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CLASS NUMBER PARITY OF A QUADRATIC TWIST OF A CYCLOTOMIC FIELD OF PRIME POWER CONDUCTOR

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Abstract

Let p be a fixed odd prime number and K_n the p^{n+1} -st cyclotomic field. For a fixed integer $d \in \mathbf{Z}$ with $\sqrt{d} \notin K_0$, denote by L_n the imaginary quadratic subextension of the biquadratic extension $K_n(\sqrt{d})/K_n^+$ with $L_n \neq K_n$. Let h_n^* and h_n^- be the relative class numbers of K_n and L_n , respectively. We give an explicit constant n_d depending on p and d such that (i) for any integer $n \geq n_d$, the ratio h_n^-/h_{n-1}^- is odd if and only if h_n^*/h_{n-1}^* is odd and (ii) for $1 \leq n < n_d$, h_n^-/h_{n-1}^- is even.

1. Introduction

Let p be a fixed odd prime number. Let $K_n = \mathbf{Q}(\zeta_{p^{n+1}})$ be the p^{n+1} -st cyclotomic field for an integer $n \geq 0$, and $K_\infty = \bigcup_n K_n$. Let $d \in \mathbf{Z}$ be a fixed integer with $\sqrt{d} \notin K_0$. We denote by L_n the imaginary quadratic subextension of the biquadratic extension $K_n(\sqrt{d})/K_n^+$ with $L_n \neq K_n$. Here, K^+ denotes the maximal real subfield of an imaginary abelian field K . When $d < 0$, we have $L_n = K_n^+(\sqrt{d})$. We call L_n the quadratic twist of K_n associated to the integer d . The extension $L_\infty = \bigcup_n L_n$ is the cyclotomic \mathbf{Z}_p -extension over L_0 with the n -th layer L_n . We call L_∞/L_0 the quadratic twist of the cyclotomic \mathbf{Z}_p -extension K_∞/K_0 associated to d . Let h_n^* and h_n^- be the relative class numbers of K_n and L_n , respectively. It is known and easy to show that h_{n-1}^* (resp. h_{n-1}^-) divides h_n^* (resp. h_n^-) using class field theory. The parity of h_0^* behaves rather irregularly when p varies (see a table in Schoof [6]). However, it is recently shown that when $p \leq 509$, the ratio h_n^*/h_{n-1}^* is odd for all $n \geq 1$ ([3, Theorem 2]). And it might be possible that the ratio is odd for any prime p and any $n \geq 1$. The purpose of this paper is to study the parity of the ratio h_n^-/h_{n-1}^- of the quadratic twist L_n . We already know that h_n^-/h_{n-1}^- is odd for sufficiently large n by a theorem of Washington [8] on the non- p -part of the class number in a cyclotomic \mathbf{Z}_p -extension. Denote by $S = S_d$ the set of prime numbers $l \neq p$ which ramify in $\mathbf{Q}(\sqrt{d})/\mathbf{Q}$. The set S is non-empty as $\sqrt{d} \notin K_0$. We define an integer $n_d \geq 1$ by

$$n_d = \max\{\text{ord}_p(l^{p-1} - 1) \mid l \in S\},$$

where $\text{ord}_p(*)$ is the normalized p -adic additive valuation. The following is the main theorem of this paper.

Theorem 1. *Under the above setting, the following assertions hold.*

- (I) *When $n \geq n_d$, the ratio h_n^-/h_{n-1}^- is odd if and only if h_n^*/h_{n-1}^* is odd.*
- (II) *When $n_d \geq 2$ and $1 \leq n < n_d$, the ratio h_n^-/h_{n-1}^- is even.*

From Theorem 1 and [3, Theorem 2], we immediately obtain the following:

Corollary 1. *Under the above setting, let p be an odd prime number with $p \leq 509$. Then the ratio $h_n^-/h_{n_d-1}^-$ is odd for all $n \geq n_d$.*

This corollary, though given in a very special setting, is an explicit version of the above mentioned theorem of Washington. In [4], we showed Theorem 1 when $d = -1$ and $L_n = K_n^+(\sqrt{-1})$ using some results of cyclotomic Iwasawa theory. In this paper, we prove Theorem 1 by using a main theorem of Conner and Hurrelbrink [1, Theorem 2.3].

