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# CHEMOTAXIS AND GROWTH SYSTEM WITH SINGULAR SENSITIVITY FUNCTION

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**ABSTRACT.** This paper continues the study of the initial value problem of a chemotaxis-growth system. In the previous paper [13], we have handled the case when the sensitivity function  $\chi(\rho)$  is regular. In this paper we are concerned with the case when the function has singularity at  $\rho = 0$  like  $\chi(\rho) = \log \rho$  or  $-\frac{1}{\rho}$ . We verify global existence of solutions and discuss some asymptotic behaviour of solutions.

Quasilinear system; Chemotaxis-growth; Singular sensitivity function; Global existence

## 1. INTRODUCTION

We study the initial value problem of a quasilinear parabolic system

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} = a\Delta u - \nabla\{u\nabla\chi(\rho)\} + f(u) & \text{in } \Omega \times (0, \infty), \\ \frac{\partial \rho}{\partial t} = b\Delta \rho - c\rho + du & \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial n} = \frac{\partial \rho}{\partial n} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x), \quad \rho(x, 0) = \rho_0(x) & \text{in } \Omega \end{array} \right. \quad (\text{CG})$$

in a bounded domain  $\Omega \subset \mathbb{R}^2$ . Here,  $u(x, t)$  and  $\rho(x, t)$  denote the population density of biological individuals and the concentration of chemical substance at a position  $x \in \Omega$  and time  $t \in [0, \infty)$ , respectively. The mobility of individuals consists of two effects, namely random walking and chemotaxis, the latter means the directed movement in a sense that they have a tendency to move toward higher concentration of the chemical substance with the rate  $\nabla\chi(\rho)$ , we refer to [1, 3, 6].  $\chi(\rho)$  is called the sensitivity function of chemotaxis.  $a > 0$  and  $b > 0$  are the diffusion rates of  $u$  and  $\rho$ , respectively.  $c > 0$  and  $d > 0$  are the degradation and production rates of  $\rho$ , respectively.  $f(u)$  is a growth term of  $u$ .

Burdrene and Berg [5] experimentally observed that bacteria called *E. coli* form complex spatio-temporal colony patterns. In order to study theoretically such chemotactic pattern, several models have been proposed by [2, 7, 8, 9, 11, 15, 18]. Among them, Mimura and Tsujikawa [10] presented the model (CG) above in which they incorporate three elemental effects, diffusion, chemotaxis, and growth of bacteria.

Our interest is to investigate a mathematical aspect of the system (CG) which is also very important for performing numerical computations. In the previous paper [13], we have studied the case where the sensitivity function is a smooth function of  $\rho \in [0, \infty)$  without singularity at  $\rho = 0$  and has uniformly bounded derivatives up to the third order (see the condition  $(\chi)$  of [13]). Under these assumptions we have constructed an

exponential attractor for the dynamical system determined by (CG) in the phase space  $\left\{\begin{pmatrix} u \\ \rho \end{pmatrix}; u \in L^2(\Omega), \rho \in H^1(\Omega)\right\}$  by using the squeezing method due to Eden, Foias, Nicolaenko, and Temam [16] and [20].

In this paper we intend to handle the left but very interesting case where  $\chi(\rho)$  has singularity at  $\rho = 0$  such as  $\log \rho$ ,  $-\rho^{-1}$  and so on.  $\chi(\rho)$  is actually assumed to be a smooth function of  $\rho \in (0, \infty)$  satisfying

$$\left| \sup_{\delta \leq \rho < \infty} \frac{d^i \chi}{d\rho^i}(\rho) \right| \leq C_\delta \quad \text{for } \delta > 0, \quad i = 1, 2, 3 \quad (\chi)$$

with some constant  $C_\delta > 0$  which is allowed to depend on  $\delta$ .

For the others we make the similar assumptions as in [13]. That is,  $\Omega \subset \mathbb{R}^2$  is a bounded domain of  $\mathcal{C}^3$  class.  $a, b, c$  and  $d$  are positive constants.  $f(u)$  is a real smooth function of  $u \in [0, \infty)$  with  $f(0) = 0$  and  $f'(0) \neq 0$  satisfying the condition

$$f(u) = (-\mu u + \nu)u \quad \text{for sufficiently large } u \quad (f)$$

with some  $\mu > 0$  and  $-\infty < \nu < \infty$ .

The initial functions are also taken as before. That is,  $u_0 \in L^2(\Omega)$  and  $\rho_0 \in H^{1+\varepsilon_0}(\Omega)$ , where  $\varepsilon_0$  is an arbitrarily fixed exponent in such a way that  $0 < \varepsilon_0 < \frac{1}{2}$ .  $u_0 \geq 0$  is nonnegative in  $\Omega$ , and in view of the singularity of  $\chi(\rho)$  we impose on  $\rho_0$  the condition

$$\inf_{x \in \Omega} \rho_0(x) > 0.$$

The space of initial values is therefore set as

$$K = \left\{ U = \begin{pmatrix} u \\ \rho \end{pmatrix}; 0 \leq u \in L^2(\Omega), 0 < \rho_0 \in H^{1+\varepsilon_0}(\Omega), \inf_{x \in \Omega} \rho_0(x) > 0 \right\}. \quad (\text{In})$$

$K$  is equipped with the distance induced by the product norm

$$d_K(U_1, U_2) = \|u_1 - u_2\|_{L^2} + \|\rho_1 - \rho_2\|_{H^{1+\varepsilon_0}}, \quad U_1, U_2 \in K. \quad (1.1)$$

