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| Title        | On the abstract linear evolution equations in Banach spaces                         |
| Author(s)    | Yagi, Atsushi   |
| Citation     | Journal of the Mathematical Society of Japan.<br>1976, 28(2), p. 290-303            |
| Version Type | VoR   |
| URL          | <a href="https://hdl.handle.net/11094/24752">https://hdl.handle.net/11094/24752</a> |
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## On the abstract linear evolution equations in Banach spaces

By Atsushi YAGI

(Received March 25, 1975)

### § 0. Introduction.

The objective of the present paper is to construct the evolution operator associated with an evolution equation

$$(E) \quad du/dt + A(t)u = f(t)$$

in a Banach space  $X$ . Here  $u = u(t)$  and  $f(t)$  are functions on  $[0, T]$  to  $X$  and  $A(t)$  is a function on  $[0, T]$  to the set of linear operators acting in  $X$ . We assume (E) is of parabolic type, that is,  $-A(t)$  are all infinitesimal generators of analytic semi-groups of bounded linear operators on  $X$ .

This problem has been considered already in many papers, for instance, [1], [2] and [3]. The main assumption of [1] is that the inequality

$$\|A(t)^\rho dA(t)^{-1}/dt\| \leq N \quad (0.1)$$

is valid with some constant  $\rho \in (0, 1]$ ; and those of [2] are that the inequality of the form

$$\begin{aligned} & \|A(t)(\lambda - A(t))^{-1}(dA(t)^{-1}/dt)A(t)(\lambda - A(t))^{-1}\| \\ & = \|\partial/\partial t(\lambda - A(t))^{-1}\| \leq N/|\lambda|^\rho \end{aligned} \quad (0.2)$$

is valid with some constant  $\rho \in (0, 1]$  and  $dA(t)^{-1}/dt$  is Hölder continuous. In [3] the following conditions are assumed: the domain of  $A(t)^\rho$  is independent of  $t$  for some  $\rho = 1/m$  where  $m$  is a positive integer, and  $A(t)^\rho A(0)^{-\rho}$  is Hölder continuous in  $t$ .

In this paper we assume the inequality

$$\|A(t)(\lambda - A(t))^{-1}dA(t)^{-1}/dt\| \leq N/|\lambda|^\rho \quad (0.3)$$

with a constant  $\rho \in (0, 1]$ . This inequality (0.3) is slightly weaker than the inequality (0.1), for (0.1) implies (0.3) by the equation

$$A(t)(\lambda - A(t))^{-1}dA(t)^{-1}/dt = A(t)^{1-\rho}(\lambda - A(t))^{-1}A(t)^\rho dA(t)^{-1}/dt$$

and the estimation

$$\|A(t)^{1-\rho}(\lambda - A(t))^{-1}\| \leq M/|\lambda|^\rho.$$

On the other hand (0.3) is rather stronger than the inequality (0.2), for (0.3) is a direct consequence of (0.2) on account of the estimation

$$\|A(t)(\lambda - A(t))^{-1}\| \leq M.$$

However we should note that in order to construct the evolution operator in [2] (0.2) alone is not sufficient and the Hölder continuity of  $dA(t)^{-1}/dt$  must be assumed in addition.

In [1] and [2] the evolution operator is constructed directly by means of E. E. Levi's method. In the present case, however, this method does not work well to prove our theorem. Instead we use Yosida's approximation of  $A(t)$  and some integral equations as was done in [3].

As a corollary of the theorem we will construct the evolution operator under the assumptions that for some constant  $\rho \in (0, 1]$  the domain of  $A(t)^\rho$  is independent of  $t$ , and  $A(t)^\rho A(0)^{-\rho}$  is strongly continuously differentiable in  $t$ . Thus we can eliminate the condition assumed in [3] that  $\rho$  must be  $1/m$  with some positive integer  $m$ . In our corollary, however, we must require the strong differentiability as for the smoothness condition of  $A(t)^\rho A(0)^{-\rho}$ , though only its Hölder continuity was sufficient in [3].

The author expresses his deep thanks to Professor H. Tanabe.

### §1. Main theorem.

As our main result we claim the following.

**THEOREM 1.** For each  $t \in [0, T]$ , let  $A(t)$  be a densely defined, closed linear operator acting in the Banach space  $X$ . We assume the following conditions:

(I) For each  $t \in [0, T]$  the resolvent set of  $A(t)$  contains a fixed closed angular domain

$$\Sigma = \{\lambda \in \mathbf{C}; \arg \lambda \in (-\theta_0, \theta_0)\},$$

where  $\theta_0$  belongs to  $(0, \pi/2)$ . For any  $t \in [0, T]$  and  $\lambda \in \Sigma$  the resolvent satisfies the inequality

$$\|(\lambda - A(t))^{-1}\| \leq N_0/(1 + |\lambda|)$$

with some positive constant  $N_0$  independent of  $t$  and  $\lambda$ ;

(II)  $A(t)^{-1}$  is strongly continuously differentiable in  $t$ , and the derivative  $dA(t)^{-1}/dt$  satisfies

$$\|A(t)(\lambda - A(t))^{-1}dA(t)^{-1}/dt\| \leq N_1/|\lambda|^\rho$$

for any  $t \in [0, T]$  and  $\lambda \in \Sigma$ , where  $N_1$  and  $\rho \in (0, 1]$  are independent of  $t$  and  $\lambda$ .

