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<th><strong>Title</strong></th>
<th>CLASS NUMBER PARITY OF A QUADRATIC TWIST OF A CYCLOTOMIC FIELD OF PRIME POWER CONDUCTOR</th>
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<td><strong>Author(s)</strong></td>
<td>Ichimura, Humio</td>
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<td>Osaka University Knowledge Archive : OUKA</td>
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CLASS NUMBER PARITY OF A QUADRATIC TWIST OF
A CYCLOTOMIC FIELD OF PRIME POWER CONDUCTOR

HUMIO ICHIMURA

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Abstract

Let $p$ be a fixed odd prime number. Let $K_n = \mathbb{Q}(\zeta_{p^n+1})$ be the $p^{n+1}$-st cyclotomic field for an integer $n \geq 0$, and $K_{\infty} = \bigcup_n K_n$. Let $d \in \mathbb{Z}$ be a fixed integer with $d \neq K_0$. We denote by $L_n$ the imaginary quadratic subextension of the biquadratic extension $K_n(\sqrt{d})/K_n^+$ with $L_n \neq K_n$. Let $h_n^+$ and $h_n^-$ be the relative class numbers of $K_n$ and $L_n$, respectively. We give an explicit constant $n_d$ depending on $p$ and $d$ such that (i) for any integer $n \geq n_d$, the ratio $h_n^+/h_{n-1}^-$ is odd if and only if $h_n^+/h_{n-1}^-$ is odd and (ii) for $1 \leq n < n_d$, $h_n^-/h_{n-1}^-$ is even.

1. Introduction

Let $p$ be a fixed odd prime number. Let $K_n = \mathbb{Q}(\zeta_{p^n+1})$ be the $p^{n+1}$-st cyclotomic field for an integer $n \geq 0$, and $K_{\infty} = \bigcup_n K_n$. Let $d \in \mathbb{Z}$ be a fixed integer with $d \neq K_0$. We denote by $L_n$ the imaginary quadratic subextension of the biquadratic extension $K_n(\sqrt{d})/K_n^+$ with $L_n \neq K_n$. Here, $K^+$ denotes the maximal real subfield of an imaginary abelian field $K$. When $d < 0$, we have $L_n = K_n^+(\sqrt{d})$. We call $L_n$ the quadratic twist of $K_n$ associated to the integer $d$. The extension $L_{\infty} = \bigcup_n L_n$ is the cyclotomic $\mathbb{Z}_p$-extension over $L_0$ with the $n$-th layer $L_n$. We call $L_{\infty}/L_0$ the biquadratic twist of the cyclotomic $\mathbb{Z}_p$-extension $K_{\infty}/K_0$ associated to $d$. Let $h_n^+$ and $h_n^-$ be the relative class numbers of $K_n$ and $L_n$, respectively. It is known and easy to show that $h_{n-1}^+$ (resp. $h_{n-1}^-$) divides $h_n^+$ (resp. $h_n^-$) using class field theory. The parity of $h_n^+$ behaves rather irregularly when $p$ varies (see a table in Schoof [6]). However, it is recently shown that when $p \leq 509$, the ratio $h_n^+/h_{n-1}^-$ is odd for all $n \geq 1$ ([3, Theorem 2]). And it might be possible that the ratio is odd for any prime $p$ and any $n \geq 1$. The purpose of this paper is to study the parity of the ratio $h_n^+/h_{n-1}^-$ of the quadratic twist $L_n$. We already know that $h_n^-/h_{n-1}^-$ is odd for sufficiently large $n$ by a theorem of Washington [8] on the non-$p$-part of the class number in a cyclotomic $\mathbb{Z}_p$-extension. Denote by $S = S_d$ the set of prime numbers $l \neq p$ which ramify in $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. The set $S$ is non-empty as $\sqrt{d} \neq K_0$. We define an integer $n_d \geq 1$ by

$$n_d = \max\{\text{ord}_p(l^{p-1} - 1) \mid l \in S\},$$

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where \( \text{ord}_p(*) \) is the normalized \( p \)-adic additive valuation. The following is the main theorem of this paper.

**Theorem 1.** Under the above setting, the following assertions hold.

(I) When \( n \geq n_d \), the ratio \( h_n^- / h_{n-1}^- \) is odd if and only if \( h_n^e / h_{n-1}^e \) is odd.