REMARK. When $p \equiv 1 \pmod{4}$ (resp. $p \equiv 3 \pmod{4}$), we can show that two integers d_1 and d_2 give the same twist L_∞/L_0 of K_∞/K_0 if and only if $d_2 = d_1x^2$ or $d_2 = pd_1x^2$ (resp. $d_2 = -pd_1x^2$) for some $x \in \mathbf{Q}^\times$. Hence, the set S_d and the integer n_d depend only on the twist L_∞/L_0 and not on the choice of d .

2. Exact hexagon of Conner and Hurrelbrink

In this section, we recall the exact hexagon of Conner and Hurrelbrink. Let k be an imaginary abelian field with 2-power degree, and F a real abelian field with $2 \nmid [F : \mathbf{Q}]$. We put $K = kF$, and

$$G = \text{Gal}(K/k) = \text{Gal}(K^+/k^+) = \text{Gal}(F/\mathbf{Q}).$$

For a number field N , let A_N be the 2-part of the ideal class group of N , \mathcal{O}_N the ring of integers, and $E_N = \mathcal{O}_N^\times$ the group of units of N . The groups A_K and E_K are naturally regarded as modules over $\text{Gal}(K/K^+)$ and at the same time as those over G . For a $\text{Gal}(K/K^+)$ -module X , denote by $H^i(X) = H^i(K/K^+; X)$ the Tate cohomology group with $i = 0, 1$. When $X = A_K$ or E_K , the group $H^i(X)$ is also regarded as G -modules. In [1, Theorem 2.3], Conner and Hurrelbrink introduced the following exact hexagon

of G -modules to study the 2-part of the class number of a relative quadratic extension.

$$\begin{array}{ccccc}
 & & H^1(A_K) & \longrightarrow & H^1(E_K) \\
 & \nearrow & & & \searrow \\
 R^0(K) & & & & R^1(K) \\
 & \nwarrow & & & \swarrow \\
 & & H^0(E_K) & \longleftarrow & H^0(A_K)
 \end{array}$$

Here, $R^i(K)$ is a certain G -module associated to K/K^+ defined in [1]. We describe the G -module structure of $R^i(K)$ following [1]. Let T_f be the set of prime ideals \wp of k^+ for which a prime ideal \mathfrak{P} of K^+ over \wp ramifies in K . Let T_∞ be the set of infinite prime divisors of k^+ . We put $T = T_f \cup T_\infty$. For each $v \in T$, let $G_v \subseteq G$ be the decomposition group of v at K^+/k^+ . When v is an infinite prime, the group G_v is trivial. We define G -modules Ω_f and Ω_∞ by

$$\Omega_f = \bigoplus_{\wp \in T_f} F_2[G/G_\wp] \quad \text{and} \quad \Omega_\infty = \bigoplus_{v \in T_\infty} F_2[G/G_v] = \bigoplus_{v \in T_\infty} F_2[G],$$

respectively, where $F_2 = \mathbb{Z}/2\mathbb{Z}$ is the finite field with two elements. (When T_f is empty, $\Omega_f = \{0\}$ by definition.) For each prime divisor w of K^+ with the restriction $w|_{k^+} \in T$ and an element $x \in (K^+)^\times$, we put $\iota_w(x) = 0$ or 1 according as $x \in N(K_w^\times)$ or not. Here, K_w is the completion of K at the unique prime divisor of K over w and $N = N_{K/K^+}$ is the norm map. For $g \in G$ and $x \in (K^+)^\times$, we see that

$$(1) \quad \iota_{w^g}(x) = \iota_w(x^{g^{-1}})$$

by local class field theory. For a prime ideal \mathfrak{P} of K^+ with $\mathfrak{P} \cap k^+ \in T_f$, let $\tilde{\mathfrak{P}}$ be the unique prime ideal of K over \mathfrak{P} . For an ideal \mathfrak{A} of K , writing $\mathfrak{A} = \tilde{\mathfrak{P}}^e \mathfrak{B}$ with an integer e and an ideal \mathfrak{B} relatively prime to $\tilde{\mathfrak{P}}$, we put $\text{ord}_{\tilde{\mathfrak{P}}}(\mathfrak{A}) = e$.

