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Osaka University
CLASS NUMBER PARITY OF A QUADRATIC TWIST OF
A CYCLOTOMIC FIELD OF PRIME POWER CONDUCTOR

HUMIO ICHIMURA

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Abstract

Let $p$ be a fixed odd prime number. Let $K_n = \mathbb{Q}(\zeta_{p^n})$ be the $p^{n+1}$-st cyclotomic field for an integer $n \geq 0$, and $K_\infty = \bigcup_n K_n$. Let $d \in \mathbb{Z}$ be a fixed integer with $\sqrt{d} \not\in K_0$. We denote by $L_n$ the imaginary quadratic subextension of the biquadratic extension $K_n(\sqrt{d})/K_n^+$ with $L_n \neq K_n$. Let $h_n^+$ and $h_n^-$ be the relative class numbers of $K_n$ and $L_n$, respectively. We give an explicit constant $n_d$ depending on $p$ and $d$ such that (i) for any integer $n \geq n_d$, the ratio $h_n^-/h_{n-1}^-$ is odd if and only if $h_n^+/h_{n-1}^+$ is odd and (ii) for $1 \leq n < n_d$, $h_n^-/h_{n-1}^-$ is even.

1. Introduction

Let $p$ be a fixed odd prime number. Let $K_n = \mathbb{Q}(\zeta_{p^n})$ be the $p^{n+1}$-st cyclotomic field for an integer $n \geq 0$, and $K_\infty = \bigcup_n K_n$. Let $d \in \mathbb{Z}$ be a fixed integer with $\sqrt{d} \not\in K_0$. We denote by $L_n$ the imaginary quadratic subextension of the biquadratic extension $K_n(\sqrt{d})/K_n^+$ with $L_n \neq K_n$. Here, $K^+$ denotes the maximal real subfield of an imaginary abelian field $K$. When $d < 0$, we have $L_n = K_\infty^+(\sqrt{d})$. We call $L_n$ the quadratic twist of $K_n$ associated to the integer $d$. The extension $L_\infty = \bigcup_n L_n$ is the cyclotomic $\mathbb{Z}_p$-extension over $L_0$ with the $n$-th layer $L_n$. We call $L_\infty/L_0$ the quadratic twist of the cyclotomic $\mathbb{Z}_p$-extension $K_\infty/K_0$ associated to $d$. Let $h_n^+$ and $h_n^-$ be the relative class numbers of $K_n$ and $L_n$, respectively. It is known and easy to show that $h_{n-1}^+$ (resp. $h_{n-1}^-$) divides $h_n^+$ (resp. $h_n^-$) using class field theory. The parity of $h_n^+$ behaves rather irregularly when $p$ varies (see a table in Schoof [6]). However, it is recently shown that when $p \leq 509$, the ratio $h_n^+/h_{n-1}^+$ is odd for all $n \geq 1$ ([3, Theorem 2]). And it might be possible that the ratio is odd for any prime $p$ and any $n \geq 1$. The purpose of this paper is to study the parity of the ratio $h_n^-/h_{n-1}^-$ of the quadratic twist $L_n$. We already know that $h_n^-/h_{n-1}^-$ is odd for sufficiently large $n$ by a theorem of Washington [8] on the non-$p$-part of the class number in a cyclotomic $\mathbb{Z}_p$-extension. Denote by $S = S_d$ the set of prime numbers $l \neq p$ which ramify in $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. The set $S$ is non-empty as $\sqrt{d} \not\in K_0$. We define an integer $n_d \geq 1$ by

$$n_d = \max\{\text{ord}_p(l^{p-1} - 1) \mid l \in S\},$$

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where \( \text{ord}_p(*) \) is the normalized \( p \)-adic additive valuation. The following is the main theorem of this paper.

**Theorem 1.** Under the above setting, the following assertions hold.

(I) When \( n \geq n_d \), the ratio \( h^{-}_n/h^{-}_{n-1} \) is odd if and only if \( h^*_n/h^*_{n-1} \) is odd.

(II) When \( n_d \geq 2 \) and \( 1 \leq n < n_d \), the ratio \( h^*_n/h^*_{n-1} \) is even.

