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CIRCULAR GEOMETRY AND THE SCHWARZIAN

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Abstract

A theory of Schwarzian is developed based on a curve-theoretic quantity called the Schwarzian derivative of curves. The relationship between the Schwarzian and Möbius transformations is made clear. As application of the theory various injectivity theorems are obtained.

0. Introduction

The Schwarzian derivative of holomorphic functions,

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2,$$

has been generalized by several authors from various viewpoints ([1], [3], [7], [8], [10], [12], [13]). In this paper, we give a framework which unifies these generalizations. The theory is based on a curve-theoretic differential operator called the Schwarzian derivative of curves whose properties are investigated in Section 1. The Schwarzian derivative of a curve decomposes into two components. Roughly speaking, the 0-part of the Schwarzian derivative controls the parametrization of the curve while the 2-part controls its shape. A curve with vanishing Schwarzian, which

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we call a Möbius circle, is a "projectively parametrized" geodesic circle.

Closely related to the Schwarzian derivative is the notion of Möbius transformations. In Section 2, we define the Schwarzian for Riemannian metrics and clarify its relationship to generalized Möbius transformations. There is a subtle point about the definition of Möbius transformation. The Schwarzian vanishes for transformations which map Möbius circles to Möbius circles, whereas the term "Möbius transformations" has historically been used for those mapping geodesic circles to geodesic circles. It is shown that these two conditions are actually equivalent in dimensions greater than or equal to 2.

The approach taken in this paper allows us to define the Schwarzian of as general a map as an immersion of class C^3 between any Riemannian manifolds. To illustrate the capability of the framework we prove various injectivity theorems in the last section. In particular, we prove the differentiable version of Nehari's univalency theorem.

1. The Schwarzian Derivative of Curves on Riemannian Manifolds

Let (M, g) be a Riemannian n-manifold. Recall that the Clifford multiplication on the Grassmann algebra $\bigwedge T_xM$ is the associative bilinear operation characterized by the following property:

$$ua = u \wedge a - \iota_u a \in \bigwedge^{p-1} T_xM \oplus \bigwedge^{p+1} T_xM \quad \left(u \in T_xM,\ a \in \bigwedge^{p} T_xM \right),$$

where ι denotes the interior product. In particular, for $u, v \in T_xM$, we have

$$u^{-1} = -\frac{1}{g(u, u)} u \qquad (u \neq 0), \tag{1.1}$$

$$uv = u \wedge v - g(u, v), \tag{1.2}$$

$$uvu = g(u, u)v - 2g(u, v)u. \tag{1.3}$$

The usual convention of omitting the multiplication symbol for the Clifford product may cause a confusion when two or more Riemannian metrics are involved. If necessary, we will indicate by words which Riemannian metric the Clifford multiplication in an expression corresponds to.

Definition. Let $I \subset \mathbf{R}$ be an interval and $x : I \to (M, g)$ be a regular curve. We define $s^3x = s_g^3x : I \to TM$ by

$$s^3x = \nabla_{\dot{x}}\nabla_{\dot{x}}\dot{x} - \frac{3}{2}(\nabla_{\dot{x}}\dot{x})\dot{x}^{-1}(\nabla_{\dot{x}}\dot{x}) - \frac{R_g}{2n(n-1)}\dot{x}^3,$$

and define the *Schwarzian derivative* $s^2x = s_g^2x$ of x by

$$s^2x = (s^3x)\dot{x}^{-1} = (\nabla_{\dot{x}}\nabla_{\dot{x}}\dot{x})\dot{x}^{-1} - \frac{3}{2}((\nabla_{\dot{x}}\dot{x})\dot{x}^{-1})^2 - \frac{R_g}{2n(n-1)}\dot{x}^2,$$

where ∇ and R_g are respectively the Riemannian connection and the scalar curvature of g.

For $t \in I$, $s^2x(t)$ lies in $\bigwedge^0 T_{x(t)}M \oplus \bigwedge^2 T_{x(t)}M$, and we have a natural decomposition, $s^2x = s^2x^{(0)} + s^2x^{(2)}$, of the Schwarzian derivative into its 0-part and 2-part. Note that the 0-part and the 2-part of the Schwarzian derivative correspond to the components of s^3x tangent to $\dot{x}$ and normal to $\dot{x}$, respectively. The auxiliary quantity s^3x will play an important role in the subsequent sections.

When $n = 1$ the term $R_g/n(n-1)$ is indefinite; we adopt the following convention which proves to be useful later: $R_g/n(n-1) = r^{-2}$ for the Euclidean circle of radius r, and $R_g/n(n-1) = 0$ for the Euclidean line.

In the usual terminology of Riemannian geometry, the quantities s^3x, $s^2x^{(0)}$, $s^2x^{(2)}$ are expressed as follows; these are obtained by straightforward calculations using (1.1)-(1.3):

$$s^3x = \nabla_{\dot{x}}\nabla_{\dot{x}}\dot{x} - \frac{3}{g(\dot{x},\, \dot{x})} g(\nabla_{\dot{x}}\dot{x},\, \dot{x})\nabla_{\dot{x}}\dot{x}$$

$$+ \left(\frac{3}{2g(\dot{x},\, \dot{x})} g(\nabla_{\dot{x}}\dot{x},\, \nabla_{\dot{x}}\dot{x}) + \frac{R_g}{2n(n-1)} g(\dot{x},\, \dot{x})\right)\dot{x} \tag{1.4}$$

$$s^2x^{(0)} = \frac{1}{g(\dot{x},\, \dot{x})}\left(g(\nabla_{\dot{x}}\nabla_{\dot{x}}\dot{x},\, \dot{x}) - \frac{3}{g(\dot{x},\, \dot{x})} g(\nabla_{\dot{x}}\dot{x},\, \dot{x})^2\right.$$

$$\left. + \frac{3}{2} g(\nabla_{\dot{x}}\dot{x},\, \nabla_{\dot{x}}\dot{x}) + \frac{R_g}{2n(n-1)} g(\dot{x},\, \dot{x})^2\right) \tag{1.5}$$

$$s^2x^{(2)} = -\frac{1}{g(\dot{x},\, \dot{x})}\left(\nabla_{\dot{x}}\nabla_{\dot{x}}\dot{x} - \frac{3}{g(\dot{x},\, \dot{x})} g(\nabla_{\dot{x}}\dot{x},\, \dot{x})\nabla_{\dot{x}}\dot{x}\right) \wedge \dot{x}\,. \tag{1.6}$$

In view of the Frénet-Serret formula in the theory of curves, we have the following useful lemma, which is again obtained by direct calculations.

Lemma 1.1. *Put* $\sigma = |\dot{x}|$ *and* $\xi = \dot{x}/\sigma$. *Let* κ *denote the geodesic curvature of* x. *Then*

(1) $s^3x = 2\sigma^3\left(\dfrac{\xi\xi\sqrt{\sigma}}{\sqrt{\sigma}} + \dfrac{1}{4}\left(\kappa^2 + \dfrac{R_g}{n(n-1)}\right)\right)\xi + \sigma^3(\nabla_\xi\nabla_\xi\xi + \kappa^2\xi)$,

(2) $s^2x = 2\sigma^2\left(\dfrac{\xi\xi\sqrt{\sigma}}{\sqrt{\sigma}} + \dfrac{1}{4}\left(\kappa^2 + \dfrac{R_g}{n(n-1)}\right)\right) - \sigma^2(\nabla_\xi\nabla_\xi\xi \wedge \xi)$.

