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CLASS NUMBER PARITY OF A QUADRATIC TWIST OF A CYCLOTOMIC FIELD OF PRIME POWER CONDUCTOR

HUMIO ICHIMURA

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Abstract

Let $p$ be a fixed odd prime number and $K_n$ the $p^{n+1}$-st cyclotomic field. For a fixed integer $d \in \mathbb{Z}$ with $\sqrt{d} \notin K_0$, denote by $L_n$ the imaginary quadratic subextension of the biquadratic extension $K_n(\sqrt{d})/K_n^{+}$ with $L_n \neq K_n$. Let $h_n^{+}$ and $h_n^{-}$ be the relative class numbers of $K_n$ and $L_n$, respectively. We give an explicit constant $n_d$ depending on $p$ and $d$ such that (i) for any integer $n \geq n_d$, the ratio $h_n^{-}/h_{n-1}^{-}$ is odd if and only if $h_n^{+}/h_{n-1}^{+}$ is odd and (ii) for $1 \leq n < n_d$, $h_n^{-}/h_{n-1}^{-}$ is even.

1. Introduction

Let $p$ be a fixed odd prime number. Let $K_n = \mathbb{Q}(\zeta_{p^n+1})$ be the $p^{n+1}$-st cyclotomic field for an integer $n \geq 0$, and $K_{\infty} = \bigcup K_n$. Let $d \in \mathbb{Z}$ be a fixed integer with $\sqrt{d} \notin K_0$. We denote by $L_n$ the imaginary quadratic subextension of the biquadratic extension $K_n(\sqrt{d})/K_n^{+}$ with $L_n \neq K_n$. Here, $K^{+}$ denotes the maximal real subfield of an imaginary abelian field $K$. When $d < 0$, we have $L_n = K_n^{+}(\sqrt{d})$. We call $L_n$ the quadratic twist of $K_n$ associated to the integer $d$. The extension $L_{\infty} = \bigcup L_n$ is the cyclotomic $\mathbb{Z}_p$-extension over $\mathbb{Q}$ with the $n$-th layer $L_n$. We call $L_{\infty}/L_0$ the quadratic twist of the cyclotomic $\mathbb{Z}_p$-extension $K_{\infty}/K_0$ associated to $d$. Let $h_n^{+}$ and $h_n^{-}$ be the relative class numbers of $K_n$ and $L_n$, respectively. It is known and easy to show that $h_{n-1}^{+}$ (resp. $h_{n-1}^{-}$) divides $h_n^{+}$ (resp. $h_n^{-}$) using class field theory. The parity of $h_n^{+}$ behaves rather irregularly when $p$ varies (see a table in Schoof [6]). However, it is recently shown that when $p \leq 509$, the ratio $h_n^{+}/h_{n-1}^{+}$ is odd for all $n \geq 1$ ([3, Theorem 2]). And it might be possible that the ratio is odd for any prime $p$ and any $n \geq 1$. The purpose of this paper is to study the parity of the ratio $h_n^{-}/h_{n-1}^{-}$ of the quadratic twist $L_n$. We already know that $h_n^{-}/h_{n-1}^{-}$ is odd for sufficiently large $n$ by a theorem of Washington [8] on the non-$p$-part of the class number in a cyclotomic $\mathbb{Z}_p$-extension. Denote by $S = S_d$ the set of prime numbers $l \neq p$ which ramify in $Q(\sqrt{d})/Q$. The set $S$ is non-empty as $\sqrt{d} \notin K_0$. We define an integer $n_d \geq 1$ by

$$n_d = \max\{\text{ord}_p(l^{n-1} - 1) \mid l \in S\},$$

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where \( \text{ord}_p(*) \) is the normalized \( p \)-adic additive valuation. The following is the main theorem of this paper.

**Theorem 1.** Under the above setting, the following assertions hold.

(I) When \( n \geq n_d \), the ratio \( h_n^-/h_{n-1}^- \) is odd if and only if \( h_n^+/h_{n-1}^+ \) is odd.

(II) When \( n_d \geq 2 \) and \( 1 \leq n < n_d \), the ratio \( h_n^-/h_{n-1}^- \) is even.

