\infty}]^{12} + C_\zeta C_{\zeta'}u^{\frac{24}{5}})dx\end{aligned}$$

with any $\zeta > 0$ and any $\zeta' > 0$. In addition,

$$\|\nabla v\|_{L_\infty} \le \|\nabla v\|_{H^{13/12}} \le C\|v\|_{H^3}^{1/12}\|v\|_{H^2}^{11/12}.$$

Therefore, if the parameter ζ is fixed sufficiently small, then

$$\frac{1}{4}\frac{d}{dt}\int_\Omega u^4dx + \int_\Omega u^4dx + 3\int_\Omega au^2|\nabla u|^2dx \le p_3(U_0)\{\zeta'\|v\|_{H^3}^2 + \widetilde{C}_{\zeta'}\}.$$

Here we used a fact that

$$C_{\zeta'}u^{\frac{24}{5}} + u^4 + cu^4 \le \gamma_{11}u^5 + \widetilde{C}_{\zeta'}$$

holds for all $u \ge 0$, whatever the parameter $\zeta' > 0$ is, where $\widetilde{C}_{\zeta'} > 0$ is another constant depending on ζ'.

We add this differential inequality to (49). Taking ζ' sufficiently small, we obtain that

$$\frac{d}{dt}\psi_3(t) + \delta_1\psi_3(t) + 3\int_\Omega au^2|\nabla u|^2dx \le p_3(U_0),$$

where $\psi_3(t) = \psi_2(t) + \frac{1}{4}\|u(t)\|_{L_4}^4$, for some positive exponent $\delta_1 > 0$. Hence,

$$\psi_3(t) \le p_3(U_0)\{e^{-\delta_1 t}\psi_3(0) + 1\}, \qquad 0 \le t \le T_U.$$

In particular,

$$\|u(t)\|_{L_4}^4 \le p_3(U_0)\{e^{-\delta_1 t}\|u_0\|_{L_4}^4 + 1\}, \qquad 0 \le t \le T_U. \tag{52}$$

Step 7. We shall use the following notation

$$p_4(U_0) = p\big(\|u_0\|_{L_4} + \|v_0\|_{H^2}\big).$$

The goal is to estimate the norm $\|u(t)\|_{H^1}$ for the local solution.

Multiply the first equation in (1) by $\frac{d}{dt}P(u,v) = P_u u_t + P_v v_t$ and integrate the product in Ω. By simple calculations,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla P|^2 dx + \int_\Omega P_u u_t^2 dx = -\int_\Omega P_v u_t v_t dx \\ + \int_\Omega f\{P_u u_t + P_v v_t\} dx \le C\int_\Omega \{u|u_t||v_t| + (u^3+v^3)(|u_t|+|v_t|)\}dx.\end{aligned}$$

Here, thanks to (52),

$$\begin{aligned}\int_\Omega u|u_t||v_t|dx &\le \|u\|_{L_4}\|u_t\|_{L_2}\|v_t\|_{L_4} \le C\|u\|_{L_4}\|u_t\|_{L_2}\|v_t\|_{H^1}^{\frac{1}{2}}\|v_t\|_{L_2}^{\frac{1}{2}} \\ &\le p_4(U_0)\|u_t\|_{L_2}\|\mathit{\Delta} Q + g\|_{H^1}^{\frac{1}{2}} \le p_4(U_0)\|u_t\|_{L_2}(\|v\|_{H^3} + \|u\|_{H^1} + 1)^{\frac{1}{2}} \\ &\le \tfrac{a}{8}\|u_t\|_{L_2}^2 + p_4(U_0)(\|u\|_{H^1} + \|v\|_{H^3} + 1).\end{aligned}$$

Similarly,

$$\begin{aligned}\int_\Omega u^3|u_t|dx &\le \|u\|_{L_6}^3\|u_t\|_{L_2} \le \|u\|_{H^1}\|u\|_{L_4}^2\|u_t\|_{L_2} \\ &\le p_4(U_0)\|u\|_{H^1}\|u_t\|_{L_2} \le \tfrac{a}{8}\|u_t\|_{L_2}^2 + p_4(U_0)\|u\|_{H^1}^2\end{aligned}$$

and

$$\int_\Omega u^3|v_t|dx \le \|u\|_{L_6}^3\|v_t\|_{L_2} \le p_4(U_0)\|u\|_{H^1}.$$

Furthermore, from $\nabla P(u,v) = P_u\nabla u + P_v\nabla v$, it is immediate to see that

$$\|\nabla u\|_{L_2} \le C\{\|\nabla P\|_{L_2} + \|u\|_{L_4}\|v\|_{H^2}\}, \qquad 0 \le u,\, v \in H^2(\Omega). \tag{53}$$

In addition,

$$\|\nabla P\|_{L_2}^2 \le \|P\|_{H^1}^2 \le C\|P\|_{H^2}\|P\|_{L_2} \le \zeta\|\mathit{\Delta} P\|_{L_2}^2 + C_\zeta\|P\|_{L_2}^2, \qquad P \in H_N^2(\Omega)$$

with any $\zeta > 0$ due to

$$\|P\|_{H^2} \le C\{\|\mathit{\Delta} P\|_{L_2} + \|P\|_{L_2}\}, \qquad P \in H_N^2(\Omega) \tag{54}$$

(cf. (27)). Hence it is obtained that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla P|^2 dx + \int_\Omega |\nabla P|^2 dx + \frac{1}{2}\int_\Omega a|\mathit{\Delta} P|^2 dx \\ \le p_4(U_0)(\|v\|_{H^3} + 1) \le \zeta'\|v\|_{H^3}^2 + C_{\zeta'}p_4(U_0)\end{aligned}$$

with any $\zeta' > 0$.