From Theorem 1 and [3, Theorem 2], we immediately obtain the following:

**Corollary 1.** Under the above setting, let \( p \) be an odd prime number with \( p \equiv 509 \). Then the ratio \( h^{-}_n/h^{-}_{n-1} \) is odd for all \( n \geq n_d \).

This corollary, though given in a very special setting, is an explicit version of the above mentioned theorem of Washington. In [4], we showed Theorem 1 when \( d = -1 \) and \( L_n = K_n^+(\sqrt{-1}) \) using some results of cyclotomic Iwasawa theory. In this paper, we prove Theorem 1 by using a main theorem of Conner and Hurrelbrink [1, Theorem 2.3].

**Remark.** When \( p \equiv 1 \mod 4 \) (resp. \( p \equiv 3 \mod 4 \)), we can show that two integers \( d_1 \) and \( d_2 \) give the same twist \( L_\infty/L_0 \) of \( K_\infty/K_0 \) if and only if \( d_2 = d_1x^2 \) or \( d_2 = pd_1x^2 \) (resp. \( d_2 = -pd_1x^2 \)) for some \( x \in \mathbb{Q}^\times \). Hence, the set \( S_d \) and the integer \( n_d \) depend only on the twist \( L_\infty/L_0 \) and not on the choice of \( d \).

2. Exact hexagon of Conner and Hurrelbrink

In this section, we recall the exact hexagon of Conner and Hurrelbrink. Let \( k \) be an imaginary abelian field with 2-power degree, and \( F \) a real abelian field with \( 2 \nmid [F:\mathbb{Q}] \). We put \( K = kF \), and

\[
G = \text{Gal}(K/k) = \text{Gal}(K^+/k^+) = \text{Gal}(F/\mathbb{Q}).
\]

For a number field \( N \), let \( A_N \) be the 2-part of the ideal class group of \( N \), \( \mathcal{O}_N \) the ring of integers, and \( E_N = \mathcal{O}_N^\times \) the group of units of \( N \). The groups \( A_K \) and \( E_K \) are naturally regarded as modules over \( \text{Gal}(K/K^+) \) and at the same time as those over \( G \). For a \( \text{Gal}(K/K^+) \)-module \( X \), denote by \( H^i(X) = H^i(K/K^+;X) \) the Tate cohomology group with \( i = 0, 1 \). When \( X = A_K \) or \( E_K \), the group \( H^i(X) \) is also regarded as \( G \)-modules. In [1, Theorem 2.3], Conner and Hurrelbrink introduced the following exact hexagon
of $G$-modules to study the 2-part of the class number of a relative quadratic extension.

$$
\begin{array}{ccc}
H^1(A_K) & \longrightarrow & H^1(E_K) \\
R^0(K) & \longleftarrow & R^1(K) \\
\downarrow & & \downarrow
\end{array}
$$

Here, $R^i(K)$ is a certain $G$-module associated to $K/K^+$ defined in [1]. We describe the $G$-module structure of $R^i(K)$ following [1]. Let $T_f$ be the set of prime ideals $\wp$ of $k^+$ for which a prime ideal $\mathfrak{P}$ of $K^+$ over $\wp$ ramifies in $K$. Let $T_\infty$ be the set of infinite prime divisors of $k^+$. We put $T = T_f \cup T_\infty$. For each $v \in T$, let $G_v \subseteq G$ be the decomposition group of $v$ at $K^+/k^+$. When $v$ is an infinite prime, the group $G_v$ is trivial. We define $G$-modules $\Omega_f$ and $\Omega_\infty$ by

$$
\Omega_f = \bigoplus_{\wp \in T_f} F_2[G/G_{\wp}] \quad \text{and} \quad \Omega_\infty = \bigoplus_{v \in T_\infty} F_2[G/G_v] = \bigoplus_{v \in T_\infty} F_2[G],
$$

respectively, where $F_2 = \mathbb{Z}/2\mathbb{Z}$ is the finite field with two elements. (When $T_f$ is empty, $\Omega_f = \{0\}$ by definition.) For each prime divisor $w$ of $K^+$ with the restriction $w|k^+ \in T$ and an element $x \in (K^+)^\times$, we put $t_w(x) = 0$ or 1 according as $x \in N(K_w^\times)$ or not. Here, $K_w$ is the completion of $K$ at the unique prime divisor of $K$ over $w$ and $N = N_{K/K^+}$ is the norm map. For $g \in G$ and $x \in (K^+)^\times$, we see that

