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## ON THE GENERALIZED DUNWOODY 3-MANIFOLDS

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### Abstract

We introduce a family of orientable 3-manifolds induced by certain cyclically presented groups and show that this family of 3-manifolds contains all Dunwoody 3-manifolds by using the planar graphs corresponding to the polyhedral description of the 3-manifolds. As applications, we consider two families of cyclically presented groups, and show that these are isomorphic to the fundamental groups of the certain Dunwoody 3-manifolds  $D_n$  ( $n \geq 2$ ) which are the  $n$ -fold cyclic coverings of the 3-sphere branched over the certain two-bridge knots, and that  $D_n$  is the  $(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ -fold covering of the 3-sphere branched over two different  $\Theta$ -curves.

### 1. Preliminaries

Every closed 3-manifold has a spine, called the Heegaard diagram, from which one can obtain a presentation for the group, but not all group presentations arise from the spines of 3-manifolds. It is an open problem to determine which cyclic presentations of groups correspond to spines of closed 3-manifolds. It draws attention to determine which cyclically presented groups correspond to spines of closed 3-manifolds, in particular, for which classes of knots the cyclic branched coverings give rise to such cyclic presentations. We refer for examples to [2] and [32].

We now discuss the above questions for a new family of cyclic presentations of groups determined by some words. This family contains many classes of cyclic presentations which have appeared in recent years and there are many connections with theories of closed connected 3-manifolds as follows.

- (1) The Fibonacci group  $F(2, 2m)$ ,  $m \geq 2$ , introduced in [12] and [13] is the fundamental group of the  $m$ -fold cyclic covering of the 3-sphere branched over the figure-eight knot ([17]). The group  $F(r, n)$ ,  $r \geq 2$ ,  $n \geq 3$  as a generalization of the Fibonacci group was introduced in [20]. In particular, the group  $F(n - 1, n)$ ,  $n \geq 3$ , is the fundamental group of Seifert fibered space ([30] and [8]).
- (2) The generalized Sieradski group  $S(r, n)$ ,  $r \geq 2$ ,  $n \geq 2$ , introduced in [35] and geometrically studied in [9] is the fundamental group of the  $n$ -fold cyclic covering of the 3-sphere branched over the torus knot of type  $(2r - 1, 2)$  ([10]). Moreover the group containing the generalized Sieradski group was defined in [31].

(3) The family of cyclically presented groups containing the generalized Neuwirth groups ([37]) was considered in [19]. The groups are the fundamental groups of closed connected orientable 3-manifolds which are homeomorphic to Seifert fibered spaces and to  $n$ -fold cyclic coverings of lens spaces branched over genus one 1-bridge knots ([11]).

(4) Besides the above, recent manuscripts on groups, 3-manifolds, and the cyclic branched coverings can be found in [1], [3], [24], [28], [29], and [33]. Moreover, from our family of words in a free group of rank  $n$  which determines cyclically presented groups, we can construct a family of 3-manifolds containing Dunwoody 3-manifolds (first considered in [14]), which is obtained from the polyhedral description on a polyhedral 3-ball, whose finitely many boundary faces are glued together in certain pairs. Recent manuscripts of the Dunwoody 3-manifolds can be found in [4], [5], [6], [15], [16], [22], [23], [25], [26], and [36].

Let  $(V_1, V_2)$  be a Heegaard splitting of a 3-manifold  $M$  with genus  $n \geq 1$ . A properly embedded disc  $D$  in the handlebody  $V_2$  is called a *meridian disc* of  $V_2$  if cutting  $V_2$  along  $D$  yields a handlebody of genus  $n - 1$ . A collection of  $n$  mutually disjoint meridian discs  $\{D_i\}$  in  $V_2$  is called a *complete system* of meridian discs of  $V_2$  if cutting  $V_2$  along  $\bigcup_i D_i$  gives a 3-ball. Let  $\alpha_i$  denote the 1-sphere  $\partial D_i$  which lies in the closed orientable surface  $\partial V_1 = \partial V_2$  of genus  $n$ . The system is said to be a *Heegaard diagram* of the 3-manifold  $M$  and denoted by  $(V_1; \alpha_1, \alpha_2, \dots, \alpha_n)$ . In the other side the system  $(V_2; \beta_1, \beta_2, \dots, \beta_n)$  is called a *dual Heegaard diagram* of the 3-manifold  $M$  if  $\{D_i\}$  is a complete system of  $n$  mutually disjoint meridian discs in  $V_1$  and  $\beta_i$  is the 1-sphere  $\partial D_i$  which lies in the closed orientable surface  $\partial V_1 = \partial V_2$  of genus  $n$ . In other words  $(V_2; \beta_1, \beta_2, \dots, \beta_n)$  is the dual Heegaard diagram of  $(V_1; \alpha_1, \alpha_2, \dots, \alpha_n)$ . In particular, let  $M$  be the lens space and  $K$  a knot in  $M$ . Then the pair  $(M, K)$  has a  $(1, 1)$ -decomposition if there exists a Heegaard splitting of genus one  $(V_1, K_1) \cup_\phi (V_2, K_2)$  of  $(M, K)$  such that  $K_i \subset V_i$  is a properly embedded trivial arc for each  $i = 1, 2$  and  $\phi$  is an attaching homeomorphism. We call such a knot  $K$  a  $(1, 1)$ -knot. Moreover  $M$  is determined by the Heegaard diagram  $(V_1; \alpha_1)$  or its dual Heegaard diagram  $(V_2; \beta_1)$ .

In Section 2, we introduce a planar graph corresponding to the polyhedral description of a certain 3-manifold constructed from a family of words. The planar graph is obtained directly from the polyhedral description by the method similar to the duality of the graph. Moreover, if the planar graph is to be a Heegaard diagram of a 3-manifold  $M$ , then the cyclically presented group determined by the family of words is the fundamental group of  $M$ .

In Section 3, we discuss some conditions under which a planar graph actually is to be a Heegaard diagram of a Dunwoody 3-manifold. Furthermore we show that every Dunwoody manifold can be induced by a planar graph. For example, a certain family of Dunwoody 3-manifolds were constructed by this fashion in [25] and [7].

As applications, in Section 4, we consider two families of cyclically presented groups which are isomorphic to the fundamental groups of the Dunwoody 3-manifolds

$D_n$ . Finally we show that  $D_n$  are the  $(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ -fold coverings of the 3-sphere  $\mathbb{S}^3$  branched over two different  $\Theta$ -curves, which extends results in [23] where  $n \geq 2$ .

**2. A planar graph of an orientable 3-manifold  $M_n(r, s)$**

For each  $n \geq 1$ , let  $\mathbf{F}_n = \langle x_1, x_2, \dots, x_n \mid \rangle$  be the free group of rank  $n$  and let  $\eta: \mathbf{F}_n \rightarrow \mathbf{F}_n$  be an automorphism of order  $n$  defined by  $\eta(x_i) = x_{i+1}$ ,  $i = 1, 2, \dots, n$ , where the indices are taken under modulo  $n$ . A group is said to be *cyclically presented* if it is isomorphic to

$$G_n(w) = \langle x_1, \dots, x_n \mid w, \eta(w), \dots, \eta^{n-1}(w) \rangle$$

for some integer  $n$  and a reduced word  $w$  in  $\mathbf{F}_n$ . For the simplicity, we use the following expression instead of the above expression:

$$G_n(w) = \langle x_1, \dots, x_n \mid w = 1 \text{ indices mod } n \rangle.$$

For a word  $w = x_{i_1}^{\epsilon_{i_1}} x_{i_2}^{\epsilon_{i_2}} \dots x_{i_r}^{\epsilon_{i_r}}$  in  $\mathbf{F}_n$  where  $\epsilon_{i_k} = \pm 1$ , we define *the length*  $l(w)$  of  $w$  by  $r$  and *the exponent sum*  $\sigma(w)$  of  $w$  by  $\sum_{k=1}^r \epsilon_{i_k}$ . Two words  $w$  and  $w'$  in  $\mathbf{F}_n$  are said to be *rotational equivalent with rotational difference*  $r - j + 1$  and denoted by  $w \approx_{r-j+1} w'$  if, for some integer  $j$ ,

$$w = x_{i_1}^{\epsilon_{i_1}} x_{i_2}^{\epsilon_{i_2}} \dots x_{i_r}^{\epsilon_{i_r}} \quad \text{and} \quad w' = x_{i_j}^{\epsilon_{i_j}} x_{i_{j+1}}^{\epsilon_{i_{j+1}}} \dots x_{i_r}^{\epsilon_{i_r}} x_{i_1}^{\epsilon_{i_1}} x_{i_2}^{\epsilon_{i_2}} \dots x_{i_{j-1}}^{\epsilon_{i_{j-1}}}.$$

Moreover it should be noted that  $w' \approx_{j-1} w$ .