In this way  $K$  can not contain a pair  $O = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  of the null function of  $\Omega$ . We, however, observe that there exists a solution which converges to this boundary point  $O$  as  $t \rightarrow \infty$ . In fact, let for example  $f(u) = -u(u-1)(u-2)$  for  $0 \leq u \leq 2$  and  $u_0 \equiv \frac{1}{2}$ ,  $\rho_0 \equiv 1$ . Then (CG) reduces to a simple system of ordinary differential equations

$$\begin{cases} \frac{du}{dt} = -u(u-1)(u-2), & 0 < t < \infty, \\ \frac{d\rho}{dt} = -c\rho + du, & 0 < t < \infty, \\ u(0) = \frac{1}{2}, \quad \rho(0) = 1. \end{cases}$$

And the solution of this system clearly converges to 0 as  $t \rightarrow \infty$ . By this consideration, we notice that the dynamical system determined by (CG) in the phase space  $K$  no longer admits a global attractor in general. This is a great difference from the case where  $\chi(\rho)$  has no singularity at  $\rho = 0$ .

So we shall first verify in this paper that (CG) admits a unique global solution for each initial value from  $K$ . (In the case when  $f(u) \equiv 0$ , (CG) is called the Keller-Segel equations; some results on global existence and blow-up are obtained in [4, 12].) Second, we shall investigate asymptotic behavior of global solutions as  $t \rightarrow \infty$ . Some are shown to

stay away from the point  $O$  and possess their nonempty  $\omega$ -limit sets in  $K$ , and the others are shown to approach to the boundary point  $O$  in a suitable sense. This alternative is determined by the condition whether  $\inf_{0 \leq t < \infty} \|u(t)\|_{L^1} > 0$  or  $\inf_{0 \leq t < \infty} \|u(t)\|_{L^1} = 0$ . In the case when the solution approaches to  $O$ , namely  $\inf_{0 \leq t < \infty} \|u(t)\|_{L^1} = 0$ , if  $f'(0) < 0$ , then the solution can converge to  $O$  in a strong topology, and if  $f'(0) > 0$ , then some Sobolev norm of  $u(t)$  grows up as  $t \rightarrow \infty$ .

*Notations.*  $n(x)$  denotes the outer normal vector at a boundary point  $x \in \partial\Omega$ . For  $1 \leq p \leq \infty$ ,  $L^p(\Omega)$  is the  $L^p$  space of real valued measurable functions in  $\Omega$ , its norm is denoted by  $\|\cdot\|_{L^p}$ .  $H^k(\Omega)$ ,  $k = 0, 1, 2, \dots$ , denotes the real Sobolev space in  $\Omega$ , its norm is denoted by  $\|\cdot\|_{H^k}$ . More generally, the fractional Sobolev space is denoted by  $H^s(\Omega)$ ,  $s > 0$ , its norm is denoted by  $\|\cdot\|_{H^s}$ . For  $s > \frac{3}{2}$ ,  $H_N^s(\Omega)$  is the closed subspace of  $H^s(\Omega)$  consisting of functions which satisfy the Neumann boundary conditions on  $\partial\Omega$ .  $\mathcal{C}(\overline{\Omega})$  denotes the space of real valued continuous functions on  $\overline{\Omega}$ , its norm is denoted by  $\|\cdot\|_{\mathcal{C}}$ .

Let  $H$  be a Hilbert space and let  $I$  be an interval of  $\mathbb{R}$ .  $L^2(I; H)$  denotes the space of  $H$  valued  $L^2$  functions defined in  $I$ .  $H^1(I; H)$  denotes the space of functions in  $L^2(I; H)$  whose first derivatives are also in  $L^2(I; H)$ .  $\mathcal{C}(I; H)$  and  $\mathcal{C}^m(I; H)$ ,  $m = 1, 2, 3, \dots$ , denote the space of  $H$  valued continuous functions and of  $H$  valued  $m$ -times continuously differentiable functions, respectively.

For simplicity, we shall use a universal notation  $C$  to denote various constants which are determined in each occurrence by  $\Omega$ ,  $a$ ,  $b$ ,  $c$ ,  $d$ ,  $\chi(\cdot)$ ,  $f(\cdot)$  and so on in a specific way. In a case where  $C$  depends also on some parameter, say  $\zeta$ , it will be denoted by  $C_\zeta$ .

## 2. GLOBAL SOLUTIONS

For each pair of initial functions  $u_0, \rho_0$  in  $K$ , we shall prove that (CG) admits a unique global solution. Since  $\rho_0(x)$  does not vanish in  $\Omega$ , we can repeat the same arguments as in [13, Section 3] to construct a local solution by using the theory of abstract evolution equations (see [14, 17, 19]).

In order to extend such local solutions globally, however, we have to notice an a priori estimate of  $\rho$  from below.

