EXPONENTIAL ATTRACTORS FOR NON-AUTONOMOUS DISSIPATIVE SYSTEM

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ABSTRACT. In this paper we will introduce a version of exponential attractor for non-autonomous equations as a time dependent set with uniformly bounded finite fractal dimension which is positively invariant and attracts every bounded set at an exponential rate. This is a natural generalization of the existent notion for autonomous equations. A generation theorem will be proved under the assumption that the evolution operator is a compact perturbation of a contraction. In the second half of the paper, these results will be applied to some non-autonomous chemotaxis system.

1. INTRODUCTION

Our aim in this paper is to discuss the behavior as time goes to infinity of ordinary differential equations of the form

$$\frac{dU}{dt} = F(t, U) \tag{1.1}$$

in a Banach space X.

When the system is autonomous, i.e., when the time does not appear explicitly in (1.1) ($F(t, U) \equiv F(U)$), then, very often, the long time behavior of the system can be described in terms of the global attractor $\mathcal{A}$. More precisely, assuming that the system is well-posed, we can define the family of solving operators

$$S(t) : U_0 \mapsto U(t), \quad t \geq 0,$$

acting on X, which maps the initial datum U_0 onto the solution at time t. This family of operators satisfies

$$S(0) = I,$$
$$S(t + s) = S(t) \circ S(s), \quad \forall t, \ s \geq 0,$$

I denoting the identity operator on X, and we say that it forms a semigroup on the phase space X.

Definition 1.1. We then say that a set $\mathcal{A}$ is the global attractor for $S(t)$ in X if:

(i) It is a compact set of X.
(ii) It is an invariant set, i.e., $S(t)\mathcal{A} = \mathcal{A}, \ \forall t \geq 0$.
(iii) It attracts (uniformly) the bounded sets of initial data in the following sense:

$$\forall B \subset X \ \text{ bounded}, \ \lim_{t \to +\infty} h(S(t)B, \mathcal{A}) = 0,$$

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where $h(\cdot,\cdot)$ denotes the Hausdorff semidistance between sets, defined by

$$h(A,B) = \sup_{a\in A} \inf_{b\in B} \|a-b\|_H.$$

This is equivalent to the following : $\forall B \subset X$ bounded, $\forall \varepsilon > 0$, $\exists t_0 = t_0(B,\varepsilon)$ such that $t \geq t_0$ implies $S(t)B \subset \mathfrak{U}_\varepsilon$, where $\mathfrak{U}_\varepsilon$ denotes the ε-neighborhood of $\mathcal{A}$.

We note that it follows from (ii) and (iii) that the global attractor, if it exists, is unique. Furthermore, it follows from (i) that it is essentially thinner than the original phase space X ; indeed, here, in general, X is an infinite-dimensional function space and, in infinite dimensions, a compact set cannot contain a ball and is nowhere dense. It is not difficult to prove that the global attractor is the smallest (for the inclusion) closed set enjoying the attraction property (iii); it is also the largest bounded invariant set. Finally, in most (if not all) cases, one can prove that the global attractor has finite dimension (in the sense of covering dimensions, such as the Hausdorff and the fractal dimensions; the global attractor is not a smooth manifold in general, but it can have a very complicated geometric structure), so that, even though the initial phase space is infinite-dimensional, the dynamics, reduced to the global attractor, is, in some proper sense, finite-dimensional and can be described by a finite number of parameters. It thus follows that the global attractor appears as a suitable object in view of the study of the long time behavior of the system. We refer the reader to [5, 12, 20, 25, 27, 29] for extensive reviews on this subject.

Now, the global attractor may present some defaults. Indeed, it may attract the trajectories slowly (see, e.g., [23]). Furthermore, in general, it is very difficult, if not impossible, to express the convergence rate in terms of the physical parameters of the problem. A second drawback, which can also be seen as a consequence of the first one, is that the global attractor may be sensitive to perturbations; a given system is only an approximation of reality and it is thus essential that the objects that we study must be robust under small perturbations. Actually, in general, the global attractor is outer semicontinuous with respect to perturbations, i.e.,

$$h(\mathcal{A}_\varepsilon, \mathcal{A}_0) \to 0 \ \text{ as } \ \varepsilon \to 0,$$

where $\mathcal{A}_0$ is the global attractor associated with the nonperturbed system and $\mathcal{A}_\varepsilon$ that associated with the perturbed one, $\varepsilon > 0$ being the perturbation parameter. Now, the inner semicontinuity, i.e.,

$$h(\mathcal{A}_0, \mathcal{A}_\epsilon) \to 0 \ \text{ as } \ \varepsilon \to 0,$$

is much more difficult to prove (see, e.g., [27]). Furthermore, this property may not hold. This is in particular the case when the perturbed and nonperturbed problems do not have the same equilibria (stationary solutions). Furthermore, in many situations, the global attractor may not be observable in experiments or in numerical simulations. This can be due to the fact that it has a very complicated geometric structure, but not necessarily. Indeed, we can consider for instance the following Chafee-Infante equation in one space dimension:

$$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} + u^3 - u = 0, \quad x \in [0,1], \ \ \nu > 0,$$
$$u(0,t) = u(1,t) = -1, \quad t \geq 0.$$

Then, due to the boundary conditions, $\mathcal{A} = \{-1\}$. Now, this problem possesses many metastable "almost stationary" equilibria which live up to a time $t_\star \sim e^{\nu^{-\frac{1}{2}}}$. Thus, for ν small, one will not see the global attractor in numerical simulations. Finally, in some situations, the global attractor may fail to capture important transient behaviors. This can be observed, e.g., on some models of one-dimensional Burgers equations with a weak dissipation term (see [6]). In that case, the global attractor is trivial, it is reduced to one exponentially attracting point, but the system presents very rich and important transient behaviors, which resemble some modified version of the Kolmogorov law. We can also mention models of pattern formation equations in autonomous chemotaxis model for which one observes important transient behaviors which are not contained in the global attractor (see [2, 3, 19, 28]).

So, it follows from the above considerations that it should be useful to have a (possibly) larger object which contains the global attractor, attracts the trajectories at a fast rate, is still finite-dimensional and is more robust under perturbations.

The first attempt to study such an object, i.e., an exponential attractor for an autonomous system, was made by A. Eden, C. Foias, B. Nicolaenko and R. Temam in [14]. Indeed, let $S(t),\ t \geq 0$, be the semigroup associated with the problem

$$\begin{cases} \dfrac{dU}{dt} = F(U), \qquad 0 < t < \infty, \\ U(0) = U_0, \end{cases} \tag{1.2}$$

in a Banach space X (in particular, we assume that (1.2) is well-posed for $u_0 \in X$). We have the following definition.

Definition 1.2. A set $\mathcal{M}$ is an exponential attractor for $S(t)$ in X if:

(i) It is a compact set of X with finite fractal dimension.

(ii) It is a positively invariant set, i.e., $S(t)\mathcal{M} \subset \mathcal{M},\ \forall t \geq 0$.

(iii) It attracts exponentially fast the bounded sets of initial data in the following sense: There exist a constant $\alpha > 0$ and a monotonic function Q such that

$$\forall B \subset X \ \text{ bounded},\ h(S(t)B, \mathcal{M}) \leq Q(\|B\|_X)e^{-\alpha t},\ t \geq 0.$$

It follows from this definition that an exponential attractor always contains the global attractor (actually, it follows from the definition that, if $S(t)$ possesses an exponential attractor $\mathcal{M}$, then it also possesses the global attractor $\mathcal{A} \subset \mathcal{M}$; indeed, $\mathcal{M}$ is a compact attracting set (see, e.g., [5]; the continuity of $S(t)$, $\forall t \geq 0$, generally holds)).

Remark 1.1. (i) Actually, proving the existence of an exponential attractor is also one way of proving the finite (fractal) dimensionality of the global attractor.

(ii) The choice of the fractal dimension over other dimensions, e.g., the Hausdorff dimension, in Definition 1.2 is related, with the Mané theorem which gives some indications on the existence of a reduced finite-dimensional system which is Hölder continuous (but, unfortunately, not Lipschitz continuous) with respect to the initial data, see [14].

The main drawback of exponential attractors is however that an exponential attractor, if it exists, is not unique. Therefore, the question of the best choice, if it makes sense, of an exponential attractor is a crucial one.