$$
t_{w^t}(x) = t_w(x^{g^{-1}})
$$

by local class field theory. For a prime ideal $\mathfrak{P}$ of $K^+$ with $\mathfrak{P} \cap k^+ \in T_f$, let $\widehat{\mathfrak{P}}$ be the unique prime ideal of $K$ over $\mathfrak{P}$. For an ideal $\mathfrak{A}$ of $K$, writing $\mathfrak{A} = \widehat{\mathfrak{P}} \mathfrak{B}$ with an integer $e$ and an ideal $\mathfrak{B}$ relatively prime to $\widehat{\mathfrak{P}}$, we put $\text{ord}_{\widehat{\mathfrak{P}}}(\mathfrak{A}) = e$.

We denote by $I(K)$ the group of (fractional) ideals of $K$. Let $X$ be the subgroup of $I(K)$ consisting of ideals $\mathfrak{A}$ with $\mathfrak{A}^J = \mathfrak{A}$. Here, $J$ is the complex conjugation acting on several objects associated to $K$. Let $X_0$ be the subgroup of $X$ consisting of ideals $\mathfrak{A} \in I(K)$ with $\mathfrak{A} = x \mathfrak{B}^{1/J}$ for some $x \in (K^+)^\times$ and $\mathfrak{B} \in I(K)$. The $G$-module $R^1(K)$ is isomorphic to the quotient $X/X_0$. For this, see the lines 1–2 from the bottom of p.6 and Lemma 2.1 of [1]. For each prime ideal $\wp \in T_f$, we fix a prime ideal $\mathfrak{P}$ of $K^+$ over $\wp$. From the argument in [1, §5], we obtain the following isomorphism of $G$-modules:

$$
R^1(K) \cong \Omega_f; \quad \mathfrak{A}X_0 \rightarrow \bigoplus_{\wp \in T_f} \left( \sum_{\tilde{g}} \text{ord}_{\mathfrak{P}}(\mathfrak{A})\tilde{g} \right),
$$

where $\tilde{g}$ (with $g \in G$) runs over the quotient $G/G_{\wp}$.
Let $Y$ be the subgroup of the multiplicative group $(K^+) \times I(K)$ consisting of pairs $(x, \mathfrak{A})$ with $x \mathfrak{A}^{1 + J} = \mathcal{O}_K$. Let $Y_0$ be the subgroup of $Y$ consisting of pairs $(N(y),y^{-1} \mathfrak{B}^{1 + J})$ with $y \in K^*$ and $\mathfrak{B} \in I(K)$. By definition, $R^0(K) = Y/Y_0$. We denote by $[x, \mathfrak{A}] \in R^0(K)$ the class containing $(x, \mathfrak{A})$. The map $i_0$ in the hexagon is defined by

$$i_0: H^0(E_K) = E_{K^+}/N(E_K) \to R^0(K); \quad [\epsilon] \to [\epsilon, \mathcal{O}_K]$$

with $\epsilon \in E_{K^+}$. For each $v \in T_\infty$, we fix a prime divisor $\tilde{v}$ of $K^+$ over $v$. Using (1), we observe that the homomorphisms

$$\alpha_\infty: (K^+) \to \Omega_\infty; \quad x \to \bigoplus_{v \in T_\infty} \left( \sum_{g \in G} \lambda_{\tilde{v}}(x) g \right)$$

and

$$\alpha_f: (K^+) \to \Omega_f; \quad x \to \bigoplus_{v \in T_f} \left( \sum_{\tilde{g} \in \mathcal{G}} \lambda_{\tilde{v}}(x) \tilde{g} \right)$$

are compatible with the action of $G$. Further, $\alpha_\infty$ is nothing but the “sign” map. From the argument in [1, §4], we obtain the following exact sequence of $G$-modules:

$$(3) \quad \{0\} \to R^0(K) \xrightarrow{\alpha} \Omega_f \oplus \Omega_\infty \xrightarrow{\beta} F_2 \to \{0\}.$$ 

Here, $\alpha$ is defined by $\alpha([x, \mathfrak{A}]) = (\alpha_f(x), \alpha_\infty(x))$, $\beta$ is the argumentation map and $G$ acts trivially on $F_2$.

