CHEMOTAXIS AND GROWTH SYSTEM WITH SINGULAR SENSITIVITY FUNCTION

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Abstract. This paper continues the study of the initial value problem of a chemotaxis-growth system. In the previous paper [13], we have handled the case when the sensitivity function $\chi(\rho)$ is regular. In this paper we are concerned with the case when the function has singularity at $\rho = 0$ like $\chi(\rho) = \log \rho$ or $-\frac{1}{\rho}$. We verify global existence of solutions and discuss some asymptotic behaviour of solutions.

Quasilinear system; Chemotaxis-growth; Singular sensitivity function; Global existence

1. Introduction

We study the initial value problem of a quasilinear parabolic system

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\Delta u - \nabla\{u\nabla\chi(\rho)\} + f(u) \qquad \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial \rho}{\partial t} = b\Delta\rho - c\rho + du \qquad \text{in} \quad \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial \rho}{\partial n} = 0 \qquad \text{on} \quad \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \ \ \rho(x,0) = \rho_0(x) \qquad \text{in} \quad \Omega \end{cases} \tag{CG}$$

in a bounded domain $\Omega \subset \mathbb{R}^2$. Here, $u(x,t)$ and $\rho(x,t)$ denote the population density of biological individuals and the concentration of chemical substance at a position $x \in \Omega$ and time $t \in [0,\infty)$, respectively. The mobility of individuals consists of two effects, namely random walking and chemotaxis, the latter means the directed movement in a sense that they have a tendency to move toward higher concentration of the chemical substance with the rate $\nabla\chi(\rho)$, we refer to [1, 3, 6]. $\chi(\rho)$ is called the sensitivity function of chemotaxis. $a > 0$ and $b > 0$ are the diffusion rates of u and ρ, respectively. $c > 0$ and $d > 0$ are the degradation and production rates of ρ, respectively. $f(u)$ is a growth term of u.

Burdrene and Berg [5] experimentally observed that bacteria called *E. coli* form complex spatio-temporal colony patterns. In order to study theoretically such chemotactic pattern, several models have been proposed by [2, 7, 8, 9, 11, 15, 18]. Among them, Mimura and Tsujikawa [10] presented the model (CG) above in which they incorporate three elemental effects, diffusion, chemotaxis, and growth of bacteria.

Our interest is to investigate a mathematical aspect of the system (CG) which is also very important for performing numerical computations. In the previous paper [13], we have studied the case where the sensitivity function is a smooth function of $\rho \in [0,\infty)$ without singularity at $\rho = 0$ and has uniformly bounded derivatives up to the third order (see the condition (χ) of [13]). Under these assumptions we have constructed an exponential attractor for the dynamical system determined by (CG) in the phase space $\{\begin{pmatrix} u \\ \rho \end{pmatrix}; u \in L^2(\Omega), \rho \in H^1(\Omega)\}$ by using the squeezing method due to Eden, Foias, Nicolaenko, and Temam [16] and [20].

In this paper we intend to handle the left but very interesting case where $\chi(\rho)$ has singularity at $\rho = 0$ such as $\log \rho$, $-\rho^{-1}$ and so on. $\chi(\rho)$ is actually assumed to be a smooth function of $\rho \in (0, \infty)$ satisfying

$$\left| \sup_{\delta \le \rho < \infty} \frac{d^i \chi}{d\rho^i}(\rho) \right| \le C_\delta \qquad \text{for } \delta > 0, \ i = 1, 2, 3 \tag{χ}$$

with some constant $C_\delta > 0$ which is allowed to depend on δ.

For the others we make the similar assumptions as in [13]. That is, $\Omega \subset \mathbb{R}^2$ is a bounded domain of $\mathcal{C}^3$ class. a, b, c and d are positive constants. $f(u)$ is a real smooth function of $u \in [0, \infty)$ with $f(0) = 0$ and $f'(0) \neq 0$ satisfying the condition

$$f(u) = (-\mu u + \nu)u \quad \text{for sufficiently large } u \tag{f}$$

with some $\mu > 0$ and $-\infty < \nu < \infty$.

The initial functions are also taken as before. That is, $u_0 \in L^2(\Omega)$ and $\rho_0 \in H^{1+\varepsilon_0}(\Omega)$, where ε_0 is an arbitrarily fixed exponent in such a way that $0 < \varepsilon_0 < \frac{1}{2}$. $u_0 \ge 0$ is nonnegative in Ω, and in view of the singularity of $\chi(\rho)$ we impose on ρ_0 the condition

$$\inf_{x \in \Omega} \rho_0(x) > 0.$$

The space of initial values is therefore set as

$$K = \left\{ U = \begin{pmatrix} u \\ \rho \end{pmatrix}; 0 \le u \in L^2(\Omega), 0 < \rho_0 \in H^{1+\varepsilon_0}(\Omega), \inf_{x \in \Omega} \rho_0(x) > 0 \right\}. \tag{In}$$

K is equipped with the distance induced by the product norm

$$d_K(U_1, U_2) = \|u_1 - u_2\|_{L^2} + \|\rho_1 - \rho_2\|_{H^{1+\varepsilon_0}}, \quad U_1, U_2 \in K. \tag{1.1}$$