The first construction of exponential attractors was due to A. Eden, C. Foias, B. Nicolaenko and R. Temam [14]. This construction is based on the so-called squeezing property which, roughly speaking, says that either the higher modes are dominated by the lower ones or that the flow is contracted exponentially. It is non-constructible (indeed, Zorn's lemma is used in order to construct the appropriate exponential attractor) and is only valid in Hilbert spaces (since it makes an essential use of orthogonal projectors with finite rank). Furthermore, based on this construction, it is possible to prove the inner semicontinuity of proper exponential attractors under perturbations, but only up to some time shift, so that, essentially, one only proves that

$$h(\mathcal{A}_0, \mathcal{M}_\varepsilon) \to 0 \text{ as } \varepsilon \to 0,$$

where $\mathcal{A}_0$ is the global attractor associated with the nonperturbed system and $\mathcal{M}_\varepsilon$ an exponential attractor associated with the perturbed one, which is not satisfactory.

In [16], was proposed a second construction, valid in Banach spaces also (see also [13] for another construction of exponential attractors valid in Banach spaces; this second construction consists in adapting that of [14] to a Banach setting and has thus some of the drawbacks mentioned above). The key point in this construction is a smoothing property on the difference of two solutions which generalizes in some sense (and, in particular, to a Banach setting) techniques proposed by O.A. Ladyzhenskaya in order to prove the finite dimensionality of the global attractor, see, e.g., [24] of the form

$$\|S(\tau^*)U_0 - S(\tau^*)V_0\|_Z \le c\|U_0 - V_0\|_X, \tag{1.3}$$

where Z is a second Banach space which is compactly embedded into X, which has to hold for some $\tau^* > 0$ and on some bounded positively invariant subset of X (see [16] for generalizations and other forms of the smoothing property (1.3)). We can note that, in a Hilbert setting, i.e., when X and Z are Hilbert spaces, then (1.3) implies the squeezing property, see [15]. Furthermore, based on this construction, it is possible to construct robust (i.e., inner and outer semicontinuous with respect to perturbations) families of exponential attractors (see [17]) which satisfy in particular an estimate of the form

$$d(\mathcal{M}_\varepsilon, \mathcal{M}_0) \le c\varepsilon^\kappa, \quad c > 0, \ \kappa \in (0, 1), \tag{1.4}$$

where the constants c and κ are independent of ε and can be computed explicitly in terms of the physical parameters of the problem and where $d(\cdot, \cdot)$ denotes the symmetric Hausdorff distance between (closed) sets

$$d(A, B) = \max\{h(A, B), h(B, A)\}.$$

Of course, such constructions are obtained having in mind the nonuniqueness problem.

Remark 1.2. (i) It is in general very difficult, if not impossible, to prove an estimate of the form (1.4) for global attractors. This is possible, for instance, when the stationary solutions enjoy some hyperbolicity assumption. In that case, the global attractor is regular (see [5]) and exponential and one has an estimate of the form (1.4). However, even in that case, one cannot compute in general the constants c and κ in terms of the physical parameters of the problem.

(ii) We also refer to [4] for results on the stability of exponential attractors under numerical approximations.

Now, let us consider the non-autonomous problem

$$\text{(1.5)} \qquad \begin{cases} \dfrac{dU}{dt} = F(t,U), & s < t < \infty, \\ U(s) = U_s, & -\infty < s < \infty, \end{cases}$$

in a Banach space X. Assuming that (1.5) is well-posed for $U_s \in X$, we have the family of solving operators

$$U(t,s)U_s : U_s \mapsto U(t), \quad -\infty < s \le t < \infty.$$

The family of operators has the properties

$$\text{(1.6)} \qquad U(s,s) = I, \quad -\infty < s < \infty,$$

$$\text{(1.7)} \qquad U(t,r) \circ U(r,s) = U(t,s), \quad -\infty < s \le r \le t < \infty.$$

It is then said that $U(t,s)$ forms an evolution operator or a process on the phase space X. We especially emphasize that the theory of attractors for non-autonomous systems is less understood than that for autonomous systems. We have essentially two approaches.

The first one, initiated by A. Haraux (see [21]) and further studied and developed by V.V. Chepyzhov and M.I. Vishik (see, e.g., [11, 12]), is based on the notion of a uniform attractor. The major drawback of this approach is that it leads, for general (translation-compact, see [11]) time dependences, to an artificial infinite dimensionality of the uniform attractor. This can already be seen for the following simple linear equation:

$$\frac{\partial u}{\partial t} - \Delta u = h(t), \quad u\big|_{\partial\Omega} = 0,$$

in a bounded smooth domain Ω, whose dynamics is simple, namely, one has one exponentially attracting trajectory. However, the uniform attractor has infinite dimension and infinite topological entropy (see [12]). However, for periodic and quasiperiodic time dependences, one has in general finite-dimensional uniform attractors (i.e., if the same is true for the corresponding autonomous system, see [10, 18]). Furthermore, one can derive sharp upper and lower bounds on the dimension of the uniform attractor, so that this approach is quite relevant in that case. We can note that, as in the autonomous case, an exponential attractor in this setting always contains the uniform attractor and, again, one has, for general time dependences, an artificial infinite dimensionality.

The second approach is based on the notion of a pullback attractor (see, e.g. [8, 22] and the references therein). In that case, one has a time dependent attractor $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$, contrary to the uniform attractor which is time independent.

Definition 1.3. A family $\{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is a pullback attractor for the evolution operator $U(t,s)$ on X if:

(i) Each $\mathcal{A}(t)$ is a compact set of X.
(ii) It is invariant, i.e., $U(t,s)\mathcal{A}(s) = \mathcal{A}(t)$ for all $-\infty < s \le t < \infty$.
(iii) It satisfies the following pullback attraction property:

$$\forall B \subset X \text{ bounded}, \lim_{s\to+\infty} h(U(t,t-s)B, \mathcal{A}(t)) = 0.$$

One can prove that, in general, $\mathcal{A}(t)$ has finite dimension for every $t \in R$. We also note that it follows from the above definition that the pullback attractor, if it exists, is unique. Furthermore, if the system is autonomous, then one recovers the global attractor. Now, the attraction property essentially means that, at time t, the attractor $\mathcal{A}(t)$ attracts the bounded sets of initial data coming from the past (i.e., from $-\infty$). However, in (iii), the rate of attraction is not uniform in t, so that the forward convergence is not true in general (see nevertheless [7, 9] for cases where the forward convergence can be proven).

In this paper, we want to introduce a version of exponential attractor for non-autonomous equations as a time dependent set satisfying certain natural assumptions. Our definition is stated as follows.

Definition 1.4. A family $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$ is an exponential attractor for the evolution operator $U(t,s)$ on X if:

(i) Each $\mathcal{M}(t)$ is a compact set of X and its fractal dimension is finite and uniformly bounded, i.e., $\sup_{t\in\mathbb{R}} \dim \mathcal{M}(t) < \infty$.
(ii) It is positively invariant, i.e., $U(t,s)\mathcal{M}(s) \subset \mathcal{M}(t)$ for all $-\infty < s \le t < \infty$.
(iii) There exist an exponent $\alpha > 0$ and two monotonic functions Q and τ such that

$$\forall B \subset X \text{ bounded}, \ h(U(t,s)B, \mathcal{M}(t)) \le Q(\|B\|_X)e^{-\alpha(t-s)}, \qquad s \in \mathbb{R},\ s + \tau(\|B\|_X) \le t < \infty.$$

The first purpose of this paper is then to show construction of exponential attractors for non-autonomous systems. To this end, we will assume existence of a family of bounded sets $\mathcal{X}(t)$, $t \in \mathbb{R}$, which is positively invariant and absorbs all bounded sets, and will generalize (1.3) into the form

$$\|U(\tau^*+s,s)U_0 - U(\tau^*+s,s)V_0\|_Z \le c\|U_0 - V_0\|_X, \quad U_0, V_0 \in \mathcal{X}(s), \text{ for all } s \in \mathbb{R}, \tag{1.8}$$

where $\tau^* > 0$ is some fixed constant. (Actually our assumption will be of the more general form, see (2.1) and (2.2).) This condition together with some minor ones in fact enables us to generalize the method of construction for autonomous systems (due to [16]) for non-autonomous ones. Our exponential attractor $\mathcal{M}(t)$ then depends on t continuously if $t \ne n\tau^*$, $n \in \mathbb{Z}$, and is right continuous at $t = n\tau^*$, $n \in \mathbb{Z}$. Left discontinuity of $\mathcal{M}(t)$ at time $n\tau^*$ comes completely from a technical reason. We notice in applications that (1.8) is actually verified for any τ^* contained in some interval (τ_0, τ_1), where $0 < \tau_0 < \tau_1$, which means that, even if $\mathcal{M}(t)$ is left discontinuous at $n\tau^*$, it is possible to choose another $\overline{\tau}^*$ in order to construct another exponential attractor $\overline{\mathcal{M}}(t)$ which is now continuous at the $n\tau^*$.

