CLASS NUMBER PARITY OF A QUADRATIC TWIST OF
A CYCLOTOMIC FIELD OF PRIME POWER CONDUCTOR

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Abstract

Let \( p \) be a fixed odd prime number and \( K_n \) the \( p^{n+1} \)-st cyclotomic field. For a fixed integer \( d \in \mathbb{Z} \) with \( \sqrt{d} \notin K_0 \), denote by \( L_n \) the imaginary quadratic subextension of the biquadratic extension \( K_n(\sqrt{d})/K_n^+ \) with \( L_n \neq K_n \). Let \( h_n^+ \) and \( h_n^- \) be the relative class numbers of \( K_n \) and \( L_n \), respectively. We give an explicit constant \( n_d \) depending on \( p \) and \( d \) such that (i) for any integer \( n \geq n_d \), the ratio \( h_n^-/h_{n-1}^- \) is odd if and only if \( h_n^+/h_{n-1}^+ \) is odd and (ii) for \( 1 \leq n < n_d \), \( h_n^-/h_{n-1}^- \) is even.

1. Introduction

Let \( p \) be a fixed odd prime number. Let \( K_n = \mathbb{Q}(\zeta_{p^{n+1}}) \) be the \( p^{n+1} \)-st cyclotomic field for an integer \( n \geq 0 \), and \( K_\infty = \bigcup_n K_n \). Let \( d \in \mathbb{Z} \) be a fixed integer with \( \sqrt{d} \notin K_0 \). We denote by \( L_n \) the imaginary quadratic subextension of the biquadratic extension \( K_n(\sqrt{d})/K_n^+ \) with \( L_n \neq K_n \). Here, \( K^+ \) denotes the maximal real subfield of an imaginary abelian field \( K \). When \( d < 0 \), we have \( L_n = K_n^+ (\sqrt{d}) \). We call \( L_n \) the quadratic twist of \( K_n \) associated to the integer \( d \). The extension \( L_\infty = \bigcup_n L_n \) is the cyclotomic \( \mathbb{Z}_p \)-extension over \( L_0 \) with the \( n \)-th layer \( L_n \). We call \( L_\infty/L_0 \) the quadratic twist of the cyclotomic \( \mathbb{Z}_p \)-extension \( K_\infty/K_0 \) associated to \( d \). Let \( h_n^+ \) and \( h_n^- \) be the relative class numbers of \( K_n \) and \( L_n \), respectively. It is known and easy to show that \( h_n^+ \) (resp. \( h_n^- \)) divides \( h_n^+(\text{resp.} h_n^-) \) using class field theory. The parity of \( h_n^+ \) behaves rather irregularly when \( p \) varies (see a table in Schoof [6]). However, it is recently shown that when \( p \leq 509 \), the ratio \( h_n^+/h_{n-1}^+ \) is odd for all \( n \geq 1 \) ([3, Theorem 2]). And it might be possible that the ratio is odd for any prime \( p \) and any \( n \geq 1 \). The purpose of this paper is to study the parity of the ratio \( h_n^-/h_{n-1}^- \) of the quadratic twist \( L_n \). We already know that \( h_n^-/h_{n-1}^- \) is odd for sufficiently large \( n \) by a theorem of Washington [8] on the non-\( p \)-part of the class number in a cyclotomic \( \mathbb{Z}_p \)-extension. Denote by \( S = S_d \) the set of prime numbers \( l \neq p \) which ramify in \( \mathbb{Q}(\sqrt{d})/\mathbb{Q} \). The set \( S \) is non-empty as \( \sqrt{d} \notin K_0 \). We define an integer \( n_d \geq 1 \) by

\[
n_d = \max \{ \text{ord}_p(l^{p-1} - 1) \mid l \in S \},
\]
where \( \text{ord}_p(*) \) is the normalized \( p \)-adic additive valuation. The following is the main theorem of this paper.

**Theorem 1.** Under the above setting, the following assertions hold.

(I) When \( n \geq n_d \), the ratio \( h_n^*/h_{n-1}^* \) is odd if and only if \( h_n^*/h_{n-1}^* \) is odd.

(II) When \( n_d \geq 2 \) and \( 1 \leq n < n_d \), the ratio \( h_n^-/h_{n-1}^- \) is even.

