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Continuous dependence on a parameter of exponential attractors for chemotaxis-growth system

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Abstract. We study dependence on a parameter of exponential attractors. As known, exponetial attractors are not uniquely determined from a dissipative dynamical system even if they exist. But we prove in this paper that one can construct an exponential attractor which depends continuously on a parameter in the dynamical system. This result is then applied to the chemotaxis-growth system.

1. Introduction.

The study of the long time behavior of systems arising from physics, mechanics and biology is a capital issue, as it is important, for practical purposes, to understand and predict the asymptotic behavior of the system.

For many parabolic and weakly damped wave equations, one can prove the existence of the finite dimensional (in the sense of the Hausdorff or the fractal dimension) global attractor, which is a compact invariant set which attracts uniformly the bounded sets of the phase space (see [**24**] and [**30**]). Since it is the smallest set enjoying these properties, it is a suitable set.

Now, the global attractor may present two major defaults for practical purposes. Indeed, the rate of attraction of the trajectories may be small and (consequently) it may be sensible to perturbations.

In order to overcome these difficulties, Foias, Sell and Temam proposed in [**7**] the notion of inertial manifold, which is a smooth finite dimensional hyperbolic (and thus robust) positively invariant manifold which contains the global attractor and attracts exponentially the trajectories. Unfortunately, all the known constructions of inertial manifolds are based on a restrictive condition, the so-called spectral gap condition. Consequently, the existence of inertial manifolds is not known for many physically important equations (e.g. Navier-Stokes equations, even in two space dimensions). A non-existence result has even been obtained by Mallet-Paret and Sell for a reaction-diffusion equation in higher space dimensions.

Thus, as an intermediate object between the two ideal objects that the global attractor and an inertial manifold are, Eden, Foias, Nicolaenko and Temam proposed in [**26**] the notion of exponential attractor, which is a compact positively invariant set which contains the global attractor, has finite fractal dimension and attracts exponentially the trajectories. So, compared with the global attractor, an exponential attractor is more robust under perturbations and numerical approximations (see [**26**] for discussions on this subject). Another motivation for the study of exponential attractors comes from the fact that the global attractor may be trivial (say, reduced

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to one point) and may thus fail to capture important transient behaviors. We note however that, contrarily to the global attractor, an exponential attractor is not necessarily unique, so that actual/concrete choice of an exponential attractor is in a sense artificial.

Exponential attractors have been constructed for a large class of equations (see [**4**], [**26**], [**13**], [**14**], [**15**], [**19**], and the references therein). The known constructions of exponential attractors (see for instance [**4**] and [**26**]) make an essential use of orthogonal projectors with finite rank (in order to prove the so-called squeezing property) and are thus valid in Hilbert spaces only. Recently, Efendiev, Miranville and Zelik gave in [**6**] a construction of exponential attractors that is no longer based on the squeezing property and that is thus valid in a Banach setting. So, exponential attractors are as general as global attractors.

Let us come back to the robustness of the global attractor. Generally, global attractors are only upper semicontinuous with respect to perturbations. The lower semicontinuity property is much more delicate and can be established only for some particular cases (see for instance [**28**], [**24**] and [**29**]); for instance, it is true when the semigroup possesses a global Lyapunov function and all the equilibria are hyperbolic. In this particular case, the corresponding global attractor (the so-called regular attractor) is exponential and is robust under perturbations (i.e. it is upper and lower semicontinuous with respect to perturbations). Moreover, if $\mathscr{A}_\xi$ is the regular attractor of a perturbed system and $\mathscr{A}_0$ corresponds to the unperturbed one, then under natural assumptions on the perturbations, we have

$$d(\mathscr{A}_\xi, \mathscr{A}_0) \le C\xi^{\kappa},$$

where $d(\cdot,\cdot)$ denotes the symmetric distance between two sets (defined by (1.1)), $\kappa \in (0,1)$ is some exponent and ξ is the perturbation parameter (see [**24**]). As already mentioned, exponential attractors are more robust objects. In particular, one can prove the continuity of exponential attractors under perturbations (see [**26**]), for the continuity for classical Galerkin approximations (see [**2**]), even when this property is violated or is not known for the global attractor. However, in so far known papers, the continuity is obtained only up to a time shift. In the present paper, we give conditions on the semigroup which provide the continuity of exponential attractors without such time shifts. Moreover, we obtain analogous (to the case of regular attractors) estimates for symmetric distance between the perturbed exponential attractor $\mathscr{M}_\xi$ and the unperturbed $\mathscr{M}_0$:

$$d(\mathscr{M}_\xi, \mathscr{M}_0) \le C'\xi^{\kappa'},$$

without assuming that the system under consideration possesses a global Lyapunov functional and that all the equilibria be hyperbolic. Note that, in contrast to the case of regular attractors, our approach allows to compute the constants C' and κ' in terms of the corresponding physical parameters in specific applications.

In the second half of this paper, we apply our general result to the chemotaxis-growth model and show continuous dependence of an exponential on the parameter of chemotaxis. In 1991, E. O. Budrene and H. C. Berg found out in [**25**] that Escherichia coli form remarkable aggregating patterns by chemotaxis and growth. After this epoch-making result, mathematical biologists tried to describe the process of pattern formation by mathematical models, see [**16**], [**18**] and [**23**]. Among others, Mimura and Tsujikawa presented in [**18**] a very simple model which is based only on diffusion, chemotaxis and growth. We are here interested in their model. As a matter of fact it is already known that one can construct exponential attractors for Mimura-Tsujikawa model by the papers Osaki, Tsujikawa, Yagi and Mimura [**22**] and Aida, Efendiev and Yagi

[1]. Our result then shows that an exponential attractor varies continuously with the chemotactic parameter in the model equation.

According to some numerical computations (see [3] and [18]), it is also known that the chemotaxis-growth model contains various types of pattern solutions which depend dramatically on the chemotactic parameter and that the pattern formations are often performed for a long term without any periodicity. So it is natural that one corresponds the pattern solutions to some trajectories in (at least in neighborhoods of) exponential attractors rather than the global attractor. Our result then shows that in the view point of exponential attractors, the structure of pattern solutions changes continuously with a change of the chemotactic parameter even though its change may be very dramatic.