In this way K can not contain a pair $O = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ of the null function of Ω. We, however, observe that there exists a solution which converges to this boundary point O as $t \to \infty$. In fact, let for example $f(u) = -u(u-1)(u-2)$ for $0 \le u \le 2$ and $u_0 \equiv \frac{1}{2}$, $\rho_0 \equiv 1$. Then (CG) reduces to a simple system of ordinary differential equations

$$\begin{cases} \dfrac{du}{dt} = -u(u-1)(u-2), & 0 < t < \infty, \\ \dfrac{d\rho}{dt} = -c\rho + du, & 0 < t < \infty, \\ u(0) = \frac{1}{2}, \ \rho(0) = 1. \end{cases}$$

And the solution of this system clearly converges to 0 as $t \to \infty$. By this consideration, we notice that the dynamical system determined by (CG) in the phase space K no longer admits a global attractor in general. This is a great difference from the case where $\chi(\rho)$ has no singularity at $\rho = 0$.

So we shall first verify in this paper that (CG) admits a unique global solution for each initial value from K. (In the case when $f(u) \equiv 0$, (CG) is called the Keller-Segel equations; some results on global existence and blow-up are obtained in [4, 12].) Second, we shall investigate asymptotic behavior of global solutions as $t \to \infty$. Some are shown to stay away from the point O and possess their nonempty ω-limit sets in K, and the others are shown to approach to the boundary point O in a suitable sense. This alternative is determined by the condition whether $\inf_{0\le t<\infty}\|u(t)\|_{L^1}>0$ or $\inf_{0\le t<\infty}\|u(t)\|_{L^1}=0$. In the case when the solution approaches to O, namely $\inf_{0\le t<\infty}\|u(t)\|_{L^1}=0$, if $f'(0)<0$, then the solution can converge to O in a strong topology, and if $f'(0)>0$, then some Sobolev norm of $u(t)$ grows up as $t\to\infty$.

Notations. $n(x)$ denotes the outer normal vector at a boundary point $x\in\partial\Omega$. For $1\le p\le\infty$, $L^p(\Omega)$ is the L^p space of real valued measurable functions in Ω, its norm is denoted by $\|\cdot\|_{L^p}$. $H^k(\Omega)$, $k=0,1,2,\dots$, denotes the real Sobolev space in Ω, its norm is denoted by $\|\cdot\|_{H^k}$. More generally, the fractional Sobolev space is denoted by $H^s(\Omega)$, $s>0$, its norm is denoted by $\|\cdot\|_{H^s}$. For $s>\frac{3}{2}$, $H^s_N(\Omega)$ is the closed subspace of $H^s(\Omega)$ consisting of functions which satisfy the Neumann boundary conditions on $\partial\Omega$. $\mathcal{C}(\overline{\Omega})$ denotes the space of real valued continuous functions on $\overline{\Omega}$, its norm is denoted by $\|\cdot\|_{\mathcal{C}}$.

Let H be a Hilbert space and let I be an interval of $\mathbb{R}$. $L^2(I;H)$ denotes the space of H valued L^2 functions defined in I. $H^1(I;H)$ denotes the space of functions in $L^2(I;H)$ whose first derivatives are also in $L^2(I;H)$. $\mathcal{C}(I;H)$ and $\mathcal{C}^m(I;H)$, $m=1,2,3,\dots$, denote the space of H valued continuous functions and of H valued m-times continuously differentiable functions, respectively.

For simplicity, we shall use a universal notation C to denote various constants which are determined in each occurrence by Ω, a, b, c, d, $\chi(\cdot)$, $f(\cdot)$ and so on in a specific way. In a case where C depends also on some parameter, say ζ, it will be denoted by C_ζ.

2. Global solutions

For each pair of initial functions u_0, ρ_0 in K, we shall prove that (CG) admits a unique global solution. Since $\rho_0(x)$ does not vanish in Ω, we can repeat the same arguments as in [13, Section 3] to construct a local solution by using the theory of abstract evolution equations (see [14, 17, 19]).

In order to extend such local solutions globally, however, we have to notice an a priori estimate of ρ from below.

Proposition 2.1. *Let u, ρ be any local solution to* (CG) *such that*

$$\begin{cases} 0\le u\in\mathcal{C}([0,T_{u,\rho}];\,L^2(\Omega))\cap\mathcal{C}^1((0,T_{u,\rho}];\,L^2(\Omega))\cap\mathcal{C}((0,T_{u,\rho}];\,H^2_N(\Omega)),\\ 0<\rho\in\mathcal{C}([0,T_{u,\rho}];\,H^{1+\varepsilon_0}(\Omega))\cap\mathcal{C}^1((0,T_{u,\rho}];\,H^1(\Omega))\cap\mathcal{C}((0,T_{u,\rho}];\,H^3_N(\Omega))\end{cases}$$

with initial functions u_0, ρ_0 in K. Then, ρ satisfies

$$\inf_{x\in\Omega}\rho(x,t)\ge\delta_0 e^{-ct}\quad\textit{for every }0\le t\le T_{u,\rho},\tag{2.1}$$

where $\delta_0=\inf_{x\in\Omega}\rho_0(x)>0$.

