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NON-COMMUTATIVE KRULL MONOIDS: A DIVISOR THEORETIC APPROACH AND THEIR ARITHMETIC

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Abstract

A (not necessarily commutative) Krull monoid—as introduced by Wauters—is defined as a completely integrally closed monoid satisfying the ascending chain condition on divisorial two-sided ideals. We study the structure of these Krull monoids, both with ideal theoretic and with divisor theoretic methods. Among others we characterize normalizing Krull monoids by divisor theories. Based on these results we give a criterion for a Krull monoid to be a bounded factorization monoid, and we provide arithmetical finiteness results in case of normalizing Krull monoids with finite Davenport constant.

1. Introduction

The arithmetic concept of a divisor theory has its origin in early algebraic number theory. Axiomatic approaches to more general commutative domains and monoids were formulated by Clifford [17], by Borewicz and Šafarevič [8], and then by Skula [61] and Gundlach [33]. The theory of divisorial ideals was developed in the first half of the 20th century by Prüfer, Krull and Lorenzen [56, 44, 45, 46, 48], and its presentation in the book of Gilmer [31] strongly influenced the development of multiplicative ideal theory. The concept of a commutative Krull monoid (defined as completely integrally closed commutative monoids satisfying the ascending chain condition on divisorial ideals) was introduced by Chouinard [16] 1981 in order to study the Krull ring property of commutative semigroup rings.

Fresh impetus came from the theory of non-unique factorizations in the 1990s. Halter-Koch observed that the concept of monoids with divisor theory coincides with the concept of Krull monoids [34], and Krause [43] proved that a commutative domain is a Krull domain if and only if its multiplicative monoid of non-zero elements is a Krull monoid. Both, the concepts of divisor theories and of Krull monoids, were widely generalized, and a presentation can be found in the monographs [36, 29] (for a recent survey see [37]).

The search for classes of non-commutative rings having an arithmetical ideal theory—generalizing the classical theory of commutative Dedekind and Krull domains—was

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started with the pioneering work of Asano [3, 4, 5, 6]. It lead to a theory of Dedekind-like rings, including Asano prime rings and Dedekind prime rings. Their ideal theory and also their connection with classical maximal orders over Dedekind domains in central simple algebras is presented in [53].

From the 1970s on a large number of concepts of non-commutative Krull rings has been introduced (see the contributions of Brungs, Bruyn, Chamarie, Dubrovin, Jespers, Marubayashi, Miyashita, Rehm and Wauters, cited in the references). Always the commutative situation was used as a model, and all these generalizations include Dedekind prime rings as a special case (see the survey of Jespers [38], and Section 5 for more details). The case of semigroup rings has received special attention, and the reader may want to consult the monograph of Jespers and Okniński [40].

In 1984 Wauters [63] introduced non-commutative Krull monoids generalizing the concept of Chouinard to the non-commutative setting. His focus was on normalizing Krull monoids, and he showed, among others, that a prime polynomial identity ring is a Chamarie–Krull ring if and only if its monoid of regular elements is a Krull monoid (see Section 5).

In the present paper we study non-commutative Krull monoids in the sense of Wauters, which are defined as completely integrally closed monoids satisfying the ascending chain condition on divisorial two-sided ideals. In Section 3 we develop the theory of divisorial two-sided ideals in analogy to the commutative setting (as it is done in [36, 29]). In Section 4 we introduce divisor theoretic concepts, and provide a characterization of normalizing Krull monoids in divisor theoretic terms (Theorem 4.13). Although many results and their proofs are very similar either to those for commutative monoids or to those for non-commutative rings, we provide full proofs. In Section 5 we discuss examples of commutative and non-commutative Krull monoids with an emphasis on the connection to ring theory. The existence of a suitable divisor homomorphism is crucial for the investigation of arithmetical finiteness properties in commutative Krull monoids (see [29, Section 3.4]). Based on the results in Sections 3 and 4 we can do some first steps towards a better understanding of the arithmetic of non-commutative Krull monoids. Among others, we generalize the concept of transfer homomorphisms, give a criterion for a Krull monoid to be a bounded-factorization monoid, and we provide arithmetical finiteness results in case of normalizing Krull monoids with finite Davenport constant (Theorem 6.5).

2. Basic concepts

Let $\mathbb{N}$ denote the set of positive integers, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For integers $a, b \in \mathbb{Z}$, we set $[a, b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$. If A, B are sets, then $A \subset B$ means that A is contained in B but may be equal to B.

By a *semigroup* we always mean an associative semigroup with unit element. If not denoted otherwise, we use multiplicative notation. Let H be a semigroup. We say that H is *cancellative* if for all elements $a, b, c \in H$, the equation $ab = ac$ implies $b = c$ and the equation $ba = ca$ implies $b = c$. Clearly, subsemigroups of groups are cancellative. A group Q is called a *left quotient group* of H (a *right quotient group* of H, resp.) if $H \subset Q$ and every element of Q can be written in the form $a^{-1}b$ with $a, b \in H$ (or in the form ba^{-1}, resp.).

We say that H satisfies the *right Ore condition* (*left Ore condition*, resp.) if $aH \cap bH \neq \emptyset$ ($Ha \cap Hb \neq \emptyset$, resp.) for all $a, b \in H$. A cancellative semigroup has a left quotient group if and only if it satisfies the left Ore condition, and if this holds, then the left quotient group is unique up to isomorphism (see [18, Theorems 1.24 and 1.25]). Moreover, a semigroup is embeddable in a group if and only if it is embeddable in a left (resp. right) quotient group (see [19, Section 12.4]).

If H is cancellative and satisfies the left and right Ore condition, then every right quotient group Q of H is also a left quotient group and conversely. In this case, Q will simply be called a *quotient group* of H (indeed, if Q is a right quotient group and $s = ax^{-1} \in Q$ with $a, x \in H$, then the left Ore condition implies the existence of $b, y \in H$ such that $ya = bx$ and hence $s = ax^{-1} = y^{-1}b$; thus Q is a left quotient group).

Throughout this paper, a *monoid* means a cancellative semigroup which satisfies the left and the right Ore condition, and every monoid homomorphism $\varphi\colon H \to D$ satisfies $\varphi(1_H) = 1_D$.

Let H be a monoid. We denote by $\mathsf{q}(H)$ a quotient group of H. If $\varphi\colon H \to D$ is a monoid homomorphism, then there is a unique homomorphism $\mathsf{q}(\varphi)\colon \mathsf{q}(H) \to \mathsf{q}(D)$ satisfying $\mathsf{q}(\varphi) \mid H = \varphi$. If S is a semigroup with $H \subset S \subset \mathsf{q}(H)$, then S is cancellative, $\mathsf{q}(H)$ is a quotient group of S, and hence S is a monoid. Every such monoid S with $H \subset S \subset \mathsf{q}(H)$ will be called an *overmonoid* of H. Let H^{op} denote the *opposite monoid* of H (H^{op} is a semigroup on the set H, where multiplication $H^{\mathrm{op}} \times H^{\mathrm{op}} \to H^{\mathrm{op}}$ is defined by $(a, b) \mapsto ba$ for all $a, b \in H$; clearly, H^{op} is a monoid in the above sense). We will encounter many statements on left and right ideals (quotients, and so on) in the monoid H. Since every right-statement (r) in H is a left-statement (l) in H^{op}, it will always be sufficient to prove the left-statement.

Let $a, b \in H$. The element a is said to be *invertible* if there exists an $a' \in H$ such that $aa' = a'a = 1$. The set of invertible elements of H will be denoted by $H^\times$, and it is a subgroup of H. We say that H is reduced if $H^\times = \{1\}$. A straightforward calculation shows that $aH = bH$ if and only if $aH^\times = bH^\times$.

We say that a is a *left divisor* (*right divisor*, resp.) if $b \in aH$ ($b \in Ha$, resp.), and we denote this by $a \mid_l b$ ($a \mid_r b$, resp.). If $b \in aH \cap Ha$, then we say that a is a *divisor* of b, and then we write $a \mid b$.

The element a is called an *atom* if $a \notin H^\times$ and, for all $u, v \in H$, $a = uv$ implies $u \in H^\times$ or $v \in H^\times$. The set of atoms of H is denoted by $\mathcal{A}(H)$. H is said to be *atomic* if every $u \in H \setminus H^\times$ is a product of finitely many atoms of H.

For a set P, we denote by $\mathcal{F}(P)$ the *free abelian monoid* with basis P. Then every $a \in \mathcal{F}(P)$ has a unique representation in the form

$$a = \prod_{p \in P} p^{\mathsf{v}_p(a)}, \quad \text{where} \quad \mathsf{v}_p(a) \in \mathbb{N}_0 \quad \text{and} \quad \mathsf{v}_p(a) = 0 \quad \text{for almost all} \quad p \in P,$$

and we call $|a| = \sum_{p \in P} \mathsf{v}_p(a) \in \mathbb{N}_0$ the *length* of a. If $H = \mathcal{F}(P)$ is free abelian with basis P, then H is reduced, atomic with $\mathcal{A}(H) = P$ and $\mathsf{q}(H) \cong (\mathbb{Z}^{(P)}, +)$. We use all notations and conventions concerning greatest common divisors in commutative monoids as in [36, Chapter 10].

3. Divisorial ideals in monoids

In this section we develop the theory of divisorial ideals in monoids as far as it is needed for the divisor theoretic approach in Section 4 and the arithmetical results in Section 6. An ideal will always be a two-sided ideal. We follow the presentation in the commutative setting (as given in [36, 29]) with the necessary adjustments. The definition of a Krull monoid (as given in Definition 3.11) is due to Wauters [63]. For Asano orders H (see Section 5), the commutativity of the group $\mathcal{F}_v(H)^\times$ (Proposition 3.12) dates back to the classical papers of Asano and can also be found in [52, Chapter II, §2].

Our first step is to introduce modules (following the terminology of [37]), fractional ideals and divisorial fractional ideals. Each definition will be followed by a simple technical lemma.

DEFINITION 3.1. Let H be a monoid and $A, B \subset \mathsf{q}(H)$ subsets.
1. We say that A is a *left module* (resp. *right module*) if $HA = A$ (resp. $AH = A$), and denote by $\mathcal{M}_l(H)$ (resp. $\mathcal{M}_r(H)$) the set of all left (resp. right) modules. The elements of $\mathcal{M}(H) = \mathcal{M}_l(H) \cap \mathcal{M}_r(H)$ are called *modules* (of H).
2. We set $AB = \{ab \mid a \in A, b \in B\}$, and define the *left* and *right quotient* of A and B by

$$(A :_l B) = \{x \in \mathsf{q}(H) \mid xB \subset A\} \quad \text{and} \quad (A :_r B) = \{x \in \mathsf{q}(H) \mid Bx \subset A\}.$$

If $B = \{b\}$, then $(A :_l b) = (A :_l B)$ and $(A :_r b) = (A :_r B)$.

The following lemma gathers some simple properties which will be used without further mention (most of them have a symmetric left or right variant).

Lemma 3.2. *Let H be a monoid, $A, B, C \subset \mathsf{q}(H)$ subsets, and $c \in H$.*
1. $(A :_l c) = Ac^{-1}$, $(cA :_l B) = c(A :_l B)$, $(Ac :_l B) = (A :_l Bc^{-1})$, *and* $(A :_l cB) = c^{-1}(A :_l B)$.
2. $(A :_l B) = \bigcap_{b \in B} (A :_l b) = \bigcap_{b \in B} Ab^{-1}$.

3. $(A :_l BC) = ((A :_l C) :_l B)$ *and* $((A :_l B) :_r C) = ((A :_r C) :_l B)$.
4. $A \subset (H :_l (H :_r A)) = \bigcap_{c \in \mathsf{q}(H), A \subset Hc} Hc$ *and* $A \subset (H :_r (H :_l A)) = \bigcap_{c \in \mathsf{q}(H), A \subset cH} cH$.
5. (a) *If* $A \in \mathcal{M}_l(H)$, *then* $(A :_l B) \in \mathcal{M}_l(H)$.
   (b) *If* $A \in \mathcal{M}_r(H)$, *then* $(A :_l B) = (A :_l BH)$.
   (c) *If* $B \in \mathcal{M}_l(H)$, *then* $(A :_l B) \in \mathcal{M}_r(H)$.