(II) When \( n_d \geq 2 \) and \( 1 \leq n < n_d \), the ratio \( h_n^- / h_{n-1}^- \) is even.

From Theorem 1 and [3, Theorem 2], we immediately obtain the following:

**Corollary 1.** Under the above setting, let \( p \) be an odd prime number with \( p \leq 509 \). Then the ratio \( h_n^- / h_{n-1}^- \) is odd for all \( n \geq n_d \).

This corollary, though given in a very special setting, is an explicit version of the above mentioned theorem of Washington. In [4], we showed Theorem 1 when \( d \neq 1 \) and \( L_n = K_n^+(\sqrt{-1}) \) using some results of cyclotomic Iwasawa theory. In this paper, we prove Theorem 1 by using a main theorem of Conner and Hurrelbrink [1, Theorem 2.3].

**Remark.** When \( p \equiv 1 \mod 4 \) (resp. \( p \equiv 3 \mod 4 \)), we can show that two integers \( d_1 \) and \( d_2 \) give the same twist \( L_\infty / L_0 \) of \( K_\infty / K_0 \) if and only if \( d_2 = d_1 x^2 \) or \( d_2 = p d_1 x^2 \) (resp. \( d_2 = -p d_1 x^2 \)) for some \( x \in \mathcal{O}^\times \). Hence, the set \( S_d \) and the integer \( n_d \) depend only on the twist \( L_\infty / L_0 \) and not on the choice of \( d \).

2. **Exact hexagon of Conner and Hurrelbrink**

In this section, we recall the exact hexagon of Conner and Hurrelbrink. Let \( k \) be an imaginary abelian field with 2-power degree, and \( F \) a real abelian field with \( 2 \nmid [F : \mathcal{O}] \). We put \( K = kF \), and

\[
G = \text{Gal}(K/k) = \text{Gal}(K^+/k^+) = \text{Gal}(F/\mathcal{O}).
\]

For a number field \( N \), let \( A_N \) be the 2-part of the ideal class group of \( N \), \( \mathcal{O}_N \) the ring of integers, and \( E_N = \mathcal{O}_N^\times \) the group of units of \( N \). The groups \( A_K \) and \( E_K \) are naturally regarded as modules over \( \text{Gal}(K/K^+) \) and at the same time as those over \( G \). For a \( \text{Gal}(K/K^+) \)-module \( X \), denote by \( H^i(X) = H^i(K/K^+; X) \) the Tate cohomology group with \( i = 0, 1 \). When \( X = A_K \) or \( E_K \), the group \( H^i(X) \) is also regarded as \( G \)-modules.

In [1, Theorem 2.3], Conner and Hurrelbrink introduced the following exact hexagon
of $G$-modules to study the 2-part of the class number of a relative quadratic extension.

$$
\begin{array}{ccc}
R^0(K) & \xrightarrow{i_0} & R^1(K) \\
\downarrow H^0(E_K) & & \downarrow H^0(A_K) \\
R^1(A_K) & \xrightarrow{i_0} & H^1(A_K) \\
\end{array}
$$

Here, $R^i(K)$ is a certain $G$-module associated to $K/K^+$ defined in [1]. We describe the $G$-module structure of $R^i(K)$ following [1]. Let $T_f$ be the set of prime ideals $\wp$ of $k^+$ for which a prime ideal $\mathfrak{P}$ of $K^+$ over $\wp$ ramifies in $K$. Let $T_\infty$ be the set of infinite prime divisors of $k^+$. We put $T = T_f \cup T_\infty$. For each $v \in T$, let $G_v \subseteq G$ be the decomposition group of $v$ at $K^+/k^+$. When $v$ is an infinite prime, the group $G_v$ is trivial. We define $G$-modules $\Omega_f$ and $\Omega_\infty$ by

$$
\Omega_f = \bigoplus_{\wp \in T_f} F_2[G/G_{\wp}] \quad \text{and} \quad \Omega_\infty = \bigoplus_{v \in T_\infty} F_2[G/G_v] = \bigoplus_{v \in T_\infty} F_2[G],
$$