This lemma leads to the following geometric interpretation of the 2-part of the Schwarzian derivative, $s^2x^{(2)} = (1/g(\dot{x},\, \dot{x}))\,\dot{x} \wedge s^3x$.

Proposition 1.2. *Let* κ *denote the geodesic curvature of* x, *and* τ *the torsion vector of* x. *Then* $s^2x^{(2)}(t) = 0$ *if and only if* $\dot{\kappa}(t) = \tau(t) = 0$.

Proof. From the Frénet-Serret formula, we have $\nabla_\xi\nabla_\xi\xi = -\kappa^2\xi + (\nabla_\xi\kappa)\nu + \tau$, where ν is the unit normal vector of x. It follows from Lemma 1.1 that $s^2x^{(2)}(t) = 0$ if and only if $\nabla_\xi\nabla_\xi\xi = -\kappa^2\xi$, which then implies our assertion.

This result may be found in Yano [14]; it is essentially the classical observation due to G. Pick [9] on the relationship between the imaginary part of the Schwarzian derivative and vertices of a planar curve (cf. [2]). Anyhow a curve x with $s^2x^{(2)} = 0$ is a so-called *geodesic circle*, which is by definition a curve of constant curvature with vanishing torsion (cf. [14]).

Definition. We call a regular curve $x : I \to M$ satisfying $s^2x = 0$, or equivalently $s^3x = 0$, a *Möbius circle*.

Every geodesic circle is a Möbius circle if appropriately parametrized. In this sense the 0-part $s^2x^{(0)}$ of the Schwarzian derivative may be considered to control the parametrization of the curve x. One can see from (1.4) that given $X, Y \in T_{x_0}M$ $(X \neq 0)$, there exists a unique Möbius circle x with $x(0) = x_0$, $\dot{x}(0) = X$ and $\nabla_{\dot{x}}\dot{x}(0) = Y$.

We can determine all the Möbius circles in the Euclidean space $\mathbf{R}^n$ as follows. Since Möbius circles have zero torsion, it is sufficient to consider the case of the plane $\mathbf{R}^2$. Then, one can easily verify that the curve

$$x(t) = \frac{at + b}{ct + d} \qquad (a, b, c, d \in \mathbf{C},\ ad - bc \neq 0)$$

provide all the solutions to the equation $s^2x = 0$. Namely, a Möbius circle in $\mathbf{R}^2 = \mathbf{C}$ is the image of a straight line of constant speed by a complex linear fractional transformation. In any case, a Möbius circle x has a unique limit point $x(+\infty) = x(-\infty)$ in $\mathbf{R}^n \cup \{\infty\}$. The limit point can be expressed as

$$x(\infty) = x(t) - 2\dot{x}(t)\,\ddot{x}(t)^{-1}\dot{x}(t). \tag{1.7}$$

In fact we can see that the derivative of the right hand side with respect to t is zero since $\dddot{x} - (3/2)\ddot{x}\dot{x}^{-1}\ddot{x} = 0$. It is then easy to check that the right hand side of (1.7) equals $x(\infty)$.

A geodesic parametrized by arclength is a Möbius circle if $R_g = 0$. However, it is not the case, in general. Let us consider the case of the Euclidean n-sphere (S^n, g), for which $R_g/n(n-1) = 1$. Let $x : \mathbf{R} \to S^n$ be a great circle parametrized by t, and s be an arclength parameter for x so that $ds/dt > 0$. Then $\sigma = |\dot{x}| = ds/dt$, $\xi = \dot{x}/\sigma = d/ds$, and $\kappa = 0$. Therefore by Lemma 1.1, the condition $s^2x^{(0)} = 0$ reads $(d/ds)^2\sqrt{\sigma} + \sqrt{\sigma}/4 = 0$. It follows that $\sigma = 2C_1\cos^2(s/2 + C_2)$ for some constants C_1, C_2, and we have $C_0 + C_1 t = \tan(s/2 + C_2)$ for some C_0. In particular, $t = \tan(s/2)$, that is, $s = 2\arctan t$ $(t \in \mathbf{R})$, is a solution. Namely, if $\widetilde{x} : \mathbf{R} \to S^n$ is a great circle parametrized by arclength, $x(t) = \widetilde{x}(2\arctan t)$ is a Möbius circle. It should be noted that by the stereographic projection from the point $x(+\infty) = x(-\infty)$, the Möbius circle x is mapped onto a straight line of constant speed in the Euclidean n-space. One can sense from this example why the scalar curvature term enters into our Schwarzian derivative; we will come back to this question in the next section.

As for the 0-part of the Schwarzian derivative, we have the following theorem. The formula (1.8) below gives a geometric meaning of $s^2x^{(0)}$.

Theorem 1.3. *A regular curve $x : I \to \mathbf{R}^n$ in the Euclidean space satisfying $s^2x^{(0)} \leq 0$ is injective.*

We will see in the next section that the Euclidean space $\mathbf{R}^n$ in the above may be replaced by the Euclidean n-sphere S^n. One may compare this result to the well-known theorem by Kneser [4] on vertex-free planar curves stating that a regular curve $x : I \to \mathbf{R}^2$ in the Euclidean plane satisfying $s^2x^{(2)} \neq 0$ is injective.

Proof. For each $t \in I$, take the Möbius circle $m : \mathbf{R} \to \mathbf{R}^n$ approximating the curve x to the second order; i.e., $m(0) = x(t)$, $\dot{m}(0) = \dot{x}(t)$, $\ddot{m}(0) = \ddot{x}(t)$.

Let $S(t)$ denote the unique hypersphere, or possibly hyperplane, which intersects x perpendicularly at $x(t)$ and passes through $m(\infty)$. Then using (1.7), we can see that the radius $r(t)$ and the center $C(t)$ of $S(t)$ are written as

$$r(t) = \frac{|\dot{x}(t)|^3}{\langle \dot{x}(t), \ddot{x}(t)\rangle},$$

$$C(t) = x(t) - r(t)\frac{\dot{x}(t)}{|\dot{x}(t)|}.$$

In terms of $r(t)$ and $C(t)$, the 0-part of the Schwarzian can be expressed as

$$s^2x^{(0)}(t) = \frac{|\dot{C}(t)|^2 - \dot{r}(t)^2}{2r(t)^2}. \tag{1.8}$$

This can be proved as follows. Using the notation of Lemma 1.1, $r(t)$ can be rewritten as

$$r = \frac{\sigma}{\xi\sigma}.$$

Hence we have

$$\begin{aligned}\dot{r} &= \sigma\xi r\\ &= \sigma - r^2\xi\xi\sigma.\end{aligned}$$

On the other hand, from

$$\dot{C} = (\sigma - \dot{r})\,\xi - r\sigma\nabla_\xi\xi,$$

we obtain

$$|\dot{C}|^2 = (\sigma - \dot{r})^2 + r^2\sigma^2\kappa^2,$$

and

$$\begin{aligned}|\dot{C}|^2 - \dot{r}^2 &= -2\dot{r}\sigma + \sigma^2 + r^2\sigma^2\kappa^2\\ &= 2r^2\left(\sigma\xi\xi\sigma - \frac{1}{2}(\xi\sigma)^2 + \frac{1}{2}\sigma^2\kappa^2\right)\end{aligned}$$

$$= 2r^2\left[2\sigma^2\left(\frac{\xi\xi\sqrt{\sigma}}{\sqrt{\sigma}} + \frac{1}{4}\kappa^2\right)\right]$$

$$= 2r^2 s^2 x^{(0)}.$$

By (1.8), the assumption $s^2x^{(0)} \le 0$ implies $|\dot{C}| \le |\dot{r}|^2$. Therefore, we have

$$|C(t_1) - C(t_2)| \le |r(t_1) - r(t_2)|$$

for $t_1 < t_2$. In particular, $S(t_1)$ and $S(t_2)$ share at most one point. From this we can conclude that x is injective, since x is a regular curve. This completes the proof of Theorem 1.3.

