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Osaka University
EULER CHARACTERISTICS ON
A CLASS OF FINITELY GENERATED NILPOTENT GROUPS

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Abstract
A finitely generated torsion free nilpotent group is called an $\mathcal{F}$-group. To each $\mathcal{F}$-group $\Gamma$ there is associated a connected, simply connected nilpotent Lie group $G_{\mathcal{F}}$. Let $\text{TUF}$ be the class of all $\mathcal{F}$-group $\Gamma$ such that $G_{\mathcal{F}}$ is totally unimodular. A group in $\text{TUF}$ is called $\text{TUF}$-group. In this paper, we are interested in finding non-zero Euler characteristic on the class $\text{TUF}$ and therefore, on $\text{TUFF}$, the class of groups $K$ having a subgroup $\Gamma$ of finite index in $\text{TUF}$. An immediate consequence we obtain that any two isomorphic finite index subgroups of a $\text{TUFF}$-group have the same index. As applications, we give two results, the first is a generalization of Belgradek’s result, in which we prove that every $\text{TUFF}$-group is co-hopfian. The second is a known result due to G.C. Smith, asserting that every $\text{TUFF}$-group is not compressible.

1. Introduction and main results

We follow [5, p. 222] in defining an Euler characteristic on a class of groups as follows (see also [2, p. 1]).

DEFINITION 1.1 (Euler characteristic). Let $\mathcal{X}$ be a class of groups closed under taking subgroups of finite index. By an Euler characteristic on $\mathcal{X}$ it meant a function $\chi : \mathcal{X} \to \mathbb{R}$ satisfying

(Ec1) If $K$ and $H$ are in $\mathcal{X}$, and $K$ is isomorphic to $H$, then $\chi(K) = \chi(H)$.

(Ec2) If $K$ is in $\mathcal{X}$, and $H$ is a subgroup of $K$ of finite index, then $\chi(H) = [K : H]\chi(K)$, where $[K : H]$ denotes the index of $H$ in $K$.

In this paper, we are interested in finding non zero Euler characteristics defined on a class of finitely generated nilpotent groups.

Let $G$ be a connected Lie group and $\text{Aut}(G)$ its group of continuous automorphisms. Let $\mu$ be a Haar measure on $G$. For every $\alpha \in \text{Aut}(G)$ we have

$$\alpha^{-1}_* \mu = \Delta(\alpha) \mu.$$

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where \(a^{-1}\mu\) is the push forward of \(\mu\) under \(a^{-1}\), and \(\Delta: \text{Aut}(G) \to \mathbb{R}_+^*\) is a homomorphism of \(\text{Aut}(G)\) into the multiplicative group of the positive reals. If \(G\) is a connected, simply connected nilpotent Lie group, then

\[
\Delta(a) = |\det(a)|.
\]

**Definition 1.2** ([10, p. 627]). A connected, simply connected nilpotent Lie group \(G\) is called totally unimodular if the image of \(\Delta\) is \(\{1\}\).

Let \(\text{TULG}\) be the class of connected, simply connected totally unimodular nilpotent Lie groups.

A real Lie algebra is called *characteristically nilpotent* if all its derivations are nilpotent ([4, p. 157], [6, p. 623]). We note that a characteristically nilpotent Lie algebra is nilpotent. Let \(\text{CNLG}\) be the class of connected, simply connected nilpotent Lie groups \(G = \exp \mathfrak{g}\) such that \(\mathfrak{g}\) is a characteristically nilpotent Lie algebra.

**Proposition 1.3** ([10, (1.1)]). We have

\[
\text{CNLG} \subset \text{TULG}.
\]

A finitely generated torsion free nilpotent group is called an \(\mathcal{F}\)-group. Any \(\mathcal{F}\)-group \(\Gamma\) is isomorphic to a discrete uniform subgroup of a connected, simply connected nilpotent Lie group \(G_{\Gamma}\) whose Lie algebra \(\mathfrak{g}_{\Gamma}\) has rational structure constants ([8, Theorem 6]). Let \(\text{TUF}\) be the class of all \(\mathcal{F}\)-groups \(\Gamma\) such that \(G_{\Gamma} \in \text{TULG}\). We call a group \(\Gamma\) a \(\text{TUF}\)-group if \(\Gamma \in \text{TUF}\). For every integer \(n \geq 7\) there exists a \(n\)-dimensional characteristically nilpotent Lie algebra with rational structure ([14, Theorem 5]). By the Mal’cev rationality criterion (Theorem 2.1) we derive the following.