respectively, where $F_2 = \mathbb{Z}/2\mathbb{Z}$ is the finite field with two elements. (When $T_f$ is empty, $\Omega_f = \{0\}$ by definition.) For each prime divisor $w$ of $K^+$ with the restriction $w_{|k^+} \in T$ and an element $x \in (K^+)^\times$, we put $t_w(x) = 0$ or 1 according as $x \in N(K_{w}^\times)$ or not. Here, $K_w$ is the completion of $K$ at the unique prime divisor of $K$ over $w$ and $N = N_{K/K^+}$ is the norm map. For $g \in G$ and $x \in (K^+)^\times$, we see that

$$
t_{w^g}(x) = t_w(x^{g^{-1}})
$$

by local class field theory. For a prime ideal $\mathfrak{P}$ of $K^+$ with $\mathfrak{P} \cap k^+ \in T_f$, let $\mathfrak{P}$ be the unique prime ideal of $K$ over $\mathfrak{P}$. For an ideal $\mathfrak{A}$ of $K$, writing $\mathfrak{A} = \mathfrak{P} \mathfrak{B}$ with an integer $e$ and an ideal $\mathfrak{B}$ relatively prime to $\mathfrak{P}$, we put $\text{ord}_\mathfrak{P}(\mathfrak{A}) = e$.

We denote by $I(K)$ the group of (fractional) ideals of $K$. Let $X$ be the subgroup of $I(K)$ consisting of ideals $\mathfrak{A}$ with $\mathfrak{A}^I = \mathfrak{A}$. Here, $J$ is the complex conjugation acting on several objects associated to $K$. Let $X_0$ be the subgroup of $X$ consisting of ideals $\mathfrak{A} \in I(K)$ with $\mathfrak{A} = x \mathfrak{B}^{1+J}$ for some $x \in (K^+)^\times$ and $\mathfrak{B} \in I(K)$. The $G$-module $R^1(K)$ is isomorphic to the quotient $X/X_0$. For this, see the lines 1–2 from the bottom of p. 6 and Lemma 2.1 of [1]. For each prime ideal $\wp \in T_f$, we fix a prime ideal $\mathfrak{P}$ of $K^+$ over $\wp$. From the argument in [1, §5], we obtain the following isomorphism of $G$-modules:

$$
R^1(K) \cong \Omega_f; \quad \mathfrak{A}X_0 \rightarrow \bigoplus_{\wp \in T_f} \left( \sum_{\tilde{g}} \text{ord}_{\mathfrak{P}}(\mathfrak{A}) \tilde{g} \right),
$$

where $\tilde{g}$ (with $g \in G$) runs over the quotient $G/G_{\wp}$. 

Let $Y$ be the subgroup of the multiplicative group $(K^+)^\times \times I(K)$ consisting of pairs $(x, A)$ with $x A^{1+J} = \mathcal{O}_K$. Let $Y_0$ be the subgroup of $Y$ consisting of pairs $(N(y), y^{-1} A^{1-J})$ with $y \in K^+$ and $A \in I(K)$. By definition, $R_0^0(K) = Y/Y_0$. We denote by $[x, A] \in R_0^0(K)$ the class containing $(x, A)$. The map $i_0$ in the hexagon is defined by

$$i_0 : H^0(E_K) = E_K^+ / N(E_K) \rightarrow R_0^0(K); \quad [\varepsilon] \rightarrow [\varepsilon, \mathcal{O}_K]$$

with $\varepsilon \in E_K^+$. For each $v \in T_{\infty}$, we fix a prime divisor $v$ of $K^+$ over $v$. Using (1), we observe that the homomorphisms

$$\alpha_\infty : (K^+)^\times \rightarrow \Omega_\infty; \quad x \mapsto \bigoplus_{v \in T_{\infty}} \left( \sum_{g \in G} t_v^\infty(x) g \right)$$

and

$$\alpha_f : (K^+)^\times \rightarrow \Omega_f; \quad x \mapsto \bigoplus_{v \in T_f} \left( \sum_{g \in G} t_v^f(x) g \right)$$

are compatible with the action of $G$. Further, $\alpha_\infty$ is nothing but the “sign” map. From the argument in [1, §4], we obtain the following exact sequence of $G$-modules:

$$\{0\} \rightarrow R_0^0(K) \xrightarrow{\alpha} \Omega_f \oplus \Omega_\infty \xrightarrow{\beta} F_2 \rightarrow \{0\}.$$  

Here, $\alpha$ is defined by $\alpha([x, A]) = (\alpha_f(x), \alpha_\infty(x))$, $\beta$ is the argumentation map and $G$ acts trivially on $F_2$.

