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Osaka University
EULER CHARACTERISTICS ON
A CLASS OF FINITELY GENERATED NILPOTENT GROUPS

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Abstract

A finitely generated torsion free nilpotent group is called an \( \mathcal{F} \)-group. To each \( \mathcal{F} \)-group \( \Gamma \) there is associated a connected, simply connected nilpotent Lie group \( G_\Gamma \). Let \( TUF \) be the class of all \( \mathcal{F} \)-group \( \Gamma \) such that \( G_\Gamma \) is totally unimodular. A group in \( TUF \) is called \( TUF \)-group. In this paper, we are interested in finding non-zero Euler characteristic on the class \( TUF \) and therefore, on \( TUFF \), the class of groups \( K \) having a subgroup \( \Gamma \) of finite index in \( TUF \). An immediate consequence we obtain that any two isomorphic finite index subgroups of a \( TUFF \)-group have the same index. As applications, we give two results, the first is a generalization of Belegradek’s result, in which we prove that every \( TUFF \)-group is co-hopfian. The second is a known result due to G.C. Smith, asserting that every \( TUFF \)-group is not compressible.

1. Introduction and main results

We follow [5, p.222] in defining an Euler characteristic on a class of groups as follows (see also [2, p.1]).

**Definition 1.1 (Euler characteristic).** Let \( \mathcal{X} \) be a class of groups closed under taking subgroups of finite index. By an Euler characteristic on \( \mathcal{X} \) it meant a function \( \chi : \mathcal{X} \to \mathbb{R} \) satisfying

1. (Ec1) If \( K \) and \( H \) are in \( \mathcal{X} \), and \( K \) is isomorphic to \( H \), then \( \chi(K) = \chi(H) \).
2. (Ec2) If \( K \) is in \( \mathcal{X} \), and \( H \) is a subgroup of \( K \) of finite index, then \( \chi(H) = [K : H] \chi(K) \), where \([K : H]\) denotes the index of \( H \) in \( K \).

In this paper, we are interested in finding non zero Euler characteristics defined on a class of finitely generated nilpotent groups.

Let \( G \) be a connected Lie group and \( \text{Aut}(G) \) its group of continuous automorphisms. Let \( \mu \) be a Haar measure on \( G \). For every \( \alpha \in \text{Aut}(G) \) we have

\[
\alpha^{-1}_* \mu = \Delta(\alpha) \mu.
\]

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where $\alpha^{-1}\mu$ is the push forward of $\mu$ under $\alpha^{-1}$, and $\Delta: \text{Aut}(G) \to \mathbb{R}^+_{\times}$ is a homomorphism of $\text{Aut}(G)$ into the multiplicative group of the positive reals. If $G$ is a connected, simply connected nilpotent Lie group, then

$$\Delta(\alpha) = |\det(\alpha)|.$$

**Definition 1.2 ([10, p. 627]).** A connected, simply connected nilpotent Lie group $G$ is called totally unimodular if the image of $\Delta$ is $\{1\}$.

Let $\text{TULG}$ be the class of connected, simply connected totally unimodular nilpotent Lie groups.

A real Lie algebra is called *characteristically nilpotent* if all its derivations are nilpotent ([4, p. 157], [6, p. 623]). We note that a characteristically nilpotent Lie algebra is nilpotent. Let $\text{CNLG}$ be the class of connected, simply connected nilpotent Lie groups $G = \exp \mathfrak{g}$ such that $\mathfrak{g}$ is a characteristically nilpotent Lie algebra.

**Proposition 1.3 ([10, (1.1)]).** We have

$$\text{CNLG} \subset \text{TULG}.$$

A finitely generated torsion free nilpotent group is called an $\mathcal{F}$-group. Any $\mathcal{F}$-group $\Gamma$ is isomorphic to a discrete uniform subgroup of a connected, simply connected nilpotent Lie group $G_\Gamma$ whose Lie algebra $\mathfrak{g}_\Gamma$ has rational structure constants ([8, Theorem 6]). Let $\text{TUF}$ be the class of all $\mathcal{F}$-groups $\Gamma$ such that $G_\Gamma \in \text{TULG}$. We call a group $\Gamma$ a $\text{TUF}$-group if $\Gamma \in \text{TUF}$. For every integer $n \geq 7$ there exists a $n$-dimensional characteristically nilpotent Lie algebra with rational structure ([14, Theorem 5]). By the Mal’cev rationality criterion (Theorem 2.1) we derive the following.