Let  $a, b, c$ , and  $u$  be words in  $\mathbf{F}_n$  such that, for some integers  $s$  and  $r$ ,

$$u\eta^s(c)\eta^{s-1}(b)\eta^{-1}(u^{-1}) \approx_r abc\eta^{-1}(a^{-1}).$$

Then  $w = u\eta^s(c)\eta^{s-1}(b)\eta^{-1}(u^{-1})$  or  $abc\eta^{-1}(a^{-1})$  induces the cyclically presented group

$$G_n(w) = \langle x_1, x_2, \dots, x_n \mid w = 1 \text{ indices mod } n \rangle.$$

We now define the polygons  $F_0$  and  $\bar{F}_0$  as shown in Fig. 1 determined by words  $w = abc\eta^{-1}(a^{-1})$  and  $w' = u\eta^s(c)\eta^{s-1}(b)\eta^{-1}(u^{-1})$  in clockwise and anticlockwise orientation respectively, where  $N$  and  $S$  are starting points. Similarly, if we repeat for all  $i = 1, \dots, n - 1$ , we have two polygons  $F_i$  and  $\bar{F}_i$  determined by words  $\eta^i(w)$  and  $\eta^i(w')$ , respectively. Since  $F_s$  and  $\bar{F}_0$  have a common subword  $\eta^s(c)$ , we can glue  $F_s$  and  $\bar{F}_0$  along  $\eta^s(c)$  to get an oriented polygon  $K_0$ . Similarly we can get  $n$  oriented polygons

$$K_0, \eta(K_0) = K_1, \dots, \eta^{n-1}(K_0) = K_{n-1}.$$

For each  $i = 0, 1, \dots, n - 2$ ,  $K_i$  and  $K_{i+1}$  have a common edge  $\eta^i(\eta^s(a)\eta^s(b)u^{-1})$ , and  $K_{n-1}$  and  $K_0$  have a common edge  $\eta^{s-1}(a)\eta^{s-1}(b)\eta^{-1}(u^{-1})$ . By gluing all of  $K_0, K_1, \dots, K_{n-1}$

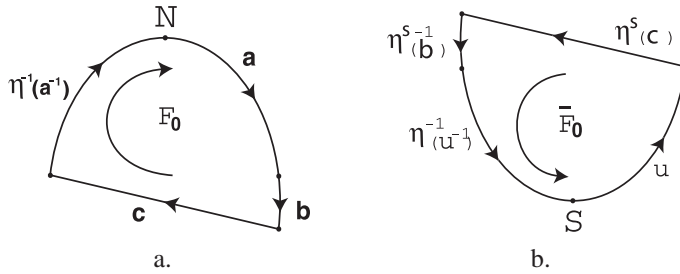


Fig. 1. A pair of two polygons determined by words.

along the common edges, we have a cellular decomposition which realizes the tessellation of the boundary of a 3-ball consisting of  $2n$  faces, called *the polyhedral description* and denoted by  $P_n(r, s)$ . Since the tessellation of the boundary consists of  $2n$  faces labelled by  $F_i$  and  $\bar{F}_i$ , we will get a 3-dimensional complex  $\mathcal{T}$  which gives a cellular decomposition of a closed orientable pseudo-manifold  $\mathcal{M}$  by identifying the faces  $F_i$  and  $\bar{F}_i$  of the boundary of the 3-ball.

By the construction,  $\mathcal{T}$  consists of edges corresponding to the generators  $x_1, x_2, \dots, x_n$  in  $\mathbf{F}_n$ ,  $n$  faces  $F_0, \dots, F_{n-1}$ , and one 3-cell. If  $l(b) = l(c)$ , then  $l(b) + l(c)$  is even and so  $\mathcal{M}$  contains the restricted information that will be known from the result of Lemma 3.5 later on. Thus, in this section, let  $l(b) \neq l(c)$ . If the Euler characteristic of  $\mathcal{M}$  vanishes, then it is a closed orientable 3-manifold, which will be denoted by  $M_n(r, s)$  ([34]). Moreover there exists a closed orientable 3-manifold  $M_n(r, s)$  such that its fundamental group is associated with the finite presentations

$$\langle x_1, \dots, x_n \mid abc\eta^{-1}(a^{-1}) = 1, \text{ indices mod } n \rangle.$$

Thus, in modified form of Theorem 1 in [25], we have proved the following corollary.

**Corollary 2.1.** *For each  $n \geq 1$ , let  $a, b, c$ , and  $u$  be words in  $\mathbf{F}_n$  such that  $l(b) \neq l(c)$  and for some integers  $s$  and  $r$ ,*

$$u\eta^s(c)\eta^{s-1}(b)\eta^{-1}(u^{-1}) \approx_r abc\eta^{-1}(a^{-1}).$$

*Then there exists a closed orientable 3-manifold  $M_n(r, s)$  induced by  $a, b, c$ , and  $u$ . Moreover, for each  $n \geq 1$ , there exists a closed orientable 3-manifold  $M_n(r, s)$  uniformized by the following cyclically presented group:*

$$\langle x_1, \dots, x_n \mid abc\eta^{-1}(a^{-1}) = 1, \text{ indices mod } n \rangle.$$

Naturally there exist closed orientable 3-manifolds whose fundamental groups are not uniformized by the cyclically presented groups of Corollary 2.1, but we are interested in conditions of  $M_n(r, s)$  for the affirmative point of view.

We now construct the planar graph  $H_n(r, s)$  corresponding to  $M_n(r, s)$ , which is dual to the polyhedral description  $P_n(r, s)$ ,  $n \geq 1$ , as follows. Let

$$u\eta^s(c)\eta^{s-1}(b)\eta^{-1}(u^{-1}) = x_{i_1}^{\epsilon_{i_1}} x_{i_2}^{\epsilon_{i_2}} \cdots x_{i_d}^{\epsilon_{i_d}}$$

and

$$abc\eta^{-1}(a^{-1}) = x_{i_{d-r+1}}^{\epsilon_{i_{d-r+1}}} x_{i_{d-r+2}}^{\epsilon_{i_{d-r+2}}} \cdots x_{i_d}^{\epsilon_{i_d}} x_{i_1}^{\epsilon_{i_1}} x_{i_2}^{\epsilon_{i_2}} \cdots x_{i_{d-r}}^{\epsilon_{i_{d-r}}}$$

be rotational equivalent with rotational difference  $r$  or  $d - r$ , say  $r$ , that is,  $u\eta^s(c)\eta^{s-1}(b)\eta^{-1}(u^{-1}) \approx_r abc\eta^{-1}(a^{-1})$ . Then  $H_n(r, s)$  is constructed by corresponding relations of the equivalent pairs and edges in the polyhedral description  $P_n(r, s)$ , in other words, oriented circles  $C_i$  and  $\bar{C}_i$  and oriented arrows with initial points or final points on  $C_i$  or  $\bar{C}_i$  in  $H_n(r, s)$  are defined by the equivalent pairs  $\bar{F}_i$  and  $F_i$  and adjoining oriented edges  $x_{i_j}^{\epsilon_{i_j}}$  with  $\bar{F}_i$  or  $F_i$  in  $P_n(r, s)$  as the following:

- (R1) oriented polygons  $F_i$  correspond to  $C_i$ ,
- (R2) oriented polygons  $\bar{F}_i$  correspond to  $\bar{C}_i$ ,
- (R3) oriented edges  $x_{i_j}^{\epsilon_{i_j}}$  correspond to transversal oriented arrows determined by right hand law, where  $1 \leq j \leq d$  and by the right hand law we mean that each oriented arrow of  $C_i$  has final point in  $C_i$  if each exponent  $\epsilon_{i_j}$  of  $abc\eta^{-1}(a^{-1})$  in  $F_i$  is +, initial point otherwise; each oriented arrow of  $\bar{C}_i$  has initial point in  $\bar{C}_i$  if each exponent  $\epsilon_{i_j}$  of  $u\eta^s(c)\eta^{s-1}(b)\eta^{-1}(u^{-1})$  in  $\bar{F}_i$  is +, final point otherwise, and
- (R4) the identifications of  $n$  equivalent pairs  $\bar{F}_i$  and  $F_i$  are represented by the same numbers on  $\bar{C}_i$  and  $C_i$  determined by a bijective function  $\bar{\beta}: C_i \rightarrow \bar{C}_i$ , defined by

$$\bar{\beta}(j) = \begin{cases} -j + r & \text{if } r < j, \\ -j + r - d & \text{otherwise,} \end{cases}$$

where 1 (resp.  $-1$ ) is the fixed starting point  $N$  (resp.  $S$ ) on  $C_i$  with oriented clockwise (resp.  $\bar{C}_i$  with oriented anticlockwise) corresponding to  $x_{i_{d-r+1}}^{\epsilon_{i_{d-r+1}}}$  (resp.  $x_{i_1}^{\epsilon_{i_1}}$ ) and  $\bar{\beta}(r) = -d$ .

Since the above construction is independent to selection of  $i$ , it is sufficient to consider  $i = 0$ . See Fig. 2 for  $i = 0$ . Moreover if  $C_i$  or  $\bar{C}_i$  and transversal oriented arrows are obtained by (R1)–(R4), the others follow by the rotational symmetry  $\eta$  of order  $n$  as  $\eta(C_i) = C_{i+1}$  and  $\eta(\bar{C}_i) = \bar{C}_{i+1}$ .

