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Osaka University
CLASS NUMBER PARITY OF A QUADRATIC TWIST OF
A CYCLOTOMIC FIELD OF PRIME POWER CONDUCTOR

HUMIO ICHIMURA

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Abstract

Let $p$ be a fixed odd prime number. Let $K_n = \mathbb{Q}(\zeta_{p^n+1})$ be the $p^{n+1}$-st cyclotomic field for an integer $n \geq 0$, and $K_\infty = \bigcup_n K_n$. Let $d \in \mathbb{Z}$ be a fixed integer with $\sqrt{d} \notin K_0$. We denote by $L_n$ the imaginary quadratic subextension of the biquadratic extension $K_n(\sqrt{d})/K_n^+$ with $L_n \neq K_n$. Let $h_n^+$ and $h_n^-$ be the relative class numbers of $K_n$ and $L_n$, respectively. We give an explicit constant $\alpha_d$ depending on $p$ and $d$ such that (i) for any integer $n \geq n_d$, the ratio $h_n^-/h_{n-1}^-$ is odd if and only if $h_n^+/h_{n-1}^+$ is even and (ii) for $1 \leq n < n_d$, $h_n^-/h_{n-1}^-$ is even.

1. Introduction

Let $p$ be a fixed odd prime number. Let $K_n = \mathbb{Q}(\zeta_{p^n+1})$ be the $p^{n+1}$-st cyclotomic field for an integer $n \geq 0$, and $K_\infty = \bigcup_n K_n$. Let $d \in \mathbb{Z}$ be a fixed integer with $\sqrt{d} \notin K_0$. We denote by $L_n$ the imaginary quadratic subextension of the biquadratic extension $K_n(\sqrt{d})/K_n^+$ with $L_n \neq K_n$. Here, $K^+$ denotes the maximal real subfield of an imaginary abelian field $K$. When $d < 0$, we have $L_n = K_n^+(\sqrt{d})$. We call $L_n$ the quadratic twist of $K_n$ associated to the integer $d$. The extension $L_\infty = \bigcup_n L_n$ is the cyclotomic $\mathbb{Z}_p$-extension over $L_0$ with the $n$-th layer $L_n$. We call $L_\infty/L_0$ the quadratic twist of the cyclotomic $\mathbb{Z}_p$-extension $K_\infty/K_0$ associated to $d$. Let $h_n^+$ and $h_n^-$ be the relative class numbers of $K_n$ and $L_n$, respectively. It is known and easy to show that $h_{n-1}^+$ (resp. $h_{n-1}^-$) divides $h_n^+$ (resp. $h_n^-$) using class field theory. The parity of $h_n^+$ behaves rather irregularly when $p$ varies (see a table in Schoof [6]). However, it is recently shown that when $p \leq 509$, the ratio $h_n^+/h_{n-1}^-$ is odd for all $n \geq 1$ ([3, Theorem 2]). And it might be possible that the ratio is odd for any prime $p$ and any $n \geq 1$. The purpose of this paper is to study the parity of the ratio $h_n^-/h_{n-1}^-$ of the quadratic twist $L_n$. We already know that $h_n^-/h_{n-1}^-$ is odd for sufficiently large $n$ by a theorem of Washington [8] on the non-$p$-part of the class number in a cyclotomic $\mathbb{Z}_p$-extension. Denote by $S = S_d$ the set of prime numbers $l \neq p$ which ramify in $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. The set $S$ is non-empty as $\sqrt{d} \notin K_0$. We define an integer $n_d \geq 1$ by

$$n_d = \max\{\ord_p(l^{p-1} - 1) \mid l \in S\},$$

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where \( \text{ord}_p(*) \) is the normalized \( p \)-adic additive valuation. The following is the main theorem of this paper.

**Theorem 1.** Under the above setting, the following assertions hold.

(I) When \( n \geq n_d \), the ratio \( h_n^-/h_{n-1}^- \) is odd if and only if \( h_n^+/h_{n-1}^+ \) is odd.

(II) When \( n_d \geq 2 \) and \( 1 \leq n < n_d \), the ratio \( h_n^-/h_{n-1}^- \) is even.

