Title	FINE IRREDUCIBILITY AND UNIQUENESS OF STATIONARY DISTRIBUTION
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Citation	Osaka Journal of Mathematics. 50(2) P.417-P.423
Issue Date	2013-06
Text Version	publisher
URL	https://doi.org/10.18910/25080
DOI	10.18910/25080
rights	
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He, P. and Ying, J.
Osaka J. Math.
50 (2013), 417–423

FINE IRREDUCIBILITY AND UNIQUENESS OF STATIONARY DISTRIBUTION

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(Received March 8, 2011, revised August 22, 2011)

Abstract

The uniqueness of invariant measure is one of the most interesting problems in theory of Markov processes. In this paper, we shall prove that the irreducibility in the sense of fine topology implies the uniqueness of invariant probability measures. It is also proven that this irreducibility is strictly weaker than the strong Feller property plus irreducibility in the sense of original topology, which is the usual uniqueness condition.

1. Introduction

The main purpose of this paper is to give a sufficient condition for the uniqueness of stationary distribution of a general Markov process. The existence and uniqueness of invariant measures have been one of the most important problems in theory of Markov processes. Let (P_t) be a transition semigroup of kernels on a measurable space $(E, \mathcal{E})$, i.e., it satisfies

$$P_{t+s}(x, dz) = \int_{y\in E} P_t(x, dy)P_s(y, dz).$$

A σ-finite measure μ is invariant if $\mu P_t = \mu$ for any $t > 0$, where

$$\mu P_t(A) := \int \mu(dx)P_t(x, A).$$

For example, a sufficient condition of uniqueness was given in [8] for Lévy processes in strong duality, such as symmetric stable processes, and it was proved in [10] that a Radon invariant measure of a Lévy process must be a multiple of Lebesgue measure if and only if its Lévy exponent has unique zero. An invariant probability measure is also called an invariant distribution or stationary distribution. The existence of an invariant distribution usually means the positive recurrence and the uniqueness means ergodicity.

There are numerous papers which discuss invariant measures in various concrete models, but not so many general results. It is well-known (see e.g. [2], [3], [7]) and

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2000 Mathematics Subject Classification. 60J45.

*Supported by National Natural Science Foundation of China (Grant No. 11271240).
†Supported in part by NSFC Grant No. 11071044.

also very useful that for a nice Markov process on a nice topological space, the strong Feller property (P_t takes bounded measurable function to continuous function) together with the irreducibility (any point can reach any open set) implies the uniqueness of invariant distribution. Usually the irreducibility is intuitive and not very hard to check. However it seems that the strong Feller is really strong in many cases especially in degenerate cases. Besides, two conditions involve the topology much more than the invariant measure itself does, and therefore are not so essential. For example, in [5], the authors investigate a class of degenerate diffusion as a solution of a SPDE and introduce an asymptotic Feller property which is weaker than strong Feller property but can replace it when discussing invariant distribution.

In this paper we are going to introduce another irreducibility which is more natural in some sense. For example it depends on the topology induced by the process itself. We shall prove that the irreducibility implies the uniqueness of invariant distribution and also prove that it is really weaker than the strong Feller property plus the irreducibility (in the sense of original topology). We also give some characterizations which are easy to check.

Though invariant probability measures are discussed in this paper, we would like to say a few words about invariant measures. The uniqueness of general invariant measures is more complicated. It is well-known that a Brownian motion has unique invariant measure, but a drift Brownian motion has not, although both of them are irreducible. Moreover an example in [10] shows that a Lévy process could have unique Radon invariant measure but not σ-finite invariant measure.

2. Main results

Let

$$X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \mathbf{P}^x)$$

be a right Markov process on $(E,\mathcal{E})$ (say, Polish), with transition semigroup (P_t), where $\mathbf{P}^x$ denotes the law of process starting from x. Roughly speaking, a right Markov process means a right continuous process with strong Markovian property. The precise definition and related properties of right Markov processes and probabilistic potential theory used in this paper, please refer to [1], [9] and [6]. It is too much to explain notions such as 'universally measurable', 'nearly Borel' here. A right Markov process is more than what a transition semigroup asks. In other words, it is still unknown what makes a transition semigroup be the one of a right Markov process, which is usually called a right semigroup. The best result in this direction is the one known 50 years ago: a Feller semigroup (on a locally compact space with countable base or LCCB in short) is a right semigroup. The main difference between a usual semigroup and a right semigroup is that we can employ powerful probabilistic potential theory for a right semigroup as what we do in this paper.