3. Consequences

In this section, we derive some consequences of the exact hexagon and (2), (3). All of them are $G$-decomposed versions of the corresponding results in [1]. We work under the setting of Section 2. Denote by $\tilde{A}_{K^+}$ the 2-part of the narrow class group of $K^+$. Letting $K^+_{>0}$ be the group of totally positive elements of $K^+$, we have an exact sequence

$$\{0\} \rightarrow (K^+)^\times / (K^+_{>0} E_{K^+}) \rightarrow \tilde{A}_{K^+} \rightarrow A_{K^+} \rightarrow \{0\}$$

of $G$-modules. We define the minus class group $A^-_{K}$ to be the kernel of the norm map $A_K \rightarrow A_{K^+}$. Let $\chi$ be a $\bar{Q}_2$-valued character of $G = \text{Gal}(K/k) = \text{Gal}(F/Q)$, which we also regard as a primitive Dirichlet character. For a module $M$ over $\mathbb{Z}_2[G]$, we denote by $M(\chi)$ the $\chi$-part of $M$. Here, $\mathbb{Z}_2$ is the ring of $2$-adic integers and $\bar{Q}_2$ is a fixed algebraic closure of the $2$-adic rationals $Q_2$. (For the definition of the $\chi$-part and some of its properties, see Tsuji [7, §2].) Denote by $S_K$ the set of prime numbers lying
below some prime ideal in \( T_f \). In all what follows, we assume that \( \chi \) is a nontrivial character. The following is a version of [1, Theorem 13.8].

**Theorem 2.** Under the above setting, the groups \( H^i(K/K^+;A_K)(\chi) \) with \( i = 0 \) and \( 1 \) are trivial if and only if

(i) \( \chi(l) \neq 1 \) for all \( l \in S_K \) and

(ii) \( |\tilde{A}_K(\chi)| = |A_K(\chi)| \).

The following corollary is a version of [1, Corollary 13.10] and Hasse [2, Satz 45].

**Corollary 2.** Under the above setting, the group \( A_K(\chi) \) is trivial if and only if

(i) \( \chi(l) \neq 1 \) for all \( l \in S_K \) and

(ii) \( \tilde{A}_K(\chi) \) is trivial.

Let \( \tilde{h}_M \) be the class number in the narrow sense of a number field \( M \). When \( M \) is an imaginary abelian field, let \( h^-_M \) be the relative class number of \( M \). We can easily show that \( h^-_K \) (resp. \( \tilde{h}_{K^+} \)) divides \( h^-_k \) (resp. \( \tilde{h}_{K^+} \)) using class field theory. The following is an immediate consequence of Corollary 2.

**Corollary 3.** Under the above setting, the ratio \( h^-_K/h^-_k \) is odd if and only if

(i) no prime number \( l \) in \( S_K \) splits in \( F \) and

(ii) \( \tilde{h}_{K^+}/\tilde{h}_{k^+} \) is odd.

To prove these assertions, we prepare the following two lemmas. For a number field \( L \), let \( \mu(L) \) be the group of roots of unity in \( L \) and \( \mu_2(L) \) the 2-part of \( \mu(L) \).

**Lemma 1.** The group \( H^1(K/K^+;E_K)(\chi) \) is trivial.

**Proof.** Let \( N_{E_K} \) be the group of units \( \epsilon \in E_K \) with \( N(\epsilon) = \epsilon^{1+j} = 1 \). We have \( N(\epsilon) = 1 \) if and only if \( \epsilon \in \mu(K) \) by a theorem on units of a CM-field (cf. Washington [9, Theorem 4.12]). Since \( \mu(K)^2 = \mu(K)^{1+j} \subseteq E_K^{1-j} \), we obtain a surjection

\[
\mu(K)/\mu(K)^2 \to H^1(K/K^+;E_K) = N_{E_K}/E_K^{1-j}
\]

of \( G \)-modules. However, as \( [K:k] \) is odd, we have

\[
\mu(K)/\mu(K)^2 = \mu_2(K)/\mu_2(K)^2 = \mu_2(k)/\mu_2(k)^2.
\]

Since \( \chi \) is nontrivial, the \( \chi \)-part \((\mu_2(k)/\mu_2(k)^2)(\chi)\) is trivial. Hence, we obtain the assertion.