For some integers  $r$  and  $s$ , there are disjoint simple closed curves under the quotient of  $H_n(r, s)$  by the identification of each pair  $C_i$  and  $\bar{C}_i$ . By the number of curves in  $H_n(r, s)$ , we mean the number of disjoint simple closed curves under the quotient of  $H_n(r, s)$ . In particular, we denote the number of curves in  $H_1(r, 0)$  by  $L$ .

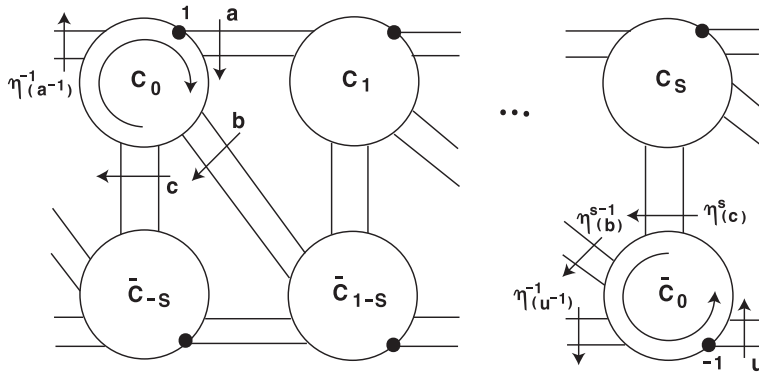


Fig. 2. A planar graph  $H_n(r, s)$  for  $i = 0$ .

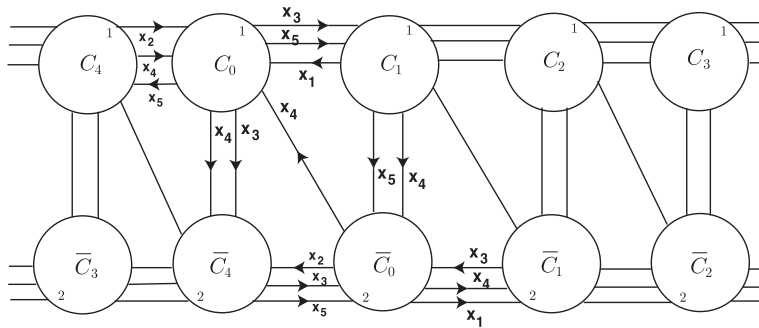


Fig. 3. The planar graph  $H_5(2, 1)$ .

EXAMPLE 2.1. Let  $a = x_3^{-1}x_5^{-1}x_1$ ,  $b = x_4$ ,  $c = x_3^{-1}x_4^{-1}$ , and  $u = x_1x_4x_3^{-1}$  be words in  $\mathbf{F}_5$  such that, for  $r = 2$  and  $s = 1$ ,

$$\begin{aligned} u\eta(c)b\eta^{-1}(u^{-1}) &= x_1x_4x_3^{-1}x_4^{-1}x_5^{-1}x_4x_2x_3^{-1}x_5^{-1} \\ &\approx_2 x_3^{-1}x_5^{-1}x_1x_4x_3^{-1}x_4^{-1}x_5^{-1}x_4x_2 = abc\eta^{-1}(a^{-1}). \end{aligned}$$

Then these words determine the polyhedral description  $P_5(2, 1)$  and 3-manifold  $M_5(2, 1)$  by Corollary 2.1. From the face identification of  $P_5(2, 1)$ , we have the cyclically presented group  $G_5(2, 1)$  as

$$\langle x_1, \dots, x_5 \mid x_{i+3}^{-1}x_{i+5}^{-1}x_{i+1}x_{i+4}x_{i+3}^{-1}x_{i+4}^{-1}x_{i+5}^{-1}x_{i+4}x_{i+2} = 1, \text{ indices mod } 5 \rangle.$$

The planar graph  $H_5(2, 1)$  of  $M_5(2, 1)$  obtained from  $P_5(2, 1)$  is depicted in Fig. 3. The number 1 on each  $C_i$  is the starting point corresponding to  $\eta^i(x_3^{-1})$  in  $\eta^i(abc\eta^{-1}(a^{-1}))$  for  $0 \leq i \leq 4$ . The number 2 on each  $\bar{C}_i$  is defined by  $\bar{\beta}(2) = -9$  in rule of (R4). Indeed  $M_5(2, 1)$  is the quotient space of  $H_5(2, 1)$  by the identifications of each pairs

$\bar{C}_i$  and  $C_i$  along the same numbers and all oriented arrows with the same words. These identifications are generated by 1-handles and 2-handles. In this example, the cyclically presented group  $G_5(2, 1)$  is the fundamental group of  $M_5(2, 1)$ .

**3. Generalized Dunwoody 3-manifolds  $M_n(r, s)$**

We recall that words  $a, b, c,$  and  $u$  are in  $\mathbf{F}_n, n \geq 1,$  such that for some integers  $r$  and  $s,$

$$u\eta^s(c)\eta^{s-1}(b)\eta^{-1}(u^{-1}) \approx_r abc\eta^{-1}(a^{-1})$$

determine an orientable 3-manifold  $M_n(r, s)$  and that  $H_n(r, s)$  is the planar graph corresponding to  $M_n(r, s)$ . We now consider the rotation  $\mathbb{Z}_n$  by an angle  $(2\pi/n)$  around the axis  $\epsilon$  connecting  $N$  and  $S$  in  $P_n(r, s)$  as an orientation preserving homeomorphism on  $M_n(r, s)$ . By  $(M_1(r, 0), K)$  we denote the quotient space of  $M_n(r, s)$  by  $\mathbb{Z}_n,$  where  $K$  is a singular set with branching index  $n$  as the image of the rotation axis  $\epsilon$ . Under this quotient, the planar graph  $H_1(r, 0)$  is also obtained as the image of  $H_n(r, s)$  by  $\mathbb{Z}_n$ . Moreover the identifications of  $H_n(r, s)$  corresponding to  $M_n(r, s)$  induce the identifications of  $H_1(r, 0)$  corresponding to  $M_1(r, 0)$ .

For a word  $w = x_{i_1}^{\epsilon_{i_1}} x_{i_2}^{\epsilon_{i_2}} \cdots x_{i_r}^{\epsilon_{i_r}}$  in  $\mathbf{F}_n$  where  $\epsilon_{i_k} = \pm 1,$  we define  $\bar{w}$  in  $\mathbf{F}_1$  by  $\bar{w} = x_1^{\epsilon_{i_1}} x_1^{\epsilon_{i_2}} \cdots x_1^{\epsilon_{i_r}}.$  That is,  $\bar{w}$  is obtained from  $w$  by replacing all  $x_i$  in  $w$  with  $x_1$ . Then the exponent sum  $\sigma(w)$  of  $w$  is invariant, that is,  $\sigma(w) = \sigma(\bar{w})$  for a word  $w \in \mathbf{F}_n.$  We also note that when two words  $w$  and  $w'$  in  $\mathbf{F}_n$  are rotational equivalent with rotational difference  $r,$  two words  $\bar{w}$  and  $\bar{w}'$  in  $\mathbf{F}_1$  are rotational equivalent with rotational difference  $r.$  Throughout this section when we consider words  $\bar{a}, \bar{b}, \bar{c},$  and  $\bar{u}$  in  $\mathbf{F}_1$  such that,

$$\bar{u}\bar{c}\bar{b}\bar{u}^{-1} \approx_r \bar{a}\bar{b}\bar{c}\bar{a}^{-1},$$

we always mean that the words  $a, b, c,$  and  $u$  in  $\mathbf{F}_n, n \geq 1,$  satisfy for some integer  $s,$

$$u\eta^s(c)\eta^{s-1}(b)\eta^{-1}(u^{-1}) \approx_r abc\eta^{-1}(a^{-1}).$$

The canonical form of  $H_1(r, 0)$  is depicted in Fig. 4, where we use the same characters in the planar graph as  $a = l(\bar{a}), b = l(\bar{b}), c = l(\bar{c})$  and  $d = 2l(\bar{a}) + l(\bar{b}) + l(\bar{c})$  if not confusing, and  $K$  is the singular set with branching index  $n$  as the image of the rotation axis connecting  $N$  and  $S$  in  $P_n(r, s)$ .

**Theorem 3.1.** *Let  $\bar{a}, \bar{b}, \bar{c},$  and  $\bar{u}$  be words in  $\mathbf{F}_1$  such that, for some integer  $r,$*

$$\bar{u}\bar{c}\bar{b}\bar{u}^{-1} \approx_r \bar{a}\bar{b}\bar{c}\bar{a}^{-1}.$$

*If the number of curves in  $H_1(r, 0)$  is  $L,$  then there are two permutations  $\alpha$  and  $\beta$  of  $X = \{\pm 1, \dots, \pm d\}$  such that  $2L = |\beta\alpha|,$  where  $|\cdot|$  means the number of disjoint cycles in a permutation.*



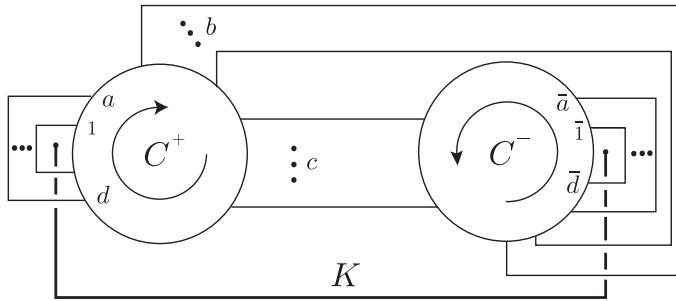


Fig. 4. The planar graph  $H_1(r, 0)$  and the singular set  $K$ .