From Theorem 1 and [3, Theorem 2], we immediately obtain the following:

**Corollary 1.** Under the above setting, let \( p \) be an odd prime number with \( p \leq 509 \). Then the ratio \( h_n^-/h_{n-1}^- \) is odd for all \( n \geq n_d \).

This corollary, though given in a very special setting, is an explicit version of the above mentioned theorem of Washington. In [4], we showed Theorem 1 when \( d = -1 \) and \( L_n = K_n^+(\sqrt{-1}) \) using some results of cyclotomic Iwasawa theory. In this paper, we prove Theorem 1 by using a main theorem of Conner and Hurrelbrink [1, Theorem 2.3].

**Remark.** When \( p \equiv 1 \mod 4 \) (resp. \( p \equiv 3 \mod 4 \)), we can show that two integers \( d_1 \) and \( d_2 \) give the same twist \( L_\infty/L_0 \) of \( K_\infty/K_0 \) if and only if \( d_2 = d_1 x^2 \) or \( d_2 = pd_1 x^2 \) (resp. \( d_2 = -pd_1 x^2 \)) for some \( x \in \mathbb{Q}^\times \). Hence, the set \( S_d \) and the integer \( n_d \) depend only on the twist \( L_\infty/L_0 \) and not on the choice of \( d \).

2. **Exact hexagon of Conner and Hurrelbrink**

In this section, we recall the exact hexagon of Conner and Hurrelbrink. Let \( k \) be an imaginary abelian field with 2-power degree, and \( F \) a real abelian field with \( 2 \nmid [F : \mathbb{Q}] \). We put \( K = kF \), and

\[
G = \text{Gal}(K/k) = \text{Gal}(K^+/k^+) = \text{Gal}(F/\mathbb{Q}).
\]

For a number field \( N \), let \( A_N \) be the 2-part of the ideal class group of \( N \), \( \mathcal{O}_N \) the ring of integers, and \( E_N = \mathcal{O}_N^\times \) the group of units of \( N \). The groups \( A_K \) and \( E_K \) are naturally regarded as modules over \( \text{Gal}(K/K^+) \) and at the same time as those over \( G \). For a \( \text{Gal}(K/K^+) \)-module \( X \), denote by \( H^i(X) = H^i(K/K^+; X) \) the Tate cohomology group with \( i = 0, 1 \). When \( X = A_K \) or \( E_K \), the group \( H^i(X) \) is also regarded as \( G \)-modules. In [1, Theorem 2.3], Conner and Hurrelbrink introduced the following exact hexagon...
of $G$-modules to study the 2-part of the class number of a relative quadratic extension.

$$
\begin{array}{c}
R^0(K) \\
\downarrow_{i_0} \\
R^1(K) \\
\end{array} 
\begin{array}{c}
H^0(K) \\
\downarrow \quad \quad \downarrow \\
H^1(K) \\
\end{array} 
\begin{array}{c}
H^1(A_K) \\
\end{array}
$$

Here, $R^i(K)$ is a certain $G$-module associated to $K/K^+$ defined in [1]. We describe the $G$-module structure of $R^i(K)$ following [1]. Let $T_f$ be the set of prime ideals $\wp$ of $k^+$ for which a prime ideal $\mathfrak{P}$ of $K^+$ over $\wp$ ramifies in $K$. Let $T_{\infty}$ be the set of infinite prime divisors of $k^+$. We put $T = T_f \cup T_{\infty}$. For each $v \in T$, let $G_v \subseteq G$ be the decomposition group of $v$ at $K^+/k^+$. When $v$ is an infinite prime, the group $G_v$ is trivial. We define $G$-modules $\Omega_f$ and $\Omega_{\infty}$ by

$$
\Omega_f = \bigoplus_{\wp \in T_f} F_2[G/G_{\wp}] \quad \text{and} \quad \Omega_{\infty} = \bigoplus_{v \in T_{\infty}} F_2[G/G_v] = \bigoplus_{v \in T_{\infty}} F_2[G],
$$