Proof. We verify only the statements 3. and 4., as the remaining ones follow immediately from the definitions.

3. We have

$$\begin{aligned}(A :_l BC) &= \{x \in \mathsf{q}(H) \mid xBC \subset A\} = \{x \in \mathsf{q}(H) \mid xB \subset (A :_l C)\} \\ &= ((A :_l C) :_l B),\end{aligned}$$

and

$$\begin{aligned}((A :_l B) :_r C) &= \{x \in \mathsf{q}(H) \mid Cx \subset (A :_l B)\} = \{x \in \mathsf{q}(H) \mid CxB \subset A\} \\ &= \{x \in \mathsf{q}(H) \mid xB \subset (A :_r C)\} = ((A :_r C) :_l B).\end{aligned}$$

4. We check only the first equality. Let a be an element of the given intersection. We have to show that $a(H :_r A) \subset H$, whence for all $b \in (H :_r A)$ we have to verify that $ab \in H$. If $b \in (H :_r A)$, then $Ab \subset H$ implies that $A \subset Hb^{-1}$. Thus we obtain that

$$a \in \bigcap_{c \in \mathsf{q}(H), A \subset Hc} Hc \subset Hb^{-1},$$

and thus $ab \in H$. Conversely, suppose that $a \in (H :_l (H :_r A))$. We have to verify that $a \in Hc$ for all $c \in \mathsf{q}(H)$ with $A \subset Hc$. If $A \subset Hc$, then $Ac^{-1} \subset H$ implies that $c^{-1} \in (H :_r A)$. Thus we get $ac^{-1} \in H$ and $a \in Hc$. □

DEFINITION 3.3. Let H be a monoid and $A \subset \mathsf{q}(H)$ a subset. Then A is said to be

- *left* (*resp. right*) *H-fractional* if there exist $a \in H$ such that $Aa \subset H$ (resp. $aA \subset H$).
- *H-fractional* if A is left and right H-fractional.
- a *fractional left* (*resp. right*) *ideal* (of H) if A is left H-fractional and a left module (resp. right H-fractional and a right module).
- a *left* (*resp. right*) *ideal* (of H) if A is a fractional left ideal (resp. right ideal) and $A \subset H$.
- a (*fractional*) *ideal* if A is a (fractional) left and right ideal.

We denote by $\mathcal{F}_s(H)$ the set of fractional ideals of H, and by $\mathcal{I}_s(H)$ the set of ideals of H.

Note that the empty set is an ideal of H. Let $A \subset \mathsf{q}(H)$ be a subset. Then A is

- left H-fractional if and only if $(H :_r A) \neq \emptyset$ if and only if $(H :_r A) \cap H \neq \emptyset$.

- right H-fractional if and only if $(H :_l A) \neq \emptyset$ if and only if $(H :_l A) \cap H \neq \emptyset$.

Thus, if A is non-empty, then Lemma 3.2 (items 4. and 5.) shows that $(H :_l A)$ is a fractional left ideal and $(H :_r A)$ is a fractional right ideal.

Lemma 3.4. *Let H be a monoid.*

1. *If $(\mathfrak{a}_i)_{i\in I}$ is a family of fractional left ideals (resp. right ideals or ideals) and $J \subset I$ is finite, then $\bigcap_{i\in I} \mathfrak{a}_i$ and $\prod_{i\in J} \mathfrak{a}_i$ are fractional left ideals (resp. right ideals or ideals).*
2. *Equipped with usual multiplication, $\mathcal{F}_s(H)$ is a semigroup with unit element H.*
3. *If $\mathfrak{a} \in \mathcal{F}_s(H)^\times$, then $(H :_l \mathfrak{a})\mathfrak{a} = H = \mathfrak{a}(H :_r \mathfrak{a})$ and $(H :_l \mathfrak{a}) = (H :_r \mathfrak{a}) \in \mathcal{F}_s(H)$.*
4. *For every $a \in \mathsf{q}(H)$, we have $(H :_l aH) = Ha^{-1}$, $(H :_r Ha) = a^{-1}H$, $(H :_l (H :_r Ha)) = Ha$ and $(H :_r (H :_l aH)) = aH$.*
5. *If $A \subset \mathsf{q}(H)$, then $(H :_l (H :_r A))$ is a fractional left ideal and $(H :_r (H :_l A))$ is a fractional right ideal.*
6. *If $A \subset \mathsf{q}(H)$, $\mathfrak{a} = (H :_l A)$ and $\mathfrak{b} = (H :_r A)$, then $\mathfrak{a} = (H :_l (H :_r \mathfrak{a}))$ and $\mathfrak{b} = (H :_r (H :_l \mathfrak{b}))$.*

Proof. 1. Since $\bigcap_{i\in I} \mathfrak{a}_i \subset \mathfrak{a}_j$, $\prod_{i\in J} \mathfrak{a}_i \subset \mathfrak{a}_j$ for some $j \in J$ and subsets of left (resp. right) H-fractional sets are left (resp. right) H-fractional, the given intersection and product are left (resp. right) H-fractional, and then clearly they are fractional left ideals (resp. fractional right ideals or ideals).

2. Obvious.

3. Let $\mathfrak{a} \in \mathcal{F}_s(H)^\times$ and $\mathfrak{b} \in \mathcal{F}_s(H)$ with $\mathfrak{b}\mathfrak{a} = \mathfrak{a}\mathfrak{b} = H$. Then $\mathfrak{b} \subset (H :_l \mathfrak{a})$ and hence $H = \mathfrak{b}\mathfrak{a} \subset (H :_l \mathfrak{a})\mathfrak{a} \subset H$, which implies that $(H :_l \mathfrak{a})\mathfrak{a} = H$. Similarly, we obtain that $\mathfrak{a}(H :_r \mathfrak{a}) = H$, and therefore $(H :_l \mathfrak{a}) = \mathfrak{b} = (H :_r \mathfrak{a}) \in \mathcal{F}_s(H)$.

4. Let $a \in \mathsf{q}(H)$. The first two equalities follow directly from the definitions. Using them we infer that

$$(H :_l (H :_r Ha)) = (H :_l a^{-1}H) = Ha$$

and

$$(H :_r (H :_l aH)) = (H :_r Ha^{-1}) = aH.$$

5. This follows from 1. and from Lemma 3.2 4.

6. By Lemma 3.2 4., we have $\mathfrak{a} \subset (H :_l (H :_r \mathfrak{a}))$. Conversely, if $q \in (H :_l (H :_r \mathfrak{a}))$, then

$$qA \subset q(H :_r (H :_l A)) \subset q(H :_r \mathfrak{a}) \subset H,$$

and hence $q \in (H :_l A) = \mathfrak{a}$. $\square$

DEFINITION 3.5. Let H be a monoid and $A \subset \mathsf{q}(H)$ a subset.

1. A is called a *divisorial fractional left ideal* if $A = (H :_l (H :_r A))$, and a *divisorial fractional right ideal* if $A = (H :_r (H :_l A))$.

2. If $(H :_l A) = (H :_r A)$, then we set $A^{-1} = (H : A) = (H :_l A)$.
3. If $(H :_l (H :_r A)) = (H :_r (H :_l A))$, then we set $A_v = (H :_l (H :_r A))$, and A is said to be a *divisorial fractional ideal* (or a *fractional v-ideal*) if $A = A_v$. The set of such ideals will be denoted by $\mathcal{F}_v(H)$, and $\mathcal{I}_v(H) = \mathcal{F}_v(H) \cap \mathcal{I}_s(H)$ is the set of *divisorial ideals* of H (or the set of v-ideals of H).
4. Suppose that $(H :_l \mathfrak{c}) = (H :_r \mathfrak{c})$ for all fractional ideals $\mathfrak{c}$ of H.
   (a) For fractional ideals $\mathfrak{a}, \mathfrak{b}$ we define $\mathfrak{a} \cdot_v \mathfrak{b} = (\mathfrak{a}\mathfrak{b})_v$, and we call $\mathfrak{a} \cdot_v \mathfrak{b}$ the *v-product* of $\mathfrak{a}$ and $\mathfrak{b}$.
   (b) A fractional v-ideal $\mathfrak{a}$ is called *v-invertible* if $\mathfrak{a} \cdot_v \mathfrak{a}^{-1} = \mathfrak{a}^{-1} \cdot_v \mathfrak{a} = H$. We denote by $\mathcal{I}_v^*(H)$ the set of all v-invertible v-ideals.

Lemma 3.4 5. shows that a divisorial fractional left ideal is indeed a fractional left ideal, and the analogous statement holds for divisorial fractional right ideals and for divisorial fractional ideals. Furthermore, Lemma 3.4 4. shows that, for every $a \in \mathsf{q}(H)$, Ha is a divisorial fractional left ideal. We will see that the assumption of Definition 3.5 4. holds in completely integrally closed monoids (Definition 3.11) and in normalizing monoids (Lemma 4.5).

Lemma 3.6. *Suppose that $(H :_l \mathfrak{c}) = (H :_r \mathfrak{c})$ for all fractional ideals $\mathfrak{c}$ of H, and let $\mathfrak{a}, \mathfrak{b}$ be fractional ideals of H.*
1. *We have $\mathfrak{a} \subset \mathfrak{a}_v = (\mathfrak{a}_v)_v$ and $(\mathfrak{a}_v)^{-1} = \mathfrak{a}^{-1} = (\mathfrak{a}^{-1})_v$. In particular, $\mathfrak{a}^{-1}, \mathfrak{a}_v \in \mathcal{F}_v(H)$.*
2. $(\mathfrak{a}\mathfrak{a}^{-1})_v = (\mathfrak{a}_v : \mathfrak{a})^{-1}$.
3. *If $\mathfrak{a}, \mathfrak{b} \in \mathcal{F}_v(H)$, then $\mathfrak{a} \cdot_v \mathfrak{b} \in \mathcal{F}_v(H)$ and $\mathfrak{a} \cap \mathfrak{b} \in \mathcal{F}_v(H)$, and if $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_v(H)$, then $\mathfrak{a} \cdot_v \mathfrak{b} \in \mathcal{I}_v(H)$, $\mathfrak{a} \cap \mathfrak{b} \in \mathcal{I}_v(H)$, and $\mathfrak{a} \cdot_v \mathfrak{b} \subset \mathfrak{a} \cap \mathfrak{b}$.*
4. *If $d \in \mathsf{q}(H)$ with $d\mathfrak{a} \subset \mathfrak{b}$, then $d\mathfrak{a}_v \subset \mathfrak{b}_v$. Similarly, $\mathfrak{a}d \subset \mathfrak{b}$ implies that $\mathfrak{a}_v d \subset \mathfrak{b}$.*
5. *We have $(\mathfrak{a}\mathfrak{b})_v = (\mathfrak{a}_v\mathfrak{b})_v = (\mathfrak{a}_v\mathfrak{b}_v)_v$.*
6. *Equipped with v-multiplication, $\mathcal{F}_v(H)$ is a semigroup with unit element H, and $\mathcal{I}_v(H)$ is a subsemigroup. Furthermore, if $\mathfrak{a} \in \mathcal{F}_v(H)$, then $\mathfrak{a}$ is v-invertible if and only if $\mathfrak{a} \in \mathcal{F}_v(H)^\times$, and hence $\mathcal{I}_v^*(H) = \mathcal{I}_v(H) \cap \mathcal{F}_v(H)^\times$.*

Proof. 1. By Lemma 3.4 5., we have $\mathfrak{a} \subset \mathfrak{a}_v$. Therefore it follows that

$$[(\mathfrak{a}^{-1})^{-1}]^{-1} = (\mathfrak{a}_v)^{-1} \subset \mathfrak{a}^{-1} \subset (\mathfrak{a}^{-1})_v = [(\mathfrak{a}^{-1})^{-1}]^{-1},$$

hence $(\mathfrak{a}_v)^{-1} = \mathfrak{a}^{-1} = (\mathfrak{a}^{-1})_v$ and $(\mathfrak{a}_v)_v = ((\mathfrak{a}_v)^{-1})^{-1} = (\mathfrak{a}^{-1})^{-1} = \mathfrak{a}_v$.