**Proposition 2.1.** *Let  $u, \rho$  be any local solution to (CG) such that*

$$\begin{cases} 0 \leq u \in \mathcal{C}([0, T_{u,\rho}]; L^2(\Omega)) \cap \mathcal{C}^1((0, T_{u,\rho}]; L^2(\Omega)) \cap \mathcal{C}((0, T_{u,\rho}]; H_N^2(\Omega)), \\ 0 < \rho \in \mathcal{C}([0, T_{u,\rho}]; H^{1+\varepsilon_0}(\Omega)) \cap \mathcal{C}^1((0, T_{u,\rho}]; H^1(\Omega)) \cap \mathcal{C}((0, T_{u,\rho}]; H_N^3(\Omega)) \end{cases}$$

*with initial functions  $u_0, \rho_0$  in  $K$ . Then,  $\rho$  satisfies*

$$\inf_{x \in \Omega} \rho(x, t) \geq \delta_0 e^{-ct} \quad \text{for every } 0 \leq t \leq T_{u,\rho}, \quad (2.1)$$

*where  $\delta_0 = \inf_{x \in \Omega} \rho_0(x) > 0$ .*

*Proof.* We introduce a decreasing convex  $\mathcal{C}^2$  function  $H(\rho)$  of  $\rho \in (-\infty, \infty)$  such that  $H(\rho) = 0$  for  $\rho \geq 0$  and  $H(\rho) > 0$  for  $\rho < 0$ . Consider a continuous function

$$\varphi(t) = \int_{\Omega} H(\rho(x, t) - \delta_0 e^{-ct}) dx, \quad 0 \leq t \leq T_{u,\rho}.$$

It is observed that

$$\begin{aligned}\frac{d\varphi}{dt}(t) &= \int_{\Omega} H'(\rho(t) - \delta_0 e^{-ct}) \left( \frac{\partial \rho}{\partial t} + c\delta_0 e^{-ct} \right) dx \\ &= -b \int_{\Omega} H''(\rho - \delta_0 e^{-ct}) |\nabla \rho|^2 dx + d \int_{\Omega} H'(\rho - \delta_0 e^{-ct}) u dx \\ &\quad - c \int_{\Omega} H'(\rho - \delta_0 e^{-ct}) (\rho - \delta_0 e^{-ct}) dx.\end{aligned}$$

Since  $H'(\rho) \leq 0$ ,  $H'(\rho)\rho \geq 0$ , and  $H''(\rho) \geq 0$ , it follows that  $\varphi'(t) \leq 0$  for every  $0 < t \leq T_{u,\rho}$ . Therefore,  $0 \leq \varphi(t) \leq \varphi(0) = 0$ . This means that  $\rho(t) - \delta_0 e^{-ct} \geq 0$  for every  $0 \leq t \leq T_{u,\rho}$ .  $\square$

This proposition jointed with [13, Theorem 4.5] then yields the global existence of solution.

**Theorem 2.1.** *For each pair of initial functions  $u_0, \rho_0$  in  $K$ , there exists a unique global solution to (CG) in the function space*

$$\begin{cases} 0 \leq u \in \mathcal{C}([0, \infty); L^2(\Omega)) \cap \mathcal{C}^1((0, \infty); L^2(\Omega)) \cap \mathcal{C}((0, \infty); H_N^2(\Omega)), \\ 0 < \rho \in \mathcal{C}([0, \infty); H^{1+\varepsilon_0}(\Omega)) \cap \mathcal{C}^1((0, \infty); H^1(\Omega)) \cap \mathcal{C}((0, \infty); H_N^3(\Omega)). \end{cases} \quad (2.2)$$

*Proof.* Let  $T > 0$  be arbitrary positive time, and set  $\delta = \delta_0 e^{-cT}$  with  $\delta_0 = \inf_{x \in \Omega} \rho_0(x)$ . We consider a smooth sensitivity function  $\chi_\delta(\rho)$  of  $\rho \in [0, \infty)$  such that  $\chi_\delta(\rho) = \chi(\rho)$  for  $\rho \in [\delta, \infty)$ ; obviously,  $\chi_\delta(\rho)$  satisfies the condition  $(\chi)$  of [13]. And we consider an auxiliary initial value problem  $(CG_\delta)$  by substituting  $\chi_\delta(\rho)$  for  $\chi(\rho)$ . Then, by virtue of [13, Theorem 4.5], there exists a global solution  $u_\delta, \rho_\delta$  to the problem  $(CG_\delta)$ . Set, further, that  $T_\delta = \sup\{\tau; \inf_{0 \leq t \leq \tau, x \in \Omega} \rho_\delta(x, t) \geq \delta\}$ . By definition,  $\rho_\delta(t) \geq \delta$  on the interval  $[0, T_\delta]$ ; this in turn means that  $u_\delta, \rho_\delta$  is also a local solution of the original problem (CG) on the interval  $[0, T_\delta]$ . Meanwhile we see that  $T_\delta \geq T$ . Indeed, if  $T_\delta < T$ , then by Proposition 2.1 we have  $\rho_\delta(T_\delta) \geq \delta_0 e^{-cT_\delta} > \delta$ . But this contradicts to the maximality of  $T_\delta$  since  $\rho_\delta$  is a function belonging to  $\mathcal{C}([0, \infty); H^{1+\varepsilon_0}(\Omega)) \subset \mathcal{C}([0, \infty); \mathcal{C}(\overline{\Omega}))$ .