We denote by $I(K)$ the group of (fractional) ideals of K . Let X be the subgroup of $I(K)$ consisting of ideals \mathfrak{A} with $\mathfrak{A}^J = \mathfrak{A}$. Here, J is the complex conjugation acting on several objects associated to K . Let X_0 be the subgroup of X consisting of ideals $\mathfrak{A} \in I(K)$ with $\mathfrak{A} = x\mathfrak{B}^{1+J}$ for some $x \in (K^+)^\times$ and $\mathfrak{B} \in I(K)$. The G -module $R^1(K)$ is isomorphic to the quotient X/X_0 . For this, see the lines 1–2 from the bottom of p. 6 and Lemma 2.1 of [1]. For each prime ideal $\wp \in T_f$, we fix a prime ideal \mathfrak{P} of K^+ over \wp . From the argument in [1, §5], we obtain the following isomorphism of G -modules:

$$(2) \quad R^1(K) \cong \Omega_f; \quad \mathfrak{A}X_0 \rightarrow \bigoplus_{\wp \in T_f} \left(\sum_{\bar{g}} \text{ord}_{\mathfrak{P}^g}(\mathfrak{A}) \bar{g} \right),$$

where \bar{g} (with $g \in G$) runs over the quotient G/G_\wp .

Let Y be the subgroup of the multiplicative group $(K^+)^{\times} \times I(K)$ consisting of pairs (x, \mathfrak{A}) with $x\mathfrak{A}^{1+J} = \mathcal{O}_K$. Let Y_0 be the subgroup of Y consisting of pairs $(N(y), y^{-1}\mathfrak{B}^{1-J})$ with $y \in K^{\times}$ and $\mathfrak{B} \in I(K)$. By definition, $R^0(K) = Y/Y_0$. We denote by $[x, \mathfrak{A}] \in R^0(K)$ the class containing (x, \mathfrak{A}) . The map i_0 in the hexagon is defined by

$$i_0: H^0(E_K) = E_{K^+}/N(E_K) \rightarrow R^0(K); \quad [\epsilon] \rightarrow [\epsilon, \mathcal{O}_K]$$

with $\epsilon \in E_{K^+}$. For each $v \in T_{\infty}$, we fix a prime divisor \tilde{v} of K^+ over v . Using (1), we observe that the homomorphisms

$$\alpha_{\infty}: (K^+)^{\times} \rightarrow \Omega_{\infty}; \quad x \rightarrow \bigoplus_{v \in T_{\infty}} \left(\sum_{g \in G} \iota_{\tilde{v}g}(x)g \right)$$

and

$$\alpha_f: (K^+)^{\times} \rightarrow \Omega_f; \quad x \rightarrow \bigoplus_{\varphi \in T_f} \left(\sum_{\tilde{g}} \iota_{\mathfrak{P}^{\varphi}}(x)\tilde{g} \right)$$

are compatible with the action of G . Further, α_{∞} is nothing but the “sign” map. From the argument in [1, §4], we obtain the following exact sequence of G -modules:

$$(3) \quad \{0\} \rightarrow R^0(K) \xrightarrow{\alpha} \Omega_f \oplus \Omega_{\infty} \xrightarrow{\beta} F_2 \rightarrow \{0\}.$$

Here, α is defined by $\alpha([x, \mathfrak{A}]) = (\alpha_f(x), \alpha_{\infty}(x))$, β is the argumentation map and G acts trivially on F_2 .