Then there exists a family  $\{U(t, s); 0 \leq s \leq t \leq T\}$  of bounded operators on  $X$  having the following properties:

- 1)  $U(t, s)$  is strongly continuous in  $(t, s)$ ;
- 2)  $U(t, r)U(r, s) = U(t, s)$ ,  $0 \leq s \leq r \leq t \leq T$ ,  $U(s, s) = I$ ;
- 3) For  $s < t$ ,  $R(U(t, s))$  (the range of  $U(t, s)$ ) is contained in  $D(A(t))$  (the domain of  $A(t)$ ), and the estimates

$$\|A(t)U(t, s)A(s)^{-1}\| \leq C_0$$

and

$$\|A(t)U(t, s)\| \leq C_1/(t-s)$$

hold, where the constants  $C_0$  and  $C_1$  are determined by  $\theta_0$ ,  $\rho$ ,  $T$ ,  $N_0$  and  $N_1$  alone. Moreover  $A(t)U(t, s)A(s)^{-1}$  is strongly continuous in  $0 \leq s \leq t \leq T$ , and  $A(t)U(t, s)$  is strongly continuous in  $0 \leq s < t \leq T$ ;

- 4) If  $f$  is a continuous function with values in  $X$ , then any strict solution  $u$  of (E) on  $[s, T]$  with its initial value  $u(s) \in X$  can be expressed in the form

$$u(t) = U(t, s)u(s) + \int_s^t U(t, r)f(r)dr; \quad (1.1)$$

- 5) Conversely if  $f$  is Hölder continuous, then any  $u$  defined by (1.1) with any  $u(s) \in X$  gives the strict solution of (E) on  $[s, T]$ .

By a strict solution of (E) on  $[s, T]$  we mean a continuous function defined on  $[s, T]$  which is continuously differentiable on  $(s, T]$  and satisfies (E) on  $(s, T]$ .

PROOF OF THE THEOREM. For any integer  $n \geq 1$  and  $t \in [0, T]$  let us define

$$A_n(t) = A(t)(I + n^{-1}A(t))^{-1}.$$

It is well known that the resolvent set of  $A_n(t)$  contains  $\Sigma$  and the estimate

$$\|(\lambda - A_n(t))^{-1}\| \leq M_0/(1 + |\lambda|) \quad (1.2)$$

still holds for any  $t \in [0, T]$ ,  $\lambda \in \Sigma$  and  $n$  with some constant  $M_0$  depending only on  $N_0$  and  $\theta_0$  (e. g. see [3]). We know also that  $(\lambda - A_n(t))^{-1}$  converges strongly to  $(\lambda - A(t))^{-1}$  as  $n \rightarrow \infty$  for each  $t \in [0, T]$  and  $\lambda \in \Sigma$ . Since  $A_n(t)^{-1} = A(t)^{-1} + n^{-1}$ ,  $A_n(t)^{-1}$  is strongly continuously differentiable in  $t$ , and the estimate

$$\|A_n(t)(\lambda - A_n(t))^{-1}dA_n(t)^{-1}/dt\| \leq M_1/|\lambda|^p \quad (1.3)$$

is valid for any  $t \in [0, T]$ ,  $\lambda \in \Sigma$  and  $n$  with some constant  $M_1$  dependent only on  $\theta_0$ ,  $\rho$  and  $N_1$ .

Throughout this section  $M_2, M_3, \dots, M_{14}$  denote constants determined by  $\theta_0$ ,  $\rho$ ,  $T$ ,  $N_0$  and  $N_1$  alone.

From (1.2),  $-A_n(t)$  generates an analytic semi-group  $\exp(-rA_n(r))$  which is represented by

$$\exp(-rA_n(t)) = \frac{-1}{2\pi i} \int_{\Gamma} e^{-\lambda r} (\lambda - A_n(t))^{-1} d\lambda \quad (1.4)$$

where  $\Gamma$  is a smooth path running in  $\Sigma$  from  $\infty e^{-\theta_0 i}$  to  $\infty e^{\theta_0 i}$ .

By the differentiability of  $(\lambda - A_n(t))^{-1}$  and (1.4),  $\exp(-(t-s)A_n(s))$  is strongly continuously differentiable in  $s \in [0, t)$  and the derivative is expressed by the following integral

$$\begin{aligned} & (\partial/\partial s) \exp(-(t-s)A_n(s)) \\ &= \frac{-1}{2\pi i} \int_{\Gamma} e^{-\lambda(t-s)} \left\{ \frac{\partial}{\partial s} (\lambda - A_n(s))^{-1} + \lambda (\lambda - A_n(s))^{-1} \right\} d\lambda. \end{aligned} \tag{1.5}$$