For a σ-finite measure μ and a (universally) measurable function f on E, define

$$\mu P_t := \int \mu(dx) P_t(x, \cdot), \quad P_t f := \int P_t(\cdot, dy) f(y).$$

We say (P_t) satisfies strong Feller property if $P_t f$ is continuous for any $t > 0$ and bounded measurable function f on E. Note that strong Feller property is not actually stronger than Feller property which means that P_t is a map on the space of continuous functions on E (assumed to be LCCB) vanishing at infinity.

We now introduce fine topology, which is induced from the process and contains many intrinsic and delicate properties of the process. For any (nearly) Borel set B, T_B always denotes the hitting time of B. A set B is finely open if $\mathbf{P}^x(T_{B^c} > 0) = 1$ for any $x \in B$. Note that the above probability is either 0 or 1 due to Blumenthal zero-one law. Intuitively it means that the process starting from a point in B is impossible to leave B immediately. Clearly any open set is finely open due to the right continuity of sample path.

DEFINITION 2.1. X is called irreducible if $\mathbf{P}^x(T_G < \infty) > 0$ for any $x \in E$ and non-trivial open G, finely irreducible if $\mathbf{P}^x(T_G < \infty) > 0$ for any $x \in E$ and non-trivial finely open G.

This definition is intuitive. The fine irreducibility is stronger than the usual irreducibility, but they are surely not equivalent as the following example shows.

EXAMPLE 1. Let ν be a probability measure charging on all non-zero rationals and X the compound Poisson process with Lévy measure ν. Then any point is finely open. Since rational numbers are dense, X is irreducible, but not finely irreducible because the process X, staring from 0, can only reach rational numbers.

It is known that for one-dimensional Brownian motion, the fine topology is the same as the Euclidean topology and for higher-dimensional, it is strictly finer. However in many cases it is hard to characterize finely open sets. Hence it is usually difficult to use the above definition. The following lemma gives a criterion which use the resolvent to characterize the fine irreducibility. For any $\alpha \geq 0$, the α-resolvent (or potential operator) U^α is defined to be

$$U^\alpha(x, A) = \int_0^\infty e^{-\alpha t} P_t(x, A)\, dt = \mathbf{E}^x \int_0^\infty e^{-\alpha t} 1_{\{X_t \in A\}}\, dt,$$

which is the (weighted) average time of X, starting at x, staying in A. Another vital notion in potential analysis is 'excessive' (measures and functions). Given $\alpha \geq 0$. A σ-finite measure μ is called α-excessive, if $e^{-\alpha t}\mu P_t \leq \mu$ for all $t > 0$ or equivalently $\beta\mu U^{\alpha+\beta} \leq \mu$ for all $\beta > 0$. Similarly, a (universally) measurable function $f \geq 0$ is called α-excessive if $e^{-\alpha t}P_t f \le f$ for all $t > 0$. Any excessive function is finely continuous (continuous in fine topology). Actually the fine topology is generated by all α-excessive functions.

Theorem 2.1. *The following statements are equivalent.*

1. *X is finely irreducible*;
2. *For any $A \in \mathcal{E}$, $U^\alpha 1_A$ is either 0 identically or positive everywhere*;
3. *All non-trivial excessive measures are equivalent.*

As we have seen, the fine topology, which is not easy to characterize, is not shown superficially in the condition 2.

Proof of Theorem 2.1. The approach we use here is the classical probabilistic potential theory. The most useful concept is so-called hitting distribution, P_A^α, defined to be

$$P_A^\alpha(x, dy) := \mathbf{P}^x(e^{-\alpha T_A} 1_{\{X(T_A)\in dy\}};\ T_A < \infty).$$

P_A^α is also called sweeping-out operator which describes how a positive charge at x makes a distribution of negative charge on surface of a metal A. A useful assertion is $P_A^\alpha h \le h$ for any α-excessive function h (see Chapter II, [1]), where the left side is called the reduit of h, a notion originated from H. Poincaré. We may assume $\alpha = 0$. Suppose (1) is true. If $U1_A$ is not identically zero, then there exists $\delta > 0$ such that $D := \{U1_A > \delta\}$ is non-empty. Since $U1_A$ is excessive and thus finely continuous, D is finely open and the fine closure of D is contained in $\{U1_A \ge \delta\}$. Then by Proposition II.2.8 and Theorem I.11.4 [1],

$$U1_A(x) \ge P_D U1_A(x) = \mathbf{E}^x(U1_A(X_{T_D})) \ge \delta \mathbf{P}^x(T_D < \infty) > 0.$$

Conversely suppose (2) is true. Then for any finely open set D, by the right continuity of X, $U1_D(x) > 0$ for any $x \in D$. Therefore $U1_D$ is positive everywhere on E.