From Theorem 1 and [3, Theorem 2], we immediately obtain the following:

**Corollary 1.** Under the above setting, let \( p \) be an odd prime number with \( p \equiv 509 \). Then the ratio \( h_n^-/h_{n-1}^- \) is odd for all \( n \geq n_d \).

This corollary, though given in a very special setting, is an explicit version of the above mentioned theorem of Washington. In [4], we showed Theorem 1 when \( d = 1 \) and \( L_n = K_n^+(\sqrt{-1}) \) using some results of cyclotomic Iwasawa theory. In this paper, we prove Theorem 1 by using a main theorem of Conner and Hurrelbrink [1, Theorem 2.3].

**Remark.** When \( p \equiv 1 \mod 4 \) (resp. \( p \equiv 3 \mod 4 \)), we can show that two integers \( d_1 \) and \( d_2 \) give the same twist \( L_\infty/L_0 \) of \( K_\infty/K_0 \) if and only if \( d_2 = d_1x^2 \) or \( d_2 = pd_1x^2 \) (resp. \( d_2 = -pd_1x^2 \)) for some \( x \in \mathbb{Q}^\times \). Hence, the set \( S_d \) and the integer \( n_d \) depend only on the twist \( L_\infty/L_0 \) and not on the choice of \( d \).

2. Exact hexagon of Conner and Hurrelbrink

In this section, we recall the exact hexagon of Conner and Hurrelbrink. Let \( k \) be an imaginary abelian field with 2-power degree, and \( F \) a real abelian field with \( 2 \nmid [F : \mathbb{Q}] \). We put \( K = kF \), and

\[
G = \text{Gal}(K/k) = \text{Gal}(K^+/k^+) = \text{Gal}(F/\mathbb{Q}).
\]

For a number field \( N \), let \( A_N \) be the 2-part of the ideal class group of \( N \), \( \mathcal{O}_N \) the ring of integers, and \( E_N = \mathcal{O}_N^\times \) the group of units of \( N \). The groups \( A_K \) and \( E_K \) are naturally regarded as modules over \( \text{Gal}(K/K^+) \) and at the same time as those over \( G \). For a \( \text{Gal}(K/K^+) \)-module \( X \), denote by \( H^i(X) = H^i(K/K^+; X) \) the Tate cohomology group with \( i = 0, 1 \). When \( X = A_K \) or \( E_K \), the group \( H^i(X) \) is also regarded as \( G \)-modules. In [1, Theorem 2.3], Conner and Hurrelbrink introduced the following exact hexagon
of $G$-modules to study the 2-part of the class number of a relative quadratic extension.

$$
\begin{align*}
H^1(A_K) &\longrightarrow H^1(E_K) \\
R^0(K) &\longleftarrow i_0 & R^1(K) \\
H^0(E_K) &\longleftarrow H^0(A_K)
\end{align*}
$$

Here, $R^i(K)$ is a certain $G$-module associated to $K/K^+$ defined in [1]. We describe the $G$-module structure of $R^i(K)$ following [1]. Let $T_f$ be the set of prime ideals $\wp$ of $k^+$ for which a prime ideal $\mathfrak{P}$ of $K^+$ over $\wp$ ramifies in $K$. Let $T_\infty$ be the set of infinite prime divisors of $k^+$. We put $T = T_f \cup T_\infty$. For each $v \in T$, let $G_v \subseteq G$ be the decomposition group of $v$ at $K^+/k^+$. When $v$ is an infinite prime, the group $G_v$ is trivial. We define $G$-modules $\Omega_f$ and $\Omega_\infty$ by

$$
\Omega_f = \bigoplus_{\wp \in T_f} F_2[G/G_\wp] \quad \text{and} \quad \Omega_\infty = \bigoplus_{v \in T_\infty} F_2[G] = \bigoplus_{v \in T_\infty} F_2[G],
$$

respectively, where $F_2 = \mathbb{Z}/2\mathbb{Z}$ is the finite field with two elements. (When $T_f$ is empty, $\Omega_f = \{0\}$ by definition.) For each prime divisor $w$ of $K^+$ with the restriction $w_{|K^+} \in T$ and an element $x \in (K^+)^\times$, we put $t_w(x) = 0$ or 1 according as $x \in N(K_w^\times)$ or not. Here, $K_w$ is the completion of $K$ at the unique prime divisor of $K$ over $w$ and $N = N_{K/K^+}$ is the norm map. For $g \in G$ and $x \in (K^+)^\times$, we see that

