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Osaka University
CHEMOTAXIS AND GROWTH SYSTEM
WITH SINGULAR SENSITIVITY FUNCTION

MASASHI AIDA, KOICHI OSAKI, TOHRU TSUJIKAWA,
ATSUSHI YAGI, MASAYASU MIMURA

Abstract. This paper continues the study of the initial value problem of a chemotaxis-
growth system. In the previous paper [13], we have handled the case when the sensitivity
function $\chi(\rho)$ is regular. In this paper we are concerned with the case when the function
has singularity at $\rho = 0$ like $\chi(\rho) = \log \rho$ or $-\frac{1}{\rho}$. We verify global existence of solutions
and discuss some asymptotic behaviour of solutions.

Quasilinear system; Chemotaxis-growth; Singular sensitivity function; Global existence

1. Introduction

We study the initial value problem of a quasilinear parabolic system

$$\begin{cases}
\frac{\partial u}{\partial t} = a \Delta u - \nabla \{u \nabla \chi(\rho)\} + f(u) & \text{in } \Omega \times (0, \infty), \\
\frac{\partial \rho}{\partial t} = b \Delta \rho - c \rho + du & \text{in } \Omega \times (0, \infty), \\
\frac{\partial \rho}{\partial n} = 0 & \text{on } \partial \Omega \times (0, \infty), \\
u(x, 0) = u_0(x), \ \rho(x, 0) = \rho_0(x) & \text{in } \Omega,
\end{cases}$$

(CG)

in a bounded domain $\Omega \subset \mathbb{R}^2$. Here, $u(x, t)$ and $\rho(x, t)$ denote the population density of
biological individuals and the concentration of chemical substance at a position $x \in \Omega$ and
time $t \in [0, \infty)$, respectively. The mobility of individuals consists of two effects, namely
random walking and chemotaxis, the latter means the directed movement in a sense that
they have a tendency to move toward higher concentration of the chemical substance with
the rate $\nabla \chi(\rho)$, we refer to [1, 3, 6]. $\chi(\rho)$ is called the sensitivity function of chemotaxis.
$a > 0$ and $b > 0$ are the diffusion rates of $u$ and $\rho$, respectively. $c > 0$ and $d > 0$ are the
degradation and production rates of $\rho$, respectively. $f(u)$ is a growth term of $u$.

Burdrene and Berg [5] experimentally observed that bacteria called $E. coli$ form complex
spatio-temporal colony patterns. In order to study theoretically such chemotactic pattern,
several models have been proposed by [2, 7, 8, 9, 11, 15, 18]. Among them, Mimura and
Tsujikawa [10] presented the model (CG) above in which they incorporate three elemental
effects, diffusion, chemotaxis, and growth of bacteria.

Our interest is to investigate a mathematical aspect of the system (CG) which is also
very important for performing numerical computations. In the previous paper [13], we
have studied the case where the sensitivity function is a smooth function of $\rho \in [0, \infty)$
without singularity at $\rho = 0$ and has uniformly bounded derivatives up to the third
order (see the condition (\(\chi\)) of [13]). Under these assumptions we have constructed an
exponential attractor for the dynamical system determined by (CG) in the phase space \( \{(u, \rho) ; u \in L^2(\Omega), \rho \in H^1(\Omega)\} \) by using the squeezing method due to Eden, Foias, Nicolaenko, and Temam [16] and [20].

In this paper we intend to handle the left but very interesting case where \( \chi(\rho) \) has singularity at \( \rho = 0 \) such as \( \log \rho, -\rho^{-1} \) and so on. \( \chi(\rho) \) is actually assumed to be a smooth function of \( \rho \in (0, \infty) \) satisfying

\[
\left| \sup_{\delta \leq \rho < \infty} \left| \frac{d^i \chi}{d \rho^i}(\rho) \right| \right| \leq C_\delta \quad \text{for} \quad \delta > 0, \quad i = 1, 2, 3
\]

with some constant \( C_\delta > 0 \) which is allowed to depend on \( \delta \).

For the others we make the similar assumptions as in [13]. That is, \( \Omega \subset \mathbb{R}^2 \) is a bounded domain of \( C^1 \) class. \( a, b, c \) and \( d \) are positive constants. \( f(u) \) is a real smooth function of \( u \in [0, \infty) \) with \( f(0) = 0 \) and \( f'(0) \neq 0 \) satisfying the condition

\[
f(u) = (-\mu u + \nu)u \quad \text{for sufficiently large} \quad u
\]

with some \( \mu > 0 \) and \( -\infty < \nu < \infty \).