The second purpose is to apply this construction to some non-autonomous chemotaxis system. For autonomous chemotaxis systems, we have already constructed exponential attractors in the papers [1, 26] (cf. also [30, Chapter 12]). In [2] we estimated their fractal dimensions from below and showed that, if the chemotaxis parameter becomes large, then the fractal dimensions also increase and finally tend to infinity. Meanwhile, in [17] we proved that the exponential attractor can depend continuously with respect to the chemotaxis parameter. In this paper, we will consider a time dependent sensitivity function. Under reasonable assumptions on the function, our general result will be applied for constructing exponential attractors as before. Our result seems to be in good agreement with the former ones in the sense that the dimension of $\mathcal{M}(t)$ is uniformly bounded and is continuous with respect to the variable t.

2. Construction of Exponential Attractors

Let X be a Banach space with norm $\|\cdot\|_X$. Let $\mathcal{K}$ be a subset of X which is a metric space equipped with the distance $d(U,V) = \|U - V\|_X$. We consider a family of nonlinear operators $U(t,s)$ acting on $\mathcal{K}$ defined for

$$(t,s) \in \Delta = \{(t,s);\ -\infty < s \le t < \infty\}.$$

We assume that $U(t,s)$ has the properties (1.6) and (1.7) on $\mathcal{K}$. A family of $U(t,s)$ having these properties is called an evolution operator or a process on the space $\mathcal{K}$. We assume also that $U(t,s)$ is continuous in the sense that

$$\text{the mapping}\quad G\colon \Delta \times \mathcal{K} \to \mathcal{K},\ ((t,s),U_0) \mapsto U(t,s)U_0 \quad \text{is continuous.}$$

Such an evolution operator is said simply to be continuous on $\mathcal{K}$. When $U(t,s)$ is a continuous evolution operator on $\mathcal{K}$, the triplet $(U(t,s),\mathcal{K},X)$ is called a non-autonomous dynamical system, and $\mathcal{K}$ and X are called the phase space and the universal space, respectively. The trace of a function $U(\cdot,s)U_0$ for $t \in [s,\infty)$ in the space $\mathcal{K}$ is called a trajectory starting from $U_0 \in \mathcal{K}$ at initial time $s \in \mathbb{R}$.

We now restate the definition of exponential attractors. (Note that in Definition 1.4, $\mathcal{K}$ coincides with X).

Definition 2.1. A family $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$ of subsets of $\mathcal{K}$ is called an exponential attractor for $(U(t,s),\mathcal{K},X)$ if:

- (i) Each $\mathcal{M}(t)$ is a compact set of X and its fractal dimension is finite and uniformly bounded, i.e., $\sup_{t\in\mathbb{R}} \dim \mathcal{M}(t) < \infty$.
- (ii) It is positively invariant, i.e., $U(t,s)\mathcal{M}(s) \subset \mathcal{M}(t)$ for all $(t,s) \in \Delta$.
- (iii) There exist an exponent $\alpha > 0$ and two monotonic functions Q and τ such that

$$\forall B \subset \mathcal{K} \text{ bounded},\ h(U(t,s)B,\mathcal{M}(t)) \le Q(\|B\|_X)e^{-\alpha(t-s)},\qquad s \in \mathbb{R},\ s + \tau(\|B\|_X) \le t < \infty.$$

In order to construct exponential attractors, we have to assume existence of a family $\{\mathcal{X}(t)\}_{t\in\mathbb{R}}$ of bounded closed subsets of $\mathcal{K}$ with the following properties:

- (1) The diameter $\|\mathcal{X}(t)\|_X$ of $\mathcal{X}(t)$ is uniformly bounded, i.e., $\sup_{t\in\mathbb{R}} \|\mathcal{X}(t)\|_X = R < \infty$.
- (2) It is positively invariant, i.e., $U(t,s)\mathcal{X}(s) \subset \mathcal{X}(t)$ for all $(t,s) \in \Delta$.
- (3) It is absorbing in the sense that there is a monotonic function σ such that

$$\forall B \subset \mathcal{K} \text{ bounded},\ U(t,s)B \subset \mathcal{X}(t),\qquad s \in \mathbb{R},\ s + \sigma(\|B\|_X) \le t < \infty.$$

- (4) There is $\tau^* > 0$ such that, for every $s \in \mathbb{R}$, $U(\tau^* + s, s)$ is a compact perturbation of contraction on $\mathcal{X}(s)$ in the sense that

$$\|U(\tau^*+s,s)U_0 - U(\tau^*+s,s)V_0\|_X \le \delta\|U_0 - V_0\|_X + \|K(s)U_0 - K(s)V_0\|_X,\qquad U_0,\ V_0 \in \mathcal{X}(s), \tag{2.1}$$

where δ is a constant such that $0 \le \delta < \frac{1}{2}$ and where $K(s)$ is an operator from $\mathcal{X}(s)$ into another Banach space Z which is embedded compactly in X and satisfies a Lipschitz condition

$$\|K(s)U_0 - K(s)V_0\|_Z \le L_1\|U_0 - V_0\|_X,\qquad U_0,\ V_0 \in \mathcal{X}(s), \tag{2.2}$$

with some constant $L_1 > 0$ independent of s.

(5) It holds for any $s \in \mathbb{R}$ and any $\tau \in [0, \tau^*]$ that

$$\|U(\tau + s, s)U_0 - U(\tau + s, s)V_0\|_X \le L_2 \|U_0 - V_0\|_X, \qquad U_0, V_0 \in \mathcal{X}(s), \tag{2.3}$$

with some constant $L_2 > 0$ independent of s and τ.

Theorem 2.1. *Let $(U(t, s), \mathcal{K}, X)$ be a non-autonomous dynamical system in X. Assume that the conditions (1)$\sim$(5) be satisfied. Then, one can construct an exponential attractor $\{\mathcal{M}(t)\}_{t \in \mathbb{R}}$ for $(U(t, s), \mathcal{K}, X)$.*

Proof. For $m, n \in \mathbb{Z}$ with $m \le n$, put $U^*(n, m) = U(n\tau^*, m\tau^*)$. Let us first consider a discrete non-autonomous dynamical system $(U^*(n, m), \mathcal{K}, X)$. In the first three steps of proof, we will construct an exponential attractor for the discrete system $(U^*(n, m), \mathcal{K}, X)$.

For $n \in \mathbb{Z}$, put $\mathcal{X}^*(n) = \mathcal{X}(n\tau^*)$. A family $\mathcal{M}^*(n)$, $n \in \mathbb{Z}$, of compact sets of X is called an exponential attractor for $(U^*(n, m), \mathcal{K}, X)$ if $\mathcal{M}(n)$ satisfies:

(1) The fractal dimension of $\mathcal{M}^*(n)$ is uniformly bounded for n, i.e., $\sup_{n \in \mathbb{Z}} \dim \mathcal{M}^*(n) \le c_1^*$.
(2) $U^*(n, m)\mathcal{M}^*(m) \subset \mathcal{M}^*(n)$ for all $-\infty < m \le n < \infty$.
(3) For some $0 < a < 1$, it holds true that

$$h(U^*(n, m)\mathcal{X}^*(m), \mathcal{M}^*(n)) \le Ra^{n-m}, \qquad -\infty < m \le n < \infty.$$

The last step will be devoted to handling the continuous case.

Step 1. Let θ be any number such that $0 < \theta < \frac{1-2\delta}{2L_1}$ and let $a_\theta = 2(\delta + \theta L_1)$. Clearly, $0 < a_\theta < 1$. The purpose of this step is to construct, for any $-\infty < m \le n < \infty$, a covering of $U^*(n, m)\mathcal{X}^*(m)$ by N_θ^{n-m}-closed balls of X with uniform radius Ra_θ^{n-m} centered at points in $U^*(n, m)\mathcal{X}^*(m)$, where N_θ is a minimal number of closed balls of X with radius θ which cover the closed unit ball $\overline{B}^Z(0; 1)$ of Z centered at 0. That is, for $-\infty < m \le n < \infty$,

$$U^*(n, m)\mathcal{X}^*(m) \subset \bigcup_{i=1}^{N_\theta^{n-m}} \overline{B}(W_{n,m,i}; Ra^{n-m}), \qquad \text{where } a = a_\theta, \tag{2.4}$$

with $W_{n,m,i} \in U^*(n, m)\mathcal{X}^*(m)$, $1 \le i \le N_\theta^{n-m}$.