Proof. We consider  $H_1(r, 0)$  determined by  $\bar{u}\bar{c}\bar{b}\bar{u}^{-1} \approx_r \bar{a}\bar{b}\bar{c}\bar{a}^{-1}$  (see Fig. 4). Let  $X^+ = \{1, 2, \dots, d\}$  and  $X^- = \{-d, -d+1, \dots, -1\}$  be sets of  $d$  points in  $C^+$  and  $C^-$ , respectively. Then we define a permutation  $\alpha$  by a product of  $d$  2-cycles corresponding to the ends points of line segments on  $H_1(r, 0)$  as follows:

$$\alpha(j) = \begin{cases} d - j + 1 & \text{if } 1 \leq j \leq l(a), \\ \overline{j + l(c)} & \text{if } l(a) + 1 \leq j \leq l(a) + l(b), \\ \overline{j - l(b)} & \text{if } l(a) + l(b) + 1 \leq j \leq l(a) + l(b) + l(c), \\ \overline{d + j + 1} & \text{if } -l(a) \leq j \leq -1 \end{cases}$$

and a permutation  $\beta$  by a product of  $d$  2-cycles connecting  $C^+$  and  $C^-$  on  $H_1(r, 0)$  as follows:

$$\beta(j) = \begin{cases} \overline{j - r} & \text{if } r < j, \\ j = \overline{j - r + d} & \text{otherwise.} \end{cases}$$

We now define an equivalence relation  $\sim$  on  $X = X^+ \cup X^-$  by

$$x \sim y \text{ if } y = (\beta\alpha)^i(x) \text{ or } y = \alpha(\beta\alpha)^i(x) \text{ for some } i$$

and call the equivalence classes of  $X$  under the relation the orbits of  $\beta\alpha$ . Let  $l$  be a simple closed curve in  $H_1(r, 0)$  and  $x$  be a point on  $C^+$  meeting  $l$ . Then  $l$  is determined by the repeated applications of  $\alpha$  and  $\beta$  as follows:

$$x, \alpha(x), \beta\alpha(x), \alpha\beta\alpha(x), \dots, \alpha\beta \cdots \alpha(x),$$

which forms exactly an orbit of  $\beta\alpha$ . Conversely each orbit of  $\beta\alpha$  determines a simple closed curve in  $H_1(r, 0)$ . Let  $Y_1, \dots, Y_L$  be orbits of  $\beta\alpha$ . If  $x \in Y_i$  and  $d_i$  is the smallest positive integer such that  $(\beta\alpha)^{d_i}(x) = x$ , then on  $Y_i$ ,  $\beta\alpha$  is expressed as a product  $\beta_i\alpha_i$  of two disjoint permutations  $\alpha_i$  and  $\beta_i$  of the same length:

$$\alpha_i = (x, \beta\alpha(x), (\beta\alpha)^2(x), \dots, (\beta\alpha)^{d_i-1}(x))$$

and

$$\beta_i = (\alpha(x), \alpha\beta\alpha(x), \dots, (\alpha\beta)^{d_i-1}\alpha(x)).$$

Furthermore the  $\beta_i\alpha_i$  are pairwise disjoint and

$$\beta\alpha = (\beta_L\alpha_L) \cdots (\beta_2\alpha_2)(\beta_1\alpha_1),$$

which means that

$$|\beta\alpha| = |\beta_L\alpha_L| + \cdots + |\beta_1\alpha_1| = 2L. \quad \square$$

In particular, if  $L = 1$  in Theorem 3.1, then  $H_1(r, 0)$  is the Heegaard diagram of the lens space  $M_1(r, 0)$ . Note that all lens spaces assume to include  $\mathbb{S}^3$  but not  $\mathbb{S}^1 \times \mathbb{S}^2$ . We remark that in this setting  $K$  is a  $(1, 1)$ -knot in the lens space  $M_1(r, 0)$ .

**Theorem 3.2.** *Let  $H_1(r, 0)$  be determined by words  $\bar{a}$ ,  $\bar{b}$ ,  $\bar{c}$  and  $\bar{u}$  in  $\mathbf{F}_1$  such that, for some integer  $r$ ,*

$$\bar{u}\bar{c}\bar{b}\bar{u}^{-1} \approx_r \bar{a}\bar{b}\bar{c}\bar{a}^{-1},$$

*and let  $p = -(\sigma(\bar{b}) + \sigma(\bar{c})) \neq 0$ ,  $q = -\sigma(\bar{a}) - \sigma(\bar{b}) + \sigma(\bar{u})$ . Then, for each  $n \geq 2$ ,  $H_n(r, s)$  is the Heegaard diagram of the 3-manifold  $M_n(r, s)$  if and only if  $-sp + q \equiv 0 \pmod n$  and  $L = 1$ .*

*Proof.* Let  $M_1(r, 0)$  be the lens space and  $K$  be the  $(1, 1)$ -knot. Then  $(M_1(r, 0), K)$  admits a  $(1, 1)$ -decomposition satisfying  $(V_1; \alpha_1) = H_1(r, 0)$ , which can be represented by Fig. 4. It is well known that the  $n$ -fold cyclic covering of  $M_1(r, 0)$  branched over  $K$  is completely defined by an epimorphism  $C: H_1(M_1(r, 0) - K) \rightarrow \mathbb{Z}_n$ , called *monodromy*, where  $\mathbb{Z}_n$  is the cyclic group of order  $n$  and  $n \geq 2$ . Let  $r_1$  be a generator of  $\partial V_1$ , which is the boundary of the meridian disk meeting with  $K$  at one point and  $r_2$  a generator of  $\partial V_1$ , which is the longitude curve meeting with  $r_1$  at one point. Let  $\alpha$ ,  $\beta$  be as in Theorem 3.1. Then every curve of  $\partial V_1$  determined by two permutations  $\alpha$  and  $\beta$  is generated by  $r_1$  and  $r_2$ . In other words, the curve on  $H_1(r, 0)$  is generated by  $r_1$  and  $r_2$ . Given  $p$  and  $q$ , we have  $\pi_1(M_1(r, 0)) = \langle x \mid x^{\pm p} \rangle = \mathbb{Z}_{|p|}$  and  $H_1(M_1(r, 0) - K) = \langle r_1, r_2 \mid pr_2 + qr_1 \rangle = \mathbb{Z} \oplus \mathbb{Z}_{gcd(p, q)}$ . Note that an  $n$ -fold cyclic covering  $f$  of  $M$  branched over a  $(1, 1)$ -knot  $K$  is called *strongly-cyclic* if the branching index of  $K$  is  $n$ , that is, the fiber  $f^{-1}(x)$  of each point  $x \in K$  contains a single point. Therefore the homology class of a meridian loop  $r_1$  around  $K$  is mapped by  $C$  in a generator of  $\mathbb{Z}_n$ , say  $C(r_1) = 1$ , and so there exists an  $n$ -fold strongly-cyclic covering space  $M_n(r, s)$  of  $M_1(r, 0)$  branched over  $K$  if and only if there is  $s = C(r_2) \in \mathbb{Z}_n$  such that  $-sp + q \equiv 0 \pmod n$ .  $\square$

From now on we let two integers  $p$  and  $q$  be as in Theorem 3.2.

**Corollary 3.3.** *For each  $n \geq 2$ ,  $H_n(r, s)$  can not be a Heegaard diagram of a 3-manifold  $M_n(r, s)$  if  $p \nmid q$  and  $p|n$ .*

Proof. If  $H_n(r, s)$  is a Heegaard diagram of a 3-manifold, then  $-sp + q \equiv 0 \pmod n$  by Theorem 3.2. Hence  $-q \equiv 0 \pmod p$ , a contradiction.  $\square$

**Corollary 3.4.** *Let  $H_1(r, 0)$  be the planar graph induced by words  $\bar{a}$ ,  $\bar{b}$ ,  $\bar{c}$ , and  $\bar{u}$  in  $\mathbf{F}_1$  such that  $\bar{u}\bar{c}\bar{b}\bar{u}^{-1} \approx_r \bar{a}\bar{b}\bar{c}\bar{a}^{-1}$ . Then*

- (i)  $M_1(r, 0)$  is a lens space if and only if  $H_1(r, 0)$  has  $L = 1$ .
- (ii)  $M_1(r, 0)$  is the 3-sphere if and only if  $H_1(r, 0)$  has  $L = 1$  and  $|p| = 1$ .
- (iii) If  $L = 1$ , then  $M_n(r, s)$ ,  $n \geq 2$ , is the  $n$ -fold strongly cyclic covering of a lens space branched over a  $(1, 1)$ -knot.
- (iv) If  $\gcd(p, q) = 1$  and  $-sp + q = 0$ , then  $M_n(r, s)$ ,  $n \geq 2$ , is the  $n$ -fold cyclic covering of the 3-sphere branched over a  $(1, 1)$ -knot.