**Proposition 1.4.** For every integer $n \geq 7$ there exists a $\text{TUF}$-group with Hirsch length $n$.

The main result of this paper is the following.

**Theorem 1.5.** The class $\text{TUF}$ admits Euler characteristics.

In Section 3, we give an explicit Euler characteristic on $\text{TUF}$.

By [5, p. 222] (see also [15], [2]) every Euler characteristic $\chi$ on $\text{TUF}$ can be extended to $\text{TUFF}$, the class of groups $K$ having a subgroup $\Gamma$ of finite index in $\text{TUF}$, by setting

$$\chi(K) = \frac{1}{[K : \Gamma]} \chi(\Gamma).$$

As an immediate consequence we have the following.
Proposition 1.6. Any two isomorphic finite index subgroups of a TUFF-group have the same index.

Definition 1.7 (Co-hopfian group). A group is called co-hopfian if it satisfies the following equivalent conditions:
1. It is not isomorphic to any proper subgroup.
2. Every injective endomorphism of the group is an automorphism.

As an easy consequence of Proposition 1.6, we obtain a generalization for I. Belegradek’s result ([1, Corollary 2.4]).

Proposition 1.8. Every TUFF-group is co-hopfian.

We introduce the following definition due to G.C. Smith ([13, Definition 1]).

Definition 1.9 (Compressible group). A group $G$ is called compressible if any finite index subgroup of $G$ contains a finite index subgroup isomorphic to $G$.

The following proposition which is due to G.C. Smith ([13, Proposition 4]) is an immediate consequence of Proposition 1.6.

Proposition 1.10. Every TUFF-group is not compressible.

2. Rational structures and discrete uniform subgroups

General references for the material in this section are [3] and [11] as well as the original paper of Mal’cev [8].

Let $G$ be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then the exponential map $\exp: \mathfrak{g} \to G$ is a diffeomorphism. Let $\log: G \to \mathfrak{g}$ denote the inverse of $\exp$.

2.1. Rational structures. Let $G$ be a nilpotent, connected and simply connected real Lie group and let $\mathfrak{g}$ be its Lie algebra. We say that $\mathfrak{g}$ (or $G$) has a rational structure if there is a Lie algebra $\mathfrak{g}(\mathbb{Q})$ over $\mathbb{Q}$ such that $\mathfrak{g} \cong \mathfrak{g}(\mathbb{Q}) \otimes \mathbb{R}$. It is clear that $\mathfrak{g}$ has a rational structure if and only if $\mathfrak{g}$ has an $\mathbb{R}$-basis $(X_1, \ldots, X_n)$ with rational structure constants.

2.2. Uniform subgroups. A discrete subgroup $\Gamma$ is called uniform in $G$ if the quotient space $G/\Gamma$ is compact. A proof of the next result can be found in Theorem 7 of [8] or in Theorem 2.12 of [11].
**Theorem 2.1** (The Malcev rationality criterion). Let $G$ be a simply connected nilpotent Lie group, and let $g$ be its Lie algebra. Then $G$ admits a uniform subgroup $\Gamma$ if and only if $g$ admits a basis $(X_1, \ldots, X_n)$ such that

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ijk} X_k, \quad (\forall 1 \leq i, j \leq n),$$

where the constants $c_{ijk}$ are all rational.