2. Using Lemma 3.2 3. we infer that

$$(\mathfrak{a}\mathfrak{a}^{-1})^{-1} = (H : \mathfrak{a}\mathfrak{a}^{-1}) = ((H : \mathfrak{a}^{-1}) : \mathfrak{a}) = (\mathfrak{a}_v : \mathfrak{a}),$$

and hence $(\mathfrak{a}\mathfrak{a}^{-1})_v = (\mathfrak{a}_v : \mathfrak{a})^{-1}$.

3. Let $\mathfrak{a}, \mathfrak{b} \in \mathcal{F}_v(H)$. Then $\mathfrak{a} \cdot_v \mathfrak{b} = (\mathfrak{a}\mathfrak{b})_v$ is a divisorial fractional ideal by 1. Clearly, we have $\mathfrak{a} \cap \mathfrak{b} \subset (\mathfrak{a} \cap \mathfrak{b})_v \subset \mathfrak{a}_v \cap \mathfrak{b}_v = \mathfrak{a} \cap \mathfrak{b}$. The remaining statements are clear.

4. If $d\mathfrak{a} \subset \mathfrak{b}$, then we get

$$d\mathfrak{a}_v = d \bigcap_{c \in \mathsf{q}(H), \mathfrak{a} \subset cH} cH = \bigcap_{c \in \mathsf{q}(H), d\mathfrak{a} \subset dcH} dcH = \bigcap_{e \in \mathsf{q}(H), d\mathfrak{a} \subset eH} eH$$
$$= (H :_r (H :_l d\mathfrak{a})) \subset (H :_r (H :_l \mathfrak{b})) = \mathfrak{b}_v .$$

If $\mathfrak{a}d \subset \mathfrak{b}$, we argue similarly.

5. We have $(\mathfrak{a}\mathfrak{b})_v \subset (\mathfrak{a}_v\mathfrak{b})_v \subset (\mathfrak{a}_v\mathfrak{b}_v)_v$. To obtain the reverse inclusion it is sufficient to verify that

$$(\mathfrak{a}\mathfrak{b})^{-1} \subset (\mathfrak{a}_v\mathfrak{b}_v)^{-1} .$$

Let $d \in (\mathfrak{a}\mathfrak{b})^{-1}$. Then $d\mathfrak{a}\mathfrak{b} \subset H$ and hence $da\mathfrak{b} \subset H$ for all $a \in \mathfrak{a}$. Then 4. implies that $da\mathfrak{b}_v \subset H_v = H$ for all $a \in \mathfrak{a}$ and hence $d\mathfrak{a}\mathfrak{b}_v \subset H$. Since $\mathfrak{a}\mathfrak{b}_v$ is a fractional ideal, it follows that $\mathfrak{a}\mathfrak{b}_v d \subset H$ and hence $\mathfrak{a}bd \subset H$ for all $b \in \mathfrak{b}_v$. Again 4. implies that $\mathfrak{a}_v bd \subset H$ for all $b \in \mathfrak{b}_v$ and hence $\mathfrak{a}_v\mathfrak{b}_v d \subset H$.

6. Using 5. we obtain to first assertion. We provide the details for the furthermore statement. Let $\mathfrak{a} \in \mathcal{F}_v(H)$. Then $\mathfrak{a}^{-1} \in \mathcal{F}_v(H)$, and thus, if $\mathfrak{a}$ is v-invertible, then $\mathfrak{a} \in \mathcal{F}_v(H)^\times$. Conversely, suppose that $\mathfrak{a} \in \mathcal{F}_v(H)^\times$ and let $\mathfrak{b} \in \mathcal{F}_v(H)$ such that $\mathfrak{a} \cdot_v \mathfrak{b} = \mathfrak{b} \cdot_v \mathfrak{a} = H$. Then $\mathfrak{a}\mathfrak{b} \subset H$, hence $\mathfrak{b} \subset (H : \mathfrak{a})$ and $\mathfrak{a}\mathfrak{b} \subset \mathfrak{a}(H : \mathfrak{a}) \subset H$. This implies that $H = (\mathfrak{a}\mathfrak{b})_v \subset \mathfrak{a} \cdot_v \mathfrak{a}^{-1} \subset H$. Similarly, we get $\mathfrak{a}^{-1} \cdot_v \mathfrak{a} = H$, and hence $\mathfrak{a}$ is v-invertible. □

The next topic are prime ideals and their properties.

Lemma 3.7. *Let H be a monoid and $\mathfrak{p} \subset H$ an ideal. Then the following statements are equivalent*:

(a) *If $\mathfrak{a}, \mathfrak{b} \subset H$ are ideals with $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$, then $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$.*
(b) *If $\mathfrak{a}, \mathfrak{b} \subset H$ are right ideals with $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$, then $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$.*
(c) *If $\mathfrak{a}, \mathfrak{b} \subset H$ are left ideals with $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$, then $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$.*
(d) *If $a, b \in H$ with $aHb \subset \mathfrak{p}$, then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.*

Proof. (a) ⇒ (b) If $\mathfrak{a}, \mathfrak{b} \subset H$ are right ideals with $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$, then $H\mathfrak{a}, H\mathfrak{b} \subset H$ are ideals with $(H\mathfrak{a})(H\mathfrak{b}) = H\mathfrak{a}\mathfrak{b} \subset H\mathfrak{p} = \mathfrak{p}$, and hence $\mathfrak{a} \subset H\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset H\mathfrak{b} \subset \mathfrak{p}$.

(b) ⇒ (d) If $a, b \in H$ with $aHb \subset \mathfrak{p}$, then $(aH)(bH) \subset \mathfrak{p}H = \mathfrak{p}$, and hence $a \in aH \subset \mathfrak{p}$ or $b \in bH \subset \mathfrak{p}$.

(d) ⇒ (a) If $\mathfrak{a} \not\subset \mathfrak{p}$ and $\mathfrak{b} \not\subset \mathfrak{p}$, then there exist $a \in \mathfrak{a} \setminus \mathfrak{p}$, $b \in \mathfrak{b} \setminus \mathfrak{p}$, and hence $aHb \not\subset \mathfrak{p}$, which implies that $\mathfrak{a}\mathfrak{b} \not\subset \mathfrak{p}$.

The proof of the implications (a) ⇒ (c) ⇒ (d) ⇒ (a) runs along the same lines. □

An ideal $\mathfrak{p} \subset H$ is called *prime* if $\mathfrak{p} \neq H$ and if it satisfies the equivalent statements in Lemma 3.7. We denote by s-spec(H) the set of prime ideals of H, and by

v-spec$(H) = s$-spec$(H) \cap \mathcal{I}_v(H)$ the set of divisorial prime ideals of H. Following ring theory ([47, Definition 10.3]), we call a subset $S \subset H$ an *m-system* if, for any $a, b \in S$, there exists an $h \in H$ such that $ahb \in S$. Thus Lemma 3.7 (d) shows that an ideal $\mathfrak{p} \subset H$ is prime if and only if $H \setminus \mathfrak{p}$ is an m-system.

A subset $\mathfrak{m} \subset H$ is called a *v-maximal v-ideal* if $\mathfrak{m}$ is a maximal element of $\mathcal{I}_v(H) \setminus \{H\}$ (with respect to the inclusion). We denote by v-max(H) the set of all v-maximal v-ideals of H.

Lemma 3.8. *Suppose that $(H :_l \mathfrak{c}) = (H :_r \mathfrak{c})$ for all fractional ideals $\mathfrak{c}$ of H.*
1. *If $S \subset H$ is an m-system and $\mathfrak{p}$ is maximal in the set $\{\mathfrak{a} \in \mathcal{I}_v(H) \mid \mathfrak{a} \cap S = \emptyset\}$, then $\mathfrak{p} \in v$-spec(H).*
2. *v-max$(H) \subset v$-spec(H).*

Proof. 1. Assume to the contrary that $\mathfrak{p} \in \mathcal{I}_v(H)$ is maximal with respect to $\mathfrak{p} \cap S = \emptyset$, but $\mathfrak{p}$ is not prime. Then there exist elements $a, b \in H \setminus \mathfrak{p}$ such that $aHb \subset \mathfrak{p}$. By the maximal property of $\mathfrak{p}$, we have $S \cap (\mathfrak{p} \cup HaH)_v \neq \emptyset$ and $S \cap (\mathfrak{p} \cup HbH)_v \neq \emptyset$. If $s \in S \cap (\mathfrak{p} \cup HaH)_v$ and $t \in S \cap (\mathfrak{p} \cup HbH)_v$, then $sht \in S$ for some $h \in H$, and using Lemma 3.6 5. we obtain that

$$\begin{aligned} sht &\in (\mathfrak{p} \cup HaH)_v H (\mathfrak{p} \cup HbH)_v \\ &\subset [(\mathfrak{p} \cup HaH) H (\mathfrak{p} \cup HbH)]_v \subset [\mathfrak{p} \cup HaHbH]_v = \mathfrak{p}_v = \mathfrak{p}, \end{aligned}$$

a contradiction.

2. If $\mathfrak{m} \in v$-max(H), then $\mathfrak{m} \in \mathcal{I}_v(H)$ is maximal with respect to $\mathfrak{m} \cap \{1\} = \emptyset$, and therefore $\mathfrak{m}$ is prime by 1. □

Our next step is to introduce completely integrally closed monoids.

Lemma 3.9. *Let H be a monoid and H' an overmonoid of H.*
1. *If $I = (H :_r H')$, then $H' \subset (I :_l I)$.*
2. *Let $a, b \in H$ with $aH'b \subset H$. Then there exists a monoid H'' with $H \subset H'' \subset H'$ such that $(H :_r H'') \neq \emptyset$ and $(H'' :_l H') \neq \emptyset$.*

Proof. 1. Since $H'(H'I) = H'I \subset H$, it follows that $H'I \subset (H :_r H') = I$ and hence $H' \subset (I :_l I)$.

2. We set $H'' = HaH' \cup H$, and obtain that $H \subset H'' \subset H'$, $H''H'' = H''$, $H''b \subset H$ and $aH' \subset H''$. □

Lemma 3.10. *Let H be a monoid.*
1. *The following statements are equivalent*:
   (a) *There is no overmonoid H' of H with $H \subsetneq H' \subset \mathsf{q}(H)$ and $aH'b \subset H$ for some $a, b \in H$.*

(b) $(\mathfrak{a} :_l \mathfrak{a}) = (\mathfrak{b} :_r \mathfrak{b}) = H$ *for all non-empty left modules* $\mathfrak{a}$ *of* H *which are right* H*-fractional and for all non-empty right modules* $\mathfrak{b}$ *of* H *which are left* H*-fractional.*

(c) $(\mathfrak{a} :_l \mathfrak{a}) = (\mathfrak{a} :_r \mathfrak{a}) = H$ *for all non-empty ideals* $\mathfrak{a}$ *of* H.

2. *Suppose that* H *satisfies one of the equivalent conditions in* 1. *Then* $(H :_l \mathfrak{a}) = (H :_r \mathfrak{a})$ *and* $(H :_l (H :_r \mathfrak{a})) = (H :_r (H :_l \mathfrak{a}))$ *for all non-empty fractional ideals* $\mathfrak{a}$ *of* H.

Proof. 1. If $H = \mathsf{q}(H)$, then all statements are fulfilled. Suppose that H is not a group.

(a) $\Rightarrow$ (b) Let $\emptyset \neq \mathfrak{a} \subset \mathsf{q}(H)$ and $a \in H$ with $H\mathfrak{a} = \mathfrak{a}$ and $a\mathfrak{a} \subset H$. Then $H' = (\mathfrak{a} :_l \mathfrak{a})$ is an overmonoid of H. If $b \in \mathfrak{a} \cap H$, then $aH'b \subset a\mathfrak{a} \subset H$ and hence $H' = H$ by 1.

(b) $\Rightarrow$ (c) Obvious.