Thus (CG) has been shown to possess a local solution on an arbitrarily finite interval  $[0, T]$ . In other words, (CG) admits a global solution.  $\square$

We conclude this section by noting some estimates  $u, \rho$  which hold independently of  $\delta_0 = \inf_{x \in \Omega} \rho_0(x)$ . From  $f(0) = 0$  and (f), we can take two positive constants  $\mu'$  and  $\nu'$  in such a way that

$$f(u) \leq (-\mu' u + \nu') u, \quad u \geq 0. \quad (2.3)$$

Then, by integrating the first equation of (CG) in  $\Omega$ , we have

$$\frac{d}{dt} \int_{\Omega} u dx = \int_{\Omega} f(u) dx \leq \int_{\Omega} (\nu' u - \mu' u^2) dx,$$

therefore

$$\|u(t)\|_{L^1} \leq C(e^{-t} \|u_0\|_{L^1} + 1), \quad 0 \leq t < \infty \quad (2.4)$$

(see Step 1 of the proof of [13, Proposition 4.1]). As well it is observed that

$$\left| \int_0^t \int_{\Omega} f(u) dx ds \right| \leq C(\|u_0\|_{L^1} + 1), \quad 0 \leq t < \infty. \quad (2.5)$$

Multiplying the second equation of (CG) by  $\rho$  and integrating the product in  $\Omega$ , we have

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho^2 dx + b \int_{\Omega} |\nabla \rho|^2 dx + c \int_{\Omega} \rho^2 dx = d \int_{\Omega} u \rho dx \leq \frac{c}{2} \|\rho\|_{L^2}^2 + \frac{d^2}{2c} \|u\|_{L^2}^2.$$

Here, it holds that

$$u^2 \leq -(\mu')^{-1} f(u) + (\mu')^{-1} \nu' u, \quad u \geq 0.$$

Therefore,

$$\int_{\Omega} \rho^2 dx \leq e^{-ct} \|\rho_0\|_{L^2}^2 + \int_0^t e^{-c(t-s)} \left\{ C \|u(s)\|_{L^1} - C \int_{\Omega} f(u(s)) dx \right\} ds.$$

Applying the second mean value theorem of integration in view of (2.4) and (2.5), we obtain that

$$\|\rho(t)\|_{L^2}^2 \leq C(e^{-ct} \|\rho_0\|_{L^2}^2 + \|u_0\|_{L^1} + 1), \quad 0 \leq t < \infty.$$

Next we multiply the second equation of (CG) by  $\Delta \rho$  and integrate the product in  $\Omega$ . Then,

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \rho|^2 dx + b \int_{\Omega} |\Delta \rho|^2 dx + c \int_{\Omega} |\nabla \rho|^2 dx = -d \int_{\Omega} u \Delta \rho dx \leq \frac{b}{2} \|\Delta \rho\|_{L^2}^2 + \frac{d^2}{2b} \|u\|_{L^2}^2.$$

Repeating the same argument as above, we obtain that

$$\|\nabla \rho(t)\|_{L^2}^2 \leq C(e^{-2ct} \|\rho_0\|_{H^1}^2 + \|u_0\|_{L^1} + 1), \quad 0 \leq t < \infty.$$

Finally we conclude that

$$\|\rho(t)\|_{H^1}^2 \leq C(e^{-ct} \|\rho_0\|_{H^1}^2 + \|u_0\|_{L^1} + 1), \quad 0 \leq t < \infty. \quad (2.6)$$

### 3. CONTINUOUS DEPENDENCE IN INITIAL VALUES

As shown in the preceding section, for each  $U_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in K$ , there exists a unique global solution  $U = \begin{pmatrix} u \\ \rho \end{pmatrix}$  to (CG) in the solution space (2.2). This section is then devoted to noting continuous dependence of solutions with respect to initial values.

**Theorem 3.1.** *Let  $U_0 \in K$  and let  $U(t)$  be the solution to (CG) with an initial value  $U_0$ . Let  $\{U_{0,n}\}_{n=1,2,3,\dots}$  be a sequence of initial values in  $K$  and let  $\{U_n\}_{n=1,2,3,\dots}$  be the sequence of corresponding solutions. If  $U_{0,n} \rightarrow U_0$  in  $K$  as  $n \rightarrow \infty$ , then  $U_n(t) \rightarrow U(t)$  in  $K$  for each fixed time  $0 \leq t < \infty$ .*

*Proof.* Since  $\rho_{0,n} \rightarrow \rho_0$  in  $H^{1+\varepsilon_0}(\Omega) \subset \mathcal{C}(\overline{\Omega})$ . There exists some positive constant  $\delta_0 > 0$  such that  $\inf_{x \in \Omega} \rho_{0,n}(x) \geq \delta_0$  for all  $n$ . For fixed time  $0 \leq t < \infty$ , set  $\delta = \delta_0 e^{-ct} > 0$ . Then, by virtue of Proposition 2.1,  $U_n$  are all local solutions on an interval  $[0, t]$  to the auxiliary problem  $(CG_{\delta})$  where the sensitivity function is substituted with  $\chi_{\delta}(\rho)$  which is a smooth function of  $\rho \in [0, \infty)$  coinciding with  $\chi(\rho)$  for  $\rho \in [\delta, \infty)$ . Therefore, we obtain the desired result, see [13, Theorem 3.2].  $\square$

#### 4. ASYMPTOTIC BEHAVIOR OF GLOBAL SOLUTIONS

For each  $0 \leq t < \infty$ , define a transform  $S(t)$  on  $K$  by the formula  $S(t)U_0 = \begin{pmatrix} u(t) \\ \rho(t) \end{pmatrix}$ ,  $U_0 \in K$ , where  $u, \rho$  denotes the global solution to the problem (CG) with the initial value  $U_0$ . By Theorems 2.1 and 3.1,  $\{S(t)\}_{t \geq 0}$  defines a nonlinear semigroup on  $K$ , namely  $S(\cdot)U_0$  is a continuous function of  $t \in [0, \infty)$  with values in  $K$  and  $S(t)$  is a continuous mapping from  $K$  into itself.

In this section we shall be concerned with asymptotic behavior of  $S(t)U_0$  as  $t \rightarrow \infty$ . We begin with noting the following proposition.