**Proposition 1.4.** For every integer \(n \geq 7\) there exists a \(\text{TUF}\)-group with Hirsch length \(n\).

The main result of this paper is the following.

**Theorem 1.5.** The class \(\text{TUF}\) admits Euler characteristics.

In Section 3, we give an explicit Euler characteristic on \(\text{TUF}\).

By [5, p. 222] (see also [15], [2]) every Euler characteristic \(\chi\) on \(\text{TUF}\) can be extended to \(\text{TUFF}\), the class of groups \(K\) having a subgroup \(\Gamma\) of finite index in \(\text{TUF}\), by setting

\[
\chi(K) = \frac{1}{[K: \Gamma]}\chi(\Gamma).
\]

As an immediate consequence we have the following.
Proposition 1.6. Any two isomorphic finite index subgroups of a TUFF-group have the same index.

Definition 1.7 (Co-hopfian group). A group is called co-hopfian if it satisfies the following equivalent conditions:
(1) It is not isomorphic to any proper subgroup.
(2) Every injective endomorphism of the group is an automorphism.

As an easy consequence of Proposition 1.6, we obtain a generalization for I. Belegradek’s result ([1, Corollary 2.4]).

Proposition 1.8. Every TUFF-group is co-hopfian.

We introduce the following definition due to G.C. Smith ([13, Definition 1]).

Definition 1.9 (Compressible group). A group $G$ is called compressible if any finite index subgroup of $G$ contains a finite index subgroup isomorphic to $G$.

The following proposition which is due to G.C. Smith ([13, Proposition 4]) is an immediate consequence of Proposition 1.6.

Proposition 1.10. Every TUFF-group is not compressible.

2. Rational structures and discrete uniform subgroups

General references for the material in this section are [3] and [11] as well as the original paper of Mal’cev [8].

Let $G$ be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then the exponential map $\exp: \mathfrak{g} \to G$ is a diffeomorphism. Let $\log: G \to \mathfrak{g}$ denote the inverse of $\exp$.

2.1. Rational structures. Let $G$ be a nilpotent, connected and simply connected real Lie group and let $\mathfrak{g}$ be its Lie algebra. We say that $\mathfrak{g}$ (or $G$) has a rational structure if there is a Lie algebra $\mathfrak{g}(\mathbb{Q})$ over $\mathbb{Q}$ such that $\mathfrak{g} \cong \mathfrak{g}(\mathbb{Q}) \otimes \mathbb{R}$. It is clear that $\mathfrak{g}$ has a rational structure if and only if $\mathfrak{g}$ has an $\mathbb{R}$-basis $(X_1, \ldots, X_n)$ with rational structure constants.

2.2. Uniform subgroups. A discrete subgroup $\Gamma$ is called uniform in $G$ if the quotient space $G/\Gamma$ is compact. A proof of the next result can be found in Theorem 7 of [8] or in Theorem 2.12 of [11].
Theorem 2.1 (The Malcev rationality criterion). Let $G$ be a simply connected nilpotent Lie group, and let $\mathfrak{g}$ be its Lie algebra. Then $G$ admits a uniform subgroup $\Gamma$ if and only if $\mathfrak{g}$ admits a basis $(X_1, \ldots, X_n)$ such that

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ijk} X_k, \quad (\forall 1 \leq i, j \leq n),$$

where the constants $c_{ijk}$ are all rational.