3. Consequences

In this section, we derive some consequences of the exact hexagon and (2), (3). All of them are G -decomposed versions of the corresponding results in [1]. We work under the setting of Section 2. Denote by \tilde{A}_{K^+} the 2-part of the narrow class group of K^+ . Letting $K_{>0}^+$ be the group of totally positive elements of K^+ , we have an exact sequence

$$(4) \quad \{0\} \rightarrow (K^+)^{\times}/(K_{>0}^+E_{K^+}) \rightarrow \tilde{A}_{K^+} \rightarrow A_{K^+} \rightarrow \{0\}$$

of G -modules. We define the minus class group A_K^- to be the kernel of the norm map $A_K \rightarrow A_{K^+}$. Let χ be a $\tilde{\mathcal{Q}}_2$ -valued character of $G = \text{Gal}(K/k) = \text{Gal}(F/\mathcal{Q})$, which we also regard as a primitive Dirichlet character. For a module M over $\mathbf{Z}_2[G]$, we denote by $M(\chi)$ the χ -part of M . Here, \mathbf{Z}_2 is the ring of 2-adic integers and $\tilde{\mathcal{Q}}_2$ is a fixed algebraic closure of the 2-adic rationals \mathcal{Q}_2 . (For the definition of the χ -part and some of its properties, see Tsuji [7, §2].) Denote by S_K the set of prime numbers lying

below some prime ideal in T_f . In all what follows, we assume that χ is a *nontrivial* character. The following is a version of [1, Theorem 13.8].

Theorem 2. *Under the above setting, the groups $H^i(K/K^+; A_K)(\chi)$ with $i = 0$ and 1 are trivial if and only if*

- (i) $\chi(l) \neq 1$ for all $l \in S_K$ and
- (ii) $|\tilde{A}_{K^+}(\chi)| = |A_{K^+}(\chi)|$.

The following corollary is a version of [1, Corollary 13.10] and Hasse [2, Satz 45].

Corollary 2. *Under the above setting, the group $A_K^-(\chi)$ is trivial if and only if*

- (i) $\chi(l) \neq 1$ for all $l \in S_K$ and
- (ii) $\tilde{A}_{K^+}(\chi)$ is trivial.

Let \tilde{h}_M be the class number in the narrow sense of a number field M . When M is an imaginary abelian field, let h_M^- be the relative class number of M . We can easily show that h_k^- (resp. \tilde{h}_{k^+}) divides h_K^- (resp. \tilde{h}_{K^+}) using class field theory. The following is an immediate consequence of Corollary 2.

Corollary 3. *Under the above setting, the ratio h_K^-/h_k^- is odd if and only if*

- (i) no prime number l in S_K splits in F and
- (ii) $\tilde{h}_{K^+}/\tilde{h}_{k^+}$ is odd.

To prove these assertions, we prepare the following two lemmas. For a number field L , let $\mu(L)$ be the group of roots of unity in L and $\mu_2(L)$ the 2-part of $\mu(L)$.

Lemma 1. *The group $H^1(K/K^+; E_K)(\chi)$ is trivial.*

Proof. Let ${}_N E_K$ be the group of units $\epsilon \in E_K$ with $N(\epsilon) = \epsilon^{1+J} = 1$. We have $N(\epsilon) = 1$ if and only if $\epsilon \in \mu(K)$ by a theorem on units of a CM-field (cf. Washington [9, Theorem 4.12]). Since $\mu(K)^2 = \mu(K)^{1-J} \subseteq E_K^{1-J}$, we obtain a surjection

$$\mu(K)/\mu(K)^2 \rightarrow H^1(K/K^+; E_K) = {}_N E_K / E_K^{1-J}$$

of G -modules. However, as $[K : k]$ is odd, we have

$$\mu(K)/\mu(K)^2 = \mu_2(K)/\mu_2(K)^2 = \mu_2(k)/\mu_2(k)^2.$$

Since χ is nontrivial, the χ -part $(\mu_2(k)/\mu_2(k)^2)(\chi)$ is trivial. Hence, we obtain the assertion. \square