NOTATIONS. Let X be a Banach space with norm $\|\cdot\|_X$ and $\mathscr{X}$ be a subset of X. $\mathscr{X}$ is a metric space with the induced distance $d(U,V) = \|U-V\|_X$ $(U,V \in \mathscr{X})$. For $U \in \mathscr{X}$ and a set $B \subset \mathscr{X}$, $d(U,B)$ is defined by $d(U,B) = \inf_{V\in B} d(U,V)$. For two sets $B_1, B_2 \subset \mathscr{X}$, their distance $d(B_1,B_2)$ is defined by

$$d(B_1,B_2) = \max\{h(B_1,B_2), h(B_2,B_1)\}, \tag{1.1}$$

where $h(B_1,B_2)$ denotes the Hausdorff pseudodistance given by

$$h(B_1,B_2) = \sup_{U\in B_1} d(U,B_2) = \sup_{U\in B_1} \inf_{V\in B_2} d(U,V).$$

Let X be a Banach space and let I be an interval. $\mathscr{C}(I;X)$ and $\mathscr{C}^1(I;X)$ denote the space of X-valued continuous functions and continuously differentiable functions equipped with the usual function norms, respectively.

2. Discrete dynamical systems.

Let X be a Banach space with norm $\|\cdot\|_X$ and let B be a compact subset of X, B being a metric space equipped with the distance $d(U,V) = \|U-V\|_X$ for $U, V \in B$.

Let S_ξ, $0 \le \xi \le 1$, be a family of continuous mappings from B into itself. We then consider a family of discrete dynamical systems (S_ξ^n, B, X), $0 \le \xi \le 1$, with uniform phase space B in the universal space X.

We assume that there exists a second Banach space Z with norm $\|\cdot\|_Z$ such that Z is compactly embedded in X and that all S_ξ, $0 \le \xi \le 1$, satisfy a Lipschitz condition of the form

$$\|S_\xi U - S_\xi V\|_Z \le L\|U-V\|_X, \qquad U, V \in B \tag{2.1}$$

with some uniform constant $L > 0$. We assume also that S_ξ converges to S_0 as $\xi \to 0$ with the rate

$$\sup_{U\in B} \|S_\xi U - S_0 U\|_X \le K\xi, \quad 0 < \xi \le 1, \tag{2.2}$$

$K \ge 1$ being some constant.

Then we obtain the following result.

THEOREM 2.1. *Under* (2.1) *and* (2.2), *there exist exponential attractors* $\mathscr{M}_\xi^*$ *for the dynamical systems* (S_ξ^n, B, X), $0 \le \xi \le 1$, *respectively for which the estimate*

$$d(\mathscr{M}_\xi^*, \mathscr{M}_0^*) \le C\xi^\kappa, \qquad 0 < \xi \le 1 \tag{2.3}$$

holds with some exponent $0 < \kappa < 1$ *and some constant* C.

PROOF. We remember the method which has been employed in [**6**] for constructing an exponential attractor for the discrete dynamical system (S_0^n, B, X).

Let R be the diameter of the compact set B. Let us define for $n = 0, 1, 2, \ldots$, a $R/2^n$-covering of $S_0^n B$ by a finite number of closed balls of X with centers belonging to $S_0^n B$. More precisely,

$$S_0^n B \subset \bigcup_{i=1}^{P^n} \overline{B}^X(x_{n,i}; R/2^n), \qquad x_{n,i} \in S_0^n B, \tag{2.4}$$

where P is the minimal number of balls of X with radii $1/(4L)$ which cover the unit closed ball $\overline{B}^Z(0;1)$ of Z centered in the zero. In fact, when $n = 0$, it suffices to fix $x_0 \in B = (S_0)^0 B$ arbitrarily.

Assume that the covering (2.4) is defined for n. For each $i \in \{1, \ldots, P^n\}$, the Lipschitz condition (2.1) implies $S_0 \overline{B}^X(x_{n,i}; R/2^n) \subset \overline{B}^Z(S_0 x_{n,i}; LR/2^n)$. By scaling it is deduced that this ball of Z can be covered by the number P of $R/(4 \cdot 2^n)$-balls of X. In this way we see that $S_0^{n+1} B$ is covered by the number P^{n+1} of $R/2^{n+2}$-balls of X. Moreover, increasing the radii of balls twice if necessary we may construct the $R/2^{n+1}$-covering with centers belonging to $S_0^{n+1} B$. Thus, we have constructed the desired covering (2.4) for $n+1$.

Let us now define a sequence of sets E_0^n by $E_0^0 = \{x_0\}$ and

$$E_0^{n+1} = (S_0 E_0^n) \bigcup \{x_{n+1,i};\, 1 \le i \le P^{n+1}\}.$$

It is easily observed that these sets enjoy the following properties: 1. $E_0^n \subset S_0^n B$; 2. $S_0 E_0^n \subset E_0^{n+1}$; 3. $\# E_0^n \le P^{n+1}$; and 4. $d(S_0^n B, E_0^n) \le R/2^n$. Furthermore, it is easily deduced that the set

$$\mathscr{M}_0^* = \overline{\bigcup_{n=0}^{\infty} E_0^n} \qquad \text{(closure in the topology of } X\text{)}$$

is an exponential attractor of (S_0^n, B, X). For the detailed arguments, see [**6**].

Our next goal is then to construct exponential attractors $\mathscr{M}_\xi^*$ for (S_ξ^n, B, X), $0 < \xi \le 1$, respectively in such a way that (2.3) will be satisfied. To this end we will essentially use the sets E_0^n.

Since $E_0^n \subset S_0^n B$, there exist sets $\widetilde{E}_0^n \subset B$ so that $\# \widetilde{E}_0^n = \# E_0^n$ and $S_0^n \widetilde{E}_0^n = E_0^n$. Fix a ξ so that $0 < \xi \le 1$. For $n = 0, 1, 2, \ldots$, let us set $E_\xi^n = S_\xi^n \widetilde{E}_0^n$. Here we notice the following lemma.