$$
t_{w\cdot x}(x) = t_w(x^{e_1})
$$

by local class field theory. For a prime ideal $\mathfrak{P}$ of $K^+$ with $\mathfrak{P} \cap k^+ = T_f$, let $\tilde{\mathfrak{P}}$ be the unique prime ideal of $K$ over $\mathfrak{P}$. For an ideal $\mathfrak{A}$ of $K$, writing $\mathfrak{A} = \tilde{\mathfrak{P}}\mathfrak{B}$ with an integer $e$ and an ideal $\mathfrak{B}$ relatively prime to $\tilde{\mathfrak{P}}$, we put $\text{ord}_{\mathfrak{P}}(\mathfrak{A}) = e$.

We denote by $I(K)$ the group of (fractional) ideals of $K$. Let $X$ be the subgroup of $I(K)$ consisting of ideals $\mathfrak{A}$ with $\mathfrak{A}^J = \mathfrak{A}$. Here, $J$ is the complex conjugation acting on several objects associated to $K$. Let $X_0$ be the subgroup of $X$ consisting of ideals $\mathfrak{A} \in I(K)$ with $\mathfrak{A} = x\mathfrak{B}^{1+J}$ for some $x \in (K^+)^\times$ and $\mathfrak{B} \in I(K)$. The $G$-module $R^1(K)$ is isomorphic to the quotient $X/X_0$. For this, see the lines 1–2 from the bottom of p.6 and Lemma 2.1 of [1]. For each prime ideal $\wp \in T_f$, we fix a prime ideal $\mathfrak{P}$ of $K^+$ over $\wp$. From the argument in [1, §5], we obtain the following isomorphism of $G$-modules:

$$
R^1(K) \cong \Omega_f; \quad \mathfrak{A}X_0 \rightarrow \bigoplus_{\wp \in T_f} \left( \sum_{\mathfrak{g}} \text{ord}_{\mathfrak{P}}(\mathfrak{A})\tilde{\mathfrak{g}} \right),
$$

where $\tilde{\mathfrak{g}}$ (with $g \in G$) runs over the quotient $G/G_\wp$. 
Let \( Y \) be the subgroup of the multiplicative group \((K^+)\times I(K)\) consisting of pairs \((x, \mathfrak{A})\) with \(x \mathfrak{A}^{1-1} = \mathcal{O}_K\). Let \( Y_0 \) be the subgroup of \( Y \) consisting of pairs \((N(y), y^{-1}\mathfrak{A}^{1-1})\) with \(y \in K^+\) and \(\mathfrak{A} \in I(K)\). By definition, \( R^0(K) = Y/Y_0 \). We denote by \([x, \mathfrak{A}] \in R^0(K)\) the class containing \((x, \mathfrak{A})\). The map \(i_0\) in the hexagon is defined by

\[
i_0 : H^0(E_K) = E_{K^+}/N(E_K) \to R^0(K); \quad [\epsilon] \to [\epsilon, \mathcal{O}_K]
\]

with \(\epsilon \in E_{K^+}\). For each \(v \in T_\infty\), we fix a prime divisor \(\mathfrak{p}\) of \(K^+\) over \(v\). Using (1), we observe that the homomorphisms

\[
\alpha_\infty : (K^+) \to \Omega; \quad x \to \bigoplus_{v \in T_\infty} \left( \sum_{\epsilon \in \mathcal{O}_K} \tau_v(x) g \right)
\]

and

\[
\alpha_f : (K^+) \to \Omega_f; \quad x \to \bigoplus_{v \in T_f} \left( \sum_{\mathfrak{g} \in \mathcal{O}_K} \tau_v(x) g \right)
\]

are compatible with the action of \(G\). Further, \(\alpha_\infty\) is nothing but the “sign” map. From the argument in [1, §4], we obtain the following exact sequence of \(G\)-modules:

\[
\begin{array}{c}
\{0\} \to R^0(K) \xrightarrow{\alpha} \Omega_f \oplus \Omega_{\infty} \xrightarrow{\beta} F_2 \to \{0\}.
\end{array}
\]

Here, \(\alpha\) is defined by \(\alpha([x, \mathfrak{A}]) = (\alpha_f(x), \alpha_\infty(x))\), \(\beta\) is the argumentation map and \(G\) acts trivially on \(F_2\).