2.3. The Malcev rigidity theorem. The following is a theorem of Mal’cev ([8, Theorem 5]); see also ([9, Theorem 4]).

Theorem 2.2 (Malcev rigidity theorem). Let $G_1$ and $G_2$ be connected simply connected nilpotent Lie groups and $\Gamma_1$, $\Gamma_2$ discrete uniform subgroups of $G_1$ and $G_2$. Any abstract group isomorphism $\phi$ between $\Gamma_1$ and $\Gamma_2$ extends uniquely to an isomorphism $M(\phi)$ of $G_1$ on $G_2$; that is, the following diagram

$$(2.1)$$

$$\begin{array}{ccc}
\Gamma_1 & \xrightarrow{\phi} & \Gamma_2 \\
\downarrow i & & \downarrow i \\
G_1 & \xrightarrow{M(\phi)} & G_2
\end{array}$$

is commutative, where $i$ is the inclusion mapping. The isomorphism $M(\phi)$ is called the Mal’cev extension of $\phi$.

3. An explicit Euler characteristic on TUF. Proof of Theorem 1.5

Let $G$ be a connected Lie group, $\mathcal{S}(G)$ be the space of discrete uniform (i.e., cocompact) subgroups of $G$. Let $\mu$ be a right Haar measure of $G$. Let $\Gamma \in \mathcal{S}(G)$, the measure $\mu$ induces a finite measure $\tilde{\mu}$ over the homogeneous space $G/\Gamma$. Let

$$V^\mu_G : \mathcal{S}(G) \to \mathbb{R}_+$$

defined for $\Gamma \in \mathcal{S}(G)$ by

$$V^\mu_G(\Gamma) = \tilde{\mu}(G/\Gamma).$$

Remark 3.1. We recall that if $F$ is a fundamental domain for $G/\Gamma$ then $\tilde{\mu}(G/\Gamma) = \mu(F)$ ([7, p. 430]).

The notation $H \leq_f K$ signifies that $H$ is a finite index subgroup of the group $K$. A proof of the following proposition can be found in Lemma 3.2 of [7].
Proposition 3.2. If \( H, K \in \mathcal{S}(G) \) and if \( H \leq F K \) then we have

\[
V_G^\mu(H) = [K : H]V_G^\mu(K).
\]

Proposition 3.3. Let \( G \) in \( \text{TULG} \) and \( \mu \) a Haar measure on \( G \). Let \( \Gamma_1, \Gamma_2 \) be two isomorphic subgroups of \( \mathcal{S}(G) \). Then we have

\[
(3.2) \quad V_G^\mu(\Gamma_1) = V_G^\mu(\Gamma_2).
\]

Proof. Let \( \phi \) be an isomorphism of \( \Gamma_1 \) onto \( \Gamma_2 \). Let \( F \) be a fundamental domain of \( G/\Gamma_1 \) and compute

\[
V_G^\mu(\Gamma_1) = \mu(F)
= \mu(M(\phi)(F))
= V_G^\mu(\Gamma_2).
\]

We define an equivalence relation \( \simeq \) on \( \text{TULG} \) by

\[
G_1 \simeq G_2 \iff G_1, G_2 \text{ are isomorphic.}
\]

For \( G \in \text{TULG} \), let \( [G] \) be the equivalence class containing \( G \). Let \( T \) be a transversal for the equivalence relation \( \simeq \).

Let \( H, K \) be two groups (resp. Lie groups), the set of all isomorphisms (resp. Lie groups isomorphisms) of \( H \) onto \( K \) is denoted by \( \mathcal{A}(H, K) \).

Lemma 3.4. Let \( G_0 \in T \) and \( G \in [G_0] \). For every \( \phi, \psi \in \mathcal{A}(G_0, G) \) we have

\[
\phi_*\mu_0 = \psi_*\mu_0,
\]

where \( \phi_*\mu_0 \) (resp. \( \psi_*\mu_0 \)) is the push forward of \( \mu_0 \) under \( \phi \) (resp. \( \psi \)).

Proof. Let \( F \) be a measurable set and compute

\[
\phi_*\mu_0(F) = \mu_0(\phi^{-1}(F))
= \mu_0(\psi^{-1}\phi(\phi^{-1}(F)))
= \mu_0(\psi^{-1}(F))
= \psi_*\mu_0(F).
\]

Let \( G_0 \in T \) and \( \mu_0 \) a Haar measure on \( G_0 \). Let \( G \in [G_0] \) and \( \Gamma \in \mathcal{S}(G) \).