The initial functions are also taken as before. That is, \( u_0 \in L^2(\Omega) \) and \( \rho_0 \in H^{1+\varepsilon_0}(\Omega) \), where \( \varepsilon_0 \) is an arbitrarily fixed exponent in such a way that \( 0 < \varepsilon_0 < \frac{1}{2} \). \( u_0 \geq 0 \) is nonnegative in \( \Omega \), and in view of the singularity of \( \chi(\rho) \) we impose on \( \rho_0 \) the condition

\[
\inf_{x \in \Omega} \rho_0(x) > 0.
\]

The space of initial values is therefore set as

\[
K = \left\{ U = \begin{pmatrix} u \\ \rho \end{pmatrix} ; 0 \leq u \in L^2(\Omega), 0 < \rho \in H^{1+\varepsilon_0}(\Omega), \inf_{x \in \Omega} \rho_0(x) > 0 \right\}.
\]

\( K \) is equipped with the distance induced by the product norm

\[
d_K(U_1, U_2) = \| u_1 - u_2 \|_{L^2} + \| \rho_1 - \rho_2 \|_{H^{1+\varepsilon_0}}, \quad U_1, U_2 \in K.
\]

In this way \( K \) can not contain a pair \( O = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) of the null function of \( \Omega \). We, however, observe that there exists a solution which converges to this boundary point \( O \) as \( t \to \infty \). In fact, let for example \( f(u) = -u(u-1)(u-2) \) for \( 0 \leq u \leq 2 \) and \( u_0 \equiv \frac{1}{2}, \rho_0 \equiv 1 \). Then (CG) reduces to a simple system of ordinary differential equations

\[
\begin{cases}
\frac{du}{dt} = -u(u-1)(u-2), & 0 < t < \infty, \\
\frac{d\rho}{dt} = -c\rho + du, & 0 < t < \infty, \\
u(0) = \frac{1}{2}, \quad \rho(0) = 1.
\end{cases}
\]

And the solution of this system clearly converges to 0 as \( t \to \infty \). By this consideration, we notice that the dynamical system determined by (CG) in the phase space \( K \) no longer admits a global attractor in general. This is a great difference from the case where \( \chi(\rho) \) has no singularity at \( \rho = 0 \).

So we shall first verify in this paper that (CG) admits a unique global solution for each initial value from \( K \). (In the case when \( f(u) \equiv 0 \), (CG) is called the Keller-Segel equations; some results on global existence and blow-up are obtained in [4, 12].) Second, we shall investigate asymptotic behavior of global solutions as \( t \to \infty \). Some are shown to
stay away from the point $O$ and possess their nonempty $\omega$-limit sets in $K$, and the others are shown to approach to the boundary point $O$ in a suitable sense. This alternative is determined by the condition whether $\inf_{0 \leq t < \infty} \|u(t)\|_{L^1} > 0$ or $\inf_{0 \leq t < \infty} \|u(t)\|_{L^1} = 0$. In the case when the solution approaches to $O$, namely $\inf_{0 \leq t < \infty} \|u(t)\|_{L^1} = 0$, if $f'(0) < 0$, then the solution can converge to $O$ in a strong topology, and if $f'(0) > 0$, then some Sobolev norm of $u(t)$ grows up as $t \to \infty$.

**Notations.** Let $(\Omega)$ denotes the outer normal vector at a boundary point $x \in \partial \Omega$. For $1 \leq p \leq \infty$, $L^p(\Omega)$ is the $L^p$ space of real valued measurable functions in $\Omega$, its norm is denoted by $\| \cdot \|_{L^p}$. $H^k(\Omega)$, $k = 0, 1, 2, \ldots$, denotes the real Sobolev space in $\Omega$, its norm is denoted by $\| \cdot \|_{H^k}$. More generally, the fractional Sobolev space is denoted by $H^s(\Omega)$, $s > 0$; its norm is denoted by $\| \cdot \|_{H^s}$. For $s > \frac{3}{2}$, $H^s_N(\Omega)$ is the closed subspace of $H^s(\Omega)$ consisting of functions which satisfy the Neumann boundary conditions on $\partial \Omega$. $C(\Omega)$ denotes the space of real valued continuous functions on $\overline{\Omega}$, its norm is denoted by $\| \cdot \|_{C}$.

Let $H$ be a Hilbert space and let $I$ be an interval of $\mathbb{R}$. $L^2(I; H)$ denotes the space of $H$ valued $L^2$ functions defined in $I$. $H^1(I; H)$ denotes the space of functions in $L^2(I; H)$ whose first derivatives are also in $L^2(I; H)$. $C(I; H)$ and $C^m(I; H)$, $m = 1, 2, 3, \ldots$, denote the space of $H$ valued continuous functions and of $H$ valued $m$-times continuously differentiable functions, respectively.

For simplicity, we shall use a universal notation $C$ to denote various constants which are determined in each occurrence by $\Omega$, $a$, $b$, $c$, $d$, $\chi(\cdot)$, $f(\cdot)$ and so on in a specific way. In a case where $C$ depends also on some parameter, say $\zeta$, it will be denoted by $C_\zeta$.

## 2. Global solutions

For each pair of initial functions $u_0$, $\rho_0$ in $K$, we shall prove that (CG) admits a unique global solution. Since $\rho_0(x)$ does not vanish in $\Omega$, we can repeat the same arguments as in [13, Section 3] to construct a local solution by using the theory of abstract evolution equations (see [14, 17, 19]).

In order to extend such local solutions globally, however, we have to notice an a priori estimate of $\rho$ from below.

**Proposition 2.1.** Let $u$, $\rho$ be any local solution to (CG) such that

\[
\begin{aligned}
0 \leq u & \in C([0, T_{u, \rho}]; L^2(\Omega)) \cap C^1((0, T_{u, \rho}]; L^2(\Omega)) \cap C((0, T_{u, \rho}]; H^1_N(\Omega)), \\
0 < \rho & \in C([0, T_{u, \rho}]; H^{1+\delta_0}(\Omega)) \cap C^1((0, T_{u, \rho}]; H^1(\Omega)) \cap C((0, T_{u, \rho}]; H^3_N(\Omega))
\end{aligned}
\]

with initial functions $u_0$, $\rho_0$ in $K$. Then, $\rho$ satisfies

\[
\inf_{x \in \Omega} \rho(x, t) \geq \delta_0 e^{-ct} \quad \text{for every} \quad 0 \leq t \leq T_{u, \rho},
\]

where $\delta_0 = \inf_{x \in \Omega} \rho_0(x) > 0$.