**Lemma 2.** The natural map \( A_K(\chi) \to A_K(\chi) \) is injective.

\[ \square \]
that the \( \mathfrak{A} \) is an injective \( G \)-homomorphism (\cite[Theorem 7.1]{1}). Then, from Lemma 1, we see that the \( \chi \)-part \((\ker \iota)(\chi)\) is trivial, from which we obtain the assertion.

Proof of Theorem 2. Let \( \mathfrak{p} \) be a prime ideal in \( T_f \), and \( l = \mathfrak{p} \cap \mathcal{O} \in \mathcal{S}_K \). We see that the \( \chi \)-part \( F_2[G/G_{\mathfrak{p}}](\chi) \neq \{0\} \) if and only if \( \chi \) factors through \( G/G_{\mathfrak{p}} \), which is equivalent to \( \chi(G_{\mathfrak{p}}) = \{1\} \). Since \( [k^+/K] \) is a 2-power and \( [F:K] \) is odd, we have \( \chi(G_{\mathfrak{p}}) = \{1\} \) if and only if \( \chi(l) = 1 \). Hence, we have shown that the condition (i) in Theorem 2 is equivalent to the condition \( \Omega_f(\chi) = \{0\} \). By the hexagon and Lemma 1, we see that \( H^0(A_K(\chi)) \) and \( H^1(A_K(\chi)) \) are trivial if and only if (iii) \( R^1(K)(\chi) = \{0\} \) and (iv) the map

\[
i_0: H^0(E_K)(\chi) = (E_K^+ / N(E_K))(\chi) \rightarrow R^0(K)(\chi)
\]

is an isomorphism. By (2) and the above, the condition (iii) is equivalent to (i). Under the condition (i), we see that \( R^0(K)(\chi) = \Omega_\infty(\chi) \) from the exact sequence (3), and that for each class \( [\epsilon] \in H^0(E_K)(\chi) \) with \( \epsilon \in E_K^+ \), we have \( i_0([\epsilon]) = \alpha_\infty(\epsilon) \) from the definitions of the maps \( i_0 \) and \( \alpha \). Further, the 2-rank of \( \Omega_\infty(\chi) \) is larger than or equal to that of \( H^0(E_K)(\chi) \) by a theorem of Minkowski on units of a Galois extension (cf. Narkiewicz \cite[Theorem 3.26]{5}). Therefore, under (i), we observe that the condition (iv) holds if and only if \( \alpha_\infty(E_K^+)(\chi) = \Omega_\infty(\chi) \). We see that the last condition is equivalent to the condition (ii) in Theorem 2 because of the exact sequence (4) and \( \alpha_\infty((K^+)\chi) = \Omega_\infty(\chi) \). Therefore, we obtain Theorem 2.

Proof of Corollary 2. First, we show the “only if” part assuming that \( A_K^- (\chi) \) is trivial. By Lemma 2, we can regard \( A_K^+ (\chi) \) as a subgroup of \( A_K (\chi) \). Assume that \( A_K^- (\chi) \) is nontrivial. Then there exists a class \( c \in A_K^+ (\chi) \) of order 2. We have \( c^J = c = c^{-1} \), and hence \( c \in A_K^- (\chi) \). It follows that \( A_K^- (\chi) \) is nontrivial, a contradiction. Hence, \( A_K^- (\chi) = \{0\} \). It follows that \( A_K (\chi) \) is trivial by the exact sequence

\[
\{0\} \rightarrow A_K^- (\chi) \rightarrow A_K (\chi) \rightarrow A_K^+ (\chi) \rightarrow \{0\}.
\]