respectively, where $F_2 = \mathbb{Z}/2\mathbb{Z}$ is the finite field with two elements. (When $T_f$ is empty, $\Omega_f = \{0\}$ by definition.) For each prime divisor $w$ of $K^+$ with the restriction $w_{K^+} \in T$ and an element $x \in (K^+)^\times$, we put $t_w(x) = 0$ or 1 according as $x \in N(K^+_w)$ or not. Here, $K_w$ is the completion of $K$ at the unique prime divisor of $K$ over $w$ and $N = N_{K/K^+}$ is the norm map. For $g \in G$ and $x \in (K^+)^\times$, we see that

$$
t_{w^{*}}(x) = t_w(x^{g^{-1}}) \quad \text{(1)}
$$

by local class field theory. For a prime ideal $\mathfrak{P}$ of $K^+$ with $\mathfrak{P} \cap k^+ \in T_f$, let $\mathfrak{P}$ be the unique prime ideal of $K$ over $\mathfrak{P}$. For an ideal $\mathfrak{A}$ of $K$, writing $\mathfrak{A} = \mathfrak{P} \mathfrak{B}$ with an integer $e$ and an ideal $\mathfrak{B}$ relatively prime to $\mathfrak{P}$, we put $\text{ord}_{\mathfrak{P}}(\mathfrak{A}) = e$.

We denote by $I(K)$ the group of (fractional) ideals of $K$. Let $X$ be the subgroup of $I(K)$ consisting of ideals $\mathfrak{A}$ with $\mathfrak{A}^J = \mathfrak{A}$. Here, $J$ is the complex conjugation acting on several objects associated to $K$. Let $X_0$ be the subgroup of $X$ consisting of ideals $\mathfrak{A} \in I(K)$ with $\mathfrak{A} = x \mathfrak{B}^{1+j}$ for some $x \in (K^+)^\times$ and $\mathfrak{B} \in I(K)$. The $G$-module $R^1(K)$ is isomorphic to the quotient $X/X_0$. For this, see the lines 1–2 from the bottom of p.6 and Lemma 2.1 of [1]. For each prime ideal $\wp \in T_f$, we fix a prime ideal $\mathfrak{P}$ of $K^+$ over $\wp$. From the argument in [1, §5], we obtain the following isomorphism of $G$-modules:

$$
R^1(K) \cong \Omega_f; \quad \mathfrak{A} X_0 \rightarrow \bigoplus_{\wp \in T_f} \left( \sum_{\mathfrak{g}} \text{ord}_{\mathfrak{P}}(\mathfrak{A}) \mathfrak{g} \right), \quad \text{(2)}
$$

where $\mathfrak{g}$ (with $g \in G$) runs over the quotient $G/G_{\wp}$. 
Let $Y$ be the subgroup of the multiplicative group $(K^+)^\times \times I(K)$ consisting of pairs $(x, \mathfrak{A})$ with $x \mathfrak{A}^{1+J} = \mathcal{O}_K$. Let $Y_0$ be the subgroup of $Y$ consisting of pairs $(N(y), y^{-1} \mathfrak{B}^{1-J})$ with $y \in K^+$ and $\mathfrak{B} \in I(K)$. By definition, $R^0(K) = Y/Y_0$. We denote by $[x, \mathfrak{A}] \in R^0(K)$ the class containing $(x, \mathfrak{A})$. The map $i_0$ in the hexagon is defined by

$$i_0: H^0(E_K) = E_{K^+}/N(E_K) \to R^0(K); \quad [\varepsilon] \to [\varepsilon, \mathcal{O}_K]$$

with $\varepsilon \in E_{K^+}$. For each $v \in T_\infty$, we fix a prime divisor $\mathfrak{v}$ of $K^+$ over $v$. Using (1), we observe that the homomorphisms

$$\alpha_\infty: (K^+)^\times \to \Omega_\infty; \quad x \mapsto \bigoplus_{v \in T_\infty} \left( \sum_{g \in G} \nu_\varepsilon(x) g \right)$$

and

$$\alpha_f: (K^+)^\times \to \Omega_f; \quad x \mapsto \bigoplus_{\psi \in I_f} \left( \sum_g \nu_\psi(x) \bar{g} \right)$$

are compatible with the action of $G$. Further, $\alpha_\infty$ is nothing but the “sign” map. From the argument in [1, §4], we obtain the following exact sequence of $G$-modules:

$$0 \to R^0(K) \xrightarrow{\alpha} \Omega_f \oplus \Omega_\infty \xrightarrow{\beta} F_2 \to 0.$$ 

Here, $\alpha$ is defined by $\alpha([x, \mathfrak{A}]) = (\alpha_f(x), \alpha_\infty(x))$, $\beta$ is the argumentation map and $G$ acts trivially on $F_2$.

