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65. *Fractional Powers of Operators and Evolution Equations of Parabolic Type*

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1. Introduction. In 1961 Kato and Sovolevski, one of the creators of the theory, utilized in [2] and [4] respectively the fractional powers of linear operators for studying abstract evolution equations of parabolic type

(E) $\qquad du/dt+A(t)u=f(t), \quad 0<t\le T \quad$ and $\quad u(0)=u_0$

in a Banach space X. Assuming for some $0<h<1$ independence of t of the definition domains $\mathcal{D}(A(t)^h)$ of the fractional powers $A(t)^h$ and the Hölder condition on $A(t)^h A(s)^{-h}$, they succeeded in formulating an intermediate case between the case of constant domains $\mathcal{D}(A(t))$ and that of completely variable domains. In spite of their beautiful abstract results, generally it is hard to verify the Hölder condition of $A(t)^h A(s)^{-h}$ in applications. Even now the problem remains unsolved except some favorable cases like Hilbert space (cf. [2], [4] and also [6]).

This note is also devoted to formulate the intermediate case but by another condition on $A(t)$, in our condition, the Condition (II) below, the fractional powers do not appear in any explicit form. Apparently the condition seems to be unnatural in form, but it is obtained by linking quite directly the two conditions assumed for handling the two extreme cases, that is, the Hölder condition on $A(t)A(s)^{-1}$ in the case of constant domains and the decay condition of $A(t)(\lambda-A(t))^{-1}dA(t)^{-1}/dt$ in λ in the other case of completely variable. As may be mentioned below, verifying the (II) in applications is now an easy problem. We can remark also that Kato and Sobolevski's condition really implies our condition.

The Condition (II) has been introduced by Acquistapace and Terreni [1] to construct under it the evolution operator $U(t,s)$. But their $U(t,s)$ guarantees existence of the solution of (E) only for regular u_0 and f, and their method of proof seems to be quite complicated. Our method uses the fractional powers of $A(t)$ as before to make the proof simpler and clearer.

$A(t)$, $0\le t\le T$, are closed linear operators in a Banach space X (the domains $\mathcal{D}(A(t))$ may not be dense in X). We will make the following assumptions:

(I) The resolvent sets $\rho(A(t))$ of $A(t)$ contain a sector $\sum=\{\lambda\in \boldsymbol{C};\ |\arg\lambda|\ge\theta_0\}$ where $0<\theta_0<\pi/2$, and there

$$\|(\lambda-A(t))^{-1}\|_{\mathcal{L}(X)}\le M/(|\lambda|+1), \quad \lambda\in\sum \qquad \text{with a constant } M\ge 0.$$

(II) For some $0<h,\ k\le 1$

$$\|A(t)(\lambda-A(t))^{-1}\{A(t)^{-1}-A(s)^{-1}\}\|_{\mathcal{L}(X)}\le K|t-s|^k/(|\lambda|+1)^h$$

holds for $0 \leq s,\ t \leq T$ and $\lambda \leq 0$ with a constant $K \geq 0$.

(III) $h+k>1$.

Constructing the evolution operator under these (I), (II), (III) we will prove existence and uniqueness of a strict solution of (E) where $u_0 \in X$ is arbitrary and $f: [0, T] \to X$ is at least continuous function.

Definition. A function $u: [0, T] \to X$ is called strict solution of (E) if i) $u \in L^\infty((0, T); X) \cap \mathcal{C}^1((0, T]; X)$, $Au \in \mathcal{C}((0, T]; X)$; ii) u satisfies the equation of (E) for all $0<t\leq T$; and iii) $A(0)^{-\theta}\{u(t)-u_0\} \to 0$ in X, as $t \to 0$, for every $\theta>0$.

We may remark that the initial condition $u(t) \to u_0$ fails in general in the topology of X since the semigroups $\exp(-\tau A(t))$ (which are analytic from (I)) are no longer of C_0 type.

This result can be proved, in fact, for more general operators $A(t)$. If (III) is suitably strengthened, it is possible to merely assume that $A(t)$ are the generators of infinitely differentiable semigroups. Full proof together with this generalization will be published elsewhere.

2. Evolution operator. Let $A_n(t)$, $0 \leq t \leq T$, be the Yosida approximation of the operator $A(t)$ for $n=1, 2, 3, \cdots$. It is verified, without difficulty, that $A_n(t)$ also satisfy the Assumptions (I) and (II) with the same constants h, k and with some uniform constants $\tilde{M}, \tilde{N}$ substituted for M, N.

$U_n(t, s)$ denotes the evolution operator for $A_n(t)$. Under (I), (II) and (III) we can prove:

Proposition. *For each* $0 \leq s<t \leq T$, $U_n(t, s)$ *and* $U_n(t, s)J_n(s)$ *converge, as* $n \to \infty$, *to a common limit* $U(t, s) \in \mathcal{L}(X)$ *strongly on* X, *where* $J_n(s) = (1+n^{-1}A(s))^{-1}$. *Further,* $\mathcal{R}(U(t, s)) \subset \mathcal{D}(A(t))$ *and* $A_n(t)U_n(t, s)$ *converges to* $A(t)U(t, s)$ *strongly on* X.

