OPTIMAL CONTROL OF KELLER-SEGEL EQUATIONS

Sang-Uk Ryu and Atsushi Yagi

Department of Applied Physics, Graduate School of Engineering, Osaka University, Suita, Osaka 565-0871, Japan

1. Introduction

This paper is concerned with the optimal control problem:

$$\underset{u}{\text{Minimize}} \quad J(u) \tag{P}$$

with the cost functional $J(u)$ of the form

$$J(u) = \int_0^T \|y(u) - y_d\|^2_{H^1(\Omega)} dt + \gamma \int_0^T \|u\|^2_{H^\varepsilon(\Omega)} dt, \ u \in L^2(0,T; H^\varepsilon(\Omega)),$$

where $y = y(u)$ is governed by the Keller-Segel equations

$$\begin{cases} \dfrac{\partial y}{\partial t} = a\Delta y - b\nabla\{y\nabla\rho\} & \text{in } \Omega \times (0,T], \\ \dfrac{\partial \rho}{\partial t} = d\Delta\rho + fy - g\rho + \nu u & \text{in } \Omega \times (0,T], \\ \dfrac{\partial y}{\partial n} = \dfrac{\partial \rho}{\partial n} = 0 & \text{on } \partial\Omega \times (0,T], \\ y(x,0) = y_0(x), \quad \rho(x,0) = \rho_0(x) & \text{in } \Omega. \end{cases} \tag{K–S}$$

Here, Ω is a bounded region in $\mathbb{R}^2$ of $\mathcal{C}^3$ class. a, b, d, f, $g > 0$ are given positive numbers and γ, $\nu \geq 0$ are given non negative numbers. $u \geq 0$ is a control function varying in some bounded subset $\mathcal{U}_{ad}$ of $L^2(0,T; H^\varepsilon(\Omega))$, ε being some fixed exponent such that $0 < \varepsilon < \frac{1}{2}$. $n = n(x)$ is the outer normal vector at a boundary point

$x \in \partial\Omega$ and $\frac{\partial}{\partial n}$ denotes the differentiation along the vector n. $y_0(x)$, $\rho_0(x) \geq 0$ are non negative initial functions in $L^2(\Omega)$ and in $H^{1+\varepsilon}(\Omega)$, respectively. y, ρ are unknown functions of the Cauchy problem (K–S).

The Keller-Segel equations were introduced in [10] to describe the aggregation process of the cellular slime mold by the chemical attraction. $y = y(x,t)$ denotes the concentration of amoebae in Ω at the time t, and $\rho = \rho(x,t)$ the concentration of chemical substance in Ω at the time t. The chemotactic term $-b\nabla \cdot \{y\nabla\rho\}$ indicates that the cells are sensitive to chemicals and are attracted by them, and the production term fy indicates that the chemical substance is itself emitted by cells. (K–S) is then a strongly coupled reaction diffusion system.

Several authors have already been interested in the equations, the existence and uniqueness of solution and the asymptotic behavior of solution were studied by them in the case when (K–S) has no control term, $u \equiv 0$. The second author of this paper showed in [12] the existence and uniqueness of $\mathcal{C}^1$ local solution with values in $L^2(\Omega)$ together with some norm behavior of the solutions. Nagai et al. [7] showed that, if the norm $\|y_0\|_{L^1}$ is smaller than a specific number, then (K–S) admits a global solution. On the contrary, Herrero and Verazques [6] proved in the case where Ω is a disk of $\mathbb{R}^2$ that, if y_0, ρ_0 are radial functions and $\|y_0\|_{L^1}$ is sufficiently large, then the norm $\|y(t)\|_{L^2}$ blows up in a finite time, that is, in those cases (K–S) does not admit any global solution.

Aggregation of cellular slime mold is known as a model of the self organization by cell interaction mediated by the chemical substance called cAMP. In this paper, we are concerned with the question of whether one can control the aggregation of cells by cAMP or not. For simplicity we consider a distributed, optimal control problem in the region Ω with the cost function above; other kinds of control problems may also be very interesting. Our techniques presented below will be useful even for some other control problems. Not only the existence of an optimal control, but also the first order necessary condition satisfied by the optimal controls is verified. We believe that, under suitable assumptions, the second order necessary condition will also be satisfied, but this will be disscussed in the forthcoming paper.

Many papers have already been published to study the control problems of nonlinear parabolic equations. In the books Ahmed [1] and Barbu [2], some general frameworks are given for handling the semilinear parabolic equations with monotone perturbations. In [1] the nonlinear terms are monotone functions with linear growth, and in [2] they are generalized to the multivalued maximal monotone operators determined by lower semicontinuous convex functions. Papageorgiou [11] and Casas et al. [4] have studied some quasilinear parabolic equations of monotone type. Since (K–S) is a parabolic system, this is not of monotone type in any sense; furthermore, as mentioned above, [6] shows that the global existence of solutions is not true in general. In this sense it seems that there is no general framework of controls which covers the Keller-Segel equations.

Our techniques are based on the energy estimates and the compact method. We shall establish various a priori estimates for the solutions of (K–S) in order to show that the classical compact method described systematically in Lions [8, Chap. 1] and Lions [9, Chap. III] is available. In section 2, (K–S) is formulated as a semilinear equation in a product Hilbert space. We have to choose a suitable Sobolev space to treat the chemotactic term as a lower term. The existence and uniqueness of local weak solutions to (K–S) are then proved. Section 3 is devoted to showing the global existence of weak solution provided that the norm $\|y_0\|_{L^1}$ is sufficiently small and the control u is in $L^2(0, T; H^1(\Omega))$. In Section 4, the control problem (P) is studied. We fix y_0, ρ_0, and assume that, for every $u \in \mathcal{U}_{ad}$, there exists a unique weak solutions to (K–S) on a fixed interval $[0, S]$, S being independent of $u \in \mathcal{U}_{ad}$. The existence of optimal controls to (P) is proved. Section 5 is devoted to verifying the first order necessary condition. As usual, differentiability of the state with respect to the control must be observed and the adjoint equations must be introduced.

Notations. $\mathbb{N}$ and $\mathbb{R}$ denote the sets of natural numbers and real numbers respectively, and $\mathbb{R}_+ = \{x \in \mathbb{R}; x \geq 0\}$. For a region $\Omega \subset \mathbb{R}^2$, the usual L^p space of real valued functions in Ω is denoted by $L^p(\Omega)$, $1 \leq p \leq \infty$. The Sobolev space of real valued functions in Ω with exponent $s \geq 0$ is denoted by $H^s(\Omega)$. $\mathcal{C}(\overline{\Omega})$ denotes the space of continuous functions on $\overline{\Omega}$. Let I be an interval in $\mathbb{R}$. $L^p(I; \mathcal{H})$, $1 \leq p \leq \infty$, denotes the L^p space of measurable functions in I with values in a Hilber space $\mathcal{H}$. $\mathcal{C}(I; \mathcal{H})$ denotes the space of continuous functions in I with values in $\mathcal{H}$. Let $\mathcal{D}(I)$ denote the space of $\mathcal{C}^\infty$-functions with compact support on I and $\mathcal{D}'(I)$ denote the space of distributions on I. For simplicity, we shall use a universal constant C to denote various constants which are determined in each occurrence in a specific way by $\Omega, a, b, d, f, g, \varepsilon, \nu, \delta, M$, and so forth. In a case when C depends also on some parameter, say θ, it will be denoted by C_θ.

We shall state some well known results on the Sobolev spaces and on the fractional powers of Laplacian which will be used in this paper. For the proof, we refer the reader to Triebel [13].

Interpolation theorem. Let $0 \leq s_0 < s_1 < \infty$. For $s_0 < s < s_1$, $H^s(\Omega) = [H^{s_0}(\Omega), H^{s_1}(\Omega)]_\theta$ with $s = (1-\theta)s_0 + \theta s_1$, and the following estimate holds

$$\|\cdot\|_{H^s} \leq C_{s_0,s_1} \|\cdot\|_{H^{s_0}}^{1-\theta} \|\cdot\|_{H^{s_1}}^{\theta}. \tag{1.1}$$

Embedding theorem. When $0 < s < 1$, $H^s(\Omega) \subset L^p(\Omega)$ for $\frac{1}{p} = \frac{1-s}{2}$ with the estimate

$$\| \cdot \|_{L^p} \leq C_s \| \cdot \|_{H^s}. \tag{1.2}$$

When $s = 1$, $H^1(\Omega) \subset L^q(\Omega)$ for any finite $1 \leq q < \infty$ with the estimate

$$\| \cdot \|_{L^q} \leq C_{q,p} \| \cdot \|_{H^1}^{1-p/q} \| \cdot \|_{L^p}^{p/q}, \tag{1.3}$$

where $1 \leq p < q$. When $s > 1$, $H^s(\Omega) \subset \mathcal{C}(\overline{\Omega})$ with the estimate

$$\| \cdot \|_{\mathcal{C}} \leq C_s \| \cdot \|_{H^s}. \tag{1.4}$$

From (1.3) we observe that $\| \cdot \|_{L^3}^3 \leq C \| \cdot \|_{H^1}^2 \| \cdot \|_{L^1}$. But this can be modified as follows. For any $\eta > 0$,

$$\|y\|_{L^3}^3 \leq \eta \|y\|_{H^1}^2 \|(y+1)\log(y+1)\|_{L^1} + p(\eta^{-1})\|y\|_{L^1}, \quad 0 \leq y \in H^1(\Omega), \tag{1.5}$$

here $p(\cdot)$ denotes some increasing function. For the proof, see [3, p. 1199].