### 3. Consequences

In this section, we derive some consequences of the exact hexagon and (2), (3). All of them are $G$-decomposed versions of the corresponding results in [1]. We work under the setting of Section 2. Denote by $\tilde{A}_{K^+}$ the 2-part of the narrow class group of $K^+$. Letting $K_{>0}^+$ be the group of totally positive elements of $K^+$, we have an exact sequence

$$\{0\} \to (K^+)^\times/(K_{>0}^+E_{K^+}) \to \tilde{A}_{K^+} \to A_{K^+} \to \{0\}$$

of $G$-modules. We define the minus class group $A_{K}^{-}$ to be the kernel of the norm map $A_K \to A_{K^+}$. Let $\chi$ be a $\tilde{Q}_2$-valued character of $G = \text{Gal}(K/k) = \text{Gal}(F/Q)$, which we also regard as a primitive Dirichlet character. For a module $M$ over $\mathbb{Z}_2[G]$, we denote by $M(\chi)$ the $\chi$-part of $M$. Here, $\mathbb{Z}_2$ is the ring of 2-adic integers and $\tilde{Q}_2$ is a fixed algebraic closure of the 2-adic rationals $Q_2$. (For the definition of the $\chi$-part and some of its properties, see Tsuji [7, §2].) Denote by $S_K$ the set of prime numbers lying
below some prime ideal in \( T_f \). In all what follows, we assume that \( \chi \) is a nontrivial character. The following is a version of [1, Theorem 13.8].

**Theorem 2.** Under the above setting, the groups \( H^i(K/K^+; A_K)(\chi) \) with \( i = 0 \) and 1 are trivial if and only if

(i) \( \chi(l) \neq 1 \) for all \( l \in S_K \) and

(ii) \( |\tilde{A}_K(\chi)| = |A_K(\chi)| \).

The following corollary is a version of [1, Corollary 13.10] and Hasse [2, Satz 45].

**Corollary 2.** Under the above setting, the group \( A_K(\chi) \) is trivial if and only if

(i) \( \chi(l) \neq 1 \) for all \( l \in S_K \) and

(ii) \( \tilde{A}_K(\chi) \) is trivial.

Let \( \tilde{h}_M \) be the class number in the narrow sense of a number field \( M \). When \( M \) is an imaginary abelian field, let \( h_M \) be the relative class number of \( M \). We can easily show that \( h_M \) (resp. \( \tilde{h}_M \)) divides \( h_K \) (resp. \( \tilde{h}_K \)) using class field theory. The following is an immediate consequence of Corollary 2.

**Corollary 3.** Under the above setting, the ratio \( h_K/h_M \) is odd if and only if

(i) no prime number \( l \) in \( S_K \) splits in \( F \) and

(ii) \( \tilde{h}_K/\tilde{h}_M \) is odd.

To prove these assertions, we prepare the following two lemmas. For a number field \( L \), let \( \mu(L) \) be the group of roots of unity in \( L \) and \( \mu_2(L) \) the 2-part of \( \mu(L) \).

**Lemma 1.** The group \( H^1(K/K^+; E_K)(\chi) \) is trivial.

Proof. Let \( NE_K \) be the group of units \( e \in E_K \) with \( N(e) = e^{1+J} = 1 \). We have \( N(e) = 1 \) if and only if \( e \in \mu(K) \) by a theorem on units of a CM-field (cf. Washington [9, Theorem 4.12]). Since \( \mu(K)^2 = \mu(K)^{1-J} \subseteq E_K^{1-J} \), we obtain a surjection

\[
\mu(K)/\mu(K)^2 \rightarrow H^1(K/K^+; E_K) = NE_K/E_K^{1-J}
\]

of \( G \)-modules. However, as \([K:k]\) is odd, we have

\[
\mu(K)/\mu(K)^2 = \mu_2(K)/\mu_2(K)^2 = \mu_2(k)/\mu_2(k)^2.
\]

Since \( \chi \) is nontrivial, the \( \chi \)-part \( (\mu_2(k)/\mu_2(k)^2)(\chi) \) is trivial. Hence, we obtain the assertion.