Lemma 2. *The natural map $A_{K^+}(\chi) \rightarrow A_K(\chi)$ is injective.*

Proof. Denote the natural map $A_{K^+} \rightarrow A_K$ by ι . Let \mathfrak{A} be an ideal of K^+ with the class $[\mathfrak{A}] \in \ker \iota$. Then $\mathfrak{A}\mathcal{O}_K = x\mathcal{O}_K$ for some $x \in K^\times$. We see that $\epsilon = x^{1-J}$ is a unit of K with $N(\epsilon) = 1$. It is known that the map

$$\ker \iota \rightarrow H^1(K/K^+; E_K); [\mathfrak{A}] \rightarrow x^{1-J} E_K^{1-J}$$

is an injective G -homomorphism ([1, Theorem 7.1]). Then, from Lemma 1, we see that the χ -part $(\ker \iota)(\chi)$ is trivial, from which we obtain the assertion. \square

Proof of Theorem 2. Let \wp be a prime ideal in T_f , and $l = \wp \cap \mathcal{Q} \in S_K$. We see that the χ -part $F_2[G/G_\wp](\chi) \neq \{0\}$ if and only if χ factors through G/G_\wp , which is equivalent to $\chi(G_\wp) = \{1\}$. Since $[k^+ : \mathcal{Q}]$ is a 2-power and $[F : \mathcal{Q}]$ is odd, we have $\chi(G_\wp) = \{1\}$ if and only if $\chi(l) = 1$. Hence, we have shown that the condition (i) in Theorem 2 is equivalent to the condition $\Omega_f(\chi) = \{0\}$. By the hexagon and Lemma 1, we see that $H^0(A_K)(\chi)$ and $H^1(A_K)(\chi)$ are trivial if and only if (iii) $R^1(K)(\chi) = \{0\}$ and (iv) the map

$$i_0: H^0(E_K)(\chi) = (E_{K^+}/N(E_K))(\chi) \rightarrow R^0(K)(\chi)$$

is an isomorphism. By (2) and the above, the condition (iii) is equivalent to (i). Under the condition (i), we see that $R^0(K)(\chi) = \Omega_\infty(\chi)$ from the exact sequence (3), and that for each class $[\epsilon] \in H^0(E_K)(\chi)$ with $\epsilon \in E_{K^+}$, we have $i_0([\epsilon]) = \alpha_\infty(\epsilon)$ from the definitions of the maps i_0 and α . Further, the 2-rank of $\Omega_\infty(\chi)$ is larger than or equal to that of $H^0(E_K)(\chi)$ by a theorem of Minkowski on units of a Galois extension (cf. Narkiewicz [5, Theorem 3.26]). Therefore, under (i), we observe that the condition (iv) holds if and only if $\alpha_\infty(E_{K^+})(\chi) = \Omega_\infty(\chi)$. We see that the last condition is equivalent to the condition (ii) in Theorem 2 because of the exact sequence (4) and $\alpha_\infty((K^+)^\times)(\chi) = \Omega_\infty(\chi)$. Therefore, we obtain Theorem 2. \square

Proof of Corollary 2. First, we show the “only if” part assuming that $A_K^-(\chi)$ is trivial. By Lemma 2, we can regard $A_{K^+}(\chi)$ as a subgroup of $A_K(\chi)$. Assume that $A_{K^+}(\chi)$ is nontrivial. Then there exists a class $c \in A_{K^+}(\chi)$ of order 2. We have $c^J = c = c^{-1}$, and hence $c \in A_K^-(\chi)$. It follows that $A_K^-(\chi)$ is nontrivial, a contradiction. Hence, $A_{K^+}(\chi) = \{0\}$. It follows that $A_K(\chi)$ is trivial by the exact sequence

$$\{0\} \rightarrow A_K^-(\chi) \rightarrow A_K(\chi) \xrightarrow{1+J} A_{K^+}(\chi) \rightarrow \{0\}.$$