Proof. We introduce a decreasing convex $\mathcal{C}^2$ function $H(\rho)$ of $\rho\in(-\infty,\infty)$ such that $H(\rho)=0$ for $\rho\ge0$ and $H(\rho)>0$ for $\rho<0$. Consider a continuous function

$$\varphi(t)=\int_\Omega H(\rho(x,t)-\delta_0e^{-ct})dx,\quad 0\le t\le T_{u,\rho}.$$

It is observed that

$$\frac{d\varphi}{dt}(t) = \int_\Omega H'(\rho(t) - \delta_0 e^{-ct}) \left(\frac{\partial\rho}{\partial t} + c\delta_0 e^{-ct}\right) dx$$
$$= -b\int_\Omega H''(\rho - \delta_0 e^{-ct})|\nabla\rho|^2 dx + d\int_\Omega H'(\rho - \delta_0 e^{-ct}) u\, dx$$
$$- c\int_\Omega H'(\rho - \delta_0 e^{-ct})(\rho - \delta_0 e^{-ct}) dx.$$

Since $H'(\rho) \le 0$, $H'(\rho)\rho \ge 0$, and $H''(\rho) \ge 0$, it follows that $\varphi'(t) \le 0$ for every $0 < t \le T_{u,\rho}$. Therefore, $0 \le \varphi(t) \le \varphi(0) = 0$. This means that $\rho(t) - \delta_0 e^{-ct} \ge 0$ for every $0 \le t \le T_{u,\rho}$. □

This proposition jointed with [13, Theorem 4.5] then yields the global existence of solution.

Theorem 2.1. *For each pair of initial functions u_0, ρ_0 in K, there exists a unique global solution to* (CG) *in the function space*

$$\begin{cases} 0 \le u \in \mathcal{C}([0,\infty);\, L^2(\Omega)) \cap \mathcal{C}^1((0,\infty);\, L^2(\Omega)) \cap \mathcal{C}((0,\infty);\, H^2_N(\Omega)), \\ 0 < \rho \in \mathcal{C}([0,\infty);\, H^{1+\varepsilon_0}(\Omega)) \cap \mathcal{C}^1((0,\infty);\, H^1(\Omega)) \cap \mathcal{C}((0,\infty);\, H^3_N(\Omega)). \end{cases} \tag{2.2}$$

Proof. Let $T > 0$ be arbitrary positive time, and set $\delta = \delta_0 e^{-cT}$ with $\delta_0 = \inf_{x\in\Omega} \rho_0(x)$. We consider a smooth sensitivity function $\chi_\delta(\rho)$ of $\rho \in [0,\infty)$ such that $\chi_\delta(\rho) = \chi(\rho)$ for $\rho \in [\delta,\infty)$; obviously, $\chi_\delta(\rho)$ satisfies the condition (χ) of [13]. And we consider an auxiliary initial value problem (CG$_\delta$) by substituting $\chi_\delta(\rho)$ for $\chi(\rho)$. Then, by virtue of [13, Theorem 4.5], there exists a global solution u_δ, ρ_δ to the problem (CG$_\delta$). Set, further, that $T_\delta = \sup\{\tau;\, \inf_{0\le t\le\tau,\, x\in\Omega} \rho_\delta(x,t) \ge \delta\}$. By definition, $\rho_\delta(t) \ge \delta$ on the interval $[0, T_\delta]$; this in turn means that u_δ, ρ_δ is also a local solution of the original problem (CG) on the interval $[0, T_\delta]$. Meanwhile we see that $T_\delta \ge T$. Indeed, if $T_\delta < T$, then by Proposition 2.1 we have $\rho_\delta(T_\delta) \ge \delta_0 e^{-cT_\delta} > \delta$. But this contradicts to the maximality of T_δ since ρ_δ is a function belonging to $\mathcal{C}([0,\infty);\, H^{1+\varepsilon_0}(\Omega)) \subset \mathcal{C}([0,\infty);\, \mathcal{C}(\overline{\Omega}))$.

Thus (CG) has been shown to possess a local solution on an arbitrarily finite interval $[0, T]$. In other words, (CG) admits a global solution. □

We conclude this section by noting some estimates u, ρ which hold independently of $\delta_0 = \inf_{x\in\Omega} \rho_0(x)$. From $f(0) = 0$ and (f), we can take two positive constants μ' and ν' in such a way that

$$f(u) \le (-\mu' u + \nu')u, \qquad u \ge 0. \tag{2.3}$$

Then, by integrating the first equation of (CG) in Ω, we have

$$\frac{d}{dt}\int_\Omega u\, dx = \int_\Omega f(u) dx \le \int_\Omega (\nu' u - \mu' u^2) dx,$$

therefore

$$\|u(t)\|_{L^1} \le C(e^{-t}\|u_0\|_{L^1} + 1), \qquad 0 \le t < \infty \tag{2.4}$$

(see Step 1 of the proof of [13, Proposition 4.1]). As well it is observed that

$$\left|\int_0^t \int_\Omega f(u) dx ds\right| \le C(\|u_0\|_{L^1} + 1), \qquad 0 \le t < \infty. \tag{2.5}$$