We add this differential inequality to (49). If the parameter ζ' is fixed sufficiently small, then

$$\frac{d}{dt}\psi_4(t) + \delta_2\psi_4(t) + \frac{1}{2}\int_\Omega a|\Delta P|^2 dx \le p_4(U_0), \tag{55}$$

where $\psi_4(t) = \psi_2(t) + \frac{1}{2}\|\nabla P(U(t))\|_{L_2}^2$, for some positive exponent $\delta_2 > 0$. Hence,

$$\|P(U(t))\|_{H^1}^2 \le p_4(U_0)\{e^{-\delta_2 t}\|P(U_0)\|_{H^1}^2 + 1\}, \qquad 0 \le t \le T_U. \tag{56}$$

By (53),

$$\|u(t)\|_{H^1}^2 \le p_4(U_0)\{e^{-\delta_2 t}\|P(U_0)\|_{H^1}^2 + 1\}, \qquad 0 \le t \le T_U. \tag{57}$$

Thanks to (54), integration of (55) on $(0,t)$ yields that

$$\int_0^t \|P(U(s))\|_{H^2}^2 ds \le p\big(\|P(U_0)\|_{H^1} + \|v_0\|_{H^2}\big)(t+1), \qquad 0 \le t \le T_U. \tag{58}$$

Step 8. We shall use the following notation

$$p_5(U_0) = p\big(\|P(U_0)\|_{H^1} + \|v_0\|_{H^2}\big).$$

In view of (9), differentiate the first equation of (1) in t and consider the duality product with u_t in $H^1(\Omega)^* \times H^1(\Omega)$. After some calculations,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega u_t^2 dx + \int_\Omega P_u|\nabla u_t|^2 \le \alpha_{12}\int_\Omega u|\nabla u_t||\nabla v_t|dx + C\int_\Omega (|\nabla u| + |\nabla v|)\\ \times (|u_t| + |v_t|)|\nabla u_t|dx + C\int_\Omega (u+v+1)(u_t^2 + v_t^2)dx.\end{aligned}$$

Here,

$$\int_\Omega u|\nabla u_t||\nabla v_t|dx \le \|u\|_{L_\infty}\|\nabla u_t\|_{L_2}\|\nabla v_t\|_{L_2} \le \zeta\|\nabla u_t\|_{L_2}^2 + C_\zeta\|u\|_{H^2}^2\|\nabla v_t\|_{L_2}^2$$

with any $\zeta > 0$. In addition, by Lemma 3.4 presented below,

$$\|u\|_{H^2} \le p_5(U_0)(\|\Delta P\|_{L_2} + 1) \le p_5(U_0)(\|u_t\|_{L_2} + 1) \tag{59}$$

and, by a direct calculation,

$$\|\nabla v_t\|_{L_2} \le \|\nabla(\Delta Q + g)\|_{L_2} \le p_5(U_0)(\|v\|_{H^3} + 1).$$

Similarly, by (59),

$$\begin{aligned}&\int_\Omega (|\nabla u| + |\nabla v|)(|u_t| + |v_t|)|\nabla u_t|dx\\ &\quad\le (\|\nabla u\|_{L_4} + \|\nabla v\|_{L_4})(\|u_t\|_{L_4} + \|v_t\|_{L_4})\|\nabla u_t\|_{L_2}\\ &\quad\le (\|u\|_{H^2}^{\frac{1}{2}}\|u\|_{H^1}^{\frac{1}{2}} + \|v\|_{H^2}^{\frac{1}{2}}\|v\|_{H^1}^{\frac{1}{2}})(\|u_t\|_{H^1}^{\frac{1}{2}}\|u_t\|_{L_2}^{\frac{1}{2}} + \|v_t\|_{H^1}^{\frac{1}{2}}\|v_t\|_{L_2}^{\frac{1}{2}})\|\nabla u_t\|_{L_2}\\ &\quad\le p_5(U_0)(\|u_t\|_{L_2}^{\frac{1}{2}} + 1)(\|\nabla u_t\|_{L_2}^{\frac{1}{2}}\|u_t\|_{L_2}^{\frac{1}{2}} + \|u_t\|_{L_2} + \|v\|_{H^3}^{\frac{1}{2}} + 1)\|\nabla u_t\|_{L_2}\\ &\quad\le \zeta\|\nabla u_t\|_{L_2}^2 + C_\zeta p_5(U_0)\|u_t\|_{L_2}^2(\|u_t\|_{L_2}^2 + \|v\|_{H^3}^2 + 1)\end{aligned}$$

with any $\zeta > 0$. Finally,

$$\begin{aligned}\int_\Omega (u+v+1)(u_t^2 + v_t^2)dx &\le \|u+v+1\|_{L_\infty}(\|u_t\|_{L_2}^2 + \|v_t\|_{L_2}^2)\\ &\le p_5(U_0)(\|u_t\|_{L_2}^3 + 1).\end{aligned}$$