### 3. Consequences

In this section, we derive some consequences of the exact hexagon and (2), (3). All of them are $G$-decomposed versions of the corresponding results in [1]. We work under the setting of Section 2. Denote by $\bar{A}_{K^+}$ the 2-part of the narrow class group of $K^+$. Letting $K_{0}^{+}$ be the group of totally positive elements of $K^+$, we have an exact sequence

$$(4) \quad \{0\} \to (K^+)^\times/(K_{0}^{+}E_{K^+}) \to \bar{A}_{K^+} \to A_{K^+} \to \{0\}$$

of $G$-modules. We define the minus class group $A_{K}^{-}$ to be the kernel of the norm map $A_{K} \to A_{K^+}$. Let $\chi$ be a $\bar{\mathbb{Q}}_2$-valued character of $G = \text{Gal}(K/k) = \text{Gal}(F/\mathbb{Q})$, which we also regard as a primitive Dirichlet character. For a module $M$ over $\mathbb{Z}_2[G]$, we denote by $M(\chi)$ the $\chi$-part of $M$. Here, $\mathbb{Z}_2$ is the ring of 2-adic integers and $\bar{\mathbb{Q}}_2$ is a fixed algebraic closure of the 2-adic rationals $\mathbb{Q}_2$. (For the definition of the $\chi$-part and some of its properties, see Tsuji [7, §2].) Denote by $S_K$ the set of prime numbers lying
below some prime ideal in $T_f$. In all what follows, we assume that $\chi$ is a nontrivial character. The following is a version of [1, Theorem 13.8].

**Theorem 2.** Under the above setting, the groups $H^i(K/K^i; A_K)(\chi)$ with $i = 0$ and $1$ are trivial if and only if
(i) $\chi(l) \neq 1$ for all $l \in S_K$ and
(ii) $|\tilde{A}_K(\chi)| = |A_K(\chi)|$.

The following corollary is a version of [1, Corollary 13.10] and Hasse [2, Satz 45].

**Corollary 2.** Under the above setting, the group $A^-_K(\chi)$ is trivial if and only if
(i) $\chi(l) \neq 1$ for all $l \in S_K$ and
(ii) $\tilde{A}_K(\chi)$ is trivial.

Let $\tilde{h}_M$ be the class number in the narrow sense of a number field $M$. When $M$ is an imaginary abelian field, let $h^-_M$ be the relative class number of $M$. We can easily show that $h^-_K$ (resp. $\tilde{h}_K(\chi)$) divides $\tilde{h}_K$ (resp. $\tilde{h}_K(\chi)$) using class field theory. The following is an immediate consequence of Corollary 2.

**Corollary 3.** Under the above setting, the ratio $h^-_K/h^-_K$ is odd if and only if
(i) no prime number $l$ in $S_K$ splits in $F$ and
(ii) $\tilde{h}_K(\chi)$ is odd.

To prove these assertions, we prepare the following two lemmas. For a number field $L$, let $\mu(L)$ be the group of roots of unity in $L$ and $\mu_2(L)$ the 2-part of $\mu(L)$.

**Lemma 1.** The group $H^1(K/K^+; E_K)(\chi)$ is trivial.

Proof. Let $N E_K$ be the group of units $\epsilon \in E_K$ with $N(\epsilon) = \epsilon^{1+j} = 1$. We have $N(\epsilon) = 1$ if and only if $\epsilon \in \mu(K)$ by a theorem on units of a CM-field (cf. Washington [9, Theorem 4.12]). Since $\mu(K)^2 = \mu(K)^{1+j} \subseteq E_K^{1+j}$, we obtain a surjection

$$\mu(K)/\mu(K)^2 \rightarrow H^1(K/K^+; E_K) = NE_K/\mu_2(K)^{1+j}$$

of $G$-modules. However, as $[K:k]$ is odd, we have

$$\mu(K)/\mu(K)^2 = \mu_2(K)/\mu_2(K)^2 = \mu_2(k)/\mu_2(k)^2.$$  

Since $\chi$ is nontrivial, the $\chi$-part $\mu_2(k)/\mu_2(k)^2(\chi)$ is trivial. Hence, we obtain the assertion.

