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Osaka University
CLASS NUMBER PARITY OF A QUADRATIC TWIST OF A CYCLOTOMIC FIELD OF PRIME POWER CONDUCTOR

HUMIO ICHIMURA

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Abstract

Let \( p \) be a fixed odd prime number and \( K_n \) the \( p^{n+1} \)-st cyclotomic field. For a fixed integer \( d \in \mathbb{Z} \) with \( \sqrt{d} \notin K_0 \), denote by \( L_n \) the imaginary quadratic subextension of the biquadratic extension \( K_n(\sqrt{d})/K_n^+ \) with \( L_n \neq K_n \). Let \( h_n^+ \) and \( h_n^- \) be the relative class numbers of \( K_n \) and \( L_n \), respectively. We give an explicit constant \( n_d \) depending on \( p \) and \( d \) such that (i) for any integer \( n \geq n_d \), the ratio \( h_n^-/h_{n-1}^- \) is odd if and only if \( h_n^+/h_{n-1}^+ \) is odd and (ii) for \( 1 \leq n < n_d \), \( h_n^-/h_{n-1}^- \) is even.

1. Introduction

Let \( p \) be a fixed odd prime number. Let \( K_n = \mathbb{Q}(\xi_{p^{n+1}}) \) be the \( p^{n+1} \)-st cyclotomic field for an integer \( n \geq 0 \), and \( K_{\infty} = \bigcup_n K_n \). Let \( d \in \mathbb{Z} \) be a fixed integer with \( \sqrt{d} \notin K_0 \). We denote by \( L_n \) the imaginary quadratic subextension of the biquadratic extension \( K_n(\sqrt{d})/K_n^+ \) with \( L_n \neq K_n \). Here, \( K^+ \) denotes the maximal real subfield of an imaginary abelian field \( K \). When \( d < 0 \), we have \( L_n = K_n^+(\sqrt{d}) \). We call \( L_n \) the quadratic twist of \( K_n \) associated to the integer \( d \). The extension \( L_{\infty} = \bigcup_n L_n \) is the cyclotomic \( \mathbb{Z}_p \)-extension over \( L_0 \) with the \( n \)-th layer \( L_n \). We call \( L_{\infty}/L_0 \) the quadratic twist of the cyclotomic \( \mathbb{Z}_p \)-extension \( K_{\infty}/K_0 \) associated to \( d \). Let \( h_n^+ \) and \( h_n^- \) be the relative class numbers of \( K_n \) and \( L_n \), respectively. It is known and easy to show that \( h_{n-1}^+ \) (resp. \( h_{n-1}^- \)) divides \( h_n^+ \) (resp. \( h_n^- \)) using class field theory. The parity of \( h_n^+ \) behaves rather irregularly when \( p \) varies (see a table in Schoof [6]). However, it is recently shown that when \( p \leq 509 \), the ratio \( h_n^+/h_{n-1}^+ \) is odd for all \( n \geq 1 \) ([3, Theorem 2]). And it might be possible that the ratio is odd for any prime \( p \) and any \( n \geq 1 \). The purpose of this paper is to study the parity of the ratio \( h_n^-/h_{n-1}^- \) of the quadratic twist \( L_n \). We already know that \( h_n^-/h_{n-1}^- \) is odd for sufficiently large \( n \) by a theorem of Washington [8] on the non-\( p \)-part of the class number in a cyclotomic \( \mathbb{Z}_p \)-extension. Denote by \( S = S_d \) the set of prime numbers \( l \neq p \) which ramify in \( \mathbb{Q}(\sqrt{d})/\mathbb{Q} \). The set \( S \) is non-empty as \( \sqrt{d} \notin K_0 \). We define an integer \( n_d \geq 1 \) by

\[
n_d = \max\{\text{ord}_p(l^{p-1} - 1) \mid l \in S\},
\]

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where \( \text{ord}_p(*) \) is the normalized \( p \)-adic additive valuation. The following is the main theorem of this paper.

**Theorem 1.** Under the above setting, the following assertions hold.

(I) When \( n \geq n_d \), the ratio \( h_n^-/h_{n-1}^- \) is odd if and only if \( h_n^*/h_{n-1}^* \) is odd.