Let  $U_n(t, s)$  ( $0 \leq s \leq t \leq T$ ) be the evolution operator corresponding to the equation

$$du/dt + A_n(t)u = 0,$$

and let

$$V_n(t, s) = A_n(t)U_n(t, s)A_n(s)^{-1},$$

$$W_n(t, s) = A_n(t)U_n(t, s) - A_n(s) \exp(-(t-s)A_n(s)).$$

Then we can construct three integral equations satisfied by  $U_n$ ,  $V_n$  and  $W_n$  respectively. For  $U_n$ ,

$$\begin{aligned} U_n(t, s) - \exp(-(t-s)A_n(s)) &= \int_s^t \frac{\partial}{\partial r} \{ \exp(-(t-r)A_n(r))U_n(r, s) \} dr \\ &= \int_s^t P_n(t, r)U_n(r, s)dr, \end{aligned} \tag{1.6}$$

where

$$P_n(t, r) = (\partial/\partial t + \partial/\partial r) \exp(-(t-r)A_n(r)). \tag{1.7}$$

For  $V_n$ ,

$$\begin{aligned} & V_n(t, s) - \exp(-(t-s)A_n(s)) \\ &= \int_s^t \frac{\partial}{\partial r} \{ \exp(-(t-r)A_n(r))V_n(r, s) \} dr \\ &= \int_s^t \{ P_n(t, r) + \exp(-(t-r)A_n(r))(dA_n(r)/dr)A_n(r)^{-1} \} V_n(r, s)dr \\ &= \int_s^t Q_n(t, r)V_n(r, s)dr, \end{aligned} \tag{1.8}$$

since  $(dA_n(r)/dr)A_n(r)^{-1} = -A_n(r)dA_n(r)^{-1}/dr$ , we see

$$Q_n(t, r) = P_n(t, r) - A_n(r) \exp(-(t-r)A_n(r))dA_n(r)^{-1}/dr. \tag{1.9}$$

Operating  $A_n(s)$  to (1.8) from the right, we obtain the equation for  $W_n$

$$W_n(t, s) = R_n(t, s) + \int_s^t Q_n(t, r)W_n(r, s)dr, \tag{1.10}$$

where

$$R_n(t, s) = \int_s^t Q_n(t, r) A_n(s) \exp(-(r-s)A_n(s)) dr. \tag{1.11}$$

These equations will play an important role in the proof of the theorem.

We next introduce a notation for classes of operator-valued functions which was used in [3]. By  $H(\nu, M)$  we denote the set of all operator-valued functions  $K(t, s)$ , defined and strongly continuous in  $0 \leq s < t \leq T$ , such that

$$\|K(t, s)\| \leq M(t-s)^{\nu-1}.$$

We see from (1.2) that

$$\exp(-(t-s)A_n(t)), \exp(-(t-s)A_n(s)) \in H(1, M_2), \tag{1.12}$$

$$A_n(t) \exp(-(t-s)A_n(t)), A_n(s) \exp(-(t-s)A_n(s)) \in H(0, M_3). \tag{1.13}$$

LEMMA 1. *The integral kernels  $P_n, Q_n$  and  $R_n$  belong to  $H(\rho, M_4)$ .*

To see this let us begin with  $P_n$ . By (1.5) and (1.7) we have

$$\begin{aligned} P_n(t, s) &= \frac{-1}{2\pi i} \int_{\Gamma} e^{-\lambda(t-s)} (\partial/\partial s)(\lambda - A_n(s))^{-1} d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma} e^{-\lambda(t-s)} A_n(s)(\lambda - A_n(s))^{-1} (dA_n(s)^{-1}/ds) A_n(s)(\lambda - A_n(s))^{-1} d\lambda. \end{aligned} \tag{1.14}$$

Here we have used the formula

$$\begin{aligned} &(\partial/\partial s)(\lambda - A_n(s))^{-1} \\ &= -A_n(s)(\lambda - A_n(s))^{-1} (dA_n(s)^{-1}/ds) A_n(s)(\lambda - A_n(s))^{-1}. \end{aligned} \tag{1.15}$$

From (1.3) and (1.14) we can see easily that  $P_n$  is strongly continuous in  $s < t$  and satisfies the estimate

$$\begin{aligned} \|P_n(t, s)\| &\leq \frac{(M_0+1)M_1}{2\pi} \int_{\Gamma} e^{-\operatorname{Re}\lambda(t-s)} |\lambda|^{-\rho} |d\lambda| \\ &\leq M_5(t-s)^{\rho-1}. \end{aligned}$$

By (1.4), (1.9) and (1.14) we have the representation

$$Q_n(t, s) = \frac{1}{2\pi i} \int_{\Gamma} \lambda e^{-\lambda(t-s)} A_n(s)(\lambda - A_n(s))^{-1} \frac{dA_n(s)^{-1}}{ds} (\lambda - A_n(s))^{-1} d\lambda. \tag{1.16}$$

From (1.16) we obtain  $Q_n \in H(\rho, M_6)$  by the same argument as above for  $P_n$ .