From Theorem 1 and [3, Theorem 2], we immediately obtain the following:

**Corollary 1.** Under the above setting, let \( p \) be an odd prime number with \( p \not\equiv 509 \). Then the ratio \( h_n^-/h_{n-1}^- \) is odd for all \( n \geq n_d \).

This corollary, though given in a very special setting, is an explicit version of the above mentioned theorem of Washington. In [4], we showed Theorem 1 when \( d = -1 \) and \( L_n = K_n^+(\sqrt{-1}) \) using some results of cyclotomic Iwasawa theory. In this paper, we prove Theorem 1 by using a main theorem of Conner and Hurrelbrink [1, Theorem 2.3].

**Remark.** When \( p \equiv 1 \mod 4 \) (resp. \( p \equiv 3 \mod 4 \)), we can show that two integers \( d_1 \) and \( d_2 \) give the same twist \( L_\infty/L_0 \) of \( K_\infty/K_0 \) if and only if \( d_2 = \pm d_1 x^2 \) or \( d_2 = p d_1 x^2 \) (resp. \( d_2 = -p d_1 x^2 \)) for some \( x \in \mathbb{Q}^\times \). Hence, the set \( S_d \) and the integer \( n_d \) depend only on the twist \( L_\infty/L_0 \) and not on the choice of \( d \).

2. **Exact hexagon of Conner and Hurrelbrink**

In this section, we recall the exact hexagon of Conner and Hurrelbrink. Let \( k \) be an imaginary abelian field with 2-power degree, and \( F \) a real abelian field with \( 2 \not| \{F : \mathbb{Q}\} \). We put \( K = kF \), and

\[
G = \text{Gal}(K/k) = \text{Gal}(K^+/k^+) = \text{Gal}(F/\mathbb{Q}).
\]

For a number field \( N \), let \( A_N \) be the 2-part of the ideal class group of \( N \), \( O_N \) the ring of integers, and \( E_N = O_N^\times \) the group of units of \( N \). The groups \( A_K \) and \( E_K \) are naturally regarded as modules over \( \text{Gal}(K/K^+) \) and at the same time as those over \( G \). For a \( \text{Gal}(K/K^+) \)-module \( X \), denote by \( H^i(X) = H^i(K/K^+; X) \) the Tate cohomology group with \( i = 0, 1 \). When \( X = A_K \) or \( E_K \), the group \( H^i(X) \) is also regarded as \( G \)-modules. In [1, Theorem 2.3], Conner and Hurrelbrink introduced the following exact hexagon
of $G$-modules to study the 2-part of the class number of a relative quadratic extension.

$$
\xymatrix{
H^1(A_K) \ar[r] & H^1(E_K) \\
R^0(K) \ar[u]^{i_0} \ar[r] & R^1(K) \\
H^0(E_K) \ar[r] & H^0(A_K) \ar[u]}
$$

Here, $R^i(K)$ is a certain $G$-module associated to $K/K^+$ defined in [1]. We describe the $G$-module structure of $R^i(K)$ following [1]. Let $T_f$ be the set of prime ideals $\mathfrak{P}$ of $k^+$ for which a prime ideal $\mathfrak{Q}$ of $K^+$ over $\mathfrak{P}$ ramifies in $K$. Let $T_\infty$ be the set of infinite prime divisors of $k^+$. We put $T = T_f \cup T_\infty$. For each $v \in T$, let $G_v \subseteq G$ be the decomposition group of $v$ at $K^+/k^+$. When $v$ is an infinite prime, the group $G_v$ is trivial. We define $G$-modules $\Omega_f$ and $\Omega_\infty$ by

$$
\Omega_f = \bigoplus_{\mathfrak{P} \in T_f} F_2[G/G_{\mathfrak{P}}] \quad \text{and} \quad \Omega_\infty = \bigoplus_{v \in T_\infty} F_2[G],
$$

respectively, where $F_2 = \mathbb{Z}/2\mathbb{Z}$ is the finite field with two elements. (When $T_f$ is empty, $\Omega_f = \{0\}$ by definition.) For each prime divisor $w$ of $K^+$ with the restriction $w|_{k^+} \in T$ and an element $x \in (K^+)^\times$, we put $t_w(x) = 0$ or 1 according as $x \in N(K_w^\times)$ or not. Here, $K_w$ is the completion of $K$ at the unique prime divisor of $K$ over $w$ and $N = N_{K/K^+}$ is the norm map. For $g \in G$ and $x \in (K^+)^\times$, we see that