Multiplying the second equation of (CG) by ρ and integrating the product in Ω, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^2 dx + b\int_\Omega |\nabla\rho|^2 dx + c\int_\Omega \rho^2 dx = d\int_\Omega u\rho\, dx \le \frac{c}{2}\|\rho\|_{L^2}^2 + \frac{d^2}{2c}\|u\|_{L^2}^2.$$

Here, it holds that

$$u^2 \le -(\mu')^{-1} f(u) + (\mu')^{-1}\nu' u, \qquad u \ge 0.$$

Therefore,

$$\int_\Omega \rho^2 dx \le e^{-ct}\|\rho_0\|_{L^2}^2 + \int_0^t e^{-c(t-s)}\left\{C\|u(s)\|_{L^1} - C\int_\Omega f(u(s))dx\right\} ds.$$

Applying the second mean value theorem of integration in view of (2.4) and (2.5), we obtain that

$$\|\rho(t)\|_{L^2}^2 \le C(e^{-ct}\|\rho_0\|_{L^2}^2 + \|u_0\|_{L^1} + 1), \qquad 0 \le t < \infty.$$

Next we multiply the second equation of (CG) by $\Delta\rho$ and integrate the product in Ω. Then,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\rho|^2 dx + b\int_\Omega |\Delta\rho|^2 dx + c\int_\Omega |\nabla\rho|^2 dx = -d\int_\Omega u\Delta\rho\, dx \le \frac{b}{2}\|\Delta\rho\|_{L^2}^2 + \frac{d^2}{2b}\|u\|_{L^2}^2.$$

Repeating the same argument as above, we obtain that

$$\|\nabla\rho(t)\|_{L^2}^2 \le C(e^{-2ct}\|\rho_0\|_{H^1}^2 + \|u_0\|_{L^1} + 1), \qquad 0 \le t < \infty.$$

Finally we conclude that

$$\|\rho(t)\|_{H^1}^2 \le C(e^{-ct}\|\rho_0\|_{H^1}^2 + \|u_0\|_{L^1} + 1), \qquad 0 \le t < \infty. \tag{2.6}$$

3. Continuous dependence in initial values

As shown in the preceding section, for each $U_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in K$, there exists a unique global solution $U = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (CG) in the solution space (2.2). This section is then devoted to noting continuous dependence of solutions with respect to initial values.

Theorem 3.1. *Let $U_0 \in K$ and let $U(t)$ be the solution to* (CG) *with an initial value U_0. Let $\{U_{0,n}\}_{n=1,2,3,\dots}$ be a sequence of initial values in K and let $\{U_n\}_{n=1,2,3,\dots}$ be the sequence of corresponding solutions. If $U_{0,n} \to U_0$ in K as $n \to \infty$, then $U_n(t) \to U(t)$ in K for each fixed time $0 \le t < \infty$.*

Proof. Since $\rho_{0,n} \to \rho_0$ in $H^{1+\varepsilon_0}(\Omega) \subset \mathcal{C}(\overline{\Omega})$. There exists some positive constant $\delta_0 > 0$ such that $\inf_{x\in\Omega} \rho_{0,n}(x) \ge \delta_0$ for all n. For fixed time $0 \le t < \infty$, set $\delta = \delta_0 e^{-ct} > 0$. Then, by virtue of Proposition 2.1, U_n are all local solutions on an interval $[0, t]$ to the auxiliary problem (CG_δ) where the sensitivity function is substituted with $\chi_\delta(\rho)$ which is a smooth function of $\rho \in [0, \infty)$ coinciding with $\chi(\rho)$ for $\rho \in [\delta, \infty)$. Therefore, we obtain the desired result, see [13, Theorem 3.2]. $\square$

4. ASYMPTOTIC BEHAVIOR OF GLOBAL SOLUTIONS

For each $0 \le t < \infty$, define a transform $S(t)$ on K by the formula $S(t)U_0 = \begin{pmatrix} u(t) \\ \rho(t) \end{pmatrix}$, $U_0 \in K$, where u, ρ denotes the global solution to the problem (CG) with the initial value U_0. By Theorems 2.1 and 3.1, $\{S(t)\}_{t \ge 0}$ defines a nonlinear semigroup on K, namely $S(\cdot)U_0$ is a continuous function of $t \in [0, \infty)$ with values in K and $S(t)$ is a continuous mapping from K into itself.

In this section we shall be concerned with asymptotic behavior of $S(t)U_0$ as $t \to \infty$. We begin with noting the following proposition.

Proposition 4.1. *Let u, ρ be any global solution to* (CG) *in the space* (2.2). *Then the following two assertions*

$$\inf_{0 \le t < \infty} \|u(t)\|_{L^1} > 0 \tag{4.1}$$

and

$$\inf_{0 \le t < \infty, x \in \Omega} \rho(x, t) > 0 \tag{4.2}$$

are equivalent.