Hence it is verified that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_t^2dx+\frac{1}{2}\int_\Omega a|\nabla u_t|^2dx\le p_5(U_0)(\|u_t\|_{L_2}^2+1)(\|P\|_{H^2}^2+\|v\|_{H^3}^2+1).$$

We add the previous differential inequality to (55) after multiplying both sides of (55) by a parameter $\xi>0$. Then we arrive at the main differential inequality of this step

$$\frac{d}{dt}\psi_5(t)+a\xi\psi_5(t)+\int_\Omega a|\nabla u_t|^2dx \le p_5(U_0)\{(\|P(U(t))\|_{H^2}^2+\|v(t)\|_{H^3}^2+1)\psi_5(t)+C_\xi\} \tag{60}$$

for the function $\psi_5(t)=\frac{1}{2}\|u_t(t)\|_{L_2}^2+1+\xi\psi_4(t)$. Solving this, we obtain that

$$\psi_5(t)\le e^{\int_0^t[p_5(U_0)\{\|P(U(s))\|_{H^2}^2+\|v(s)\|_{H^3}^2+1\}-a\xi]ds}\psi_5(0) + C_\xi p_5(U_0)\int_0^t e^{\int_s^t[p_5(U_0)\{\|P(U(\tau))\|_{H^2}^2+\|v(\tau)\|_{H^3}^2+1\}-a\xi]d\tau}ds.$$

Moreover, by the corresponding versions of (51) and (58) obtained by integrating (49) and (55) on (s,t), respectively, we verify that

$$\psi_5(t)\le p_5(U_0)e^{\{p_5(U_0)-a\xi\}t}\psi_5(0)+C_\xi p_5(U_0)\int_0^t e^{\{p_5(U_0)-a\xi\}(t-\tau)}ds.$$

If the parameter ξ is fixed in such a way that $p_5(U_0)-a\xi\le-1$, then

$$\psi_5(t)\le p_5(U_0)(e^{-t}\|u_0\|_{H^2}^2+1),\qquad 0\le t\le T_U.$$

In particular,

$$\|\Delta P(U(t))\|_{L_2}^2\le p_5(U_0)(e^{-t}\|u_0\|_{H^2}^2+1),\qquad 0\le t\le T_U.$$

Hence, by (59), we conclude that

$$\|u(t)\|_{H^2}^2\le p_5(U_0)(e^{-t}\|u_0\|_{H^2}^2+1),\qquad 0\le t\le T_U. \tag{61}$$

Thus, (61) together with (50) gives the desired estimate of proposition. □

Lemma 3.4. *For $0\le u$, $v\in H^2(\Omega)$ with $\frac{\partial u}{\partial n}\equiv\frac{\partial v}{\partial n}\equiv 0$ on $\partial\Omega$, it holds that*

$$\|u\|_{H^2}\le C(\|P(U)\|_{H^1}^2+\|v\|_{H^2}^2+1)(\|\Delta P(U)\|_{L_2}+\|P(U)\|_{L_2}). \tag{62}$$

Proof of Lemma. We know by (54) that $\|u\|_{H^2}\le C(\|\Delta u\|_{L_2}+\|u\|_{L_2})$. By a direct calculation, we see that

$$\Delta u=P_u^{-1}\big[\Delta P-P_{uu}P_u^{-2}|\nabla P|^2+2P_u^{-1}\{P_{uu}P_vP_u^{-1}-P_{uv}\}\nabla P\cdot\nabla v + P_vP_u^{-1}\{2P_{uv}-P_{uu}P_vP_u^{-1}\}|\nabla v|^2\big].$$

The desired estimate is then verified by the similar arguments as in the proof of Lemma 3.1. □

We can of course repeat analogous arguments to verify (29) when $\alpha_{12}=0$. Hence, under $\alpha_{12}\alpha_{21}=0$, it is proved that any local solution to (1) lying in (10) with initial value $U_0\in\mathcal{K}\cap\mathbb{H}^2(\Omega)$ satisfies (29).

As an immediate consequence of the series of estimates built up above, we obtain the dissipative estimate for U at an exponential rate. Let us apply (31) and (35) in

$[0, \frac{t}{7}]$, (39) in $[\frac{t}{7}, \frac{2t}{7}]$, (43) in $[\frac{2t}{7}, \frac{3t}{7}]$, (50) in $[\frac{3t}{7}, \frac{4t}{7}]$, (52) in $[\frac{4t}{7}, \frac{5t}{7}]$, (57) in $[\frac{5t}{7}, \frac{6t}{7}]$ and (61) in $[\frac{6t}{7}, t]$, respectively. Then we verify that

$$\|U(t)\|_{\mathbb{H}^2} \le p\big(e^{-\delta t} p(\|U_0\|_{\mathbb{H}^2}) + 1\big), \qquad 0 \le t \le T_U \tag{63}$$

choosing some continuous increasing functions $p(\cdot)$ and an exponent $\delta > 0$ in a suitable way.