3. Consequences

In this section, we derive some consequences of the exact hexagon and (2), (3). All of them are \(G\)-decomposed versions of the corresponding results in [1]. We work under the setting of Section 2. Denote by \(\mathcal{A}_{K^+}\) the 2-part of the narrow class group of \(K^+\). Letting \(K^+_{\geq 0}\) be the group of totally positive elements of \(K^+\), we have an exact sequence

\[
\begin{array}{c}
\{0\} \to (K^+) \to (K^+_{\geq 0}E_{K^+}) \to \mathcal{A}_{K^+} \to A_{K^+} \to \{0\}
\end{array}
\]

of \(G\)-modules. We define the minus class group \(A_K^-\) to be the kernel of the norm map \(A_K \to A_{K^+}\). Let \(\chi\) be a \(\mathcal{O}_2\)-valued character of \(G = \text{Gal}(K/k) = \text{Gal}(F/Q)\), which we also regard as a primitive Dirichlet character. For a module \(M\) over \(Z_2[G]\), we denote by \(M(\chi)\) the \(\chi\)-part of \(M\). Here, \(Z_2\) is the ring of 2-adic integers and \(\mathcal{O}_2\) is a fixed algebraic closure of the 2-adic rationals \(Q_2\). (For the definition of the \(\chi\)-part and some of its properties, see Tsuji [7, §2].) Denote by \(S_K\) the set of prime numbers lying
below some prime ideal in $T_f$. In all what follows, we assume that $\chi$ is a nontrivial character. The following is a version of [1, Theorem 13.8].

**Theorem 2.** Under the above setting, the groups $H^i(K/K^+; A_K)(\chi)$ with $i = 0$ and $1$ are trivial if and only if

(i) $\chi(l) \neq 1$ for all $l \in S_K$ and

(ii) $|\tilde{A}_K(\chi)| = |A_K(\chi)|$.

The following corollary is a version of [1, Corollary 13.10] and Hasse [2, Satz 45].

**Corollary 2.** Under the above setting, the group $A_K(\chi)$ is trivial if and only if

(i) $\chi(l) \neq 1$ for all $l \in S_K$ and

(ii) $\tilde{A}_K(\chi)$ is trivial.

Let $\tilde{h}_M$ be the class number in the narrow sense of a number field $M$. When $M$ is an imaginary abelian field, let $h_M$ be the relative class number of $M$. We can easily show that $h_K$ (resp. $\tilde{h}_K$) divides $h_K$ (resp. $\tilde{h}_K$) using class field theory. The following is an immediate consequence of Corollary 2.

**Corollary 3.** Under the above setting, the ratio $h_K/\tilde{h}_K$ is odd if and only if

(i) no prime number $l$ in $S_K$ splits in $F$ and

(ii) $\tilde{h}_K$ is odd.

To prove these assertions, we prepare the following two lemmas. For a number field $L$, let $\mu(L)$ be the group of roots of unity in $L$ and $\mu_2(L)$ the 2-part of $\mu(L)$.

**Lemma 1.** The group $H^1(K/K^+; E_K)(\chi)$ is trivial.

Proof. Let $N_{E_K}$ be the group of units $\epsilon \in E_K$ with $N(\epsilon) = \epsilon^{1+J} = 1$. We have $N(\epsilon) = 1$ if and only if $\epsilon \in \mu(K)$ by a theorem on units of a CM-field (cf. Washington [9, Theorem 4.12]). Since $\mu(K)^2 = \mu(K)^{1-J} \subseteq E_K^{1-J}$, we obtain a surjection

$$
\mu(K)/\mu(K)^2 \twoheadrightarrow H^1(K/K^+; E_K) = N_{E_K}/E_K^{1-J}
$$

of $G$-modules. However, as $[K:k]$ is odd, we have

$$
\mu(K)/\mu(K)^2 = \mu_2(K)/\mu_2(K)^2 = \mu_2(k)/\mu_2(k)^2.
$$

Since $\chi$ is nontrivial, the $\chi$-part $(\mu_2(k)/\mu_2(k)^2)(\chi)$ is trivial. Hence, we obtain the assertion.