**Proof.** We introduce a decreasing convex $C^2$ function $H(\rho)$ of $\rho \in (-\infty, \infty)$ such that $H(\rho) = 0$ for $\rho \geq 0$ and $H(\rho) > 0$ for $\rho < 0$. Consider a continuous function

\[
\varphi(t) = \int_\Omega H(\rho(x, t) - \delta_0 e^{-ct}) dx, \quad 0 \leq t \leq T_{u, \rho}.
\]
It is observed that
\[ \frac{d\varphi}{dt}(t) = \int_{\Omega} H'(\rho(t) - \delta_0 e^{-ct}) \left( \frac{\partial \rho}{\partial t} + c\delta_0 e^{-ct} \right) dx \]
\[ = -b \int_{\Omega} H''(\rho - \delta_0 e^{-ct})|\nabla \rho|^2 dx + d \int_{\Omega} H'(\rho - \delta_0 e^{-ct}) u dx \]
\[ - c \int_{\Omega} H'(\rho - \delta_0 e^{-ct})(\rho - \delta_0 e^{-ct})dx. \]

Since \( H'(\rho) \leq 0, H'(\rho) \geq 0, \) and \( H''(\rho) \geq 0, \) it follows that \( \varphi'(t) \leq 0 \) for every \( 0 < t \leq T_{u,\rho}. \) Therefore, \( 0 \leq \varphi(t) \leq \varphi(0) = 0. \) This means that \( \rho(t) - \delta_0 e^{-ct} \geq 0 \) for every \( 0 \leq t \leq T_{u,\rho}. \)

This proposition jointed with [13, Theorem 4.5] then yields the global existence of solution.

**Theorem 2.1.** For each pair of initial functions \( u_0, \rho_0 \) in \( K, \) there exists a unique global solution to (CG) in the function space
\[
\begin{align*}
0 &\leq u \in C([0, \infty); L^2(\Omega)) \cap C^1((0, \infty); L^2(\Omega)) \cap C((0, \infty); H^2_N(\Omega)), \\
0 &< \rho \in C([0, \infty); H^{1+\varepsilon_0}(\Omega)) \cap C^1((0, \infty); H^1(\Omega)) \cap C((0, \infty); H^3_N(\Omega)).
\end{align*}
\]

**Proof.** Let \( T > 0 \) be arbitrary positive time, and set \( \delta = \delta_0 e^{-cT} \) with \( \delta_0 = \inf_{x \in \Omega} \rho_0(x). \) We consider a smooth sensitivity function \( \chi_{\delta}(\rho) \) of \( \rho \in [0, \infty) \) such that \( \chi_{\delta}(\rho) = \chi(\rho) \) for \( \rho \in [\delta, \infty); \) obviously, \( \chi_{\delta}(\rho) \) satisfies the condition (\( \chi \)) of [13]. And we consider an auxiliary initial value problem (CG\(_{\delta} \)) by substituting \( \chi_{\delta}(\rho) \) for \( \chi(\rho) \). Then, by virtue of [13, Theorem 4.5], there exists a global solution \( u_{\delta}, \rho_{\delta} \) to the problem (CG\(_{\delta} \)). Set, further, that \( T_{\delta} = \sup\{\tau; \inf_{0 \leq t \leq \tau, x \in \Omega} \rho_{\delta}(x, t) \geq \delta\}. \) By definition, \( \rho_{\delta}(t) \geq \delta \) on the interval \([0, T_{\delta})\); this in turn means that \( u_{\delta}, \rho_{\delta} \) is also a local solution of the original problem (CG) on the interval \([0, T_{\delta})\). Meanwhile we see that \( T_{\delta} \geq T. \) Indeed, if \( T_{\delta} < T, \) then by Proposition 2.1 we have \( \rho_{\delta}(T_{\delta}) \geq \delta_0 e^{-cT_{\delta}} > \delta. \) But this contradicts to the maximality of \( T_{\delta} \) since \( \rho_{\delta} \) is a function belonging to \( C([0, \infty); H^{1+\varepsilon_0}(\Omega)) \subset C((0, \infty); \mathcal{C}(\Omega)). \)

Thus (CG) has been shown to possess a local solution on an arbitrarily finite interval \([0, T]\). In other words, (CG) admits a global solution. \( \square \)