2.3. The Malcev rigidity theorem. The following is a theorem of Mal’cev ([8, Theorem 5]); see also ([9, Theorem 4]).

**Theorem 2.2** (Malcev rigidity theorem). Let $G_1$ and $G_2$ be connected simply connected nilpotent Lie groups and $\Gamma_1$, $\Gamma_2$ discrete uniform subgroups of $G_1$ and $G_2$. Any abstract group isomorphism $\phi$ between $\Gamma_1$ and $\Gamma_2$ extends uniquely to an isomorphism $M(\phi)$ of $G_1$ on $G_2$; that is, the following diagram

(2.1)

is commutative, where $i$ is the inclusion mapping. The isomorphism $M(\phi)$ is called the Mal’cev extension of $\phi$.

3. An explicit Euler characteristic on TUF. Proof of Theorem 1.5

Let $G$ be a connected Lie group, $\mathcal{U}(G)$ be the space of discrete uniform (i.e., cocompact) subgroups of $G$. Let $\mu$ be a right Haar measure of $G$. Let $\Gamma \in \mathcal{U}(G)$, the measure $\mu$ induces a finite measure $\tilde{\mu}$ over the homogeneous space $G/\Gamma$. Let

$$V^\mu_G : \mathcal{U}(G) \to \mathbb{R}_+$$

defined for $\Gamma \in \mathcal{U}(G)$ by

$$V^\mu_G(\Gamma) = \tilde{\mu}(G/\Gamma).$$

**Remark 3.1.** We recall that if $F$ is a fundamental domain for $G/\Gamma$ then $\tilde{\mu}(G/\Gamma) = \mu(F)$ ([7, p. 430]).

The notation $H \leq_f K$ signifies that $H$ is a finite index subgroup of the group $K$. A proof of the following proposition can be found in Lemma 3.2 of [7].
Proposition 3.2. If $H, K \in \mathcal{S}(G)$ and if $H \leq_f K$ then we have

\begin{equation}
V^\mu_G(H) = [K : H]V^\mu_G(K).
\end{equation}

Proposition 3.3. Let $G$ in $\text{TULG}$ and $\mu$ a Haar measure on $G$. Let $\Gamma_1, \Gamma_2$ be two isomorphic subgroups of $\mathcal{S}(G)$. Then we have

\begin{equation}
V^\mu_G(\Gamma_1) = V^\mu_G(\Gamma_2).
\end{equation}

Proof. Let $\phi$ be an isomorphism of $\Gamma_1$ onto $\Gamma_2$. Let $F$ be a fundamental domain of $G/\Gamma_1$ and compute

\[
V^\mu_G(\Gamma_1) = \mu(F) = \mu(M(\phi)(F)) = V^\mu_G(\Gamma_2).\]

We define an equivalence relation $\simeq$ on $\text{TULG}$ by

\[G_1 \simeq G_2 \iff G_1, G_2 \text{ are isomorphic.}\]

For $G \in \text{TULG}$, let $[G]$ be the equivalence class containing $G$. Let $T$ be a transversal for the equivalence relation $\simeq$.

Let $H, K$ be two groups (resp. Lie groups), the set of all isomorphisms (resp. Lie groups isomorphisms) of $H$ onto $K$ is denoted by $\mathbb{R}(H, K)$.

Lemma 3.4. Let $G_0 \in T$ and $G \in [G_0]$. For every $\phi, \psi \in \mathbb{R}(G_0, G)$ we have

\[\phi_*\mu_0 = \psi_*\mu_0,\]

where $\phi_*\mu_0$ (resp. $\psi_*\mu_0$) is the push forward of $\mu_0$ under $\phi$ (resp. $\psi$).

Proof. Let $F$ be a measurable set and compute

\[
\phi_*\mu_0(F) = \mu_0(\phi^{-1}(F)) = \mu_0(\psi^{-1}\phi(\phi^{-1}(F))) = \mu_0(\psi^{-1}(F)) = \psi_*\mu_0(F).\]

Let $G_0 \in T$ and $\mu_0$ a Haar measure on $G_0$. Let $G \in [G_0]$ and $\Gamma \in \mathcal{S}(G)$. The function

\[\mathbb{R}(G_0, G) \to \mathbb{R}, \quad \phi \mapsto V^\phi_{G,G_0}(\Gamma)\]
is constant. In the sequel, we note

\[ V[G_0, \mu_0, G](\Gamma) = V^{\phi_*\mu_0}(\Gamma) \quad (\forall \phi \in \mathcal{R}(G_0, G)). \]