Fractional powers. Let $L = -\varDelta + 1$ be the Laplace operator acting in $L^2(\Omega)$ with the domain $\mathcal{D}(L) = \{y \in H^2(\Omega); \frac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega\}$, L is a positive definite self adjoint operator. Then, for $0 \leq \theta < \frac{3}{4}$,

$$\mathcal{D}(L^\theta) = H^{2\theta}(\Omega) \quad \text{(with norm equivalence)}. \tag{1.6}$$

For $\frac{3}{4} < \theta \leq \frac{3}{2}$,

$$\mathcal{D}(L^\theta) = H_n^{2\theta}(\Omega) = \{y \in H^{2\theta}(\Omega); \frac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega\} \quad \text{(with norm equivalence)}. \tag{1.7}$$

(1.6) and (1.7) are well known for $0 \leq \theta \leq 1$ (even for $\theta = \frac{3}{4}$, the characterization of $\mathcal{D}(L^{3/4})$ is known). Since it is assumed that Ω is of $\mathcal{C}^3$ class, $\mathcal{D}(L^{3/2}) =$

$L^{-1}(H^1(\Omega)) = H^3_n(\Omega)$. Then (1.7) for $1 \le \theta \le \frac{3}{2}$ is verfied from the fact that $\mathcal{D}(L^\theta) = [\mathcal{D}(L), \mathcal{D}(L^{3/2})]_\mu$ with $\theta = 1 + \frac{\mu}{2}$.

2. Existence and uniqueness of local weak solutions

Let $\mathcal{V}$ and $\mathcal{H}$ be two separable real Hilbert spaces with dense and compact embedding $\mathcal{V} \hookrightarrow \mathcal{H}$. Identifying $\mathcal{H}$ and its dual $\mathcal{H}'$ and denoting the dual space of $\mathcal{V}$ by $\mathcal{V}'$, we have: $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$. We denote the scalar product of $\mathcal{H}$ by $(\cdot,\cdot)$ and the norm by $|\cdot|$. The duality product between $\mathcal{V}'$ and $\mathcal{V}$ which coincides with the scalar product of $\mathcal{H}$ on $\mathcal{H} \times \mathcal{H}$ is denoted by $\langle \cdot,\cdot \rangle$, and the norms of $\mathcal{V}$ and $\mathcal{V}'$ by $\|\cdot\|$ and $\|\cdot\|_*$, respectively.

In this section, we shall first prove existence and uniqueness of a weak solution for the Cauchy problem of a semilinear abstract differential equation

$$\begin{cases} \dfrac{dY}{dt} + AY = F(Y) + U(t), \quad 0 < t \le T, \\ Y(0) = Y_0 \end{cases} \tag{E}$$

in the space $\mathcal{V}'$.

Here, A is the positive definite self adjoint operator of $\mathcal{H}$ defined by a symmetric sesquilinear form $a(Y, \tilde{Y})$ on $\mathcal{V}$, $\langle AY, \tilde{Y} \rangle = a(Y, \tilde{Y})$, which satisfies:

$$|a(Y, \tilde{Y})| \le M\|Y\|\|\tilde{Y}\|, \quad Y, \tilde{Y} \in \mathcal{V}, \tag{a.i}$$

$$a(Y, Y) \ge \delta\|Y\|^2, \quad Y \in \mathcal{V} \tag{a.ii}$$

with some δ and $M > 0$. A is also a bounded operator from $\mathcal{V}$ to $\mathcal{V}'$. $F(\cdot)$ is a given continuous function from $\mathcal{V}$ to $\mathcal{V}'$ satisfying :

(f.i) For each $\eta > 0$, there exists an increasing continuous function $\phi_\eta : [0,\infty) \to [0,\infty)$ such that

$$\|F(Y)\|_* \le \eta\|Y\| + \phi_\eta(|Y|), \quad Y \in \mathcal{V};$$

(f.ii) For each $\eta > 0$, there exists an increasing continuous function $\psi_\eta : [0,\infty) \to [0,\infty)$ such that

$$\|F(\tilde{Y}) - F(Y)\|_* \le \eta\|\tilde{Y} - Y\| + (\|\tilde{Y}\| + \|Y\| + 1)\psi_\eta(|\tilde{Y}| + |Y|)|\tilde{Y} - Y|, \quad \tilde{Y}, Y \in \mathcal{V}.$$

$U(\cdot) \in L^2(0,T;\mathcal{V}')$ is a given function and $Y_0 \in \mathcal{H}$ is an initial value.

We then verify the following theorem.

Theorem 2.1. *Let* (a.i), (a.ii), (f.i) *and* (f.ii) *be satisfied. Then, for any* $U \in L^2(0,T;\mathcal{V}')$ *and* $Y_0 \in \mathcal{H}$, *there exists a unique weak solution*

$$Y \in H^1(0,T(Y_0,U);\mathcal{V}') \cap \mathcal{C}([0,T(Y_0,U)];\mathcal{H}) \cap L^2(0,T(Y_0,U);\mathcal{V}) \tag{2.1}$$

to (E), *the number* $T(Y_0,U) > 0$ *is determined by the norms* $\|U\|_{L^2(0,T;\mathcal{V}')}$ *and* $|Y_0|$.

Proof. Let us first prove the uniqueness of the weak solution.

Let $\tilde{Y}$ and Y be two weak solutions of (E) satisfying (2.1) on $[0,T(Y_0,U)]$. Then it is seen that $W = \tilde{Y} - Y$ satisfies:

$$\begin{cases} \dfrac{dW(t)}{dt} + AW(t) = F(\tilde{Y}(t)) - F(Y(t)), \quad 0 < t \le T(Y_0,U), \\ W(0) = 0. \end{cases} \tag{2.2}$$

Taking the scalar product of the equation of (2.2) with W, we have:

$$\frac{1}{2}\frac{d}{dt}|W(t)|^2 + \langle AW(t), W(t)\rangle = \langle F(\tilde{Y}(t)) - F(Y(t)), W(t)\rangle.$$

From (a.ii) and (f.ii), it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|W(t)|^2 + \delta\|W(t)\|^2 \\ &\quad\le \eta\|W(t)\|^2 + (\|\tilde{Y}(t)\| + \|Y(t)\| + 1)\psi_\eta(|\tilde{Y}(t)| + |Y(t)|)|W(t)|\|W(t)\| \\ &\qquad\le \frac{\delta}{2}\|W(t)\|^2 + C(\|\tilde{Y}(t)\|^2 + \|Y(t)\|^2 + 1)\psi_{\frac{\delta}{4}}(|\tilde{Y}(t)| + |Y(t)|)^2|W(t)|^2. \end{aligned}$$

Therefore, by Gronwall's lemma,

$$|W(t)|^2 \leq |W(0)|^2 e^{\int_0^t C(\|\tilde{Y}(s)\|^2+\|Y(s)\|^2+1)\psi_{\frac{\delta}{4}}(|\tilde{Y}(s)|+|Y(s)|)^2 ds}.$$

Since $W(0)=0$, this implies $W(t)=0$ for every $t \in [0, T(Y_0, U)]$.

The existence is proved by several steps.

Step 1. Approximate problem. Let $\{\mathcal{V}_m\}_{m\in\mathbb{N}}$ be an increasing family of finite dimensional vector subspaces of $\mathcal{V}$ such that, for each $V \in \mathcal{V}$, there exists a sequence $\{V_m\}$ satisfying: $V_m \in \mathcal{V}_m$ and $V_m \to V$ in $\mathcal{V}$ as $m \to \infty$. In particular, since $\mathcal{V}$ is dense in $\mathcal{H}$, we can choose for $Y_0 \in \mathcal{H}$ a sequence $\{Y_{0m}\}_{m\in\mathbb{N}}$ such that

$$Y_{0m} \in \mathcal{V}_m \quad \text{and} \quad Y_{0m} \to Y_0 \ \text{ in } \ \mathcal{H} \quad \text{as} \quad m \to \infty, \tag{2.3}$$

without loss of generality, $|Y_{0m}| \leq |Y_0| + 1$.