We conclude this section by noting some estimates \( u, \rho \) which hold independently of \( \delta_0 = \inf_{x \in \Omega} \rho_0(x). \) From \( f(0) = 0 \) and (f), we can take two positive constants \( \mu' \) and \( \nu' \) in such a way that
\[ f(u) \leq (-\mu' u + \nu' u)u, \quad u \geq 0. \]
Then, by integrating the first equation of (CG) in \( \Omega, \) we have
\[ \frac{d}{dt} \int_{\Omega} u dx = \int_{\Omega} f(u)dx \leq \int_{\Omega} (\nu' u - \mu' u^2)dx, \]
therefore
\[ \|u(t)\|_{L^1} \leq C(e^{-t}\|u_0\|_{L^1} + 1), \quad 0 \leq t < \infty \]
(see Step 1 of the proof of [13, Proposition 4.1]). As well it is observed that
\[ \int_0^t \int_{\Omega} f(u)dxds \leq C(\|u_0\|_{L^1} + 1), \quad 0 \leq t < \infty. \]
Multiplying the second equation of (CG) by $\rho$ and integrating the product in $\Omega$, we have
\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho^2 dx + b \int_{\Omega} |\nabla \rho|^2 dx + c \int_{\Omega} \rho^2 dx = d \int_{\Omega} u \rho dx \leq \frac{c}{2} \| \rho \|_{L^2}^2 + \frac{d^2}{2c} \| u \|_{L^2}^2 . \]
Here, it holds that
\[ u^2 \leq -(\mu')^{-1} f(u) + (\mu')^{-1} \nu u, \quad u \geq 0 . \]
Therefore,
\[ \int_{\Omega} \rho^2 dx \leq e^{-ct} \| \rho_0 \|_{L^2}^2 + \int_0^t e^{-c(t-s)} \left\{ C \| u(s) \|_{L^1} - C \int_{\Omega} f(u(s)) dx \right\} ds . \]
Applying the second mean value theorem of integration in view of (2.4) and (2.5), we obtain that
\[ \| \rho(t) \|_{L^2}^2 \leq C(e^{-ct} \| \rho_0 \|_{L^2}^2 + \| u_0 \|_{L^1} + 1), \quad 0 \leq t < \infty . \]
Next we multiply the second equation of (CG) by $\Delta \rho$ and integrate the product in $\Omega$.
Then,
\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \rho|^2 dx + b \int_{\Omega} |\Delta \rho|^2 dx + c \int_{\Omega} |\nabla \rho|^2 dx = -d \int_{\Omega} u \Delta \rho dx \leq \frac{b}{2} \| \Delta \rho \|_{L^2}^2 + \frac{d^2}{2b} \| u \|_{L^2}^2 . \]
Repeating the same argument as above, we obtain that
\[ \| \nabla \rho(t) \|_{L^2}^2 \leq C(e^{-2ct} \| \rho_0 \|_{H^1}^2 + \| u_0 \|_{L^1} + 1), \quad 0 \leq t < \infty . \]
Finally we conclude that
\[ \| \rho(t) \|_{H^1}^2 \leq C(e^{-ct} \| \rho_0 \|_{H^1}^2 + \| u_0 \|_{L^1} + 1), \quad 0 \leq t < \infty . \quad (2.6) \]

3. CONTINUOUS DEPENDENCE IN INITIAL VALUES

As shown in the preceding section, for each $U_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in K$, there exists a unique global solution $U = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (CG) in the solution space (2.2). This section is then devoted to noting continuous dependence of solutions with respect to initial values.

**Theorem 3.1.** Let $U_0 \in K$ and let $U(t)$ be the solution to (CG) with an initial value $U_0$. Let $\{U_{0,n}\}_{n=1,2,3,\ldots}$ be a sequence of initial values in $K$ and let $\{U_n\}_{n=1,2,3,\ldots}$ be the sequence of corresponding solutions. If $U_{0,n} \to U_0$ in $K$ as $n \to \infty$, then $U_n(t) \to U(t)$ in $K$ for each fixed time $0 \leq t < \infty$.

**Proof.** Since $\rho_{0,n} \to \rho_0$ in $H^{1+\alpha}(\Omega) \subset C(\overline{\Omega})$. There exists some positive constant $\delta_0 > 0$ such that $\inf_{x \in \Omega} \rho_{0,n}(x) \geq \delta_0$ for all $n$. For fixed time $0 \leq t < \infty$, set $\delta = \delta_0 e^{-ct} > 0$. Then, by virtue of Proposition 2.1, $U_n$ are all local solutions on an interval $[0, t]$ to the auxiliary problem $(CG)_\delta$ where the sensitivity function is substituted with $\chi_\delta(\rho)$ which is a smooth function of $\rho \in [0, \infty)$ coinciding with $\chi(\rho)$ for $\rho \in [\delta, \infty)$. Therefore, we obtain the desired result, see [13, Theorem 3.2].
4. ASYMPTOTIC BEHAVIOR OF GLOBAL SOLUTIONS

For each $0 \leq t < \infty$, define a transform $S(t)$ on $K$ by the formula $S(t)U_0 = \left( \frac{u(t)}{\rho(t)} \right)$, $U_0 \in K$, where $u$, $\rho$ denotes the global solution to the problem (CG) with the initial value $U_0$. By Theorems 2.1 and 3.1, $\{S(t)\}_{t \geq 0}$ defines a nonlinear semigroup on $K$, namely $S(t)$ is a continuous function of $t \in [0, \infty)$ with values in $K$ and $S(t)$ is a continuous mapping from $K$ into itself.

In this section we shall be concerned with asymptotic behavior of $S(t)U_0$ as $t \to \infty$. We begin with noting the following proposition.

**Proposition 4.1.** Let $u, \rho$ be any global solution to (CG) in the space (2.2). Then the following two assertions

\[ \inf_{0 \leq t < 1} \| u(t) \|_{L^1} > 0 \]  \hspace{1cm} (4.1)

and

\[ \inf_{0 \leq t < \infty, x \in \Omega} \rho(x, t) > 0 \]  \hspace{1cm} (4.2)

are equivalent.