To prove  $R_n \in H(\rho, M_4)$ , we need some preparations. By (1.3) and (1.15)

$$\|(\lambda - A_n(t))^{-1} - (\lambda - A_n(s))^{-1}\| \leq (M_0+1)M_1(t-s)|\lambda|^{-\rho},$$

and therefore

$$\begin{aligned} & \|A_n(t) \exp(-(t-s)A_n(t)) - A_n(s) \exp(-(t-s)A_n(s))\| \\ & \leq M_7(t-s)^{\rho-1}. \end{aligned} \tag{1.17}$$

Using the expression (1.16), we obtain

$$\|Q_n(t, s)A_n(s)\| \leq M_8(t-s)^{\rho-2}. \tag{1.18}$$

Now, (1.11) can be rewritten in the following form

$$\begin{aligned} R_n(t, s) &= \int_s^t Q_n(t, r)A_n(r) \exp(-(r-s)A_n(r))dr \\ & \quad + \int_s^t Q_n(t, r) \{A_n(s) \exp(-(r-s)A_n(s)) \\ & \quad - A_n(r) \exp(-(r-s)A_n(r))\} dr \\ &= R_n^1(t, s) + R_n^2(t, s). \end{aligned} \tag{1.19}$$

$R_n^1(t, s)$  is strongly continuous for  $s < t$  by (1.12), (1.13), (1.18) and the fact  $Q_n \in H(\rho, M_6)$ .  $\|R_n^1(t, s)\|$  is estimated as follows

$$\begin{aligned} \|R_n^1(t, s)\| &\leq \int_{(t+s)/2}^t \|Q_n(t, r)\| \|A_n(r) \exp(-(r-s)A_n(r))\| dr \\ & \quad + \int_s^{(t+s)/2} \|Q_n(t, r)A_n(r)\| \|\exp(-(r-s)A_n(r))\| dr \\ &\leq M_9(t-s)^{\rho-1}. \end{aligned}$$

(1.17) and the fact  $Q_n \in H(\rho, M_6)$  give  $R_n^2 \in H(2\rho, M_{10})$ . Thus we have  $R_n \in H(\rho, M_{11})$ .

Finally we have only to put  $M_4 = \text{Max}\{M_5, M_6, M_{11}\}$ .

Let us define  $P(t, s)$ ,  $Q(t, s)$  and  $R(t, s)$  by the equalities (1.7), (1.9) and (1.11) with  $A_n(t)$  replaced by  $A(t)$ , respectively. Then the whole proof of Lemma 1 holds for  $P$ ,  $Q$  and  $R$ ; namely  $P$ ,  $Q$  and  $R$  are elements of  $H(\rho, M_4)$ . This fact enables us to define  $U$ ,  $V$  and  $W$  as the solutions of the following integral equations;

$$U(t, s) = \exp(-(t-s)A(s)) + \int_s^t P(t, r)U(r, s)dr, \tag{1.20}$$

$$V(t, s) = \exp(-(t-s)A(s)) + \int_s^t Q(t, r)V(r, s)dr, \tag{1.21}$$

$$W(t, s) = R(t, s) + \int_s^t Q(t, r)W(r, s)dr. \tag{1.22}$$

LEMMA 2. Let  $\varphi \in H(\nu, L)$  and  $\psi \in H(\mu, M)$  with  $\nu > 0$  and  $\mu > 0$ , and let  $Y$  be the solution of the equation

$$Y(t, s) = \varphi(t, s) + \int_s^t \psi(t, r) Y(r, s) dr,$$

then  $Y \in H(\nu, N)$  with some constant  $N$  determined by  $\nu, \mu, L, M$  and  $T$  alone.

Moreover, let  $\varphi_n \in H(\nu, L)$ ,  $\psi_n \in H(\mu, M)$  and  $Y_n$  be the solution of the equation

$$Y_n(t, s) = \varphi_n(t, s) + \int_s^t \psi_n(t, r) Y_n(r, s) dr.$$

If  $\varphi_n(t, s)$  and  $\psi_n(t, s)$  converge to  $\varphi(t, s)$  and  $\psi(t, s)$  strongly as  $n \rightarrow \infty$  for each  $0 \leq s < t \leq T$ , then  $Y_n(t, s)$  converges to  $Y(t, s)$  strongly.

The proof is a simple calculation when we use the theorem of dominated convergence (see [3]).

We want to apply Lemma 2 to the equations (1.6), (1.20), (1.8), (1.21) and (1.10), (1.22). To this end it is sufficient to check the convergence of  $P_n, Q_n$  and  $R_n$ , for we have already showed Lemma 1 and stated the remark to it.

In view of (1.14) and (1.16), we observe easily that  $P_n(t, s)$  and  $Q_n(t, s)$  are strongly convergent to  $P(t, s)$  and  $Q(t, s)$ . Rewriting  $R_n^1(t, s)$  defined by (1.19) in the form

$$R_n^1(t, s) = \left( \int_{(t+s)/2}^t + \int_s^{(t+s)/2} \right) Q_n(t, r) A_n(r) \exp(-(r-s)A_n(r)) dr,$$

we verify the strong convergence of  $R_n^1(t, s)$  to  $R^1(t, s)$ . The strong convergence of  $R_n^2(t, s)$  to  $R^2(t, s)$  is rather obvious. Here  $R^1$  and  $R^2$  are defined correspondingly. In this way we have seen that  $R_n(t, s)$  converges to  $R(t, s)$  strongly for each  $0 \leq s < t \leq T$ .