Let us construct the covering (2.4) by induction on n (m being fixed). If $n = m$, then we can take $W_{m,m,1} \in \mathcal{X}^*(m)$ arbitrarily. (Remember the condition (1) for $\mathcal{X}(t)$.) Assume that we have the covering (2.4) for $n \ge m$. Then,

$$\begin{aligned} U^*(n+1, m)\mathcal{X}^*(m) &= U^*(n+1, n)U^*(n, m)\mathcal{X}^*(m) \\ &\subset \bigcup_{i=1}^{N_\theta^{n-1-m}} U^*(n+1, n)\Big(\overline{B}(W_{n,m,i};\ Ra^{n-m}) \cap U^*(n, m)\mathcal{X}^*(m)\Big). \end{aligned}$$

So, it suffices to cover each set

$$U^*(n+1, n)\big(\overline{B}(W_{n,m,i}; Ra^{n-m}) \cap U^*(n, m)\mathcal{X}^*(m)\big)$$

by N_θ-closed balls with the radius Ra^{n+1-m} centered in $U^*(n+1, m)\mathcal{X}^*(m)$. Using (2.2) with $s = n\tau^*$, we see that

$$K(n\tau^*)\Big(\overline{B}(W_{n,m,i};\ Ra^{n-m}) \cap U^*(n, m)\mathcal{X}^*(m)\Big) \subset \overline{B}^Z(K(n\tau^*)W_{n,m,i};\ L_1 Ra^{n-m}).$$

Then, by the compactness of closed bounded balls of Z in X, the last ball can be covered by N_θ-closed balls of X in such a way that

$$\overline{B}^Z(K(n\tau^*)W_{n,m,i};\ L_1Ra^{n-m}) \subset \bigcup_{j=1}^{N_\theta} \overline{B}(\widetilde{V}_{n,m,i,j};\ \theta L_1 Ra^{n-m})$$

with centers $\widetilde{V}_{n,m,i,j} \in X$, $1 \le j \le N_\theta$, and radius $\theta L_1 Ra^{n-m}$. Therefore,

$$K(n\tau^*)\Big(\overline{B}(W_{n,m,i};\ Ra^{n-m}) \cap U^*(n,m)\mathfrak{X}^*(m)\Big) \subset \bigcup_{j=1}^{N_\theta} \overline{B}(\widetilde{V}_{n,m,i,j};\ \theta L_1 Ra^{n-m}). \tag{2.5}$$

We are here allowed to assume that

$$K(n\tau^*)\Big(\overline{B}(W_{n,m,i};\ Ra^{n-m}) \cap U^*(n,m)\mathfrak{X}^*(m)\Big) \cap \overline{B}(\widetilde{V}_{n,m,i,j};\ \theta L_1 Ra^{n-m}) \ne \emptyset$$

for every j, since, if not for some j's, we can exclude these balls from the covering. So, we can choose for each j, a point $V_{n,m,i,j}$ such that

$$\begin{aligned} V_{n,m,i,j} &\in \overline{B}(W_{n,m,i};\ Ra^{n-m}) \cap U^*(n,m)\mathfrak{X}^*(m), \\ K(n\tau^*)V_{n,m,i,j} &\in \overline{B}(\widetilde{V}_{n,m,i,j};\ \theta L_1 Ra^{n-m}). \end{aligned} \tag{2.6}$$

Therefore, from (2.5) it is deduced that

$$K(n\tau^*)\Big(\overline{B}(W_{n,m,i};\ Ra^{n-m}) \cap U^*(n,m)\mathfrak{X}^*(m)\Big) \subset \bigcup_{j=1}^{N_\theta} \overline{B}(K(n\tau^*)V_{n,m,i,j};\ 2\theta L_1 Ra^{n-m}).$$

Let now $U \in \overline{B}(W_{n,m,i};\ Ra^{n-m}) \cap U^*(n,m)\mathfrak{X}^*(m)$. Then, there is some j such that $K(n\tau^*)U \in \overline{B}(K(n\tau^*)V_{n,m,i,j}; 2\theta L_1 Ra^{n-m})$. As a consequence, it follows from (2.1) that

$$\begin{aligned} \|U^*(n+1,n)U - U^*(n+1,n)V_{n,m,i,j}\|_X &\le \delta\|U - V_{n,m,i,j}\|_X \\ &\qquad + \|K(n\tau^*)U - K(n\tau^*)V_{n,m,i,j}\|_X \\ &\le \delta\|U - V_{n,m,i,j}\|_X + 2\theta L_1 Ra^{n-m}. \end{aligned}$$

In addition, by (2.6),

$$\|U - V_{n,m,i,j}\|_X \le \|U - W_{n,m,i}\|_X + \|W_{n,m,i,j} - V_{n,m,i,j}\|_X \le 2Ra^{n-m}.$$

So that, $\|U^*(n+1,n)U - U^*(n+1,n)V_{n,m,i,j}\|_X \le 2(\delta + \theta L_1)Ra^{n-m} = Ra^{n+1-m}$. Hence, it holds that

$$\begin{aligned} U^*(n+1,n)\Big(\overline{B}(W_{n,m,i};\ Ra^{n-m}) \cap U^*(n,m)\mathfrak{X}^*(m)\Big) \\ \subset \bigcup_{j=1}^{N_\theta} \overline{B}(U^*(n+1,n)V_{n,m,i,j};\ Ra^{n+1-m}). \end{aligned} \tag{2.7}$$

We observe from (2.6) that $U^*(n+1,n)V_{n,m,i,j} \in U^*(n+1,m)\mathfrak{X}^*(m)$.

Covering of the form (2.7) can of course be constructed for all other balls. Therefore, the desired covering (2.4) for $n+1$ is obtained by locating central points as

$$\begin{aligned} \{W_{n+1,m,i};\ 1 \le i \le N_\theta^{n+1-m}\} = \{U^*(n+1,n)V_{n,m,i,j};\ 1 \le i \le N_\theta^{n-m},\ 1 \le j \le N_\theta\} \\ \subset U^*(n+1,m)\mathfrak{X}^*(m). \end{aligned}$$

Step 2. For $-\infty < m \le n < \infty$, we put

$$E_m(n) = \{U^*(n, m+k)W_{m+k,m,i_k};\ 0 \le k \le n-m,\ 1 \le i_k \le N_\theta^k\}.$$

It is clear by definition that

$$E_m(n) \subset U^*(n,m)\mathcal{X}^*(m) \subset \mathcal{X}^*(n).$$

In addition, for $n \le p < \infty$,

$$U^*(p,n)E_m(n) \subset E_m(p). \tag{2.8}$$

We then set, for each $-\infty < n < \infty$,

$$\mathcal{M}^*(n) = \overline{\bigcup_{m=-\infty}^{n} E_m(n)}. \tag{2.9}$$

This family $\mathcal{M}^*(n)$, $-\infty < n < \infty$, will indeed give an exponential attractor for $(U^*(n,m), \mathcal{K}, X)$.