Proof. For (i) it is well known in [18]. For the specific statement, see Proposition 2.1 and 2.2 in [1]. For (ii), it follows from Lemma 3 in [23] with  $n = 1$ . For (iii), it follows from the proof in Theorem 3.2. For (iv), if  $\gcd(p, q) = 1$  and  $-sp + q = 0$ , then  $|p| = 1$  and so  $d = 2a + b + c = 2a + |p|$  is odd. The result follows from (ii) and (iii). (It also follows from the results in [23] with odd  $d$ .)  $\square$

**Lemma 3.5.** *The following properties hold in  $H_1(r, s)$ :*

- (1)  $q$  and  $2l(a) + l(b)$  have the same parity.
- (2)  $d, p,$  and  $l(b) + l(c)$  have the same parity.

Proof. Let  $D$  be the number of arrows pointing down the page in  $H_1(r, 0)$  of Fig. 4,  $U$  the number of arrows pointing up,  $LR$  the number of arrows pointing from left to right, and  $RL$  the number of arrows pointing from right to left, when we turn Fig. 4 by  $90^\circ$  clockwise. From the definition of  $p$  and  $q$ , it means that  $D - U = p$  and  $LR - RL = q$ . Since  $\{D, U\}$  (resp.  $\{LR, RL\}$ ) has no relation with  $l(a)$  edges (resp.  $l(c)$  edges),  $D + U = l(b) + l(c)$  (resp.  $LR + RL = 2l(a) + l(b)$ ). Moreover  $d = 2l(a) + l(b) + l(c)$ . Then (1) and (2) follow from the fact that  $s + t$  and  $s - t$  have the same parities for integers  $s$  and  $t$ .  $\square$

**Corollary 3.6.** *The lens space  $M_1(r, 0)$  can not be the 3-sphere if  $d$  is even.*

Proof. The proof follows from results of Corollary 3.4 and Lemma 3.5.  $\square$

We recall that if  $a, b, c$ , and  $u$  are words in  $\mathbf{F}_n$ ,  $n \geq 1$ , such that, for some integers  $r$  and  $s$ ,

$$u\eta^s(c)\eta^{s-1}(b)\eta^{-1}(u^{-1}) \approx_r abc\eta^{-1}(a^{-1}),$$

then the identifications of  $H_n(r, s)$  yielding  $M_n(r, s)$  induce the identifications of  $H_1(r, 0)$  yielding  $M_1(r, 0)$  determined by words  $\bar{a}$ ,  $\bar{b}$ ,  $\bar{c}$  and  $\bar{u}$  in  $\mathbf{F}_1$  such that

$$\bar{u}\bar{c}\bar{b}\bar{u}^{-1} \approx_r \bar{a}\bar{b}\bar{c}\bar{a}^{-1}.$$

Conversely, let  $M_1(r, 0)$  be determined by words  $\bar{a}$ ,  $\bar{b}$ ,  $\bar{c}$  and  $\bar{u}$  in  $\mathbf{F}_1$  such that

$$\bar{u}\bar{c}\bar{b}\bar{u}^{-1} \approx_r \bar{a}\bar{b}\bar{c}\bar{a}^{-1}.$$

Then for a fixed  $s_0$ , we can determine all types of  $M_n(r, s_0)$  having  $M_1(r, 0)$  as the quotient space.

**Corollary 3.7.** *Let  $H_1(r, 0)$  be determined by words  $\bar{a}$ ,  $\bar{b}$ ,  $\bar{c}$  and  $\bar{u}$  in  $\mathbf{F}_1$  such that, for some integer  $r$ ,*

$$\bar{u}\bar{c}\bar{b}\bar{u}^{-1} \approx_r \bar{a}\bar{b}\bar{c}\bar{a}^{-1},$$

*and let  $p = -(\sigma(\bar{b}) + \sigma(\bar{c})) \neq 0$ ,  $q = -\sigma(\bar{a}) - \sigma(\bar{b}) + \sigma(\bar{u})$  and  $n \geq 2$ . If  $s_0$  is an integer such that  $-s_0p + q \equiv 0 \pmod n$ , then there are as many Heegaard diagrams  $H_n(r, s_0)$  having  $H_1(r, 0)$  as the quotient graph as the number of divisors of  $-s_0p + q$ .*

EXAMPLE 3.1. Let  $\bar{a} = x_1^{-1}$ ,  $\bar{b} = x_1^{-1}$ ,  $\bar{c} = x_1^{-1}x_1^{-1}x_1x_1^{-1}x_1^{-1}$ , and  $\bar{u} = x_1$  be words in  $\mathbf{F}_1$  such that  $\bar{u}\bar{c}\bar{b}\bar{u}^{-1} \approx_4 \bar{a}\bar{b}\bar{c}\bar{a}^{-1}$ . Then for the planar graph  $H_1(4, 0)$ , we have  $p = 4$ ,  $q = 3$  and  $L = 1$ . By Theorem 3.2,  $\{H_n(4, s) \mid n \mid -4s + 3\}$  is the set of Heegaard diagrams representing 3-manifolds with  $H_1(4, 0)$  as a quotient space. In particular, if  $s = -3$ , then, by Corollary 3.7, the Heegaard diagrams representing 3-manifolds with the quotient space  $H_1(4, 0)$  are  $H_3(4, -3)$ ,  $H_5(4, -3)$ , and  $H_{15}(4, -3)$ . In fact,  $H_3(4, -3)$ ,  $H_5(4, -3)$ , and  $H_{15}(4, -3)$  are determined by words

$$\begin{aligned} a &= x_1^{-1}, & b &= x_2^{-1}, & c &= x_1^{-1}x_2^{-1}x_3x_1^{-1}x_2^{-1}, & u &= x_3, \\ a &= x_1^{-1}, & b &= x_1^{-1}, & c &= x_2^{-1}x_2^{-1}x_3x_4^{-1}x_4^{-1}, & u &= x_3, \end{aligned}$$

and

$$a = x_1^{-1}, \quad b = x_{11}^{-1}, \quad c = x_7^{-1}x_2^{-1}x_3x_4^{-1}x_{14}^{-1}, \quad u = x_3$$

respectively.

We show that every Dunwoody 3-manifold can be reformulated as  $M_n(r, s)$ . Let  $D_n(a, b, c, r, -s)$  be the Heegaard diagram of the Dunwoody 3-manifold. Since  $L = 1$ ,  $\beta\alpha$  has two cycles of length  $d$  such that  $(\beta\alpha)^d(x) = x$  for all  $x$  on  $D_1(a, b, c, r)$ . It is an immediate consequence of Theorem 3.1. Suppose that the first cycle of  $\beta\alpha$  starts with the symbol 1. Since  $-sp + q \equiv 0 \pmod n$ , the path corresponding to this cycles connects from the endpoint labelled 1 in the oriented circle labelled 0 (as  $C_0$ ) to the endpoint labelled 1 in the oriented circle labelled  $\overline{-ps + q}$  (as  $\bar{C}_{-ps+q}$ ) under

mod  $n$ . Thus the condition  $-sp + q \equiv 0 \pmod n$  ensures that the path corresponding to the cycles is a simple closed curve with an orientation. Since  $D_n(a, b, c, r, -s)$  has  $n$  simple closed curves, the second simple closed curve corresponding to the cycles connects from the endpoint labelled 1 in the oriented circle labelled 1 to the endpoint labelled 1 in the oriented circle labelled  $\bar{1}$  under mod  $n$ . Repeatedly, the path which starts with the endpoint labelled 1 in the oriented circle labelled  $n-1$  corresponding to the cycles of  $\beta\alpha$  will be arrived the endpoint labelled 1 in the oriented circle labelled  $\overline{n-1}$  under mod  $(n)$ . For each  $i = 0, 1, \dots, n-1$ , let  $w(C_i)$  (resp.  $w(\bar{C}_i)$ ) be the word obtained from reading off simple closed curves around the oriented circle labelled  $i$  (resp.  $\bar{i}$ ), denoted by  $C_i$  (resp.  $\bar{C}_i$ ), by the right hand law. Thus the identification of  $C_i$  and  $\bar{C}_i$  by  $r$  on  $D_n(a, b, c, r, s)$  induce  $w(C_i) \approx_r w(\bar{C}_i)$ . In case of  $i = 0$ , we have

$$u\eta^s(c)\eta^{s-1}(b)\eta^{-1}(u^{-1}) \approx_r abc\eta^{-1}(a^{-1}).$$

For the specific example, if  $p$  is odd and  $q \equiv \pm 2 \pmod p$ , the Dunwoody 3-manifold which is the  $n$ -fold cyclic covering of the 3-sphere branched over the torus knots  $T(p, q)$  satisfies  $u\eta^s(c)\eta^{s-1}(b)\eta^{-1}(u^{-1}) \approx_r abc\eta^{-1}(a^{-1})$  ([25]). Generally we call the compact connected orientable 3-manifold  $M_n(r, s)$ (possibly with boundary) constructed from the polyhedral description in Section 2 *the generalized Dunwoody 3-manifold*.