Therefore, noting (1.12) and Lemma 1, and applying Lemma 2, we deduce that  $U_n, U, V_n$  and  $V$  are all elements of  $H(1, M_{12})$ ; and that

$$W_n, W \in H(\rho, M_{13}); \tag{1.23}$$

and that  $U_n(t, s), V_n(t, s)$  and  $W_n(t, s)$  converge strongly to  $U(t, s), V(t, s)$  and  $W(t, s)$ , respectively, for each  $0 \leq s < t \leq T$ .

Now we will show that  $U(t, s)$  has the properties 1)~5) mentioned in our theorem. First of all, letting  $n \rightarrow \infty$  in the equation

$$A_n(t)^{-1} V_n(t, s) = U_n(t, s) A_n(s)^{-1}$$

we have

$$A(t)^{-1} V(t, s) = U(t, s) A(s)^{-1},$$

which implies  $R(U(t, s)A(s)^{-1}) \subset D(A(t))$  and

$$V(t, s) = A(t)U(t, s)A(s)^{-1}.$$

The term  $\exp(-(t-s)A(s))$  in the equations (1.20) and (1.21) is strongly con-



tinuous in  $0 \leq s \leq t \leq T$ , and this property is inherited by those of the solutions  $U(t, s)$  and  $V(t, s)$ . The property 2) of  $U_n$  implies that of  $U$ . Thus we have obtained 1), 2) and a part of 3).

Next, let us prove 5), dividing the proof into two steps. For the first step we assume  $f=0$ . Letting  $n \rightarrow \infty$  in the following

$$U_n(t, s) = A_n(t)^{-1} \{W_n(t, s) + A_n(s) \exp(-(t-s)A_n(s))\},$$

we obtain that  $R(U(t, s)) \subset D(A(t))$  for  $s < t$  and

$$A(t)U(t, s) = W(t, s) + A(s) \exp(-(t-s)A(s)).$$

This shows that  $A(t)U(t, s)$  is strongly continuous in  $0 \leq s < t \leq T$  and  $A_n(t)U_n(t, s)$  converges to  $A(t)U(t, s)$  strongly. On the other hand by (1.13) and (1.23)

$$\|A_n(t)U_n(t, s)\| \leq M_{14}(t-s)^{-1}, \tag{1.24}$$

and hence

$$\|A(t)U(t, s)\| \leq M_{14}(t-s)^{-1}.$$

Therefore, for any  $\varepsilon > 0$  and  $t \geq s + \varepsilon$  it follows from

$$U_n(t, s)u(s) - U_n(s + \varepsilon, s)u(s) = - \int_{s+\varepsilon}^t A_n(r)U_n(r, s)u(s)dr$$

that

$$u(t) = U(s + \varepsilon, s)u(s) - \int_{s+\varepsilon}^t A(r)U(r, s)u(s)dr,$$

which shows that  $u$  is a strict solution on  $[s, T]$ . At the same time we have deduced the remaining part of 3).

For the second step we assume  $u(s)=0$ . Put for integer  $n \geq 1$

$$u_n(t) = \int_s^t U_n(t, r)f(r)dr, \tag{1.25}$$

then by the definition of  $U_n$  we can write

$$u_n(t) = \int_s^t \{f(r) - A_n(r)u_n(r)\} dr. \tag{1.26}$$

Multiplying (1.25) by  $A_n(t)$ , we have

$$\begin{aligned} A_n(t)u_n(t) &= \int_s^t A_n(t)U_n(t, r)[f(r) - f(t)]dr \\ &+ \left[ \int_s^t \{A_n(r) \exp(-(t-r)A_n(r)) - A_n(t) \exp(-(t-r)A_n(t))\} dr \right. \\ &\left. + \int_s^t W_n(t, r)dr + \int_s^t A_n(t) \exp(-(t-r)A_n(t))dr \right] f(t). \end{aligned} \tag{1.27}$$

From (1.27), noting

$$\int_s^t A_n(t) \exp(-(t-r)A_n(t)) dr = I - \exp(-(t-s)A_n(t))$$

and using (1.12), (1.17), (1.23), (1.24) and the Hölder continuity of  $f$ ; we conclude that

$$\|A_n(t)u_n(t)\| \leq M_{15}, \quad (1.28)$$

where  $M_{15}$  depends on  $f$  besides  $\theta_0$ ,  $\rho$ ,  $T$ ,  $N_0$  and  $N_1$ , and that  $A_n(t)u_n(t)$  converges to a function  $v(t)$  pointwise.  $v$  is then defined on  $[s, T]$  by the right hand side of (1.27) with  $A_n$ ,  $U_n$  and  $W_n$  replaced by  $A$ ,  $U$  and  $W$ , respectively.  $v$  is continuous on  $[s, T]$ . In a similar way to  $V_n$  we can see  $u(t) \in D(A(t))$  and  $A(t)u(t) = v(t)$  for any  $t \in [s, T]$ . Therefore if  $n$  tends to infinity in (1.26), the pointwise convergence of  $A_n(r)u_n(r)$  and (1.28) give us

$$u(t) = \int_s^t \{f(r) - A(r)u(r)\} dr,$$

which means that  $u$  is a strict solution on  $[s, T]$ . Now the proof of 5) is complete with the above two steps.