2. The Schwarzian for Riemannian Metrics

Let g and $\hat{g}$ be Riemannian metrics on M. We consider the difference between the Schwarzian derivatives with respect to the two metrics. For a regular curve $x : I \to M$, we define

$$S_g^3(\hat{g}) = s_{\hat{g}}^3 x - s_g^3 x.$$

We see from (1.4) that the third order derivatives in this expression cancel out, and $S_g^3(\hat{g})$ depends only on $X = \dot{x}$ and $Y = \nabla_{\dot{x}}\dot{x}$. Thus, it is reasonable to use the notation $S_g^3(\hat{g})(X, Y)$ for arbitrary tangent vectors $X, Y \in T_pM$ $(X \ne 0)$.

Definition. We define the *Schwarzian* $S_g^2(\hat{g})$ of $\hat{g}$ with respect to g by

$$S_g^2(\hat{g})(X, Y) = S_g^3(\hat{g})(X, Y)X^{-1} \in \bigwedge^0 T_pM \oplus \bigwedge^2 T_pM$$

for $X, Y \in T_pM$ $(X \ne 0)$. The Clifford multiplication on the right hand side is with respect to the metric $\hat{g}$.

The 0-part and the 2-part of the Schwarzian $S_g^2(\hat{g})$ are denoted by $S_g^2(\hat{g})^{(0)}$ and $S_g^2(\hat{g})^{(2)}$, respectively.

Proposition 2.1. *Let* g_i $(i = 1, 2, 3)$ *be Riemannian metrics on* M, *and* ${}^i\nabla$ *be the Riemannian connection of* g_i. *Denote the difference between* ${}^1\nabla$ *and* ${}^2\nabla$ *by* $A(X, Y) = {}^2\nabla_X Y - {}^1\nabla_X Y$. *Then,*

$$S_{g_1}^3(g_3)(X, Y) = S_{g_1}^3(g_2)(X, Y) + S_{g_2}^3(g_3)(X, Y + A(X, X)).$$

In particular, we have

$$S_{g_1}^3(g_2)(X, Y) + S_{g_2}^3(g_1)(X, Y + A(X, X)) = 0.$$

Proof. Take a curve x such that $X = \dot{x}$ and $Y = {}^1\nabla_{\dot{x}}\dot{x}$. Then we have

$${}^2\nabla_{\dot{x}}\dot{x} = Y + A(X, X),$$

and

$$S_{g_1}^3(g_3)(X, Y) = \left(s_{g_3}^3 x - s_{g_2}^3 x\right) + \left(s_{g_2}^3 x - s_{g_1}^3 x\right)$$

$$= S_{g_2}^3(g_3)(X, Y + A(X, X)) + S_{g_1}^3(g_2)(X, Y).$$

Therefore the condition $S_g^3(\hat{g}) = 0$, or equivalently $S_g^2(\hat{g}) = 0$, defines an equivalence relation between Riemannian metrics. Note that the condition $S_g^2(\hat{g})^{(2)} = 0$ also defines an equivalence relation between metrics because

$$S_g^2(\hat{g})^{(2)}(X, Y) = \frac{1}{\hat{g}(X, X)} X \wedge S_g^3(\hat{g})(X, Y).$$

Proposition 2.2.

(1) $S_g^2(\hat{g}) = 0$ *if and only if Möbius circles for* g *are Möbius circles for* $\hat{g}$.

(2) $S_g^2(\hat{g})^{(2)} = 0$ *if and only if geodesic circles for* g *are geodesic circles for* $\hat{g}$.

Proof. The first statement is obvious from the definition. To see (2), we note that

$$\begin{aligned}\hat{g}(\dot{x},\,\dot{x})\,S_g^2(\hat{g})^{(2)}(\dot{x},\,\nabla_{\dot{x}}\dot{x}) &= \dot{x}\wedge S_g^3(\hat{g})\,(\dot{x},\,\nabla_{\dot{x}}\dot{x})\\ &= \hat{g}(\dot{x},\,\dot{x})\,s_{\hat{g}}^2\,x^{(2)} - g(\dot{x},\,\dot{x})\,s_g^2\,x^{(2)}.\end{aligned}$$

Then, the assertion follows from Proposition 1.2.

Definition. For Riemannian metrics g and $\hat{g}$ on M, we say that

(1) $\hat{g}$ is *Möbius equivalent* to g if $S_g^2(\hat{g}) = 0$,

(2) $\hat{g}$ is *concircularly equivalent* to g if $S_g^2(\hat{g})^{(2)} = 0$.

Accordingly, we define Möbius and concircular transformations:

Definition. A local diffeomorphism f of a Riemannian manifold $(M,\,g)$ is called

(1) a *Möbius transformation* if f^*g is Möbius equivalent to g,

(2) a *concircular transformation* if f^*g is concircularly equivalent to g.

Some historical remarks are in order. The notion of a concircular transformation was introduced by Yano [14] in 1940 as a conformal transformation which preserves geodesic circles. In 1970, Vogel [11] showed that the conformality condition in the definition of concircular transformation is redundant; we will reconsider Vogel's result from the viewpoint of the Schwarzian. Thereby our definition of concircularity coincides with Yano's. The term "Möbius transformation," on the other hand, seems to have been used more vaguely. According to a historical remark by Ahlfors [1], Möbius transformations should be the same thing as concircular transformations. Indeed, the generalized "Möbius transformation" introduced by Osgood and Stowe [8] is a synonym of "concircular transformation." Our notion of Möbius transformation however looks stricter by definition.

In what follows, we show that the two notions are, in fact, equivalent if $\dim M \geq 2$. It thus proves a conjecture posed by the second author in [12].