LEMMA 2.1. *The condition* (2.2) *implies*

$$\sup_{U \in B} \|S_\xi^n U - S_0^n U\|_X \le \widetilde{K}^n \xi, \qquad 0 < \xi \le 1$$

for every $n = 0, 1, 2, \ldots$ *with some constant* $\widetilde{K}$ *determined by* c, L *and* K *alone.*

PROOF OF LEMMA. We have

$$S_\xi^n U - S_0^n U = \sum_{i=0}^{n-1} (S_\xi^{n-i} S_0^i U - S_\xi^{n-i-1} S_0^{i+1} U).$$

Therefore, by $\|S_\xi U - S_\xi V\|_X \le c\|S_\xi U - S_\xi V\|_Z \le cL\|U-V\|_X$ for $U,V \in B$,

$$\begin{aligned}\|S_\xi^n U - S_0^n U\|_X &\le \sum_{i=0}^{n-1}(cL)^{n-i-1}\|S_\xi S_0^i U - S_0 S_0^i U\|_X \\ &\le \sum_{i=0}^{n-1}(cL)^{n-i-1}K\xi = \{(cL)^n - 1\}(cL-1)^{-1}K\xi.\end{aligned}$$

□

From this lemma we have

$$d(S_\xi^n B, S_0^n B) \le \widetilde{K}^n \xi \quad \text{and} \quad d(S_\xi^n \widetilde{E}_0^n, S_0^n \widetilde{E}_0^n) \le \widetilde{K}^n \xi,$$

and consequently

$$d(S_\xi^n B, E_\xi^n) \le d(S_\xi^n B, S_0^n B) + d(S_0^n B, E_0^n) + d(S_0^n \widetilde{E}_0^n, S_\xi^n \widetilde{E}_0^n) \le 2\widetilde{K}^n \xi + R/2^n. \tag{2.5}$$

Note that usually $\widetilde{K} > 1$, therefore the right-hand side of (2.5) tends to infinity as $n \to \infty$ and consequently we cannot construct the exponential attractor $\mathscr{M}_\xi^*$ using *only* the sets E_ξ^n. But for n's which are not so large, the estimate (2.5) gives us a reasonable covering of the set $S_\xi^n B$. Indeed, let

$$n \le N(\xi) := \left[\frac{\ln R/(2\xi)}{\ln 2\widetilde{K}}\right],$$

then $2\xi\widetilde{K}^n \le R/2^n$; therefore it follows from (2.5) that

$$d(S_\xi^n B, E_\xi^n) \le R/2^{n-1}.$$

Moreover, for such n's, it is deduced that

$$d(E_\xi^n, E_0^n) = d(S_\xi^n \widetilde{E}_0^n, S_0^n \widetilde{E}_0^n) \le \widetilde{K}^n \xi \le \widetilde{K}^{N(\xi)} \xi \le C_1 \xi^\kappa, \quad \text{where } \kappa := \frac{\ln 2}{\ln 2 + \ln \widetilde{K}} \tag{2.6}$$

with the constant $C_1 = \widetilde{K}^{(\ln R/2)/(\ln 2\widetilde{K})}$.

We here redefine the sequence of sets F_ξ^n by the following rule: 1. $F_\xi^n = E_\xi^n$ for $0 \le n \le N(\xi)$; 2. for $n > N(\xi)$, we forget the sets E_ξ^n and construct the sets F_ξ^n by the inductive procedure using the condition (2.1) (in the same way as we have constructed the sets E_ξ^n but starting with the initial covering, generated by $E_\xi^{N(\xi)}$). Then the sets thus constructed evidently satisfy the following conditions: 1. $F_\xi^n \subset S_\xi^n B$; 2. $S_\xi F_\xi^n \subset F_\xi^{n+1}$; 3. $\# F_\xi^n \le P^{n+2}$; 4. $d(S_\xi^n B, F_\xi^n) \le R/2^{n-1}$. Then, these conditions imply as before that the set

$$\mathscr{M}_\xi^* = \overline{\bigcup_{n=0}^{\infty} F_\xi^n} \qquad \text{(closure in the topology of } X) \tag{2.7}$$

is an exponential attractor for (S_ξ^n, B, X). Moreover it is seen that the attraction property for $\mathscr{M}_\xi$ is uniform with respect to $\xi \in [0,1]$, i.e.

$$d(S_\xi^n B, \mathscr{M}_\xi^*) \le R/2^{n-1}.$$

Let us finally verify the convergence property (2.3). Indeed, for the first, let $x \in \bigcup_{n=0}^{\infty} F_\xi^n$; then, $x \in F_\xi^n$ for a certain n. If $n \le N(\xi)$, then $x \in E_\xi^n$ and $d(x, \mathscr{M}_0^*) \le d(x, E_0^n) \le C_1 \xi^\kappa$ due to (2.6). If $n > N(\xi)$, then, since $x \in S_\xi^n B \subset S_\xi^{N(\xi)} B$, there exists an element $y \in B$ such that $x = S_\xi^{N(\xi)} y$. Let $x' = S_0^{N(\xi)} y$. Then, by Lemma 2.1, $\|x - x'\|_X \le \widetilde{K}^{N(\xi)} \xi$; therefore, by the same calculation as for (2.6), we observe that $\|x - x'\|_X \le C_1 \xi^\kappa$. From the other side, it is already known that

$$d(x', \mathscr{M}_0^*) \le d(S_0^{N(\xi)} y, E_0^{N(\xi)}) \le R/2^{N(\xi)} \le R 2^{1 - (\ln R/(2\xi))/(\ln 2\widetilde{K})} = C_2 \xi^\kappa$$

with the constant $C_2 = 2^{1 - (\ln R/2)/(\ln 2\widetilde{K})} R$. Therefore, $d(x, \mathscr{M}_0^*) \le (C_1 + C_2)\xi^\kappa$. Since x is arbitrary in $\bigcup_{n=0}^{\infty} F_\xi^n$, it follows that $h(\bigcup_{n=0}^{\infty} F_\xi^n, \mathscr{M}_0^*) \le (C_1 + C_2)\xi^\kappa$. In view of (2.7),

$$h(\mathscr{M}_\xi^*, \mathscr{M}_0^*) \le (C_1 + C_2)\xi^\kappa.$$

The opposite estimate

$$h(\mathscr{M}_0^*, \mathscr{M}_\xi^*) \le C\xi^\kappa$$

can also be verified in a completely analogous way. Thus the theorem has been proved. □

3. Continuous dynamical systems.

Let X be a Banach space and let $\mathscr{X}$ be a subset of X. Let $S(t)$, $0 \le t < \infty$, be a family of continuous mappings from $\mathscr{X}$ into itself with the properties: i) $S(0) = 1$ (the identity mapping) and ii) $S(t)S(s) = S(t+s)$, $0 \le t, s < \infty$ (the semigroup property). Such a family is called a (nonlinear) semigroup acting on $\mathscr{X}$. For each $U_0 \in \mathscr{X}$, $S(\cdot)U_0$ defines a function for $t \in [0, \infty)$ with values in $\mathscr{X}$; this function is called a trajectory starting from U_0. The space of all trajectories is called a dynamical system with phase space $\mathscr{X}$ in the universal space X and is denoted by $(S(t), \mathscr{X}, X)$.