Finally we will prove 4). The strong continuity of  $V(t, s)$  at  $t=s$  implies the strong right differentiability of  $U(t, s)A(s)^{-1}$  in  $t$  at  $t=s$ . And then 2), 3) and this property give us

$$\lim_{\Delta s \rightarrow +0} \frac{U(t, s+\Delta s)u_0 - U(t, s)u_0}{\Delta s} = U(t, s)A(s)u_0$$

for any  $u_0 \in D(A(s))$  and  $t > s$ . Hence for any strict solution  $u$  on  $[s, T]$ ,  $U(t, r)u(r)$  is continuously differentiable from the right in  $r \in (s, t)$  and

$$(\partial^+ / \partial r)U(t, r)u(r) = U(t, r)f(r). \quad (1.29)$$

Integrating (1.29) on  $(s, t)$ , we conclude

$$u(t) = U(t, s)u(s) + \int_s^t U(t, r)f(r)dr.$$

## § 2. A consequence of the main theorem.

As a consequence of Theorem 1 we have

**THEOREM 2.** *Under the same situation as the theorem we assume the following conditions:*

(I') *The same condition as (I) stated in the theorem;*

(II') *There exists a constant  $\rho \in (0, 1]$  such that the domain of  $A(t)^\rho$  is independent of  $t$  and  $A(t)^\rho A(0)^{-\rho}$  is strongly continuously differentiable in  $t$ .*

*Then there exists a family  $\{U(t, s); 0 \leq s \leq t \leq T\}$  of bounded operators having the properties 1)~5) stated in the theorem. In this case the constants*

$C_0$  and  $C_1$  which appear in the statement of 3) are determined by  $\theta_0, \rho, T, N_0, N_2 = \sup_{t \in [0, T]} \|dA(t)^\rho A(0)^{-\rho}/dt\|$  and  $N_3 = \sup_{t \in [0, T]} \|A(0)^\rho A(t)^{-\rho}\|$  alone, the existence of finite  $N_2$  and  $N_3$  being ensured by (II'). Moreover under these assumptions  $U$  have the following property:

6) For any  $u_0 \in D(A(0)^\rho)$ ,  $U(t, s)u_0$  is continuously differentiable in  $s \in [0, t)$  and

$$\partial/\partial s U(t, s)u_0 = \overline{U(t, s)A(s)^{1-\rho}A(s)^\rho}u_0.$$

PROOF. By the assumption (II') we shall show

$$\|A(t)^\rho dA(t)^{-1}/dt\| \leq M_{16}. \tag{2.1}$$

Unless there is any specification,  $M_{16} \cdots M_{21}$  denote the constants depending only on  $\theta_0, \rho, T, N_0, N_2$  and  $N_3$ . Actually if  $\rho=1$ ,  $A(t)^{-1}$  is strongly continuously differentiable with its derivative

$$dA(t)^{-1}/dt = -A(t)^{-1}(dA(t)A(0)^{-1}/dt)A(0)A(t)^{-1},$$

which implies (2.1). If  $0 < \rho < 1$ , then we can write

$$A(t)^{-1} = \frac{-1}{2\pi i} \int_{\Gamma} \lambda^{-\frac{1}{\rho}} (\lambda - A(t)^\rho)^{-1} d\lambda.$$

Since  $(\lambda - A(t)^\rho)^{-1}$  is strongly continuously differentiable in  $t$ , so is  $A(t)^{-1}$ , whose derivative is expressed by

$$dA(t)^{-1}/dt = \frac{-1}{2\pi i} \int_{\Gamma} \lambda^{-\frac{1}{\rho}} (\lambda - A(t)^\rho)^{-1} \frac{dA(t)^\rho A(0)^{-\rho}}{dt} A(0)^\rho (\lambda - A(t)^\rho)^{-1} d\lambda.$$

From this expression we obtain (2.1). Thus we have proved that the condition (II') implies (2.1), and hence (II).

Now we have only to show 6). To this end we need a lemma concerned with the fractional powers of  $A_n(t)$ .

LEMMA 3. Let  $\delta \in (0, \rho)$ , then  $A_n(t)^{\rho-\delta} A(t)^{-\rho}$  converges to  $A(t)^{-\delta}$  strongly as  $n \rightarrow \infty$ .