(II) When \( n_d \geq 2 \) and \( 1 \leq n < n_d \), the ratio \( h_n^-/h_{n-1}^- \) is even.

From Theorem 1 and [3, Theorem 2], we immediately obtain the following:

**Corollary 1.** Under the above setting, let \( p \) be an odd prime number with \( p \equiv 509 \). Then the ratio \( h_n^-/h_{n-1}^- \) is odd for all \( n \geq n_d \).

This corollary, though given in a very special setting, is an explicit version of the above mentioned theorem of Washington. In [4], we showed Theorem 1 when \( d = -1 \) and \( L_n = K_n^+(\sqrt{-1}) \) using some results of cyclotomic Iwasawa theory. In this paper, we prove Theorem 1 by using a main theorem of Conner and Hurrelbrink [1, Theorem 2.3].

**Remark.** When \( p \equiv 1 \mod 4 \) (resp. \( p \equiv 3 \mod 4 \)), we can show that two integers \( d_1 \) and \( d_2 \) give the same twist \( L_\infty/L_0 \) of \( K_\infty/K_0 \) if and only if \( d_2 = d_1 x^2 \) or \( d_2 = pd_1 x^2 \) (resp. \( d_2 = -pd_1 x^2 \)) for some \( x \in \mathbb{Q}^\times \). Hence, the set \( S_d \) and the integer \( n_d \) depend only on the twist \( L_\infty/L_0 \) and not on the choice of \( d \).

**2. Exact hexagon of Conner and Hurrelbrink**

In this section, we recall the exact hexagon of Conner and Hurrelbrink. Let \( k \) be an imaginary abelian field with 2-power degree, and \( F \) a real abelian field with \( 2 \nmid [F : \mathbb{Q}] \). We put \( K = kF \), and

\[
G = \text{Gal}(K/k) = \text{Gal}(K^+/k^+) = \text{Gal}(F/\mathbb{Q}).
\]

For a number field \( N \), let \( A_N \) be the 2-part of the ideal class group of \( N \), \( \mathcal{O}_N \) the ring of integers, and \( E_N = \mathcal{O}_N^\times \) the group of units of \( N \). The groups \( A_K \) and \( E_K \) are naturally regarded as modules over \( \text{Gal}(K/K^+) \) and at the same time as those over \( G \). For a \( \text{Gal}(K/K^+) \)-module \( X \), denote by \( H^i(X) = H^i(K/K^+; X) \) the Tate cohomology group with \( i = 0, 1 \). When \( X = A_K \) or \( E_K \), the group \( H^i(X) \) is also regarded as \( G \)-modules. In [1, Theorem 2.3], Conner and Hurrelbrink introduced the following exact hexagon.
of $G$-modules to study the 2-part of the class number of a relative quadratic extension.

$$
\xymatrix{
H^1(A_K) \ar[r] & H^1(E_K) \ar[dl]_{i_0} & \\
R^0(K) & H^0(E_K) & R^1(K)
}
$$

Here, $R^i(K)$ is a certain $G$-module associated to $K/K^+$ defined in [1]. We describe the $G$-module structure of $R^i(K)$ following [1]. Let $T_f$ be the set of prime ideals $\mathfrak{p}$ of $k^+$ for which a prime ideal $\mathfrak{P}$ of $K^+$ over $\mathfrak{p}$ ramifies in $K$. Let $T_\infty$ be the set of infinite prime divisors of $k^+$. We put $T = T_f \cup T_\infty$. For each $v \in T$, let $G_v \subseteq G$ be the decomposition group of $v$ at $K^+/k^+$. When $v$ is an infinite prime, the group $G_v$ is trivial. We define $G$-modules $\Omega_f$ and $\Omega_\infty$ by

$$
\Omega_f = \bigoplus_{\mathfrak{p} \in T_f} F_2[G/G_{\mathfrak{p}}] \quad \text{and} \quad \Omega_\infty = \bigoplus_{v \in T_\infty} F_2[G/G_v] = \bigoplus_{v \in T_\infty} F_2[G],
$$