(c) $\Rightarrow$ (a) Let H' be an overmonoid of H with $aH'b \subset H$ for some $a, b \in H$. We have to show that $H' = H$. By Lemma 3.9 2., there exists a monoid H'' with $H \subset H'' \subset H'$ such that $\mathfrak{a} = (H :_r H'') \neq \emptyset$ and $\mathfrak{b} = (H'' :_l H') \neq \emptyset$. Then Lemma 3.9 1. implies that $H'' \subset (\mathfrak{a} :_l \mathfrak{a}) = H$ and $H' \subset (\mathfrak{b} :_r \mathfrak{b}) = H$.

2. If $\mathfrak{a} \subset \mathsf{q}(H)$ is a non-empty fractional ideal, then Lemma 3.2 3. and 1. imply that

$$(H :_l \mathfrak{a}) = ((\mathfrak{a} :_r \mathfrak{a}) :_l \mathfrak{a}) = ((\mathfrak{a} :_l \mathfrak{a}) :_r \mathfrak{a}) = (H :_r \mathfrak{a}).$$

Since $(H :_l \mathfrak{a}) = (H :_r \mathfrak{a})$ is a non-empty fractional ideal, the previous argument implies that $(H :_l (H :_r \mathfrak{a})) = (H :_r (H :_l \mathfrak{a}))$. □

DEFINITION 3.11. A monoid H is said to be

- *completely integrally closed* if it satisfies the equivalent conditions of Lemma 3.10 1.
- v*-noetherian* if it satisfies the ascending chain condition on v-ideals of H.
- a *Krull monoid* if it is completely integrally closed and v-noetherian.

If H is a commutative monoid, then the above notion of being completely integrally closed coincides with the usual one (see [29, Section 2.3]). We need a few notions from the theory of po-groups (we follow the terminology of [62]). Let $Q = (Q, \cdot)$ be a multiplicatively written group with unit element $1 \in Q$, and let $\le$ be a partial order on Q. Then $(Q, \cdot, \le)$ is said to be

- a *po-group* if $x \le y$ implies that $axb \le ayb$ for all $x, y, a, b \in Q$.
- *directed* if each two element subset of Q has an upper and a lower bound.
- *integrally closed* if for all $a, b \in Q$, $a^n \le b$ for all $n \in \mathbb{N}$ implies that $a \le 1$.

Proposition 3.12. *Let* H *be a completely integrally closed monoid.*

1. *Every non-empty fractional* v*-ideal is* v*-invertible, and* $v\text{-max}(H) = v\text{-spec}(H) \setminus \{\emptyset\}$.

2. *Equipped with the set-theoretical inclusion as a partial order and v-multiplication as group operation, the group* $\mathcal{F}_v(H)^\times$ *is a directed integrally closed po-group.*
3. $\mathcal{I}_v^*(H)$ *is a commutative monoid with quotient group* $\mathcal{F}_v(H)^\times$.
4. *If* $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_v^*(H)$, *then* $\mathfrak{a} \supset \mathfrak{b}$ *if and only if* $\mathfrak{a} \mid \mathfrak{b}$ *in* $\mathcal{I}_v^*(H)$. *In particular,* $(\mathfrak{a} \cup \mathfrak{b})_v = \gcd(\mathfrak{a}, \mathfrak{b})$ *in* $\mathcal{I}_v^*(H)$, *and* $\mathcal{I}_v^*(H)$ *is reduced.*

Proof. 1. Let $\emptyset \neq \mathfrak{a} \in \mathcal{F}_v(H)$.

Using Lemma 3.6 2. and that H is completely integrally closed, we obtain that $(\mathfrak{a}\mathfrak{a}^{-1})_v = (\mathfrak{a}_v : \mathfrak{a})^{-1} = (\mathfrak{a} : \mathfrak{a})^{-1} = H^{-1} = H$. Since $\mathfrak{a}^{-1} \in \mathcal{F}_v(H)$, we may apply this relation for $\mathfrak{a}^{-1}$ and get $(\mathfrak{a}^{-1}\mathfrak{a})_v = H$. Therefore it follows that

$$\mathfrak{a} \cdot_v \mathfrak{a}^{-1} = (\mathfrak{a}\mathfrak{a}^{-1})_v = H = (\mathfrak{a}^{-1}\mathfrak{a})_v = \mathfrak{a}^{-1} \cdot_v \mathfrak{a}.$$

By Lemma 3.8 2., we have $v\text{-max}(H) \subset v\text{-spec}(H) \setminus \{\emptyset\}$. Assume to the contrary that there are $\mathfrak{p}, \mathfrak{q} \in v\text{-spec}(H)$ with $\emptyset \neq \mathfrak{p} \subsetneq \mathfrak{q} \subset H$. Since $\mathfrak{q}$ is v-invertible, we get $\mathfrak{p} = \mathfrak{q} \cdot_v \mathfrak{a}$ with $\mathfrak{a} = \mathfrak{q}^{-1} \cdot_v \mathfrak{p} \subset H$. Since $\mathfrak{p}$ is a prime ideal and $\mathfrak{q} \not\subset \mathfrak{p}$, it follows that $\mathfrak{a} \subset \mathfrak{p}$. Then $\mathfrak{a} = \mathfrak{b} \cdot_v \mathfrak{p}$ with $\mathfrak{b} = \mathfrak{a} \cdot_v \mathfrak{p}^{-1} \subset H$, whence $\mathfrak{p} = \mathfrak{q} \cdot_v \mathfrak{b} \cdot_v \mathfrak{p}$ and thus $H = \mathfrak{q} \cdot_v \mathfrak{b}$, a contradiction.

2. Clearly, $(\mathcal{F}_v(H)^\times, \cdot_v, \subset)$ is a po-group. In order to show that it is directed, consider $\mathfrak{a}, \mathfrak{b} \in \mathcal{F}_v(H)^\times$. Then $\mathfrak{a} \cdot_v \mathfrak{b} \in \mathcal{F}_v^\times(H)$ is a lower bound of $\{\mathfrak{a}, \mathfrak{b}\}$, and $(\mathfrak{a} \cup \mathfrak{b})_v$ is an upper bound. In order to show that it is integrally closed, let $\mathfrak{a}, \mathfrak{b} \in \mathcal{F}_v(H)^\times$ be given such that $\mathfrak{a}^n \subset \mathfrak{b}$ for all $n \in \mathbb{N}$. We have to show that $\mathfrak{a} \subset H$. The set

$$\mathfrak{a}_0 = \bigcup_{n \geq 1} \mathfrak{a}^n \subset \mathfrak{b}$$

is a non-empty fractional ideal, and we get $\mathfrak{a} \subset (\mathfrak{a}_0 :_l \mathfrak{a}_0) = H$, since H is completely integrally closed.

3. Since $(\mathcal{F}_v(H)^\times, \cdot_v, \subset)$ is a directed integrally closed po-group by 2., $\mathcal{F}_v(H)^\times$ is a commutative group by [62, Theorem 2.3.9]. Since $\mathcal{I}_v^*(H) = \mathcal{F}_v(H)^\times \cap \mathcal{I}_v(H)$ by Lemma 3.6 6., it follows that $\mathcal{I}_v^*(H)$ is a commutative monoid. In order to show that $\mathcal{F}_v(H)^\times$ is a quotient group of $\mathcal{I}_v^*(H)$, let $\mathfrak{c} \in \mathcal{F}_v(H)^\times$ be given. We have to find some $\mathfrak{a} \in \mathcal{I}_v^*(H)$ such that $\mathfrak{a} \cdot_v \mathfrak{c} \in \mathcal{I}_v^*(H)$, and for that it suffices to verify that $\mathfrak{a} \cdot_v \mathfrak{c} \subset H$. Now, since $\mathfrak{c}$ is a fractional ideal, there exists some $c \in H$ such that $c\mathfrak{c} \subset H$, thus $(HcH)_v \in \mathcal{I}_v^*(H)$ and, by Lemma 3.6 5.,

$$(HcH)_v \cdot_v \mathfrak{c} = ((HcH)_v\mathfrak{c})_v = (Hc\mathfrak{c})_v \subset H_v = H.$$

4. Note that $\mathcal{I}_v^*(H)$ is commutative by 3., and hence the greatest common divisor is formed in a commutative monoid. Thus the in particular statements follow immediately from the main statement. In order to show that divisibility is equivalent to containment, we argue as before. Let $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_v^*(H)$. If $\mathfrak{a} \mid \mathfrak{b}$ in $\mathcal{I}_v^*(H)$, then $\mathfrak{b} = \mathfrak{a} \cdot_v \mathfrak{c}$ for some $\mathfrak{c} \in \mathcal{I}_v^*(H)$, and therefore $\mathfrak{b} \subset \mathfrak{a}$. If $\mathfrak{b} \subset \mathfrak{a}$, then $\mathfrak{b} \cdot_v \mathfrak{a}^{-1} \subset \mathfrak{a} \cdot_v \mathfrak{a}^{-1} = H$, and

thus $\mathfrak{b} \cdot_v \mathfrak{a}^{-1} \in \mathcal{F}_v(H)^\times \cap \mathcal{I}_v(H) = \mathcal{I}_v^*(H)$. The relation $\mathfrak{b} = (\mathfrak{b} \cdot_v \mathfrak{a}^{-1}) \cdot_v \mathfrak{a}$ shows that $\mathfrak{a} \mid \mathfrak{b}$ in $\mathcal{I}_v^*(H)$. $\square$

The missing parts are ideal theoretic properties of v-noetherian monoids.

Proposition 3.13. *Suppose that* $(H :_l \mathfrak{c}) = (H :_r \mathfrak{c})$ *for all fractional ideals* $\mathfrak{c}$ *of* H.

1. *The following statements are equivalent*:
   (a) *H is v-noetherian.*
   (b) *Every non-empty set of v-ideals of H has a maximal element* (*with respect to the inclusion*).
   (c) *Every non-empty set of fractional v-ideals of H with non-empty intersection has a minimal element* (*with respect to the inclusion*).
   (d) *For every non-empty ideal $\mathfrak{a} \subset H$, there exists a finite subset $E \subset \mathfrak{a}$ such that* $(HEH)^{-1} = \mathfrak{a}^{-1}$.
2. *If H is v-noetherian and $\mathfrak{a} \in \mathcal{I}_v^*(H)$, then there exists a finite set $E \subset \mathfrak{a}$ such that* $\mathfrak{a} = (HEH)_v$.
3. *If H is v-noetherian and $a \in H$, then the set* $\{\mathfrak{p} \in v\text{-spec}(H) \mid a \in \mathfrak{p}\}$ *is finite.*

Proof. 1. (a) $\Rightarrow$ (b) If $\emptyset \ne \Omega \subset \mathcal{I}_v(H)$ has no maximal element, then every $\mathfrak{a} \in \Omega$ is properly contained in some $\mathfrak{a}' \in \Omega$. If $\mathfrak{a}_0 \in \Omega$ is arbitrary and the sequence $(\mathfrak{a}_n)_{n\ge 0}$ is recursively defined by $\mathfrak{a}_{n+1} = \mathfrak{a}_n'$ for all $n \ge 0$, then $(\mathfrak{a}_n)_{n\ge0}$ is an ascending sequence of v-ideals not becoming stationary.

(b) $\Rightarrow$ (c) Suppose that $\emptyset \ne \Omega \subset \mathcal{F}_v(H)$ and $a \in \mathfrak{a}$ for all $\mathfrak{a} \in \Omega$. Then the set $\Omega^* = \{a\mathfrak{a}^{-1} \mid \mathfrak{a} \in \Omega\} \subset \mathcal{I}_v(H)$ has a maximal element $a\mathfrak{a}_0^{-1}$ with $\mathfrak{a}_0 \in \Omega$, and then $\mathfrak{a}_0$ is a minimal element of Ω.