We take a basis $\{W_{jm}, j = 1, ..., d_m\}$ of $\mathcal{V}_m$, where $d_m = \dim \mathcal{V}_m$, and define an approximate solution of (E) by $Y_m(t) = \sum_{j=1}^{d_m} g_{jm}(t) W_{jm}$. Here, the $g_{jm}(t)$ are chosen so that $Y_m(t)$ satisfies the system of differential equations

$$\begin{cases} \langle \dfrac{dY_m}{dt}, W_{jm}\rangle + \langle AY_m, W_{jm}\rangle = \langle F(Y_m), W_{jm}\rangle + \langle U(t), W_{jm}\rangle, \quad 1 \leq j \leq d_m, \\ Y_m(0) = Y_{0m}. \end{cases} \tag{2.4}$$

This system is obviously equivalent to

$$\begin{cases} \mathcal{B}_m \dfrac{d\overrightarrow{g}_m}{dt} + \mathcal{A}_m \overrightarrow{g}_m = \mathcal{F}_m(\overrightarrow{g}_m) + \mathcal{U}_m(t), \quad 0 < t \leq T, \\ \overrightarrow{g}_m(0) = \big(g_{1m}(0), ..., g_{d_m m}(0)\big). \end{cases} \tag{2.5}$$

Here, $\overrightarrow{g}_m = \overrightarrow{g}_m(t) = (g_{1m}(t), ..., g_{d_m m}(t))$. $\mathcal{B}_m = (\beta_{ijm})$ and $\mathcal{A}_m = (\alpha_{ijm})$ are two $d_m \times d_m$ matrices whose elements are given by $\beta_{ijm} = \langle W_{im}, W_{jm}\rangle$ and

$\alpha_{ijm} = \langle AW_{im}, W_{jm}\rangle$, respectively. $\mathcal{F}_m(\cdot) : \mathbb{R}^{d_m} \to \mathbb{R}^{d_m}$ is defined by $\mathcal{F}_m(\overrightarrow{g}_m) = \big(F_1(\overrightarrow{g}_m), ..., F_{d_m}(\overrightarrow{g}_m)\big)$ with $F_j(\overrightarrow{g}_m) = \langle F(\sum_{i=1}^{d_m} g_{im}W_{im}),\ W_{jm}\rangle,\ j = 1, ..., d_m$, and $\mathcal{U}_m(t) = \big(\langle U(t), W_{1m}\rangle, ..., \langle U(t), W_{d_m m}\rangle\big)$. $g_{jm}(0)$ are chosen so that $\sum_{j=1}^{d_m} g_{jm}(0)W_{jm} = Y_{0m}$. Clearly $\det \mathcal{B}_m \neq 0$, and $\mathcal{F}_m(\cdot)$ is Lipschitz continuous from $\mathbb{R}^{d_m}$ to $\mathbb{R}^{d_m}$. Therefore, by the theory of ordinary differential equations, (2.5) admits a local solution $\overrightarrow{g}_m(t)$.

Step 2. A priori estimate. Multiplying the equation of (2.4) by $g_{jm}(t)$ and summing up the products from 1 to d_m, we obtain the equality

$$\frac{1}{2}\frac{d}{dt}|Y_m(t)|^2 + \langle AY_m(t), Y_m(t)\rangle = \langle F(Y_m(t)), Y_m(t)\rangle + \langle U(t), Y_m(t)\rangle.$$

Then, from (a.ii) and (f.i),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|Y_m(t)|^2 + \delta\|Y_m(t)\|^2 &\leq \eta\|Y_m(t)\|^2 + \big\{\phi_\eta(|Y_m(t)|) + \|U(t)\|_*\big\}\|Y_m(t)\| \\ &\leq \frac{\delta}{2}\|Y_m(t)\|^2 + \tilde{\phi}(|Y_m(t)|^2) + \frac{4}{\delta}\|U(t)\|_*^2\end{aligned}$$

with an increasing, locally Lipschitz continuous function $\tilde{\phi}$: $[0, \infty) \to [0, \infty)$. Therefore,

$$\frac{d}{dt}|Y_m(t)|^2 \leq 2\tilde{\phi}(|Y_m(t)|^2) + G(t),$$

where $G(t) = \frac{8}{\delta}\|U(t)\|_*^2$. Here, we consider the following differential equation:

$$\begin{cases} \dfrac{dZ}{dt} = 2\tilde{\phi}(Z) + G(t), \quad 0 < t \leq T, \\ Z(0) = (|Y_0| + 1)^2. \end{cases}$$

By Caratheodory's theorem there exists a solution $Z(t)$ on an interval $[0, T(Y_0, U)]$, where $T(Y_0, U)$ is determined by the norms $|Y_0|$, $\|U\|_{L^2(0,T;V')}$ and $\tilde{\phi}_\eta$. Since $|Y_{0m}|^2 \leq (|Y_0| + 1)^2$, the comparision theorem then yields that the solution $Y_m(t)$

exists at least on $[0, T(Y_0, U)]$ and the estimate $|Y_m(t)|^2 \leq Z(t), \;\; 0 \leq t \leq T(Y_0, U),$

holds. Therefore,

$$\begin{aligned}
&\frac{1}{2}|Y_m(t)|^2 + \frac{\delta}{2}\int_0^t \|Y_m(s)\|^2 ds \\
&\qquad \leq \frac{1}{2}|Y_{0m}|^2 + \int_0^t \tilde{\phi}(Z(s))ds + \frac{4}{\delta}\int_0^t \|U(s)\|_*^2 ds, \quad 0 < t \leq T(Y_0, U).
\end{aligned}$$

In particular, we have:

$$|Y_m(t)|^2 \leq C\Big[|Y_0|^2 + \int_0^{T(Y_0,U)} \{\tilde{\phi}(Z(s)) + \|U(s)\|_*^2\}ds\Big], \quad 0 < t \leq T(Y_0, U),$$

and

$$\int_0^{T(Y_0,U)} \|Y_m(s)\|^2 ds \leq C\Big[|Y_0|^2 + \int_0^{T(Y_0,U)} \{\tilde{\phi}(Z(s)) + \|U(s)\|_*^2\}ds\Big]. \tag{2.6}$$

Since

$$\begin{aligned}
\left\|\frac{dY_m(t)}{dt}\right\|_*^2 &\leq C\big\{\|AY_m(t)\|_*^2 + \|F(Y_m(t))\|_*^2 + \|U(t)\|_*^2\big\} \\
&\leq C\big\{\|Y_m(t)\|^2 + \phi_1(|Y_m(t)|)^2 + \|U(t)\|_*^2\big\},
\end{aligned}$$

it follows from (2.6) that

$$\int_0^{T(Y_0,U)} \left\|\frac{dY_m(s)}{dt}\right\|_*^2 ds \leq C\Big[|Y_0|^2 + \int_0^{T(Y_0,U)} \{\tilde{\phi}(Z(s)) + \|U(s)\|_*^2\}ds\Big].$$

Step 3. Convergence. We can now extract a subsequence $\{Y_{m'}\}$ of $\{Y_m\}$ such that

$$\begin{gathered}
Y_{m'} \to Y \quad \text{weakly in} \quad L^2(0, T(Y_0, U); \mathcal{V}), \\
\frac{dY_{m'}}{dt} \to \frac{dY}{dt} \quad \text{weakly in} \quad L^2(0, T(Y_0, U); \mathcal{V}'), \\
Y_{m'} \to Y \quad \text{in weak star topology of} \quad L^\infty(0, T(Y_0, U); \mathcal{H}), \\
AY_{m'} \to AY \quad \text{weakly in} \quad L^2(0, T(Y_0, U); \mathcal{V}').
\end{gathered}$$

Moreover, by [8, Chap. 1, Theorem 5.1] it is shown that

$$Y_{m'} \to Y \quad \text{strongly in} \quad L^2(0, T(Y_0, U); \mathcal{H}). \tag{2.7}$$

Let us verify that this Y is a solution to (E). Let $\xi \in \mathcal{D}(0, T(Y_0, U))$ and $V \in \mathcal{V}$, and put $\varPhi_m = \xi(t)V_m$ and $\varPhi = \xi(t)V$, where $V_m \in \mathcal{V}_m$ and $V_m \to V$ in $\mathcal{V}$ as $m \to \infty$. We have particularly $\varPhi_m \to \varPhi$ strongly in $L^2(0, T(Y_0, U); \mathcal{V})$ and $\varPhi'_m = \frac{d\varPhi_m}{dt} \to \varPhi'$ strongly in $L^2(0, T(Y_0, U); \mathcal{H})$. From (2.4), we obtain that

$$\begin{aligned}\int_0^{T(Y_0,U)} \langle Y'_{m'}(t), \varPhi_{m'}(t)\rangle dt + \int_0^{T(Y_0,U)} \langle AY_{m'}(t), \varPhi_{m'}(t)\rangle dt \\ = \int_0^{T(Y_0,U)} \langle F(Y_{m'}(t)), \varPhi_{m'}(t)\rangle dt + \int_0^{T(Y_0,U)} \langle U(t), \varPhi_{m'}(t)\rangle dt\end{aligned} \tag{2.8}$$