respectively, where $F_2 = \mathbb{Z}/2\mathbb{Z}$ is the finite field with two elements. (When $T_f$ is empty, $\Omega_f = \{0\}$ by definition.) For each prime divisor $w$ of $K^+$ with the restriction $w_{K^+} \in T$ and an element $x \in (K^+)^\times$, we put $t_w(x) = 0$ or $1$ according as $x \in N(K_\infty^+/K_\infty)$ or not. Here, $K_w$ is the completion of $K$ at the unique prime divisor of $K$ over $w$ and $N = N_{K/K^+}$ is the norm map. For $g \in G$ and $x \in (K^+)^\times$, we see that

$$
t_{w,f}(x) = t_w(x^{e_{\mathfrak{p}}})
$$

by local class field theory. For a prime ideal $\mathfrak{P}$ of $K^+$ with $\mathfrak{P} \cap k^+ \in T_f$, let $\mathfrak{P}$ be the unique prime ideal of $K$ over $\mathfrak{P}$. For an ideal $\mathfrak{A}$ of $K$, writing $\mathfrak{A} = \mathfrak{P}\mathfrak{B}$ with an integer $e$ and an ideal $\mathfrak{B}$ relatively prime to $\mathfrak{P}$, we put $\text{ord}_{\mathfrak{P}}(\mathfrak{A}) = e$.

We denote by $I(K)$ the group of (fractional) ideals of $K$. Let $X$ be the subgroup of $I(K)$ consisting of ideals $\mathfrak{A}$ with $\mathfrak{A}^{-J} = \mathfrak{A}$. Here, $J$ is the complex conjugation acting on several objects associated to $K$. Let $X_0$ be the subgroup of $X$ consisting of ideals $\mathfrak{A} \in I(K)$ with $\mathfrak{A} = x\mathfrak{B}^{1+J}$ for some $x \in (K^+)^*$ and $\mathfrak{B} \in I(K)$. The $G$-module $R^1(K)$ is isomorphic to the quotient $X/X_0$. For this, see the lines 1–2 from the bottom of p.6 and Lemma 2.1 of [1]. For each prime ideal $\mathfrak{p} \in T_f$, we fix a prime ideal $\mathfrak{P}$ of $K^+$ over $\mathfrak{p}$. From the argument in [1, §5], we obtain the following isomorphism of $G$-modules:

$$
R^1(K) \cong \Omega_f; \quad \mathfrak{A}X_0 \rightarrow \bigoplus_{\mathfrak{p} \in T_f} \left( \sum_{\tilde{g}} \text{ord}_{\mathfrak{P}}(\mathfrak{A}) \tilde{g} \right),
$$

where $\tilde{g}$ (with $g \in G$) runs over the quotient $G/G_{\mathfrak{p}}$. 


Let $Y$ be the subgroup of the multiplicative group $(K^+)\times I(K)$ consisting of pairs $(x, \mathfrak{A})$ with $x\mathfrak{A}^{1+J} = \mathcal{O}_K$. Let $Y_0$ be the subgroup of $Y$ consisting of pairs $(N(y), y^{-1}\mathfrak{B}^{1-J})$ with $y \in K^\times$ and $\mathfrak{B} \in I(K)$. By definition, $R^0(K) = Y/Y_0$. We denote by $[x, \mathfrak{A}] \in R^0(K)$ the class containing $(x, \mathfrak{A})$. The map $i_0$ in the hexagon is defined by

$$i_0: H^0(E_K) = E_{K^+}/N(E_K) \to R^0(K): [\varepsilon] \to [\varepsilon, \mathcal{O}_K]$$

with $\varepsilon \in E_{K^+}$. For each $v \in T_\infty$, we fix a prime divisor $\mathfrak{v}$ of $K^+$ over $v$. Using (1), we observe that the homomorphisms

$$\alpha_\infty: (K^+)\times \to \Omega_\infty; \quad x \to \bigoplus_{v \in T_\infty} \left( \sum_{g \in G} t_v^g(x)\mathfrak{g} \right)$$

and

$$\alpha_f: (K^+)\times \to \Omega_f; \quad x \to \bigoplus_{\mathfrak{v} \in T_f} \left( \sum_{\mathfrak{g} \in \mathcal{O}_K} t_{\mathfrak{v}}^{\mathfrak{g}}(x)\mathfrak{g} \right)$$

are compatible with the action of $G$. Further, $\alpha_\infty$ is nothing but the “sign” map. From the argument in [1, §4], we obtain the following exact sequence of $G$-modules:

$$\{0\} \to R^0(K) \overset{\alpha}{\to} \Omega_f \oplus \Omega_\infty \overset{\beta}{\to} F_2 \to \{0\}.$$  