**Lemma 2.** The natural map \( A_K(\chi) \rightarrow A_K(\chi) \) is injective.
Proof. Denote the natural map \( A_{K^+} \to A_K \) by \( \iota \). Let \( \mathfrak{A} \) be an ideal of \( K^+ \) with the class \([\mathfrak{A}] \in \ker \iota\). Then \( \mathfrak{A}O_K = xO_K \) for some \( x \in K^\times \). We see that \( \epsilon = x^{1-I} \) is a unit of \( K \) with \( N(\epsilon) = 1 \). It is known that the map

\[
\ker \iota \to H^1(K/K^+; E_K); \ [\mathfrak{A}] \to x^{1-I}E_K^{1-I}
\]

is an injective \( G \)-homomorphism ([1, Theorem 7.1]). Then, from Lemma 1, we see that the \( \chi \)-part \( (\ker \iota)(\chi) \) is trivial, from which we obtain the assertion. \( \square \)

Proof of Theorem 2. Let \( \varphi \) be a prime ideal in \( T_f \), and let \( l = \varphi \cap \mathcal{Q} \in S_K \). We see that the \( \chi \)-part \( F_2[G/G_\varphi](\chi) \neq \{0\} \) if and only if \( \chi \) factors through \( G/G_\varphi \), which is equivalent to \( \chi(G_\varphi) = \{1\} \). Since \([k^+ : \mathcal{Q}]\) is a 2-power and \([F : \mathcal{Q}]\) is odd, we have \( \chi(G_\varphi) = \{1\} \) if and only if \( \chi(l) = 1 \). Hence, we have shown that the condition (i) in Theorem 2 is equivalent to the condition \( \Omega_f(\chi) = \{0\} \). By the hexagon and Lemma 1, we see that \( H^0(A_K)(\chi) \) and \( H^1(A_K)(\chi) \) are trivial if and only if (iii) \( R^1(K)(\chi) = \{0\} \) and (iv) the map

\[
i_0: H^0(E_K)(\chi) = (E_{K^+}/N(E_K))(\chi) \to R^0(K)(\chi)
\]

is an isomorphism. By (2) and the above, the condition (iii) is equivalent to (i). Under the condition (i), we see that \( R^0(K)(\chi) = \Omega_{\infty}(\chi) \) from the exact sequence (3), and that for each class \([\epsilon] \in H^0(E_K)(\chi)\) with \( \epsilon \in E_{K^+} \), we have \( i_0([\epsilon]) = \alpha_{\infty}(\epsilon) \) from the definitions of the maps \( i_0 \) and \( \alpha \). Further, the 2-rank of \( \Omega_{\infty}(\chi) \) is larger than or equal to that of \( H^0(E_K)(\chi) \) by a theorem of Minkowski on units of a Galois extension (cf. Narkiewicz [5, Theorem 3.26]). Therefore, under (i), we observe that the condition (iv) holds if and only if \( \alpha_{\infty}(E_{K^+})(\chi) = \Omega_{\infty}(\chi) \). We see that the last condition is equivalent to the condition (ii) in Theorem 2 because of the exact sequence (4) and \( \alpha_{\infty}((K^+)^\times)(\chi) = \Omega_{\infty}(\chi) \). Therefore, we obtain Theorem 2. \( \square \)

Proof of Corollary 2. First, we show the “only if” part assuming that \( A_K^-(\chi) \) is trivial. By Lemma 2, we can regard \( A_K^+(\chi) \) as a subgroup of \( A_K(\chi) \). Assume that \( A_K^+(\chi) \) is nontrivial. Then there exists a class \( c \in A_K^+(\chi) \) of order 2. We have \( c^f = c = c^{-1} \), and hence \( c \in A_K^-(\chi) \). It follows that \( A_K^-(\chi) \) is nontrivial, a contradiction. Hence, \( A_K^-(\chi) = \{0\} \). It follows that \( A_K(\chi) \) is trivial by the exact sequence

\[
\{0\} \to A_K^-(\chi) \to A_K(\chi) \xrightarrow{1+j} A_K^+(\chi) \to \{0\}.
\]