$$
t_{w^e}(x) = t_w(x^{w^{-1}})
$$

by local class field theory. For a prime ideal $\mathfrak{P}$ of $K^+$ with $\mathfrak{P} \cap k^+ \in T_f$, let $\mathfrak{Q}$ be the unique prime ideal of $K$ over $\mathfrak{P}$. For an ideal $\mathfrak{A}$ of $K$, writing $\mathfrak{A} = \mathfrak{P} \mathfrak{B}$ with an integer $e$ and an ideal $\mathfrak{B}$ relatively prime to $\mathfrak{P}$, we put $\text{ord}_{\mathfrak{Q}}(\mathfrak{A}) = e$.

We denote by $I(K)$ the group of (fractional) ideals of $K$. Let $X$ be the subgroup of $I(K)$ consisting of ideals $\mathfrak{A}$ with $\mathfrak{A}^J = \mathfrak{A}$. Here, $J$ is the complex conjugation acting on several objects associated to $K$. Let $X_0$ be the subgroup of $X$ consisting of ideals $\mathfrak{A} \in I(K)$ with $\mathfrak{A} = x \mathfrak{B}^{1+J}$ for some $x \in (K^+)^\times$ and $\mathfrak{B} \in I(K)$. The $G$-module $R^1(K)$ is isomorphic to the quotient $X/X_0$. For this, see the lines 1–2 from the bottom of p. 6 and Lemma 2.1 of [1]. For each prime ideal $\mathfrak{P} \in T_f$, we fix a prime ideal $\mathfrak{Q}$ of $K^+$ over $\mathfrak{P}$. From the argument in [1, §5], we obtain the following isomorphism of $G$-modules:

$$
R^1(K) \cong \Omega_f; \quad \mathfrak{A} X_0 \to \bigoplus_{\mathfrak{P} \in T_f} \left( \sum_{\bar{g}} \text{ord}_{\mathfrak{Q}}(\mathfrak{A})\bar{g} \right),
$$

where $\bar{g}$ (with $g \in G$) runs over the quotient $G/G_{\mathfrak{P}}$.  


Let $Y$ be the subgroup of the multiplicative group $(K^+) \times I(K)$ consisting of pairs $(x, \mathfrak{A})$ with $x\mathfrak{A}^{1-\mathfrak{B}} = \mathcal{O}_K$. Let $Y_0$ be the subgroup of $Y$ consisting of pairs $(N(y), y^{-1}\mathfrak{B}^{1-\mathfrak{J}})$ with $y \in K^\times$ and $\mathfrak{B} \in I(K)$. By definition, $R^0(K) = Y/Y_0$. We denote by $[x, \mathfrak{A}] \in R^0(K)$ the class containing $(x, \mathfrak{A})$. The map $i_0$ in the hexagon is defined by

$$i_0: H^0(E_K) = E_{K^+}/N(E_K) \to R^0(K); \quad [\epsilon] \mapsto [\epsilon, \mathcal{O}_K]$$

with $\epsilon \in E_{K^+}$. For each $v \in T_\infty$, we fix a prime divisor $\mathfrak{v}$ of $K^+$ over $v$. Using (1), we observe that the homomorphisms

$$\alpha_\infty: (K^+) \to \Omega_\infty; \quad x \mapsto \bigoplus_{v \in T_\infty} \left( \sum_{\ell \in G} \ell v^\ell(x) \hat{g} \right)$$

and

$$\alpha_f: (K^+) \to \Omega_f; \quad x \mapsto \bigoplus_{\mathfrak{v} \in T_f} \left( \sum_{\mathfrak{q} \in G} \ell q^\ell(x) \hat{g} \right)$$

are compatible with the action of $G$. Further, $\alpha_\infty$ is nothing but the “sign” map. From the argument in [1, §4], we obtain the following exact sequence of $G$-modules:

$$[0] \to R^0(K) \xrightarrow{\alpha} \Omega_f \oplus \Omega_\infty \xrightarrow{\beta} F_2 \to [0].$$

Here, $\alpha$ is defined by $\alpha([x, \mathfrak{A}]) = (\alpha_f(x), \alpha_\infty(x))$, $\beta$ is the argumentation map and $G$ acts trivially on $F_2$.