Here, $\alpha$ is defined by $\alpha([x, \mathfrak{A}]) = (\alpha_f(x), \alpha_\infty(x))$, $\beta$ is the argumentation map and $G$ acts trivially on $F_2$.

### 3. Consequences

In this section, we derive some consequences of the exact hexagon and (2), (3). All of them are $G$-decomposed versions of the corresponding results in [1]. We work under the setting of Section 2. Denote by $A_{K^+}$ the 2-part of the narrow class group of $K^+$. Letting $K_{2,0}^+$ be the group of totally positive elements of $K^+$, we have an exact sequence

$$\{0\} \to (K^+)\times/(K_{2,0}^+ E_{K^+}) \to A_{K^+} \to A_{K^+} \to \{0\}$$

of $G$-modules. We define the minus class group $A_{K}^{-}$ to be the kernel of the norm map $A_K \to A_{K^+}$. Let $\chi$ be a $\mathcal{O}_2$-valued character of $G = \text{Gal}(K/k) = \text{Gal}(F/Q)$, which we also regard as a primitive Dirichlet character. For a module $M$ over $\mathbb{Z}_2[G]$, we denote by $M(\chi)$ the $\chi$-part of $M$. Here, $\mathbb{Z}_2$ is the ring of 2-adic integers and $\mathcal{O}_2$ is a fixed algebraic closure of the 2-adic rationals $\mathcal{O}_2$. (For the definition of the $\chi$-part and some of its properties, see Tsuji [7, §2].) Denote by $S_K$ the set of prime numbers lying
below some prime ideal in $T_f$. In all what follows, we assume that $\chi$ is a nontrivial character. The following is a version of [1, Theorem 13.8].

**Theorem 2.** Under the above setting, the groups $H^i(K/K^+; A_K)(\chi)$ with $i = 0$ and 1 are trivial if and only if

1. $\chi(l) \neq 1$ for all $l \in S_K$ and
2. $|\tilde{A}_K(\chi)| = |A_K(\chi)|$.

The following corollary is a version of [1, Corollary 13.10] and Hasse [2, Satz 45].

**Corollary 2.** Under the above setting, the group $A_K^-(\chi)$ is trivial if and only if

1. $\chi(l) \neq 1$ for all $l \in S_K$ and
2. $\tilde{A}_K^+(\chi)$ is trivial.

Let $\tilde{h}_M$ be the class number in the narrow sense of a number field $M$. When $M$ is an imaginary abelian field, let $h_M^-$ be the relative class number of $M$. We can easily show that $h_M^-$ (resp. $\tilde{h}_K^+$) divides $h_K^-$ (resp. $\tilde{h}_K^+$) using class field theory. The following is an immediate consequence of Corollary 2.

**Corollary 3.** Under the above setting, the ratio $h_K^-/h_K^-$ is odd if and only if

1. no prime number $l$ in $S_K$ splits in $F$ and
2. $\tilde{h}_K^+/\tilde{h}_K^-$ is odd.

To prove these assertions, we prepare the following two lemmas. For a number field $L$, let $\mu(L)$ be the group of roots of unity in $L$ and $\mu_2(L)$ the 2-part of $\mu(L)$.