(c) $\Rightarrow$ (d) If $\emptyset \ne E \subset \mathfrak{a}$, then $\emptyset \ne \mathfrak{a}^{-1} \subset (HEH)^{-1} \in \mathcal{F}_v(H)$. Thus the set $\Omega = \{(HEH)^{-1} \mid \emptyset \ne E \subset \mathfrak{a}, E \text{ finite}\}$ has a minimal element $(HE_0H)^{-1}$, where $E_0 \subset \mathfrak{a}$ is a finite non-empty subset. Then $(HE_0H)^{-1} \supset \mathfrak{a}^{-1}$, and we assert that equality holds. Assume to the contrary that there exists some $u \in (HE_0H)^{-1} \setminus \mathfrak{a}^{-1}$. Then there exists an element $a \in \mathfrak{a}$ such that $ua \notin H$, and if $E_1 = E_0 \cup \{a\}$, then $u \notin (HE_1H)^{-1}$ and consequently $(HE_1H)^{-1} \subsetneq (HE_0H)^{-1}$, a contradiction.

(d) $\Rightarrow$ (a) Let $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \ldots$ be an ascending sequence of v-ideals. Then

$$\mathfrak{a} = \bigcup_{n\ge1} \mathfrak{a}_n \subset H$$

is an ideal of H, and we pick a finite non-empty subset $E \subset \mathfrak{a}$ such that $(HEH)^{-1} = \mathfrak{a}^{-1}$. Then there exists some $m \ge 0$ such that $E \subset \mathfrak{a}_m$. For all $n \ge m$ we obtain $\mathfrak{a}_n \subset \mathfrak{a} \subset \mathfrak{a}_v = (HEH)_v \subset \mathfrak{a}_m$ and hence $\mathfrak{a}_n = \mathfrak{a}_m$.

2. Let H be a v-noetherian and $\mathfrak{a} \in \mathcal{I}_v^*(H)$. By 1., there exists a finite subset $E \subset \mathfrak{a}$ such that $(HEH)^{-1} = \mathfrak{a}^{-1}$ and therefore $(HEH)_v = \mathfrak{a}_v = \mathfrak{a}$.

3. Assume to the contrary that H is v-noetherian and that there exists some $a \in H$ such that the set $\Omega = \{\mathfrak{p} \in v\text{-spec}(H) \mid a \in \mathfrak{p}\}$ is infinite. Then 1. implies that there

is a sequence $(\mathfrak{p}_n)_{n\geq 0}$ in Ω such that, for all $n \geq 0$, $\mathfrak{p}_n$ is maximal in $\Omega \setminus \{\mathfrak{p}_0, \ldots, \mathfrak{p}_{n-1}\}$, and again by 1., the set $\{\mathfrak{p}_0 \cap \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n \mid n \in \mathbb{N}_0\}$ has a minimal element. Hence there exists some $n \in \mathbb{N}_0$ such that $\mathfrak{p}_0 \cap \cdots \cap \mathfrak{p}_n = \mathfrak{p}_0 \cap \cdots \cap \mathfrak{p}_{n+1} \subset \mathfrak{p}_{n+1}$. Since $\mathfrak{p}_{n+1}$ is a prime ideal, Lemma 3.7 implies that there exists some $i \in [0, n]$ such that $\mathfrak{p}_i \subset \mathfrak{p}_{n+1}$. Since now $\mathfrak{p}_{n+1} \in \Omega \setminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} \subset \Omega \setminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{i-1}\}$ and $\mathfrak{p}_i$ is maximal in the larger set, it follows that $\mathfrak{p}_{n+1} \subset \mathfrak{p}_i$, and hence $\mathfrak{p}_{n+1} = \mathfrak{p}_i \in \Omega \setminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, a contradiction. □

In contrast to the commutative setting the set $\{\mathfrak{p} \in v\text{-spec}(H) \mid a \in \mathfrak{p}\}$ can be empty. We will provide an example in Section 5 after having established the relationship between Krull monoids and Krull rings (see Example 5.2).

Theorem 3.14 (Ideal theory of Krull monoids). *Let H be a Krull monoid. Then $\mathcal{I}_v^*(H)$ is a free abelian monoid with basis $v\text{-max}(H) = v\text{-spec}(H) \setminus \{\emptyset\}$.*

Proof. Since H is v-noetherian and since divisibility in $\mathcal{I}_v^*(H)$ is equivalent to containment (by Proposition 3.12 4.), $\mathcal{I}_v^*(H)$ is reduced and satisfies the divisor chain condition. Therefore, it is atomic by [29, Proposition 1.1.4]. Again by the equivalence of divisibility and containment, the set of atoms of $\mathcal{I}_v^*(H)$ equals $v\text{-max}(H)$, and by Proposition 3.12, we have $v\text{-max}(H) = v\text{-spec}(H) \setminus \{\emptyset\}$. Since every non-empty prime v-ideal is a prime element of $\mathcal{I}_v^*(H)$, every atom of $\mathcal{I}_v^*(H)$ is a prime element, and thus $\mathcal{I}_v^*(H)$ is a free abelian monoid with basis $v\text{-max}(H)$ by [29, 1.1.10 and 1.2.2]. □

4. Divisor homomorphisms and normalizing monoids

The classic concept of a divisor theory was first presented in an abstract (commutative) setting by Skula [61], and after that it was generalized in many steps (see e.g. [27], and the presentations in [36, 29]). In this section we investigate divisor homomorphisms and divisor theories in a non-commutative setting. We study normal elements and normalizing submonoids of rings and monoids as introduced by Wauters [63] and Cohn [20, Section 3.1]. For the role of normal elements in ring theory see [32, Chapter 12] and [53, Chapter 10]. The normalizing monoid $\mathsf{N}(H)$ of a monoid H plays a crucial role in the study of semigroup algebras $K[H]$ (see [40]). In this context, Jespers and Okniński showed that completely integrally closed monoids, whose quotient groups are finitely generated torsion-free nilpotent groups and which satisfy the ascending chain condition on right ideals, are normalizing (see [39, Theorem 2]). Recall that, if R is a prime ring and $a \in R \setminus \{0\}$ is a normal element, then a is a regular element. The main results in this section are the divisor theoretic characterization of normalizing Krull monoids together with its consequences (Theorem 4.13 and Corollary 4.14).

DEFINITION 4.1. 1. A homomorphism of monoids $\varphi\colon H \to D$ is called a

- (left and right) *divisor homomorphism* if $\varphi(u) \mid_l \varphi(v)$ implies that $u \mid_l v$ and $\varphi(u) \mid_r \varphi(v)$ implies that $u \mid_r v$ for all $u, v \in H$.

- (left and right) *cofinal* if for every $a \in D$ there exist $u, v \in H$ such that $a \mid_l \varphi(u)$ and $a \mid_r \varphi(v)$ (equivalently, $aD \cap \varphi(H) \neq \emptyset$ and $Da \cap \varphi(H) \neq \emptyset$).

2. A *divisor theory* (for H) is a divisor homomorphism $\varphi\colon H \to D$ such that $D = \mathcal{F}(P)$ for some set P and, for every $p \in P$, there exists a finite subset $\emptyset \neq X \subset H$ satisfying $p = \gcd(\varphi(X))$.
3. A submonoid $H \subset D$ is called
   - *cofinal* if the embedding $H \hookrightarrow D$ is cofinal.
   - *saturated* if the embedding $H \hookrightarrow D$ is a divisor homomorphism.

DEFINITION 4.2. Let H be a cancellative semigroup.

1. An element $a \in H$ is said to be *normal* (or *invariant*) if $aH = Ha$. The subset $\mathsf{N}(H) = \{a \in H \mid aH = Ha\} \subset H$ is called the *normalizing submonoid* (or *invariant submonoid*) of H, and H is said to be *normalizing* if $\mathsf{N}(H) = H$ (Lemma 4.3 will show that $\mathsf{N}(H)$ is indeed a normalizing submonoid).
2. An element $a \in H$ is said to be *weakly normal* if $aH^\times = H^\times a$. The subset $H^{\mathsf{w}} = \{a \in H \mid aH^\times = H^\times a\} \subset H$ is called the *weakly normal submonoid* of H, and H is said to be *weakly normal* if $H^{\mathsf{w}} = H$.
3. Two elements $a, b \in H$ are said to be *associated* if $a \in H^\times b H^\times$ (we write $a \simeq b$, and note that this is an equivalence relation on H).
4. We denote by $\mathcal{P}(H) = \{aH \mid a \in H\}$ the set of principal right ideals, by $\mathcal{P}^{\mathsf{n}}(H) = \{aH \mid a \in \mathsf{N}(H)\}$ the set of normalizing principal ideals, by $\mathsf{C}(H) = \{a \in H \mid ab = ba \text{ for all } b \in H\}$ the *center* of H, and we set $H_{\text{red}} = \{aH^\times \mid a \in H^{\mathsf{w}}\}$,

Lemma 4.3. *Let H be a cancellative semigroup.*

1. *If H is normalizing, then H is a monoid.*
2. $\mathsf{N}(H)$ *is a subsemigroup with* $H^\times \subset \mathsf{N}(H)$, *and if H is a monoid, then* $\mathsf{N}(H) \subset H$ *is a normalizing saturated submonoid.*
3. $\mathsf{C}(H) \subset \mathsf{N}(H)$ *is a commutative saturated submonoid.*

Proof. 1. Let H be a normalizing semigroup. If $a, b \in H$, then $ab \in aH = Ha$ implies the existence of an element $c \in H$ such that $ab = ca$ and hence $Ha \cap Hb \neq \emptyset$. Similarly, we get that $aH \cap bH \neq \emptyset$. Thus the left and right Ore condition is satisfied, and H is a monoid.

2. If $a, b \in H$ with $aH = Ha$ and $bH = Hb$, then $abH = aHb = Hab$. Since $1 \in \mathsf{N}(H)$, it follows that $\mathsf{N}(H) \subset H$ is a subsemigroup. Since $\varepsilon H = H = H\varepsilon$ for all $\varepsilon \in H^\times$, we have $H^\times \subset \mathsf{N}(H)$.

Suppose that H is a monoid. In order to show that $\mathsf{N}(H)$ is normalizing, we have to verify that $a\mathsf{N}(H) = \mathsf{N}(H)a$ for all $a \in \mathsf{N}(H)$. Let $a, b \in \mathsf{N}(H)$. Since $ab \in aH = Ha$, there exists some $c \in H$ such that $ab = ca$. Since H is a monoid, $a \in H$ is invertible in $\mathsf{q}(H)$, and we get $cH = aba^{-1}H = Haba^{-1} = Hc$, which shows shows that $c \in \mathsf{N}(H)$. This implies that $a\mathsf{N}(H) \subset \mathsf{N}(H)a$, and by repeating the argument we obtain equality.

In order to show that $\mathsf{N}(H) \subset H$ is saturated, let $a, b \in \mathsf{N}(H)$ be given such that $a \mid_l b$ in H. Then there exists an element $c \in H$ such that $b = ac$. Since $cH = a^{-1}bH = Ha^{-1}b = Hc$, it follows that $c \in \mathsf{N}(H)$, and hence $a \mid_l b$ in $\mathsf{N}(H)$. If $a, b \in \mathsf{N}(H)$ such that $a \mid_r b$ in H, then we similarly get that $a \mid_r b$ in $\mathsf{N}(H)$. Thus $\mathsf{N}(H) \subset H$ is a saturated submonoid.