**Proof.** 1) Let us first verify that (4.1) implies (4.2). Put $\inf_{0 \leq t < 1} \| u(t) \|_{L^1} = \ell > 0$. We here introduce the realization $L$ of the Laplace operator $-b\Delta$ in $L^2(\Omega)$ under the Neumann boundary conditions on $\partial \Omega$. $L$ is a nonnegative self-adjoint operator in $L^2(\Omega)$. From the second equation of (CG), $u(t)$ is written as

\[ u(t) = e^{-t(L+c)}\rho_0 + \int_0^t e^{-(t-s)(L+c)}u(s)ds. \]  \hspace{1cm} (4.3)

Set a time $t_0 \geq 2$. For every $t \geq 2t_0$, we have

\[ \rho(t) \geq d \int_0^{t-t_0} e^{-(t-s)(L+c)}\{\pi(s) + u_m(s)\}ds. \]

Here, $u = \pi + u_m$ denotes the orthogonal decomposition of $u \in L^2(\Omega)$ such that $\pi = |\Omega|^{-1} \int_\Omega u \, dx$ and

\[ u_m \in L^2_m(\Omega) = \left\{ u \in L^2(\Omega); \int_\Omega u \, dx = 0 \right\}. \]

Since $\pi(t) \geq |\Omega|^{-1} \ell$ and $e^{-(t-s)L}\pi(s) = \pi(s)$, it is seen that

\[ \int_0^{t-t_0} e^{-(t-s)(L+c)}\pi(s)ds \geq \frac{\ell}{|\Omega|} \int_0^{t-t_0} e^{-c(t-s)}ds \]

\[ = \frac{\ell}{c|\Omega|} \{e^{-ct_0} - e^{-ct} \} \geq \frac{\ell e^{-ct_0}}{c|\Omega|} \{1 - e^{-ct}\}, \hspace{1cm} t \geq 2t_0. \]  \hspace{1cm} (4.4)

On the other hand, the part $L_m$ of $L$ in the component $L^2_m(\Omega)$ is a positive definite self-adjoint operator in $L^2_m(\Omega)$. Therefore, there exists some $\lambda_m > 0$ such that $L_m \geq \lambda_m$. Then, using the fact that

\[ e^{-L} \in \mathcal{L}(L^2(\Omega), \mathcal{C}(\Omega)) \cap \mathcal{L}(L^1(\Omega), L^2(\Omega)), \]
Therefore, we can observe that
\[
\left\| \int_0^{t-t_0} e^{-(t-s)(L+c)} u_m(s) ds \right\|_{L^2} \\
\leq \left\| e^{-L} \right\|_{L(L^2,\mathbb{C})} \int_0^{t-t_0} e^{-(t-s)-2L-L(t-s)} e^{-L} u_m(s) ds \right\|_{L^2} \\
\leq \left\| e^{-L} \right\|_{L(L^2,\mathbb{C})} \int_0^{t-t_0} e^{-(t-s)-2\lambda_m e^{-c(t-s)}} e^{-L} u_m(s) ds \right\|_{L^2} \\
\leq \left\| e^{-L} \right\|_{L(L^2,\mathbb{C})} \left\| e^{-L} \right\|_{L(L^1,\mathbb{C})} \int_0^{t-t_0} e^{(c+\lambda_m)(s-t)} \| u_m(s) \|_{L^1} ds.
\] (4.5)

Since \( \| u_m(s) \|_{L^1} \leq 2 \| u(s) \|_{L^1} \) and since (2.4) holds, the norm is furthermore estimated by
\[
\leq C(\| u_0 \|_{L^1} + 1) e^{-c(t-s)} u_m(t), \quad t \geq 2t_0.
\]

From (4.4) and (4.5) it is therefore verified that
\[
\rho(t) \geq \frac{d(e^{-ct_0})}{e^{\| u \|_{L^1}}} \{ 1 - e^{-ct_0} - C(\| u_0 \|_{L^1} + 1) e^{-\lambda_m t_0} \}, \quad t \geq 2t_0.
\]

This obviously shows that, if \( t_0 \) is taken sufficiently large, then
\[
\inf_{2t_0 \leq t < \infty, x \in \Omega} \rho(x, t) > 0.
\]

Since (2.1) has been verified, we conclude (4.2).

II) Let us next verify that (4.2) implies (4.1). We assume that
\[
\inf_{0 \leq t < \infty, x \in \Omega} \rho(x, t) = \delta > 0.
\] (4.6)

As done above, we consider an auxiliary initial value problem \((CG_\delta)\) in which a sensitivity function \( \chi_\delta(\rho) \) is substituted for \( \chi(\rho) \), \( \chi_\delta(\rho) \) is a smooth function of \( \rho \in [0, \infty) \) coinciding with \( \chi(\rho) \) for all \( \rho \in [\delta, \infty) \). Then, \( u, \rho \) is clearly a global solution to the problem \((CG_\delta)\). Therefore, as a global solution to \((CG_\delta)\), all the results obtained in [13] are available.