**Lemma 1.** The group $H^1(K/K^+; E_K)(\chi)$ is trivial.

Proof. Let $N E_K$ be the group of units $\epsilon \in E_K$ with $N(\epsilon) = \epsilon^{1/J} = 1$. We have $N(\epsilon) = 1$ if and only if $\epsilon \in \mu(K)$ by a theorem on units of a CM-field (cf. Washington [9, Theorem 4.12]). Since $\mu(K)^2 = \mu(K)^{1/J} \subseteq E_K^{1/J}$, we obtain a surjection

$$\mu(K)/\mu(K)^2 \to H^1(K/K^+; E_K) = N E_K/E_K^{1/J}$$

of $G$-modules. However, as $[K:k]$ is odd, we have

$$\mu(K)/\mu(K)^2 = \mu_2(K)/\mu_2(K)^2 = \mu_2(k)/\mu_2(k)^2.$$ 

Since $\chi$ is nontrivial, the $\chi$-part $(\mu_2(k)/\mu_2(k)^2)(\chi)$ is trivial. Hence, we obtain the assertion.

**Lemma 2.** The natural map $A_K^+(\chi) \to A_K(\chi)$ is injective.
Proof. Denote the natural map \( A_K^+ \to A_K \) by \( \iota \). Let \( \mathfrak{A} \) be an ideal of \( K^+ \) with the class \([\mathfrak{A}] \in \ker \iota\). Then \( \mathfrak{A}O_K = xO_K \) for some \( x \in K^\times \). We see that \( \epsilon = x^{1-J} \) is a unit of \( K \) with \( N(\epsilon) = 1 \). It is known that the map

\[
\ker \iota \to H^1(K/K^+; E_K); \ [\mathfrak{A}] \to x^{1-J}E_K^{1-J}
\]

is an injective \( G \)-homomorphism ([1, Theorem 7.1]). Then, from Lemma 1, we see that the \( \chi \)-part \( (\ker \iota)(\chi) \) is trivial, from which we obtain the assertion.

Proof of Theorem 2. Let \( \wp \) be a prime ideal in \( T_f \), and \( l = \wp \cap Q \in S_K \). We see that the \( \chi \)-part \( F_2[G/G_{\wp}](_{\chi}) \neq \{0\} \) if and only if \( \chi \) factors through \( G/G_{\wp} \), which is equivalent to \( \chi(G_{\wp}) = \{1\} \). Since \([k^+: Q] \) is a 2-power and \([F: Q] \) is odd, we have \( \chi(G_{\wp}) = \{1\} \) if and only if \( \chi(l) = 1 \). Hence, we have shown that the condition (i) in Theorem 2 is equivalent to the condition \( \Omega_f(\chi) = \{0\} \). By the hexagon and Lemma 1, we see that \( H^0(A_K(\chi)) \) and \( H^1(A_K(\chi)) \) are trivial if and only if (iii) \( R^1(K)(\chi) = \{0\} \) and (iv) the map

\[
i_0: H^0(E_K(\chi)) = (E_K^+/N(E_K))(\chi) \to R^0(K)(\chi)
\]

is an isomorphism. By (2) and the above, the condition (iii) is equivalent to (i). Under the condition (i), we see that \( R^0(K)(\chi) = \Omega_{\infty}(\chi) \) from the exact sequence (3), and that for each class \( [\epsilon] \in H^0(E_K)(\chi) \) with \( \epsilon \in E_K^+ \), we have \( i_0([\epsilon]) = \alpha_{\infty}(\epsilon) \) from the definitions of the maps \( i_0 \) and \( \alpha \). Further, the 2-rank of \( \Omega_{\infty}(\chi) \) is larger than or equal to that of \( H^0(E_K)(\chi) \) by a theorem of Minkowski on units of a Galois extension (cf. Narkiewicz [5, Theorem 3.26]). Therefore, under (i), we observe that the condition (iv) holds if and only if \( \alpha_{\infty}(E_K^+)(\chi) = \Omega_{\infty}(\chi) \). We see that the last condition is equivalent to the condition (ii) in Theorem 2 because of the exact sequence (4) and \( \alpha_{\infty}((K^+)^\times)(\chi) = \Omega_{\infty}(\chi) \). Therefore, we obtain Theorem 2.