Proof. I) Let us first verify that (4.1) implies (4.2). Put $\inf_{0 \le t < \infty} \|u(t)\|_{L^1} = \ell > 0$. We here introduce the realization L of the Laplace operator $-b\Delta$ in $L^2(\Omega)$ under the Neumann boundary conditions on $\partial\Omega$. L is a nonnegative self-adjoint operator in $L^2(\Omega)$. From the second equation of (CG), $\rho(t)$ is written as

$$\rho(t) = e^{-t(L+c)}\rho_0 + d \int_0^t e^{-(t-s)(L+c)} u(s)ds. \tag{4.3}$$

Set a time $t_0 \ge 2$. For every $t \ge 2t_0$, we have

$$\rho(t) \ge d \int_0^{t-t_0} e^{-(t-s)(L+c)} \{\overline{u}(s) + u_m(s)\} ds.$$

Here, $u = \overline{u} + u_m$ denotes the orthogonal decomposition of $u \in L^2(\Omega)$ such that $\overline{u} = |\Omega|^{-1} \int_\Omega u\, dx$ and

$$u_m \in L^2_m(\Omega) = \left\{ u \in L^2(\Omega);\ \int_\Omega u\, dx = 0 \right\}.$$

Since $\overline{u}(t) \ge |\Omega|^{-1}\ell$ and $e^{-(t-s)L}\overline{u}(s) = \overline{u}(s)$, it is seen that

$$\begin{aligned} \int_0^{t-t_0} e^{-(t-s)(L+c)} \overline{u}(s) ds &\ge \frac{\ell}{|\Omega|} \int_0^{t-t_0} e^{-c(t-s)} ds \\ &= \frac{\ell}{c|\Omega|} \{e^{-ct_0} - e^{-ct}\} \ge \frac{\ell e^{-ct_0}}{c|\Omega|} \{1 - e^{-ct_0}\}, \qquad t \ge 2t_0. \end{aligned} \tag{4.4}$$

On the other hand, the part L_m of L in the component $L^2_m(\Omega)$ is a positive definite self-adjoint operator in $L^2_m(\Omega)$. Therefore, there exists some $\lambda_m > 0$ such that $L_m \ge \lambda_m$. Then, using the fact that

$$e^{-L} \in \mathcal{L}(L^2(\Omega), \mathcal{C}(\Omega)) \cap \mathcal{L}(L^1(\Omega), L^2(\Omega)),$$

we can observe that

$$
\begin{aligned}
&\left\| \int_0^{t-t_0} e^{-(t-s)(L+c)} u_m(s) ds \right\|_{\mathcal{C}} \\
&\qquad \le \|e^{-L}\|_{\mathcal{L}(L^2,\mathcal{C})} \left\| \int_0^{t-t_0} e^{-(t-s-2)L} e^{-c(t-s)} e^{-L} u_m(s) ds \right\|_{L^2} \\
&\qquad \le \|e^{-L}\|_{\mathcal{L}(L^2,\mathcal{C})} \int_0^{t-t_0} e^{-(t-s-2)\lambda_m} e^{-c(t-s)} \|e^{-L} u_m(s)\|_{L^2} ds \\
&\qquad \le \|e^{-L}\|_{\mathcal{L}(L^2,\mathcal{C})} \|e^{-L}\|_{\mathcal{L}(L^1,L^2)} e^{2\lambda_m} \int_0^{t-t_0} e^{(c+\lambda_m)(s-t)} \|u_m(s)\|_{L^1} ds. \qquad (4.5)
\end{aligned}
$$

Since $\|u_m(s)\|_{L^1} \le 2\|u(s)\|_{L^1}$ and since (2.4) holds, the norm is furthermore estimated by

$$
\le C(\|u_0\|_{L^1} + 1) e^{-(c+\lambda_m)t_0}, \quad t \ge 2t_0.
$$

From (4.4) and (4.5) it is therefore verified that

$$
\rho(t) \ge \frac{d\ell e^{-ct_0}}{c|\Omega|} \{1 - e^{-ct_0} - C(\|u_0\|_{L^1} + 1) e^{-\lambda_m t_0}\}, \quad t \ge 2t_0.
$$

This obviously shows that, if t_0 is taken sufficiently large, then

$$
\inf_{2t_0 \le t < \infty, x \in \Omega} \rho(x,t) > 0.
$$

Since (2.1) has been verified, we conclude (4.2).

II) Let us next verify that (4.2) implies (4.1). We assume that

$$
\inf_{0 \le t < \infty, x \in \Omega} \rho(x,t) = \delta > 0. \qquad (4.6)
$$

As done above, we consider an auxiliary initial value problem (CG$_\delta$) in which a sensitivity function $\chi_\delta(\rho)$ is substituted for $\chi(\rho)$, $\chi_\delta(\rho)$ is a smooth function of $\rho \in [0,\infty)$ coinciding with $\chi(\rho)$ for all $\rho \in [\delta,\infty)$. Then, u, ρ is clearly a global solution to the problem (CG$_\delta$). Therefore, as a global solution to (CG$_\delta$), all the results obtained in [13] are available.