In this section we consider a family of dynamical systems $(S_\xi(t), \mathscr{X}_\xi, X)$ which are defined for $0 \le \xi \le 1$ with compact phase spaces $\mathscr{X}_\xi$ of X for all $0 \le \xi \le 1$.

Assume that there exists a second Banach space Z which is compactly embedded in X and that, for some $t^* > 0$, all the mappings $S_\xi(t^*)$, $0 \le \xi \le 1$, satisfy a compact Lipschitz condition

$$\|S_\xi(t^*)U - S_\xi(t^*)V\|_Z \le L\|U - V\|_X, \quad U, V \in \mathscr{X}_\xi \tag{3.1}$$

with some uniform constant $L > 0$. And assume also that all $S_\xi(t)$, $0 \le \xi \le 1$, satisfy a Lipschitz condition

$$\|S_\xi(t)U - S_\xi(s)V\|_X \le D(|t - s| + \|U - V\|_X), \quad 0 \le s, t \le t^*,\ U, V \in \mathscr{X}_\xi \tag{3.2}$$

on the interval $[0, t^*]$ with some uniform constant $D > 0$.

For the dynamical systems we assume that there exists a uniform absorbing set $B \subset \cap_{0 \le \xi \le 1} \mathscr{X}_\xi$ which satisfies

$$S_\xi(t)\mathscr{X}_\xi \subset B \qquad \text{for every} \quad t \ge t^* \tag{3.3}$$

for all $0 \le \xi \le 1$, and that

$$\sup_{U\in B}\sup_{0\le t\le t^*}\|S_\xi(t)U-S_0(t)U\|_X\le K\xi \tag{3.4}$$

with some constant $K\ge 1$ for all $0\le\xi\le 1$. It is easy to see that, if the conditions (3.3) and (3.4) are satisfied by a set B, then they are satisfied by $\overline{B}$ also. So we can assume that B is a closed set of X without loss of generality.

Then we prove the following convergence result.

THEOREM 3.1. *Under* (3.1), (3.2), (3.3), *and* (3.4)*, there exist exponential attractors* $\mathscr{M}_\xi$ *for* $(S_\xi(t),\mathscr{X}_\xi,X)$, $0\le\xi\le 1$, *respectively for which the estimate*

$$d(\mathscr{M}_\xi,\mathscr{M}_0)\le C\xi^\kappa,\qquad 0<\xi\le 1$$

holds with some exponent $0<\kappa<1$ *and constant* $C>0$.

PROOF. For $0\le\xi\le 1$, let $S^*_\xi=S_\xi(t^*)$. Then, since (3.3) implies $S^*_\xi B\subset B$, $((S^*_\xi)^n,B,X)$ are discrete dynamical systems with the uniform phase space B which is a compact set of X. Therefore we can apply Theorem 2.1 in the preceding section to conclude that there exist exponential attractors $\mathscr{M}^*_\xi$ for the systems $((S^*_\xi)^n,B,X)$ respectively which satisfy the estimate

$$d(\mathscr{M}^*_\xi,\mathscr{M}^*_0)\le C_1\xi^\kappa \tag{3.5}$$

with some exponent $0<\kappa<1$ and constant $C_1>0$.

We now set for each $0\le\xi\le 1$, $\mathscr{M}_\xi=\cup_{0\le t\le t^*}S(t)\mathscr{M}^*_\xi$. According to [**26**, Theorem 3.1], these $\mathscr{M}_\xi$ are then exponential attractors for the continuous systems $(S_\xi(t),\mathscr{X}_\xi,X)$ respectively.

Let $U_\xi\in\mathscr{M}_\xi$ be any element. Then, $U_\xi=S_\xi(t)U^*_\xi$ with some $0\le t\le t^*$ and some $U^*_\xi\in\mathscr{M}^*_\xi$. From (3.5), there exists an element $U^*_0\in\mathscr{M}^*_0$ such that $d(U^*_\xi,U^*_0)\le 2C_1\xi^\kappa$. Set $U_0=S_0(t)U^*_0\in\mathscr{M}_0$. Then, by (3.2) and (3.4),

$$\begin{aligned}d(U_\xi,\mathscr{M}_0)\le d(U_\xi,U_0)=d(S_\xi(t)U^*_\xi,S_0(t)U^*_0)&\le d(S_\xi(t)U^*_\xi,S_0(t)U^*_\xi)\\&\quad+d(S_0(t)U^*_\xi,S_0(t)U^*_0)\le K\xi+2C_1D\xi^\kappa.\end{aligned}$$

Thus we obtain that

$$h(\mathscr{M}_\xi,\mathscr{M}_0)\le(K+2C_1D)\xi^\kappa.$$

By the same argument we can obtain also $h(\mathscr{M}_0,\mathscr{M}_\xi)\le(K+2C_1D)\xi^\kappa$. Hence the theorem has been proved. $\square$

4. Application to chemotaxis-growth model.

We shall apply our abstract results obtained in the preceding section to the chemotaxis-growth model due to Mimura and Tsujikawa [**18**].

In this section we use the following notations. Ω is a bounded domain in the plane. For $1\le p\le\infty$, $L^p(\Omega)$ is the Banach space of all L^p functions with norm $\|\cdot\|_{L^p}$. For $m=1,2,3,\dots$, $H^m(\Omega)$ denotes the Sobolev space, its norm being denoted by $\|\cdot\|_{H^m}$ (see [**27**, Chapter 1] and [**31**]). For $m\ge 2$, $H^m_N(\Omega)$ is a closed subspace of $H^m(\Omega)$ which consists of the function $u\in H^m(\Omega)$ satisfying the Neumann boundary conditions $\partial u/\partial n=0$ on the boundary of Ω.