**Proposition 4.1.** *Let  $u, \rho$  be any global solution to (CG) in the space (2.2). Then the following two assertions*

$$\inf_{0 \leq t < \infty} \|u(t)\|_{L^1} > 0 \quad (4.1)$$

and

$$\inf_{0 \leq t < \infty, x \in \Omega} \rho(x, t) > 0 \quad (4.2)$$

are equivalent.

*Proof.* I) Let us first verify that (4.1) implies (4.2). Put  $\inf_{0 \leq t < \infty} \|u(t)\|_{L^1} = \ell > 0$ . We here introduce the realization  $L$  of the Laplace operator  $-b\Delta$  in  $L^2(\Omega)$  under the Neumann boundary conditions on  $\partial\Omega$ .  $L$  is a nonnegative self-adjoint operator in  $L^2(\Omega)$ . From the second equation of (CG),  $\rho(t)$  is written as

$$\rho(t) = e^{-t(L+c)}\rho_0 + d \int_0^t e^{-(t-s)(L+c)}u(s)ds. \quad (4.3)$$

Set a time  $t_0 \geq 2$ . For every  $t \geq 2t_0$ , we have

$$\rho(t) \geq d \int_0^{t-t_0} e^{-(t-s)(L+c)}\{\bar{u}(s) + u_m(s)\}ds.$$

Here,  $u = \bar{u} + u_m$  denotes the orthogonal decomposition of  $u \in L^2(\Omega)$  such that  $\bar{u} = |\Omega|^{-1} \int_{\Omega} u dx$  and

$$u_m \in L_m^2(\Omega) = \left\{ u \in L^2(\Omega); \int_{\Omega} u dx = 0 \right\}.$$

Since  $\bar{u}(t) \geq |\Omega|^{-1}\ell$  and  $e^{-(t-s)L}\bar{u}(s) = \bar{u}(s)$ , it is seen that

$$\begin{aligned} \int_0^{t-t_0} e^{-(t-s)(L+c)}\bar{u}(s)ds &\geq \frac{\ell}{|\Omega|} \int_0^{t-t_0} e^{-c(t-s)}ds \\ &= \frac{\ell}{c|\Omega|} \{e^{-ct_0} - e^{-ct}\} \geq \frac{\ell e^{-ct_0}}{c|\Omega|} \{1 - e^{-ct_0}\}, \quad t \geq 2t_0. \end{aligned} \quad (4.4)$$

On the other hand, the part  $L_m$  of  $L$  in the component  $L_m^2(\Omega)$  is a positive definite self-adjoint operator in  $L_m^2(\Omega)$ . Therefore, there exists some  $\lambda_m > 0$  such that  $L_m \geq \lambda_m$ . Then, using the fact that

$$e^{-L} \in \mathcal{L}(L^2(\Omega), \mathcal{C}(\Omega)) \cap \mathcal{L}(L^1(\Omega), L^2(\Omega)),$$

we can observe that

$$\begin{aligned}
& \left\| \int_0^{t-t_0} e^{-(t-s)(L+c)} u_m(s) ds \right\|_c \\
& \leq \|e^{-L}\|_{\mathcal{L}(L^2, \mathbb{C})} \left\| \int_0^{t-t_0} e^{-(t-s-2)L} e^{-c(t-s)} e^{-L} u_m(s) ds \right\|_{L^2} \\
& \leq \|e^{-L}\|_{\mathcal{L}(L^2, \mathbb{C})} \int_0^{t-t_0} e^{-(t-s-2)\lambda_m} e^{-c(t-s)} \|e^{-L} u_m(s)\|_{L^2} ds \\
& \leq \|e^{-L}\|_{\mathcal{L}(L^2, \mathbb{C})} \|e^{-L}\|_{\mathcal{L}(L^1, L^2)} e^{2\lambda_m} \int_0^{t-t_0} e^{(c+\lambda_m)(s-t)} \|u_m(s)\|_{L^1} ds. \quad (4.5)
\end{aligned}$$

Since  $\|u_m(s)\|_{L^1} \leq 2\|u(s)\|_{L^1}$  and since (2.4) holds, the norm is furthermore estimated by

$$\leq C(\|u_0\|_{L^1} + 1)e^{-(c+\lambda_m)t_0}, \quad t \geq 2t_0.$$

From (4.4) and (4.5) it is therefore verified that

$$\rho(t) \geq \frac{d\ell e^{-ct_0}}{c|\Omega|} \{1 - e^{-ct_0} - C(\|u_0\|_{L^1} + 1)e^{-\lambda_m t_0}\}, \quad t \geq 2t_0.$$

This obviously shows that, if  $t_0$  is taken sufficiently large, then

$$\inf_{2t_0 \leq t < \infty, x \in \Omega} \rho(x, t) > 0.$$

Since (2.1) has been verified, we conclude (4.2).

II) Let us next verify that (4.2) implies (4.1). We assume that

$$\inf_{0 \leq t < \infty, x \in \Omega} \rho(x, t) = \delta > 0. \quad (4.6)$$

As done above, we consider an auxiliary initial value problem  $(CG_\delta)$  in which a sensitivity function  $\chi_\delta(\rho)$  is substituted for  $\chi(\rho)$ ,  $\chi_\delta(\rho)$  is a smooth function of  $\rho \in [0, \infty)$  coinciding with  $\chi(\rho)$  for all  $\rho \in [\delta, \infty)$ . Then,  $u, \rho$  is clearly a global solution to the problem  $(CG_\delta)$ . Therefore, as a global solution to  $(CG_\delta)$ , all the results obtained in [13] are available.