We now apply the a priori estimates established by [13, Proposition 4.1] to \( u, \rho \) on the interval \([1, \infty)\). Then there must exist some constant \( C_u > 0 \) such that
\[
\| u(t) \|_{H^1} \leq C_u, \quad t \geq 1.
\]

In addition, we note that for any \( 0 < \varepsilon \leq 1 \) it holds that
\[
\| u \|_{\varepsilon} \leq C \| u \|_{H^{1+\frac{\varepsilon}{2}}} \leq C \| u \|_{H^{1+\frac{\varepsilon}{2}}} \| u \|_{H^{1+\frac{\varepsilon}{2}}} \| u \|_{L^{1+\frac{\varepsilon}{2}}} \| u \|_{L^{1+\frac{\varepsilon}{2}}}, \quad u \in H^{1+\varepsilon}(\Omega)
\] (4.7)

(from [13, (2.1~4)]). Using this estimate with \( \varepsilon = 1 \), we observe that
\[
\| u(t) \|_{\varepsilon} \leq C_u \| u(t) \|_{L^1}, \quad t \geq 1.
\] (4.8)

To prove (4.1), we first notice that \( u(s) \) can not vanish in any finite time \( s \). Indeed, suppose that \( u(s) = 0 \) at some time \( s \). Then, by the uniqueness of solution, \( u(t) = 0 \) for every \( t \in [s, \infty) \). On the other hand, \( \rho(t) \) must be determined by
\[
\frac{\partial \rho}{\partial t} = b \Delta \rho - c \rho \quad \text{in } \Omega \times (s, \infty).
\]

Therefore, \( \rho(t) \) must converge to 0 as \( t \to \infty \). But this contradicts to (4.6).
To verify that \( u(t) \) does not vanish as \( t \to \infty \), neither, we shall use the condition \( f'(0) \neq 0 \). First, let \( f'(0) < 0 \), then there are some constants \( \nu > 0 \) and \( \ell > 0 \) such that

\[
f(u) \leq -\nu u \quad \text{holds for all } u \in [0, \ell].
\]

We shall then verify that \( \|u(t)\|_{L^1} \geq (\ell C_u^{-1})^8 \) for every \( t \geq 1 \), where \( C_u \) is the constant appearing in (4.8). Indeed, if once \( \|u(s)\|_{L^1} < (\ell C_u^{-1})^8 \) for some \( s \geq 1 \), then \( \|u(s)\|_{L^1} < \ell \) and therefore

\[
\frac{d}{ds} \|u(s)\|_{L^1} = \int_{\Omega} f(u(s))dx \leq -\nu \|u(s)\|_{L^1}.
\]

Hence, \( \|u(s)\|_{L^1} \) is decreasing at \( s \), and this implies that \( \|u(t)\|_{L^1} \) is less than \( (\ell C_u^{-1})^8 \) for any \( t \geq s \). In this way, \( \frac{d}{dt} \|u(t)\|_{L^1} \leq -\nu \|u(t)\|_{L^1} \) and \( \|u(t)\|_{L^1} \leq e^{-\nu (t-s)} \|u(s)\|_{L^1} \) for all \( t \geq s \). Thus, we conclude that \( \|u(t)\|_{L^1} \to 0 \) as \( t \to \infty \).

While, integrating the second equation of (CG) in \( \Omega \), we see that

\[
\frac{d}{dt} \|\rho(t)\|_{L^1} = -c\|\rho(t)\|_{L^1} + d\|u(t)\|_{L^1},
\]

as a consequence

\[
\|\rho(t)\|_{L^1} = e^{-ct} \|\rho_0\|_{L^1} + d \int_0^t e^{-c(t-\tau)} \|u(\tau)\|_{L^1} d\tau.
\]

This together with the vanishing of \( \|u(t)\|_{L^1} \) implies that \( \|\rho(t)\|_{L^1} \) also vanishes as \( t \to \infty \). But this again contradicts to (4.6).

Let now \( f'(0) > 0 \). In this case there exist two positive numbers \( \mu'' \) and \( \nu'' \) such that

\[
f(u) \geq -\mu'' u^2 + \nu'' u, \quad u \geq 0.
\]

From (4.8) we have

\[
\frac{d}{dt} \|u(t)\|_{L^1} = \int_{\Omega} f(u(t))dx \geq (\nu'' - \mu'' \|u(t)\|_{L^1}) \|u(t)\|_{L^1}
\geq (\nu'' - \mu'' C_u \|u(t)\|_{L^1}^{\frac{1}{\mu''}}) \|u(t)\|_{L^1}, \quad t \geq 1.
\]

If \( \|u(s)\|_{L^1} < (\nu''/\mu'' C_u)^8 \) at some \( s \geq 1 \), then \( \|u(s)\|_{L^1} \) is increasing at the time. Then, if once \( \|u(s')\|_{L^1} < (\nu''/\mu'' C_u)^8 \) at some time \( s' \geq 1 \), then this differential inequality shows that \( \|u(t)\|_{L^1} \) is never less than \( (\nu''/\mu'' C_u)^8 \) for any \( t \geq s' \).

We can now prove the main results of the paper.

**Theorem 4.1.** For each \( U_0 \in K \), let \( S(\cdot)U_0 = \begin{pmatrix} u \\ \rho \end{pmatrix} \). If \( \inf_{0 \leq t < \infty} \|u(t)\|_{L^1} > 0 \) or equivalently \( \inf_{0 \leq t < \infty, x \in \Omega} \rho(x,t) > 0 \), then its \( \omega \)-limit set \( \omega(U_0) = \bigcap_{t \geq 0} \bigcup_{t \geq t} S(\tau)U_0 \) in \( K \) is nonempty and is actually contained in the product space \( \{ \begin{pmatrix} u \\ \rho \end{pmatrix} ; u \in H^2(\Omega), \rho \in H^3(\Omega) \} \).