On the other hand, (f.ii) implies that, for each $Z \in \mathcal{C}([0, T(Y_0, U)]; \mathcal{V})$,

$$\begin{aligned}&\int_0^{T(Y_0,U)} \big|\langle F(Y_{m'}(t)) - F(Y(t)), Z(t)\rangle\big| dt \\ &\leq \int_0^{T(Y_0,U)} \big\{(\|Y_{m'}(t)\| + \|Y(t)\| + 1)\psi_\eta(|Y_{m'}(t)| + |Y(t)|)|Y_{m'}(t) - Y(t)|\|Z(t)\| \\ &\qquad + \eta\|Y_{m'}(t) - Y(t)\|\|Z(t)\|\big\} dt = I_{1m'} + I_{2m'}.\end{aligned} \tag{2.9}$$

Then, it follows from (2.7) that $\overline{\lim}_{m' \to \infty} I_{1m'} = 0$. Similarly, $\overline{\lim}_{m' \to \infty} I_{2m'} \leq C\eta\|Z\|_{L^2(0,T(Y_0,U);\mathcal{V})}$. Since $\eta > 0$ is arbitrary, this shows that $F(Y_{m'})$ is weakly convergent to $F(Y)$ in $L^2(0, T(Y_0, U); \mathcal{V}')$. Letting $m' \to \infty$ in (2.8), we see that

$$\begin{aligned}\int_0^{T(Y_0,U)} \langle Y'(t), V\rangle \xi(t) dt + \int_0^{T(Y_0,U)} \langle AY(t), V\rangle \xi(t) dt \\ = \int_0^{T(Y_0,U)} \langle F(Y(t), V\rangle \xi(t) dt + \int_0^{T(Y_0,U)} \langle U(t), V\rangle \xi(t) dt,\end{aligned}$$

therefore

$$\langle \frac{dY(\cdot)}{dt}, V\rangle + \langle AY(\cdot), V\rangle = \langle F(Y(\cdot)), V\rangle + \langle U(\cdot), V\rangle \tag{2.10}$$

in the sense of $\mathcal{D}'(0, T(Y_0, U))$. From [5, Chap. XVIII, Theorem 1], it is known that $Y \in H^1(0, T(Y_0, U); \mathcal{V}') \cap L^2(0, T(Y_0, U); \mathcal{V}) \subset \mathcal{C}([0, T(Y_0, U)]; \mathcal{H})$.

Finally, we verify that Y satisfies the initial condition. Let ξ be a real valued $\mathcal{C}^\infty$ function on $[0, T(Y_0, U)]$ such that $\xi(0) = 1$ and that $\xi(t) = 0$ in a neighbourhood of $T(Y_0, U)$. Multiplying (2.10) by $\xi(t)$ and integrating the product by parts, we have:

$$\begin{aligned} &-\int_0^{T(Y_0,U)} \langle Y(t), V\rangle \xi'(t)dt + \int_0^{T(Y_0,U)} \langle AY(t), V\rangle \xi(t)dt \\ &\quad = \big(Y(0), V\big) + \int_0^{T(Y_0,U)} \langle F(Y(t)), V\rangle \xi(t)dt + \int_0^{T(Y_0,U)} \langle U(t), V\rangle \xi(t)dt. \quad (2.11) \end{aligned}$$

On the other hand, integrating the first term of (2.8) by parts and letting $m' \to \infty$, we see that

$$\begin{aligned} &-\int_0^{T(Y_0,U)} \langle Y(t), V\rangle \xi'(t)dt + \int_0^{T(Y_0,U)} \langle AY(t), V\rangle \xi(t)dt \\ &\qquad = \big(Y_0, V\big) + \int_0^{T(Y_0,U)} \langle F(Y(t)), V\rangle \xi(t)dt + \int_0^{T(Y_0,U)} \langle U(t), V\rangle \xi(t)dt. \end{aligned}$$

Comparing this with (2.11), we see that $\big(Y(0), V\big) = \big(Y_0, V\big)$ for all $V \in \mathcal{V}$; hence, $Y(0) = Y_0$. Thus, $Y(\cdot)$ has been shown to be the desired weak solution. $\square$

We shall now construct a local weak solution to (K–S) by applying Theorem 2.1.

Let $A_1 = -a\Delta + a$ and $A_2 = -d\Delta + g$ with the same domain $\mathcal{D}(A_i) = \{z \in H^2(\Omega); \frac{\partial z}{\partial n} = 0 \text{ on } \partial\Omega\}$ $(i = 1, 2)$. Then, A_i are two positive definite self adjoint operators in $L^2(\Omega)$. As noticed in (1.6) and (1.7), $\mathcal{D}(A_i^\theta) = H^{2\theta}(\Omega)$ for $0 \le \theta < \frac{3}{4}$, and $\mathcal{D}(A_i^\theta) = H_n^{2\theta}(\Omega)$ for $\frac{3}{4} < \theta \le \frac{3}{2}$. We set two product Hilbert spaces $\mathcal{V} \subset \mathcal{H}$ as $\mathcal{V} = H^1(\Omega) \times \mathcal{D}(A_2^{1+\varepsilon/2})$ and $\mathcal{H} = L^2(\Omega) \times \mathcal{D}(A_2^{(1+\varepsilon)/2})$, respectively, with some fixed $0 < \varepsilon < \frac{1}{2}$. By identifying $\mathcal{H}$ with its dual space, we consider $\mathcal{V} \subset \mathcal{H} =$

$\mathcal{H}' \subset \mathcal{V}'$. It is then seen that $\mathcal{V}' = (H^1(\Omega))' \times \mathcal{D}(A_2^{\varepsilon/2})$ with the duality product $\langle \Phi, Y\rangle_{\mathcal{V}'\times\mathcal{V}} = \langle \zeta, y\rangle_{(H^1)'\times H^1} + \left(A_2^{\varepsilon/2}\varphi, A_2^{1+\varepsilon/2}\rho\right)_{L^2}$, $\Phi = \binom{\zeta}{\varphi}$, $Y = \binom{y}{\rho}$. We set also a symmetric sesquilinear form on $\mathcal{V}\times\mathcal{V}$:

$$a(Y,\tilde{Y}) = \left(A_1^{1/2}y, A_1^{1/2}\tilde{y}\right)_{L^2} + \left(A_2^{1+\varepsilon/2}\rho, A_2^{1+\varepsilon/2}\tilde{\rho}\right)_{L^2}, \quad Y = \begin{pmatrix} y \\ \rho \end{pmatrix}, \tilde{Y} = \begin{pmatrix} \tilde{y} \\ \tilde{\rho} \end{pmatrix} \in \mathcal{V}.$$

Clearly, $a(\cdot,\cdot)$ satisfies (a.i) and (a.ii). This form in fact defines a linear isomorphism $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ from $\mathcal{V}$ to $\mathcal{V}'$, and A becomes a positive definite self adjoint operator in $\mathcal{H}$.

(K–S) is, then, formulated as an abstract equation

$$\begin{cases} \dfrac{dY}{dt} + AY = F(Y) + U(t), & 0 < t \le T, \\ Y(0) = Y_0 \end{cases} \tag{2.12}$$

in the space $\mathcal{V}'$. Here, $F(\cdot): \mathcal{V} \to \mathcal{V}'$ is the mapping

$$F(Y) = \begin{pmatrix} -b\nabla\{y\nabla\rho\} + ay \\ fy \end{pmatrix}, \qquad Y = \begin{pmatrix} y \\ \rho \end{pmatrix} \in \mathcal{V}. \tag{2.13}$$

$U(t)$ and Y_0 are defined by $U(t) = \binom{0}{\nu u(t)}$ and $Y_0 = \binom{y_0}{\rho_0}$, respectively.

Verification of (f.i) is direct. Indeed, since $Y = \binom{y}{\rho} \in \mathcal{V}$ implies that $\frac{\partial\rho}{\partial n} = 0$ on $\partial\Omega$, we have:

$$\begin{aligned} \|\nabla\{y\nabla\rho\}\|_{(H^1)'} &= \sup_{\|v\|_{H^1}\le 1} \left|\langle \nabla\{y\nabla\rho\}, v\rangle_{(H^1(\Omega))'\times H^1(\Omega)}\right| \\ &= \sup_{\|v\|_{H^1}\le 1} \left|\int_\Omega \{y\nabla\rho\}\cdot\nabla v dx\right| \le C\|y\|_{L^4}\|\nabla\rho\|_{L^4} \\ &\le C\|y\|_{L^2}^{1/2}\|y\|_{H^1}^{1/2}\|\rho\|_{H^1}^{1/2}\|\rho\|_{H^2}^{1/2} \qquad \text{(by (1.3))} \\ &\le C\|y\|_{L^2}^{1/2}\|y\|_{H^1}^{1/2}\|\rho\|_{H^{1+\varepsilon}}^{(1+\varepsilon)/2}\|\rho\|_{H^{2+\varepsilon}}^{(1-\varepsilon)/2} \qquad \text{(by (1.1))} \\ &\le C|Y|^{1+\varepsilon/2}\|Y\|^{1-\varepsilon/2}, \quad Y = \begin{pmatrix} y \\ \rho \end{pmatrix} \in \mathcal{V}. \end{aligned}$$

In addition,

$$\|y\|_{H^\varepsilon} \le C\|y\|_{H^1}^{\varepsilon}\|y\|_{L^2}^{1-\varepsilon} \le C\|Y\|^{\varepsilon}|Y|^{1-\varepsilon}.$$

Hence, the condition (f.i) is fulfilled.