Therefore, the “only if” part follows from Theorem 2. Next, assume that the conditions (i) and (ii) in Corollary 2 are satisfied. Then, $A_{K^+}(\chi) = \{0\}$, and the groups $H^i(A_K)(\chi)$ ($i = 0, 1$) are trivial by Theorem 2. As the cohomology groups are trivial, we obtain an exact sequence

$$\{0\} \rightarrow A_{K^+}(\chi) \hookrightarrow A_K(\chi) \xrightarrow{1-J} A_K^{1-J}(\chi) = A_K^-(\chi) \rightarrow \{0\}.$$

Since $A_{K^+}(\chi) = \{0\}$, we see that $A_K(\chi) = A_K^-(\chi)$, and

$$A_K^-(\chi) = A_K^-(\chi)^{1-J} = A_K^-(\chi)^2$$

from the above exact sequence. Therefore, $A_K^-(\chi)$ is trivial. \square

4. Proof of Theorem 1

We use the same notation as in Section 1. In particular, $d \in \mathbf{Z}$ is a fixed integer with $\sqrt{d} \notin K_0$ and L_n is the quadratic twist of K_n associated to d . We have $L_n^+ = K_n^+$. Let k (resp. k_d) be the maximal intermediate field of K_0/\mathbf{Q} (resp. L_0/\mathbf{Q}) of 2-power degree, and let F_0 be the maximal subfield of $K_0^+ = L_0^+$ of odd degree over \mathbf{Q} . Then k and k_d are imaginary abelian fields with $k^+ = k_d^+$. Let \mathbf{B}_n/\mathbf{Q} be the real abelian field with conductor p^{n+1} and $[\mathbf{B}_n : \mathbf{Q}] = p^n$. We put $F_n = F_0\mathbf{B}_n$. Then $L_n = k_d F_n$ and $K_n = k F_n$. The triples (k_d, F_n, L_n) and (k, F_n, K_n) correspond to (k, F, K) in Sections 2 and 3. We see that

$$(5) \quad S_{L_n} = S_d \quad \text{or} \quad S_d \cup \{p\}$$

and $S_{K_n} = \{p\}$. We put

$$G_n = \text{Gal}(F_n/\mathbf{Q}) = \text{Gal}(L_n/k_d) = \text{Gal}(K_n/k),$$

and

$$\Delta = \text{Gal}(F_0/\mathbf{Q}), \quad \Gamma_n = \text{Gal}(F_n/F_0) = \text{Gal}(\mathbf{B}_n/\mathbf{Q}).$$

Then we have a natural decomposition $G_n = \Delta \times \Gamma_n$. For characters φ and ψ of Δ and Γ_n respectively, we regard $\varphi\psi = \varphi \times \psi$ as a character of G_n . Further, we regard φ , ψ and $\varphi\psi$ also as primitive Dirichlet characters. The class groups $A_{L_n}^-$, $A_{K_n}^-$ and $\tilde{A}_{K_n^+}$ are modules over G_n . We can naturally regard $A_{L_{n-1}}^-$ as a subgroup of $A_{L_n}^-$ since L_n/L_{n-1} is a cyclic extension of degree $p \neq 2$ and $A_{L_{n-1}}^-$ is the 2-part of the class group. Actually, it is a direct summand of $A_{L_n}^-$ (cf. [9, Lemma 16.15]). We see that

$$(6) \quad A_{L_n}^-/A_{L_{n-1}}^- = \bigoplus_{\varphi, \psi_n} A_{L_n}^-(\varphi\psi_n)$$

where φ (resp. ψ_n) runs over a complete set of representatives of the \mathbf{Q}_2 -conjugacy classes of the $\bar{\mathbf{Q}}_2$ -valued characters of Δ (resp. Γ_n of order p^n). Regarding $A_{K_{n-1}}^-$ as a subgroup of $A_{K_n}^-$, we have a similar decomposition for $A_{K_n}^-/A_{K_{n-1}}^-$. As $S_{K_n} = \{p\}$ and $(\varphi\psi_n)(p) = 0$, we obtain the following assertion from Corollary 2 for the triple (k, F_n, K_n) .