Therefore, the “only if” part follows from Theorem 2. Next, assume that the conditions (i) and (ii) in Corollary 2 are satisfied. Then, \( A_K^-(\chi) = \{0\} \), and the groups \( H^i(A_K)(\chi) \) \( (i = 0, 1) \) are trivial by Theorem 2. As the cohomology groups are trivial, we obtain an exact sequence

\[
\{0\} \to A_K^+(\chi) \to A_K(\chi) \xrightarrow{1-j} A_K^-(\chi) = A_K^-(\chi) \to \{0\}.
\]
Since \( A_{K^+}(\chi) = \{0\} \), we see that \( A_{K}(\chi) = A_{K}^{-}(\chi) \), and
\[
A_{K}^{-}(\chi) = A_{K}^{-}(\chi)^{1-J} = A_{K}^{-}(\chi)^2
\]
from the above exact sequence. Therefore, \( A_{K}^{-}(\chi) \) is trivial. 

4. Proof of Theorem 1

We use the same notation as in Section 1. In particular, \( d \in \mathbb{Z} \) is a fixed integer with \( \sqrt{d} \notin K_0 \) and \( L_n \) is the quadratic twist of \( K_n \) associated to \( d \). We have \( L_n^+ = K_n^+ \). Let \( k \) (resp. \( k_d \)) be the maximal intermediate field of \( K_0/\mathcal{Q} \) (resp. \( L_0/\mathcal{Q} \)) of 2-power degree, and let \( F_0 \) be the maximal subfield of \( K_0^+ = L_0^+ \) of odd degree over \( \mathcal{Q} \). Then \( k \) and \( k_d \) are imaginary abelian fields with \( k^+ = k_d^- \). Let \( B_n/\mathcal{Q} \) be the real abelian field with conductor \( p^{n+1} \) and \([B_n : \mathcal{Q}] = p^n \). We put \( F_n = F_0B_n \). Then \( L_n = k_dF_n \) and \( K_n = kF_n \). The triples \((k_d, F_n, L_n)\) and \((k, F, K_n)\) correspond to \((k, F, K)\) in Sections 2 and 3. We see that

\[
S_{L_n} = S_d \text{ or } S_d \cup \{p\}
\]
and \( S_{K_n} = \{p\} \). We put
\[
G_n = \text{Gal}(F_n/\mathcal{Q}) = \text{Gal}(L_n/k_d) = \text{Gal}(K_n/k),
\]
and
\[
\Delta = \text{Gal}(F_0/\mathcal{Q}), \quad \Gamma_n = \text{Gal}(F_n/F_0) = \text{Gal}(B_n/\mathcal{Q}).
\]
Then we have a natural decomposition \( G_n = \Delta \times \Gamma_n \). For characters \( \varphi \) and \( \psi \) of \( \Delta \) and \( \Gamma_n \) respectively, we regard \( \varphi \psi = \varphi \times \psi \) as a character of \( G_n \). Further, we regard \( \varphi \), \( \psi \) and \( \varphi \psi \) also as primitive Dirichlet characters. The class groups \( A_{L_n^+} \), \( A_{K_n}^{-} \) and \( \tilde{A}_{K_n^+} \) are modules over \( G_n \). We can naturally regard \( A_{L_n^+}^{-} \) as a subgroup of \( A_{K_n}^{-} \) since \( L_n/L_{n-1} \) is a cyclic extension of degree \( p \neq 2 \) and \( A_{L_n^+}^{-} \) is the 2-part of the class group. Actually, it is a direct summand of \( A_{L_n}^{-} \) (cf. [9, Lemma 16.15]). We see that

\[
A_{L_n^+}^{-}/A_{L_{n-1}^+}^{-} \cong \bigoplus_{\varphi, \psi_n} A_{L_n}^{-}(\varphi \psi_n)
\]
where \( \varphi \) (resp. \( \psi_n \)) runs over a complete set of representatives of the \( \mathbb{Q}_2 \)-conjugacy classes of the \( \mathbb{Q}_2 \)-valued characters of \( \Delta \) (resp. \( \Gamma_n \) of order \( p^n \)). Regarding \( \tilde{A}_{K_n^+} \) as a subgroup of \( A_{K_n}^{-} \), we have a similar decomposition for \( \tilde{A}_{K_n^+}/A_{K_{n-1}^+}^{-} \). As \( S_{K_n} = \{p\} \) and \((\varphi \psi_n)(p) = 0\), we obtain the following assertion from Corollary 2 for the triple \((k, F_n, K_n)\).