**Theorem 3.8.** *The Dunwoody 3-manifold which is the strongly-cyclic branched covering of a lens space is a closed orientable 3-manifold  $M_n(r, s)$  uniformized by the cyclically presented groups  $\langle x_1, \dots, x_n \mid abc\eta^{-1}(a^{-1}) = 1, \text{ indices mod } n \rangle$ , where  $n \geq 2$ .*

The following shows the existence of 3-manifolds  $M_n(r, s)$  which are not Dunwoody 3-manifolds.

EXAMPLE 3.2. Let  $a = x_1^{-1}x_1^{-1}$ ,  $b = x_5x_1^{-1}$ ,  $c = x_2x_3x_4$ , and  $u = x_1^{-1}x_2$  be words in  $\mathbf{F}_5$  such that, for  $r = 3$  and  $s = 1$ ,

$$\begin{aligned} u\eta(c)b\eta^{-1}(u^{-1}) &= x_1^{-1}x_2x_3x_4x_5x_5x_1^{-1}x_1^{-1}x_5 \\ &\approx_3 x_1^{-1}x_1^{-1}x_5x_1^{-1}x_2x_3x_4x_5x_5 = abc\eta^{-1}(a^{-1}). \end{aligned}$$

Then the cyclically presented group

$$G_5(3, 1) = \langle x_1, \dots, x_5 \mid x_i^{-1}x_{i+1}^{-1}x_ix_{i+1}^{-1}x_{i+2}x_{i+3}x_{i+4}x_ix_i = 1 \text{ indices mod } 5 \rangle$$

determines the polyhedral description  $P_5(3, 1)$  and its planar graph  $H_5(3, 1)$ . Moreover we have words  $\bar{a} = x_1^{-1}x_1^{-1}$ ,  $\bar{b} = x_1x_1^{-1}$ ,  $\bar{c} = x_1x_1x_1$ ,  $\bar{u} = x_1^{-1}x_1$ , in  $\mathbf{F}_1$ , satisfying  $\bar{u}\bar{c}\bar{b}\bar{u}^{-1} \approx_3 \bar{a}\bar{b}\bar{c}\bar{a}^{-1}$ . However the number  $L$  of curves in  $H_1(3, 0)$  is 2 and so  $|\alpha\beta| = 4$ . By Theorem 3.2,  $H_5(3, 1)$  is not a Heegaard diagram of  $M_5(3, 1)$ . In fact,  $M_5(3, 1)$  is not Dunwoody 3-manifold because the planar graph  $H_1(3, 0)$  is representing a link in a lens space  $L(3, 2)$  and Corollary 3.4 (iii).

EXAMPLE 3.3. Let  $a = x_1^{-1}$ ,  $b = x_1x_2^{-1}x_2x_3^{-1}$ ,  $c = x_1^{-1}$ , and  $u = x_3^{-1}$  be words in  $\mathbf{F}_3$  such that, for  $r = 4$  and  $s = 0$ ,

$$uc\eta^{-1}(b)\eta^{-1}(u^{-1}) \approx_4 abc\eta^{-1}(a^{-1}).$$

Then the cyclically presented group  $G_3(4, 0)$  determines the polyhedral description  $P_3(4,0)$  and its planar graph  $H_3(4,0)$ . However the number  $L$  of curves in  $H_1(4,0)$  is 2 and by Theorem 3.2,  $H_3(4,0)$  is not a Heegaard diagram of  $M_3(4,0)$ . In fact,  $M_3(4,0)$  is not Dunwoody 3-manifold because the planar graph  $H_1(4,0)$  is representing the link  $\mathcal{G}_1^2$  in  $\mathbb{S}^3$  (see the Rolfsen-table in [32]).

**4. On the cyclically presented groups**

We review the properties of the quotient spaces of 3-manifolds and dual Heegaard diagrams. Let  $M_n(r, s)$  be the Dunwoody 3-manifold and its Heegaard diagram  $D_n(a, b, c, r, s)$ . Then the quotient space  $M_1(r, 0)$  is the lens space with its Heegaard diagram  $H_1(r, 0)$ . Moreover  $M_1(r, 0)$  admits a  $(1, 1)$ -decomposition  $D_1(a, b, c, r)$  of the certain  $(1, 1)$ -knot  $K(a, b, c, r)$ . As the dual Heegaard diagram we have the dual  $(1, 1)$ -decomposition of  $D_1(a, b, c, r)$  and denote it by  $Du(a, b, c, r)$  or  $D_1(a', b', c', r')$ , where the dual  $(1, 1)$ -decomposition can be understood from the attaching homeomorphism onto the dual Heegaard diagram  $(V_2; \beta_1)$  consisting of the meridian curve  $m'$  and simple closed curve  $l'$  on  $V_2$  that are the images of simple closed curve  $l$  and meridian curve  $m$  on  $V_1$ , respectively. We now introduce an algorithm for the dual  $(1, 1)$ -decomposition of  $D_1(a, 0, 1, r)$  as follows.

On  $D_1(a, 0, 1, r)$ , there are *three types of region* as follows (See Fig. 5 where all indices are taken under modulo  $2a + 1$ ):

- (1) two bigons bounded by the edges  $[1, 2a + 1]$  and  $[\bar{r}, \overline{r + 1}]$  at  $C^+$  and  $C^-$  respectively,
- (2)  $2(a - 1)$  quadrilaterals bounded by the edges  $[j, i]$  and  $[j - 1, i + 1]$  at  $C^+$  and the edges  $[\overline{r - i}, \overline{r + i + 1}]$  and  $[\overline{r - i + 1}, \overline{r + i}]$  at  $C^-$  respectively, where  $1 \leq i \leq a - 1$  and  $i + j = 2a + 2$ , and
- (3) an octagon bounded by the edges  $[a, a + 2]$ ,  $[\overline{r + a}, \overline{r - a + 1}]$ ,  $[a + 1, \overline{r + a + 1}]$ , and  $[a + 1, \overline{r - a}]$  of  $l$  and parts of  $C^+$  and  $C^-$ .

Through process that changes  $\{m, l\}$  on  $V_1$  into the curve-system  $\{l', m'\}$  on  $V_2$ , the role of  $m$  (resp.  $l$ ) in each region will be changed into  $l'$  (resp.  $m'$ ) in  $Du(a, 0, 1, r)$ , so if 1, in the bigon bounded by  $[1, 2a + 1]$ , is a starting point in  $D_1(a, 0, 1, r)$ , then  $2a + 1$  will be starting point in  $Du(a, 0, 1, r)$ . Similarly, the point  $r + 1$ , in the bigon bounded by  $[\bar{r}, \overline{r + 1}]$ , is going to situate on  $r'$  in  $Du(a, 0, 1, r)$ . On  $D_1(a, 0, 1, r)$ , if  $a$  is the number of regions of type (1) or (2) that is connected from the bigon bounded by  $[1, 2a + 1]$  to a quadrilateral with  $[a, a + 2]$  which is to be a side of an octagon, then on  $Du(a, 0, 1, r)$ ,  $a'$  determined by  $l'$  is equal to the number of regions of type (1) or (2) that is connected along parts of  $m$  from the bigon bounded by  $[2a + 1, 1]$  to a quadrilateral which is connected with a side of an octagon. Since each region is

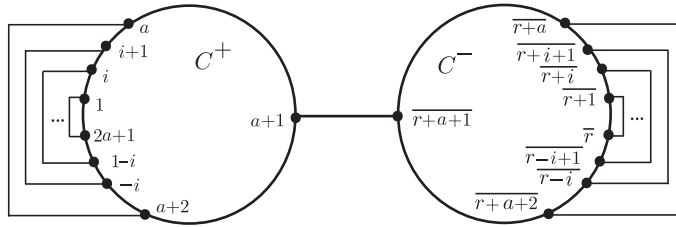


Fig. 5. Three types of region on  $D_1(a, 0, 1, r)$ .

preserved in  $D_1(a, 0, 1, r)$  and  $Du(a, 0, 1, r)$ , we have  $c' = 2a + 1 - a'$ . Moreover we see that  $r + 1$  or  $\overline{r + 1}$  appears at the  $r'$ th term of the following cycle:

$$0 = 2a + 1 \rightarrow \dots \rightarrow r + 1 \quad \text{or} \quad \overline{r + 1} \rightarrow r \rightarrow \dots \rightarrow 1,$$

which is a cycle sequence along  $l$  starting from  $2a + 1$  and determining  $m'$  on  $Du(a, 0, 1, r)$ . Vice versa we can obtain  $Du(a, 0, c, r)$  from  $D_1(a, 0, c, r)$  by the dual process above. (For detail, see [26].)

For positive integers  $n (\geq 1)$  and  $t$ , we consider two families of cyclically presented groups as follows:

$$G(n, t) = \langle y_0, y_1, \dots, y_{n-1} \mid (y_i^{-1}y_{i+1})^{t/2}(y_i^{-1}y_{i+1})(y_{i+2}^{-1}y_{i+1})^{t/2}(y_{i+2}^{-1}) = 1 \text{ indices mod } n \rangle,$$

where  $t$  is even, and

$$G(n, t) = \langle y_0, y_1, \dots, y_{n-1} \mid (y_i)^{(t+1)/2}(y_{i+1}^{-1})^{(t+1)/2}(y_{i+2})^{(t+1)/2}(y_{i+1}^{-1})^{(t+3)/2} = 1 \text{ indices mod } n \rangle,$$

where  $t$  is odd.