**Lemma 2.** The natural map $A^-_K(\chi) \rightarrow A^-_K(\chi)$ is injective.
Proof. Denote the natural map $A_{K^+} \to A_K$ by $i$. Let $\mathfrak{A}$ be an ideal of $K^+$ with the class $[\mathfrak{A}] \in \text{ker} \ i$. Then $\mathfrak{A}O_K = xO_K$ for some $x \in K^\times$. We see that $\epsilon = x^{1-\lambda}$ is a unit of $K$ with $N(\epsilon) = 1$. It is known that the map

$$\text{ker} \ i \to H^1(K/K^+; E_K); [\mathfrak{A}] \to x^{1-\lambda}E_K^{1-\lambda}$$

is an injective $G$-homomorphism ([1, Theorem 7.1]). Then, from Lemma 1, we see that the $\chi$-part $(\text{ker} \ i)(\chi)$ is trivial, from which we obtain the assertion.  \[\square\]

Proof of Theorem 2. Let $\wp$ be a prime ideal in $T_\ell$, and $l = \wp \cap Q \in S_K$. We see that the $\chi$-part $F_2[G/G_{\wp}]$ of $Q$ is $[0] \neq 0$ if and only if $\chi$ factors through $G/G_{\wp}$, which is equivalent to $\chi(G_{\wp}) = \{1\}$. Since $[k^+: Q]$ is a 2-power and $[F : Q]$ is odd, we have $\chi(G_{\wp}) = \{1\}$ if and only if $\chi(l) = 1$. Hence, we have shown that the condition (i) in Theorem 2 is equivalent to the condition $\Omega_f(\chi) = \{0\}$. By the hexagon and Lemma 1, we see that $H^0(A_K)(\chi)$ and $H^1(A_K)(\chi)$ are trivial if and only if (iii) $R^1(K)(\chi) = \{0\}$ and (iv) the map

$$i_0 : H^0(E_K)(\chi) = (E_{K^+}/N(E_K))(\chi) \to R^0(K)(\chi)$$

is an isomorphism. By (2) and the above, the condition (iii) is equivalent to (i). Under the condition (i), we see that $R^0(K)(\chi) = \Omega_\infty(\chi)$ from the exact sequence (3), and that for each class $[\epsilon] \in H^0(E_K)(\chi)$ with $\epsilon \in E_{K^+}$, we have $i_0([\epsilon]) = \alpha_\infty(\epsilon)$ from the definitions of the maps $i_0$ and $\alpha$. Further, the 2-rank of $\Omega_\infty(\chi)$ is larger than or equal to that of $H^0(E_K)(\chi)$ by a theorem of Minkowski on units of a Galois extension (cf. Narkiewicz [5, Theorem 3.26]). Therefore, under (i), we observe that the condition (iv) holds if and only if $\alpha_\infty(E_{K^+})(\chi) = \Omega_\infty(\chi)$. We see that the last condition is equivalent to the condition (ii) in Theorem 2 because of the exact sequence (4) and $\alpha_\infty((K^+)\chi) = \Omega_\infty(\chi)$. Therefore, we obtain Theorem 2.  \[\square\]

Proof of Corollary 2. First, we show the “only if” part assuming that $A_K^-(\chi)$ is trivial. By Lemma 2, we can regard $A_K^+(\chi)$ as a subgroup of $A_K(\chi)$. Assume that $A_K^+(\chi)$ is nontrivial. Then there exists a class $c \in A_K^+(\chi)$ of order 2. We have $c^2 = c = c^{-1}$, and hence $c \in A_K^-(\chi)$. It follows that $A_K^-(\chi)$ is nontrivial, a contradiction. Hence, $A_K^+(\chi) = \{0\}$. It follows that $A_K(\chi)$ is trivial by the exact sequence