**Proof.** From the assumption there exists some positive constant \( \delta > 0 \) for which (4.6) holds. As above, introducing a smooth function \( \chi_\delta(\rho) \) of \( \rho \in [0, \infty) \) which coincides with \( \chi(\rho) \) for \( \rho \in [\delta, \infty) \), we regard \( S(t)U_0 \) as the global solution to the initial value problem (CG) in which \( \chi_\delta(\rho) \) substitutes for \( \chi(\rho) \). Then [13, Theorem 4.6] is available for \( u, \rho \).
to conclude that there exists some constant $C_{U_0}$ such that $\|u(t)\|_{H^2} + \|\rho(t)\|_{H^3} \leq C_{U_0}$ for $1 \leq t < \infty$. Since the set

$$\left\{ \left( \frac{u}{\rho} \right) ; \|u\|_{H^2} + \|\rho\|_{H^3} \leq C_{U_0}, \ \inf_{x \in \Omega} \rho(x) \geq \delta \right\}$$

is a compact set of $K$, we verify that the solution $S(t)U_0$ admits a nonempty $\omega$-limit set in the topology (1.1) of $K$.

In the case when $\inf_{0 \leq t < \infty} \|u(t)\|_{L^1} = 0$, we prove the following theorem.

**Theorem 4.2.** Let $U_0 \in K$ and $S(\cdot)U_0 = \begin{pmatrix} u \\ \rho \end{pmatrix}$. Assume that $\inf_{0 \leq t < \infty} \|u(t)\|_{L^1} = 0$ or equivalently $\inf_{0 \leq t < \infty, x \in \Omega} \rho(x, t) = 0$. Then there exists a sequence $t_n$ tending to $\infty$ such that $S(t_n)U_0$ converges to the boundary point $O = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ of $K$ in the norm $\|u\|_{L^1} + \|\rho\|_{L^p}$ with any $1 \leq p < \infty$.

Furthermore, when $f'(0) > 0$, $\|u_0\|_{L^1} \neq 0$ implies that $\sup_{1 \leq t < \infty} \|u(t)\|_{H^1+\varepsilon} = \infty$ with an arbitrary $\varepsilon > 0$. On the other hand, when $f'(0) < 0$, $\sup_{1 \leq t < \infty} \|u(t)\|_{H^1+\varepsilon} < \infty$ with some $\varepsilon > 0$ implies that $S(t)U_0$ converges to $O$ in the distance (1.1).

**Proof.** If $u(s) = 0$ at some finite time $s$, then $u(t) = 0$ for every $t \geq s$ and $\rho(t) \to 0$ as $t \to \infty$; therefore, $S(t)U_0$ converges to $O$ in the distance (1.1).

So let us consider the case when $\|u(t)\|_{L^1} > 0$ for every $t$ and $\inf_{0 \leq t < \infty} \|u(t)\|_{L^1} = 0$. Then there exists an increasing sequence $\{s_n\}_{n=1,2,3,\ldots}$ tending to $\infty$ such that $\lim_{n \to \infty} \|u(s_n)\|_{L^1} = 0$ and $0 < \|u(s_n)\|_{L^1} < 1$. We here set another increasing sequence $\{t_n\}_{n=1,2,3,\ldots}$ by the formula

$$t_n = s_n - \frac{1}{2\nu'} \log \|u(s_n)\|_{L^1}, \quad n = 1, 2, 3, \ldots,$$

where $\nu'$ is the positive constant appearing in (2.3). Since

$$\frac{d}{dt} \|u(t)\|_{L^1} = \int_{\Omega} f(u(t))dx \leq \nu' \|u(t)\|_{L^1},$$

it follows that

$$\|u(t)\|_{L^1} \leq e^{\nu'(t-s_n)} \|u(s_n)\|_{L^1}, \quad s_n \leq t < \infty. \quad (4.12)$$

Therefore,

$$\|u(t_n)\|_{L^1} \leq e^{\nu'(t_n-s_n)} \|u(s_n)\|_{L^1} = \sqrt[3]{\|u(s_n)\|_{L^1}},$$

and hence $\lim_{n \to \infty} \|u(t_n)\|_{L^1} = 0$.

Next we notice from (4.10) that

$$\|\rho(t_n)\|_{L^1} = e^{-c\tau_n} \|\rho_0\|_{L^1} + d \int_0^{s_n} e^{-c(t_n-s)} \|u(s)\|_{L^1} ds + e^{-c(t_n-s_n)} \|u(s)\|_{L^1} ds.$$

Here, by (2.4),

$$\int_0^{s_n} e^{-c(t_n-s)} \|u(s)\|_{L^1} ds \leq C(\|u_0\|_{L^1} + 1) \int_0^{s_n} e^{-c(t_n-s)} ds$$

$$\leq C(\|u_0\|_{L^1} + 1) e^{-c(t_n-s_n)} = C(\|u_0\|_{L^1} + 1) \|u(s_n)\|_{L^1}.$$


In addition, by (4.12),
\[
\int_{t_n}^{t_{n+1}} e^{-c(t_n-s)} \|u(s)\|_{L^1} ds \leq \int_{s_n}^{t_n} e^{c'(s-s_n)} ds \|u(s_n)\|_{L^1} \leq \frac{1}{\rho^2} \sqrt{\|u(s_n)\|_{L^1}}.
\]
Hence, we conclude that \(\lim_{n \to \infty} \|\rho(t_n)\|_{L^1} = 0\).