By using (1.2) and (1.4), we obtain that

$$\begin{aligned}\Big|\int_\Omega \{(\tilde{y}-y)\nabla\tilde{\rho}\}\cdot\nabla v dx\Big| &\le C\|\tilde{y}-y\|_{L^2}\|\tilde{\rho}\|_{H^{2+\varepsilon}}\|v\|_{H^1},\\ \Big|\int_\Omega \{y\nabla(\tilde{\rho}-\rho)\}\cdot\nabla v dx\Big| &\le C\|y\|_{H^1}\|\tilde{\rho}-\rho\|_{H^{1+\varepsilon}}\|v\|_{H^1}.\end{aligned}$$

In addition,

$$\|\tilde{y}-y\|_{H^\varepsilon} \le C\|\tilde{Y}-Y\|^{\varepsilon}|\tilde{Y}-Y|^{1-\varepsilon} \le \eta\|\tilde{Y}-Y\| + C_\eta|\tilde{Y}-Y|,$$

where $\eta > 0$ is arbitrary. Hence, $F(\cdot)$ fulfills (f.ii) also.

We can now state the main result of this section.

Theorem 2.2. *Let* $0 \le y_0 \in L^2(\Omega)$, $0 \le \rho_0 \in H^{1+\varepsilon}(\Omega)$, *and let* $0 \le u \in L^2(0,T;H^\varepsilon(\Omega))$. *Then,* (K–S) *possesses a unique non negative local solution*

$$0 \le y \in H^1(0,S;(H^1(\Omega))') \cap \mathcal{C}([0,S];L^2(\Omega)) \cap L^2(0,S;H^1(\Omega)),$$

$$0 \le \rho \in H^1(0,S;H^\varepsilon(\Omega)) \cap \mathcal{C}([0,S];H^{1+\varepsilon}(\Omega)) \cap L^2(0,S;H^{2+\varepsilon}_n(\Omega)),$$

the time $S \in (0,T]$ *is determined by the norms* $\|u\|_{L^2(0,T);H^\varepsilon(\Omega))}$, $\|y_0\|_{L^2(\Omega)}$ *and* $\|\rho_0\|_{H^{1+\varepsilon}(\Omega)}$.

Proof. The existence and uniqueness of a local solution y, ρ to (K–S) is an immediate consequence of Theorem 2.1. Therefore, the only thing to be proved here is that the solution y, ρ is non negative.

According to the result in [12, Sec. 4], it is known that, for $0 \le u \in \mathcal{C}^\sigma([0,T];H^\varepsilon(\Omega))$, (K–S) admits a non negative solution. Then, as in the proof of Theorem 2.1, the non negativity of the solution y, ρ for the general $0 \le u \in L^2(0,T;H^\varepsilon(\Omega))$ is verified by considering a sequence $0 \le u_n \in \mathcal{C}^\sigma([0,T];H^\varepsilon(\Omega))$ such that $u_n \to u$ in $L^2(0,T;H^\varepsilon(\Omega))$. $\square$

3. Global existence

In the case when the initial function y_0 is sufficiently small, we can obtain some a priori estimates for the weak solution and show the global existence.

Theorem 3.1. *There exists some constant $\ell > 0$ such that, if $\|y_0\|_{L^1(\Omega)} \le \ell$, then, for any $0 \le u \in L^2(0,T;H^1(\Omega))$, the weak solution y, ρ in Theorem 2.2 can be extended as weak solution on the whole interval $[0,T]$.*

Proof. Let y, ρ be any weak solution as in Theorem 2.2 on an interval $[0,S]$. We shall establish a priori estimates by three steps.

Step 1. It is easy to see that

$$\begin{aligned}\frac{d}{dt}\int_\Omega y dx = \langle \frac{dy}{dt}, 1\rangle_{(H^1)'\times H^1} &= a\langle \Delta y, 1\rangle_{(H^1)'\times H^1} \\ &\quad - b\langle \nabla\{y\nabla\rho\}, 1\rangle_{(H^1)'\times H^1} = 0 \qquad \text{a. e. } t\in(0,S).\end{aligned}$$

Since $y \ge 0$,

$$\|y(t)\|_{L^1(\Omega)} = \|y_0\|_{L^1(\Omega)} \qquad \text{for all } t\in[0,S]. \tag{3.1}$$

Step 2. We consider the function $\log(y+1)$; since $\nabla\log(y+1) = \frac{\nabla y}{y+1}$, it follows that $\log(y+1)\in L^2(0,S;H^1(\Omega))$. Noting that

$$\frac{d}{dt}\int_\Omega \{(y(t)+1)\log(y(t)+1) - y(t)\}dx = \langle \frac{dy}{dt}(t), \log(y(t)+1)\rangle_{(H^1)'\times H^1},$$

we obtain from the first equation in (K–S) that

$$\frac{d}{dt}\int_\Omega\{(y(t)+1)\log(y(t)+1)-y(t)\}dx \\ + 4a\int_\Omega|\nabla\sqrt{y(t)+1}|^2dx = b\int_\Omega\{\log(y(t)+1)-y(t)\}\Delta\rho(t)dx.$$

Therefore,

$$\frac{d}{dt}\|(y(t)+1)\log(y(t)+1)\|_{L^1}+4a\|\nabla\sqrt{y(t)+1}\|_{L^2}^2 \\ \leq \eta\|\Delta\rho(t)\|_{L^2}^2+C\eta^{-1}\|y(t)\|_{L^2}^2 \quad (3.2)$$

with an arbitrary $\eta>0$.

On the other hand, from the second equation of (K–S), we obtain the following energy equalities

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\rho(t)^2dx+d\int_\Omega|\nabla\rho(t)|^2dx+g\int_\Omega\rho(t)^2dx \\ = f\int_\Omega y(t)\rho(t)dx+\nu\int_\Omega u(t)\rho(t)dx$$

and

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla\rho(t)|^2dx+d\int_\Omega|\Delta\rho(t)|^2dx+g\int_\Omega|\nabla\rho(t)|^2dx \\ = f\int_\Omega y(t)\Delta\rho(t)dx+\nu\int_\Omega u(t)\Delta\rho(t)dx.$$

Therefore, it follows that

$$\frac{1}{2}\frac{d}{dt}\|\rho(t)\|_{L^2}^2+d\|\nabla\rho(t)\|_{L^2}^2+\frac{g}{2}\|\rho(t)\|_{L^2}^2\leq C\left\{\|y(t)\|_{L^2}^2+\|u(t)\|_{L^2}^2\right\}$$

and

$$\frac{1}{2}\frac{d}{dt}\|\nabla\rho(t)\|_{L^2}^2+\frac{d}{2}\|\Delta\rho(t)\|_{L^2}^2+g\|\nabla\rho(t)\|_{L^2}^2\leq C\left\{\|y(t)\|_{L^2}^2+\|u(t)\|_{L^2}^2\right\},$$

respectively. By addition,

$$\frac{d}{dt}\|\rho(t)\|_{H^1}^2 + c\|\rho(t)\|_{H^2}^2 \le C\left\{\|y(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2\right\}, \tag{3.3}$$

where $c = \min\{d, g\}$. Here, we notice, applying (1.3) with $p = 2$, $q = 8$, that

$$\begin{aligned}\|y\|_{L^2} &\le \|y\|_{L^1}^{1/3}\|y\|_{L^4}^{2/3} \le \|y\|_{L^1}^{1/3}\|\sqrt{y+1}\|_{L^8}^{4/3} \\ &\le C\|y\|_{L^1}^{1/3}\|\sqrt{y+1}\|_{L^2}^{1/3}\left(\|\nabla\sqrt{y+1}\|_{L^2} + \|\sqrt{y+1}\|_{L^2}\right), \quad 0 \le y \in H^1(\Omega).\end{aligned}$$

Similarly,

$$\begin{aligned}\|(y+1)\log(y+1)\|_{L^1} &\le \|\sqrt{y+1}\|_{L^4}^4 \\ &\le C\|\sqrt{y+1}\|_{L^2}^2(\|\nabla\sqrt{y+1}\|_{L^2}^2 + \|\sqrt{y+1}\|_{L^2}^2), \quad 0 \le y \in H^1(\Omega).\end{aligned}$$

Therefore, from (3.1),

$$\|y(t)\|_{L^2}^2 \le C\left\{\|y_0\|_{L^1}^{2/3}(\|y_0\|_{L^1}+1)^{1/3}\|\nabla\sqrt{y(t)+1}\|_{L^2}^2 + \|y_0\|_{L^1}^2 + 1\right\} \tag{3.4}$$

and

$$\|(y(t)+1)\log(y(t)+1)\|_{L^1} \le C\{(\|y_0\|_{L^1}+1)\|\nabla\sqrt{y(t)+1}\|_{L^2}^2 + \|y_0\|_{L^1}^2 + 1\}. \tag{3.5}$$