Since

$$A_n(t)^{\rho-\delta} = \frac{\sin(\rho-\delta)\pi}{\pi} \int_0^\infty \mu^{\rho-\delta-1} (\mu + A_n(t))^{-1} d\mu A_n(t), \tag{2.2}$$

we see that  $A_n(t)^{\rho-\delta} A(t)^{-\rho} x$  converges to  $A(t)^{-\delta} x$  for any  $x \in D(A(t)^{1-\rho})$ . Thus it suffices to show

$$\|A_n(t)^{\rho-\delta} A(t)^{-\rho}\| \leq M_{17}. \tag{2.3}$$

If  $\rho=1$ , this is clear from (2.2). If  $0 < \rho < 1$ , it is obvious from the equation

$$A_n(t)^{\rho-\delta} A(t)^{-\rho} = \frac{\sin \rho\pi}{\pi} \int_0^\infty \mu^{-\rho} A_n(t)^{\rho-\delta} (\mu + A(t))^{-1} d\mu$$

and the well-known inequality

$$\|A_n(t)^{\rho-\delta}(\mu+A(t))^{-1}\| \leq M_{18} \|(\mu+A(t))^{-1}\|^{(1-\rho+\delta)} \|A_n(t)(\mu+A(t))^{-1}\|^{(\rho-\delta)}.$$

Thus we have proved (2.3).  $M_{17}$  and  $M_{18}$  are dependent also on  $\delta$ .

Let  $\eta \in [0, 1)$ . Operating  $A_n(s)^\eta$  to (1.6) from the right, we have

$$\begin{aligned} U_n(t, s)A_n(s)^\eta &= A_n(s)^\eta \exp(-(t-s)A_n(s)) \\ &\quad + \int_s^t P_n(t, r)U_n(r, s)A_n(s)^\eta dr. \end{aligned} \quad (2.4)$$

In (2.4), the term  $A_n(s)^\eta \exp(-(t-s)A_n(s))$  belongs to  $H(1-\eta, M_{19})$  and converges to  $A(s)^\eta \exp(-(t-s)A(s))$  strongly, therefore Lemma 2 is applicable to (2.4) again. Hence it follows that

$$U_n(t, s)A_n(s)^\eta \in H(1-\eta, M_{20}) \quad (2.5)$$

and  $U_n(t, s)A_n(s)^\eta$  is strongly convergent to the bounded extension

$$\overline{U(t, s)A(s)^\eta} \in H(1-\eta, M_{20}). \quad (2.6)$$

Here  $M_{19}$  and  $M_{20}$  depend on  $\eta$ .

Now let  $\delta \in (0, \rho)$  be fixed and write  $\eta = 1 + \delta - \rho$ . By the definition of  $U_n$

$$u_0 - U_n(t, s)u_0 = \int_s^t U_n(t, r)A_n(r)^\eta A_n(r)^{\rho-\delta} A(r)^{-\rho} A(r)^\rho u_0 dr.$$

Letting  $n \rightarrow \infty$  in this equation with the aid of Lemma 3, (2.3) and (2.5); we obtain

$$u_0 - U(t, s)u_0 = \int_s^t \overline{U(t, r)A(r)^\eta} A(r)^{-\delta} A(r)^\rho u_0 dr.$$

Taking  $\eta = 1 - \rho$  in (2.6), we have

$$\overline{U(t, r)A(r)^\eta} A(r)^{-\delta} = \overline{U(t, r)A(r)^{1-\rho}} \in H(\rho, M_{21})$$

and the desired relation.

### § 3. Remark.

Professor M. Watanabe informed the author that our main theorem can be applied to the initial-boundary value problems for parabolic equations.

Let  $\Omega$  be a bounded region in  $R^n$  with a sufficiently smooth boundary,

$$A(x, t, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x, t) D^\alpha, \quad x \in \bar{\Omega}, t \in [0, T]$$

be uniformly strongly elliptic differential operators with smooth coefficients,

and

$$B_j(x, t, D) = \sum_{|\beta| \leq m_j} b_{j\beta}(x, t) D^\beta, \quad j=1, \dots, m$$

be normal boundary differential operators having smooth coefficients. We consider the following mixed problem

$$\begin{cases} \partial u(x, t)/\partial t + A(x, t, D)u(x, t) = f(x, t), & x \in \Omega, t \in (0, T] \\ u(x, 0) = u_0(x), & x \in \Omega \\ B_j(x, t, D)u(x, t) = 0, & x \in \partial\Omega, t \in (0, T] \end{cases}$$

in  $L^p(\Omega)$ ,  $1 < p < \infty$ . For each  $t \in [0, T]$ , we define the linear operator  $A(t)$  acting in  $L^p(\Omega)$  as follows,

$$D(A(t)) = \{u \in W_p^{2m}(\Omega); B_j(x, t, D)u(x) = 0, x \in \partial\Omega, j=1, \dots, m\}$$

and for  $u \in D(A(t))$

$$(A(t)u)(x) = A(x, t, D)u(x).$$

We denote by  $W_p^l(\Omega)$  the set of all complex-valued functions defined in  $\Omega$  whose distribution derivatives  $D^\alpha u$  belong to  $L^p(\Omega)$  for any  $\alpha$  with  $0 \leq |\alpha| \leq l$ .

Then, it can be shown that  $A(t)$  satisfies the conditions (I) and (II) of the theorem provided that we add a sufficiently large positive number to  $A(t)$  if necessary.