**Lemma 2.** The natural map $A_K(\chi) \rightarrow A_K(\chi)$ is injective.
Proof. Denote the natural map \( A_{K^+} \to A_K \) by \( \iota \). Let \( \mathfrak{A} \) be an ideal of \( K^+ \) with the class \([\mathfrak{A}] \in \ker \iota \). Then \( \mathfrak{A}O_K = xO_K \) for some \( x \in K^\times \). We see that \( \varepsilon = x^{1-J} \) is a unit of \( K \) with \( N(\varepsilon) = 1 \). It is known that the map

\[
\ker \iota \to H^1(K/K^+; E_K); [\mathfrak{A}] \to x^{1-J} E_K^{1-J}
\]

is an injective \( G \)-homomorphism ([1, Theorem 7.1]). Then, from Lemma 1, we see that the \( \chi \)-part \((\ker \iota)(\chi) \) is trivial, from which we obtain the assertion.

Proof of Theorem 2. Let \( \wp \) be a prime ideal in \( T_f \), and \( l = \wp \cap \mathcal{Q} \in S_K \). We see that the \( \chi \)-part \( F_2[G/G_\wp](\chi) \neq \{0\} \) if and only if \( \chi \) factors through \( G/G_\wp \), which is equivalent to \( \chi(G_\wp) = \{1\} \). Since \( [k^+: \mathcal{Q}] \) is a 2-power and \([F: \mathcal{Q}] \) is odd, we have \( \chi(G_\wp) = \{1\} \) if and only if \( \chi(l) = 1 \). Hence, we have shown that the condition (i) in Theorem 2 is equivalent to the condition \( \Omega_f(\chi) = \{0\} \). By the hexagon and Lemma 1, we see that \( H^0(A_K(\chi)) \) and \( H^1(A_K(\chi)) \) are trivial if and only if (iii) \( R^1(K)(\chi) = \{0\} \) and (iv) the map

\[
i_0; H^0(E_K(\chi)) = (E_{K^+}/N(E_K))(\chi) \to R^0(K)(\chi)
\]

is an isomorphism. By (2) and the above, the condition (iii) is equivalent to (i). Under the condition (i), we see that \( R^0(K)(\chi) = \Omega_\infty(\chi) \) from the exact sequence (3), and that for each class \([\varepsilon] \in H^0(E_K(\chi)) \) with \( \varepsilon \in E_{K^+} \), we have \( i_0([\varepsilon]) = \alpha_{\infty}(\varepsilon) \) from the definitions of the maps \( i_0 \) and \( \alpha \). Further, the 2-rank of \( \Omega_\infty(\chi) \) is larger than or equal to that of \( H^0(E_K(\chi)) \) by a theorem of Minkowski on units of a Galois extension (cf. Narkiewicz [5, Theorem 3.26]). Therefore, under (i), we observe that the condition (iv) holds if and only if \( \alpha_{\infty}(E_{K^+})(\chi) = \Omega_{\infty}(\chi) \). We see that the last condition is equivalent to the condition (ii) in Theorem 2 because of the exact sequence (4) and \( \alpha_{\infty}(E_{K^+})(\chi) = \Omega_{\infty}(\chi) \). Therefore, we obtain Theorem 2.

Proof of Corollary 2. First, we show the “only if” part assuming that \( A^-_{K}(\chi) \) is trivial. By Lemma 2, we can regard \( A_{K^+}(\chi) \) as a subgroup of \( A_K(\chi) \). Assume that \( A_{K^+}(\chi) \) is nontrivial. Then there exists a class \( c \in A_{K^+}(\chi) \) of order 2. We have \( c^f = c = c^{-1} \), and hence \( c \in A^-_{K}(\chi) \). It follows that \( A^-_{K}(\chi) \) is nontrivial, a contradiction. Hence, \( A_{K^+}(\chi) = \{0\} \). It follows that \( A_{K^+}(\chi) \) is trivial by the exact sequence

\[
\{0\} \to A^-_{K}(\chi) \to A_K(\chi) \xrightarrow{1+J} A_{K^+}(\chi) \to \{0\}.
\]