We now apply the a priori estimates established by [13, Proposition 4.1] to  $u, \rho$  on the interval  $[1, \infty)$ . Then there must exist some constant  $C_u > 0$  such that

$$\|u(t)\|_{H^2} \leq C_u, \quad t \geq 1.$$

In addition, we note that for any  $0 < \varepsilon \leq 1$  it holds that

$$\|u\|_c \leq C\|u\|_{H^{1+\frac{\varepsilon}{2}}} \leq C\|u\|_{H^{1+\varepsilon}}^{\frac{2+\varepsilon}{2(1+\varepsilon)}} \|u\|_{H^1}^{\frac{\varepsilon}{4(1+\varepsilon)}} \|u\|_{L^1}^{\frac{\varepsilon}{4(1+\varepsilon)}}, \quad u \in H^{1+\varepsilon}(\Omega) \quad (4.7)$$

(from [13, (2.1~4)]). Using this estimate with  $\varepsilon = 1$ , we observe that

$$\|u(t)\|_c \leq C_u \|u(t)\|_{L^1}^{\frac{1}{8}}, \quad t \geq 1. \quad (4.8)$$

To prove (4.1), we first notice that  $u(s)$  can not vanish in any finite time  $s$ . Indeed, suppose that  $u(s) = 0$  at some time  $s$ . Then, by the uniqueness of solution,  $u(t) = 0$  for every  $t \in [s, \infty)$ . On the other hand,  $\rho(t)$  must be determined by

$$\frac{\partial \rho}{\partial t} = b\Delta \rho - c\rho \quad \text{in } \Omega \times (s, \infty).$$

Therefore,  $\rho(t)$  must converge to 0 as  $t \rightarrow \infty$ . But this contradicts to (4.6).



To verify that  $u(t)$  does not vanish as  $t \rightarrow \infty$ , neither, we shall use the condition  $f'(0) \neq 0$ . First, let  $f'(0) < 0$ , then there are some constants  $\tilde{\nu} > 0$  and  $\ell > 0$  such that

$$f(u) \leq -\tilde{\nu}u \quad \text{holds for all } u \in [0, \ell]. \quad (4.9)$$

We shall then verify that  $\|u(t)\|_{L^1} \geq (\ell C_u^{-1})^8$  for every  $t \geq 1$ , where  $C_u$  is the constant appearing in (4.8). Indeed, if once  $\|u(s)\|_{L^1} < (\ell C_u^{-1})^8$  for some  $s \geq 1$ , then  $\|u(s)\|_c < \ell$  and therefore

$$\frac{d}{ds}\|u(s)\|_{L^1} = \int_{\Omega} f(u(s))dx \leq -\tilde{\nu}\|u(s)\|_{L^1}.$$

Hence,  $\|u(s)\|_{L^1}$  is decreasing at  $s$ , and this implies that  $\|u(t)\|_{L^1}$  is less than  $(\ell C_u^{-1})^8$  for any  $t \geq s$ . In this way,  $\frac{d}{dt}\|u(t)\|_{L^1} \leq -\tilde{\nu}\|u(t)\|_{L^1}$  and  $\|u(t)\|_{L^1} \leq e^{-\tilde{\nu}(t-s)}\|u(s)\|_{L^1}$  for all  $t \geq s$ . Thus, we conclude that  $\|u(t)\|_{L^1} \rightarrow 0$  as  $t \rightarrow \infty$ .

While, integrating the second equation of (CG) in  $\Omega$ , we see that

$$\frac{d}{dt}\|\rho(t)\|_{L^1} = -c\|\rho(t)\|_{L^1} + d\|u(t)\|_{L^1},$$

as a consequence

$$\|\rho(t)\|_{L^1} = e^{-ct}\|\rho_0\|_{L^1} + d \int_0^t e^{-c(t-\tau)}\|u(\tau)\|_{L^1}d\tau. \quad (4.10)$$

This together with the vanishing of  $\|u(t)\|_{L^1}$  implies that  $\|\rho(t)\|_{L^1}$  also vanishes as  $t \rightarrow \infty$ . But this again contradicts to (4.6).

Let now  $f'(0) > 0$ . In this case there exist two positive numbers  $\mu''$  and  $\nu''$  such that

$$f(u) \geq -\mu''u^2 + \nu''u, \quad u \geq 0. \quad (4.11)$$

From (4.8) we have

$$\begin{aligned} \frac{d}{dt}\|u(t)\|_{L^1} &= \int_{\Omega} f(u(t))dx \geq (\nu'' - \mu''\|u(t)\|_c)\|u(t)\|_{L^1} \\ &\geq (\nu'' - \mu''C_u\|u(t)\|_{L^1}^{\frac{1}{8}})\|u(t)\|_{L^1}, \quad t \geq 1. \end{aligned}$$

If  $\|u(s)\|_{L^1} < (\nu''/\mu''C_u)^8$  at some  $s \geq 1$ , then  $\|u(s)\|_{L^1}$  is increasing at the time. Then, if once  $\|u(s')\|_{L^1} \geq (\nu''/\mu''C_u)^8$  at some time  $s' \geq 1$ , then this differential inequality shows that  $\|u(t)\|_{L^1}$  is never less than  $(\nu''/\mu''C_u)^8$  for any  $t \geq s'$ .  $\square$

We can now prove the main results of the paper.