4.1. Chemotaxis-growth system.

We consider the Cauchy problem for a nonlinear diffusion system

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\Delta u - \nabla \cdot \{u\nabla\chi(\rho)\} + f(u) & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial \rho}{\partial t} = b\Delta\rho - c\rho + du & \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \rho}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \;\; \rho(x,0) = \rho_0(x) & \text{in} \quad \Omega \end{cases} \tag{4.1}$$

in a bounded domain $\Omega \subset \boldsymbol{R}^2$ with $\mathscr{C}^3$ boundary. Here, $u(x,t)$ and $\rho(x,t)$ denote the population density of biological individuals and the concentration of chemical substance at a position $x \in \Omega$ and time $t \in [0,\infty)$, respectively. The chemotactic term $\nabla \cdot \{u\nabla\chi(u)\}$ shows that u flows under the influence of the chemical substance by a sensitivity function $\chi(\rho)$. The growth rate of u is given by a growth function $f(u)$. $a > 0$ and $b > 0$ are the diffusion rates of u and ρ respectively. $c > 0$ and $d > 0$ are the degradation and production rates of ρ respectively.

The sensitivity function $\chi(\rho)$ is a real smooth function of $\rho \in [0,\infty)$ which is assumed to satisfy the condition

$$\sup_{0\le\rho<\infty} \left| \frac{d^i\chi}{d\rho^i}(\rho) \right| \le D \qquad \text{for } \; i = 1,2,3 \tag{4.2}$$

with some constant $D > 0$. Prototypes of $\chi(\rho)$ are ρ, $\log(\rho+1)$, $\rho/(\rho+1)$ and so on.

The growth function $f(u)$ is a real smooth function of $u \in [0,\infty)$ with $f(0) = 0$ which is assumed to satisfy the condition

$$f(u) = (-\mu u + \nu)u \qquad \text{for sufficiently large } \; u \tag{4.3}$$

with some $\mu > 0$ and $-\infty < \nu < \infty$.

For simplicity, we shall use a universal notation C to denote constants which are determined in each occurrence by the initial constants a, b, c, d, D, μ and ν, and by the domain Ω.

As verified by [**22**, Theorem 4.4], for any pair of initial functions $0 \le u_0 \in H^2_N(\Omega)$ and $0 \le \rho_0 \in H^3_N(\Omega)$, the problem (4.1) possesses a unique global solution in the function space

$$\begin{cases} 0 \le u \in \mathscr{C}([0,\infty); H^2_N(\Omega)) \cap \mathscr{C}^1([0,\infty); L^2(\Omega)), \\ 0 \le \rho \in \mathscr{C}([0,\infty); H^3_N(\Omega)) \cap \mathscr{C}^1([0,\infty); H^1(\Omega)). \end{cases} \tag{4.4}$$

Moreover by [**22**, Proposition 4.1], the solution satisfies the estimate

$$\|u(t)\|_{H^2} + \|\rho(t)\|_{H^3} \le p(\|u_0\|_{H^2} + \|\rho_0\|_{H^3}), \qquad 0 \le t < \infty \tag{4.5}$$

with some continuous increasing function $p(\cdot)$ which is determined by the initial constants a, b, c, d, D, μ and ν, and by the domain Ω.

In order to have this global existence of solutions, the condition (4.3) on $f(u)$ plays an important role. In the case where $f(u) \equiv 0$, the model (4.1) is called the Keller-Segel equations which were presented by Keller and Segel in [**17**] to describe the aggregation process of slime mold. In Keller-Segel equations the blowups of solutions can take place as verified by Gajewski, Jäger and Koshelev [**8**], Gajewski and Zacharias [**9**], [**10**], Herrero and Velázquez [**11**], Nagai and Senba [**20**], Nagai, Senba and Suzuki [**21**], and so on. For a full list of Keller-Segel equations, see [**12**].

4.2. Dynamical system.

We set a universal space X by

$$X = \left\{ \begin{pmatrix} u \\ \rho \end{pmatrix};\ u \in L^2(\Omega) \text{ and } \rho \in H^1(\Omega) \right\}.$$

We set also a space of initial functions K by

$$K = \left\{ \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix};\ 0 \le u_0 \in H^2_N(\Omega) \text{ and } 0 \le \rho_0 \in H^3_N(\Omega) \right\}.$$

Since (4.1) possesses a unique global solution in the space (4.4) and since the solution is continuous with respect to the initial functions in the topology of X, we obtain a dynamical system $(S(t), K, X)$ which is determined from (4.1) with phase space K.

According to [**22**, Proposition 5.1], there exists a constant $R > 0$ which is determined by the initial constants a, b, c, d, D, μ, and ν and by the domain Ω, and the following statement is true. The set

$$B = \left\{ \begin{pmatrix} u \\ \rho \end{pmatrix};\ u \in H^2_N(\Omega) \text{ and } \rho \in H^3_N(\Omega) \text{ with } \|u\|_{H^2} + \|\rho\|_{H^3} \le R \right\} \tag{4.6}$$

is an absorbing set. That is, for any $0 < r < \infty$, there exists a time $t_r > 0$ such that the set

$$K_r = \left\{ \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix};\ 0 \le u_0 \in H^2_N(\Omega) \text{ and } 0 \le \rho_0 \in H^3_N(\Omega) \text{ with } \|u_0\|_{H^2} + \|\rho_0\|_{H^3} \le r \right\},$$

is absorbed by B in the sense that

$$S(t)K_r \subset B \text{ for every } t \ge t_r. \tag{4.7}$$

As B itself is absorbed by B, $S(t)B \subset B$ for every $t \ge t_R$.