3. It follows by the definition that $\mathsf{C}(H) \subset \mathsf{N}(H)$ is a commutative submonoid. In order to show that $\mathsf{C}(H) \subset \mathsf{N}(H)$ is saturated, let $a, b \in \mathsf{C}(H)$ be given such that $a \mid_l b$ in $\mathsf{N}(H)$. Then there exists an element $c \in \mathsf{N}(H)$ such that $b = ac$. For every $d \in H$, we have $cd = a^{-1}bd = da^{-1}b = dc$, hence $c \in \mathsf{C}(H)$ and $a \mid_l b$ in $\mathsf{C}(H)$. We argue similarly in case of right divisibility and obtain that $\mathsf{C}(H) \subset \mathsf{N}(H)$ is saturated. □

Lemma 4.4. *Let H be a monoid.*

1. *H^{w} is a monoid with $H^\times \subset \mathsf{N}(H) \subset H^{\mathsf{w}} \subset H$. To be associated is a congruence relation on H^{w}, and $[a]_\simeq = aH^\times = H^\times a$ for all $a \in H^{\mathsf{w}}$.*
2. *The quotient semigroup $H^{\mathsf{w}}/\simeq\, = H_{\mathrm{red}}$ is a monoid with quotient group $\mathsf{q}(H^{\mathsf{w}})/H^\times$. Moreover, H is normalizing if and only if $H = H^{\mathsf{w}}$ and H_{red} is normalizing.*
3. *Let D be a monoid and $\varphi\colon H \to D$ a monoid homomorphism. Then there exists a unique homomorphism $\varphi_{\mathrm{red}}\colon H_{\mathrm{red}} \to D_{\mathrm{red}}$ satisfying $\varphi_{\mathrm{red}}(aH^\times) = \varphi(a)D^\times$ for all $a \in H^{\mathsf{w}}$.*
4. *The map $f\colon \mathcal{I}_s(H^{\mathsf{w}}) \to \mathcal{I}_s(H_{\mathrm{red}})$, $I \mapsto \bar{I} = \{uH^\times \mid u \in I\}$ is an inclusion preserving bijection. Moreover, I is a principal right ideal or a divisorial ideal if and only if $\bar{I}$ has the same property.*

Proof. 1. If $a, b \in H$ are weakly normal, then $abH^\times = aH^\times b = H^\times ab$, and hence ab is weakly normal. Next we show that every normal element is weakly normal. Let $a \in H$ be normal. If $\varepsilon \in H^\times$, then $a\varepsilon = ba \in aH = Ha$ with $b \in H$ and hence $a\varepsilon a^{-1} \in H$. Similarly, we get $a\varepsilon^{-1}a^{-1} \in H$, hence $a\varepsilon a^{-1} \in H^\times$, and $a\varepsilon = (a\varepsilon a^{-1})a \in H^\times a$. This shows that $aH^\times \subset H^\times a$, and by symmetry we get $aH^\times = H^\times a$.

By Lemma 4.3, we infer that H^{w} is a monoid with $H^\times \subset \mathsf{N}(H) \subset H^{\mathsf{w}} \subset H$. Clearly, $\simeq$ is a congruence relation on H^{w} and $[a]_\simeq = aH^\times = H^\times a$ for all $a \in H^{\mathsf{w}}$.

2. The group $\mathsf{q}(H^{\mathsf{w}})/H^\times$ is a quotient group of H_{red}, and hence H_{red} is a monoid.

Suppose that H is normalizing. Then $\mathsf{N}(H) \subset H^{\mathsf{w}} \subset H = \mathsf{N}(H)$, and we verify that H_{red} is normalizing. Since

$$\{ac \mid c \in H\} = aH = Ha = \{ca \mid c \in H\},$$

it follows that

$$\begin{aligned}(aH^\times)H_{\mathrm{red}} &= \{aH^\times cH^\times \mid c \in H\} = \{acH^\times \mid c \in H\} = \{caH^\times \mid c \in H\} \\ &= \{cH^\times aH^\times \mid c \in H\} = H_{\mathrm{red}}(aH^\times),\end{aligned}$$

and thus H_{red} is normalizing.

Conversely, suppose that $H = H^{\mathrm{w}}$ and that H_{red} is normalizing. Let $a \in H$. By symmetry it suffices to verify that $aH \subset Ha$. Let $c \in H$. Since

$$acH^{\times} \in \{(aH^{\times})(dH^{\times}) = adH^{\times} \mid d \in H\} = \{(dH^{\times})(aH^{\times}) = daH^{\times} \mid d \in H\},$$

there exist $d \in H$ and $\varepsilon \in H^{\times}$ such that $ac = da\varepsilon$. Since $aH^{\times} = H^{\times}a$, there is an $\eta \in H^{\times}$ such that $a\varepsilon = \eta a$, and hence $ac = (d\eta)a \in Ha$.

3. If $b, c \in H^{\mathrm{w}}$ with $bH^{\times} = cH^{\times}$, then $\varphi(b)D^{\times} = \varphi(c)D^{\times}$. Hence we can define a map $\varphi_{\mathrm{red}}\colon H_{\mathrm{red}} \to D_{\mathrm{red}}$ satisfying $\varphi_{\mathrm{red}}(aH^{\times}) = \varphi(a)D^{\times}$. Obviously, φ_{red} is uniquely determined and a homomorphism.

4. We define a map $g\colon \mathcal{I}_s(H_{\mathrm{red}}) \to \mathcal{I}_s(H^{\mathrm{w}})$ by setting $g(J) = \{v \in H^{\mathrm{w}} \mid vH^{\times} \in J\}$ for all $J \in \mathcal{I}_s(H_{\mathrm{red}})$. Obviously, f and g are inclusion preserving, inverse to each other, and hence f is bijective.

If $I = aH^{\mathrm{w}}$, then $f(I) = \{abH^{\times} = (aH^{\times})(bH^{\times}) \mid b \in H^{\mathrm{w}}\} = (aH^{\times})H_{\mathrm{red}}$, and if $J = (aH^{\times})H_{\mathrm{red}}$, then $g(J) = aH^{\mathrm{w}}$.

If $A \subset \mathsf{q}(H^{\mathrm{w}})$, then

$$\begin{aligned}(H^{\mathrm{w}} :_l A)H^{\times} &= \{uH^{\times} \mid u \in \mathsf{q}(H^{\mathrm{w}}),\ uA \subset H^{\mathrm{w}}\}\\ &= \{uH^{\times} \mid u \in \mathsf{q}(H^{\mathrm{w}}),\ u\{aH^{\times} \mid a \in A\} \subset H_{\mathrm{red}}\}\\ &= (H_{\mathrm{red}} :_l \{aH^{\times} \mid a \in A\}).\end{aligned}$$

The analogous statement is true for right quotients, and thus the assertion for divisorial ideals follows. □

Lemma 4.5. *Let H be a monoid. Then the following statements are equivalent*:

(a) *H is normalizing.*
(b) *For all $X \subset \mathsf{q}(H)$, $(H :_l X) = (H :_r X)$.*
(c) *For all $X \subset \mathsf{q}(H)$, $HX = XH$.*
(d) *Every (fractional) left ideal is a (fractional) ideal.*
(e) *Every divisorial (fractional) left ideal is a divisorial (fractional) ideal.*
(f) *For every $a \in \mathsf{q}(H)$, Ha is a fractional ideal.*

REMARK. Of course, the statements on right ideals, symmetric to (d), (e) and (f), are also equivalent.

Proof of Lemma 4.5. (a) $\Rightarrow$ (b) If $X \subset \mathsf{q}(H)$, then

$$(H :_l X) = \bigcap_{a \in X}(H :_l a) = \bigcap_{a \in X}(H :_l aH) = \bigcap_{a \in X} Ha^{-1} = \bigcap_{a \in X} a^{-1}H = (H :_r X).$$

(b) $\Rightarrow$ (c) If $X \subset \mathsf{q}(H)$, then

$$HX = \bigcup_{a \in X} Ha = \bigcup_{a \in X} (H :_l a^{-1}H) = \bigcup_{a \in X} (H :_l a^{-1})$$
$$= \bigcup_{a \in X} (H :_r a^{-1}) = \bigcup_{a \in X} aH = XH.$$

(c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) Obvious.

(f) $\Rightarrow$ (a) Let $a \in H$. Then $Ha = HaH \supset aH$, $Ha^{-1} = Ha^{-1}H \supset a^{-1}H$ and hence $aH \supset Ha$, which implies that $aH = Ha$. $\square$

Lemma 4.6. *Let H be a weakly normal monoid, $\pi\colon H \to H_{\mathrm{red}}$ the canonical epimorphism, and let $\varphi\colon H \to D$ be a homomorphism to a monoid D.*

1. *If φ is a divisor homomorphism and $\psi\colon D \to D'$ is a divisor homomorphism to a monoid D', then $\psi \circ \varphi\colon H \to D'$ is a divisor homomorphism.*
2. *π is a cofinal divisor homomorphism, and φ is a divisor homomorphism if and only if $\varphi_{\mathrm{red}}\colon H_{\mathrm{red}} \to D_{\mathrm{red}}$ is a divisor homomorphism. If φ is a divisor homomorphism, then φ_{red} is injective, $H_{\mathrm{red}} \cong \varphi_{\mathrm{red}}(H_{\mathrm{red}})$ and $\varphi_{\mathrm{red}}(H_{\mathrm{red}}) \subset D_{\mathrm{red}}$ is a saturated submonoid.*
3. *If $D = \mathcal{F}(P)$, then φ is a divisor theory if and only if $\varphi_{\mathrm{red}}\colon H_{\mathrm{red}} \to D$ is a divisor theory.*

Proof. 1. Suppose that φ and ψ are divisor homomorphisms, and let $a, b \in H$ such that $\psi(\varphi(a)) \mid_l \psi(\varphi(b))$. Since ψ is a divisor homomorphism, we infer that $\varphi(a) \mid_l \varphi(b)$, and since φ is a divisor homomorphism, we obtain that $a \mid_l b$. The analogous argument works for right divisibility.

2. The first statements are clear. Now suppose that φ is a divisor homomorphism, and let $a,b \in H$ with $\varphi(a) = \varphi(b)$. Then $\varphi(a) \mid \varphi(b)$, $\varphi(b) \mid \varphi(a)$, hence $a \mid b$, $b \mid a$, and thus $aH^\times = bH^\times$. Thus φ_{red} is injective, $H_{\mathrm{red}} \cong \varphi_{\mathrm{red}}(H_{\mathrm{red}})$, and since φ_{red} is a divisor homomorphism, $\varphi_{\mathrm{red}}(H_{\mathrm{red}}) \subset D_{\mathrm{red}}$ is saturated.

3. By 2., it remains to verify that φ satisfies the condition involving the greatest common divisor if and only if φ_{red} does. Indeed, if $a_1, \ldots, a_n \in H$, then $\varphi_{\mathrm{red}}(a_i H^\times) = \varphi(a_i)$ for all $i \in [1, n]$ and hence

$$\gcd(\varphi(a_1), \ldots, \varphi(a_n)) = \gcd(\varphi_{\mathrm{red}}(a_1 H^\times), \ldots, \varphi_{\mathrm{red}}(a_n H^\times)),$$

which implies the assertion. $\square$

Lemma 4.7. *Let H be a monoid.*

1. *If $a, b \in \mathsf{N}(H)$, then aH, bH are divisorial ideals of H, and $(aH) \cdot_v (bH) = (aH)(bH) = abH$. Thus the usual ideal multiplication coincides with the v-multiplication.*
2. *Equipped with usual ideal multiplication, $\mathcal{P}^{\mathsf{n}}(H)$ is a normalizing monoid. It is a saturated submonoid of $\mathcal{I}_v^*(H)$, and the inclusion is cofinal if and only if $\mathfrak{a} \cap \mathsf{N}(H) \neq \emptyset$ for all $\mathfrak{a} \in \mathcal{I}_v^*(H)$.*

3. *The map* $f\colon \mathsf{N}(H)_{\mathrm{red}} \to \mathcal{P}^{\mathsf{n}}(H)$, *defined by* $aH^\times = a\mathsf{N}(H)^\times \mapsto aH$ *for all* $a \in \mathsf{N}(H)$, *is an isomorphism.*
4. *If* H *is normalizing, then the map* $\partial\colon H \to \mathcal{I}_v^*(H)$, *defined by* $\partial(a) = aH$ *for all* $a \in H$, *is a cofinal divisor homomorphism.*

Proof. 1. If $c \in \mathsf{N}(H)$, then cH is an ideal of H by definition, and it is divisorial by Lemma 3.4 4. If $a, b \in \mathsf{N}(H)$, then

$$(aH) \cdot_v (bH) = ((aH)(bH))_v = (abH)_v = abH.$$

2. and 3. Let $a, b \in H$. Since $aH = bH$ if and only if $aH^\times = bH^\times$, f is injective, and obviously f is a semigroup epimorphism. Since $\mathsf{N}(H)$ is normalizing by Lemma 4.3, its associated reduced monoid $\mathsf{N}(H)_{\mathrm{red}}$ is normalizing, and thus $\mathcal{P}^{\mathsf{n}}(H)$ is a normalizing monoid. By 1., it is a submonoid of $\mathcal{I}_v^*(H)$.