Therefore, the “only if” part follows from Theorem 2. Next, assume that the conditions (i) and (ii) in Corollary 2 are satisfied. Then, \( A_{K^+}(\chi) = \{0\} \), and the groups \( H^i(A_K(\chi)) \) \((i = 0, 1) \) are trivial by Theorem 2. As the cohomology groups are trivial, we obtain an exact sequence

\[
\{0\} \to A_{K^+}(\chi) \to A_{K^+}(\chi) \xrightarrow{1-J} A_{K^-}(\chi) \equiv A^-_{K}(\chi) \to \{0\}.
\]
Since $A_K^+(\chi) = \{0\}$, we see that $A_K(\chi) = A_K^-(\chi)$, and

$$A_K^-(\chi) = A_K^-(\chi)^{1 - J} = A_K^-(\chi)^2$$

from the above exact sequence. Therefore, $A_K^-(\chi)$ is trivial.

4. Proof of Theorem 1

We use the same notation as in Section 1. In particular, $d \in \mathbb{Z}$ is a fixed integer with $\sqrt{d} \not\in K_0$ and $L_n$ is the quadratic twist of $K_n$ associated to $d$. We have $L_n^+ = K_n^+$. Let $k$ (resp. $k_d$) be the maximal intermediate field of $K_0/Q$ (resp. $L_0/Q$) of 2-power degree, and let $F_0$ be the maximal subfield of $K_0^+ = L_0^+$ of odd degree over $Q$. Then $k$ and $k_d$ are imaginary abelian fields with $k^+ = k_d^+$. Let $B_n/Q$ be the real abelian field with conductor $p^{n+1}$ and $[B_n : Q] = p^n$. We put $F_n = F_0B_n$. Then $L_n = k_dF_n$ and $K_n = kF_n$. The triples $(k_d, F_n, L_n)$ and $(k, F, K)$ correspond to $(k, F, K)$ in Sections 2 and 3. We see that

$$S_{L_n} = S_d \quad \text{or} \quad S_d \cup \{p\}$$

and $S_{K_n} = \{p\}$. We put

$$G_n = \text{Gal}(F_n/Q) = \text{Gal}(L_n/k_d) = \text{Gal}(K_n/k),$$

and

$$\Delta = \text{Gal}(F_0/Q), \quad \Gamma_n = \text{Gal}(F_n/F_0) = \text{Gal}(B_n/Q).$$

Then we have a natural decomposition $G_n = \Delta \times \Gamma_n$. For characters $\varphi$ and $\psi$ of $\Delta$ and $\Gamma_n$ respectively, we regard $\varphi \psi = \varphi \times \psi$ as a character of $G_n$. Further, we regard $\varphi$, $\psi$ and $\varphi \psi$ also as primitive Dirichlet characters. The class groups $A_{L_n}$, $A_{K_n}$ and $\tilde{A}_{K_n}^+$ are modules over $G_n$. We can naturally regard $A_{L_n}^-$ as a subgroup of $A_{L_n}$ since $L_n/L_{n-1}$ is a cyclic extension of degree $p \neq 2$ and $A_{L_n}^-$ is the 2-part of the class group. Actually, it is a direct summand of $A_{L_n}$ (cf. [9, Lemma 16.15]). We see that

$$A_{L_n}^- / A_{L_{n-1}}^- = \bigoplus_{\varphi, \psi_n} A_{L_n}^-(\varphi \psi_n)$$

where $\varphi$ (resp. $\psi_n$) runs over a complete set of representatives of the $Q_2$-conjugacy classes of the $\tilde{Q}_2$-valued characters of $\Delta$ (resp. $\Gamma_n$ of order $p^n$). Regarding $A_{K_n}^+$ as a subgroup of $A_{K_n}^+$, we have a similar decomposition for $A_{K_n}^- / A_{K_{n-1}}^-$. As $S_{K_n} = \{p\}$ and $(\varphi \psi_n)(p) = 0$, we obtain the following assertion from Corollary 2 for the triple $(k, F_n, K_n)$.