The function

\[
\mathcal{A}(G_0, G) \to \mathbb{R}, \quad \phi \to V_G^{\phi_*\mu_0}(\Gamma)
\]
is constant. In the sequel, we note

\[ V[G_0, \mu_0, G](\Gamma) = V_G^{\phi_0}(\Gamma) \quad (\forall \phi \in \mathcal{R}(G_0, G)). \]

Let \( G_1, G_2 \in [G_0] \). For every \( \psi \in \mathcal{R}(G_0, G_2) \) and \( \phi \in \mathcal{R}(G_1, G_2) \), we note

\[ V[G_0, \mu_0, G_1] \ast \phi = V[G_0, \mu_0, G_2] \circ \phi^*, \]
\[ \psi \ast V[G_0, \mu_0, G_1] = V[\psi(G_0), \psi_* \mu_0, G_1], \]

where \( \phi^*: \mathcal{J}(G_1) \to \mathcal{J}(G_2), \Gamma \mapsto \phi(\Gamma) \).

**Proposition 3.5.** With the same notation as above we have:

\[(3.3a) \quad V[G_0, \mu_0, G_1] \ast \phi = V[G_0, \mu_0, G_1], \]
\[(3.3b) \quad \psi \ast V[G_0, \mu_0, G_1] = V[G_0, \mu_0, G_1]. \]

**Proof.** Let \( \Gamma \in \mathcal{J}(G_1) \). Let \( F \) be a fundamental domain for \( G_1/\Gamma \) and compute

\[ V[G_0, \mu_0, G_1] \ast \phi(\Gamma) = V[G_0, \mu_0, G_2](\phi(\Gamma)) \]
\[ = V_{G_2}^{\phi_0}(\phi(\Gamma)) \quad (\phi \in \mathcal{R}(G_0, G_2)) \]
\[ = \varphi_* \mu_0(\phi(F)) \]
\[ = (\phi^{-1} \varphi)_* \mu_0(F) \]
\[ = V_{G_1}^{(\phi^{-1} \varphi)_* \mu_0}(F) \]
\[ = V[G_0, \mu_0, G_1](\Gamma). \]

Similarly, we prove (3.3b). \( \square \)

We come now to the principal theorem of this paper, in which we give an explicit Euler characteristic on \( \text{TUF} \).

**Theorem 3.6.** The mapping

\[ \chi: \text{TUF} \to \mathbb{R}, \quad \Gamma \mapsto V[G_0, \mu_0, G_{\Gamma}](\Gamma), \]

where \( \{G_0\} = \{G_{\Gamma}\} \cap T \), is an Euler characteristic on \( \text{TUF} \), which does not depend of the choice of transversal.

**Proof.** Let \( \Gamma \) be a \( \text{TUF} \)-group and \( \Gamma_0 \leq \gamma \Gamma \). Then \( \Gamma_0 \) and \( \Gamma \) have the same Hirsch length, it follows that \( g_{\Gamma_0} = g_{\Gamma} \) and hence the class \( \text{TUF} \) is closed under taking subgroups of finite index. Let \( \Gamma_1 \) and \( \Gamma_2 \) be two isomorphic \( \text{TUF} \)-groups. By Theorem 2.2,
the Lie groups $G_{\Gamma_1}$ and $G_{\Gamma_2}$ are isomorphic and hence $[G_{\Gamma_1}] = [G_{\Gamma_2}]$. Let

$$\{G_0\} = [G_{\Gamma_1}] \cap T.$$  

Let $\phi \in \mathcal{R}(\Gamma_1, \Gamma_2)$ and compute

$$\chi(\Gamma_2) = V[G_0, \mu_0, G_{\Gamma_2}](\Gamma_2)$$

$$= V[G_0, \mu_0, G_{\Gamma_1}] \ast M(\phi)(\Gamma_1)$$

$$= V[G_0, \mu_0, G_{\Gamma_1}](\Gamma_1) \quad \text{(by (3.3a))}$$

$$= \chi(\Gamma_1).$$

This completes the proof of (Ec1). (Ec2) follows from (3.1). Finally, the formula (3.3b) implies that the mapping $\chi$ is independent of the choice of the transversal $T$. \qed 

References

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