We begin by rewriting $S_g^3(\hat{g})$ and $S_g^2(\hat{g})$ without using auxiliary curves. First note for a vector field X that

$$\hat{\nabla}_X X = \nabla_X X + A(X, X),$$

$$\hat{\nabla}_X \hat{\nabla}_X X = \nabla_X \nabla_X X + 3A(\nabla_X X, X) + ((\nabla_X A)(X, X) + A(X, A(X, X))),$$

where $A(X, Y) = \hat{\nabla}_X Y - \nabla_X Y$. We define formal expressions D_1, D_2, and D_3 of the variables $X, Y, Z \in T_pM$ by

$$D_1(X, Y, Z) = Z,$$

$$D_2(X, Y, Z) = Y + A(X, X),$$

$$D_3(X, Y, Z) = Z + 3A(X, Y) + ((\nabla_X A)(X, X) + A(X, A(X, X))),$$

and define $\Sigma_g^3(\hat{g})$ and $\Sigma_g^2(\hat{g})$ by

$$\Sigma_g^3(\hat{g}) = D_3 - \frac{3}{2} D_2 D_1^{-1} D_2 - \frac{R_g}{2n(n-1)} D_1^3,$$

$$\Sigma_g^2(\hat{g}) = \Sigma_g^3(\hat{g}) D_1^{-1} = D_3 D_1^{-1} - \frac{3}{2} (D_2 D_1^{-1})^2 - \frac{R_g}{2n(n-1)} D_1^2,$$

where the Clifford multiplication is with respect to $\hat{g}$. We then have

$$S_g^3(\hat{g})(X, Y) = \Sigma_g^3(\hat{g})\left(X, Y, \frac{3}{2} Y X^{-1} Y + \frac{R_g}{2n(n-1)} X^3\right), \tag{2.1}$$

$$S_g^2(\hat{g})(X, Y) = \Sigma_g^2(\hat{g})\left(X, Y, \frac{3}{2} Y X^{-1} Y + \frac{R_g}{2n(n-1)} X^3\right). \tag{2.2}$$

This time, the Clifford multiplication in the third arguments of the right hand sides is with respect to g.

From this it follows that $S_g^3(\hat{g})(X, Y)$ and $S_g^2(\hat{g})(X, Y)$ are polynomials of degree at most 2 in Y. Let us write $S_g^3(\hat{g})(X, Y)$ as

$$S_g^3(\hat{g})(X, Y) = Q^3(X, Y) + T^3(X, Y) + S_g^3(\hat{g})(X, 0), \tag{2.3}$$

where $Q^3(X, Y)$ and $T^3(X, Y)$ are respectively the quadratic part and the linear part, with respect to Y, of $S_g^3(\hat{g})(X, Y)$. The following is immediate from Proposition 2.1.

Proposition 2.3. *Denote by Q_{ij} and T_{ij}, the Q-part and the T-part of $S_{g_i}^3(g_j)$, respectively, and by $A_{ij} = {}^j\nabla - {}^i\nabla$ the difference between the connections. Then, we have*

(1) $Q_{ik}(X, Y) = Q_{ij}(X, Y) + Q_{jk}(X, Y)$,

(2) $T_{ik}(X, Y) = T_{ij}(X, Y) + T_{jk}(X, Y) + Q_{jk}(X, Y + A_{ij}(X, X))$

$$- Q_{jk}(X, A_{ij}(X, X)) - Q_{jk}(X, Y).$$

In the same way as in (2.3), we have the following decompositions of $S_g^2(\hat{g})^{(0)}$ and $S_g^2(\hat{g})^{(2)}$:

$$S_g^2(\hat{g})^{(0)} = Q^{2(0)}(X, Y) + T^{2(0)}(X, Y) + S_g^2(\hat{g})^{(0)}(X, 0),$$

$$S_g^2(\hat{g})^{(2)} = Q^{2(2)}(X, Y) + T^{2(2)}(X, Y) + S_g^2(\hat{g})^{(2)}(X, 0).$$

Here we write down these terms explicitly:

Lemma 2.4.

$$Q^3(X, Y) = \frac{3}{2}\left(\frac{\hat{g}(Y, Y)}{\hat{g}(X, X)} - \frac{g(Y, Y)}{g(X, X)}\right)X - 3\left(\frac{\hat{g}(X, Y)}{\hat{g}(X, X)} - \frac{g(X, Y)}{g(X, X)}\right)Y$$

$$Q^{2(0)}(X, Y) = \frac{3}{2}\left(\frac{\hat{g}(Y, Y)}{\hat{g}(X, X)} - \frac{g(Y, Y)}{g(X, X)}\right) - 3\left(\frac{\hat{g}(X, Y)}{\hat{g}(X, X)} - \frac{g(X, Y)}{g(X, X)}\right)\frac{\hat{g}(X, Y)}{\hat{g}(X, X)}$$

$$Q^{2(2)}(X, Y) = -\frac{3}{\hat{g}(X, X)}\left(\frac{\hat{g}(X, Y)}{\hat{g}(X, X)} - \frac{g(X, Y)}{g(X, X)}\right)X \wedge Y$$

$$T^3(X, Y) = 3\left(A(X, Y) + \frac{\hat{g}(A(X, X), Y)}{\hat{g}(X, X)}X - \frac{\hat{g}(A(X, X), X)}{\hat{g}(X, X)}Y\right.$$
$$\left. - \frac{\hat{g}(X, Y)}{\hat{g}(X, X)}A(X, X)\right)$$

$$T^{2(0)}(X, Y) = 3\frac{\hat{g}(A(X, Y), X)}{\hat{g}(X, X)} + 3\frac{\hat{g}(Z(X, X), Y)}{\hat{g}(X, X)}$$
$$- 6\frac{\hat{g}(A(X, X), X)\hat{g}(X, Y)}{\hat{g}(X, X)^2}$$

$$T^{2(2)}(X, Y) = \frac{3}{\hat{g}(X, X)}X \wedge \left(A(X, Y) - \frac{\hat{g}(A(X, X), X)}{\hat{g}(X, X)}\right.$$
$$\left. - \frac{\hat{g}(X, Y)}{\hat{g}(X, X)}A(X, X)\right).$$

Proof. The expressions for $Q^3(X, Y)$ and $T^3(X, Y)$ are obtained by a straightforward computation from (2.1). The others are also easily obtained by using

$$Q^{2(0)}(X, Y) = \frac{1}{\hat{g}(X, X)}\hat{g}(Q(X, Y), X),$$

$$Q^{2(2)}(X, Y) = \frac{1}{\hat{g}(X, X)}X \wedge Q(X, Y),$$

$$T^{2(0)}(X, Y) = \frac{1}{\hat{g}(X, X)}\hat{g}(T(X, Y), X),$$

$$T^{2(2)}(X, Y) = \frac{1}{\hat{g}(X, X)}X \wedge T(X, Y).$$

From Lemma 2.4, we immediately have

Proposition 2.5. *The following are equivalent*:

(1) $\hat{g}$ *is conformal to* g,

(2) $Q^3 = 0$,

(3) $Q^{2(0)} = 0$.

If $\dim M \geq 2$, *these conditions are also equivalent to*

(4) $Q^{2(2)} = 0$.

The second statement is the Vogel's argument [11] for showing that concircularity implies conformality. Kühnel [5] gives a proof of the same result from a slightly different viewpoint. Our approach is in line with the original proof by Vogel.

Proposition 2.6. *The following are equivalent*:

(1) ∇ *is a conformal connection of* $\hat{g}$; *i.e., there exists a* 1-*form* λ *such that* $\nabla_X \hat{g} = \lambda(X)\hat{g}$,

(2) $T^3 = 0$,

(3) $T^{2(0)} = 0$,

(4) $T^{2(2)} = 0$.

Proof. Under the condition (1), the difference between the connections is written as

$$A(X, Y) = \frac{1}{2}(\lambda(X)Y + \lambda(Y)X - \hat{g}(X, Y)\lambda^{\#}),$$

where $\lambda^{\#}$ is the vector field satisfying $\hat{g}(\lambda^{\#}, Z) = \lambda(Z)$ for any Z. Using this, one can easily verify by Lemma 2.4 that the conditions (2), (3) and (4) hold. We leave the proof of the other implications as an exercise for the reader.