3. Consequences

In this section, we derive some consequences of the exact hexagon and (2), (3). All of them are $G$-decomposed versions of the corresponding results in [1]. We work under the setting of Section 2. Denote by $\hat{A}_{K^+}$ the 2-part of the narrow class group of $K^+$. Letting $K^+_{>0}$ be the group of totally positive elements of $K^+$, we have an exact sequence

$$[0] \to (K^+) \to A_{K^+} \to \hat{A}_{K^+} \to A_{K^+} \to \{0\}$$

of $G$-modules. We define the minus class group $A_{K}^\times$ to be the kernel of the norm map $A_K \to A_{K^+}$. Let $\chi$ be a $\hat{Q}_2$-valued character of $G = \text{Gal}(K/k) = \text{Gal}(F/Q)$, which we also regard as a primitive Dirichlet character. For a module $M$ over $Z_2[G]$, we denote by $M(\chi)$ the $\chi$-part of $M$. Here, $Z_2$ is the ring of 2-adic integers and $\hat{Q}_2$ is a fixed algebraic closure of the 2-adic rationals $Q_2$. (For the definition of the $\chi$-part and some of its properties, see Tsuji [7, §2].) Denote by $S_K$ the set of prime numbers lying
below some prime ideal in $T_f$. In all what follows, we assume that $\chi$ is a nontrivial character. The following is a version of [1, Theorem 13.8].

**Theorem 2.** Under the above setting, the groups $H^i(K/K^+; A_K(\chi))$ with $i = 0$ and 1 are trivial if and only if

(i) $\chi(l) \neq 1$ for all $l \in S_K$ and

(ii) $|\tilde{A}_K(\chi)| = |A_K(\chi)|$.

The following corollary is a version of [1, Corollary 13.10] and Hasse [2, Satz 45].

**Corollary 2.** Under the above setting, the group $A_K(\chi)$ is trivial if and only if

(i) $\chi(l) \neq 1$ for all $l \in S_K$ and

(ii) $\tilde{A}_K(\chi)$ is trivial.

Let $\tilde{h}_M$ be the class number in the narrow sense of a number field $M$. When $M$ is an imaginary abelian field, let $h_M^-$ be the relative class number of $M$. We can easily show that $h_M^-$ (resp. $\tilde{h}_K^+$) divides $h_K^-$ (resp. $\tilde{h}_K^+$) using class field theory. The following is an immediate consequence of Corollary 2.

**Corollary 3.** Under the above setting, the ratio $h_K^-/h_K^-$ is odd if and only if

(i) no prime number $l$ in $S_K$ splits in $F$ and

(ii) $\tilde{h}_K^+/\tilde{h}_K^+$ is odd.

To prove these assertions, we prepare the following two lemmas. For a number field $L$, let $\mu(L)$ be the group of roots of unity in $L$ and $\mu_2(L)$ the 2-part of $\mu(L)$.

**Lemma 1.** The group $H^1(K/K^+; E_K)(\chi)$ is trivial.

Proof. Let $N_{E_K}$ be the group of units $e \in E_K$ with $N(e) = e^{1+J} = 1$. We have $N(e) = 1$ if and only if $e \in \mu(K)$ by a theorem on units of a CM-field (cf. Washington [9, Theorem 4.12]). Since $\mu(K)^2 = \mu(K)^{1-J} \subseteq E_{K}^{1-J}$, we obtain a surjection

$$\mu(K)/\mu(K)^2 \to H^1(K/K^+; E_K) = N_{E_K}/E_{K}^{1-J}$$

of $G$-modules. However, as $[K:k]$ is odd, we have

$$\mu(K)/\mu(K)^2 = \mu_2(K)/\mu_2(K)^2 = \mu_2(k)/\mu_2(k)^2.$$

Since $\chi$ is nontrivial, the $\chi$-part $(\mu_2(k)/\mu_2(k)^2)(\chi)$ is trivial. Hence, we obtain the assertion.