We then set a phase space $\mathscr{X}$ by

$$\mathscr{X} = \bigcup_{0 \le t < \infty} S(t)B = \bigcup_{0 \le t \le t_R} S(t)B. \tag{4.8}$$

Then $\mathscr{X}$ is such that $\mathscr{X} \supset B$, is a positively invariant set, i.e. $S(t)\mathscr{X} \subset \mathscr{X}$ for every $t \ge 0$, and is seen without difficulty to be a compact set of X. Thus $(S(t), \mathscr{X}, X)$ defines a second dynamical system. We may notice that every trajectory starting from K enters to $B \subset \mathscr{X}$ in finite time.

We finally list some properties of the phase space $\mathscr{X}$ which are used in what follows.

(1) $\mathscr{X}$ is a compact set of X;
(2) $B \subset \mathscr{X} \subset K_{p(R)}$ (due to (4.5));
(3) $S(t)\mathscr{X} \subset B$ for every $t \ge t_{p(R)}$ (due to (4.7)).

4.3. Compact Lipschitz condition.

Let u_0, ρ_0 and $\widetilde{u}_0, \widetilde{\rho}_0$ be two pairs of initial functions in $\mathscr{X}$, and let u, ρ and $\widetilde{u}, \widetilde{\rho}$ be the corresponding solutions respectively. Since $\mathscr{X} \subset K_{p(R)}$, it follows that

$$\|u(t)\|_{H^2} + \|\rho(t)\|_{H^3} \le p(R), \quad 0 \le t < \infty,$$

$$\|\widetilde{u}(t)\|_{H^2} + \|\widetilde{\rho}(t)\|_{H^3} \le p(R), \quad 0 \le t < \infty.$$

For $w = u - \widetilde{u}$ and $\eta = \rho - \widetilde{\rho}$, we have

$$\begin{cases} \dfrac{\partial w}{\partial t} = a\Delta w - \nabla \cdot \{w\nabla\chi(\rho)\} - \nabla \cdot [\widetilde{u}\nabla\{\chi(\rho) - \chi(\widetilde{\rho})\}] + f(u) - f(\widetilde{u}), \\ \dfrac{\partial \eta}{\partial t} = b\Delta\eta - c\eta + dw. \end{cases} \tag{4.9}$$

Multiply the first equation by w and integrate the product in Ω. Then,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 dx + a\int_\Omega |\nabla w|^2 dx \\ &\quad = \int_\Omega w\nabla w \cdot \nabla\chi(\rho)\, dx + \int_\Omega \widetilde{u}\nabla w \cdot \nabla\{\chi(\rho) - \chi(\widetilde{\rho})\}\, dx + \int_\Omega \{f(u) - f(\widetilde{u})\} w\, dx \\ &\quad \le C\{\|\chi(\rho)\|_{H^{2+\varepsilon}}\|w\|_{L^2}\|\nabla w\|_{L^2} + \|\widetilde{u}\|_{H^{1+\varepsilon}}\|\nabla w\|_{L^2}\|\chi(\rho) - \chi(\widetilde{\rho})\|_{H^1} \\ &\qquad\quad + \|f(u) - f(\widetilde{u})\|_{L^2}\|w\|_{L^2}\} \le C\{(\|w\|_{L^2} + \|\eta\|_{H^1})\|\nabla w\|_{L^2} + \|w\|_{L^2}^2\} \end{aligned}$$

with an arbitrarily fixed exponent $0 < \varepsilon < 1/2$. Here we used [**22**, (2.10) and (2.14)] and

$$\|\chi(\rho) - \chi(\widetilde{\rho})\|_{H^1} \le C(\|\rho\|_{H^{1+\varepsilon}} + \|\widetilde{\rho}\|_{H^{1+\varepsilon}} + 1)\|\rho - \widetilde{\rho}\|_{H^1}, \qquad \rho, \widetilde{\rho} \in H^{1+\varepsilon}(\Omega)$$

(instead of [**22**, (2.15)]). Therefore we obtain that

$$\frac{d}{dt}\int_\Omega w^2 dx + a\int_\Omega |\nabla w|^2 dx \le C(\|\eta\|_{H^1}^2 + \|w\|_{L^2}^2). \tag{4.10}$$

Multiply next the second equation of (4.9) by $(\eta - \Delta\eta)$ and integrate the product in Ω. Then,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega (\eta^2 + |\nabla\eta|^2)\, dx + \int_\Omega \{b|\Delta\eta|^2 + (b+c)|\nabla\eta|^2 + c\eta^2\}\, dx \\ &\quad = d\int_\Omega w(\eta - \Delta\eta)\, dx \le d\|w\|_{L^2}(\|\eta\|_{L^2} + \|\Delta\eta\|_{L^2}). \end{aligned}$$

Therefore,

$$\frac{d}{dt}\int_\Omega (\eta^2 + |\nabla\eta|^2)\, dx + \int_\Omega \{b|\Delta\eta|^2 + (b+c)|\nabla\eta|^2 + c\eta^2\}\, dx \le C\|w\|_{L^2}^2.$$

This inequality jointed with (4.10) then provides that

$$\frac{d}{dt}(\|w(t)\|_{L^2}^2 + \|\eta\|_{H^1}^2) + \delta(\|w(t)\|_{H^1}^2 + \|\eta(t)\|_{H^2}^2) \le C(\|w(t)\|_{L^2}^2 + \|\eta(t)\|_{H^1}^2)$$

with some constant $\delta > 0$. Solving this differential inequality, we conclude that

$$\|w(t)\|_{L^2}^2 + \|\eta(t)\|_{H^1}^2 \le e^{Ct}(\|w(0)\|_{L^2}^2 + \|\eta(0)\|_{H^1}^2), \qquad 0 \le t < \infty.$$

Furthermore,

$$\begin{aligned}
\delta \int_0^t (\|w(s)\|_{L^2}^2 + \|\eta(s)\|_{H^2}^2) ds &\le \|w(0)\|_{L^2}^2 + \|\eta(0)\|_{H^1}^2 \\
+C \int_0^t (\|w(s)\|_{L^2}^2 + \|\eta(s)\|_{H^1}^2) ds &\le Ce^{Ct}(\|w(0)\|_{L^2}^2 + \|\eta(0)\|_{H^1}^2), \qquad 0 \le t < \infty. \quad (4.11)
\end{aligned}$$