In truth it is possible to show with the aid of the interpolation theory that such  $A(t)$  satisfies the stronger condition (0.1) with a sufficiently small  $\rho > 0$ ; however, the proof is not elementary. We give below a simple proof that (II) is satisfied with

$$\rho = \begin{cases} 1, & \text{if } m=1 \text{ and } m_1=0 \\ \text{Min } \{m_j/2m; m_j \geq 1, j=1, \dots, m\}, & \text{otherwise.} \end{cases}$$

For any  $f \in L^p(\Omega)$ , we have by the definition

$$A(x, t, D)A(t)^{-1}f(x) = f(x), \quad x \in \Omega \tag{3.1}$$

and

$$B_j(x, t, D)A(t)^{-1}f(x) = 0, \quad x \in \partial\Omega. \tag{3.2}$$

Since  $A(t)^{-1}$ , the bounded operator from  $L^p(\Omega)$  into  $W_p^{2m}(\Omega)$ , is strongly continuously differentiable, (3.1) and (3.2) give

$$A(x, t, D)(dA(t)^{-1}/dt)f(x) = -\dot{A}(x, t, D)A(t)^{-1}f(x), \quad x \in \Omega, \tag{3.3}$$

and

$$B_j(x, t, D)(dA(t)^{-1}/dt)f(x) = -\dot{B}_j(x, t, D)A(t)^{-1}f(x), \quad x \in \partial\Omega, \tag{3.4}$$

where  $\dot{A}$  and  $\dot{B}_j$  are defined by

$$\dot{A}(x, t, D) = \sum_{|\alpha| \leq 2m} (\partial a_\alpha(x, t) / \partial t) D^\alpha$$

and

$$\dot{B}_j(x, t, D) = \sum_{|\beta| \leq m_j} (\partial b_{j\beta}(x, t) / \partial t) D^\beta.$$

Next, for any  $\lambda \in \Sigma$  and  $u \in W_p^{2m}(\Omega)$

$$\begin{aligned} & (\lambda - A(x, t, D))A(t)(\lambda - A(t))^{-1}u \\ &= (\lambda - A(x, t, D))\{\lambda(\lambda - A(t))^{-1} - 1\}u \\ &= A(x, t, D)u, \end{aligned}$$

therefore using (3.3), we have

$$\begin{aligned} & (\lambda - A(x, t, D))A(t)(\lambda - A(t))^{-1}(dA(t)^{-1}/dt)f(x) \\ &= -\dot{A}(x, t, D)A(t)^{-1}f(x), \quad x \in \Omega. \end{aligned} \quad (3.5)$$

Similarly for  $\lambda \in \Sigma$  and  $u \in W_p^{2m}(\Omega)$

$$B_j(x, t, D)A(t)(\lambda - A(t))^{-1}u = -B_j(x, t, D)u$$

on  $\partial\Omega$ , and hence in view of (3.4) we get

$$\begin{aligned} & B_j(x, t, D)A(t)(\lambda - A(t))^{-1}(dA(t)^{-1}/dt)f(x) \\ &= \dot{B}_j(x, t, D)A(t)^{-1}f(x), \quad x \in \partial\Omega. \end{aligned} \quad (3.6)$$

It is known that for any  $\lambda \in \Sigma$  and  $t \in [0, T]$  the estimate

$$\begin{aligned} & |\lambda| \|u\|_p + \|u\|_{2m,p} \leq M_{22} \{ \|(\lambda - A(x, t, D))u\|_p \\ &+ \sum_{j=1}^m |\lambda|^{(1-m_j/2m)} \|g_j\|_p + \sum_{j=1}^m \|g_j\|_{2m-m_j,p} \} \end{aligned}$$

holds for  $u \in W_p^{2m}(\Omega)$  with some constant  $M_{22}$  independent of  $\lambda$  and  $t$ . Here  $g_j$  is an arbitrary function which belongs to  $W_p^{2m-m_j}(\Omega)$  and coincides with  $B_j(x, t, D)u(x)$  on  $\partial\Omega$ . Substituting  $A(t)(\lambda - A(t))^{-1}(dA(t)^{-1}/dt)f$  for  $u$  in this estimate, we obtain from (3.5) and (3.6)

$$\begin{aligned} & |\lambda| \|A(t)(\lambda - A(t))^{-1}(dA(t)^{-1}/dt)f\|_p \\ & \leq M_{23} \{ \|A(t)^{-1}f\|_{2m,p} + \sum_{j=1}^m |\lambda|^{(1-m_j/2m)} \|h_j\|_p \}, \end{aligned}$$

where  $h_j \in W_p^{2m-m_j}(\Omega)$  and

$$h_j(x) = \dot{B}_j(x, t, D)A(t)^{-1}f(x), \quad x \in \partial\Omega.$$

If  $m_j=0$ , then  $B_j(x, t, D)=1$ , and therefore we can take  $h_j=0$ . Thus we conclude

$$\|A(t)(\lambda - A(t))^{-1}dA(t)^{-1}/dt\| \leq M_{24}/|\lambda|^\rho$$

with

$$\rho = \begin{cases} 1, & \text{if } m=1 \text{ and } m_1=0 \\ \text{Min}\{m_j/2m; m_j \geq 1, j=1, \dots, m\}, & \text{otherwise.} \end{cases}$$

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