Proof of Corollary 2. First, we show the “only if” part assuming that \( A_K^-(\chi) \) is trivial. By Lemma 2, we can regard \( A_K^-(\chi) \) as a subgroup of \( A_K(\chi) \). Assume that \( A_K^+(\chi) \) is nontrivial. Then there exists a class \( c \in A_K^+(\chi) \) of order 2. We have \( c^J = c = c^{-1} \), and hence \( c \in A_K^-(\chi) \). It follows that \( A_K^-(\chi) \) is nontrivial, a contradiction. Hence, \( A_K^+(\chi) = \{0\} \). It follows that \( A_K(\chi) \) is trivial by the exact sequence

\[
\{0\} \to A_K^-(\chi) \to A_K(\chi) \xrightarrow{1+J} A_K^+(\chi) \to \{0\}.
\]

Therefore, the “only if” part follows from Theorem 2. Next, assume that the conditions (i) and (ii) in Corollary 2 are satisfied. Then, \( A_K^+(\chi) = \{0\} \), and the groups \( H^i(A_K(\chi)) \) \( (i = 0, 1) \) are trivial by Theorem 2. As the cohomology groups are trivial, we obtain an exact sequence

\[
\{0\} \to A_K^+(\chi) \to A_K(\chi) \xrightarrow{1-J} A_K^{-J}(\chi) \to A_K^-(\chi) \to \{0\}.
\]
Since $A_{K^+}(\chi) = \{0\}$, we see that $A_k(\chi) = A_{\overline{K}}(\chi)$, and

$$A_{\overline{K}}(\chi) = A_{\overline{K}}(\chi)^{1-f} = A_{\overline{K}}(\chi)^2$$

from the above exact sequence. Therefore, $A_{\overline{K}}(\chi)$ is trivial.

**4. Proof of Theorem 1**

We use the same notation as in Section 1. In particular, $d \in \mathbb{Z}$ is a fixed integer with $\sqrt{d} \notin K_0$ and $L_n$ is the quadratic twist of $K_n$ associated to $d$. We have $L_n^+ = K_n^+$. Let $k$ (resp. $k_d$) be the maximal intermediate field of $K_0/\mathbb{Q}$ (resp. $L_0/\mathbb{Q}$) of 2-power degree, and let $F_0$ be the maximal subfield of $K_0^+ = L_0^+$ of odd degree over $\mathbb{Q}$. Then $k$ and $k_d$ are imaginary abelian fields with $k^+ = k_d^+$. Let $B_n/\mathbb{Q}$ be the real abelian field with conductor $p^{n+1}$ and $[B_n : \mathbb{Q}] = p^n$. We put $F_n = F_0B_n$. Then $L_n = k_dF_n$ and $K_n = kF_n$. The triples $(k_d, F_n, L_n)$ and $(k, F, K)$ correspond to $(k, F, K)$ in Sections 2 and 3. We see that

$$S_n = S_d \quad \text{or} \quad S_d \cup \{p\}$$

and $S_{K_n} = \{p\}$. We put

$$G_n = \text{Gal}(F_n/\mathbb{Q}) = \text{Gal}(L_n/k_d) = \text{Gal}(K_n/k),$$

and

$$\Delta = \text{Gal}(F_0/\mathbb{Q}), \quad \Gamma_n = \text{Gal}(F_n/F_0) = \text{Gal}(B_n/\mathbb{Q}).$$

Then we have a natural decomposition $G_n = \Delta \times \Gamma_n$. For characters $\varphi$ and $\psi$ of $\Delta$ and $\Gamma_n$ respectively, we regard $\varphi \psi' = \varphi \times \psi'$ as a character of $G_n$. Further, we regard $\varphi$, $\psi$ and $\varphi \psi$ also as primitive Dirichlet characters. The class groups $A_{L_n}$, $A_{\overline{K}_n}$ and $\tilde{A}_{K_n^+}$ are modules over $G_n$. We can naturally regard $A_{L_{n-1}}^-$ as a subgroup of $A_{L_n}^-$ since $L_n/L_{n-1}$ is a cyclic extension of degree $p \neq 2$ and $A_{L_{n-1}}^-$ is the 2-part of the class group. Actually, it is a direct summand of $A_{L_n}^-$ (cf. [9, Lemma 16.15]). We see that

$$A_{L_n}^-/A_{L_{n-1}}^- = \bigoplus_{\varphi, \psi_n} A_{L_n}^- (\varphi \psi_n)$$

where $\varphi$ (resp. $\psi_n$) runs over a complete set of representatives of the $\mathbb{Q}_2$-conjugacy classes of the $\mathbb{Q}_2$-valued characters of $\Delta$ (resp. $\Gamma_n$ of order $p^n$). Regarding $A_{K_n^-}$ as a subgroup of $A_{\overline{K}_n^+}$, we have a similar decomposition for $A_{K_n^-}/A_{K_{n-1}^-}$. As $S_{K_n} = \{p\}$ and $(\varphi \psi_n)(p) = 0$, we obtain the following assertion from Corollary 2 for the triple $(k, F_n, K_n)$.