$$\{0\} \to A_K^-(\chi) \to A_K(\chi) \xrightarrow{1+J} A_K^+(\chi) \to \{0\}.$$

Therefore, the “only if” part follows from Theorem 2. Next, assume that the conditions (i) and (ii) in Corollary 2 are satisfied. Then, $A_K^+(\chi) = \{0\}$, and the groups $H^i(A_K)(\chi)$ ($i = 0, 1$) are trivial by Theorem 2. As the cohomology groups are trivial, we obtain an exact sequence

$$\{0\} \to A_K^+(\chi) \to A_K(\chi) \xrightarrow{1-J} A_K^- (\chi) = A_K^- (\chi) \to \{0\}.$$
Since $A_{K^+}(\chi) = \{0\}$, we see that $A_K(\chi) = A_{K^+}(\chi)$, and

$$A_{K^+}(\chi) = A_{K^+}(\chi)^{1-J} = A_K(\chi)^2$$

from the above exact sequence. Therefore, $A_K(\chi)$ is trivial. \hfill \Box

4. Proof of Theorem 1

We use the same notation as in Section 1. In particular, $d \in \mathbb{Z}$ is a fixed integer with $\sqrt{d} \not\in K_0$ and $L_n$ is the quadratic twist of $K_n$ associated to $d$. We have $L_n^+ = K_n^+$. Let $k$ (resp. $k_d$) be the maximal intermediate field of $K_0/\mathbb{Q}$ (resp. $L_0/\mathbb{Q}$) of 2-power degree, and let $F_0$ be the maximal subfield of $K_0^+ = L_0^+$ of odd degree over $\mathbb{Q}$. Then $k$ and $k_d$ are imaginary abelian fields with $k^+ = k_d^+$. Let $B_n/\mathbb{Q}$ be the real abelian field with conductor $p^{n+1}$ and $[B_n : \mathbb{Q}] = p^n$. We put $F_n = F_0B_n$. Then $L_n = kdB_n$ and $K_n = kF_n$. The triples $(k_d, F_n, L_n)$ and $(k, F, K)$ correspond to $(k, F, K)$ in Sections 2 and 3. We see that

\begin{equation}
S_{L_n} = S_d \quad \text{or} \quad S_d \cup \{p\}
\end{equation}

and $S_{K_n} = \{p\}$. We put

$$G_n = \text{Gal}(F_n/\mathbb{Q}) = \text{Gal}(L_n/k_d) = \text{Gal}(K_n/k),$$

and

$$\Delta = \text{Gal}(F_0/\mathbb{Q}), \quad \Gamma_n = \text{Gal}(F_n/F_0) = \text{Gal}(B_n/\mathbb{Q}).$$

Then we have a natural decomposition $G_n = \Delta \times \Gamma_n$. For characters $\varphi$ and $\psi$ of $\Delta$ and $\Gamma_n$ respectively, we regard $\varphi\psi = \varphi \times \psi$ as a character of $G_n$. Further, we regard $\varphi$, $\psi$ and $\varphi\psi$ also as primitive Dirichlet characters. The class groups $A_{L_n}^-$, $A_{K_n}^-$ and $\tilde{A}_{K_n}^+$ are modules over $G_n$. We can naturally regard $A_{L_n}^-/A_{L_{n-1}}^-$ as a subgroup of $A_{K_n}^-$ since $L_n/L_{n-1}$ is a cyclic extension of degree $p \neq 2$ and $A_{L_{n-1}}^-$ is the 2-part of the class group. Actually, it is a direct summand of $A_{K_n}^-$ (cf. [9, Lemma 16.15]). We see that

\begin{equation}
A_{L_n}^-/A_{L_{n-1}}^- = \bigoplus_{\varphi, \psi_n} A_{L_n}^-(\varphi\psi_n)
\end{equation}

where $\varphi$ (resp. $\psi_n$) runs over a complete set of representatives of the $\mathbb{Q}_2$-conjugacy classes of the $\mathbb{Q}_2$-valued characters of $\Delta$ (resp. $\Gamma_n$ of order $p^n$). Regarding $A_{K_n}^-$ as a subgroup of $A_{K_n}^-$, we have a similar decomposition for $A_{K_{n-1}}^-/A_{K_{n-1}}^-$. As $S_{K_n} = \{p\}$ and $(\varphi\psi_n)(p) = 0$, we obtain the following assertion from Corollary 2 for the triple $(k, F_n, K_n)$.