Proof. Fix a ρ such that $1-k<\rho<h$, and define

$$V_n(t, s) = U_n(t, s)A_n(s)^{1-\rho} \qquad \text{for } 0 \leq s<t \leq T.$$

Then $V_n(t, s)$ is shown to satisfy the integral equation

$$(1) \qquad V_n(t, s) = A_n(s)^{1-\rho} \exp(-(t-s)A_n(s)) + \int_s^t V_n(t, \tau)A_n(\tau)^\rho \{A_n(\tau)^{-1}-A_n(s)^{-1}\}A_n(s)^{2-\rho} \exp(-(\tau-s)A_n(s))d\tau.$$

After some calculation it is verified that the integral kernels of (1) have weak and integrable singularities at $t=s$ which are uniform in n and that they are strongly convergent on X. Consequently the solution $V_n(t, s)$ is also strongly convergent on X, *a fortiori* $U_n(t, s)$. Let $U(t, s)$, $V(t, s)$ be the limits of $U_n(t, s)$ and $V_n(t, s)$ respectively; obviously, $V(t, s) \supset U(t, s)A(s)^{1-\rho}$. Convergence of $U_n(t, s)J_n(s)$ to $U(t, s)$ results from the fact that

$$A_n(s)^\theta J_n(s) \exp(-\tau A_n(s)) \longrightarrow A(s)^\theta \exp(-\tau A(s))$$

strongly on X, as $n \to \infty$, for every $\theta \geq 0$, $\tau>0$ (see [7, Theorem 2.1]). The second assertion is deduced by a similar argument of another integral equation on $A_n(t)U_n(t, s) - A_n(t) \exp(-(t-s)A_n(t))$ obtained from

$$A_n(t)U_n(t, s) = A_n(t) \exp(-(t-s)A_n(t)) + \int_s^t A_n(t)^2 \exp(-(t-\tau)A_n(t))\{A_n(\tau)^{-1}-A_n(t)^{-1}\}A_n(\tau)U_n(\tau, s)d\tau.$$

3. Existence and uniqueness results. Let (I), (II) and (III) be satisfied, then we prove the following two theorems:

Theorem A. *Let* $u_0 \in X$ *and* $f \in C^{\sigma}([0, T]; X)$, $\sigma > 0$. *Then the function defined by*

$$u(t) = U(t, 0)u_0 + \int_0^t U(t, \tau)f(\tau)d\tau, \qquad 0 < t \leq T \tag{2}$$

is a strict solution of the problem (E).

Proof. We have to verify two things that the u satisfies the equation of (E) for every $0 < t \leq T$, and satisfies the initial condition. But it is very standard to verify the first one, so let us observe here how the initial condition is satisfied. Since

$$\int_0^t U(t, \tau)f(\tau)d\tau \longrightarrow 0,$$

as $t \to 0$, in the topology of X, it suffices to prove that $A(0)^{-\theta}\{U(t, 0) - 1\}u_0 \to 0$ in X for every $\theta > 0$. Insert a quantity $A(0)^{-\theta} \exp(-tA(0))u_0$, then we have: $\| U(t, 0) - \exp(-tA(0)) \|_{\mathcal{L}(X)} \leq Ct^{\rho + k - 1}$ and $\| A(0)^{-\theta} \{\exp(-tA(0)) - 1\} \|_{\mathcal{L}(X)} \leq C_\theta t^\theta$.

Theorem B. *Conversely let* u *be any strict solution of* (E) *with an initial value* $u_0 \in X$ *and a function* $f \in C([0, T]; X)$, *then necessarily* u *must be of the form* (2).

Proof. From the proposition we can verify that for any $\varepsilon > 0$

$$u(t) = U(t, \varepsilon)u(\varepsilon) + \int_\varepsilon^t U(t, \tau)f(\tau)d\tau, \qquad \varepsilon \leq t \leq T.$$

So the proof is to show that $U(t, \varepsilon)u(\varepsilon) \to U(t, 0)u_0$ in X, as $\varepsilon \to 0$. Write $V(t, \varepsilon)A(\varepsilon)^{\rho - 1}u(\varepsilon)$; then $V(t, \varepsilon) \to V(t, 0)$ strongly on X; on the other hand, $A(\varepsilon)^{\rho-1} - A(0)^{\rho - 1} \to 0$ in $\mathcal{L}(X)$ from (II) (since $h > \rho > 0$), and $A(0)^{\rho-1}\{u(\varepsilon) - u_0\} \to 0$ in X by definition.

4. Comment on applications. Consider, as an example, a second order partial differential operator $\sum_{|\alpha| \leq 2} a_\alpha(t, x)D^\alpha$ in $\Omega \subset \bar{\boldsymbol{R}}^n$ and a boundary operator $\sum_{|\beta| \leq 1} b_\beta(t, x)D^\beta$ on $\partial\Omega$, the coefficients being Hölder continuous in t. In applications $A(t)$ denotes realization of the operator $\sum_{|\alpha| \leq 2} a_\alpha(t, x)D^\alpha$ under the boundary condition: $\sum_{|\beta| \leq 1} b_\beta(t, x)D^\beta u = 0$, X is an underlying Banach space like $L^1(\Omega)$, $L^p(\Omega)$ $(1 < p < \infty)$, $C(\bar{\Omega})$. When $X = L^p(\Omega)$ $(1 < p < \infty)$, the Condition (II) is valid for any $h < (p+1)/2p$; this is verified from the fact that $\mathcal{D}(A(t)^h) = H_p^{2h}(\Omega)$ for (and only for) $h < (p+1)/2p$. For $h < 1/2$, however, it can be verified by another quite immediate way devised by Watanabe [5]. When $X = C(\bar{\Omega})$, we notice that for $h < 1/2$, (II) is true in every $L^p(\Omega)$, and really verify from this the validity of (II) for any $h < 1/2$; see [8]. On the contrary, the problem becomes much more delicate in $L^1(\Omega)$, because there is no a priori estimate of elliptic operators available in $L^1(\Omega)$. Owing to the theory of the integral kernels of semigroups by Tanabe [9], we can anyway verify (II) for small h; indeed, for $h < 1/2$ by Park [3].

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