**Lemma 3.** Let $n \geq 1$ be an integer, and the characters $\varphi$ and $\psi_n$ be as in (6). Then $A_{\overline{K}_n}^-(\varphi \psi_n) = \{0\}$ if and only if $\tilde{A}_{K_n^+}(\varphi \psi_n) = \{0\}.$
Proof of Theorem 1 (I). Let $\varphi$ and $\psi_n$ be as in (6). As the orders of $\varphi$ and $\psi_n$ are relatively prime to each other, we have $(\varphi \psi_n)(l) = 1$ if and only if $\varphi(l) = \psi_n(l) = 1$ for a prime number $l$. Let $n$ be an integer with $n \geq n_d$. Then we have $\psi_n(l) \neq 1$ and hence $(\varphi \psi_n)(l) \neq 1$ for all prime numbers $l \in S = S_d$. Further, we have $(\varphi \psi_n)(p) = 0$. Hence, by (5), the condition (i) in Corollary 2 for the triple $(k_d, F_n, L_n)$ is satisfied. It follows that the condition $A^{-}_L(\varphi \psi_n) = \{0\}$ is equivalent to $\hat{A}^{-}_{K_n^+}(\varphi \psi_n) = \{0\}$. (Note that $L_n^+ = K_n^+$. Therefore, we obtain Theorem 1 (I) from Lemma 3.

To show Theorem 1 (II), assume that $n_d \geq 2$ and let $n$ be an integer with $1 \leq n < n_d$. We put

$$S^{(n)} = \{l \in S = S_d \mid \text{ord}_n(l^{p-1} - 1) \geq n + 1\}.$$ 

From the definition, we see that

$$S \supseteq S^{(1)} \supseteq S^{(2)} \supseteq \cdots \supseteq S^{(n-1)}$$

and that each $S^{(n)}$ is non-empty. Let $\varphi$ (resp. $\psi_n$) be a $\hat{Q}_2$-valued character of $\Delta$ (resp. of $\Gamma_n$ of order $p^n$). Denote by $\varphi_0$ the trivial character of $\Delta$. Theorem 1 (II) is a consequence of the following assertion.

Proposition 1. Under the above setting, the following hold.

(I) The class group $A^{-}_L(\varphi \psi_n)$ is nontrivial if $\varphi(l) = 1$ for some $l \in S^{(n)}$. In particular, $A^{-}_L(\varphi_0 \psi_n)$ is nontrivial.

(II) If $A^{-}_L(\varphi \psi_n) = \{0\}$, the converse of the first assertion of (I) holds.

Proof. Applying Corollary 2 for the triple $(k_d, F_n, L_n)$, we see from Lemma 3 that $A^{-}_L(\varphi \psi_n) = \{0\}$ if and only if (i) $(\varphi \psi_n)(l) \neq 1$ for all $l \in S = S_d$ and (ii) $A^{-}_L(\varphi_0 \psi_n) = \{0\}$. We have $\psi_n(l) = 1$ for $l \in S^{(n)}$, and $\psi_n(l) \neq 1$ for $l \in S \setminus S^{(n)}$. Therefore, we see that the condition (i) is satisfied if and only if $\varphi(l) \neq 1$ for all $l \in S^{(n)}$ noting that the orders of $\varphi$ and $\psi_n$ are relatively prime. From this, we obtain the proposition.

We put $M_n = K_n(\sqrt{d}) = K_nL_n$. On the relative class number $h^{-}_{M_n}$ of $M_n$, the following assertion holds.

Proposition 2. (I) When $n \geq n_d$, the ratio $h^{-}_{M_n}/h^{-}_{M_n \omega^i}$ is odd if and only if $h^{*}_{n}/h^{*}_{n-1}$ is odd.