Let us estimate in this step the fractal dimension of $\mathcal{M}^*(n)$. Let n be fixed and let $0 < \varepsilon < 1$ be any number. Let m_ε $(\le n)$ be the largest integer such that $Ra^{n-m} \le \varepsilon$, i.e., $m_\varepsilon \le \frac{\log(R^{-1}a^{-n}\varepsilon)}{-\log a}$. For all m's such that $-\infty < m \le m_\varepsilon$, we have

$$E_m(n) \subset U^*(n,m)\mathcal{X}^*(m) \subset U^*(n,m_\varepsilon)U^*(m_\varepsilon,m)\mathcal{X}^*(m) \subset U^*(n,m_\varepsilon)\mathcal{X}^*(m_\varepsilon).$$

Therefore, by (2.4), we deduce that the set $\overline{\bigcup_{m=-\infty}^{m_\varepsilon} E_m(n)}$ is covered by $N_\theta^{n-m_\varepsilon}$-closed balls with radius ε. Meanwhile, for each $m_\varepsilon < m \le n$, $E_m(n)$ is a finite set. Hence, $\mathcal{M}^*(n)$ is a precompact set of X and actually is a compact set of X. Denote by $N(\varepsilon)$ the minimal number of balls with radius ε which can cover $\mathcal{M}^*(n)$. Then,

$$\begin{aligned} N(\varepsilon) &\le N_\theta^{n-m_\varepsilon} + \sum_{m=m_\varepsilon+1}^{n} \#E_m(n) \\ &= N_\theta^{n-m_\varepsilon} + \sum_{m=m_\varepsilon+1}^{n}\sum_{k=0}^{n-m} N_\theta^k \le (n-m_\varepsilon)N_\theta^{n-m_\varepsilon}. \end{aligned}$$

Since $\varepsilon < Ra^{n-m_\varepsilon-1}$, it follows that

$$\frac{\log N(\varepsilon)}{-\log \varepsilon} \le \frac{(n-m_\varepsilon)\log N_\theta + \log(n-m_\varepsilon)}{-(n-m_\varepsilon-1)\log a - \log R}.$$

Letting $\varepsilon \to 0$, we conclude that $\dim \mathcal{M}^*(n) \le \frac{\log N_\theta}{-\log a}$.

Step 3. It is seen by (2.8) that

$$\begin{aligned} U^*(p,n)\mathcal{M}^*(n) &= U^*(p,n)\overline{\bigcup_{m=-\infty}^{n} E_m(n)} \\ &\subset \overline{U^*(p,n)\bigcup_{m=-\infty}^{n} E_m(n)} \subset \overline{\bigcup_{m=-\infty}^{n} E_m(p)} \subset \mathcal{M}^*(p). \end{aligned} \tag{2.10}$$

Meanwhile, it is seen by (2.4) that $h(U^*(n,m)\mathcal{X}^*(m), \mathcal{M}^*(n)) \le Ra^{n-m}$ since $W_{n,m,i} \in \mathcal{M}^*(n)$ for $1 \le i \le N_\theta^{n-m}$.

We have thus verified that $\mathcal{M}^*(n)$ is an exponential attractor for $(U^*(n,m), \mathcal{K}, X)$.

Step 4. Let us now consider the continuous dynamical system $(U(t,s), \mathcal{K}, X)$. For $-\infty < t < \infty$, let n be the integer such that $n\tau^* \le t < (n+1)\tau^*$. We then set

$$\mathcal{M}(t) = U(t, n\tau^*)\mathcal{M}^*(n), \qquad n\tau^* \le t < (n+1)\tau^*.$$

Since $U(t, n\tau^*)$ is a continuous mapping from $\mathcal{K}$ into X, the image $\mathcal{M}(t)$ of a compact set $\mathcal{M}^*(n)$ by $U(t, n\tau^*)$ is also a compact set of X. Similarly, since $U(t, n\tau^*)$ is Lipschitz continuous due to (2.3), the fractal dimension of $\mathcal{M}(t)$ is finite and does not exceed $\dim \mathcal{M}^*(n)$, namely, $\dim \mathcal{M}(t) \le \frac{\log N_\theta}{-\log a}$ for any t.

For $-\infty < s < t < \infty$, let $m\tau^* \le s < (m+1)\tau^*$ and $n\tau^* \le t < (n+1)\tau^*$ with integers $m \le n$. Then, by (2.10),

$$\begin{aligned} U(t,s)\mathcal{M}(s) &= U(t, n\tau^*)U(n\tau^*, s)U(s, m\tau^*)\mathcal{M}^*(m) \\ &= U(t, n\tau^*)U^*(n,m)\mathcal{M}^*(m) \subset U(t, n\tau^*)\mathcal{M}^*(n) = \mathcal{M}(t). \end{aligned}$$

Let $U_s \in \mathcal{X}(s)$. We write

$$d(U(t,s)U_s, \mathcal{M}(t)) = d(U(t, n\tau^*)U(n\tau^*, (m+1)\tau^*)U((m+1)\tau^*, s)U_s, U(t, n\tau^*)\mathcal{M}^*(n)).$$

Noting that $U_{m+1} = U((m+1)\tau^*, s)U_s \in \mathcal{X}^*(m+1)$, we obtain that

$$d(U(t,s)U_s, \mathcal{M}(t)) \le L_2 d(U^*(n, m+1)U_{m+1}, \mathcal{M}^*(n)) \le L_2 R a^{n-m-1}.$$

Hence it holds true that

$$h(U(t,s)\mathcal{X}(s), \mathcal{M}(t)) \le L_2 R a^{-2} e^{-\alpha(t-s)}, \qquad -\infty < s \le t < \infty,$$

with $\alpha = \frac{-\log a}{\tau^*}$.

We have thus verified that the family of sets $\mathcal{M}(t)$, $-\infty < t < \infty$, enjoys the desired properties. □

3. Continuous Dependence of $\mathcal{M}(t)$ in t

We are concerned with continuity of $\mathcal{M}(t)$ with respect to the variable t. We make the following assumptions. For each fixed $-\infty < t < \infty$,

$$\lim_{t' \searrow t} \sup_{U_t \in \mathcal{X}(t)} \|[U(t', t) - 1]U_t\|_X = 0. \tag{3.1}$$

For each fixed $-\infty < t < \infty$,

$$\lim_{t' \nearrow t} \sup_{U_{t'} \in \mathcal{X}(t')} \|[U(t, t') - 1]U_{t'}\|_X = 0. \tag{3.2}$$

Theorem 3.1. *Let $\{\mathcal{M}(t)\}_{t\in\mathbb{R}}$ be the exponential attractor constructed in Theorem 2.1. Let $U(t,s)$ satisfy* (3.1) *and* (3.2)*, too. Then, $\mathcal{M}(t)$ is right continuous at any $t \in \mathbb{R}$, i.e., $\lim_{t'\searrow t} d(\mathcal{M}(t'), \mathcal{M}(t)) = 0$. If $t \ne n\tau^*$ for any $n \in \mathbb{Z}$, then $\mathcal{M}(t)$ is left continuous, too, i.e., $\lim_{t'\nearrow t} d(\mathcal{M}(t'), \mathcal{M}(t)) = 0$. If $t = n\tau^*$ with some $n \in \mathbb{Z}$, then $\mathcal{M}(t)$ is at least left outer continuous, i.e., $\lim_{t'\nearrow t} h(\mathcal{M}(t'), \mathcal{M}(t)) = 0$.*

Proof. Let $n\tau^* \le t < t' < (n+1)\tau^*$. Then, $\mathcal{M}(t') = U(t', t)\mathcal{M}(t)$. For any $U_{t'} \in \mathcal{M}(t')$, there is a point $U_t \in \mathcal{M}(t)$ such that $U_{t'} = U(t', t)U_t$. Therefore,

$$d(U_{t'}, \mathcal{M}(t)) \le d(U(t', t)U_t, U_t) \le \sup_{U_t \in \mathcal{M}(t)} \|[U(t', t) - 1]U_t\|_X.$$

Consequently,

$$d(\mathcal{M}(t'), \mathcal{M}(t)) \leq \sup_{U_t \in \mathcal{X}(t)} \|[U(t', t) - 1]U_t\|_X.$$

In the meantime, let $U_t \in \mathcal{M}(t)$. Then,

$$d(U_t, \mathcal{M}(t')) \leq d(U_t, U(t', t)U_t) \leq \sup_{U_t \in \mathcal{M}(t)} \|[U(t', t) - 1]U_t\|_X.$$

Consequently,

$$d(\mathcal{M}(t), \mathcal{M}(t')) \leq \sup_{U_t \in \mathcal{X}(t)} \|[U(t', t) - 1]U_t\|_X.$$

Therefore, (3.1) implies $\lim_{t' \to t} d(\mathcal{M}(t'), \mathcal{M}(t)) = 0$.

Let $t \neq n\tau^*$ for any $n \in \mathbb{Z}$. Let indeed $n\tau^* < t' < t < (n+1)\tau^*$. Then, we have $\mathcal{M}(t) = U(t, t')\mathcal{M}(t')$. By the same arguments as above, we can conclude from (3.2) that $\lim_{t' \nearrow t} d(\mathcal{M}(t'), \mathcal{M}(t)) = 0$.