Lemma 3. Let \( n \geq 1 \) be an integer, and the characters \( \varphi \) and \( \psi_n \) be as in (6). Then \( A_{K_n}^{-}(\varphi \psi_n) = \{0\} \) if and only if \( \tilde{A}_{K_n^+}(\varphi \psi_n) = \{0\} \).
Proof of Theorem 1 (I). Let $\varphi$ and $\psi_n$ be as in (6). As the orders of $\varphi$ and $\psi_n$ are relatively prime to each other, we have $(\varphi \psi_n)(l) = 1$ if and only if $\varphi(l) = \psi_n(l) = 1$ for a prime number $l$. Let $n$ be an integer with $n \geq n_d$. Then we have $\psi_n(l) \neq 1$ and hence $(\varphi \psi_n)(l) \neq 1$ for all prime numbers $l \in S = S_d$. Further, we have $(\varphi \psi_n)(p) = 0$. Hence, by (5), the condition (i) in Corollary 2 for the triple $(k_d, F_n, L_n)$ is satisfied. It follows that the condition $A_{\alpha_n}(\varphi \psi_n) = \{0\}$ is equivalent to $A_{K_n^+}(\varphi \psi_n) = \{0\}$. (Note that $L_n^+ = K_n^+$. Therefore, we obtain Theorem 1(I) from Lemma 3.

To show Theorem 1 (II), assume that $n_d \geq 2$ and let $n$ be an integer with $1 \leq n < n_d$. We put

$$S^{(n)} = \{l \in S = S_d \mid \text{ord}_p(l^{p^n-1} - 1) \geq n + 1\}.$$  

From the definition, we see that

$$S \supseteq S^{(1)} \supseteq S^{(2)} \supseteq \cdots \supseteq S^{(n_d-1)}$$

and that each $S^{(n)}$ is non-empty. Let $\varphi$ (resp. $\psi_n$) be a $\hat{Q}_2$-valued character of $\Delta$ (resp. of $\Gamma_n$ of order $p^n$). Denote by $\varphi_0$ the trivial character of $\Delta$. Theorem 1 (II) is a consequence of the following assertion.

**Proposition 1.** Under the above setting, the following hold.

(I) The class group $A_{\Delta_n}(\varphi \psi_n)$ is nontrivial if $\varphi(l) = 1$ for some $l \in S^{(n)}$. In particular, $A_{\Delta_n}(\varphi \psi_n)$ is nontrivial.

(II) If $A_{\Delta_n}(\varphi \psi_n) = \{0\}$, the converse of the first assertion of (I) holds.

Proof. Applying Corollary 2 for the triple $(k_d, F_n, L_n)$, we see from Lemma 3 that $A_{\Delta_n}(\varphi \psi_n) = \{0\}$ if and only if (i) $(\varphi \psi_n)(l) \neq 1$ for all $l \in S = S_d$ and (ii) $A_{\Delta_n}(\varphi \psi_n) = \{0\}$. We have $\psi_n(l) = 1$ for $l \in S^{(n)}$, and $\psi_n(l) \neq 1$ for $l \in S \setminus S^{(n)}$. Therefore, we see that the condition (i) is satisfied if and only if $\varphi(l) \neq 1$ for all $l \in S^{(n)}$ noting that the orders of $\varphi$ and $\psi_n$ are relatively prime. From this, we obtain the proposition.

We put $M_n = K_n(\sqrt{d}) = K_n L_n$. On the relative class number $h_{M_n}$ of $M_n$, the following assertion holds.

**Proposition 2.** (I) When $n \geq n_d$, the ratio $h_{M_n} / h_{M_{n-1}}$ is odd if and only if $h_{n}^* / h_{n-1}^*$ is odd.