4. Global solutions.

By Proposition 1 and Proposition 2, we can deduce the global existence of solution. Indeed, we know that, for any initial value $U_0 \in \mathcal{K}$, there exists a nonnegative local solution at least on some interval $[0, T_U]$. Let $0 < t_1 < T_U$. Then, $0 \le U_1 = U(t_1) \in \mathbb{H}^2_N(\Omega)$. We next consider the problem of form (1) but with the initial value U_1. Proposition 1 when $0 < \alpha_{12}\alpha_{21} \le 64\alpha_{11}\alpha_{22}$ and Proposition 2 when $\alpha_{12}\alpha_{21} = 0$ ensure that any local solution starting from U_1 stays at any time in a fixed ball $B^{\mathbb{H}^2}(0; p(\|U_1\|_{\mathbb{H}^2}))$ of $\mathbb{H}^2(\Omega)$. This shows that the a priori estimates of form (78) are valid. Therefore, by virtue of Theorem 6.3 in Section 6, Problem (1) possesses a unique global solution.

More precisely, under (4), for any $U_0 \in \mathcal{K}$, there exists a unique nonnegative global solution to (1) in the function space:

$$\begin{aligned} 0 \le U \in \mathcal{C}((0,\infty); \mathbb{H}^2_N(\Omega)) &\cap \mathcal{C}([0,\infty); \mathbb{H}^{1+\varepsilon}(\Omega)) \\ &\cap \mathcal{C}^1((0,\infty); \mathbb{H}^{2\theta}(\Omega)) \cap \mathcal{C}^2((0,\infty); \mathbb{H}^1(\Omega)^*), \qquad 0 \le \theta < 1. \end{aligned} \tag{64}$$

It is as well observed from (28) and (63) that there exist continuous increasing functions $p(\cdot)$ and a positive exponent $\delta > 0$ such that

$$\|U(t)\|_{\mathbb{H}^2} \le p\big(t^{\frac{\varepsilon-1}{2}} e^{-\delta t} p(\|U_0\|_{\mathbb{H}^{1+\varepsilon}}) + 1\big), \qquad 0 < t < \infty;\ U_0 \in \mathcal{K} \tag{65}$$

is valid for every global solution. As well,

$$\|U(t)\|_{\mathbb{H}^{1+\varepsilon}} \le p(\|U_0\|_{\mathbb{H}^{1+\varepsilon}}), \qquad 0 \le t < \infty;\ U_0 \in \mathcal{K}. \tag{66}$$

5. Dynamical system.

5.1. Construction of dynamical system.

Let us construct a dynamical system determined from (1) in the universal space $X = \mathbb{L}_2(\Omega)$. For this purpose we will follow the general strategy described in Subsection 6.2.

Define a semigroup $S(t)$ acting on $\mathcal{K}$ by setting $S(t)U_0 = U(t; U_0)$ for $U_0 \in \mathcal{K}$, where $U(t; U_0)$ denotes the global solution of (1). For any $0 < R < \infty$, let $\mathcal{K}_R = \{U_0 \in \mathcal{K};\ \|U_0\|_{\mathbb{H}^{1+\varepsilon}} < R\}$. In view of (66), we observe that $\cup_{0 \le t < \infty} S(t)\mathcal{K}_R \subset \mathcal{K}_{p(R)}$. Put

$$\mathcal{B}_R = \overline{\bigcup_{0 \le t < \infty} S(t)\mathcal{K}_R} \qquad \text{(closure in the norm } \|\cdot\|_X).$$

Then, as shown in Corollary 2, $\mathcal{B}_R$, where $\mathcal{K}_R \subset \mathcal{B}_R \subset \mathcal{K}_{p(R)}$, is a positively invariant set of $S(t)$ and the mapping $G(t, U_0) = S(t)U_0$ is continuous from $[0,\infty) \times \mathcal{B}_R$ into X with respect to the X-norm.

Therefore, under (4), Problem (1) defines, for each $0 < R < \infty$, a dynamical system $(S(t), \mathcal{B}_R, X)$ in which phase space contains the subset $\mathcal{K}_R$.

5.2. **Compact absorbing set.** On account of (65), for any $0 < R < \infty$, there exists a suitable time $t_R > 0$ such that

$$S(t)\big(\mathcal{B}_R\big) \subset \mathcal{K} \cap \{U \in \mathbb{H}^2(\Omega); \|U\|_{\mathbb{H}^2} \leq p(2)\} \qquad \text{for all } t \geq t_R.$$

This means that the estimate (82) in Section 6 is valid with $\widetilde{C} = p(2)$. We can then apply Theorem 6.4 to construct compact absorbing set.