Lemma 3. *Let $n \geq 1$ be an integer, and the characters φ and ψ_n be as in (6). Then $A_{K_n}^-(\varphi\psi_n) = \{0\}$ if and only if $\tilde{A}_{K_n^+}(\varphi\psi_n) = \{0\}$.*

Proof of Theorem 1 (I). Let φ and ψ_n be as in (6). As the orders of φ and ψ_n are relatively prime to each other, we have $(\varphi\psi_n)(l) = 1$ if and only if $\varphi(l) = \psi_n(l) = 1$ for a prime number l . Let n be an integer with $n \geq n_d$. Then we have $\psi_n(l) \neq 1$ and hence $(\varphi\psi_n)(l) \neq 1$ for all prime numbers $l \in S = S_d$. Further, we have $(\varphi\psi_n)(p) = 0$. Hence, by (5), the condition (i) in Corollary 2 for the triple (k_d, F_n, L_n) is satisfied. It follows that the condition $A_{L_n}^-(\varphi\psi_n) = \{0\}$ is equivalent to $\tilde{A}_{K_n^+}(\varphi\psi_n) = \{0\}$. (Note that $L_n^+ = K_n^+$.) Therefore, we obtain Theorem 1(I) from Lemma 3. \square

To show Theorem 1 (II), assume that $n_d \geq 2$ and let n be an integer with $1 \leq n < n_d$. We put

$$S^{(n)} = \{l \in S = S_d \mid \text{ord}_p(l^{p-1} - 1) \geq n + 1\}.$$

From the definition, we see that

$$S \supseteq S^{(1)} \supseteq S^{(2)} \supseteq \dots \supseteq S^{(n_d-1)}$$

and that each $S^{(n)}$ is non-empty. Let φ (resp. ψ_n) be a $\bar{\mathcal{Q}}_2$ -valued character of Δ (resp. of Γ_n of order p^n). Denote by φ_0 the trivial character of Δ . Theorem 1 (II) is a consequence of the following assertion.

Proposition 1. *Under the above setting, the following hold.*

- (I) *The class group $A_{L_n}^-(\varphi\psi_n)$ is nontrivial if $\varphi(l) = 1$ for some $l \in S^{(n)}$. In particular, $A_{L_n}^-(\varphi_0\psi_n)$ is nontrivial.*
- (II) *If $A_{K_n}^-(\varphi\psi_n) = \{0\}$, the converse of the first assertion of (I) holds.*

Proof. Applying Corollary 2 for the triple (k_d, F_n, L_n) , we see from Lemma 3 that $A_{L_n}^-(\varphi\psi_n) = \{0\}$ if and only if (i) $(\varphi\psi_n)(l) \neq 1$ for all $l \in S = S_d$ and (ii) $A_{K_n}^-(\varphi\psi_n) = \{0\}$. We have $\psi_n(l) = 1$ for $l \in S^{(n)}$, and $\psi_n(l) \neq 1$ for $l \in S \setminus S^{(n)}$. Therefore, we see that the condition (i) is satisfied if and only if $\varphi(l) \neq 1$ for all $l \in S^{(n)}$ noting that the orders of φ and ψ_n are relatively prime. From this, we obtain the proposition. \square

We put $M_n = K_n(\sqrt{d}) = K_n L_n$. On the relative class number $h_{M_n}^-$ of M_n , the following assertion holds.

- Proposition 2.** (I) *When $n \geq n_d$, the ratio $h_{M_n}^-/h_{M_{n-1}}^-$ is odd if and only if h_n^*/h_{n-1}^* is odd.*
- (II) *When $n_d \geq 2$ and $1 \leq n < n_d$, $h_{M_n}^-/h_{M_{n-1}}^-$ is even.*

To prove this proposition, we need to show the following lemma. For an imaginary abelian field N , we put

$$\mathcal{E}_N = E_N/\mu(N)E_{N^+}.$$

It is well known that the unit index $\mathcal{Q}_N = |\mathcal{E}_N|$ is 1 or 2 ([9, Theorem 4.12]).