Therefore, the “only if” part follows from Theorem 2. Next, assume that the conditions (i) and (ii) in Corollary 2 are satisfied. Then, \( A_K^- (\chi) = \{0\} \), and the groups \( H^i(A_K)(\chi) \) \((i = 0, 1)\) are trivial by Theorem 2. As the cohomology groups are trivial, we obtain an exact sequence

\[
\{0\} \rightarrow A_K^+ (\chi) \rightarrow A_K (\chi) \rightarrow A_K^- (\chi) \rightarrow \{0\}.
\]
Since $A_K^+(\chi) = \{0\}$, we see that $A_K(\chi) = A_K^-(\chi)$, and

$$A_K^-(\chi) = A_K^-(\chi)^{1-J} = A_K^-(\chi)^2$$

from the above exact sequence. Therefore, $A_K^-(\chi)$ is trivial. \(\square\)

4. Proof of Theorem 1

We use the same notation as in Section 1. In particular, $d \in \mathbb{Z}$ is a fixed integer with $\sqrt{d} \notin K_0$ and $L_n$ is the quadratic twist of $K_n$ associated to $d$. We have $L_n^+ = K_n^+$. Let $k$ (resp. $k_d$) be the maximal intermediate field of $K_0/Q$ (resp. $L_0/Q$) of 2-power degree, and let $F_0$ be the maximal subfield of $K_0^+ = L_0^+$ of odd degree over $Q$. Then $k$ and $k_d$ are imaginary abelian fields with $k^+ = k_d^+$. Let $B_n/Q$ be the real abelian field with conductor $p^{n+1}$ and $[B_n : Q] = p^n$. We put $F_n = F_0B_n$. Then $L_n = k_dF_n$ and $K_n = kF_n$. The triples $(k_d, F_n, L_n)$ and $(k, F, K)$ correspond to $(k, F, K)$ in Sections 2 and 3. We see that

\begin{equation}
S_{L_n} = S_d \quad \text{or} \quad S_d \cup \{p\}
\end{equation}

and $S_{K_n} = \{p\}$. We put

$$G_n = \text{Gal}(F_n/Q) = \text{Gal}(L_n/k_d) = \text{Gal}(K_n/k),$$

and

$$\Delta = \text{Gal}(F_0/Q), \quad \Gamma_n = \text{Gal}(F_n/F_0) = \text{Gal}(B_n/Q).$$

Then we have a natural decomposition $G_n = \Delta \times \Gamma_n$. For characters $\varphi$ and $\psi$ of $\Delta$ and $\Gamma_n$ respectively, we regard $\varphi\psi = \varphi \times \psi$ as a character of $G_n$. Further, we regard $\varphi$, $\psi$ and $\varphi\psi$ as also primitive Dirichlet characters. The class groups $A_{L_n}$, $A_{K_n}$ and $\tilde{A}_{K_n^+}$ are modules over $G_n$. We can naturally regard $A_{L_n}^{-}$ as a subgroup of $A_{L_n}^-$ since $L_n/L_{n-1}$ is a cyclic extension of degree $p \neq 2$ and $A_{L_n}^{-}$ is the 2-part of the class group. Actually, it is a direct summand of $A_{L_n}^-$ (cf. [9, Lemma 16.15]). We see that

\begin{equation}
A_{L_n}^- / A_{L_{n-1}}^- = \bigoplus_{\varphi, \psi_n} A_{L_n}^- (\varphi\psi_n)
\end{equation}

where $\varphi$ (resp. $\psi_n$) runs over a complete set of representatives of the $Q_2$-conjugacy classes of the $\tilde{Q}_2$-valued characters of $\Delta$ (resp. $\Gamma_n$ of order $p^n$). Regarding $A_{K_n}^-$ as a subgroup of $A_{K_n}$, we have a similar decomposition for $A_{K_n}^- / A_{K_{n-1}}^-$. As $S_{K_n} = \{p\}$ and $(\varphi\psi_n)(p) = 0$, we obtain the following assertion from Corollary 2 for the triple $(k, F, K_n)$.