**Lemma 2.** The natural map $A_K^+(\chi) \to A_K(\chi)$ is injective.
Proof. Denote the natural map $A_{K+} \to A_K$ by $\iota$. Let $\mathfrak{A}$ be an ideal of $K^+$ with the class $[\mathfrak{A}] \in \ker \iota$. Then $\mathfrak{A}O_K = xO_K$ for some $x \in K^\times$. We see that $\epsilon = x^{1-J}$ is a unit of $K$ with $N(\epsilon) = 1$. It is known that the map 

$$\ker \iota \to H^1(K/K^+; E_K); [\mathfrak{A}] \to x^{1-J}E_K^{1-J}$$

is an injective $G$-homomorphism ([1, Theorem 7.1]). Then, from Lemma 1, we see that the $\chi$-part $(\ker \iota)(\chi)$ is trivial, from which we obtain the assertion.

Proof of Theorem 2. Let $\mathfrak{p}$ be a prime ideal in $T/K$, and $l = \mathfrak{p} \cap Q \in S_K$. We see that the $\chi$-part $F_2[G/G_{\mathfrak{p}}](\chi) \neq \{0\}$ if and only if $\chi$ factors through $G/G_{\mathfrak{p}}$, which is equivalent to $\chi(G_{\mathfrak{p}}) = \{1\}$. Since $[k^+:Q]$ is a $2$-power and $[F:Q]$ is odd, we have $\chi(G_{\mathfrak{p}}) = \{1\}$ if and only if $\chi(l) = 1$. Hence, we have shown that the condition (i) in Theorem 2 is equivalent to the condition $\Omega_I(\chi) = \{0\}$. By the hexagon and Lemma 1, we see that $H^0(A_K(\chi))$ and $H^1(A_K(\chi))$ are trivial if and only if (iii) $R^1(K)(\chi) = \{0\}$ and (iv) the map 

$$i_0: H^0(E_K)(\chi) = (E_K+/N(E_K))(\chi) \to R^0(K)(\chi)$$

is an isomorphism. By (2) and the above, the condition (iii) is equivalent to (i). Under the condition (i), we see that $R^0(K)(\chi) = \Omega_\infty(\chi)$ from the exact sequence (3), and that for each class $[\epsilon] \in H^0(E_K)(\chi)$ with $\epsilon \in E^+_K$, we have $i_0([\epsilon]) = \alpha_{\infty}(\epsilon)$ from the definitions of the maps $i_0$ and $\alpha$. Further, the 2-rank of $\Omega_\infty(\chi)$ is larger than or equal to that of $H^0(E_K)(\chi)$ by a theorem of Minkowski on units of a Galois extension (cf. Narkiewicz [5, Theorem 3.26]). Therefore, under (i), we observe that the condition (iv) holds if and only if $\alpha_{\infty}(E_K)(\chi) = \Omega_\infty(\chi)$. We see that the last condition is equivalent to the condition (ii) in Theorem 2 because of the exact sequence (4) and $\alpha_{\infty}(K^+)(\chi) = \Omega_\infty(\chi)$. Therefore, we obtain Theorem 2.

Proof of Corollary 2. First, we show the “only if” part assuming that $A_{K}^-(\chi)$ is trivial. By Lemma 2, we can regard $A_K^+(\chi)$ as a subgroup of $A_K(\chi)$. Assume that $A_K^+(\chi)$ is nontrivial. Then there exists a class $c \in A_K^+(\chi)$ of order 2. We have $c^{-J} = c = c^{-1}$, and hence $c \in A_{K}^-(\chi)$. It follows that $A_{K}^-(\chi)$ is nontrivial, a contradiction. Hence, $A_{K}^+(\chi) = \{0\}$. It follows that $A_{K}^-(\chi)$ is trivial by the exact sequence 

$$\{0\} \to A_{K}^- (\chi) \to A_K(\chi) \xrightarrow{1-J} A_{K}^+(\chi) \to \{0\}.$$ 

Therefore, the “only if” part follows from Theorem 2. Next, assume that the conditions (i) and (ii) in Corollary 2 are satisfied. Then, $A_{K}^+(\chi) = \{0\}$, and the groups $H^i(A_K^+(\chi)) (i = 0, 1)$ are trivial by Theorem 2. As the cohomology groups are trivial, we obtain an exact sequence 

$$\{0\} \to A_{K}^+(\chi) \leftrightarrow A_K(\chi) \xrightarrow{1-J} A_{K}^-(\chi) \equiv A_{K}^-(\chi) \to \{0\}.$$
Since \( A_{K^+}(\chi) = \{0\} \), we see that \( A_K(\chi) = A_{K,\chi}^{-} \), and
\[
A_{K,\chi}^{-}(\chi) = A_{K,\chi}^{-}(\chi)^{1 - J} = A_{K,\chi}^{-}(\chi)^2
\]
from the above exact sequence. Therefore, \( A_{K,\chi}^{-}(\chi) \) is trivial.