We next establish energy estimates of higher order. Multiply the first equation of (4.9) by $-\Delta w$ and integrate the product in Ω. Then,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2 dx + a\int_\Omega |\Delta w|^2 dx \\
&= \int_\Omega \big(\nabla \cdot \{w\nabla\chi(\rho)\} + \nabla \cdot [\widetilde{u}\nabla\{\chi(\rho) - \chi(\widetilde{\rho})\}]\big)\Delta w\, dx + \int_\Omega \{f(u) - f(\widetilde{u})\}\Delta w\, dx \\
&\le C(\|\nabla \cdot \{w\nabla\chi(\rho)\}\|_{L^2} + \|\nabla \cdot [\widetilde{u}\nabla\{\chi(\rho) - \chi(\widetilde{\rho})\}]\|_{L^2} + \|f(u) - f(\widetilde{u})\|_{L^2})\|\Delta w\|_{L^2} \\
&\le C(\|w\|_{H^1}\|\chi(\rho)\|_{H^{2+\varepsilon}} + \|\widetilde{u}\|_{H^{1+\varepsilon}}\|\chi(\rho) - \chi(\widetilde{\rho})\|_{H^2} + \|w\|_{L^2})\|\Delta w\|_{L^2} \\
&\le C(\|w\|_{H^1} + \|\eta\|_{H^2})\|\Delta w\|_{L^2}
\end{aligned}$$

with an arbitrarily fixed $0 < \varepsilon < 1/2$. Here we used [**22**, (2.11), (2.14) and (2.15)]. Therefore,

$$\frac{d}{dt}\int_\Omega |\nabla w|^2 dx + a\int_\Omega |\Delta w|^2 dx \le C(\|w\|_{H^1}^2 + \|\eta\|_{H^2}^2).$$

Multiply next the second equation of (4.9) by $\Delta^2\eta$ and integrate the product in Ω. Then,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta\eta|^2 dx + \int_\Omega (b|\nabla\Delta\eta|^2 + c|\Delta\eta|^2)\, dx = -d\int_\Omega \nabla w \cdot \nabla\Delta\eta\, dx \le d\|w\|_{H^1}\|\nabla\Delta\eta\|_{L^2}.$$

Consequently,

$$\frac{d}{dt}\int_\Omega |\Delta\eta|^2 dx + b\int_\Omega |\nabla\Delta\eta|^2 dx \le C\|w\|_{H^1}^2.$$

In this way we obtain that

$$\frac{d}{dt}(\|w(t)\|_{H^1}^2 + \|\eta(t)\|_{H^2}^2) + \delta(\|w(t)\|_{H^2}^2 + \|\eta\|_{H^3}^2) \le C(\|w(t)\|_{H^1}^2 + \|\eta(t)\|_{H^2}^2)$$

with some constant $\delta > 0$. Solving this, we conclude that

$$\|w(t)\|_{H^1}^2 + \|\eta(t)\|_{H^2}^2 \le Ce^{Ct}(\|w(0)\|_{H^1}^2 + \|\eta(0)\|_{H^2}^2), \qquad 0 \le t < \infty.$$

We now notice that this estimate is valid for any pair of $s \le t$. In other words, it is true that

$$\|w(t)\|_{H^1}^2 + \|\eta(t)\|_{H^2}^2 \le Ce^{C(t-s)}(\|w(s)\|_{H^1}^2 + \|\eta(s)\|_{H^2}^2), \qquad 0 \le s < t < \infty.$$

Integrating this inequality in $s \in (0,t)$, we observe by (4.11) that

$$t(\|w(t)\|_{H^1}^2 + \|\eta(t)\|_{H^2}^2) \le C\int_0^t e^{C(t-s)}(\|w(s)\|_{H^1}^2 + \|\eta(s)\|_{H^2}^2)ds$$
$$\le C(e^{Ct})^2(\|w(0)\|_{L^2}^2 + \|\eta(0)\|_{H^1}^2), \qquad 0 < t < \infty.$$

Thus we have arrived at the estimate

$$\|u(t) - \widetilde{u}(t)\|_{H^1} + \|\rho(t) - \widetilde{\rho}(t)\|_{H^2}$$
$$\le C(e^{Ct}/\sqrt{t})(\|u(0) - \widetilde{u}(0)\|_{L^2} + \|\rho(0) - \widetilde{\rho}(0)\|_{H^1}), \quad 0 < t < \infty.$$

Therefore if we set a second Hilbert space by

$$Z = \left\{ \begin{pmatrix} u \\ \rho \end{pmatrix};\ u \in H^1(\Omega) \text{ and } \rho \in H^2(\Omega) \right\},$$

then the semigroup $S(t)$ satisfies the compact Lipschitz condition

$$\|S(t)U_0 - S(t)\widetilde{U}_0\|_Z \le C(e^{Ct}/\sqrt{t})\|U_0 - \widetilde{U}_0\|_X, \qquad U_0,\, \widetilde{U}_0 \in \mathscr{X}.$$

In addition it is easily verified that $S(t)U_0$ satisfies the Lipschitz condition (3.2), cf. [**22**, p. 142]. Hence, as mentioned in Theorem 3.1, an exponential attractor $\mathscr{M}$ for $(S(t), \mathscr{X}, X)$ can be constructed by employing the method presented in [**6**].

4.4. Estimate of convergence of semigroup.

We consider a family of sensitivity functions $\chi_\xi(\cdot)$ depending on a parameter $0 \le \xi \le 1$. They are assumed to satisfy the condition (4.2) with some uniform constant D. In addition we assume that for any $0 < r < \infty$,

$$\sup_{0\le\rho\le r} |\chi_\xi'(\rho) - \chi_0'(\rho)| \le D_r\xi, \qquad 0 < \xi \le 1, \tag{4.12}$$

with some constant $D_r > 0$.