Corollary 2.7. *If* $\hat{g}$ *is conformal to* g, $S_g^3(\hat{g})(X, Y)$ *and* $S_g^2(\hat{g})(X, Y)$ *do not depend on* Y.

Thus we may use the notation $S_g^3(\hat{g})(X)$ and $S_g^2(\hat{g})(X)$ when $\hat{g}$ is conformal to g.

It is clear from Propositions 2.5 and 2.6 that conformal transformations are of special importance to the Schwarzian. Let us assume that $\hat{g}$ is conformal to g, and is written as $\hat{g} = e^{2\varphi}g$. The difference between the connections, $A_\varphi = \hat{\nabla} - \nabla$, is then explicitly written as

$$A_\varphi(X, Y) = -\frac{1}{2}(X(\nabla\varphi)Y + Y(\nabla\varphi)X).$$

We define P_φ by

$$P_\varphi = \nabla^2\varphi - d\varphi \otimes d\varphi + \frac{1}{2}|\,d\varphi\,|^2 g$$

$$= -\frac{\nabla^2 e^{-\varphi}}{e^{-\varphi}} + \frac{1}{2}\left|\frac{de^{-\varphi}}{e^{-\varphi}}\right|^2 g.$$

Note that if $n = \dim M = 1$, P_φ has a form analogous to the classical Schwarzian,

$$P_\varphi = \frac{\nabla^2 e^{\varphi}}{e^{\varphi}} - \frac{3}{2}\left|\frac{de^{\varphi}}{e^{\varphi}}\right|^2 g.$$

If $n \geq 3$, we have

$$P_\varphi = -\frac{1}{n-2}(L_{\hat{g}} - L_g),$$

where

$$L_g = \mathrm{Ric}_g - \frac{R_g}{2(n-1)}g.$$

In particular,

$$P_\varphi^\circ = -\frac{1}{n-2}\left(\mathrm{Ric}_{\hat{g}}^\circ - \mathrm{Ric}_g^\circ\right), \tag{2.4}$$

$$(\mathrm{tr}_g P_\varphi)\,g = -\frac{1}{2(n-1)}(R_{\hat g}\hat g - R_g g),$$

where $\circ$ stands for the traceless part of symmetric 2-tensor. We note that the last equality holds even if $n = 2$.

Theorem 2.8. *Suppose that $n \geq 2$ and $\hat g$ is conformal to g. Then, using the notation as above, we have*

(1) $S_g^2(\hat g)(X) = P_\varphi^\circ(X, X) + e^{-2\varphi}(P_\varphi^\circ \cdot X) \wedge X$,

(2) $S_g^3(\hat g)(X) = -\,g(X, X)\,P_\varphi^\circ \cdot X$.

Proof. Let $x : I \to M$ be a regular curve such that $\dot x = X$. Since $\hat g$ is conformal to g, we have

$$S_g^2(\hat g)^{(0)}(X) = s_{\hat g}^2 x^{(0)} - s_g^2 x^{(0)},$$

$$S_g^2(\hat g)^{(2)}(X) = s_{\hat g}^2 x^{(2)} - e^{-2\varphi} s_g^2 x^{(2)}.$$

As in Lemma 1.1, let σ, ξ, κ respectively be the speed, the unit tangent vector, and the geodesic curvature of x with respect to g, and $\hat\sigma$, $\hat\xi$, $\hat\kappa$ those with respect to $\hat g$. Then, we have $\hat\sigma = e^{\varphi}\sigma$ and $\hat\xi = e^{-\varphi}\xi$. As for the geodesic curvature, we have

$$\hat\kappa^2 = e^{-2\varphi}(\kappa^2 - (\xi\varphi)^2 + g(d\varphi, d\varphi) - 2\nabla_\xi \xi\varphi).$$

This formula is given in [14], in which Yano also gives

$$\hat\nabla_{\hat\xi}\hat\nabla_{\hat\xi}\hat\xi \wedge \hat\xi = e^{-4\varphi}(\nabla_\xi\nabla_\xi\xi - P_\varphi^\circ \cdot \xi) \wedge \xi.$$

Hence by Lemma 1.1, we obtain

$$S_g^2(\hat g)^{(2)}(X) = e^{-2\varphi}(P_\varphi^\circ \cdot X) \wedge X.$$

The computation for the 0-part is as follows. Note that the second equality holds only if $n \geq 2$.

$$s_{\hat{g}}^2 x^{(0)} = 2\hat{\sigma}^2\left(\frac{\hat{\xi}\hat{\xi}\sqrt{\hat{\sigma}}}{\sqrt{\hat{\sigma}}} + \frac{1}{4}\hat{\kappa}^2\right) + \hat{\sigma}^2 \frac{R_{\hat{g}}}{2n(n-1)}$$

$$= 2\sigma^2\left(\frac{\xi\xi\sqrt{\sigma}}{\sqrt{\sigma}} + \frac{1}{4}\kappa^2 + \frac{1}{2}P_{\varphi}(\xi, \xi)\right) + \sigma^2\left(\frac{R_g}{2n(n-1)} - \frac{\mathrm{tr}_g P_{\varphi}}{n}\right)$$

$$= s_g^2 x^{(0)} + \sigma^2 P_{\varphi}^{\circ}(\xi, \xi).$$

Thus we obtain

$$S_g^2(\hat{g})^{(0)}(X) = P_{\varphi}^{\circ}(X, X).$$

This completes the proof of (1). The assertion (2) follows immediately from (1).

Yano [14] shows that two conformal metrics g and $\hat{g}$ are concircularly equivalent if and only if $P_{\varphi}^{\circ} = 0$.

Osgood and Stowe [8] consider P_{φ}° to be the generalized Schwarzian derivative, while Carne [3] regards P_{φ} as a generalization of the Schwarzian derivative. As is seen from the above proof, the 2-part of the Schwarzian $S_g^2(\hat{g})(X)$ involves P_{φ}° whether or not we put the scalar curvature term in our definition of the Schwarzian derivative of curves. On the other hand, the 0-part of the Schwarzian would be $P_{\varphi}(X, X)$ rather than $P_{\varphi}^{\circ}(X, X)$ if we omitted the scalar curvature compensation term.

Let us consider the stereographic projection

$$f : S^n \setminus \{p\} \to \mathbf{R}^n$$

of the Euclidean sphere (S^n, g_r) of radius r to the Euclidean space $(\mathbf{R}^n, g_0)$. By Theorem 2.8, we have $S_{g_r}^2(f^* g_0) = 0$. This holds even if $n = 1$. Therefore, the Euclidean space in Theorem 1.3 may be replaced by the Euclidean sphere of radius r.

Theorem 2.8, together with Proposition 2.5, implies:

Theorem 2.9. *If $n \geq 2$, two metrics g and $\hat{g}$ are Möbius equivalent if and only if they are concircularly equivalent.*

In the level of definition, the relationship between Möbius geometry and concircular geometry is analogous to that between affine geometry and projective geometry. In reality, however, there is no such difference between the two circular geometries if $n \geq 2$; therefore Möbius transformations and concircular transformations are the same thing.

In the Euclidean space $\mathbf{R}^n$ $(n \geq 3)$, Möbius and concircular transformations also coincide with conformal transformations by (2.4) and Theorem 2.8; they are nothing but Möbius transformations in the usual sense.