**Lemma 3.** Let $n \geq 1$ be an integer, and the characters $\varphi$ and $\psi_n$ be as in (6). Then $A_{K_n}^-(\varphi\psi_n) = \{0\}$ if and only if $\tilde{A}_{K_n^+}(\varphi\psi_n) = \{0\}$. 
Proof of Theorem 1 (I). Let $\varphi$ and $\psi_n$ be as in (6). As the orders of $\varphi$ and $\psi_n$ are relatively prime to each other, we have $(\varphi\psi_n)(l) = 1$ if and only if $\varphi(l) = \psi_n(l) = 1$ for a prime number $l$. Let $n$ be an integer with $n \geq n_d$. Then we have $\psi_n(l) \neq 1$ and hence $(\varphi\psi_n)(l) \neq 1$ for all prime numbers $l \in S = S_d$. Further, we have $(\varphi\psi_n)(p) = 0$. Hence, by (5), the condition (i) in Corollary 2 for the triple $(k_d, F_n, L_n)$ is satisfied. It follows that the condition $A_{L_n}^{-}(\varphi\psi_n) = \{0\}$ is equivalent to $\hat{A}_{K_n}^{-}(\varphi\psi_n) = \{0\}$. (Note that $L_n^{+} = K_n^{+}$.) Therefore, we obtain Theorem 1 (I) from Lemma 3.

To show Theorem 1 (II), assume that $n_d \geq 2$ and let $n$ be an integer with $1 \leq n < n_d$. We put

$$S^{(n)} = \{ l \in S = S_d \mid \text{ord}_p(l^{p^n-1} - 1) \geq n + 1 \}.$$  

From the definition, we see that

$$S \supseteq S^{(1)} \supseteq S^{(2)} \supseteq \cdots \supseteq S^{(n_d-1)}$$

and that each $S^{(n)}$ is non-empty. Let $\varphi$ (resp. $\psi_n$) be a $\hat{\mathbb{Q}}_2$-valued character of $\Delta$ (resp. of $\Gamma_n$ of order $p^n$). Denote by $\varphi_0$ the trivial character of $\Delta$. Theorem 1 (II) is a consequence of the following assertion.

**Proposition 1.** Under the above setting, the following hold.

(I) The class group $A_{L_n}^{-}(\varphi\psi_n)$ is nontrivial if $\varphi(l) = 1$ for some $l \in S^{(n)}$. In particular, $A_{L_n}^{-}(\varphi\psi_n)$ is nontrivial.

(II) If $A_{K_n}^{-}(\varphi\psi_n) = \{0\}$, the converse of the first assertion of (I) holds.

Proof. Applying Corollary 2 for the triple $(k_d, F_n, L_n)$, we see from Lemma 3 that $A_{L_n}^{-}(\varphi\psi_n) = \{0\}$ if and only if (i) $(\varphi\psi_n)(l) \neq 1$ for all $l \in S = S_d$ and (ii) $A_{L_n}^{-}(\varphi\psi_n) = \{0\}$. We have $\psi_n(l) = 1$ for $l \in S^{(n)}$, and $\psi_n(l) \neq 1$ for $l \in S \setminus S^{(n)}$. Therefore, we see that the condition (i) is satisfied if and only if $\varphi(l) \neq 1$ for all $l \in S^{(n)}$ noting that the orders of $\varphi$ and $\psi_n$ are relatively prime. From this, we obtain the proposition.

We put $M_n = K_n(\sqrt{d}) = K_n L_n$. On the relative class number $h_{M_n}^{-}$ of $M_n$, the following assertion holds.

**Proposition 2.**

(I) When $n \geq n_d$, the ratio $h_{M_n}^{-}/h_{M_n}^{-1}$ is odd if and only if $h_{n}^{*}/h_{n-1}^{*}$ is odd.

(II) When $n_d \geq 2$ and $1 \leq n < n_d$, $h_{M_n}^{-}/h_{M_n}^{-1}$ is even.

To prove this proposition, we need to show the following lemma. For an imaginary abelian field $N$, we put

$$\mathcal{E}_N = E_N/\mu(N)E_N^{\times}.$$  

It is well known that the unit index $Q_N = |\mathcal{E}_N|$ is 1 or 2 ([9, Theorem 4.12]).
Lemma 4. Let \( T \) and \( N \) be imaginary abelian fields with \( N \subseteq T \). If the degree \( [T : N] \) is odd, then \( Q_T = Q_N \).