We now apply the a priori estimates established by [13, Proposition 4.1] to u, ρ on the interval $[1,\infty)$. Then there must exist some constant $C_u > 0$ such that

$$
\|u(t)\|_{H^2} \le C_u, \qquad t \ge 1.
$$

In addition, we note that for any $0 < \varepsilon \le 1$ it holds that

$$
\|u\|_{\mathcal{C}} \le C\|u\|_{H^{1+\frac{\varepsilon}{2}}} \le C\|u\|_{H^{1+\varepsilon}}^{\frac{2+\varepsilon}{2(1+\varepsilon)}} \|u\|_{H^1}^{\frac{\varepsilon}{4(1+\varepsilon)}} \|u\|_{L^1}^{\frac{\varepsilon}{4(1+\varepsilon)}}, \quad u \in H^{1+\varepsilon}(\Omega) \qquad (4.7)
$$

(from [13, (2.1$\sim$4)]). Using this estimate with $\varepsilon = 1$, we observe that

$$
\|u(t)\|_{\mathcal{C}} \le C_u \|u(t)\|_{L^1}^{\frac{1}{8}}, \qquad t \ge 1. \qquad (4.8)
$$

To prove (4.1), we first notice that $u(s)$ can not vanish in any finite time s. Indeed, suppose that $u(s) = 0$ at some time s. Then, by the uniqueness of solution, $u(t) = 0$ for every $t \in [s,\infty)$. On the other hand, $\rho(t)$ must be determined by

$$
\frac{\partial \rho}{\partial t} = b\Delta\rho - c\rho \qquad \text{in } \Omega \times (s,\infty).
$$

Therefore, $\rho(t)$ must converge to 0 as $t \to \infty$. But this contradicts to (4.6).

To verify that $u(t)$ does not vanish as $t \to \infty$, neither, we shall use the condition $f'(0) \neq 0$. First, let $f'(0) < 0$, then there are some constants $\tilde{\nu} > 0$ and $\ell > 0$ such that

$$f(u) \leq -\tilde{\nu} u \quad \text{holds for all } \; u \in [0, \ell]. \tag{4.9}$$

We shall then verify that $\|u(t)\|_{L^1} \geq (\ell C_u^{-1})^8$ for every $t \geq 1$, where C_u is the constant appearing in (4.8). Indeed, if once $\|u(s)\|_{L^1} < (\ell C_u^{-1})^8$ for some $s \geq 1$, then $\|u(s)\|_{\mathcal{C}} < \ell$ and therefore

$$\frac{d}{ds}\|u(s)\|_{L^1} = \int_\Omega f(u(s))dx \leq -\tilde{\nu}\|u(s)\|_{L^1}.$$

Hence, $\|u(s)\|_{L^1}$ is decreasing at s, and this implies that $\|u(t)\|_{L^1}$ is less than $(\ell C_u^{-1})^8$ for any $t \geq s$. In this way, $\frac{d}{dt}\|u(t)\|_{L^1} \leq -\tilde{\nu}\|u(t)\|_{L^1}$ and $\|u(t)\|_{L^1} \leq e^{-\tilde{\nu}(t-s)}\|u(s)\|_{L^1}$ for all $t \geq s$. Thus, we conclude that $\|u(t)\|_{L^1} \to 0$ as $t \to \infty$.

While, integrating the second equation of (CG) in Ω, we see that

$$\frac{d}{dt}\|\rho(t)\|_{L^1} = -c\|\rho(t)\|_{L^1} + d\|u(t)\|_{L^1},$$

as a consequence

$$\|\rho(t)\|_{L^1} = e^{-ct}\|\rho_0\|_{L^1} + d\int_0^t e^{-c(t-\tau)}\|u(\tau)\|_{L^1}d\tau. \tag{4.10}$$

This together with the vanishing of $\|u(t)\|_{L^1}$ implies that $\|\rho(t)\|_{L^1}$ also vanishes as $t \to \infty$. But this again contradicts to (4.6).

Let now $f'(0) > 0$. In this case there exist two positive numbers μ'' and ν'' such that

$$f(u) \geq -\mu'' u^2 + \nu'' u, \qquad u \geq 0. \tag{4.11}$$

From (4.8) we have

$$\begin{aligned}\frac{d}{dt}\|u(t)\|_{L^1} = \int_\Omega f(u(t))dx &\geq (\nu'' - \mu''\|u(t)\|_{\mathcal{C}})\|u(t)\|_{L^1} \\ &\geq (\nu'' - \mu'' C_u \|u(t)\|_{L^1}^{\frac{1}{8}})\|u(t)\|_{L^1}, \quad t \geq 1.\end{aligned}$$

If $\|u(s)\|_{L^1} < (\nu''/\mu'' C_u)^8$ at some $s \geq 1$, then $\|u(s)\|_{L^1}$ is increasing at the time. Then, if once $\|u(s')\|_{L^1} \geq (\nu''/\mu'' C_u)^8$ at some time $s' \geq 1$, then this differential inequality shows that $\|u(t)\|_{L^1}$ is never less than $(\nu''/\mu'' C_u)^8$ for any $t \geq s'$. □

We can now prove the main results of the paper.