Let us consider the 2-dimensional case. We equip the complex plane $\mathbf{C}$ with the Euclidean metric $g = |dz|^2$, and consider a complex analytic function f defined on some domain on which $f'(z) \neq 0$. The function f is conformal, and we can write $\hat{g} = f^*g = e^{2\varphi}g$, where $\varphi = \log|f'|$. Put $\omega_g = -2\sqrt{-1}\,\partial_z \wedge \partial_{\bar{z}}$, and $\omega_{\hat{g}} = f^*\omega_g = e^{-2\varphi}\omega_g$. Then, we have

$$S_g^3(\hat{g})(X) = \operatorname{Re} S_f(X, X)X + \operatorname{Im} S_f(X, X)JX,$$

$$S_g^2(\hat{g})(X) = \operatorname{Re} S_f(X, X) + \operatorname{Im} S_f(X, X)\omega_{\hat{g}},$$

where J is the almost complex structure of $\mathbf{C}$, and

$$S_f = \left(\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right) dz \otimes dz$$

is the classical Schwarzian differential. A proof of this will be given in the next section (also see [8]).

So far we are mainly concerned with the case where $n \geq 2$. In dimension 1, every transformation is clearly concircular. On the other hand, Möbius transformations are restricted to some extent. Recall that we put $R_g/n(n-1) = r^{-2}$ for the Euclidean circle of radius r, and $R_g/n(n-1) = 0$ for the Euclidean line. Under this convention, Möbius transformations are linear fractional transformations.

The proof of Theorem 2.8 actually shows the following.

Corollary 2.10. *Suppose that* $n \geq 2$ *and* $\hat{g} = e^{2\varphi} g$. *Then, we have*

(1) $s_{\hat{g}}^2 x^{(0)} = s_g^2 x^{(0)} + P_\varphi^\circ(\dot{x}, \dot{x})$,

(2) $s_{\hat{g}}^2 x^{(2)} = e^{-2\varphi}(s_g^2 x^{(2)} + (P_\varphi^\circ \cdot \dot{x}) \wedge \dot{x})$,

(3) $s_{\hat{g}}^3 x = s_g^3 x + P_\varphi^\circ(\dot{x}, \dot{x})\dot{x} + (P_\varphi^\circ(\dot{x}, \dot{x})\dot{x} - |\dot{x}|^2 P_\varphi^\circ \cdot \dot{x})$.

Corollary 2.11. *The quantities* $s_g^2 x^{(0)}$, $s_g^2 x^{(2)} \otimes g$, $\left| s_g^2 x^{(2)} \right|$ *and* $s_g^3 x$ *are concircular invariants.*

This is a generalization of the observation by Yano [15] that the orthogonal component of $s_g^3 x$ to $\dot{x}$, which is equivalent to $s_g^2 x^{(2)} \otimes g$, is concircularly invariant.

We now consider a modification of the scalar curvature term in the definition of the Schwarzian derivative of curves. Namely, for a regular curve $x : I \to M$, we define

$$\tilde{s}_g^3 x = \nabla_{\dot{x}} \nabla_{\dot{x}} \dot{x} - \frac{3}{2}(\nabla_{\dot{x}} \dot{x})\dot{x}^{-1}(\nabla_{\dot{x}} \dot{x}) - \frac{1}{n-2}\dot{x}(L_g \cdot \dot{x})\dot{x},$$

$$\tilde{s}_g^2 x = (\nabla_{\dot{x}} \nabla_{\dot{x}} \dot{x})\dot{x}^{-1} - \frac{3}{2}((\nabla_{\dot{x}} \dot{x})\dot{x}^{-1})^2 - \frac{1}{n-2}\dot{x}(L_g \cdot \dot{x}).$$

Corollary 2.12. *Suppose that* $n \geq 3$ *and* $\hat{g} = e^{2\varphi} g$. *Then, we have*

(1) $\tilde{s}_{\hat{g}}^2 x = \tilde{s}_g^2 x^{(0)} + e^{-2\varphi} \tilde{s}_g^2 x^{(2)}$,

(2) $\tilde{s}_{\hat{g}}^3 x = \tilde{s}_g^3 x$.

Proof. Note that

$$\tilde{s}_g^2 x = s_g^2 x - \frac{1}{n-2}\dot{x}(\mathrm{Ric}_g^\circ \cdot \dot{x})$$

$$= s_g^2 x - \frac{1}{n-2}(\mathrm{Ric}_g^\circ(\dot{x}, \dot{x}) + (\mathrm{Ric}_g^\circ \cdot \dot{x}) \wedge \dot{x}),$$

and

$$\tilde{s}_g^3 x = s_g^3 x - \frac{1}{n-2}(g(\dot{x}, \dot{x})\mathrm{Ric}_g^\circ \cdot \dot{x} - 2\mathrm{Ric}_g^\circ(\dot{x}, \dot{x})\dot{x}).$$

Then, apply Corollary 2.10.

Corollary 2.13. *The quantities* $\tilde{s}_g^2 x^{(0)}$, $\tilde{s}_g^2 x^{(2)} \otimes g$, $\left|\tilde{s}_g^2 x^{(2)}\right|$ *and* $\tilde{s}_g^3 x$ *are conformal invariants.*

3. The Schwarzian of Immersions and Injectivity Theorems

Let (M, g_M) and (N, g_N) be Riemannian manifolds, and

$$f : M \to N$$

an immersion of class C^3. For a regular curve $x : I \to M$, we have $s^3 x(t) \in T_{x(t)}M$. The image of x under f is again a regular curve, $y = f \circ x : I \to N$, and we also have $s^3 y(t) \in T_{y(t)}N$.

Definition. We define $S^3 f$ by

$$S^3 f = s^3 y - f_*(s^3 x) \in T_y N,$$

where $f_* : T_x M \to T_y N$ is the tangential map of f. The Schwarzian of f is then defined to be

$$S^2 f = (S^3 f)\dot{y}^{-1} \in \bigwedge^0 T_y N \oplus \bigwedge^2 T_y N.$$

As before, $S^3 f$ and $S^2 f$ depend only on the first derivative $X = \dot{x}$ and the second derivative $Y = \nabla_{\dot{x}}\dot{x}$ of x, and may be denoted respectively by $S^3 f(X, Y)$ and $S^2 f(X, Y)$.