**Theorem 4.1.** *For each  $U_0 \in K$ , let  $S(\cdot)U_0 = \begin{pmatrix} u \\ \rho \end{pmatrix}$ . If  $\inf_{0 \leq t < \infty} \|u(t)\|_{L^1} > 0$  or equivalently  $\inf_{0 \leq t < \infty, x \in \Omega} \rho(x, t) > 0$ , then its  $\omega$ -limit set  $\omega(U_0) = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} S(\tau)U_0}$  in  $K$  is nonempty and is actually contained in the product space  $\left\{ \begin{pmatrix} u \\ \rho \end{pmatrix}; u \in H^2(\Omega), \rho \in H^3(\Omega) \right\}$ .*

*Proof.* From the assumption there exists some positive constant  $\delta > 0$  for which (4.6) holds. As above, introducing a smooth function  $\chi_{\delta}(\rho)$  of  $\rho \in [0, \infty)$  which coincides with  $\chi(\rho)$  for  $\rho \in [\delta, \infty)$ , we regard  $S(t)U_0$  as the global solution to the initial value problem  $(CG_{\delta})$  in which  $\chi_{\delta}(\rho)$  substitutes for  $\chi(\rho)$ . Then [13, Theorem 4.6] is available for  $u, \rho$

to conclude that there exists some constant  $C_{U_0}$  such that  $\|u(t)\|_{H^2} + \|\rho(t)\|_{H^3} \leq C_{U_0}$  for  $1 \leq t < \infty$ . Since the set

$$\left\{ \begin{pmatrix} u \\ \rho \end{pmatrix}; \|u\|_{H^2} + \|\rho\|_{H^3} \leq C_{U_0}, \quad u \geq 0, \quad \inf_{x \in \Omega} \rho(x) \geq \delta \right\}$$

is a compact set of  $K$ , we verify that the solution  $S(t)U_0$  admits a nonempty  $\omega$ -limit set in the topology (1.1) of  $K$ .  $\square$

In the case when  $\inf_{0 \leq t < \infty} \|u(t)\|_{L^1} = 0$ , we prove the following theorem.

**Theorem 4.2.** *Let  $U_0 \in K$  and  $S(\cdot)U_0 = \begin{pmatrix} u \\ \rho \end{pmatrix}$ . Assume that  $\inf_{0 \leq t < \infty} \|u(t)\|_{L^1} = 0$  or equivalently  $\inf_{0 \leq t < \infty, x \in \Omega} \rho(x, t) = 0$ . Then there exists a sequence  $t_n$  tending to  $\infty$  such that  $S(t_n)U_0$  converges to the boundary point  $O = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  of  $K$  in the norm  $\|u\|_{L^1} + \|\rho\|_{L^p}$  with any  $1 \leq p < \infty$ .*

*Furthermore, when  $f'(0) > 0$ ,  $\|u_0\|_{L^1} \neq 0$  implies that  $\sup_{1 \leq t < \infty} \|u(t)\|_{H^{1+\varepsilon}} = \infty$  with an arbitrary  $\varepsilon > 0$ . On the other hand, when  $f'(0) < 0$ ,  $\sup_{1 \leq t < \infty} \|u(t)\|_{H^{1+\varepsilon}} < \infty$  with some  $\varepsilon > 0$  implies that  $S(t)U_0$  converges to  $O$  in the distance (1.1).*

*Proof.* If  $u(s) = 0$  at some finite time  $s$ , then  $u(t) = 0$  for every  $t \geq s$  and  $\rho(t) \rightarrow 0$  as  $t \rightarrow \infty$ ; therefore,  $S(t)U_0$  converges to  $O$  in the distance (1.1).

So let us consider the case when  $\|u(t)\|_{L^1} > 0$  for every  $t$  and  $\inf_{0 \leq t < \infty} \|u(t)\|_{L^1} = 0$ . Then there exists an increasing sequence  $\{s_n\}_{n=1,2,3,\dots}$  tending to  $\infty$  such that  $\lim_{n \rightarrow \infty} \|u(s_n)\|_{L^1} = 0$  and  $0 < \|u(s_n)\|_{L^1} < 1$ . We here set another increasing sequence  $\{t_n\}_{n=1,2,3,\dots}$  by the formula

$$t_n = s_n - \frac{1}{2\nu'} \log \|u(s_n)\|_{L^1}, \quad n = 1, 2, 3, \dots,$$

where  $\nu'$  is the positive constant appearing in (2.3). Since

$$\frac{d}{dt} \|u(t)\|_{L^1} = \int_{\Omega} f(u(t)) dx \leq \nu' \|u(t)\|_{L^1},$$

it follows that

$$\|u(t)\|_{L^1} \leq e^{\nu'(t-s_n)} \|u(s_n)\|_{L^1}, \quad s_n \leq t < \infty. \quad (4.12)$$

Therefore,

$$\|u(t_n)\|_{L^1} \leq e^{\nu'(t_n-s_n)} \|u(s_n)\|_{L^1} = \sqrt{\|u(s_n)\|_{L^1}},$$

and hence  $\lim_{n \rightarrow \infty} \|u(t_n)\|_{L^1} = 0$ .