Theorem 4.1. *For each $U_0 \in K$, let $S(\cdot)U_0 = \begin{pmatrix} u \\ \rho \end{pmatrix}$. If $\inf_{0 \leq t < \infty}\|u(t)\|_{L^1} > 0$ or equivalently $\inf_{0 \leq t < \infty, x \in \Omega}\rho(x,t) > 0$, then its ω-limit set $\omega(U_0) = \bigcap_{t \geq 0}\overline{\bigcup_{\tau \geq t} S(\tau)U_0}$ in K is nonempty and is actually contained in the product space $\{\begin{pmatrix} u \\ \rho \end{pmatrix}; u \in H^2(\Omega), \rho \in H^3(\Omega)\}$.*

Proof. From the assumption there exists some positive constant $\delta > 0$ for which (4.6) holds. As above, introducing a smooth function $\chi_\delta(\rho)$ of $\rho \in [0, \infty)$ which coincides with $\chi(\rho)$ for $\rho \in [\delta, \infty)$, we regard $S(t)U_0$ as the global solution to the initial value problem (CG$_\delta$) in which $\chi_\delta(\rho)$ substitutes for $\chi(\rho)$. Then [13, Theorem 4.6] is available for u, ρ

to conclude that there exists some constant C_{U_0} such that $\|u(t)\|_{H^2} + \|\rho(t)\|_{H^3} \le C_{U_0}$ for $1 \le t < \infty$. Since the set

$$\left\{ \begin{pmatrix} u \\ \rho \end{pmatrix};\ \|u\|_{H^2} + \|\rho\|_{H^3} \le C_{U_0},\ \ u \ge 0,\ \ \inf_{x\in\Omega} \rho(x) \ge \delta \right\}$$

is a compact set of K, we verify that the solution $S(t)U_0$ admits a nonempty ω-limit set in the topology (1.1) of K. $\square$

In the case when $\inf_{0\le t<\infty} \|u(t)\|_{L^1} = 0$, we prove the following theorem.

Theorem 4.2. *Let $U_0 \in K$ and $S(\cdot)U_0 = \begin{pmatrix} u \\ \rho \end{pmatrix}$. Assume that $\inf_{0\le t<\infty} \|u(t)\|_{L^1} = 0$ or equivalently $\inf_{0\le t<\infty, x\in\Omega} \rho(x,t) = 0$. Then there exists a sequence t_n tending to ∞ such that $S(t_n)U_0$ converges to the boundary point $O = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ of K in the norm $\|u\|_{L^1} + \|\rho\|_{L^p}$ with any $1 \le p < \infty$.*

Furthermore, when $f'(0) > 0$, $\|u_0\|_{L^1} \ne 0$ implies that $\sup_{1\le t<\infty} \|u(t)\|_{H^{1+\varepsilon}} = \infty$ with an arbitrary $\varepsilon > 0$. On the other hand, when $f'(0) < 0$, $\sup_{1\le t<\infty} \|u(t)\|_{H^{1+\varepsilon}} < \infty$ with some $\varepsilon > 0$ implies that $S(t)U_0$ converges to O in the distance (1.1).

Proof. If $u(s) = 0$ at some finite time s, then $u(t) = 0$ for every $t \ge s$ and $\rho(t) \to 0$ as $t \to \infty$; therefore, $S(t)U_0$ converges to O in the distance (1.1).

So let us consider the case when $\|u(t)\|_{L^1} > 0$ for every t and $\inf_{0\le t<\infty} \|u(t)\|_{L^1} = 0$. Then there exists an increasing sequence $\{s_n\}_{n=1,2,3,\dots}$ tending to ∞ such that $\lim_{n\to\infty} \|u(s_n)\|_{L^1} = 0$ and $0 < \|u(s_n)\|_{L^1} < 1$. We here set another increasing sequence $\{t_n\}_{n=1,2,3,\dots}$ by the formula

$$t_n = s_n - \frac{1}{2\nu'} \log \|u(s_n)\|_{L^1}, \quad n = 1, 2, 3, \dots,$$

where ν' is the positive constant appearing in (2.3). Since

$$\frac{d}{dt}\|u(t)\|_{L^1} = \int_\Omega f(u(t))dx \le \nu'\|u(t)\|_{L^1},$$

it follows that

$$\|u(t)\|_{L^1} \le e^{\nu'(t-s_n)}\|u(s_n)\|_{L^1}, \qquad s_n \le t < \infty. \tag{4.12}$$

Therefore,

$$\|u(t_n)\|_{L^1} \le e^{\nu'(t_n-s_n)}\|u(s_n)\|_{L^1} = \sqrt{\|u(s_n)\|_{L^1}},$$

and hence $\lim_{n\to\infty} \|u(t_n)\|_{L^1} = 0$.