4. Proof of Theorem 1

We use the same notation as in Section 1. In particular, \( d \in \mathbb{Z} \) is a fixed integer with \( \sqrt{d} \notin K_0 \) and \( L_n \) is the quadratic twist of \( K_n \) associated to \( d \). We have \( L_n^+ = K_n^+ \).

Let \( k \) (resp. \( k_d \)) be the maximal intermediate field of \( K_0/\mathbb{Q} \) (resp. \( L_0/\mathbb{Q} \)) of 2-power degree, and let \( F_0 \) be the maximal subfield of \( K_0^+ = L_0^+ \) of odd degree over \( \mathbb{Q} \). Then \( k \) and \( k_d \) are imaginary abelian fields with \( k^+ = k_d^+ \). Let \( B_n/\mathbb{Q} \) be the real abelian field with conductor \( p^{n+1} \) and \( [B_n : \mathbb{Q}] = p^n \). We put \( F_n = F_0 B_n \). Then \( L_n = k_d F_n \) and \( K_n = k F_n \). The triples \((k_d, F_n, L_n)\) and \((k, F_n, K_n)\) correspond to \((k, F, K)\) in Sections 2 and 3. We see that

\[
S_{L_n} = S_d \quad \text{or} \quad S_d \cup \{p\}
\]

and \( S_{K_n} = \{p\} \). We put
\[
G_n = \text{Gal}(F_n/\mathbb{Q}) = \text{Gal}(L_n/k_d) = \text{Gal}(K_n/k),
\]
and
\[
\Delta = \text{Gal}(F_0/\mathbb{Q}), \quad \Gamma_n = \text{Gal}(F_n/F_0) = \text{Gal}(B_n/\mathbb{Q}).
\]

Then we have a natural decomposition \( G_n = \Delta \times \Gamma_n \). For characters \( \varphi \) and \( \psi \) of \( \Delta \) and \( \Gamma_n \) respectively, we regard \( \varphi \psi = \varphi \times \psi \) as a character of \( G_n \). Further, we regard \( \varphi \), \( \psi \) and \( \varphi \psi \) also as primitive Dirichlet characters. The class groups \( A_{L_n}^+, A_{K,n}^+ \) and \( \tilde{A}_{K,n}^+ \) are modules over \( G_n \). We can naturally regard \( A_{L_n^{-1}}^- \) as a subgroup of \( A_{L_n}^- \) since \( L_n/L_{n^{-1}} \) is a cyclic extension of degree \( p \neq 2 \) and \( A_{L_n^{-1}}^- \) is the 2-part of the class group. Actually, it is a direct summand of \( A_{L_n}^- \) (cf. [9, Lemma 16.15]). We see that

\[
A_{L_n}^-/A_{L_{n^{-1}}}^- = \bigoplus_{\varphi, \psi_n} A_{L_n}^- (\varphi \psi_n)
\]

where \( \varphi \) (resp. \( \psi_n \)) runs over a complete set of representatives of the \( \mathbb{Q}_2 \)-conjugacy classes of the \( \tilde{Q}_2 \)-valued characters of \( \Delta \) (resp. \( \Gamma_n \) of order \( p^n \)). Regarding \( A_{K,n}^- \) as a subgroup of \( A_{K,n}^+ \), we have a similar decomposition for \( A_{K,n}^-/A_{K,n^{-1}}^- \). As \( S_{K,n} = \{p\} \) and \((\varphi \psi_n)(p) = 0\), we obtain the following assertion from Corollary 2 for the triple \((k, F_n, K_n)\).