We now sum up (3.2) and (3.3) and use (3.4). Then,

$$\begin{aligned}&\frac{d}{dt}\{\|(y(t)+1)\log(y(t)+1)\|_{L^1} + \|\rho(t)\|_{H^1}^2\} \\ &\quad + \{4a - C\eta^{-1}\|y_0\|_{L^1}^{2/3}(\|y_0\|_{L^1}+1)^{1/3}\}\|\nabla\sqrt{y(t)+1}\|_{L^2}^2 + \{c-\eta\}\|\rho(t)\|_{H^2}^2 \\ &\qquad \le C\left\{\eta^{-1}(\|y_0\|_{L^1}^2+1) + \|u(t)\|_{L^2}^2\right\}.\end{aligned}$$

Take η, ℓ so that $\eta = \frac{c}{2}$, $\ell^{2/3}(\ell+1)^{1/3} = \frac{2a\eta}{C}$, respectively, and use (3.5). Then, if $\|y_0\|_{L^1} \le \ell$, the estimate

$$\begin{aligned}
&\frac{d}{dt}\{\|(y(t)+1)\log(y(t)+1)\|_{L^1} + \|\rho(t)\|_{H^1}^2\} \\
&\qquad\qquad + \frac{2a}{C(\ell+1)}\|(y(t)+1)\log(y(t)+1)\|_{L^1} + \frac{c}{2}\|\rho(t)\|_{H^1}^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad \le C\left\{\ell^2 + 1 + \|u(t)\|_{L^2}^2\right\}
\end{aligned}$$

holds for a. e. $t \in (0, S)$. Hence,

$$\begin{aligned}
&\|(y(t)+1)\log(y(t)+1)\|_{L^1} + \|\rho(t)\|_{H^1}^2 \\
&\quad \le \|(y_0+1)\log(y_0+1)\|_{L^1} + \|\rho_0\|_{H^1}^2 + C\left\{\|u\|_{L^2(0,T;L^2(\Omega))}^2 + \ell^2 + 1\right\} \qquad (3.6)
\end{aligned}$$

holds for all $t \in [0, S]$, C being independent of S.

Step 3. Take $t_1 \in (0, S)$ so that $\rho(t_1) \in \mathcal{D}(A_2^{1+\varepsilon/2})$, and set $y_1 = y(t_1)$, $\rho_1 = \rho(t_1)$. In this step, t varies in $[t_1, S]$. From the first equation in (K–S),

$$\frac{1}{2}\frac{d}{dt}\int_\Omega y(t)^2 dx + a\int_\Omega |\nabla y(t)|^2 dx = \frac{b}{2}\int_\Omega y(t)^2 \varDelta\rho(t) dx,$$

so that

$$\frac{1}{2}\frac{d}{dt}\|y(t)\|_{L^2}^2 + a\|\nabla y(t)\|_{L^2}^2 \le \eta\|\varDelta\rho(t)\|_{L^3}^3 + C\eta^{-1/2}\|y(t)\|_{L^3}^3 \qquad (3.7)$$

with an arbitrary $\eta > 0$.

On the other hand, we consider ρ as solution of the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\rho = -A_2\rho + fy + \nu u, \quad t_1 < t < S, \\ \rho(t_1) = \rho_1 \end{cases}$$

in the space $\mathcal{D}(A_2^{1/2}) = H^1(\Omega)$. Since $fy + \nu u \in L^2(t_1, S; H^1(\Omega))$ and $\rho_1 \in \mathcal{D}(A_2^{1+\varepsilon/2})$, it follows that $\rho \in L^2(t_1, S; \mathcal{D}(A_2^{3/2})) \cap H^1(t_1, S; \mathcal{D}(A_2^{1/2}))$ and

$$\frac{d}{dt} A_2^{1/2} \rho = -A_2^{3/2} \rho + f A_2^{1/2} y + \nu A_2^{1/2} u.$$

Therefore,

$$\frac{1}{2} \frac{d}{dt} \|A_2 \rho(t)\|_{L^2}^2 + \frac{1}{2} \|A_2^{3/2} \rho(t)\|_{L^2}^2 \leq C\{\|A_2^{1/2} y(t)\|_{L^2}^2 + \|A_2^{1/2} u(t)\|_{L^2}^2\},$$

or, since $\mathcal{D}(A_2^{3/2}) \subset H^3(\Omega)$ (from (1.7)),

$$\frac{d}{dt} \|A_2 \rho(t)\|_{L^2}^2 + \delta \|\rho(t)\|_{H^3}^2 \leq C\{\|y(t)\|_{H^1}^2 + \|u(t)\|_{H^1}^2\} \tag{3.8}$$

with some $\delta > 0$.

From (1.1) and (1.3) it is verified that

$$\|\Delta \rho\|_{L^3} \leq C \|\rho\|_{H^3}^{1/3} \|\rho\|_{H^2}^{2/3} \leq C \|\rho\|_{H^3}^{2/3} \|\rho\|_{H^1}^{1/3}, \quad \rho \in H^3(\Omega).$$

Therefore, (3.6) together with this yields that

$$\|\Delta \rho(t)\|_{L^3}^3 \leq C \|\rho(t)\|_{H^3}^2.$$

In addition, using (1.5), we verify from (3.6) that

$$\|y(t)\|_{L^3}^3 \leq C_{u,\ell} \zeta \|y(t)\|_{H^1}^2 + p(\zeta^{-1}) \ell,$$

where $\zeta > 0$ is an arbitrary number. Similarly, from

$$\|y\|_{L^2} \leq \tfrac{1}{2} \|\nabla y\|_{L^2} + C \|y\|_{L^1}, \quad y \in H^1(\Omega),$$

it follows that

$$\|y(t)\|_{H^1}^2 \le 2\|\nabla y(t)\|_{L^2}^2 + C\ell^2.$$

We now sum up (3.7) which is multipled by a constant $\frac{4C}{a}$, where this $C > 0$ denotes the constant appearing in (3.8), and (3.8). Then, it follows that

$$\frac{d}{dt}\left\{\frac{2C}{a}\|y(t)\|_{L^2}^2 + \|A_2\rho(t)\|_{L^2}^2\right\} + \{2C - C_{u,\ell}\zeta\eta^{-1/2}\}\|y(t)\|_{H^1}^2 \\ + (\delta - C\eta)\|\rho(t)\|_{H^3}^2 \le C\{\|u(t)\|_{H^1}^2 + \eta^{-1/2}p(\zeta^{-1}) + 1\}.$$

Hence we conclude that

$$\int_{t_1}^{S}\{\|y(t)\|_{H^1}^2 + \|\rho(t)\|_{H^3}^2\}dt \le C_{u,\ell}\{\|y_1\|_{L^2}^2 + \|\rho_1\|_{H^2}^2 + \int_0^T \|u(t)\|_{H^1}^2 dt + 1\}$$

with some constant $C_{u,\ell}$ independent of S.

Completion of the proof. By the a priori estimates established above, we have verified that the norms $\|y\|_{L^2(t_1,S;H^1)}$ and $\|\rho\|_{L^2(t_1,S;H^3)}$ do not depend on S. As a consequence, the norms $\|y\|_{H^1(t_1,S;(H^1)')}$ and $\|\rho\|_{H^1(t_1,S;H^1)}$, and hence those of $\|y\|_{\mathcal{C}([t_1,S];L^2)}$ and $\|\rho\|_{\mathcal{C}([t_1,S];H^2)}$, do not depend on S. In particular, this shows that the solution y, ρ can be extended as weak solution beyond the S. By the standard argument on the extension of weak solutions, we can then prove the desired result. $\square$

4. Existence of optimal control

In this section, we shall deal with the Problem (P) described in Introduction. If we set $\mathcal{U} = L^2(0,T;\mathcal{V}')$ and

$$\mathcal{U}_{ad} = \left\{\begin{pmatrix}0\\u\end{pmatrix} \in \mathcal{U}; u \in L^2(0,T;H^\varepsilon(\Omega)), u \ge 0, \|u\|_{L^2(0,T;H^\varepsilon)} \le C\right\},$$

then $\mathcal{U}_{ad}$ is closed, bounded and convex subset of $\mathcal{U}$. The problem (P) is obviously formulated as follows:

$$\text{Minimize } J(U), \tag{$\overline{P}$}$$

where the cost functional $J(U)$ is of the form

$$J(U) = \int_0^S \|Y(U) - Y_d\|^2 dt + \gamma \int_0^S \|U\|_*^2 dt, \quad U \in \mathcal{U}_{ad}.$$

Here, $Y(U)$, $U \in \mathcal{U}_{ad}$, is the weak solution to (2.12) and is assumed to exist on a fixed interval $[0, S]$. $Y_d = \binom{y_d}{0}$ is a fixed element of $L^2(0, S; \mathcal{V})$ with $y_d \in L^2(0, T; H^1(\Omega))$. γ is a non negative constant.