Next we notice from (4.10) that

$$\|\rho(t_n)\|_{L^1} = e^{-ct_n}\|\rho_0\|_{L^1} + d\int_0^{s_n} e^{-c(t_n-s)}\|u(s)\|_{L^1}ds + d\int_{s_n}^{t_n} e^{-c(t_n-s)}\|u(s)\|_{L^1}ds.$$

Here, by (2.4),

$$\begin{aligned}\int_0^{s_n} e^{-c(t_n-s)}\|u(s)\|_{L^1}ds &\le C(\|u_0\|_{L^1}+1)\int_0^{s_n} e^{-c(t_n-s)}ds \\ &\le C(\|u_0\|_{L^1}+1)e^{-c(t_n-s_n)} = C(\|u_0\|_{L^1}+1)\|u(s_n)\|_{L^1}^{\frac{c}{2\nu'}}.\end{aligned}$$

In addition, by (4.12),

$$\int_{s_n}^{t_n} e^{-c(t_n-s)}\|u(s)\|_{L^1}ds \le \int_{s_n}^{t_n} e^{\nu'(s-s_n)}ds\,\|u(s_n)\|_{L^1} \le \frac{1}{\nu'}\sqrt{\|u(s_n)\|_{L^1}}.$$

Hence, we conclude that $\lim_{n\to\infty}\|\rho(t_n)\|_{L^1} = 0$.

Furthermore, in view of (2.6), we verify by using [13, (2.3)] that

$$\|\rho(t_n)\|_{L^p} \le C_p\|\rho(t_n)\|_{L^1}^{\frac{1}{p}}$$

for any $1 \le p < \infty$. Therefore, $\lim_{n\to\infty}\|\rho(t_n)\|_{L^p} = 0$.

Thus we have proved the first assertion of the theorem.

Consider now the case when $f'(0) > 0$. We suppose that $\|u_0\|_{L^1} \ne 0$ and $\sup_{1\le t<\infty} \|u(t)\|_{H^{1+\varepsilon}} = C_u < \infty$ with some $\varepsilon > 0$. Then, in view of (4.7) and (4.11), we have

$$\frac{d}{dt}\|u(t)\|_{L^1} \ge (\nu'' - \mu''C_u\|u(t)\|_{L^1}^{\frac{\varepsilon}{4(1+\varepsilon)}})\|u(t)\|_{L^1}, \quad 1 \le t < \infty.$$

This, however, shows that

$$\liminf_{t\to\infty}\|u(t)\|_{L^1} \ge \left(\frac{\nu''}{C_u\mu''}\right)^{\frac{4(1+\varepsilon)}{\varepsilon}} > 0,$$

which contradicts to the assumption.

When $f'(0) < 0$, we have (4.9). Then $\sup_{1\le t<\infty}\|u(t)\|_{H^{1+\varepsilon}} = C_u < \infty$ jointed with (4.7) implies that

$$\|u(t)\|_{\mathcal{C}} \le C_u\|u(t)\|_{L^1}^{\frac{\varepsilon}{4(1+\varepsilon)}}, \qquad 1 \le t < \infty.$$

Therefore, at sufficiently large t_n, we have

$$\left[\frac{d}{dt}\|u(t)\|_{L^1}\right]_{|t=t_n} \le -\tilde{\nu}\|u(t_n)\|_{L^1}.$$

This means that $\|u(t)\|_{L^1}$ is decreasing for every $t \ge t_n$; and as a consequence, it follows that $\lim_{t\to\infty}\|u(t)\|_{L^1} = 0$. Noting again (4.7), we have $\lim_{t\to\infty}\|u(t)\|_{\mathcal{C}} = 0$. From the formula (4.3), it is also verified that $\lim_{t\to\infty}\|\rho(t)\|_{H^{1+\varepsilon_0}} = 0$. □

5. Numerical Simulation

In view of Theorem 4.2, extremely interesting is the question of whether there exists a solution to (CG) which tends to $O = \begin{pmatrix}0\\0\end{pmatrix}$ as $t \to \infty$ or not, if $f'(0) > 0$. We shall present here some numerical results concerning this question.

Let $\Omega = (0, 4)$ be an open interval. The coefficients are fixed as $a = 0.25$, $b = 1$, $c = 6.25$, except that d is a parameter. The sensitivity function and the growth function are taken as

$$\chi(\rho) = -\tfrac{0.125}{\rho}, \qquad f(u) = u(1-u).$$

The spatial variable is discretized by the finite element method with the step size $\Delta x = 2^{-10}$ and the time variable by the implicit Runge-Kutta method (two-stage Radau IIA) with the step size $\Delta t = 2^{-12}$.

For $d \geq 0.8$, we found a numerically stable stationary solution $\overline{U}_d = \begin{pmatrix} \overline{u}_d \\ \overline{\rho}_d \end{pmatrix}$, Fig. 1 and 2. When $d = 0.7$, we computed the solution $U_{0.7} = \begin{pmatrix} u_{0.7} \\ \rho_{0.7} \end{pmatrix}$ which starts from $\overline{U}_{0.8}$. $U_{0.7}$ are seen to approach to O for a while with L^1-norm of $u_{0.7}$ decaying as t, cf. Fig. 3. But when t is about 79.4, our computation of $U_{0.7}$ had lost its stability.

This may not be satisfactory evidence to draw the conclusion that no stable stationary solution $\overline{U}_d$ exists for $d = 0.7$ and the solution $U_{0.7}$ tends to O as $t \to \infty$. But, we could say at least that $U_{0.7}$ does get close to O and that $U_{0.7} \to O$ as $t \to \infty$ if and only if the stable stationary solution $\overline{U}_d$ does not exist for $d = 0.7$.

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