Lemma 3. Let $n \geq 1$ be an integer, and the characters $\varphi$ and $\psi_n$ be as in (6). Then $A_{K_n}^-(\varphi \psi_n) = \{0\}$ if and only if $\tilde{A}_{K_n}^+(\varphi \psi_n) = \{0\}$.
Proof of Theorem 1 (I). Let \( \varphi \) and \( \psi_n \) be as in (6). As the orders of \( \varphi \) and \( \psi_n \) are relatively prime to each other, we have \((\varphi \psi_n)(l) = 1\) if and only if \(\varphi(l) = \psi_n(l) = 1\) for a prime number \(l\). Let \(n\) be an integer with \(n \geq n_d\). Then we have \(\psi_n(l) \neq 1\) and hence \((\varphi \psi_n)(l) \neq 1\) for all prime numbers \(l \in S = S_d\). Further, we have \((\varphi \psi_n)(p) = 0\).

Hence, by (5), the condition (i) in Corollary 2 for the triple \((k_d, F_n, L_n)\) is satisfied. It follows that the condition \(A_{K_n}^-(\varphi \psi_n) = \{0\}\) is equivalent to \(\hat{A}_{K_n^+}(\varphi \psi_n) = \{0\}\). (Note that \(L_n^+ = K_n^+\).) Therefore, we obtain Theorem 1 (I) from Lemma 3.

To show Theorem 1 (II), assume that \(n_d \geq 2\) and let \(n\) be an integer with \(1 \leq n < n_d\). We put

\[
S^{(n)} = \{l \in S = S_d \mid \text{ord}_l(l^{p^n-1} - 1) \geq n + 1\}.
\]

From the definition, we see that

\[
S \supseteq S^{(1)} \supseteq S^{(2)} \supseteq \cdots \supseteq S^{(n-1)}
\]

and that each \(S^{(n)}\) is non-empty. Let \(\varphi\) (resp. \(\psi_n\)) be a \(\hat{Q}_2\)-valued character of \(\Delta\) (resp. of \(\Gamma_n\) of order \(p^n\)). Denote by \(\varphi_0\) the trivial character of \(\Delta\). Theorem 1 (II) is a consequence of the following assertion.

**Proposition 1.** Under the above setting, the following hold.

(I) The class group \(A_{K_n}^-(\varphi \psi_n)\) is nontrivial if \(\varphi(l) = 1\) for some \(l \in S^{(n)}\). In particular, \(A_{K_n}^-(\varphi_0 \psi_n)\) is nontrivial.

(II) If \(A_{K_n}^-(\varphi \psi_n) = \{0\}\), the converse of the first assertion of (I) holds.

Proof. Applying Corollary 2 for the triple \((k_d, F_n, L_n)\), we see from Lemma 3 that \(A_{K_n}^-(\varphi \psi_n) = \{0\}\) if and only if (i) \((\varphi \psi_n)(l) \neq 1\) for all \(l \in S = S_d\) and (ii) \(A_{K_n}^-(\varphi \psi_n) = \{0\}\). We have \(\psi_n(l) = 1\) for \(l \in S^{(n)}\), and \(\psi_n(l) \neq 1\) for \(l \in S \setminus S^{(n)}\). Therefore, we see that the condition (i) is satisfied if and only if \(\varphi(l) \neq 1\) for all \(l \in S^{(n)}\) noting that the orders of \(\varphi\) and \(\psi_n\) are relatively prime. From this, we obtain the proposition.

We put \(M_n = K_n(\sqrt{d}) = K_nL_n\). On the relative class number \(h_{M_n}^-\) of \(M_n\), the following assertion holds.

**Proposition 2.** (I) When \(n \geq n_d\), the ratio \(h_{M_n}^-/h_{M_n,1}\) is odd if and only if \(h_{n}^*/h_{n-1}^*\) is odd.

(II) When \(n_d \geq 2\) and \(1 \leq n < n_d\), \(h_{M_n}^-/h_{M_n,1}\) is even.