**Lemma 3.** Let \( n \geq 1 \) be an integer, and the characters \( \varphi \) and \( \psi_n \) be as in (6). Then \( A_{K,n}^- (\varphi \psi_n) = \{0\} \) if and only if \( \tilde{A}_{K,n}^+ (\varphi \psi_n) = \{0\} \).
Proof of Theorem 1 (I). Let \( \varphi \) and \( \psi_n \) be as in (6). As the orders of \( \varphi \) and \( \psi_n \) are relatively prime to each other, we have \((\varphi \psi_n)(l) = 1\) if and only if \( \varphi(l) = \psi_n(l) = 1 \) for a prime number \( l \). Let \( n \) be an integer with \( n \geq n_d \). Then we have \( \psi_n(l) \neq 1 \) and hence \((\varphi \psi_n)(l) \neq 1\) for all prime numbers \( l \in S = S_d \). Further, we have \((\varphi \psi_n)(p) = 0\). Hence, by (5), the condition (i) in Corollary 2 for the triple \((k_d, F_n, L_n)\) is satisfied. It follows that the condition \( A^+_n(\varphi \psi_n) = \{0\} \) is equivalent to \( \hat{A}^+_n(\varphi \psi_n) = \{0\} \). (Note that \( L^+_n = K^+_n \).) Therefore, we obtain Theorem 1 (I) from Lemma 3.

To show Theorem 1 (II), assume that \( n_d \geq 2 \) and let \( n \) be an integer with \( 1 \leq n < n_d \). We put

\[ S^{(n)} = \{ l \in S = S_d \mid \text{ord} \,(l^{p-1} - 1) \geq n + 1 \}. \]

From the definition, we see that

\[ S \supseteq S^{(1)} \supseteq S^{(2)} \supseteq \cdots \supseteq S^{(n-1)} \]

and that each \( S^{(n)} \) is non-empty. Let \( \varphi \) (resp. \( \psi_n \)) be a \( \hat{Q}_2 \)-valued character of \( \Delta \) (resp. of \( \Gamma_n \) of order \( p^n \)). Denote by \( \varphi_0 \) the trivial character of \( \Delta \). Theorem 1 (II) is a consequence of the following assertion.

**Proposition 1.** Under the above setting, the following hold.

(I) The class group \( A^+_{L_n}(\varphi \psi_n) \) is nontrivial if \( \varphi(l) = 1 \) for some \( l \in S^{(n)} \). In particular, \( A^+_{L_n}(\varphi_0 \psi_n) \) is nontrivial.

(II) If \( A^+_{K_n}(\varphi \psi_n) = \{0\} \), the converse of the first assertion of (I) holds.

Proof. Applying Corollary 2 for the triple \((k_d, F_n, L_n)\), we see from Lemma 3 that \( A^+_{L_n}(\varphi \psi_n) = \{0\} \) if and only if (i) \((\varphi \psi_n)(l) \neq 1\) for all \( l \in S = S_d \) and (ii) \( A^+_{L_n}(\varphi \psi_n) = \{0\} \). We have \( \psi_n(l) = 1 \) for \( l \in S^{(n)} \), and \( \psi_n(l) \neq 1 \) for \( l \in S \setminus S^{(n)} \). Therefore, we see that the condition (i) is satisfied if and only if \( \varphi(l) \neq 1 \) for all \( l \in S^{(n)} \) noting that the orders of \( \varphi \) and \( \psi_n \) are relatively prime. From this, we obtain the proposition.

We put \( M_n = K_n(\sqrt{d}) = K_n L_n \). On the relative class number \( h_{M_n}^- \) of \( M_n \), the following assertion holds.

**Proposition 2.** (I) When \( n \geq n_d \), the ratio \( h_{M_n}^- / h_{M_n^{v_1}}^- \) is odd if and only if \( h_{n}^n / h_{n-1}^n \) is odd.

(II) When \( n_d \geq 2 \) and \( 1 \leq n < n_d \), \( h_{M_n}^- / h_{M_n^{v_1}}^- \) is even.

To prove this proposition, we need to show the following lemma. For an imaginary abelian field \( N \), we put

\[ E_N = E_N / \mu(N)E_N^+ \]

It is well known that the unit index \( Q_N = \vert E_N \vert \) is 1 or 2 ([9, Theorem 4.12]).
Lemma 4. Let \( T \) and \( N \) be imaginary abelian fields with \( N \subseteq T \). If the degree \([T : N]\) is odd, then \( Q_T = Q_N \).