Next we notice from (4.10) that

$$\|\rho(t_n)\|_{L^1} = e^{-ct_n} \|\rho_0\|_{L^1} + d \int_0^{s_n} e^{-c(t_n-s)} \|u(s)\|_{L^1} ds + d \int_{s_n}^{t_n} e^{-c(t_n-s)} \|u(s)\|_{L^1} ds.$$

Here, by (2.4),

$$\begin{aligned} \int_0^{s_n} e^{-c(t_n-s)} \|u(s)\|_{L^1} ds &\leq C(\|u_0\|_{L^1} + 1) \int_0^{s_n} e^{-c(t_n-s)} ds \\ &\leq C(\|u_0\|_{L^1} + 1) e^{-c(t_n-s_n)} = C(\|u_0\|_{L^1} + 1) \|u(s_n)\|_{L^1}^{\frac{c}{2\nu'}}. \end{aligned}$$

In addition, by (4.12),

$$\int_{s_n}^{t_n} e^{-c(t_n-s)} \|u(s)\|_{L^1} ds \leq \int_{s_n}^{t_n} e^{\nu'(s-s_n)} ds \|u(s_n)\|_{L^1} \leq \frac{1}{\nu'} \sqrt{\|u(s_n)\|_{L^1}}.$$

Hence, we conclude that  $\lim_{n \rightarrow \infty} \|\rho(t_n)\|_{L^1} = 0$ .

Furthermore, in view of (2.6), we verify by using [13, (2.3)] that

$$\|\rho(t_n)\|_{L^p} \leq C_p \|\rho(t_n)\|_{L^1}^{\frac{1}{p}}$$

for any  $1 \leq p < \infty$ . Therefore,  $\lim_{n \rightarrow \infty} \|\rho(t_n)\|_{L^p} = 0$ .

Thus we have proved the first assertion of the theorem.

Consider now the case when  $f'(0) > 0$ . We suppose that  $\|u_0\|_{L^1} \neq 0$  and  $\sup_{1 \leq t < \infty} \|u(t)\|_{H^{1+\varepsilon}} = C_u < \infty$  with some  $\varepsilon > 0$ . Then, in view of (4.7) and (4.11), we have

$$\frac{d}{dt} \|u(t)\|_{L^1} \geq (\nu'' - \mu'' C_u \|u(t)\|_{L^1}^{\frac{\varepsilon}{4(1+\varepsilon)}}) \|u(t)\|_{L^1}, \quad 1 \leq t < \infty.$$

This, however, shows that

$$\liminf_{t \rightarrow \infty} \|u(t)\|_{L^1} \geq \left( \frac{\nu''}{C_u \mu''} \right)^{\frac{4(1+\varepsilon)}{\varepsilon}} > 0,$$

which contradicts to the assumption.

When  $f'(0) < 0$ , we have (4.9). Then  $\sup_{1 \leq t < \infty} \|u(t)\|_{H^{1+\varepsilon}} = C_u < \infty$  jointed with (4.7) implies that

$$\|u(t)\|_{\mathcal{C}} \leq C_u \|u(t)\|_{L^1}^{\frac{\varepsilon}{4(1+\varepsilon)}}, \quad 1 \leq t < \infty.$$

Therefore, at sufficiently large  $t_n$ , we have

$$\left[ \frac{d}{dt} \|u(t)\|_{L^1} \right]_{t=t_n} \leq -\tilde{\nu} \|u(t_n)\|_{L^1}.$$

This means that  $\|u(t)\|_{L^1}$  is decreasing for every  $t \geq t_n$ ; and as a consequence, it follows that  $\lim_{t \rightarrow \infty} \|u(t)\|_{L^1} = 0$ . Noting again (4.7), we have  $\lim_{t \rightarrow \infty} \|u(t)\|_{\mathcal{C}} = 0$ . From the formula (4.3), it is also verified that  $\lim_{t \rightarrow \infty} \|\rho(t)\|_{H^{1+\varepsilon_0}} = 0$ .  $\square$

## 5. NUMERICAL SIMULATION

In view of Theorem 4.2, extremely interesting is the question of whether there exists a solution to (CG) which tends to  $O = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  as  $t \rightarrow \infty$  or not, if  $f'(0) > 0$ . We shall present here some numerical results concerning this question.

Let  $\Omega = (0, 4)$  be an open interval. The coefficients are fixed as  $a = 0.25$ ,  $b = 1$ ,  $c = 6.25$ , except that  $d$  is a parameter. The sensitivity function and the growth function are taken as

$$\chi(\rho) = -\frac{0.125}{\rho}, \quad f(u) = u(1 - u).$$

The spatial variable is discretized by the finite element method with the step size  $\Delta x = 2^{-10}$  and the time variable by the implicit Runge-Kutta method (two-stage Radau IIA) with the step size  $\Delta t = 2^{-12}$ .

For  $d \geq 0.8$ , we found a numerically stable stationary solution  $\bar{U}_d = \begin{pmatrix} \bar{u}_d \\ \bar{\rho}_d \end{pmatrix}$ , Fig. 1 and 2. When  $d = 0.7$ , we computed the solution  $U_{0.7} = \begin{pmatrix} u_{0.7} \\ \rho_{0.7} \end{pmatrix}$  which starts from  $\bar{U}_{0.8}$ .  $U_{0.7}$  are seen to approach to  $O$  for a while with  $L^1$ -norm of  $u_{0.7}$  decaying as  $t$ , cf. Fig. 3. But when  $t$  is about 79.4, our computation of  $U_{0.7}$  had lost its stability.

This may not be satisfactory evidence to draw the conclusion that no stable stationary solution  $\bar{U}_d$  exists for  $d = 0.7$  and the solution  $U_{0.7}$  tends to  $O$  as  $t \rightarrow \infty$ . But, we could say at least that  $U_{0.7}$  does get close to  $O$  and that  $U_{0.7} \rightarrow O$  as  $t \rightarrow \infty$  if and only if the stable stationary solution  $\bar{U}_d$  does not exist for  $d = 0.7$ .

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