Let \( G_1, G_2 \in [G_0] \). For every \( \psi \in \mathcal{R}(G_0, G_2) \) and \( \phi \in \mathcal{R}(G_1, G_2) \), we note

\[ V[G_0, \mu_0, G_1] * \phi = V[G_0, \mu_0, G_2] \circ \phi^*, \]

\[ \psi \ast V[G_0, \mu_0, G_1] = V[\psi(G_0), \psi_*\mu_0, G_1], \]

where \( \phi^* : \mathcal{R}(G_1) \to \mathcal{R}(G_2), \Gamma \mapsto \phi(\Gamma) \).

**Proposition 3.5.** With the same notation as above we have:

\begin{align}
(3.3a) & \quad V[G_0, \mu_0, G_1] * \phi = V[G_0, \mu_0, G_1], \\
(3.3b) & \quad \psi \ast V[G_0, \mu_0, G_1] = V[G_0, \mu_0, G_1].
\end{align}

**Proof.** Let \( \Gamma \in \mathcal{S}(G_1) \). Let \( F \) be a fundamental domain for \( G_1/\Gamma \) and compute

\[ V[G_0, \mu_0, G_1] * \phi(\Gamma) = V[G_0, \mu_0, G_2](\phi(\Gamma)) \]

\[ = V^{\phi_*\mu_0}(\phi(\Gamma)) \quad (\phi \in \mathcal{R}(G_0, G_2)) \]

\[ = \varphi_*\mu_0(\phi(F)) \]

\[ = (\phi^{-1}\varphi)_*\mu_0(F) \]

\[ = V^{\phi_*\mu_0}(\phi(F)) \]

\[ = V[G_0, \mu_0, G_1](\Gamma). \]

Similarly, we prove (3.3b). \qed

We come now to the principal theorem of this paper, in which we give an explicit Euler characteristic on \( \text{TUF} \).

**Theorem 3.6.** The mapping

\[ \chi : \text{TUF} \to \mathbb{R}, \quad \Gamma \mapsto V[G_0, \mu_0, G_\Gamma](\Gamma), \]

where \( \{G_0\} = \{G_\Gamma\} \cap T \), is an Euler characteristic on \( \text{TUF} \), which does not depend of the choice of transversal.

**Proof.** Let \( \Gamma \) be a \( \text{TUF} \)-group and \( \Gamma_0 \leq_T \Gamma \). Then \( \Gamma_0 \) and \( \Gamma \) have the same Hirsch length, it follows that \( g_{\Gamma_0} = g_{\Gamma} \) and hence the class \( \text{TUF} \) is closed under taking subgroups of finite index. Let \( \Gamma_1 \) and \( \Gamma_2 \) be two isomorphic \( \text{TUF} \)-groups. By Theorem 2.2,
the Lie groups $G_{\Gamma_1}$ and $G_{\Gamma_2}$ are isomorphic and hence $[G_{\Gamma_1}] = [G_{\Gamma_2}]$. Let

$$\{G_0\} = [G_{\Gamma_1}] \cap T.$$ 

Let $\phi \in R(\Gamma_1, \Gamma_2)$ and compute

$$\chi(\Gamma_2) = V[G_0, \mu_0, G_{\Gamma_2}](\Gamma_2)$$
$$= V[G_0, \mu_0, G_{\Gamma_1}] \ast M(\phi)(\Gamma_1)$$
$$= V[G_0, \mu_0, G_{\Gamma_1}](\Gamma_1)$$  \hspace{1cm} (by (3.3a))
$$= \chi(\Gamma_1).$$

This completes the proof of (Ec1). (Ec2) follows from (3.1). Finally, the formula (3.3b) implies that the mapping $\chi$ is independent of the choice of the transversal $T$. □

References

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