Proof. We first show that the inclusion map \( N \to T \) induces an injection \( \mathcal{E}_N \to \mathcal{E}_T \). For a unit \( \epsilon \) of \( N \), assume that \( \epsilon = \zeta \eta \) for some \( \zeta \in \mu(T) \) and \( \eta \in E_T^+ \). Let \( \rho \) be a nontrivial element of the Galois group \( G = \text{Gal}(T/N) \). Then, as \( \epsilon = \epsilon^\rho \), we see that \( \zeta^{1-\rho} = \eta^{\rho-1} \in \mu(T) \cap E_T^+ \). Hence, \( \zeta^{1-\rho} = \pm 1 \). However, as \( N_{T/N}(\zeta^{1-\rho}) = 1 \) and \([T : N]\) is odd, the case \( \zeta^{1-\rho} = -1 \) does not happen. Hence, \( \zeta^{1-\rho} = 1 \) for all \( \rho \in G \). It follows that \( \zeta \in \mu(N) \) and hence \( \eta \in E_N^+ \). Therefore, we can regard \( \mathcal{E}_N \) as a subgroup of \( \mathcal{E}_T \). In particular, \( Q_N \) divides \( Q_T \).

Assume that \( Q_N \neq Q_T \). Then we have \( [\mathcal{E}_T] = [\mathcal{E}_T/\mathcal{E}_N] = 2 \). Regarding \( \mathcal{E}_T \) as a module over \( G \), we have a canonical decomposition

\[
\mathcal{E}_T = \mathcal{E}_T/\mathcal{E}_N = \bigoplus_{\chi} \mathcal{E}_T(\chi)
\]

where \( \chi \) runs over a complete set of representatives of the \( \mathbb{Q}_2 \)-conjugacy classes of the nontrivial \( \mathbb{Q}_2 \)-valued characters of \( G \). Hence, \([\mathcal{E}_T(\chi)] = 2\) for some such \( \chi \). Let \( \mathbb{Z}_2[\chi] \) be the subring of \( \mathbb{Q}_2 \) generated by the values of \( \chi \) over \( \mathbb{Z}_2 \). The group \( \mathcal{E}_T(\chi) \) is naturally regarded as a module over the principal ideal domain \( \mathbb{Z}_2[\chi] \). Since the order of \( \chi \) is odd and \( \geq 3 \), we observe that \( \mathbb{Z}_2[\chi] \cong \mathbb{Z}_2^d \) as \( \mathbb{Z}_2 \)-modules for some \( d \geq 2 \). Hence, \([\mathcal{E}_n(\chi)]\) is a multiple of \( 2^d \), which contradicts \([\mathcal{E}_n(\chi)] = 2\). Therefore, we obtain \( Q_N = Q_T \). \( \square \)

Proof of Proposition 2. By Lemma 4, we have \( Q_{M_n} = Q_{M_{n-1}} \) and \( Q_{L_n} = Q_{L_{n-1}} \) for all \( n \geq 1 \). Therefore, using the class number formula [9, Theorem 4.17], we see that

\[
h_{M_n}/h_{M_{n-1}} = p \prod_{\sigma} \prod_{\psi_n} \left( -\frac{1}{2} B_{1,\sigma,\psi_n} \right)
\]

where \( \sigma \) runs over the odd Dirichlet characters associated to \( M_0 \), and \( \psi_n \) over the even characters of conductor \( p^{n+1} \) and order \( p^n \). Further, \( B_{1,\sigma,\psi_n} \) denotes the generalized Bernoulli number. We easily see that \( \sigma \) equals an odd Dirichlet character associated to \( K_0 \) or \( L_0 \) since \( M_0/K_0^+ \) is an imaginary biquadratic extension with the imaginary quadratic subextensions \( K_0 \) and \( L_0 \). Hence, using the class number formulas for \( L_n \), \( K_n \) and \( Q_{L_n} = Q_{L_{n-1}} \), we obtain

\[
h_{M_n}/h_{M_{n-1}} = h^*_n/h^*_n \times h_n^*/h_{n-1}^*.
\]

Therefore, the assertion follows from Theorem 1. \( \square \)
References


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