Let ξ be a non-trivial excessive measure. Since $\alpha\xi U^\alpha \le \xi$, $\xi(A) = 0$ implies that $\xi U^\alpha(A) = 0$. However ξ is non-trivial. Thus it follows from (2) that $U^\alpha 1_A \equiv 0$, i.e., A is potential zero. Conversely if A is potential zero, then $\xi(A) = 0$ for any excessive measure ξ. Therefore (2) and (3) are equivalent. □

It is well-known that the strong Feller property and irreducibility together imply the uniqueness of invariant distribution, which implies the ergodicity. By the strong Feller property, we mean that P_t takes bounded measurable functions to continuous functions. A condition obviously weaker than strong Feller is called LSC, which means that for any measurable set B, $U^\alpha(\cdot, B)$ is lower semi-continuous. The Brownian motion is strong Feller, but compound Poisson process is not strong Feller.

Lemma 2.1. *If X satisfies LSC and irreducibility, then it is finely irreducible.*

Proof. Let $A \in \mathcal{E}$. $U^\alpha 1_A \neq 0$ identically. There is $b > 0$ such that $G = \{U^\alpha 1_A > b\} \neq \emptyset$ and is open due to the property LSC. Since $U^\alpha 1_A$ is α-excessive, we have by Proposition II.2.8 [1] for any $x \in E$,

$$U^\alpha 1_A(x) \geq P_G^\alpha U^\alpha 1_A(x) = \mathbf{P}^x(e^{-\alpha T_G} \cdot U^\alpha 1_A(X(T_G))).$$

But $X(T_G) \in \bar{G}$ by Theorem I.11.4[1] and then $X(T_G) \geq b$. Hence by the irreducibility, we have

$$U^\alpha 1_A(x) \geq b\mathbf{E}^x(e^{-\alpha T_G}, T_G < \infty) > 0. \qquad \square$$

EXAMPLE 2. Let $N = (N_t)$ be a Poisson process with parameter $\lambda > 0$ and $X_t = N_t - t$. Then X does not satisfy strong Feller but satisfies LSC. Hence it is finely irreducible. Since X jumps forward and drifts backward, any set such as $(a, b]$ is finely open. It may be shown that it satisfies a stronger irreducibility or pointwise irreducibility: $\mathbf{P}^x(T_y < \infty) > 0$ for any $x, y \in \mathbb{R}$.

The strong Feller property is certainly too much and hence is not essential for uniqueness of stationary distribution as indicated in the following very simple example.

EXAMPLE 3. Consider the uniform translation X on unit circle. Then X is not strong Fellerian. However the uniform distribution on circle is the only stationary distribution of X.

It seems that in dealing with uniqueness problems, the fine irreducibility is more natural than the irreducibility under the original topology. Here we shall prove the uniqueness of invariant distribution under irreducibility.

Theorem 2.2. *The fine irreducibility of X implies the uniqueness of invariant distribution.*

Proof. It is stated in ([3], Theorem 3.2.4) that if a probability measure μ is invariant, then μ is ergodic if and only if for any $A \in \mathcal{E}$ and $t > 0$, $P_t 1_A = 1_A$, μ-a.s. implies that $\mu(A) = 0$ or 1.

We claim that if (P_t) is finely irreducible, then any invariant distribution is ergodic. In fact for any $A \in \mathcal{E}$ and $t > 0$, $P_t 1_A = 1_A$, μ-a.s. Then for any $B \in \mathcal{E}$,

$$\langle 1_B, P_t 1_A \rangle_\mu = \langle 1_B, 1_A \rangle_\mu$$

for any $t > 0$. It follows by Fubini's theorem that

$$\langle 1_B, \alpha U^\alpha 1_A \rangle_\mu = \langle 1_B, 1_A \rangle_\mu$$

for any $\alpha > 0$. Hence $\alpha U^\alpha 1_A = 1_A$, μ-a.s. The fine irreducibility is equivalent to $U^\alpha 1_A$ is either 0 identically or positive everywhere. This implies that $\mu(A) = 0$ or 1.

Now if μ and ν are two different invariant distributions, then they are ergodic. By Theorem 3.2.5 in [3], they are singular to each other. However the fine irreducibility implies that all excessive measures are equivalent to each other and this leads to a contradiction. $\square$

After reading the theorem above, Pat Fitzsimmons gives another simple proof using ratio ergodic theorem.

ACKNOWLEDGEMENT. The authors would like to thank Professor Dong, Zhao of Academia Sinica for stimulating discussion and Professor Pat Fitzsimmons of UCSD for a valuable remark.

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