If $\dim M = \dim N$ so that f is a local diffeomorphism, $S^3 f$ and $S^2 f$ can be expressed in terms of the Schwarzian of metrics defined in the last section as $f_*\left(S^3_{g_M}\left(f^* g_N\right)\right)$ and $f_*\left(S^2_{g_M}\left(f^* g_N\right)\right)$, respectively. In particular, $S^2 f$ generalizes the classical Schwarzian differential of holomorphic functions in the following sense. Let us identify the complex numbers $\mathbf{C}$ with $\bigwedge^{ev} \mathbf{R}^2 = \mathbf{R} \oplus \mathbf{R}\mathbf{e}_1\mathbf{e}_2$, where $\mathbf{e}_1, \mathbf{e}_2$ denote the canonical basis for $\mathbf{R}^2$. In order to apply our framework to a holomorphic function f, we also need to identify $\mathbf{C}$ with $\mathbf{R}^2 = \bigwedge^1 \mathbf{R}^2$. This identification may be given by multiplication by $\mathbf{e}_1$ from the right:

$$R_{\mathbf{e}_1} : \mathbf{C} = \bigwedge^{ev} \mathbf{R}^2 \to \bigwedge^{1} \mathbf{R}^2.$$

Now, let z, w be regular curves in $\mathbf{C}$ with $w(t) = f(z(t))$. Then a straightforward computation shows

$$\dot{w}\mathbf{e}_1 = f'(z)\dot{z}\mathbf{e}_1,$$

$$\ddot{w}\mathbf{e}_1 = f''(z)\dot{z}^2\mathbf{e}_1 + f'(z)\ddot{z}\mathbf{e}_1,$$

$$\dddot{w}\mathbf{e}_1 = f'''(z)\dot{z}^3\mathbf{e}_1 + 3f''(z)\ddot{z}\dot{z}\mathbf{e}_1 + f'(z)\dddot{z}\mathbf{e}_1,$$

and

$$s^3 w = \dddot{w}\mathbf{e}_1 - \frac{3}{2}(\ddot{w}\mathbf{e}_1)(\dot{w}\mathbf{e}_1)^{-1}(\ddot{w}\mathbf{e}_1)$$

$$= \left(f'''(z) - \frac{3}{2} f''(z)^2 f'(z)^{-1}\right)\dot{z}^3\mathbf{e}_1 + f'(z)\, s^3 z.$$

Therefore,

$$S^3 f = \left(f'''(z) - \frac{3}{2} f''(z)^2 f'(z)^{-1}\right)\dot{z}^3\mathbf{e}_1 \in \bigwedge^{1} \mathbf{R}^2,$$

and

$$S^2 f = S_f(z)\dot{z}^2 \in \mathbf{C} = \bigwedge^{ev} \mathbf{R}^2,$$

where

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is the classical Schwarzian derivative of f.

Before stating the injectivity theorems, we need to introduce two new notions about Riemannian manifolds. Let (M, g) be a Riemannian manifold.

Definition. A function $\alpha : M \to \mathbf{R}$ is called a *connectivity function* for M if every pair of points in M can be joined by a regular curve $x : I \to M$ satisfying

$$s^2 x^{(0)} \leq \alpha(x)\, g(\dot{x}, \dot{x})$$

for all $t \in I$. A constant connectivity function is called a *connectivity constant.*

Definition. A function $\beta : M \to \mathbf{R}$ is called an *injectivity function* for M if a regular curve $x : I \to M$ is injective whenever

$$s^2 x^{(0)} \leq \beta(x)\, g(\dot{x}, \dot{x})$$

for all $t \in I$. A constant injectivity function is called an *injectivity constant.*

Theorem 1.3 states that 0 is an injectivity constant for the Euclidean space $\mathbf{R}^n$. By the remark of Theorem 2.8, 0 is also an injectivity constant for the Euclidean sphere S^n. Using the following lemma together with Lemma 1.1, one can show that $-1/2r^2$ but no values larger than that is an injectivity constant for the cylinder of radius r.

Lemma 3.1. *A (open) curve x of length l can be reparametrized so that*

$$\frac{s^2 x^{(0)}}{g(\dot{x}, \dot{x})} = \frac{1}{2}\left(\kappa^2 - \frac{4\pi^2}{l^2} + \frac{R_g}{n(n-1)}\right),$$

where κ is the geodesic curvature of x.

Proof. Take an arclength parameter s for x so that $-l/2 < s < l/2$. Then replace the parameter s with $t = \tan \frac{\pi s}{l}$.

Let us first consider the case of a conformal immersion $f : M \to N$. By the conformality assumption, we have

$$S^2 f = s^2 y - f_* s^2 x \tag{3.1}$$

for regular curves $x : I \to M$ and $y : I \to N$ with $y = f \circ x$. In particular, we have

$$S^2 f^{(0)} = s^2 y^{(0)} - s^2 x^{(0)} \tag{3.2}$$

in this case.

Proposition 3.2. *Let $f : M \to N$ be a conformal immersion. Then, $S^2 f^{(0)}(X, Y)$ does not depend on Y.*

Proof. Since $S^2 f^{(0)}$ is a local quantity, we may assume that f is an embedding, and maps M conformally onto a submanifold V of N. Let $\hat{g}$ denote the restriction of the metric g_N to V. For regular curves $x : I \to M$ and $y : I \to V \subset N$ satisfying $y = f \circ x$, we have

$$\begin{aligned} S^2 f &= \left(s^3_{g_N} y - s^3_{\hat{g}} y\right) \dot{y}^{-1} + \left(s^3_{\hat{g}} y - f_*\left(s^3_{g_M} x\right)\right) \dot{y}^{-1} \\ &= s^2_{g_N} y - s^2_{\hat{g}} y + f_*\left(S^2_{g_M}\left(f^* g_N\right)\right). \end{aligned} \tag{3.3}$$

Note that the last term of (3.3) does not depend on Y by Corollary 2.7.

Put $\sigma = \sqrt{g_N(\dot{y}, \dot{y})}$ and $\xi = \dot{y}/\sigma$. Write the decomposition of ${}^{g_N}\nabla_\xi \xi$ into its tangential and normal components to V as

$${}^{g_N}\nabla_\xi \xi = {}^{\hat{g}}\nabla_\xi \xi + h(\xi, \xi).$$

Then, the geodesic curvatures κ_{g_N} and $\kappa_{\hat{g}}$ of y in N and in V respectively satisfy

$$\kappa_{g_N}^2 = \kappa_{\hat{g}}^2 + |h(\xi, \xi)|^2.$$

Therefore applying Lemma 1.1 to (3.3), we have

$$S^2 f^{(0)} = \frac{1}{2}\sigma^2\left(\kappa_{g_N}^2 - \kappa_{\hat{g}}^2 + \frac{R_{g_N}}{n(n-1)} - \frac{R_{\hat{g}}}{m(m-1)}\right) + S_{g_M}^2 (f^* g_N)^{(0)}$$

$$= \frac{1}{2}\sigma^2\left(|h(\xi, \xi)|^2 + \frac{R_{g_N}}{n(n-1)} - \frac{R_{\hat{g}}}{m(m-1)}\right) + S_{g_M}^2 (f^* g_N)^{(0)},$$

where $n = \dim N$ and $m = \dim M$. In particular, $S^2 f^{(0)}$ does not depend on Y.

Therefore, we can use the notation $S^2 f^{(0)}(X)$. On the contrary, the 2-part $S^2 f^{(2)}$ of the Schwarzian depends on Y as well as on X, in general.

We now state the main theorem.

Theorem 3.3. *Let (M, g_M) and (N, g_N) be Riemannian manifolds, α be a connectivity function for M, and β be an injectivity function for N. If a conformal immersion*

$$f : M \to N$$

of class C^3 satisfies

$$S^2 f^{(0)}(X) \leq \beta(f(x))\, g_N(f_*(X), f_*(X)) - \alpha(x)\, g_M(X, X) \tag{3.4}$$

for all tangent vectors $X \in T_x M$ at each point $x \in M$, then f is injective.