To prove this proposition, we need to show the following lemma. For an imaginary abelian field \(N\), we put

\[
E_N = E_N/\mu(N)E_{N^+}.
\]

It is well known that the unit index \(Q_N = |E_N|\) is 1 or 2 ([9, Theorem 4.12]).
Lemma 4. Let $T$ and $N$ be imaginary abelian fields with $N \subseteq T$. If the degree $[T : N]$ is odd, then $Q_T = Q_N$.

Proof. We first show that the inclusion map $N \to T$ induces an injection $\mathcal{E}_N \hookrightarrow \mathcal{E}_T$. For a unit $\epsilon$ of $N$, assume that $\epsilon = \zeta \eta$ for some $\zeta \in \mu(T)$ and $\eta \in E_T$. Let $\rho$ be a nontrivial element of the Galois group $G = \text{Gal}(T/N)$. Then, as $\epsilon = e^\rho$, we see that $\zeta^{1-\rho} = \eta^{\rho-1} \in \mu(T) \cap E_T$. Hence, $\zeta^{1-\rho} = \pm 1$. However, as $N_{T/N}(\zeta^{1-\rho}) = 1$ and $[T : N]$ is odd, the case $\zeta^{1-\rho} = -1$ does not happen. Hence, $\zeta^{1-\rho} = 1$ for all $\rho \in G$. It follows that $\zeta \in \mu(N)$ and hence $\eta \in E_{N^+}$. Therefore, we can regard $\mathcal{E}_N$ as a subgroup of $\mathcal{E}_T$. In particular, $Q_N$ divides $Q_T$.

Assume that $Q_N \neq Q_T$. Then we have $|\mathcal{E}_T| = |\mathcal{E}_T/\mathcal{E}_N| = 2$. Regarding $\mathcal{E}_T$ as a module over $G$, we have a canonical decomposition

$$\mathcal{E}_T = \mathcal{E}_T/\mathcal{E}_N = \bigoplus_{\chi} \mathcal{E}_T(\chi)$$

where $\chi$ runs over a complete set of representatives of the $Q_2$-conjugacy classes of the nontrivial $Q_2$-valued characters of $G$. Hence, $|\mathcal{E}_T(\chi)| = 2$ for some such $\chi$. Let $\mathbb{Z}_2[\chi]$ be the subring of $\mathbb{Q}_2$ generated by the values of $\chi$ over $\mathbb{Z}_2$. The group $\mathcal{E}_T(\chi)$ is naturally regarded as a module over the principal ideal domain $\mathbb{Z}_2[\chi]$. Since the order of $\chi$ is odd and $\geq 3$, we observe that $\mathbb{Z}_2[\chi]$ is an $\mathbb{Z}_2$-module for some $d \geq 2$. Hence, $|\mathcal{E}_N(\chi)|$ is a multiple of $2^d$, which contradicts $|\mathcal{E}_N(\chi)| = 2$. Therefore, we obtain $Q_N = Q_T$. \hfill $\Box$

Proof of Proposition 2. By Lemma 4, we have $Q_{M_n} = Q_{M_{n-1}}$ and $Q_{L_n} = Q_{L_{n-1}}$ for all $n \geq 1$. Therefore, using the class number formula [9, Theorem 4.17], we see that

$$h_{M_n}/h_{M_{n-1}} = p \prod_{\sigma \psi_n} \left( -\frac{1}{2} B_{1,\sigma,\psi_n} \right)$$

where $\sigma$ runs over the odd Dirichlet characters associated to $M_0$, and $\psi_n$ over the even characters of conductor $p^{n+1}$ and order $p^n$. Further, $B_{1,\sigma,\psi_n}$ denotes the generalized Bernoulli number. We easily see that $\sigma$ equals an odd Dirichlet character associated to $K_0$ or $L_0$ since $M_0/K_0^+$ is an imaginary biquadratic extension with the imaginary quadratic subextensions $K_0$ and $L_0$. Hence, using the class number formulas for $L_{n}$, $K_n$ and $Q_{L_n} = Q_{L_{n-1}}$, we obtain

$$h_{M_n}/h_{M_{n-1}} = h_n^*/h_{n-1}^* \times h_n^-/h_{n-1}^-.$$

Therefore, the assertion follows from Theorem 1. \hfill $\Box$
References


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