**Lemma 3.** Let $n \geq 1$ be an integer, and the characters $\varphi$ and $\psi_n$ be as in (6). Then $A_{K_n}^-(\varphi\psi_n) = \{0\}$ if and only if $\tilde{A}_{K_n}^+(\varphi\psi_n) = \{0\}$. 
Proof of Theorem 1 (I). Let $\varphi$ and $\psi_n$ be as in (6). As the orders of $\varphi$ and $\psi_n$ are relatively prime to each other, we have $(\varphi \psi_n)(l) = 1$ if and only if $\varphi(l) = \psi_n(l) = 1$ for a prime number $l$. Let $n$ be an integer with $n \geq n_d$. Then we have $\psi_n(l) \neq 1$ and hence $(\varphi \psi_n)(l) \neq 1$ for all prime numbers $l \in S = S_d$. Further, we have $(\varphi \psi_n)(p) = 0$. Hence, by (5), the condition (i) in Corollary 2 for the triple $(k_d, F_n, L_n)$ is satisfied.

It follows that the condition $A_{L_d}(\varphi \psi_n) = \{0\}$ is equivalent to $A_{K_n}(\varphi \psi_n) = \{0\}$. (Note that $L_n^{+} = K_n^{+}$.) Therefore, we obtain Theorem 1 (I) from Lemma 3.

To show Theorem 1 (II), assume that $n_d \geq 2$ and let $n$ be an integer with $1 \leq n < n_d$. We put

$$S^{(n)} = \{l \in S = S_d \mid \text{ord}_p(l^p-1) \geq n + 1\}.$$  

From the definition, we see that

$$S \supseteq S^{(1)} \supseteq S^{(2)} \supseteq \cdots \supseteq S^{(n_d-1)}$$

and that each $S^{(n)}$ is non-empty. Let $\varphi$ (resp. $\psi_n$) be a $\bar{Q}_2$-valued character of $\Delta$ (resp. of $\Gamma_n$ of order $p^n$). Denote by $\varphi_0$ the trivial character of $\Delta$. Theorem 1 (II) is a consequence of the following assertion.

**Proposition 1.** Under the above setting, the following hold.

(I) The class group $A_{L_d}(\varphi \psi_n)$ is nontrivial if $\varphi(l) = 1$ for some $l \in S^{(n)}$. In particular, $A_{L_d}(\varphi_0 \psi_n)$ is nontrivial.

(II) If $A_{K_n}(\varphi \psi_n) = \{0\}$, the converse of the first assertion of (I) holds.

Proof. Applying Corollary 2 for the triple $(k_d, F_n, L_n)$, we see from Lemma 3 that $A_{L_d}(\varphi \psi_n) = \{0\}$ if and only if (i) $(\varphi \psi_n)(l) \neq 1$ for all $l \in S = S_d$ and (ii) $A_{L_d}(\varphi \psi_n) = \{0\}$. We have $\psi_n(l) = 1$ for $l \in S^{(n)}$, and $\psi_n(l) \neq 1$ for $l \in S \setminus S^{(n)}$. Therefore, we see that the condition (i) is satisfied if and only if $\varphi(l) \neq 1$ for all $l \in S^{(n)}$ noting that the orders of $\varphi$ and $\psi_n$ are relatively prime. From this, we obtain the proposition.

We put $M_n = K_n(\sqrt{d}) = K_nL_n$. On the relative class number $h_{M_n}$ of $M_n$, the following assertion holds.

**Proposition 2.** (I) When $n \geq n_d$, the ratio $h_{M_n}/h_{M_n}$ is odd if and only if $h_{n}/h_{n-1}$ is odd.

(II) When $n_d \geq 2$ and $1 \leq n < n_d$, $h_{M_n}/h_{M_n}$ is even.