Furthermore, in view of (2.6), we verify by using [13, (2.3)] that
\[
\|\rho(t_n)\|_{L^p} \leq C_p \|\rho(t_n)\|_{L^1}^{\frac{1}{p}}
\]
for any \(1 \leq p < \infty\). Therefore, \(\lim_{n \to \infty} \|\rho(t_n)\|_{L^p} = 0\).

Thus we have proved the first assertion of the theorem.

Consider now the case when \(f'(0) > 0\). We suppose that \(\|u_0\|_{L^1} = 0\) and \(\sup_{1 \leq t < \infty} \|u(t)\|_{H^{1+\varepsilon}} = C_u < \infty\) with some \(\varepsilon > 0\). Then, in view of (4.7) and (4.11), we have
\[
\frac{d}{dt} \|u(t)\|_{L^1} \geq (\nu'' - \mu'' C_u \|u(t)\|_{L^1}^{\frac{\varepsilon}{1+\varepsilon}}) \|u(t)\|_{L^1}, \quad 1 \leq t < \infty.
\]
This, however, shows that
\[
\lim_{t \to \infty} \|u(t)\|_{L^1} \geq \left( \frac{\nu''}{C_u \mu''} \right)^{\frac{1}{\varepsilon}} > 0,
\]
which contradicts to the assumption.

When \(f'(0) < 0\), we have (4.9). Then \(\sup_{1 \leq t < \infty} \|u(t)\|_{H^{1+\varepsilon}} = C_u < \infty\) jointed with (4.7) implies that
\[
\|u(t)\|_{C} \leq C_u \|u(t)\|_{L^1}^{\frac{\varepsilon}{1+\varepsilon}}, \quad 1 \leq t < \infty.
\]
Therefore, at sufficiently large \(t_n\), we have
\[
\left[ \frac{d}{dt} \|u(t)\|_{L^1} \right]_{t=t_n} \leq -\nu \|u(t_n)\|_{L^1}.
\]
This means that \(\|u(t)\|_{L^1}\) is decreasing for every \(t \geq t_n\); and as a consequence, it follows that \(\lim_{t \to \infty} \|u(t)\|_{L^1} = 0\). Noting again (4.7), we have \(\lim_{t \to \infty} \|u(t)\|_{C} = 0\). From the formula (4.3), it is also verified that \(\lim_{t \to \infty} \|\rho(t)\|_{H^{1+\varepsilon} \cap C} = 0\). \(\square\)

5. Numerical Simulation

In view of Theorem 4.2, extremely interesting is the question of whether there exists a solution to (CG) which tends to \(O = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\) as \(t \to \infty\) or not, if \(f'(0) > 0\). We shall present here some numerical results concerning this question.

Let \(\Omega = (0, 4)\) be an open interval. The coefficients are fixed as \(a = 0.25, b = 1, c = 6.25\), except that \(d\) is a parameter. The sensitivity function and the growth function are taken as
\[
\chi(\rho) = -\frac{0.125}{\rho}, \quad f(u) = u(1 - u).
\]

The spatial variable is discretized by the finite element method with the step size \(\Delta x = 2^{-10}\) and the time variable by the implicit Runge-Kutta method (two-stage Radau IIA) with the step size \(\Delta t = 2^{-12}\).
For \( d \geq 0.8 \), we found a numerically stable stationary solution \( \overline{U}_d = \left( \frac{u_d}{p_d} \right) \), Fig. 1 and 2. When \( d = 0.7 \), we computed the solution \( U_{0.7} = \left( \frac{u_{0.7}}{p_{0.7}} \right) \) which starts from \( \overline{U}_{0.8} \). \( U_{0.7} \) are seen to approach to \( O \) for a while with \( L^1 \)-norm of \( u_{0.7} \) decaying as \( t \), cf. Fig. 3. But when \( t \) is about 79.4, our computation of \( U_{0.7} \) had lost its stability.

This may not be satisfactory evidence to draw the conclusion that no stable stationary solution \( \overline{U}_d \) exists for \( d = 0.7 \) and the solution \( U_{0.7} \) tends to \( O \) as \( t \to \infty \). But, we could say at least that \( U_{0.7} \) does get close to \( O \) and that \( U_{0.7} \to O \) as \( t \to \infty \) if and only if the stable stationary solution \( \overline{U}_d \) does not exist for \( d = 0.7 \).

References

(Masashi Aida, Atsushi Yagi) Department of Applied Physics, Osaka University, Suita, Osaka 565-0871, Japan

(Koichi Osaki) Department of Mathematics, Ube National College of Technology, Ube, Yamaguchi 755-8555, Japan

(Tohru Tsujikawa) Faculty of Engineering, Miyazaki University, Miyazaki 889-2192, Japan

(Masayasu Mimura) Department of Mathematical and Life Science, Hiroshima University, Higashi-Hiroshima 739-8526, Japan