In fact, the set

$$\mathcal{X}_1 = \mathcal{K} \cap \{U \in \mathbb{H}^2(\Omega);\ \|U\|_{\mathbb{H}^2} \leq \widetilde{C}\}$$

is an absorbing set of $(S(t), \mathcal{B}_R, X)$ in the sense that $S(t)\mathcal{B}_R \subset \mathcal{X}_1$ for every $t \geq t_R$, of course $\mathcal{X}_1$ being determined independently of R. So, if we put

$$\mathcal{X} = \overline{\bigcup_{0 \leq t < \infty} S(t)\mathcal{X}_1} = \overline{\bigcup_{0 \leq t \leq t_{\mathcal{X}_1}} S(t)\mathcal{X}_1} \qquad (\text{closure in the norm } \|\cdot\|_X),$$

then $\mathcal{X}$ is a positively invariant set of $S(t)$. In addition, $\mathcal{X}$ is a subset of $\mathcal{K}$, is a bounded subset of $\mathbb{H}^2(\Omega)$, and is a compact set of X. For the detail, see the proof of Theorem 6.4.

Thus, $(S(t), \mathcal{X}, X)$ defines a dynamical system. Every dynamical system of the form $(S(t), \mathcal{B}_R, X)$ is reduced to it in the sense that $S(t)\mathcal{B}_R \subset \mathcal{X}_1 \subset \mathcal{X}$ for $t \geq t_R$, t_R being a suitable time depending on R.

5.3. **Exponential attractors.** It now suffices to apply Theorem 6.5 to construct exponential attractors for $(S(t), \mathcal{X}, X)$. Thus, under (4), the dynamical system $(S(t), \mathcal{X}, X)$ possesses exponential attractors.

6. Quasilinear Abstract Parabolic Evolution Equations.

6.1. **Local problem.** Consider the Cauchy problem for an abstract evolution equation

$$\begin{cases} \dfrac{dU}{dt} + A(U)U = F(U), & 0 < t < \infty, \\ U(0) = U_0 \end{cases} \tag{67}$$

in a Banach space X. Let Z be a second Banach space which is continuously embedded in X, and let K be an open ball of Z such that

$$K = \{U \in Z;\ \|U\|_Z < R\}, \qquad 0 < R < \infty.$$

For each $U \in K$, $A(U)$ is a densely defined closed linear operator of X with a uniform domain $\mathcal{D}(A(U))$ independent of $U \in K$. The operator F is a nonlinear operator from K into X. And, U_0 is an initial value at least in K.

We make the following structural assumptions, i.e., (68) $\sim$ (71). The spectral set $\sigma(A(U))$ is contained in a fixed open sectorial domain

$$\sigma(A(U)) \subset \Sigma_\phi = \{\lambda \in \mathbb{C};\ |\arg \lambda| < \phi\}, \quad 0 < \phi < \tfrac{\pi}{2},$$

and the resolvent satisfies

$$\|(\lambda - A(U))^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda| + 1}, \quad \lambda \notin \Sigma_\phi, U \in K. \tag{68}$$

The domain $\mathcal{D}(A(U)) \equiv \mathcal{D}$ is independent of $U \in K$, $\mathcal{D}$ being a Banach space with a graph norm $\|\cdot\|_{\mathcal{D}} = \|A(0)\cdot\|_X$. The operator-valued function $A(\cdot)$ satisfies a Lipschitz condition of the form

$$\|A(U)\{A(U)^{-1} - A(V)^{-1}\}\|_{\mathcal{L}(X)} \leq N\|U - V\|_Y, \quad U, V \in K, \tag{69}$$

where Y is a third Banach space such that $Z \subset Y \subset X$ with continuous embedding.

The nonlinear operator F also satisfies a usual Lipschitz condition

$$\|F(U) - F(V)\|_X \le L\|U - V\|_Y, \qquad U, V \in K. \tag{70}$$

There are two exponents $0 \le \alpha < \beta < 1$ such that $\mathcal{D}(A(U)^\alpha) \subset Y$ and $\mathcal{D}(A(U)^\beta) \subset Z$ for every $U \in K$ with the estimates

$$\begin{cases} \|\widetilde{U}\|_Y \le D_1\|A(U)^\alpha \widetilde{U}\|_X, & \widetilde{U} \in \mathcal{D}(A(U)^\alpha),\ U \in K, \\ \|\widetilde{U}\|_Z \le D_2\|A(U)^\beta \widetilde{U}\|_X, & \widetilde{U} \in \mathcal{D}(A(U)^\beta),\ U \in K, \end{cases} \tag{71}$$

$D_i\,(i = 1, 2)$ being some constants independent of $U \in K$.

For the initial value $U_0 \in K$, we assume a compatibility condition of the form

$$U_0 \in \mathcal{D}(A(U_0)^\beta) \quad \text{with the same } \beta \text{ as above.} \tag{72}$$

The following result on local existence is then proved, see [2, Theorem 1].