To prove this proposition, we need to show the following lemma. For an imaginary abelian field $N$, we put

$$E_N = E_N/\mu(N)E_N.$$  

It is well known that the unit index $Q_N = |E_N|$ is 1 or 2 ([9, Theorem 4.12]).
**Lemma 4.** Let $T$ and $N$ be imaginary abelian fields with $N \subseteq T$. If the degree $[T : N]$ is odd, then $Q_T = Q_N$.

Proof. We first show that the inclusion map $N \to T$ induces an injection $\mathcal{E}_N \to \mathcal{E}_T$. For any unit $e$ of $N$, assume that $e = \zeta \eta$ for some $\zeta \in \mu(T)$ and $\eta \in \mathfrak{E}_T^+$. Let $\rho$ be a nontrivial element of the Galois group $G = \text{Gal}(T/N)$. Then, as $e = e^\rho$, we see that $\zeta^{1-\rho} = \eta^{\rho-1} \in \mu(T) \cap \mathfrak{E}_T^+$. Hence, $\zeta^{1-\rho} = \pm 1$. However, as $N_{T/N}(\zeta^{1-\rho}) = 1$ and $[T : N]$ is odd, the case $\zeta^{1-\rho} = -1$ does not happen. Hence, $\zeta^{1-\rho} = 1$ for all $\rho \in G$. It follows that $\zeta \in \mu(N)$ and hence $\eta \in \mathfrak{E}_N^+$. Therefore, we can regard $\mathcal{E}_N$ as a subgroup of $\mathcal{E}_T$. In particular, $Q_N$ divides $Q_T$.

Assume that $Q_N \neq Q_T$. Then we have $|\mathcal{E}_T| = |\mathcal{E}_T/\mathcal{E}_N| = 2$. Regarding $\mathcal{E}_T$ as a module over $G$, we have a canonical decomposition

$$\mathcal{E}_T = \mathcal{E}_T/\mathcal{E}_N = \bigoplus_{\chi} \mathcal{E}_T(\chi)$$

where $\chi$ runs over a complete set of representatives of the $\mathbb{Q}_2$-conjugacy classes of the nontrivial $\mathbb{Q}_2$-valued characters of $G$. Hence, $|\mathcal{E}_T(\chi)| = 2$ for some such $\chi$. Let $\mathbb{Z}_2[\chi]$ be the subring of $\mathbb{Q}_2$ generated by the values of $\chi$ over $\mathbb{Z}_2$. The group $\mathcal{E}_T(\chi)$ is naturally regarded as a module over the principal ideal domain $\mathbb{Z}_2[\chi]$. Since the order of $\chi$ is odd and $\geq 3$, we observe that $\mathbb{Z}_2[\chi] \cong \mathbb{Z}_2^d$ as $\mathbb{Z}_2$-modules for some $d \geq 2$. Hence, $|\mathcal{E}_n(\chi)|$ is a multiple of $2^d$, which contradicts $|\mathcal{E}_n(\chi)| = 2$. Therefore, we obtain $Q_N = Q_T$.

Proof of Proposition 2. By Lemma 4, we have $Q_{M_n} = Q_{M_{n-1}}$ and $Q_{L_n} = Q_{L_{n-1}}$ for all $n \geq 1$. Therefore, using the class number formula [9, Theorem 4.17], we see that

$$h_{M_n}/h_{M_{n-1}} = p \prod_{\sigma} \prod_{\psi_n} \left( -\frac{1}{2} B_{1, \sigma, \psi_n} \right)$$

where $\sigma$ runs over the odd Dirichlet characters associated to $M_0$, and $\psi_n$ over the even characters of conductor $p^{n+1}$ and order $p^n$. Further, $B_{1, \sigma, \psi_n}$ denotes the generalized Bernoulli number. We easily see that $\sigma$ equals an odd Dirichlet character associated to $K_0$ or $L_0$ since $M_0/K_0^+$ is an imaginary biquadratic extension with the imaginary quadratic subextensions $K_0$ and $L_0$. Hence, using the class number formulas for $L_n$, $K_n$ and $Q_{L_n} = Q_{L_{n-1}}$, we obtain

$$h_{M_n}/h_{M_{n-1}} = h_n^+/h_{n-1}^+ \times h_n^-/h_{n-1}^-.$$
References