Let $t = n\tau^*$ with $n \in \mathbb{Z}$. Let $(n-1)\tau^* < t' < t = n\tau^*$. Since $U(t, t')\mathcal{M}(t') \subset \mathcal{M}(t)$, we deduce from (3.2) that $\lim_{t' \nearrow t} h(\mathcal{M}(t'), \mathcal{M}(t)) = 0$. □

4. Non-autonomous Chemotaxis System

We consider the initial-boundary values problem for non-autonomous chemotaxis growth equations

$$\text{(4.1)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = a\Delta u - \nabla \cdot [u \nabla \chi(t, \rho)] + f(t, u) & \text{in} \quad \Omega \times (s, \infty), \\ \dfrac{\partial \rho}{\partial t} = b\Delta \rho - c\rho + \nu u & \text{in} \quad \Omega \times (s, \infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \rho}{\partial n} = 0 & \text{on} \quad \partial\Omega \times (s, \infty), \\ u(x, s) = u_s(x), \quad \rho(x, s) = \rho_s(x) & \text{in} \quad \Omega, \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^2$ with initial time $s \in \mathbb{R}$.

We assume that Ω is a two-dimensional bounded domain with sufficiently smooth boundary $\partial\Omega$, say of $\mathcal{C}^4$ class. For each t, the sensitivity function $\chi(t, \rho)$ is a $\mathcal{C}^3$ function for $0 \leq \rho < \infty$ satisfying

$$\text{(4.2)} \qquad \left| \frac{\partial^i \chi}{\partial \rho^i}(t, \rho) \right| \leq C_1, \qquad -\infty < t < \infty,\ 0 \leq \rho < \infty,\ i = 1, 2, 3,$$

with some constant $C_1 > 0$. The partial derivatives also satisfy uniform Lipschitz conditions

$$\text{(4.3)} \qquad \left| \frac{\partial^i \chi}{\partial \rho^i}(s, \rho) - \frac{\partial^i \chi}{\partial \rho^i}(t, \rho) \right| \leq C_2 |t - s|, \qquad -\infty < s, t < \infty, 0 \leq \rho < \infty, i = 1, 2, 3,$$

with some constant $C_2 > 0$. The growth function $f(t, u)$ is a continuous function for $(t, u) \in \mathbb{R} \times \mathbb{R}_+$ satisfying

$$\text{(4.4)} \qquad c_1 u - c_2 u^2 \leq f(t, u) \leq c_3 u - c_4 u^2$$

with some positive constants $c_i > 0\,(i = 1, 2, 3, 4)$. We assume also a Lipschitz condition of the form

$$(4.5)\quad |f(s,u) - f(t,v)| \le C_3(u+v+1) \times [(u+v+1)|t-s| + |u-v|], \qquad -\infty < s,\, t < \infty,\, 0 \le u,\, v < \infty,$$

with some constant $C_3 > 0$.

We will treat this problem in the product space

$$X = \left\{ U = \begin{pmatrix} u \\ \rho \end{pmatrix};\ u \in L_2(\Omega) \quad \text{and} \quad \rho \in H_N^2(\Omega) \right\}. \tag{4.6}$$

As for the space of initial functions, we set

$$\mathcal{K} = \left\{ U_s = \begin{pmatrix} u_s \\ \rho_s \end{pmatrix};\ 0 \le u_s \in L_2(\Omega) \quad \text{and} \quad 0 \le \rho_s \in H_N^2(\Omega) \right\}. \tag{4.7}$$

4.1. **Local Solutions.** We want to appeal to the theory of nonlinear abstract parabolic evolution equations (see [30]). Problem (4.1) is formulated as the Cauchy problem for a non-autonomous semilinear evolution equation

$$\begin{cases} \dfrac{dU}{dt} + AU = F(t,U), \qquad s < t < \infty, \\ U(s) = U_s, \end{cases} \tag{4.8}$$

in the product space X given by (4.6). Here, A is a matrix linear operator of X given by

$$A = \begin{pmatrix} A_1 & 0 \\ -\nu & A_2 \end{pmatrix},$$

where A_1 (resp. A_2) is a realization of the elliptic operator $-a\Delta + 1$ (resp. $-b\Delta + c$) in $L_2(\Omega)$ under the Neumann boundary conditions on $\partial\Omega$ and is a positive definite self-adjoint operator of $L_2(\Omega)$ with domain $\mathcal{D}(A_1) = \mathcal{D}(A_2) = H_N^2(\Omega)$. But, since the underlying space for the equation of ρ is the space $H_N^2(\Omega)$ (see (4.6)), A_2 is actually an operator from $\mathcal{D}(A_2^2)$ into $\mathcal{D}(A_2)$. The nonlinear operator $F(t,U)$ is given by

$$F(t,U) = \begin{pmatrix} -\nabla\cdot[u\nabla\chi(t,\operatorname{Re}\rho)] + f(t,\operatorname{Re}u) + u \\ 0 \end{pmatrix}, \qquad U = \begin{pmatrix} u \\ \rho \end{pmatrix}.$$

The initial value is given $U_s = {}^t(u_s, \rho_s) \in \mathcal{K}$.

We shall use the standard techniques of reducing the non-autonomous problems to autonomous ones by introducing a new unknown function $\tau = \tau(t)$. Namely, we rewrite (4.8) into the form

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} \tau \\ U \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} \tau \\ U \end{pmatrix} = \begin{pmatrix} \tau + 1 \\ F(\operatorname{Re}\tau, U) \end{pmatrix}, \qquad s < t < \infty, \\ \begin{pmatrix} \tau \\ U \end{pmatrix}(s) = \begin{pmatrix} s \\ U_s \end{pmatrix}, \end{cases}$$

in the product space $\mathbb{X}$ of $\mathbb{C}$ and X. Then, we have the Cauchy problem of the form

$$\begin{cases} \dfrac{d\widetilde{U}}{dt} + \widetilde{A}\widetilde{U} = \widetilde{F}(\widetilde{U}), & s < t < \infty, \\ \widetilde{U}(s) = \widetilde{U}_s. \end{cases} \tag{4.9}$$

Here, $\widetilde{U} = {}^t(\tau, U) \in \mathbb{X}$ and $\widetilde{A} = \operatorname{diag}\{1, A\}$ is a matrix operator of $\mathbb{X}$. The nonlinear operator $\widetilde{F}$ is defined by

$$\widetilde{F}(\widetilde{U}) = \begin{pmatrix} \tau + 1 \\ F(\operatorname{Re}\tau, U) \end{pmatrix}, \qquad \widetilde{U} = \begin{pmatrix} \tau \\ U \end{pmatrix}. \tag{4.10}$$

Finally, the initial value is given by $\widetilde{U}_s = {}^t(s, U_s) \in \mathbb{R} \times \mathcal{K}$.

Let us now apply the general results for semilinear abstract parabolic evolution equations, say [30, Theorem 4.4], in order to construct local solutions to (4.9). In fact, we can verify that $\widetilde{F}(U)$ defined by (4.10) fulfills the condition [30, (4.21)] with $\frac{1}{2} < \eta < 1$ by the analogous arguments as in [1, Section 6] or [30, Section 12.1.2]. As a result, we conclude that, for any initial value $\widetilde{U}_s \in \mathbb{R} \times \mathcal{K}$, (4.9) has a unique local solution in the function space:

$$\widetilde{U} \in \mathcal{C}((s, s + T_{U_s}]; \mathcal{D}(\widetilde{A})) \cap \mathcal{C}([s, s + T_{U_s}]; \mathbb{X}) \cap \mathcal{C}^1((s, s + T_{U_s}]; \mathbb{X}),$$

here $T_{U_s} > 0$ is determined by the norm $\|U_s\|_X$ alone. In addition,

$$(t - s)\|\widetilde{A}\widetilde{U}(t)\|_{\mathbb{X}} + \|\widetilde{U}(t)\|_{\mathbb{X}} \le C_{U_s}, \qquad s < t \le s + T_{U_s}. \tag{4.11}$$

By definition, the local solution to (4.9) and hence to the original problem (4.8) (equally, (4.1)) is given by

$$U(t) = \operatorname{pr}_2 \widetilde{U}(t), \qquad s < t \le s + T_{U_s}, \tag{4.12}$$

where $\operatorname{pr}_2 : \mathbb{X} \to X$ is the projection from $\mathbb{X}$ onto X. (4.11) then yields that

$$(t - s)\|AU(t)\|_X + \|U(t)\|_X \le C_{U_s}, \qquad s < t \le s + T_{U_s}. \tag{4.13}$$

It is easy to verify that $u_s \ge 0$ and $\rho_s \ge 0$ imply that the local solution to (4.8) also satisfies $u(t) \ge 0$ and $\rho(t) \ge 0$ for every $s < t \le s + T_{U_s}$, see [26, Theorem 3.5] or [30, Section 12.1.3].