In order to show that $\mathcal{P}^{\mathsf{n}}(H) \subset \mathcal{I}_v^*(H)$ is saturated, let $a, b \in \mathsf{N}(H)$ such that $aH \mid_l bH$ in $\mathcal{I}_v^*(H)$. Then there exists some $\mathfrak{a} \in \mathcal{I}_v^*(H)$ such that $bH = aH \cdot_v \mathfrak{a}$, and hence $a^{-1}b \in a^{-1}bH = (a^{-1}H)bH = (a^{-1}H) \cdot_v (aH) \cdot_v \mathfrak{a} = \mathfrak{a} \subset H$. The argument for divisibility on the right side is similar.

If $\mathfrak{a} \in \mathcal{I}_v^*(H)$ and $a \in \mathfrak{a} \cap \mathsf{N}(H)$, then $\mathfrak{a} \cdot_v \mathfrak{a}^{-1} = \mathfrak{a}^{-1} \cdot_v \mathfrak{a} = H$, $aH \subset \mathfrak{a}$, and hence $\mathfrak{a} \cdot_v (\mathfrak{a}^{-1} \cdot_v aH) = aH = (aH \cdot_v \mathfrak{a}^{-1}) \cdot_v \mathfrak{a}$. This shows that, if $\mathfrak{a} \cap \mathsf{N}(H) \neq \emptyset$ for all $\mathfrak{a} \in \mathcal{I}_v^*(H)$, then $\mathcal{P}^{\mathsf{n}}(H) \subset \mathcal{I}_v^*(H)$ is cofinal. An analogous argument shows the converse.

4. If H is normalizing, then $H = \mathsf{N}(H)$ is weakly normal. Using 2., 3., and Lemma 4.6 we infer that

$$\partial\colon H \xrightarrow{\pi} H_{\mathrm{red}} \cong \mathcal{P}^{\mathsf{n}}(H) = \mathcal{P}(H) \hookrightarrow \mathcal{I}_v^*(H)$$

is a cofinal divisor homomorphism, because it is a composition of such homomorphisms. $\square$

The following characterization of a divisor homomorphism will be used without further mention.

Lemma 4.8. *Let* $\varphi\colon H \to D$ *be a monoid homomorphism, and set* $\phi = \mathsf{q}(\varphi)\colon \mathsf{q}(H) \to \mathsf{q}(D)$. *Then the following statements are equivalent*:
(a) φ *is a divisor homomorphism.*
(b) $\phi^{-1}(D) = H$.
In particular, if $\varphi = (H \hookrightarrow D)$, *then* $H \subset D$ *is saturated if and only if* $H = \mathsf{q}(H) \cap D$.

Proof. (a) $\Rightarrow$ (b) Clearly, we have $H \subset \phi^{-1}(D)$. If $x = a^{-1}b \in \phi^{-1}(D)$ with $a, b \in H$, then $\phi(x) = \varphi(a)^{-1}\varphi(b) \in D$ and therefore $\varphi(a) \mid_l \varphi(b)$. Hence $a \mid_l b$ and $x \in H$.

(b) $\Rightarrow$ (a) Let $a, b \in H$ such that $\varphi(a) \mid_l \varphi(b)$. Then $\phi(a^{-1}b) = \varphi(a)^{-1}\varphi(b) \in D$, hence $a^{-1}b \in H$ and $a \mid_l b$. Similarly, $\varphi(a) \mid_r \varphi(b)$ implies that $a \mid_r b$.

If $\varphi = (H \hookrightarrow D)$, then $\phi^{-1}(D) = \mathsf{q}(H) \cap D$, and the assertion follows. $\square$

Lemma 4.9. *Let D be a monoid and $H \subset D$ a saturated submonoid.*
1. *If $\mathfrak{a} \subset H$ is a left ideal of H, then $D\mathfrak{a} \subset D$ is a left ideal of D, and $D\mathfrak{a} \cap H = \mathfrak{a}$ (similarly, if $\mathfrak{a} \subset H$ is a right ideal of H, then $\mathfrak{a}D \cap H = \mathfrak{a}$).*
2. *Let $\mathfrak{a} \subset H$ be an ideal. If $\mathfrak{a}$ is a divisorial left ideal, then $\big(D :_l (H :_r \mathfrak{a})\big)$ is a divisorial left ideal of D with $\mathfrak{a} = (D :_l (H :_r \mathfrak{a})) \cap H$. If $\mathfrak{a}$ is a divisorial right ideal, then $(D :_r (H :_l \mathfrak{a}))$ is a divisorial right ideal of D with $\mathfrak{a} = (D :_r (H :_l \mathfrak{a})) \cap H$.*
3. *If D satisfies the ascending chain condition on divisorial left ideals, then H is v-noetherian.*

REMARK. All quotients are formed in their respective quotient groups. So $(H :_r \mathfrak{a}) = \{q \in \mathsf{q}(H) \mid \mathfrak{a}q \subset H\}$, $(D :_l (H :_r \mathfrak{a})) = \{q \in \mathsf{q}(D) \mid q(H :_r \mathfrak{a}) \subset D\}$, and so on.

Proof of Lemma 4.9. 1. Clearly, $D\mathfrak{a} \subset D$ is a left ideal of D, and we have $\mathfrak{a} \subset D\mathfrak{a} \cap H$. If $x = uz \in H$ where $u \in D$ and $z \in \mathfrak{a} \subset H$, then $u \in \mathsf{q}(H) \cap D = H$ and hence $x \in H\mathfrak{a} = \mathfrak{a}$.

2. Let $\mathfrak{a} \subset H$ be a divisorial left ideal. Then $H \subset (H :_r \mathfrak{a})$ and $D = HD \subset (H :_r \mathfrak{a})D$ which implies that $(D :_l (H :_r \mathfrak{a})) = (D :_l (H :_r \mathfrak{a})D) \subset D$. By Lemma 3.4 6., $(D :_l (H :_r \mathfrak{a}))$ is a divisorial left ideal of D.

If $a \in \mathfrak{a}$, then $a(H :_r \mathfrak{a}) \subset H \subset D$ and hence $a \in (D :_l (H :_r \mathfrak{a}))$. If $a \in (D :_l (H :_r \mathfrak{a})) \cap H$, then $a(H :_r \mathfrak{a}) \subset D \cap \mathsf{q}(H) = H$ and hence $a \in (H :_l (H :_r \mathfrak{a})) = \mathfrak{a}$. Thus we have $\mathfrak{a} = (D :_l (H :_r \mathfrak{a})) \cap H$.

3. Let $(\mathfrak{a}_n)_{n \ge 0}$ be an ascending chain of divisorial ideals of H, and set $\mathfrak{A}_n = (D :_l (H :_r \mathfrak{a}_n))$ for all $n \ge 0$. Then $(\mathfrak{A}_n)_{n \ge 0}$ is an ascending chain of divisorial left ideals of D. If it becomes stationary, then the initial chain $(\mathfrak{a}_n)_{n \ge 0}$ becomes stationary because $\mathfrak{a}_n = \mathfrak{A}_n \cap H$ for all $n \ge 0$. $\square$

Lemma 4.10. *Let $\varphi \colon H \to D$ be a monoid homomorphism with $\varphi(H) \subset \mathsf{N}(D)$, and set $\phi = \mathsf{q}(\varphi) \colon \mathsf{q}(H) \to \mathsf{q}(D)$.*
1. *If H' is an overmonoid of H with $aH'b \subset H$ for some $a, b \in H$, then $D' = D\phi(H')$ is an overmonoid of D with $\varphi(a)D'\varphi(b) \subset D$.*
2. *Suppose that φ is a divisor homomorphism.*
   (a) *If D is completely integrally closed, then H is completely integrally closed.*
   (b) *H is normalizing.*

Proof. 1. Since $\varphi(H) \subset \mathsf{N}(D)$, we have $D\phi(H') = \phi(H')D$, and hence D' is an overmonoid of D. Furthermore, we get

$$\varphi(a)D'\varphi(b) = \varphi(a)D\phi(H')\varphi(b) = D\varphi(a)\phi(H')\varphi(b) = D\phi(aH'b) \subset D.$$

2. (a) If D is completely integrally closed and H' is an overmonoid of H as in 1., then $H' \subset \phi^{-1}(D') = \phi^{-1}(D) = H$. Thus H is completely integrally closed by Lemma 3.10.

2. (b) Let $a \in H$. We show that $aH \subset Ha$, and then by symmetry we get $aH = Ha$. If $b \in aH$, then $\varphi(b) \in \varphi(a)D = D\varphi(a)$, which implies that $\varphi(a) \mid_r \varphi(b)$, $a \mid_r b$ and hence $b \in Ha$. □

Lemma 4.11. *Let $\varphi\colon H \to D$ be a divisor homomorphism into a normalizing monoid D, and set $\phi = \mathsf{q}(\varphi)\colon \mathsf{q}(H) \to \mathsf{q}(D)$.*

1. *For every $X \subset H$ we have $X^{-1} = \phi^{-1}(\phi(X)^{-1})$.*
2. *For every $\mathfrak{a} \in \mathcal{F}_v(H)$ we have $\mathfrak{a} = \phi^{-1}(\phi(\mathfrak{a})_v)$.*
3. *If $D = \mathcal{F}(P)$, $\emptyset \neq \mathfrak{a} \in \mathcal{I}_v(H)$ and $a = \gcd(\varphi(\mathfrak{a}))$, then $\mathfrak{a} = \varphi^{-1}(aD)$.*
4. *Let φ be a divisor theory.*
   (a) *For every $a \in \mathsf{q}(D)$ there is a finite non-empty set $X \subset \mathsf{q}(H)$ such that $aD = \phi(X)_v$.*
   (b) *For every $\emptyset \neq X \subset H$, we have $\gcd(\varphi(X)) = \gcd(\varphi(X_v))$.*

Proof. We observe that H is normalizing by Lemma 4.10, and hence $(H :_l X) = (H :_r X)$ for all $X \subset \mathsf{q}(H)$ by Lemma 4.5.(b). We will need the following fact for a commutative monoid M satisfying $\mathrm{GCD}(E) \neq \emptyset$ for all $E \subset M$ (see [36, Theorem 11.5]): for any subset $X \subset M$ we have

$$(*) \qquad X_v = dM \quad \text{if and only if} \quad \mathrm{GCD}(X) = dM^{\times}.$$

1. If $x \in X^{-1}$, then $xX \subset H$, hence $\phi(x)\phi(X) = \phi(xX) \subset D$, and $\phi(x) \in \phi(X)^{-1}$, which implies $x \in \phi^{-1}(\phi(X)^{-1})$.

Conversely, if $x \in \phi^{-1}(\phi(X)^{-1})$, then $\phi(xX) = \phi(x)\phi(X) \subset D$. Hence it follows that $xX \subset \phi^{-1}(D) = H$ and $x \in X^{-1}$.

2. Let $\mathfrak{a} \in \mathcal{F}_v(H)$. Clearly, we have $\mathfrak{a} \subset \phi^{-1}(\phi(\mathfrak{a})_v)$. Conversely, let $x \in \phi^{-1}(\phi(\mathfrak{a})_v)$. Then $\phi(x) \in \phi(\mathfrak{a})_v = (\phi(\mathfrak{a})^{-1})^{-1}$, and hence by 1., we get

$$\phi(x\mathfrak{a}^{-1}) = \phi(x\phi^{-1}(\phi(\mathfrak{a})^{-1})) \subset \phi(x)\phi(\mathfrak{a})^{-1} \subset D.$$

Since $H = \phi^{-1}(D)$ by Lemma 4.8, it follows that $x\mathfrak{a}^{-1} \subset H$ and thus $x \in (\mathfrak{a}^{-1})^{-1} = \mathfrak{a}$.

3. If $a = \gcd(\varphi(\mathfrak{a}))$, then $aD = \varphi(\mathfrak{a})_v$ by $(*)$, and 2. implies that $\mathfrak{a} = \varphi^{-1}(\varphi(\mathfrak{a})_v) = \varphi^{-1}(aD)$.