Theorem 6.1. *Under (68) $\sim$ (71), let $U_0 \in K$ satisfy (72). Then, there exists a unique local solution to (67) in the function space:*

$$\begin{cases} U \in \mathcal{C}^1((0, T_{U_0}];\, X) \cap \mathcal{C}^{\beta-\alpha}([0, T_{U_0}];\, Y) \cap \mathcal{C}([0, T_{U_0}];\, Z), \\ \quad A(U)^\beta U \in \mathcal{C}([0, T_{U_0}];\, X), \quad t^{1-\beta}U \in \mathcal{C}([0, T_{U_0}];\, \mathcal{D}). \end{cases} \tag{73}$$

Here, $T_{U_0} > 0$ is determined by the norm $\|A(U_0)^\beta U_0\|_X$ and the modulus of continuity

$$\omega_{U_0}(t) = \sup_{0 \le s \le t} \|\{e^{-sA(U_0)} - 1\}U_0\|_Z \quad \text{as } t \to 0.$$

For more regular initial values such as $U_0 \in \mathcal{D}(A(U_0)^\gamma)$ with an exponent γ such that $\beta < \gamma \le 1$, we can prove a stronger result, see [2, Corollary 1] or [25, Theorem A.2].

Corollary 1. *Under (68) $\sim$ (71), let $U_0 \in K$ satisfy a stronger compatibility condition*

$$U_0 \in \mathcal{D}(A(U_0)^\gamma), \qquad \beta < \gamma \le 1. \tag{74}$$

Then, the local solution U obtained in Theorem 6.1 satisfies:

$$\begin{cases} U \in \mathcal{C}^1((0, T_{U_0}];\, X) \cap \mathcal{C}^{\gamma-\alpha}([0, T_{U_0}];\, Y) \cap \mathcal{C}^{\gamma-\beta}([0, T_{U_0}];\, Z), \\ \quad A(U)^\gamma U \in \mathcal{C}([0, T_{U_0}];\, X), \quad t^{1-\gamma}U \in \mathcal{C}([0, T_{U_0}];\, \mathcal{D}). \end{cases} \tag{75}$$

Furthermore, $T_{U_0} > 0$ is determined by the norm $\|A(U_0)^\gamma U_0\|_X$ alone.

Let us finally verify Lipschitz continuity of solutions to (67) with respect to the initial values. For this purpose we introduce a set of initial values

$$B = \{U_0 \in Z;\, \|U_0\|_Z \le R_1 \text{ and } \|A(U_0)^\gamma U_0\|_X \le C_1\}, \quad \beta < \gamma \le 1$$

with some constants $0 < R_1 < R$ and $0 < C_1 < \infty$. Then, for each $U_0 \in B$, there exists a unique local solution. Moreover, by Corollary 1, we see that (67) possesses a local solution in the space given by (75) at least over an interval $[0, T_B]$ for every initial value $U_0 \in B$.

We can then show the following theorem, see [2, Theorem 2].

Theorem 6.2. *Under (68) $\sim$ (71), let U and V be the local solutions to (67) with initial values U_0 and V_0 in the set B, respectively. Then, there exists some constant $C_B > 0$ depending on the set B such that*

$$t^\beta \|U(t) - V(t)\|_Z + t^\alpha \|U(t) - V(t)\|_Y + \|U(t) - V(t)\|_X \\ \leq C_B \|U_0 - V_0\|_X, \quad 0 \leq t \leq T_B.$$

6.2. **Global problem.** On the basis of the results presented in the preceding subsection, we shall consider the global problem in a sense that the linear operator $A(U)$ is defined for every $U \in Z$ and try to construct a global solution on the whole interval $[0, \infty)$.

Consider the Cauchy problem for an abstract parabolic evolution equation

$$\begin{cases} \dfrac{dU}{dt} + A(U)U = F(U), \qquad 0 < t < \infty, \\ U(0) = U_0 \end{cases} \tag{76}$$

in a Banach space X. For every $U \in Z$, $A(U)$ is a densely defined closed linear operator of X with a uniform domain $\mathcal{D}(A(U)) \equiv \mathcal{D}$, where $Z \subset X$ is a second Banach space with continuous embedding. The domain $\mathcal{D}$ is a Banach space with a graph norm $\|\cdot\|_{\mathcal{D}} = \|A(0)\cdot\|_X$. The operator F is a nonlinear operator from Z into X. And, U_0 is an initial value at least in Z.

For $0 < R < \infty$, let

$$K_R = \{U \in Z;\ \|U\|_Z < R\}.$$

We assume that, for each $R > 0$, the family of linear operators $A(U)$, $U \in K_R$, and the nonlinear operator $F : K_R \to X$ satisfy all the structural assumptions (68) $\sim$ (71) announced in Subsection 6.1 with a third Banach space Y such that $Z \subset Y \subset X$ and exponents $0 \leq \alpha < \beta < 1$ which are all independent of R. If it is necessary for verifying (68) and (69), however, we may replace $A(U)$ (resp. F) by $A(U) + k_R$ (resp. $F + k_R$) in the equation of (76), where k_R is some sufficiently large constant depending on R. Since

$$F(U) - A(U)U = \{F(U) + k_R U\} - \{A(U) + k_R\}U, \quad U \in \mathcal{D},$$

such replacement does not cause any essential change of equations in (76).