Remark. Let $Y_0 \in \mathcal{H}$ be fixed. By Theorem 2.1, for $U \in \mathcal{U}_{ad}$, $Y(U)$ exists on the interval $[0, T(U)]$ with $T(U) > 0$ depending on $\|U\|_{L^2(0,T;\mathcal{V}')}$. Hence, $0 < S \leq \inf\{T(U); U \in \mathcal{U}_{ad}\}$. Furthermore, by Theorem 3.1, if $\|y_0\|_{L^1}$ is sufficiently small and u is in $L^2(0, T; H^1(\Omega))$, $Y(U)$ exists on the whole interval [0,T]; hence, $S = T$.

We prove the following theorem.

Theorem 4.1. *There exists an optimal control $\overline{U} \in \mathcal{U}_{ad}$ for $(\overline{P})$ such that*

$$J(\overline{U}) = \min_{U \in \mathcal{U}_{ad}} J(U).$$

Proof. The proof is quite standard, so it will be only sketched (cf. [2, Chap. 5, Proposition 1.1] and [9, Chap. III, Theorem 15.1]). Let $\{U_n\} \subset \mathcal{U}_{ad}$ be a minimizing sequence such that $\lim_{n\to\infty} J(U_n) = \min_{U \in \mathcal{U}_{ad}} J(U)$. Since $\{U_n\}$ is bounded, we can assume that $U_n \to \overline{U}$ weakly in $L^2(0, S; \mathcal{V}')$. For simplicity, we will write Y_n instead of the solution $Y(U_n)$ of (2.12) corresponding to U_n. Using the similar estimate of the solution Y_n, we see as in the proof of Theorem 2.1 that

$$Y_n \to \overline{Y} \text{ weakly in } \ L^2(0,S;\mathcal{V}),$$
$$\frac{dY_n}{dt} \to \frac{d\overline{Y}}{dt} \quad \text{weakly in} \quad L^2(0,S;\mathcal{V}').$$

Since $\mathcal{V}$ is compactly embedded in $\mathcal{H}$, we can conclude that $Y_n \to \overline{Y}$ strongly in $L^2(0,S;\mathcal{H})$. Hence, by the uniqueness, $\overline{Y}$ is the weak solution of (2.12) corresponding to $\overline{U}$ (i.e. $\overline{Y} = Y(\overline{U})$). Since $Y(U_n) - Y_d$ is weakly convergent to $Y(\overline{U}) - Y_d$ in $L^2(0,S;\mathcal{V})$, we have: $\min\limits_{V\in\mathcal{U}_{ad}} J(V) \le J(\overline{U}) \le \underline{\lim}_{n\to\infty} J(U_n) = \min\limits_{V\in\mathcal{U}_{ad}} J(V)$. Hence, $\min\limits_{V\in\mathcal{U}_{ad}} J(V) = J(\overline{U})$. $\square$

5. First order necessary condition

In this section, we show the first order necessary condition for the Problem (P). We denote the scalar products in $\mathcal{V}$ and $\mathcal{V}'$ by $\langle\cdot,\cdot\rangle_{\mathcal{V}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{V}'}$, respectively. In order to derive the necessary condition satisfied by an optimal control $\overline{U} = \binom{0}{\nu\overline{u}}$, the mapping $F(\cdot)\colon \mathcal{V} \to \mathcal{V}'$ defined by (2.13) must be Fréchet differentiable and some estimate for the derivative $F'(Y)(\cdot)$ is necessary. It is indeed observed by a direct calculation that $F(Y)$ is Fréchet differentiable with the derivative

$$F'(Y)Z = \begin{pmatrix} -b\nabla\{y\nabla w\} - b\nabla\{z\nabla\rho\} + az \\ fz \end{pmatrix}, \quad Y = \begin{pmatrix} y \\ \rho \end{pmatrix}, Z = \begin{pmatrix} z \\ w \end{pmatrix} \in \mathcal{V}.$$

Lemma 5.1. *For each $\eta > 0$, there exists constant $C_\eta > 0$ such that*

$$|\langle F'(Y)Z, P\rangle| \le \begin{cases} \eta\|Z\|\|P\| + C_\eta(\|Y\|+1)|Z|\|P\|, & Y,Z,P \in \mathcal{V}, \quad (5.1) \\ \eta\|Z\|\|P\| + C_\eta(\|Y\|+1)\|Z\||P|, & Y,Z,P \in \mathcal{V}. \quad (5.2) \end{cases}$$

In addition, there exists a constant $C > 0$ such that

$$\|F'(\tilde{Y})Z - F'(Y)Z\|_* \le C\|Z\||\tilde{Y} - Y|, \quad \tilde{Y}, Y, Z \in \mathcal{V}. \tag{5.3}$$

Proof. Verification of (5.1) and (5.3) is immediate if we use the same estimates as in the verification of (f.ii). To prove (5.2) we notice that

$$\langle \nabla\{y\nabla w\}, p\rangle_{(H^1)'\times H^1} \le C\|\nabla\{y\nabla w\}\|_{L^2}\|p\|_{L^2} \le C\|y\|_{H^1}\|w\|_{H^{2+\varepsilon}}\|p\|_{L^2}$$

and

$$\begin{aligned} |\big(A_2^{\varepsilon/2}z, A_2^{1+\varepsilon/2}p_2\big)_{L^2}| &= |\big(A_2^{1/2}z, A_2^{1/2+\varepsilon}p_2\big)_{L^2}| \le \|A_2^{1/2}z\|_{L^2}\|A_2^{1/2+\varepsilon}p_2\|_{L^2} \\ &\le \|z\|_{H^1}\big\{C_\eta\|A_2^{(1+\varepsilon)/2}p_2\|_{L^2} + \eta\|A_2^{1+\varepsilon/2}p_2\|_{L^2}\big\} \end{aligned}$$

with an arbitrary $\eta > 0$. Then (5.2) is an immediate consequence of these estimates. $\square$

Proposition 5.2. *The mapping $Y : \mathcal{U}_{ad} \to H^1(0,S;\mathcal{V}')\cap C([0,S];\mathcal{H})\cap L^2(0,S;\mathcal{V})$ is Gâteaux differentiable with respect to U. For $V \in \mathcal{U}_{ad}$, $Y'(U)V = Z$ is the unique solution in $H^1(0,S;\mathcal{V}')\cap C([0,S];\mathcal{H})\cap L^2(0,S;\mathcal{V})$ of the problem*

$$\begin{cases} \dfrac{d}{dt}Z + AZ + F'(Y)Z = V(t), \quad 0 < t \le S, \\ Z(0) = 0. \end{cases} \tag{5.4}$$

Proof. Let $U, V \in \mathcal{U}_{ad}$ and $0 \le h \le 1$. Let Y_h and Y be the solutions of (2.12) corresponding to $U + hV$ and U, respectively.

Step 1. $Y_h \to Y$ strongly in $\mathcal{C}([0,S];\mathcal{H})$ as $h \to 0$. Let $W = Y_h - Y$. Obviously, W satisfies:

$$\begin{cases} \dfrac{d}{dt}W + AW + F(Y_h(t)) - F(Y(t)) = hV(t), \quad 0 < t \le S, \\ W(0) = 0. \end{cases} \tag{5.5}$$

Taking the scalar product of the equation (5.5) with W, we obtain that

$$\frac{1}{2}\frac{d}{dt}|W(t)|^2 + \langle AW(t), W(t)\rangle = \langle F(Y_h(t)) - F(Y(t)), W(t)\rangle + \langle hV(t), W(t)\rangle.$$

Using (a.ii) and (f.ii), we have:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|W(t)|^2 + \delta\|W(t)\|^2 \\
&\quad\leq \frac{\delta}{2}\|W(t)\|^2 + \big(\|Y_h(t)\|^2 + \|Y(t)\|^2 + 1\big)\psi_{\frac{\delta}{4}}\big(|Y_h(t)| + |Y(t)|\big)^2|W(t)|^2 \\
&\qquad\qquad + 4h^2\delta^{-1}\|V(t)\|_*^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}|W(t)|^2 + \frac{\delta}{2}\int_0^t \|W(s)\|^2 ds \\
&\quad\leq \int_0^t \big(\|Y_h(s)\|^2 + \|Y(s)\|^2 + 1\big)\psi_{\frac{\delta}{4}}\big(|Y_h(s)| + |Y(s)|\big)^2|W(s)|^2 ds \\
&\qquad\qquad + 4h^2\delta^{-1}\int_0^S \|V(s)\|_*^2 ds.
\end{aligned}$$

Using Gronwall's lemma, we obtain that

$$|W(t)|^2 \leq Ch^2\|V\|^2_{L^2(0,S;\mathcal{V}')}e^{\int_0^S (\|Y_h(s)\|^2 + \|Y(s)\|^2 + 1)\psi_{\frac{\delta}{4}}(|Y_h(s)| + |Y(s)|)^2 ds}$$

for all $t \in [0, S]$. Hence, $Y_h \to Y$ strongly in $\mathcal{C}([0, S]; \mathcal{H})$ as $h \to 0$.