Proof. Let p, q be any distinct points in M. It suffices to show that $f(p) \neq f(q)$. By the definition of connectivity function, we can take a regular curve $x : [a, b] \to M$ with $x(a) = p$, $x(b) = q$ satisfying

$$s^2 x^{(0)} \leq \alpha(x)\, g_M(\dot{x}, \dot{x}). \tag{3.5}$$

Denote the image of x under f by $y = f \circ x : [a, b] \to N$. Then (3.5) together with (3.4) for $X = \dot{x}$ implies

$$\begin{aligned} s^2 y^{(0)} &= s^2 x^{(0)} + S^2 f^{(0)}(\dot{x}) \\ &\le \beta(y)\, g_N(\dot{y}, \dot{y}). \end{aligned}$$

It then follows from the definition of injectivity function that the curve y is injective. In particular,

$$f(p) = y(a) \neq y(b) = f(q).$$

This completes the proof of Theorem 3.3.

Combining Theorem 3.3 with Lemma 3.1 one obtains the following:

Corollary 3.4. *Let (M, g) be a Riemannian manifold, and C some fixed real number. Suppose that every pair of points of M can be joined by a curve whose geodesic curvature κ and length l satisfy*

$$\frac{1}{2}\left(\kappa^2 - \frac{4\pi^2}{l^2}\right) \le -C.$$

If a conformal immersion

$$f : M \to \mathbf{R}^n \text{ (or } S^n\text{)}$$

of class C^3 satisfies

$$\frac{S^2 f^{(0)}(X)}{g(X, X)} \le C - \frac{R_g}{2n(n-1)}$$

for all tangent vectors X at each point $x \in M$, then f is injective.

Let us consider a holomorphic function of the unit disc $D^2 = \{z \in \mathbf{C} \mid |z| < 1\}$. For the Euclidean metric $ds = |dz|$, the hyperbolic metric $ds = \dfrac{2|dz|}{1 - |z|^2}$, and the spherical metric $ds = \dfrac{2|dz|}{1 + |z|^2}$, we have respectively $\dfrac{R_g}{2} = 0, -1, +1$. These metrics are Möbius equivalent to each other and we have $S^2 f = S_f(z)\dot{z}^2$ in all the three cases.

Corollary 3.5. *A holomorphic function of the unit disc $f : D^2 \to \mathbf{C}$ is univalent if any one of the following holds:*

(1) $|S_f(z)| \le \dfrac{\pi^2}{2}$ *for* $z \in D^2$,

(2) $|S_f(z)| \le \dfrac{2}{(1-|z|^2)^2}$ *for* $z \in D^2$,

(3) $|S_f(z)| \le \dfrac{6}{(1+|z|^2)^2}$ *for* $z \in D^2$.

The conditions (1) and (2) are the Nehari's sufficient conditions for univalency [6]. Another application of Corollary 3.4 is:

Corollary 3.6. *If a holomorphic function $f : U \to \mathbf{C}$ defined on a neighborhood U of the unit circle $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$ satisfies*

$$|S_f(z)| < \frac{3}{2} \quad \text{for} \quad z \in S^1,$$

then f is injective on the unit circle, hence is univalent on some neighborhood of the unit circle.

The bound $3/2$ in the above is the best possible, for the function $f(z) = z^2$ satisfies $|S_f(z)| = 3/2$.

Corollary 3.7. *Let (M, g) be a complete Riemannian manifold of dimension n with $n \ge 3$. If*

$$\mathrm{Ric}_g(X, X) - \frac{R_g}{2(n-1)} g(X, X) \le 0$$

for all tangent vectors $X \in T_xM$ at each point $x \in M$, then any conformal map of M to the sphere S^n is injective.

Proof. One can apply Theorem 3.3 using the connectivity function $\alpha(x) = \dfrac{R_g}{2n(n-1)}$ for M and the injectivity constant $\beta = 0$ for S^n. Note in this case that

$$S^2 f^{(0)}(X) = \frac{1}{n-2}\operatorname{Ric}^\circ_g(X, X)$$

by Theorem 2.8 and (2.4).

Finally, let us consider the case of nonconformal immersions. We must be careful in this case since (3.1) and (3.2) do not hold.

Theorem 3.8. *Let (M, g_M) be a Riemannian manifold of which any two points can be joined by a geodesic. Let (N, g_N) be a Riemannian manifold, and β be an injectivity function for N. If an immersion of class* C^3

$$f : M \to N$$

satisfies

$$S^2 f^{(0)}(X, 0) \leq \beta(f(x))\, g_N(f_*(X), f_*(X)) - \frac{R_{g_M}}{2n(n-1)} g_M(X, X) \quad (3.6)$$

for all tangent vectors X at each point x of M, then f is injective.

Proof. Given a pair of distinct points in M, we take a geodesic $x : I \to M$ passing through the two points. Denote the image of x under f by $y = f \circ x : I \to N$. It suffices to show that y is injective. We have

$$s^2 y = f_*(s^3 x)\, f_*(\dot{x})^{-1} + S^2 f(\dot{x}, \nabla_{\dot{x}}\dot{x}).$$

Since x is a geodesic, we have $\nabla_{\dot{x}}\dot{x} = 0$, and

$$s^3 x = \frac{R_{g_M}}{2n(n-1)} g_M(\dot{x}, \dot{x})\, \dot{x}$$

is parallel to $\dot{x}$. Therefore

$$s^2 y^{(0)} = \frac{R_{g_M}}{2n(n-1)} g_M(\dot{x}, \dot{x}) + S^2 f^{(0)}(\dot{x}, 0).$$

This together with the condition (3.6) for $X = \dot{x}$ implies

$$s^2 y^{(0)} \leq \beta(y)\, g_N(\dot{y}, \dot{y}),$$

and hence y is injective.

Let us consider a local diffeomorphism of the complex plane $f : \mathbf{C} \to \mathbf{C}$. In this case, we have

$$S^2 f(X, 0) = \frac{\mathrm{Num}}{\mathrm{Den}},$$

where

$$\begin{aligned}\mathrm{Num} = {} & \left(f_{zzz} f_z - \frac{3}{2} f_{zz}^2\right) X^4 \\ & + (f_{zzz} f_{\bar z} + 3 f_{zz\bar z} f_z - 6 f_{zz} f_{z\bar z}) X^3 \overline{X} \\ & + \left(3 f_{zz\bar z} f_{\bar z} + 3 f_{z\bar z\bar z} f_z - 3 f_{zz} f_{\bar z\bar z} - 6 f_{z\bar z}^2\right) X^2 \overline{X}^2 \\ & + (3 f_{z\bar z\bar z} f_{\bar z} + f_{\bar z\bar z\bar z} f_z - 6 f_{z\bar z} f_{\bar z\bar z}) X \overline{X}^3 \\ & + \left(f_{\bar z\bar z\bar z} f_{\bar z} - \frac{3}{2} f_{\bar z\bar z}^2\right) \overline{X}^4,\end{aligned}$$

and

$$\mathrm{Den} = f_z^2 X^2 + 2 f_z f_{\bar z} X \overline{X} + f_{\bar z}^2 \overline{X}^2.$$

The following is a generalization of Nehari's univalency theorem to local diffeomorphisms.

Corollary 3.9. *If a local diffeomorphism of the unit disc*

$$f : D^2 \to \mathbf{C}$$

of class C^3 *satisfies*

$$S^2 f^{(0)}(X, 0) \le \frac{2}{(1 - |z|^2)^2} |X|^2$$

for all $X \in \mathbf{C}$ *and* $z \in D^2$, *then f is injective.*

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