**Theorem 4.1.** *For an even  $t$ ,  $G(n, t)$  is isomorphic to the fundamental group of the Dunwoody 3-manifold represented by  $D_n(t + 1, 0, 1, (t + 2)/2, 2)$ .*

Proof. From the presentation of  $G(n, t)$ , we have

$$(y_{i-2}^{-1}y_{i-1})^{t/2}(y_{i-2}^{-1}y_{i-1})(y_i^{-1}y_{i-1})^{t/2} = y_i$$

for all  $i = 0, \dots, n-1$  under modulo  $n$ . By letting  $x_i = (y_{i-1}^{-1}y_i)^{t/2}y_{i-1}^{-1}$  for  $i = 0, 1, \dots, n-1$ , we have  $y_i = x_{i-1}x_i^{-1}$  for all  $i = 0, \dots, n-1$  under modulo  $n$ . Hence  $G(n, t)$  is isomorphic to

$$\langle x_0, \dots, x_{n-1} \mid w = 1, \text{ indices mod } n \rangle,$$

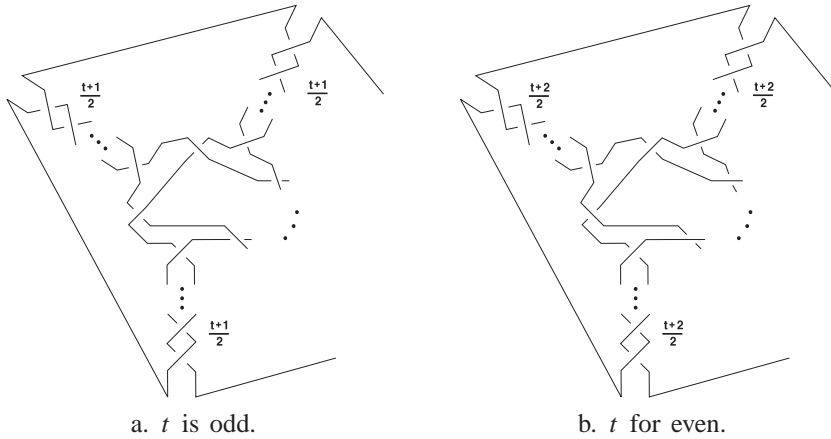


Fig. 6. The  $n$ -periodic link  $b_n(2t + 3, 2)$  for  $t \geq 0$ .

where

$$w = (x_i x_{i-1}^{-1} x_{i-2} x_{i-1}^{-1})^{t/2} x_i x_{i-2} x_{i-1}^{-1}.$$

We now consider words  $a, b, c,$  and  $u$  in  $\mathbf{F}_n$ :

$$\begin{aligned} a &= x_0^{-1} x_1, \\ b &= 1, \\ c &= x_n x_0^{-1} x_1 (x_0^{-1} x_n x_0^{-1} x_1)^{(t-2)/2}, \\ u &= x_1 x_0^{-1}. \end{aligned}$$

Then we have that

$$u \eta^0(c) \eta^{-1}(b) \eta^{-1}(u^{-1}) \approx_4 abc \eta^{-1}(a^{-1}),$$

that is,  $abc \eta^{-1}(a^{-1})$  and  $u \eta^0(c) \eta^{-1}(b) \eta^{-1}(u^{-1})$  are rotational equivalent by 4. This determines a Dunwoody 3-manifold represented by  $D_n(2, 0, 2t - 1, 4, 0)$  which is the dual Heegaard diagram of  $D_n(t + 1, 0, 1, (t + 2)/2, 2)$ . This completes the proof.  $\square$

Similar argument gives the following.

**Theorem 4.2.** *For an odd  $t, G(n, t)$  is isomorphic to the fundamental group of the Dunwoody 3-manifold represented by  $D_n(t + 1, 0, 1, 1, 0)$ .*

We recall that any link can be obtained as the closure of some braid. For coprime integers  $p$  and  $q,$  by  $\sigma_i^{p/q}$  we denote the rational  $p/q$ -tangle whose incoming arcs are  $i$ -th and  $(i + 1)$ -th strings. For an integer  $n \geq 2$  we denote by  $b_n(2t + 3, 2)$  the  $n$ -periodic



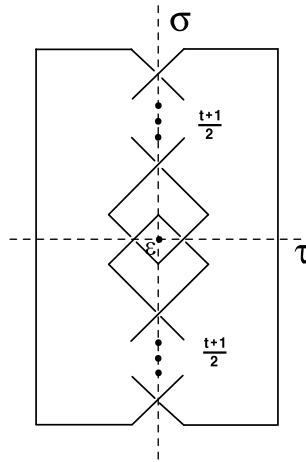


Fig. 7. The 2-bridge knot  $b_2(2t + 3, 2)$ .

link which is the closure of the rational 3-strings braid  $(\sigma_2 \sigma_1^{2/(t+2)})^n$  or  $(\sigma_2^{-1} \sigma_1^{2/(t+1)})^n$  if  $t$  is even or odd respectively. The link  $b_n(2t + 3, 2)$  is pictured in Fig. 6 where  $(t + 1)/2$  and  $(t + 2)/2$  mean the numbers of half twists. We note that  $b_n(2t + 3, 2)$  is a knot for  $n \neq 3m$ , and  $b_{3m}(2t + 3, 2)$  is a 3-component link for an odd  $(t + 1)/2$  or an odd  $(t + 2)/2$  if  $t$  is even or odd respectively.

We suppose that  $n \geq 2$  from now on. Then we obtain the  $(1, 1)$ -knots by using the certain rotation symmetries on the Dunwoody 3-manifolds as follows.

**Corollary 4.3.** *For an odd  $t$ , the Dunwoody 3-manifold represented by  $D_n(t + 1, 0, 1, 1, 0)$  is an  $n$ -fold cyclic covering of the 3-sphere  $S^3$  branched over the 2-bridge knot  $b_2(2t + 3, 2)$ . For even  $t$ , the Dunwoody 3-manifold represented by  $D_n(t + 1, 0, 1, (t + 2)/2, 2)$  is an  $n$ -fold cyclic covering of the 3-sphere  $S^3$  branched over the 2-bridge knot  $b_2(2t + 3, 2)$ .*

*Proof.* Let  $t$  be odd. We note that there is a rotation symmetry  $\mu$  of order  $n$  on  $D_n(t + 1, 0, 1, 1, 0)$ . The rotation by  $2\pi/n$  defines an action of the cyclic group  $\mathbb{Z}_n = \langle \mu \mid \mu^n = 1 \rangle$  on  $D_n(t + 1, 0, 1, 1, 0)$ . The quotient space  $D_n(t + 1, 0, 1, 1, 0)/\mathbb{Z}_n$  admits a  $(1, 1)$ -decomposition consisting of a trivial arc with branching index  $n$ . Moreover the  $(1, 1)$ -decomposition  $D_1(t + 1, 0, 1, 1)$  represents the 2-bridge knot  $b_2(2t + 3, 2)$  in  $S^3$ . (See Fig. 7 and [15] or [36].) Therefore the Dunwoody 3-manifold represented by  $D_n(t + 1, 0, 1, 1, 0)$  is an  $n$ -fold cyclic branched covering of  $S^3$ , branched over the 2-bridge knot  $b_2(2t + 3, 2)$ . Similarly, let  $t$  be even, the  $(1, 1)$ -decomposition of  $D_n(t + 1, 0, 1, (t + 2)/2, 2)/\mathbb{Z}_n$  represents the 2-bridge knot  $b_2(2t + 3, t + 2)$  which is equivalent to  $b_2(2t + 3, 2)$  because  $2(t + 2) \equiv 1 \pmod{2t + 3}$ . This completes the proof.  $\square$

Note that the result of Corollary 4.3 can also be obtained from results in [15] or [36]. We now mention some covering properties of the quotient spaces of  $D_n(t + 1, 0, 1, 1, 0)$  and  $D_n(t + 1, 0, 1, (t + 2)/2, 2)$ , which were noted implicitly in [21]. There is an involution  $\epsilon$  for  $\mathfrak{b}_2(2t + 3, 2)$  as shown in Fig. 7, and the quotient space by  $\epsilon$  is the 3-sphere  $\mathbb{S}^3$  and the image of  $\mathfrak{b}_2(2t + 3, 2)$  is the 2-component link  $\mathcal{L}(4l - 2, l - 1)$  or  $\mathcal{L}(4l - 2, l)$  (denoted by  $\mathfrak{b}(2t + 3, 2)$  briefly) obtained as the closure of the rational  $(4l - 2)/2$ -tangle or  $(4l - 2)/2$ -tangle if  $t$  is odd or even respectively. In general, if  $n$  is even, there are at least 2 involutions, denoted by  $\sigma$  and  $\tau$ , and one rotation symmetry of order  $n$ , denoted by  $\epsilon$ , on  $\mathfrak{b}_n(2t + 3, 2)$ . If  $n$  is odd, there are at least one involution and one rotation symmetry of order  $n$ , denoted by  $\epsilon$  on  $\mathfrak{b}_n(2t + 3, 2)$ . Thus the quotient space by  $\epsilon$  for  $\mathfrak{b}_n(2t + 3, 2)$  is  $\mathbb{S}^3$  and the image of  $\mathfrak{b}_n(2t + 3, 2)$  is  $\mathcal{L}(4l - 2, l - 1)$  or  $\mathcal{L}(4l - 2, l)$ . In other words,