We assume that there is a third exponent γ, where $\beta < \gamma \leq 1$, such that

$$\mathcal{D}_\gamma \equiv \mathcal{D}(A(U)^\gamma), \qquad U \in Z \tag{77}$$

holds. The space $\mathcal{D}_\gamma$ is a Banach space with a graph norm $\|\cdot\|_{\mathcal{D}_\gamma} = \|A(0)^\gamma \cdot\|_X$.

Let B be any bounded set of $\mathcal{D}_\gamma$, and take a semi-diameter R of K_R sufficiently large in such a way that $B \subset K_R$. By Corollary 1, for every $U_0 \in B$, there exists a unique local solution to (76) on a fixed interval $[0, T_B]$, $T_B > 0$ is determined by B. If we can show a priori estimates for all local solutions starting from B, then the global solutions are constructed. In fact, assume that there exist constants R_B and C_B such that the estimates

$$\begin{cases} \|U(t)\|_Z \leq R_B < R, \qquad 0 \leq t \leq T_U, \\ \quad \|U(t)\|_{\mathcal{D}_\gamma} \leq C_B, \qquad 0 \leq t \leq T_U \end{cases} \tag{78}$$

hold for every local solution U on $[0, T_U]$ with $U(0) = U_0 \in B$. Then (76) possesses a global solution on $[0, \infty)$ for any $U_0 \in B$.

Furthermore, if such a priori estimates hold for each bounded subset B of $\mathcal{D}_\gamma$, then we obtain a global existence result for (76). That is, we can state the following theorem.

Theorem 6.3. *Let the structural assumptions (68) $\sim$ (71) be satisfied for each K_R, $0 < R < \infty$. For some exponent $\gamma\,(\beta < \gamma \le 1)$ satisfying (77) let the a priori estimates of the form (78) hold for each bounded subset $B \subset \mathcal{D}_\gamma$. Then, for any $U_0 \in \mathcal{D}_\gamma$ there exists a unique global solution to (78) in the function space:*

$$U \in \mathcal{C}^{\gamma-\alpha}([0,\infty);\, Y) \cap \mathcal{C}^{\gamma-\beta}([0,\infty);\, Z) \cap \mathcal{C}([0,\infty);\, \mathcal{D}_\gamma),\ \ t^{1-\gamma}U \in \mathcal{C}([0,\infty);\, \mathcal{D}).$$

We can then define a semigroup $S(t)$ acting on $\mathcal{D}_\gamma$ by setting $S(t)U_0 = U(t;U_0)$ for $U_0 \in \mathcal{D}_\gamma$, where $U(t;U_0)$ denotes the global solution of (76).

From now we add an assumption that

$$X \text{ is a reflexive Banach space.} \tag{79}$$

Since $A(0)^\gamma$ is an isomorphism from $\mathcal{D}_\gamma = \mathcal{D}(A(0)^\gamma)$ to X, (79) immediately implies that $\mathcal{D}_\gamma$ is also reflexive.

Under the situation that Assumptions of Theorem 6.3 and (79) are all true, consider again any bounded subset B of $\mathcal{D}_\gamma$. We put

$$\mathcal{B} = \overline{\bigcup_{0\le t<\infty} S(t)B} \qquad \text{(closure in the norm } \|\cdot\|_X). \tag{80}$$

The second estimates of (78) jointed with the reflexivity of $\mathcal{D}_\gamma$ implies that $\mathcal{B}$ is also a bounded set of $\mathcal{D}_\gamma$. Then, utilizing Theorem 6.2 repeatedly (in view of (78)), we see that $S(t)$, t being fixed, is a Lipschitz continuous mapping from $\mathcal{B}$ into X with respect to the X-norm. Moreover the Lipschitz constant is uniformly bounded on any finite interval $[0,T]$. Meanwhile, it is already known that $S(t)U_0$ is continuous for $t \in [0,\infty)$ as an X-valued function. Hence, $G(t,U_0) = S(t)U_0$ is a continuous mapping from $[0,\infty) \times \mathcal{B}$ into X with respect to the X-norm. The continuity of $S(t)$ with respect to the X-norm implies that

$$S(t)\overline{\bigcup_{0\le\tau<\infty} S(\tau)B} \subset \overline{S(t)\bigcup_{0\le\tau<\infty} S(\tau)B} \subset \overline{\bigcup_{0\le\tau<\infty} S(\tau)B}.$$

This means that $\mathcal{B}$ is a positively invariant set of $S(t)$.

We are thus led to the following result.