Proof. We first show that the inclusion map \( N \to T \) induces an injection \( \mathcal{E}_N \hookrightarrow \mathcal{E}_T \). For a unit \( \epsilon \) of \( N \), assume that \( \epsilon = \zeta \eta \) for some \( \zeta \in \mu(T) \) and \( \eta \in E_T^* \). Let \( \rho \) be a nontrivial element of the Galois group \( G = \text{Gal}(T/N) \). Then, as \( \epsilon = e^\rho \), we see that \( \zeta^{1-\rho} = \eta^{p-1} \in \mu(T) \cap E_T^* \). Hence, \( \zeta^{1-\rho} = \pm 1 \). However, as \( N_{T/N}(\zeta^{1-\rho}) = 1 \) and \( [T : N] \) is odd, the case \( \zeta^{1-\rho} = -1 \) does not happen. Hence, \( \zeta^{1-\rho} = 1 \) for all \( \rho \in G \). It follows that \( \zeta \in \mu(N) \) and hence \( \eta \in E_{N^+} \). Therefore, we can regard \( \mathcal{E}_N \) as a subgroup of \( \mathcal{E}_T \). In particular, \( Q_N \) divides \( Q_T \).

Assume that \( Q_N \neq Q_T \). Then we have \( |\mathcal{E}_T| = |\mathcal{E}_T/\mathcal{E}_N| = 2 \). Regarding \( \mathcal{E}_T \) as a module over \( G \), we have a canonical decomposition

\[
\mathcal{E}_T = \mathcal{E}_T/\mathcal{E}_N = \bigoplus_\chi \mathcal{E}_T(\chi)
\]

where \( \chi \) runs over a complete set of representatives of the \( \mathbb{Q}_2 \)-conjugacy classes of the nontrivial \( \mathbb{Q}_2 \)-valued characters of \( G \). Hence, \( |\mathcal{E}_T(\chi)| = 2 \) for some such \( \chi \). Let \( \mathbb{Z}_2[\chi] \) be the subring of \( \mathbb{Q}_2 \) generated by the values of \( \chi \) over \( \mathbb{Z}_2 \). The group \( \mathcal{E}_T(\chi) \) is naturally regarded as a module over the principal ideal domain \( \mathbb{Z}_2[\chi] \). Since the order of \( \chi \) is odd and \( \geq 3 \), we observe that \( \mathbb{Z}_2[\chi] \cong \mathbb{Z}_2^d \) as \( \mathbb{Z}_2 \)-modules for some \( d \geq 2 \). Hence, \( |\mathcal{E}_T(\chi)| \) is a multiple of \( 2^d \), which contradicts \( |\mathcal{E}_T(\chi)| = 2 \). Therefore, we obtain \( Q_N = Q_T \). \( \square \)

Proof of Proposition 2. By Lemma 4, we have \( Q_{M_n} = Q_{M_{n+1}} \) and \( Q_{L_n} = Q_{L_{n+1}} \) for all \( n \geq 1 \). Therefore, using the class number formula [9, Theorem 4.17], we see that

\[
h_{M_n}^-/h_{M_{n+1}}^- = p \prod_{\sigma} \prod_{\psi_n} \left( -\frac{1}{2} B_{1, \sigma \psi_n} \right)
\]

where \( \sigma \) runs over the odd Dirichlet characters associated to \( M_0 \), and \( \psi_n \) over the even characters of conductor \( p^{n+1} \) and order \( p^n \). Further, \( B_{1, \sigma \psi_n} \) denotes the generalized Bernoulli number. We easily see that \( \sigma \) equals an odd Dirichlet character associated to \( K_0 \) or \( L_0 \) since \( M_0/K_0^+ \) is an imaginary biquadratic extension with the imaginary quadratic subextensions \( K_0 \). Hence, using the class number formulas for \( L_n, K_n \) and \( Q_{L_n} = Q_{L_{n+1}} \), we obtain

\[
h_{M_n}^-/h_{M_{n+1}}^- = h_n^*/h_{n-1}^* \times h_n^-/h_{n-1}^-.
\]

Therefore, the assertion follows from Theorem 1. \( \square \)
References


Faculty of Science
Ibaraki University
Bunkyo 2-1-1, Mito, 310-8512
Japan