(II) When $n_d \geq 2$ and $1 \leq n < n_d$, $h^{-}_{M_n}/h^{-}_{M_n \omega^i}$ is even.

To prove this proposition, we need to show the following lemma. For an imaginary abelian field $N$, we put

$$E_N = E_N/\mu(N)E_N^+.$$ 

It is well known that the unit index $Q_N = |E_N|$ is 1 or 2 ([9, Theorem 4.12]).
Lemma 4. Let $T$ and $N$ be imaginary abelian fields with $N \subseteq T$. If the degree 
$[T : N]$ is odd, then $Q_T = Q_N$.

Proof. We first show that the inclusion map $N \rightarrow T$ induces an injection $E_N \hookrightarrow E_T$. For a unit $\epsilon$ of $N$, assume that $\epsilon = \zeta \eta$ for some $\zeta \in \mu(T)$ and $\eta \in E_T^\times$. Let $\rho$ be a nontrivial element of the Galois group $G = \text{Gal}(T/N)$. Then, as $\epsilon = e^\rho$, we see that $\zeta^{1-\rho} = \eta^{\rho-1} \in \mu(T) \cap E_T^\times$. Hence, $\zeta^{1-\rho} = \pm 1$. However, as $N_{T/N}(\zeta^{1-\rho}) = 1$ and $[T : N]$ is odd, the case $\zeta^{1-\rho} = -1$ does not happen. Hence, $\zeta^{1-\rho} = 1$ for all $\rho \in G$. It follows that $\zeta \in \mu(N)$ and hence $\eta \in E_N^\times$. Therefore, we can regard $E_N$ as a subgroup of $E_T$. In particular, $Q_N$ divides $Q_T$.

Assume that $Q_N \neq Q_T$. Then we have $|E_T| = |E_T/E_N| = 2$. Regarding $E_T$ as a module over $G$, we have a canonical decomposition

$$E_T = E_T/E_N = \bigoplus_\chi E_T(\chi)$$

where $\chi$ runs over a complete set of representatives of the $Q_2$-conjugacy classes of the nontrivial $Q_2$-valued characters of $G$. Hence, $|E_T(\chi)| = 2$ for some such $\chi$. Let $Z_2[\chi]$ be the subring of $Q_2$ generated by the values of $\chi$ over $Z_2$. The group $E_T(\chi)$ is naturally regarded as a module over the principal ideal domain $Z_2[\chi]$. Since the order of $\chi$ is odd and $\geq 3$, we observe that $Z_2[\chi] \cong Z_2^d$ as $Z_2$-modules for some $d \geq 2$. Hence, $|E_n(\chi)|$ is a multiple of $2^d$, which contradicts $|E_n(\chi)| = 2$. Therefore, we obtain $Q_N = Q_T$. □

Proof of Proposition 2. By Lemma 4, we have $Q_{M_n} = Q_{M_{n-1}}$ and $Q_{L_n} = Q_{L_{n-1}}$ for all $n \geq 1$. Therefore, using the class number formula [9, Theorem 4.17], we see that

$$h_{M_n}/h_{M_{n-1}} = p \prod_{\sigma} \prod_{\psi_n} \left( -1 \right) \frac{1}{2} B_{1, \sigma \psi_n}$$

where $\sigma$ runs over the odd Dirichlet characters associated to $M_0$, and $\psi_n$ over the even characters of conductor $p^{n+1}$ and order $p^n$. Further, $B_{1, \sigma \psi_n}$ denotes the generalized Bernoulli number. We easily see that $\sigma$ equals an odd Dirichlet character associated to $K_0$ or $L_0$ since $M_0/K_0$ is an imaginary biquadratic extension with the imaginary quadratic subextensions $K_0$ and $L_0$. Hence, using the class number formulas for $L_n$, $K_n$ and $Q_{L_n} = Q_{L_{n-1}}$, we obtain

$$h_{M_n}/h_{M_{n-1}} = h_n^*/h_{n-1}^* \times h_{n-1}^-/h_{n-2}^-.$$

Therefore, the assertion follows from Theorem 1. □
References


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