Let $0 < R < \infty$. Let $\mathcal{K}_R = \mathcal{K} \cap \overline{B}^X(0; R)$, where $\overline{B}^X(0; R)$ denotes the closed ball of X centred at 0 with radius R. For each $U_s \in \mathcal{K}_R$, (4.9) has a unique local solution on an interval $[s, s + T_R]$, where $T_R > 0$ is determined by R alone. We can then verify the Lipschitz continuity of the local solutions with respect to the initial data. Thanks to [30, Theorem 4.5], we have

$$\begin{aligned} &(t - s)^\eta \|A^\eta[U_1(t) - U_2(t)]\|_X + \|U_1(t) - U_2(t)\|_X \\ &\qquad\qquad \le C_R \|U_s^1 - U_s^2\|_X, \qquad s < t \le s + T_R, \end{aligned} \tag{4.14}$$

where $U_1(t)$ (resp. $U_2(t)$) is a local solution to (4.8) for initial function $U_s^1 \in \mathcal{K}_R$ (resp. $U_s^2 \in \mathcal{K}_R$).

4.2. **Global Solutions.** We consider Problem (4.8). For any $U_s \in \mathcal{K}$, we have already constructed a local solution on an interval $[s, T_{U_s}]$. Let $U = {}^t(u, \rho)$ be any extension of this local solution in the function space:

$$0 \le u \in \mathcal{C}((s, s+T_U]; H^2_N(\Omega)) \cap \mathcal{C}([s, s+T_U]; L_2(\Omega)) \cap \mathcal{C}^1((s, s+T_U]; L_2(\Omega)),$$
$$0 \le \rho \in \mathcal{C}((s, s+T_U]; H^4_{N^2}(\Omega)) \cap \mathcal{C}([s, s+T_U]; H^2_N(\Omega)) \cap \mathcal{C}^1((s, s+T_U]; H^2_N(\Omega)),$$

U being defined on $[s, s+T_U]$. Then, repeating the similar arguments as in [26, Section 4] or [30, Section 12.3.2], we can establish a priori estimates

$$\|U(t)\|_X \le p(\|U_s\|_X), \qquad s \le t \le s + T_U, \tag{4.15}$$

here $p(\cdot)$ denotes some specific continuous increasing function which is independent of $U(\cdot)$.

This a priori estimate shows that the local solution on $[s, s+T_{U_s}]$ mentioned above can be extended on an interval $[s, s+T_{U_s}+\tau]$, $\tau > 0$ being dependent only on $p(\|U_s\|_X)$ and independent of $s + T_{U_s}$. We will repeat such a procedure. Each step the time $\tau > 0$ is determined by $p(\|U_s\|_X)$ alone. Hence, we can construct a unique global solution of (4.8) in the function space:

$$0 \le u \in \mathcal{C}((s, \infty); H^2_N(\Omega)) \cap \mathcal{C}([s, \infty); L_2(\Omega)) \cap \mathcal{C}^1((s, \infty); L_2(\Omega)),$$
$$0 \le \rho \in \mathcal{C}((s, \infty); H^4_{N^2}(\Omega)) \cap \mathcal{C}([s, \infty); H^2_N(\Omega)) \cap \mathcal{C}^1((s, \infty); H^2_N(\Omega)).$$

Moreover, as shown by [26, Proposition 5.1] or [30, (12.38)], the global solution satisfies a dissipative estimate

$$\|U(t)\|_X \le p\big((t-s+1)^{-1}\|U_0\|_X + 1\big), \qquad s < t < \infty.$$

This jointed with the local estimate (4.13) provides a stronger dissipative estimate of the form

$$\|AU(t)\|_X \le p\big((t-s)^{-1}\|U_0\|_X + 1\big), \qquad s < t < \infty. \tag{4.16}$$

4.3. **Non-autonomous Dynamical System.** Let $s \in \mathbb{R}$. For $U_s \in \mathcal{K}$, let $U(\cdot, s; U_s)$ be the global solution of (4.8). We then set

$$U(t,s)U_s = U(t,s;U_s) \qquad \text{for } (t,s) \in \Delta.$$

This $U(t,s)$ defines an evolution operator acting on $\mathcal{K}$. It is indeed clear that $U(s,s) = I$ for $s \in \mathbb{R}$ and $U(t,s) = U(t,r) \circ U(r,s)$ for (t,r), $(r,s) \in \Delta$.

Let us prove that $U(t,s)$ is a continuous evolution operator on $\mathcal{K}$.

Proposition 4.1. *Let $0 < R < \infty$ and $0 < T < \infty$ be arbitrarily fixed. For any $(t,s) \in \Delta$ such that $0 \le t-s \le T$, $U(t,s)$ satisfies*

$$\|U(t,s)U_0 - U(t,s)V_0\|_X \le L_{R,T}\|U_0 - V_0\|_X, \qquad U_0,\, V_0 \in \mathcal{K}_R, \tag{4.17}$$

$L_{R,T} > 0$ being determined by R and T alone.

Proof. We notice from (4.15) that $\|U(t,s)U_0\|_X \le p(R)$ for any $0 \le t-s < \infty$ provided $U_0 \in \mathcal{K}_R$.

In the meantime, by applying (4.14) with radius $p(R)$, we see that

$$\|U(t,s)U_1 - U(t,s)V_1\|_X \le C_{p(R)}\|U_1 - V_1\|_X, \qquad U_1,\, V_1 \in \mathcal{K}_{p(R)},$$

provided that $0 \le t-s \le T_{p(R)}$. Since $R \le p(R)$, i.e., $\mathcal{K}_R \subset \mathcal{K}_{p(R)}$, this means that the desired estimate (4.17) holds for $0 \le t-s \le T_{p(R)}$.

Let next $T_{p(R)} \le t-s \le 2T_{p(R)}$. Then,

$$\begin{aligned}
&\|U(t,s)U_0 - U(t,s)V_0\|_X \\
&\quad = \|U(t,t-T_{p(R)})U(t-T_{p(R)},s)U_0 - U(t,t-T_{p(R)})U(t-T_{p(R)},s)V_0\|_X \\
&\quad = \|U(t,t-T_{p(R)})U_1 - U(t,t-T_{p(R)})V_1\|_X \le C_{p(R)}\|U_1 - V_1\|_X \le C_{p(R)}^2\|U_0 - V_0\|_X.
\end{aligned}$$

That is, the desired estimate holds for $T_{p(R)} \le t-s \le 2T_{p(R)}$. Repeating this arguments, we see that

$$\|U(t,s)U_0 - U(t,s)V_0\|_X \le C_{p(R)}^n\|U_0 - V_0\|_X$$

for $(n-1)T_{p(R)} \le t-s \le nT_{p(R)}$, where $n = 1, 2, 3, \ldots$.

Hence, the proposition is proved. □

Proposition 4.2. *Let $U_0 \in \mathcal{K}$ be arbitrarily fixed. Then, $U(t,s)U_0$ is a continuous function for $(t,s) \in \Delta$ with values in X.*

Proof. Let $\widetilde{U}_s = {}^t(s, U_0)$. Let $\widetilde{U}(\cdot)$ be the global solution of (4.9) with the initial value $\widetilde{U}_s$. Then, $\widetilde{U}(t)$ is given by

$$\widetilde{U}(t) = e^{-(t-s)\widetilde{A}}\widetilde{U}_s + \int_s^t e^{-(t-\tau)\widetilde{A}}\widetilde{F}(\widetilde{U}(\tau))d\tau, \qquad s < t < \infty.$$

In view of (4.12), we observe that $U(t,s)U_0$ satisfies the integral equation

$$U(t,s)U_0 = e^{-(t-s)A}U_0 + \int_s^t e^{-(t-\tau)A}F(\tau, U(\tau,s)U_0)d\tau, \qquad s < t < \infty. \tag{4.18}$$

We can then verify without difficulty that $U(t,s)U_0$ is continuous for (t,s) with values in X. □

These two propositions yield that the mapping $G : \Delta \times \mathcal{K} \to X$, where $G(t,s;U_0) = U(t,s)U_0$, is continuous. Hence, $(U(t,s), \mathcal{K}, X)$ generates a non-autonomous dynamical system determined from (4.8).