By $(4.1)_\xi$, $0 \le \xi \le 1$, we denote the Cauchy problem for the chemotaxis-growth system including the sensitivity function $\chi_\xi(\cdot)$. As shown in the preceding subsection, for each $0 \le \xi \le 1$, a dynamical system $(S_\xi(t), \mathscr{X}_\xi, X)$ is determined from $(4.1)_\xi$. As $\chi_\xi(\cdot)$ satisfies (4.2) uniformly, we can take a uniform continuous function $p(\cdot)$ in (4.5), a uniform constant R in (4.6), and also a uniform absorbing set B in (4.7). Therefore, in view of (4.8),

$$B \subset \bigcap_{0\le\xi\le1} \mathscr{X}_\xi \subset \bigcup_{0\le\xi\le1} \mathscr{X}_\xi \subset K_{p(R)} \tag{4.13}$$

and

$$S_\xi(t)\mathscr{X}_\xi \subset B \text{ for every } t \ge t_{p(R)}.$$

The purpose of this subsection is to derive the condition (3.4) from (4.12). Let v_0, ζ_0 be a pair of initial functions from B, and let u_ξ, ρ_ξ be the solution to $(4.1)_\xi$, $0 \le \xi \le 1$, with the initial functions v_0, ζ_0. From (4.13) it follows that

$$\|u_\xi(t)\|_{H^2} + \|\rho_\xi(t)\|_{H^3} \le p(R), \quad 0 \le t < \infty,\ 0 \le \xi \le 1.$$

We set $w = u_\xi - u_0$ and $\eta = \rho_\xi - \rho_0$. Then,

$$\begin{cases} \dfrac{\partial w}{\partial t} = a\Delta w - \nabla \cdot \{w\nabla \chi_\xi(\rho_\xi)\} - \nabla \cdot [u_0 \nabla\{\chi_\xi(\rho_\xi) - \chi_\xi(\rho_0)\}] \\ \qquad -\nabla \cdot [u_0\nabla\{\chi_\xi(\rho_0) - \chi_0(\rho_0)\}] + f(u_\xi) - f(u_0), \\ \dfrac{\partial \eta}{\partial t} = b\Delta\eta - c\eta + dw. \end{cases} \tag{4.14}$$

Multiply the first equation by w and integrate the product in Ω. Then, by the same calculations as above,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 dx + a\int_\Omega |\nabla w|^2 dx \\ &= \int_\Omega w\nabla w \cdot \nabla \chi_\xi(\rho_\xi)\, dx + \int_\Omega u_0 \nabla w \cdot \nabla\{\chi_\xi(\rho_\xi) - \chi_\xi(\rho_0)\}\, dx \\ &\quad + \int_\Omega u_0\nabla w \cdot \nabla\{\chi_\xi(\rho_0) - \chi_0(\rho_0)\}\, dx + \int_\Omega \{f(u_\xi) - f(u_0)\} w\, dx \\ &\le C[\|\chi_\xi(\rho_\xi)\|_{H^{2+\varepsilon}}\|w\|_{L^2}\|\nabla w\|_{L^2} + \|u_0\|_{H^{1+\varepsilon}}\|\nabla w\|_{L^2}\|\chi_\xi(\rho) - \chi_\xi(\rho_0)\|_{H^1} \\ &\quad + \|u_0\|_{H^{1+\varepsilon}}\|\nabla w\|_{L^2}\|\nabla\{\chi_\xi(\rho_0) - \chi_0(\rho_0)\}\|_{L^2} + \|f(u_\xi) - f(u_0)\|_{L^2}\|w\|_{L^2}] \\ &\le C\{(\xi + \|w\|_{L^2} + \|\eta\|_{H^1})\|\nabla w\|_{L^2} + \|w\|_{L^2}^2\}. \end{aligned}$$

Here we used a fact that (4.12) implies that

$$\|\nabla\{\chi_\xi(\rho_0) - \chi_0(\rho_0)\}\|_{L^2} \le \|\chi'_\xi(\rho_0) - \chi'_0(\rho_0)\|_{L^\infty}\|\nabla\rho_0\|_{L^2} \le C\xi$$

due to $\|\rho_0\|_{L^\infty} \le C\|\rho_0\|_{H^2} \le Cp(R)$. Therefore,

$$\frac{d}{dt}\int_\Omega w^2 dx + a\int_\Omega |\nabla w|^2 dx \le C(\xi^2 + \|\eta\|_{H^1}^2 + \|w\|_{L^2}^2). \tag{4.15}$$

Multiply next the second equation of (4.14) by $(\eta - \Delta\eta)$ and integrate the product in Ω. Then, by the same calculations as above,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega (\eta^2 + |\nabla\eta|^2)\, dx + \int_\Omega \{b|\Delta\eta|^2 + (b+c)|\nabla\eta|^2 + c\eta^2\}\, dx \\ &= d\int_\Omega w(\eta - \Delta\eta)\, dx \le d\|w\|_{L^2}(\|\eta\|_{L^2} + \|\Delta\eta\|_{L^2}), \end{aligned}$$

and

$$\frac{d}{dt}\int_\Omega (\eta^2 + |\nabla\eta|^2)\, dx + \int_\Omega \{b|\Delta\eta|^2 + (b+c)|\nabla\eta|^2 + c\eta^2\}\, dx \le C\|w\|_{L^2}^2.$$

This jointed with (4.15) then yields that

$$\frac{d}{dt}(\|w(t)\|_{L^2}^2 + \|\eta\|_{H^1}^2) \le C(\xi^2 + \|w(t)\|_{L^2}^2 + \|\eta(t)\|_{H^1}^2).$$

Solving this differential inequality, we conclude that

$$\|w(t)\|_{L^2}^2 + \|\eta(t)\|_{H^1}^2 \le e^{Ct}(\|w(0)\|_{L^2}^2 + \|\eta(0)\|_{H^1}^2) + C\xi^2 \int_0^t e^{C(t-s)} \le C\xi^2 e^{Ct}, \quad 0 \le t < \infty.$$

Therefore,

$$\|S_\xi(t)U_0 - S_0(t)U_0\|_X \le C\xi e^{Ct}, \qquad 0 \le t < \infty,\ U_0 = \begin{pmatrix} v_0 \\ \zeta_0 \end{pmatrix} \in B.$$

In this way we have verified that the semigroups $S_\xi(t)$ satisfy (3.3) and (3.4) with $t^* = t_{p(R)}$. Consequently, there exist exponential attractors $\mathscr{M}_\xi$ for $(S_\xi(t), \mathscr{X}_\xi, X)$, $0 \le \xi \le 1$, for which the estimate

$$d(\mathscr{M}_\xi, \mathscr{M}_0) \le C\xi^\kappa, \qquad 0 < \xi \le 1$$

holds with some exponent $0 < \kappa < 1$.

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