(II) When $n_d \geq 2$ and $1 \leq n < n_d$, $h_{M_n} / h_{M_{n-1}}$ is even.

To prove this proposition, we need to show the following lemma. For an imaginary abelian field $N$, we put

$$E_N = E_N / \mu(N) E_{N^+}.$$  

It is well known that the unit index $Q_N = |E_N|$ is 1 or 2 ([9, Theorem 4.12]).
Lemma 4. Let $T$ and $N$ be imaginary abelian fields with $N \subseteq T$. If the degree $[T : N]$ is odd, then $Q_T = Q_N$.

Proof. We first show that the inclusion map $N \rightarrow T$ induces an injection $E_N \hookrightarrow E_T$. For a unit $\epsilon$ of $N$, assume that $\epsilon = \eta \zeta$ for some $\zeta \in \mu(T)$ and $\eta \in \mathcal{E}_T$. Let $\rho$ be a nontrivial element of the Galois group $G = \text{Gal}(T/N)$. Then, as $\epsilon = e^\rho$, we see that $\zeta^{1-\rho} = \eta^{\rho-1} \in \mu(T) \cap \mathcal{E}_T$. Hence, $\zeta^{1-\rho} = \pm 1$. However, as $N_{T/N}(\zeta^{1-\rho}) = 1$ and $[T : N]$ is odd, the case $\zeta^{1-\rho} = -1$ does not happen. Hence, $\zeta^{1-\rho} = 1$ for all $\rho \in G$. It follows that $\zeta \in \mu(N)$ and hence $\eta \in \mathcal{E}_N$. Therefore, we can regard $E_N$ as a subgroup of $E_T$. In particular, $Q_N$ divides $Q_T$.

Assume that $Q_N \neq Q_T$. Then we have $|E_T| = |E_T/E_N| = 2$. Regarding $E_T$ as a module over $G$, we have a canonical decomposition

$$E_T = E_T/E_N = \bigoplus \mathcal{E}_T(\chi)$$

where $\chi$ runs over a complete set of representatives of the $Q_2$-conjugacy classes of the nontrivial $Q_2$-valued characters of $G$. Hence, $|\mathcal{E}_T(\chi)| = 2$ for some such $\chi$. Let $Z_2[\chi]$ be the subring of $\bar{Q}_2$ generated by the values of $\chi$ over $Z_2$. The group $E_T(\chi)$ is naturally regarded as a module over the principal ideal domain $Z_2[\chi]$. Since the order of $\chi$ is odd and $\geq 3$, we observe that $Z_2[\chi] \cong Z_2^d$ as $Z_2$-modules for some $d \geq 2$. Hence, $|\mathcal{E}_n(\chi)|$ is a multiple of $2^d$, which contradicts $|\mathcal{E}_n(\chi)| = 2$. Therefore, we obtain $Q_N = Q_T$. 

Proof of Proposition 2. By Lemma 4, we have $Q_{M_n} = Q_{M_{n-1}}$ and $Q_{L_n} = Q_{L_{n-1}}$ for all $n \geq 1$. Therefore, using the class number formula [9, Theorem 4.17], we see that

$$h_{M_n}^-/h_{M_{n-1}}^- = p \prod_{\sigma} \prod_{\psi_n} \left(-\frac{1}{2} B_{1, \sigma \psi_n}\right)$$

where $\sigma$ runs over the odd Dirichlet characters associated to $M_0$, and $\psi_n$ over the even characters of conductor $p^{n+1}$ and order $p^n$. Further, $B_{1, \sigma \psi_n}$ denotes the generalized Bernoulli number. We easily see that $\sigma$ equals an odd Dirichlet character associated to $K_0$ or $L_0$ since $M_0/K_0^\sigma$ is an imaginary biquadratic extension with the imaginary quadratic subextensions $K_0$ and $L_0$. Hence, using the class number formulas for $L_n$, $K_n$ and $Q_{L_n} = Q_{L_{n-1}}$, we obtain

$$h_{M_n}^- = h_{M_{n-1}}^- \times h_{n}^+ / h_{n-1}^-$$. 

Therefore, the assertion follows from Theorem 1. 

References


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