- (i) The link  $\mathfrak{b}_n(2t + 3, 2)$  is the  $n$ -fold cyclic covering of  $\mathbb{S}^3$  branched over  $\mathcal{L}(4l - 2, l - 1)$  for odd  $t$  and  $\mathcal{L}(4l - 2, l)$  for even  $t$ .
- (ii) The Dunwoody 3-manifolds represented by  $D_n(t + 1, 0, 1, 1, 0)$  and  $D_n(t + 1, 0, 1, (t + 2)/2, 2)$  are the 2-fold cyclic coverings of  $\mathbb{S}^3$  branched over the  $n$ -periodic links  $\mathfrak{b}_n(2t + 3, 2)$ .
- (iii) The Dunwoody 3-manifolds represented by  $D_n(t + 1, 0, 1, 1, 0)$  and  $D_n(t + 1, 0, 1, (t + 2)/2, 2)$  are the  $(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ -fold cyclic branched coverings of  $\mathcal{L}(4l - 2, l - 1)$  and  $\mathcal{L}(4l - 2, l)$ , respectively.

**Theorem 4.4.** *For each  $2 \leq i \leq n - 1$ , there is a closed orientable 3-manifold  $M_{i,i+1}$  which is the  $(i + 1)$ -fold cyclic branched covering of  $\mathfrak{b}_i(2t + 3, 2)$  and the  $i$ -fold cyclic branched covering of  $\mathfrak{b}_{i+1}(2t + 3, 2)$  for any  $t \geq 0$ .*

*Proof.* For any  $t \geq 0$ ,  $(\mathbb{S}^3, \mathfrak{b}_i(2t + 3, 2))$  is the  $i$ -fold cyclic covering of  $\mathbb{S}^3$  branched over  $\mathfrak{b}(2t + 3, 2)$  by (i). Moreover there is a closed orientable 3-manifold  $M_{i,i+1}$  such that it is the  $(i + 1)$ -fold cyclic covering of  $\mathbb{S}^3$  branched over  $\mathfrak{b}_i(2t + 3, 2)$ . On the other hand,  $(\mathbb{S}^3, \mathfrak{b}_{i+1}(2t + 3, 2))$  is the  $(i + 1)$ -fold cyclic covering of  $\mathbb{S}^3$  branched over  $\mathfrak{b}(2t + 3, 2)$  by (i). By the commutativity, we obtain that  $M_{i,i+1}$  is the  $i$ -fold cyclic covering of  $\mathbb{S}^3$  branched over  $\mathfrak{b}_{i+1}(2t + 3, 2)$ . □

We note that  $M_{i,i+1}$  in Theorem 4.4 is the  $(\mathbb{Z}_{i+1} \oplus \mathbb{Z}_i)$ -fold cyclic branched covering of the link  $\mathcal{L}(4l - 2, l - 1)$  or  $\mathcal{L}(4l - 2, l)$ , where  $2 \leq i \leq n - 1$ . In case of  $n = 3$ , the manifold  $M_{2,3}$  is homeomorphic to the Dunwoody 3-manifold  $D_3(t + 1, 0, 1, 1, 0)$  or  $D_3(t + 1, 0, 1, (t + 2)/2, 2)$ .

By a  $\Theta$ -curve  $\theta(x, y, z)$ , we mean a  $\Theta$ -curve which has three edges  $x, y$  and  $z$ , each of which joins two vertices. In this case,  $x \cup y, y \cup z$  and  $z \cup x$  are called *the constituent knots of  $\Theta(x, y, z)$* .

**Theorem 4.5.** *The Dunwoody 3-manifolds represented by  $D_n(t + 1, 0, 1, 1, 0)$  and  $D_n(t + 1, 0, 1, (t + 2)/2, 2)$  are the  $(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ -fold cyclic branched coverings of two*

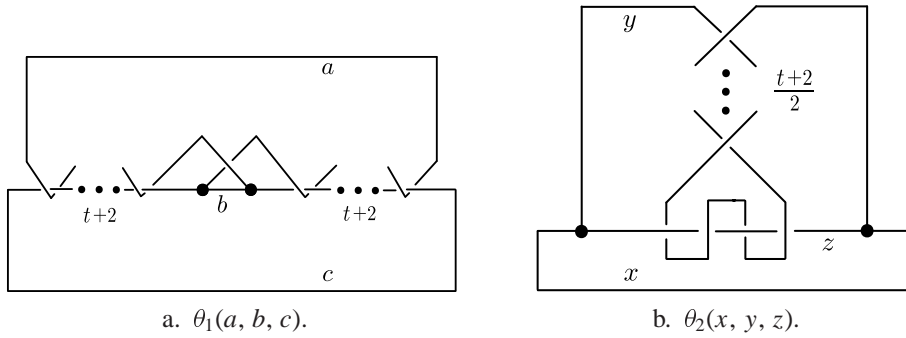


Fig. 8. Two different  $\Theta$ -curves.

different  $\Theta$ -curves in  $\mathbb{S}^3$ , where  $t \neq 1$ .

Proof. We only do for  $D_n(t + 1, 0, 1, 1, 0)$ . The same argument can be applied for the other case. By Corollary 4.3, the Dunwoody 3-manifold represented by  $D_n(t + 1, 0, 1, 1, 0)$  is an  $n$ -fold cyclic branched covering of  $\mathbb{S}^3$ , branched over  $b_2(2t + 3, 2)$ . From Fig. 7 we see that the orbifold  $b_2(2t + 3, 2)(n)$  has two rotation symmetries  $\sigma$  and  $\tau$  of order two such that each axis of the symmetry intersects the singular set of  $b_2(2t + 3, 2)(n)$  in two points. The quotient space  $b_2(2t + 3, 2)(n)/\langle \sigma \rangle$  by  $\sigma$  is an orbifold whose underlying space is  $\mathbb{S}^3$  and whose singular set is a  $\Theta$ -curve  $\theta_1(a, b, c)$  with two vertices, depicted in Fig. 8 a. Similarly  $\tau$  induces a quotient space  $b_2(2t + 3, 2)(n)/\langle \tau \rangle$  and a  $\Theta$ -curve  $\theta_2(x, y, z)$  in Fig. 8 b. We note that two  $\Theta$ -curves  $\theta_1(a, b, c)$  and  $\theta_2(x, y, z)$  are different each other. This can be shown by checking three constituent knots of each  $\Theta$ -curve. Indeed  $\theta_1(a, b, c)$  has two trivial knots and the torus knot  $T(2, 2t + 3)$  as constituent knots. However the three constituent knots of  $\theta_2(x, y, z)$  are two trivial knots and the closure of the rational  $((2t + 5)/4)$ -tangle.  $\square$

In case of  $n = 2$ , by (iii) and Theorem 4.5, the Dunwoody 3-manifolds represented by  $D_2(t + 1, 0, 1, 1, 0)$  and  $D_2(t + 1, 0, 1, (t + 2)/2, 2)$ ,  $t \neq 1$ , are the  $(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ -fold branched coverings of two different  $\Theta$ -curves and a link in  $\mathbb{S}^3$ . The results extend the corresponding ones in [23] and [22]. We also note that for an even  $n$ , the involutions  $\sigma$  and  $\tau$  for  $b_2(2t + 3, 2)$  in Theorem 4.5 can be naturally extended to the  $n$ -periodic link  $b_n(2t + 3, 2)$ . Hence there are two spatial  $\Theta$ -curves  $\theta_\sigma$  and  $\theta_\tau$  such that the  $n$ -periodic knot  $b_n(2t + 3, 2)$  is the 2-fold cyclic covering branched over  $\theta_\sigma$  and  $\theta_\tau$  in  $\mathbb{S}^3$ . By (ii), we have the following.

**Corollary 4.6.** *Let  $n$  be even and  $b_n(2t + 3, 2)$  be a knot. Then the Dunwoody 3-manifolds represented by  $D_n(t + 1, 0, 1, 1, 0)$  and  $D_n(t + 1, 0, 1, (t + 2)/2, 2)$  are the  $(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ -fold cyclic branched coverings of two spatial  $\Theta$ -curves.*

Similarly let  $n$  be odd, then there exists an involution on the  $n$ -periodic knot  $b_n(2t+3, 2)$  such that the Dunwoody 3-manifolds represented by  $D_n(t+1, 0, 1, 1, 0)$  and  $D_n(t+1, 0, 1, (t+2)/2, 2)$  are the  $(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ -fold cyclic branched coverings of a spatial  $\Theta$ -curve. As another application of the cyclically presented group, we proved the fact that the Dunwoody polynomial of  $(1, 1)$ -knot in 3-sphere is to be the Alexander polynomial (see Theorem 3.5 in [27]).

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