Step 2. $\frac{Y_h - Y}{h} \to Z$ *strongly in* $H^1(0, S; \mathcal{V}') \cap C([0, S]; \mathcal{H}) \cap L^2(0, S; \mathcal{V})$ *as* $h \to 0$. We rewrite the problem (5.5) in the form

$$\begin{cases}
\dfrac{d}{dt}\dfrac{Y_h - Y}{h} + A\dfrac{Y_h - Y}{h} + \dfrac{F(Y_h) - F(Y)}{h} = V(t), \quad 0 < t \leq S, \\
\dfrac{Y_h - Y}{h}(0) = 0.
\end{cases} \tag{5.6}$$

On the other hand, we consider the linear problem (5.4). From (a.i), (a.ii), (f.i), (f.ii) and (5.1), we can easily verify that (5.4) possesses a unique weak solution $Z \in H^1(0,S;\mathcal{V}') \cap C([0,S];\mathcal{H}) \cap L^2(0,S;\mathcal{V})$ on [0,S] (cf. [5, Chap. XVIII, Theorem 2]). Then $\widetilde{W} = \frac{Y_h - Y}{h} - Z$ satisfies:

$$\begin{cases} \frac{d}{dt}\widetilde{W}(t) + A\widetilde{W}(t) + \int_0^1 F'\big(Y + \theta(Y_h - Y)\big)\widetilde{W}(t)d\theta \\ \qquad = \int_0^1 \big\{F'\big(Y + \theta(Y_h - Y)\big) - F'(Y)\big\}Z(t)d\theta, \quad 0 < t \le S, \\ \widetilde{W}(0) = 0. \end{cases} \tag{5.7}$$

Taking the scalar product of the equation of (5.7) with $\widetilde{W}$, we obtain that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\widetilde{W}(t)|^2 + \langle A\widetilde{W}(t), \widetilde{W}(t)\rangle &= \langle \int_0^1 F'\big(Y + \theta(Y_h - Y)\big)\widetilde{W}(t)d\theta, \widetilde{W}(t)\rangle \\ &+ \langle \int_0^1 \big\{F'\big(Y + \theta(Y_h - Y)\big) - F'(Y)\big\}Z(t)d\theta, \widetilde{W}(t)\rangle. \end{aligned}$$

From (5.3), we have:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\widetilde{W}(t)|^2 + \delta\|\widetilde{W}(t)\|^2 \le \frac{\delta}{2}\|\widetilde{W}(t)\|^2 + C\big\{(\|Y(t)\|^2 + \|Y_h(t) - Y(t)\|^2 + 1)|\widetilde{W}(t)|^2 \\ + |Y_h(t) - Y(t)|^2\|Z(t)\|^2\big\}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{2}|\widetilde{W}(t)|^2 + \frac{\delta}{2}\int_0^t \|\widetilde{W}(s)\|^2 ds \le C\{\int_0^t (\|Y(s)\|^2 + \|Y_h(s)\|^2 + 1)|\widetilde{W}(s)|^2 ds \\ + |Y_h - Y|^2_{\mathcal{C}([0,S];\mathcal{H})}\int_0^S \|Z(s)\|^2 ds\}. \end{aligned}$$

Using Gronwall's lemma, we obtain that

$$\begin{aligned} &|\widetilde{W}(t)|^2 + \int_0^t \|\widetilde{W}(s)\|^2 ds \\ &\qquad \le C|Y_h - Y|^2_{\mathcal{C}([0,S];\mathcal{H})}\|Z\|^2_{L^2(0,S;\mathcal{V})} e^{\int_0^S C(\|Y(s)\|^2 + \|Y_h(s)\|^2 + 1)ds} \end{aligned}$$

for all $t \in [0, S]$. Since $Y_h \to Y$ in $\mathcal{C}([0,S];\mathcal{H})$, we conclude that $\frac{Y_h - Y}{h}$ is strongly convergent to Z in $H^1(0,S;\mathcal{V}') \cap C([0,S];\mathcal{H}) \cap L^2(0,S;\mathcal{V})$. $\square$

With the aid of this proposition, we can easily show the first order necessary condition.

Theorem 5.3. *Let $\overline{U}$ be an optimal control of $(\overline{P})$ and let $\overline{Y} \in L^2(0,S;\mathcal{V}) \cap \mathcal{C}([0,S];\mathcal{H}) \cap H^1(0,S;\mathcal{V}')$ be the optimal state, that is $\overline{Y}$ is the solution to* (2.12) *with the control $\overline{U}(t)$. Then, there exists a unique solution $P \in L^2(0,S;\mathcal{V}) \cap \mathcal{C}([0,S];\mathcal{H}) \cap H^1(0,S;\mathcal{V}')$ to the linear problem*

$$\begin{cases} -\dfrac{dP}{dt} + AP + F'(\overline{Y})^* P = \Lambda(\overline{Y} - Y_d), \quad 0 \le t < S, \\ P(S) = 0 \end{cases} \tag{5.8}$$

in $\mathcal{V}'$, where $\Lambda : \mathcal{V} \to \mathcal{V}'$ is a canonical isomorphism; moreover,

$$\int_0^S \langle \Lambda P + \gamma \overline{U}, V - \overline{U} \rangle_{\mathcal{V}'} dt \ge 0 \qquad \textit{for all } V \in \mathcal{U}_{ad}.$$

Proof. Since J is Gâteaux differentiable at $\overline{U}$ and $\mathcal{U}_{ad}$ is convex, it is seen that

$$J'(\overline{U})(V - \overline{U}) \ge 0 \qquad \text{for all } V \in \mathcal{U}_{ad}.$$

On the other hand, we verify that

$$J'(\overline{U})(V - \overline{U}) = \int_0^S \langle Y(\overline{U}) - Y_d, Z \rangle_{\mathcal{V}} dt \; + \; \gamma \int_0^S \langle \overline{U}, V - \overline{U} \rangle_{\mathcal{V}'} dt \tag{5.9}$$

with $Z = Y'(\overline{U})(V - \overline{U})$. Let P be the unique solution of (5.8) in $H^1(0,S;\mathcal{V}') \cap \mathcal{C}([0,S];\mathcal{H}) \cap L^2(0,S;\mathcal{V})$. From (a.i), (a.ii), (f.i), (f.ii) and (5.2), we can guarantee that such a solution P exists (cf. [5, Chap. XVIII, Theorem 2]). Thus, in view of Proposition 5.2 the first intergal in the right hand side of (5.9) is shown to be

$$\int_0^S \langle Y(\overline{U}) - Y_d, Z\rangle_{\mathcal{V}} dt = \int_0^S \langle \Lambda(Y(\overline{U}) - Y_d), Z\rangle dt$$
$$= \int_0^S \langle -\frac{dP}{dt} + AP + F'(\overline{Y})^* P, Z\rangle dt = \int_0^S \langle P, \frac{dZ}{dt} + AZ + F'(\overline{Y})Z\rangle dt$$
$$= \int_0^S \langle \Lambda P, V - \overline{U}\rangle_{\mathcal{V}'} dt.$$

Hence,

$$\int_0^S \langle \Lambda P + \gamma\overline{U}, V - \overline{U}\rangle_{\mathcal{V}'} dt \geq 0, \qquad \text{for all } V \in \mathcal{U}_{ad}. \quad \square$$

References

1. N. U. Ahmed and K. L. Teo, "Optimal Control of Distributed Parameter Systems," North-Holland, New York, 1981.
2. V. Barbu, "Analysis and Control of Nonlinear Infinite Dimensional Systems," Academic Press, Boston, 1993.
3. P. Biler, W. Hebisch and T. Nadzieja, The Debye system: Existence and large time behavior of solutions, *Nonlinear Anal.* **238** (1994), 1189–1209.
4. E. Casas, L. A. Fernández and J. Yong, Optimal control of quasilinear parabolic equations, *Proc. Roy. Soc. Edinburgh Sect. A* **125** (1995), 545–565.
5. R. Dautray and J. L. Lions, "Mathematical Analysis and Numerical Methods for Science and Technology," Vol. 5, Springer-Verlag, Berlin, 1992.
6. M. A. Herrero and J. J. L. Velázquez, Chemotaxis collapse for the Keller-Segel model, *J. Math. Biol.*, to appear.
7. T. Nagai, T. Senba and K. Yosida, Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis, *Funkcial. Ekvac.* **40** (1997), 411–433.
8. J. L. Lions, "Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires," Dunod/Gauthier-Villars, Paris, 1969.

9. J. L. Lions, “Optimal Control of Systems Governed by Partial Differential Equations,” Springer-Verlag, Berlin, 1971.

10. E. F. Keller and L. A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. theor. Biol.* **26** (1970), 399–415.

11. N. S. Papageorgiou, On the optimal control of strongly nonlinear evolution equations, *J. Math. Anal. Appl.* **164** (1992), 83–103.

12. A. Yagi, Norm behavior of solutions to a parabolic system of chemotaxis, *Math. Japonica* **45** (1997), 241–265.

13. H. Triebel, “Interpolation Theory, Function Spaces, Differential Operators,” North-Holland, Amsterdam, 1978.