Lemma 4. *Let T and N be imaginary abelian fields with $N \subseteq T$. If the degree $[T : N]$ is odd, then $Q_T = Q_N$.*

Proof. We first show that the inclusion map $N \rightarrow T$ induces an injection $\mathcal{E}_N \hookrightarrow \mathcal{E}_T$. For a unit ϵ of N , assume that $\epsilon = \zeta\eta$ for some $\zeta \in \mu(T)$ and $\eta \in E_{T^+}$. Let ρ be a nontrivial element of the Galois group $G = \text{Gal}(T/N)$. Then, as $\epsilon = \epsilon^\rho$, we see that $\zeta^{1-\rho} = \eta^{\rho-1} \in \mu(T) \cap E_{T^+}$. Hence, $\zeta^{1-\rho} = \pm 1$. However, as $N_{T/N}(\zeta^{1-\rho}) = 1$ and $[T : N]$ is odd, the case $\zeta^{1-\rho} = -1$ does not happen. Hence, $\zeta^{1-\rho} = 1$ for all $\rho \in G$. It follows that $\zeta \in \mu(N)$ and hence $\eta \in E_{N^+}$. Therefore, we can regard \mathcal{E}_N as a subgroup of \mathcal{E}_T . In particular, Q_N divides Q_T .

Assume that $Q_N \neq Q_T$. Then we have $|\mathcal{E}_T| = |\mathcal{E}_T/\mathcal{E}_N| = 2$. Regarding \mathcal{E}_T as a module over G , we have a canonical decomposition

$$\mathcal{E}_T = \mathcal{E}_T/\mathcal{E}_N = \bigoplus_{\chi} \mathcal{E}_T(\chi)$$

where χ runs over a complete set of representatives of the \bar{Q}_2 -conjugacy classes of the *nontrivial* \bar{Q}_2 -valued characters of G . Hence, $|\mathcal{E}_T(\chi)| = 2$ for some such χ . Let $\mathbf{Z}_2[\chi]$ be the subring of \bar{Q}_2 generated by the values of χ over \mathbf{Z}_2 . The group $\mathcal{E}_T(\chi)$ is naturally regarded as a module over the principal ideal domain $\mathbf{Z}_2[\chi]$. Since the order of χ is odd and ≥ 3 , we observe that $\mathbf{Z}_2[\chi] \cong \mathbf{Z}_2^d$ as \mathbf{Z}_2 -modules for some $d \geq 2$. Hence, $|\mathcal{E}_T(\chi)|$ is a multiple of 2^d , which contradicts $|\mathcal{E}_T(\chi)| = 2$. Therefore, we obtain $Q_N = Q_T$. \square

Proof of Proposition 2. By Lemma 4, we have $Q_{M_n} = Q_{M_{n-1}}$ and $Q_{L_n} = Q_{L_{n-1}}$ for all $n \geq 1$. Therefore, using the class number formula [9, Theorem 4.17], we see that

$$h_{M_n}^-/h_{M_{n-1}}^- = p \prod_{\varpi} \prod_{\psi_n} \left(-\frac{1}{2} B_{1, \varpi \psi_n} \right)$$

where ϖ runs over the odd Dirichlet characters associated to M_0 , and ψ_n over the even characters of conductor p^{n+1} and order p^n . Further, $B_{1, \varpi \psi_n}$ denotes the generalized Bernoulli number. We easily see that ϖ equals an odd Dirichlet character associated to K_0 or L_0 since M_0/K_0^+ is an imaginary biquadratic extension with the imaginary quadratic subextensions K_0 and L_0 . Hence, using the class number formulas for L_n , K_n and $Q_{L_n} = Q_{L_{n-1}}$, we obtain

$$h_{M_n}^-/h_{M_{n-1}}^- = h_n^*/h_{n-1}^* \times h_n^-/h_{n-1}^-.$$

Therefore, the assertion follows from Theorem 1. \square

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