Corollary 2. *Let Assumptions of Theorem 6.3 and (79) be satisfied. For each bounded subset $B \subset \mathcal{D}_\gamma$, define the set $\mathcal{B} \supset B$ by (80). Then, $(S(t), \mathcal{B}, X)$ defines a dynamical system.*

Our end goal is to construct exponential attractors. In general, consider a dynamical system $(S(t), \mathcal{X}, X)$ with phase space $\mathcal{X}$ in a Banach space X. As is well-known, see [29], a subset $\mathcal{M}$ of $\mathcal{X}$ is called the exponential attractor of $(S(t), \mathcal{X}, X)$ if $\mathcal{M}$ satisfies the following conditions:

1. $\mathcal{M}$ is a compact subset of X with finite fractal dimension;
2. $\mathcal{M}$ is a positively invariant set of $S(t)$, i.e., $S(t)\mathcal{M} \subset \mathcal{M}$ for every $t \ge 0$;
3. $\mathcal{M}$ attracts any bounded subset B of $\mathcal{X}$ at an exponential rate in the sense that

$$h(S(t)B, \mathcal{M}) \le C_B e^{-\delta t}, \qquad 0 \le t < \infty$$

with some exponent $\delta > 0$ and a constant $C_B > 0$ depending on B, where

$$h(B_1, B_2) = \sup_{U\in B_1} \inf_{V\in B_2} \|U - V\|_X$$

is the pseudo-distance of two subsets B_1 and B_2.

To this end we will furthermore make two crucial assumptions that

$$Z \text{ is compactly embedded in } X \tag{81}$$

and that there is a constant $\widetilde{C}$ such that, for every bounded set B of $\mathcal{D}_\gamma$, there is a time $t_B > 0$ for which the following estimate holds:

$$\sup_{t\ge t_B} \sup_{U_0\in B} \|S(t)U_0\|_{\mathcal{D}} \le \widetilde{C}. \tag{82}$$

Since $\mathcal{D} = \mathcal{D}(A(0)) \subset \mathcal{D}(A(0)^\beta) \subset Z$, (81) naturally implies that $\mathcal{D}$ is also compactly embedded in X. In the meantime, (82) means that the set

$$\mathcal{X}_1 = \{U \in \mathcal{D};\ \|U\|_{\mathcal{D}} \le \widetilde{C}\}$$

is an absorbing set of $(S(t), \mathcal{B}, X)$ in the sense that $S(t)\mathcal{B} \subset \mathcal{X}_1$ for every $t \ge t_{\mathcal{B}}$.

As before, we put

$$\mathcal{X} = \overline{\bigcup_{0\le t<\infty} S(t)\mathcal{X}_1} = \overline{\bigcup_{0\le t\le t_{\mathcal{X}_1}} S(t)\mathcal{X}_1} \qquad (\text{closure in the } \|\cdot\|_X \text{ norm}).$$

Repeating the same argument, we observe that $\mathcal{X}$ is a positively invariant set of $S(t)$ such that $\mathcal{X}_1 \subset \mathcal{X} \subset \mathcal{D}$, and hence $(S(t), \mathcal{X}, X)$ defines a dynamical system. The phase space $\mathcal{X}$ is a compact set of X, for $\mathcal{X}$ coincides with an image of the compact set $[0, t_{\mathcal{X}_1}] \times \mathcal{X}_1$ by the continuous mapping $G(t, U_0) = S(t)U_0$. The space $\mathcal{X}$ is a bounded subset of $\mathcal{D}$.

We thus arrive at the following result.

Theorem 6.4. *Let Assumptions of Theorem 6.3, (79), (81) and (82) be satisfied. There exists a set $\mathcal{X}$ which is a bounded subset of $\mathcal{D}$ and is a compact set of X for which $(S(t), \mathcal{X}, X)$ defines a dynamical system. In addition, for any bounded set B of $\mathcal{D}_\gamma$, the trajectories starting from B enter in $\mathcal{X}$ in a finite time, i.e., $S(t)B \subset \mathcal{X}$ for every $t \ge t_B$, t_B being determined by B.*

It now suffices to construct exponential attractors for $(S(t), \mathcal{X}, X)$. But we know for a suitable $T > 0$ that Theorem 6.2 provides

$$\|U(t; U_0) - U(t; V_0)\|_Z \le C_{\mathcal{X}} t^{-\beta} \|U_0 - V_0\|_X, \qquad U_0, V_0 \in \mathcal{X};\ 0 < t \le T.$$

Then this means that our semigroup $S(t)$ enjoys the compact smoothing property

$$\|S(T)U_0 - S(T)V_0\|_Z \le L\|U_0 - V_0\|_X, \qquad U_0, V_0 \in \mathcal{X} \tag{83}$$

introduced in [6]. In addition, we can verify that

$$\|S(t)U_0 - S(s)V_0\|_X \le \widetilde{L}\{|t-s| + \|U_0 - V_0\|_X\} \qquad 0 \le s,\ t \le T;\ U_0, V_0 \in \mathcal{X}. \tag{84}$$

For the detail, see [2, Section 5]. According to [6], under (83) and (84), we can construct a family of exponential attractors.

Theorem 6.5. *Let Assumptions of Theorem 6.3, (79), (81) and (82) be satisfied. Then, the dynamical system $(S(t), \mathcal{X}, X)$ possesses exponential attractors $\mathcal{M}$.*

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