5. Exponential attractors

We now proceed to constructing an exponential attractor. It indeed suffices to show that there exists a family of closed bounded subsets $\mathcal{X}(t)$ of X having the properties (1)~(5).

In view of the dissipative estimate (4.16), we consider a subset

$$\mathcal{B} = \mathcal{K} \cap \overline{B}^{\mathcal{D}(A)}(0; p(2)),$$

where $p(\cdot)$ is the same continuous increasing function as in (4.16). This $\mathcal{B}$ is a compact set of X and is a bounded subset of $\mathcal{D}(A)$. From (4.16) we observe that, for any bounded set B of $\mathcal{K}$, there exists a time $t_B > 0$ such that $U(t,s)B \subset \mathcal{B}$ for every $t \ge t_B + s$, here t_B is independent of s.

We here set, for each $t \in \mathbb{R}$, that

$$\mathcal{X}(t) = \bigcup_{-\infty < s \le t} U(t,s)\mathcal{B}. \tag{5.1}$$

Since $\mathcal{B}$ is a bounded subset of $\mathcal{K}$, $\mathcal{B}$ itself is absorbed by $\mathcal{B}$, i.e., $U(t,s)\mathcal{B} \subset \mathcal{B}$ for any $(t,s) \in \Delta$ such that $t \geq t_{\mathcal{B}} + s$. This means that $\mathcal{X}(t)$ is written by

$$\mathcal{X}(t) = \bigcup_{t - t_{\mathcal{B}} \leq s \leq t} U(t,s)\mathcal{B}, \tag{5.2}$$

too.

Let us see that $\mathcal{X}(t)$, $t \in \mathbb{R}$, fulfills all the desired conditions. It is clear that $\mathcal{B} \subset \mathcal{X}(t) \subset \mathcal{K}$. In addition, $\mathcal{X}(t)$ is considered as the image of a mapping $g\colon [t - t_{\mathcal{B}}, t] \times \mathcal{B} \to \mathcal{K}$ such that $g(s, U_0) = U(t,s)U_0$. Since $[t - t_{\mathcal{B}}, t] \times \mathcal{B}$ is compact and g is continuous, its image $g([t - t_{\mathcal{B}}] \times \mathcal{B}) = \mathcal{X}(t)$ is also compact. Hence, the condition (1) is fulfilled. Moreover, we have the following result.

Proposition 5.1. *The union* $\bigcup_{t\in\mathbb{R}} \mathcal{X}(t)$ *is a bounded subset of* $\mathcal{D}(A)$*. Consequently, the union is a relatively compact set of* X*.*

Proof. To prove this we have to go back to the abstract problem (4.9). Let the initial data U_s satisfy $U_s \in \mathcal{D}(A)$ such that $\|AU_s\|_X \leq p(2)$ and consequently $\widetilde{U}_s \in \mathcal{D}(\widetilde{A})$ with $\|\widetilde{A}\widetilde{U}_s\|_{\mathbb{X}} \leq p(2)$. Let $\widetilde{U}(t)$ be the global solution of (4.9). We want to use the estimates obtained in [30, Theorem 4.2] with $\gamma = \eta$ to conclude that

$$\begin{cases} \|\widetilde{A}^{\eta}\widetilde{U}(t)\|_X \leq C, & s \leq t \leq s + T, \\ \|\widetilde{A}^{\eta}[\widetilde{U}(t) - \widetilde{U}(\tau)]\|_X \leq C(t-\tau)^{1-\eta}(\tau - s)^{-\eta}, & s \leq \tau < t \leq s + T, \end{cases}$$

for the solution with the initial data U_s with some $T > 0$ and $C > 0$ depending only on $p(2)$. Therefore, by (4.12),

$$\begin{cases} \|A^{\eta}U(t,s)U_s\|_X \leq C, & s \leq t \leq s + T, \\ \|A^{\eta}[U(t,s)U_s - U(\tau,s)U_s]\|_X \leq C(t-\tau)^{1-\eta}(\tau - s)^{-\eta}, & s \leq \tau < t \leq s + T. \end{cases} \tag{5.3}$$

Let $-\infty < t < \infty$ and $t - t_{\mathcal{B}} \leq s \leq t$ and let $U_s \in \mathcal{B}$. By definition, $\|AU_s\|_X \leq p(2)$. As seen in (4.18), $U(t,s)U_s$ satisfies the integral equation

$$U(t,s)U_s = e^{-(t-s)A}U_s + \int_s^t e^{-(t-\tau)A}F(\tau, U(\tau,s)U_s)d\tau.$$

Therefore,

$$AU(t,s)U_s = e^{-(t-s)A}AU_s + \int_s^t Ae^{-(t-\tau)A}[F(\tau, U(\tau,s)U_s) - F(t, U(t,s)U_s)]d\tau$$
$$+ \int_s^t Ae^{-(t-\tau)A}F(t, U(t,s)U_s)d\tau.$$

And

$$\int_s^t Ae^{-(t-\tau)A}F(t, U(t,s)U_s)d\tau = (1 - e^{-tA})F(t, U(t,s)U_s).$$

Using (5.3), we easily obtain that

$$\|AU(t,s)U_s\|_X \leq C, \qquad s \leq t \leq s + T,$$

the constant C being determined by $p(2)$.

We have thus verified that the union $\bigcup_{t-T\leq s\leq t} AU(t,s)\mathcal{B}$ is uniformly bounded in X with respect to t. Hence, the proof is complete if $T \geq t_{\mathcal{B}}$.

Let $T < t_{\mathcal{B}}$. For (t, s) such that $T \le t - s \le t_{\mathcal{B}}$, we utilize the global estimate (4.13) to conclude that

$$\|AU(t,s)U_s\|_X \le p(T^{-1}\|U_s\|_X + 1), \qquad T + s \le t \le t_{\mathcal{B}} + s.$$

This means that the union $\bigcup_{t-t_{\mathcal{B}} \le s \le t-T} U(t,s)\mathcal{B}$ is also uniformly bounded in X with respect to t. Hence, the proof is complete even in this case. □

Let us verify the condition (2). By (5.1),

$$\mathcal{X}(s) = \bigcup_{-\infty < r \le s} U(s,r)\mathcal{B}.$$

For each $-\infty < r \le s$, it follows that $U(t,s) \circ U(s,r)\mathcal{B} = U(t,r)\mathcal{B} \subset \mathcal{X}(t)$. Hence, $U(t,s)\mathcal{X}(s) \subset \mathcal{X}(t)$.

Consider any bounded subset B of $\mathcal{K}$. Then, there exists a time t_B such that $U(t,s)B \subset \mathcal{B}$ for every $t \ge t_B + s$. Since $\mathcal{B} \subset \mathcal{X}(t)$, this means that the condition (3) is valid.

We set $Z = \mathcal{D}(A^\eta)$, where $\eta > 0$ is the exponent appearing in (4.14). By Proposition 5.1, there is $R > 0$ such that $\bigcup_{t \in \mathbb{R}} \mathcal{X}(t) \subset \mathcal{K}_R$. Then, (4.14) shows that the Lipschitz condition of (4) is valid provided $\tau^* = T_R$. The estimate provides also the Lipschitz condition of (5).

We have thus verified that all the conditions (1)∼(5) are fulfilled. Hence, Theorem 2.1 yields existence of an exponential attractor $\mathcal{M}(t)$, $-\infty < t < \infty$, for $(U(t,s), \mathcal{K}, X)$.

Let us finally verify that $U(t,s)$ satisfies (3.1) and (3.2). For $(t,s) \in \Delta$, we see from (4.8) that

$$U(t,s)U_s - U_s = \int_s^t [-AU(\tau,s)U_s + F(\tau, U(\tau,s)U_s)]d\tau, \qquad U_s \in \mathcal{X}(s).$$

Therefore,

$$\|[U(t,s) - 1]U_s\|_X \le C(t-s) \sup_{s \le \tau \le t} \|AU(\tau,s)U_s\|_X, \qquad U_s \in \mathcal{X}(s).$$

Then, Proposition 5.1 provides that

$$\sup_{U_s \in \mathcal{X}(s)} \|[U(t,s) - 1]U_s\|_X \le C(t-s), \qquad -\infty < s \le t < \infty.$$

This means that both (3.1) and (3.2) are fulfilled.

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