4. Suppose that $D = \mathcal{F}(P)$.

4. (a) First we consider an element $a \in D$. Then $a = p_1 \cdots p_l$ with $l \in \mathbb{N}_0$ and $p_1, \ldots, p_l \in P$. For every $\nu \in [1, l]$ there exists a finite non-empty set $X_\nu \subset H$ such that $p_\nu = \gcd(\varphi(X_\nu))$. Then the product set $X_1 \cdots X_l \subset H$ is finite and $a = \gcd(\varphi(X_1 \cdots X_l))$ (where we use the convention that $X_1 \cdots X_l = \{1\}$ if $l = 0$). Now $(*)$ implies that $aD = \varphi(X_1 \cdots X_l)_v$.

Let $a \in \mathsf{q}(D)$ be given. Then there is some $u \in H$ such that $\varphi(u)a \in D$. If $X \subset H$ is a finite non-empty set with $\varphi(u)aD = \varphi(X)_v$, then $aD = \phi(u^{-1}X)_v$.

4. (b) We start with the following assertion.

A. *For every $X \subset \mathsf{q}(H)$ we have $\phi(X)_v = \phi(X_v)_v$.*

Suppose that A holds, let $X \subset H$ and $a = \gcd(\varphi(X))$. Applying A and $(*)$ we infer that $aD = \varphi(X)_v = \varphi(X_v)_v$ and hence $a = \gcd(\varphi(X_v))$ by 3.

Proof of A. Let $X \subset \mathsf{q}(H)$. Clearly, we have $\phi(X)_v \subset \phi(X_v)_v$. To show the converse, we assert that $(D : \phi(X)) \subset (D : \phi(X_v))$. This implies that

$$\phi(X_v)_v = (D : \phi(X_v))^{-1} \subset (D : \phi(X))^{-1} = \phi(X)_v.$$

Let $a \in (D : \phi(X)) \subset \mathsf{q}(D)$. By 4. (a), there is a finite non-empty set $Y \subset \mathsf{q}(H)$ with $aD = \phi(Y)_v$. Then $\phi(XY) \subset \phi(X)aD \subset D$ and hence $XY \subset H$. This implies that $X_vY \subset (XY)_v \subset H$, hence $\phi(X_v)\phi(Y) = \phi(X_vY) \subset D$ and therefore $\phi(X_v)\phi(Y)_v \subset (\phi(X_v)\phi(Y))_v \subset D$. Thus it follows that $\phi(X_v)a \subset \phi(X_v)\phi(Y)_v \subset D$ and $a \in (D : \phi(X_v))$. $\square$

Corollary 4.12. *Let $\varphi\colon H \to D$ be a divisor homomorphism into a normalizing monoid D.*

1. *If D is v-noetherian, then H is v-noetherian.*
2. *If D is a Krull monoid, then H is a normalizing Krull monoid.*

Proof. 1. If $(\mathfrak{a}_n)_{n\ge0}$ is an ascending chain of divisorial ideals of H, then $(\varphi(\mathfrak{a}_n)_v)_{n\ge0}$ is an ascending chain of divisorial ideals of D. If this chain becomes stationary, then so does the initial chain in H, because $\mathfrak{a}_n = \phi^{-1}(\phi(\mathfrak{a}_n)_v)$ for all $n \ge 0$ by Lemma 4.11 2.

2. If D is a normalizing Krull monoid, then H is completely integrally closed by Lemma 4.10 2, and hence the assertion follows from 1. $\square$

Theorem 4.13 (A divisor theoretic characterization of normalizing Krull monoids)**.** *Let H be a monoid. Then the following statements are equivalent*:

(a) *The map $\partial\colon H \to \mathcal{I}_v^*(H)$, defined by $\partial(a) = aH$ for all $a \in H$, is a divisor theory.*
(b) *H has a divisor theory.*
(c) *There exists a divisor homomorphism $\varphi\colon H \to \mathcal{F}(P)$ into a free abelian monoid.*
(d) *H is a normalizing Krull monoid.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) Obvious.

(c) $\Rightarrow$ (d) Since $\mathcal{F}(P)$ is a normalizing Krull monoid, this follows from Corollary 4.12 2.

(d) $\Rightarrow$ (a) By Lemma 4.7 4, $\partial\colon H \to \mathcal{I}_v^*(H)$ is a cofinal divisor homomorphism. Theorem 3.14 shows that $\mathcal{I}_v^*(H)$ is a free abelian monoid with basis v-spec$(H) \setminus \{\emptyset\}$. Let $\mathfrak{p}$ be a non-empty divisorial prime ideal. By Proposition 3.13 2, there exists a finite set $E = \{a_1, \ldots, a_n\} \subset \mathfrak{p}$ such that $(HEH)_v = \mathfrak{p}$. Since H is normalizing, we get $HEH = a_1H \cup \cdots \cup a_nH$, where $a_1H, \ldots, a_nH$ are divisorial ideals by Lemmas 3.4 and 4.5. Now Proposition 3.12 4. implies that

$$\mathfrak{p} = (a_1H \cup \cdots \cup a_nH)_v = \gcd(\partial(a_1), \ldots, \partial(a_n)). \qquad \square$$

Corollary 4.14. *Let H be a monoid.*

1. *If H is a Krull monoid, then $\mathsf{N}(H) \subset H$ is a normalizing Krull monoid, and there is a monomorphism $f\colon \mathcal{I}_v^*(\mathsf{N}(H)) \to \mathcal{I}_v^*(H)$ which maps $\mathcal{P}(\mathsf{N}(H))$ onto $\mathcal{P}^{\mathsf{n}}(H)$.*
2. *$\mathsf{N}(H)$ is a normalizing Krull monoid if and only if $\mathsf{N}(H)_{\mathrm{red}}$ is a normalizing Krull monoid. If this holds, then both, $\mathsf{N}(H)_{\mathrm{red}} \cong \mathcal{P}^{\mathsf{n}}(H)$ and $\mathsf{C}(H)$, are commutative Krull monoids.*

Proof. We set $S = \mathsf{N}(H)$.

1. Suppose that H is a Krull monoid. By Lemma 4.3 2., $S \subset H$ is a normalizing saturated submonoid. Thus the inclusion map $S \hookrightarrow H$ satisfies the assumption of Lemma 4.10 2., and hence S is completely integrally closed.

Let $f\colon \mathcal{I}_v^*(S) \to \mathcal{I}_v^*(H)$ be defined by $f(\mathfrak{a}) = (H :_l (S :_r \mathfrak{a}))$ for all $\mathfrak{a} \in \mathcal{I}_v^*(S)$ (with the same notational conventions as in Lemma 4.9; in particular, $A = (S :_r \mathfrak{a}) \subset \mathsf{q}(S)$).

We check that $f(\mathfrak{a}) \in \mathcal{I}_v^*(H)$. If $x \in \mathsf{q}(H)$ with $xA \subset H$, then $xHA = xAH \subset H$, and thus $(H :_l A)$ is a right module of H. By Lemma 4.9 2., $(H :_l A)$ is a divisorial left ideal of H. Since H is a Krull monoid, it follows that $f(\mathfrak{a})$ is a divisorial ideal of H, and hence $f(\mathfrak{a}) \in \mathcal{I}_v^*(H)$.

Since $f(\mathfrak{a}) \cap S = \mathfrak{a}$ by Lemma 4.9 2., f is injective and S is v-noetherian because H is v-noetherian. If $a \in S$, then, by Lemma 3.4 4., we infer that

$$f(Sa) = (H :_l (S :_r Sa)) = (H :_l a^{-1}S) = (H :_l a^{-1}SH) = Ha.$$

This shows that f maps $\mathcal{P}(S)$ onto $\mathcal{P}^{\mathsf{n}}(H)$. Since $f_1\colon \mathcal{I}_v^*(S) \to \mathcal{I}_v^*(S)$, defined by $\mathfrak{a} \mapsto (S : \mathfrak{a})$, and $f_2\colon \mathcal{I}_v^*(H) \to \mathcal{I}_v^*(H)$, defined by $\mathfrak{a} \mapsto (H : \mathfrak{a})$, are homomorphisms, $f = f_2 \circ f_1$ (use Lemma 3.6) is a homomorphism.

2. We freely use Theorem 4.13. If S_{red} is a normalizing Krull monoid, then there exists a divisor homomorphism $\varphi\colon S_{\mathrm{red}} \to \mathcal{F}(P)$. If $\pi : S \to S_{\mathrm{red}}$ denotes the canonical epimorphism, then $\varphi \circ \pi : S \to \mathcal{F}(P)$ is a divisor homomorphism by Lemma 4.6 and thus S is a normalizing Krull monoid. Suppose that S is a normalizing Krull monoid. Again, by Theorem 4.13 (b) and by Lemma 4.6 3., it follows that S_{red} is a normalizing Krull monoid. Lemma 4.7 shows that S_{red} and $\mathcal{P}^{\mathsf{n}}(H)$ are isomorphic, and that $\mathcal{P}^{\mathsf{n}}(H)$ is a submonoid of the commutative monoid $\mathcal{I}_v^*(H)$. Lemma 4.3 3. implies that $\mathsf{C}(H) \subset S$ is saturated, and thus $\mathsf{C}(H)$ is a Krull monoid by Corollary 4.12 2. $\square$

Our next step is to introduce a concept of class groups, and then to show a uniqueness result for divisor theories. Let $\varphi\colon H \to D$ be a homomorphism of monoids. The group

$$\mathcal{C}(\varphi) = \mathsf{q}(D)/\mathsf{q}(\varphi(H))$$

is called the *class group* of φ. This coincides with the notion in the commutative setting (see [29, Section 2.4]), and we will point out that in case of a Krull monoid H and a divisor theory $\varphi\colon \mathsf{N}(H) \to D$ the class group $\mathcal{C}(\varphi)$ is isomorphic to the normalizing class group of H (see Equations (4.1) and (4.2) at the end of this section).

For $a \in \mathsf{q}(D)$, we denote by

$$[a]_\varphi = [a] = a\mathsf{q}(\varphi(H)) \in \mathcal{C}(\varphi)$$

the class containing a. As usual, the class group $\mathcal{C}(\varphi)$ will be written additively, that is,

$$[ab] = [a] + [b] \quad \text{for all} \quad a, b \in \mathsf{q}(D),$$

and then $[1] = 0$ is the zero element of $\mathcal{C}(\varphi)$. If $\varphi\colon H \to D$ is a divisor homomorphism, then a straightforward calculation shows that for an element $\alpha \in D$, we have $[\alpha] = 0$ if and only if $\alpha \in \varphi(H)$. If $D = \mathcal{F}(P)$ is free abelian, then $G_P = \{[p] \mid p \in P\} \subset \mathcal{C}(\varphi)$ is the set of classes containing prime divisors.

Consider the special case $H \subset D$, $\varphi = (H \hookrightarrow D)$, and suppose that $\mathsf{q}(H) \subset \mathsf{q}(D)$. Then $\mathcal{C}(\varphi) = \mathsf{q}(D)/\mathsf{q}(H)$, and we define

$$D/H = \{[a] = a\mathsf{q}(H) \mid a \in D\} \subset \mathcal{C}(\varphi).$$

Then $D/H \subset \mathcal{C}(\varphi)$ is a submonoid with quotient group $\mathcal{C}(\varphi)$, and $D/H = \mathcal{C}(\varphi)$ if and only if $H \subset D$ is cofinal.

Suppose that H is a normalizing Krull monoid, and let $\partial\colon H \to \mathcal{I}_v^*(H)$ be as in Theorem 4.13. Then $\mathcal{P}^{\mathsf{n}}(H) = \mathcal{P}(H) \subset \mathcal{I}_v^*(H)$ is cofinal, and

$$\mathcal{C}(\partial) = \mathcal{I}_v^*(H)/\mathcal{P}(H) = \mathcal{F}_v^\times(H)/\mathsf{q}(\mathcal{P}(H))$$

is called the *v-class group* of H, and will be denoted by $\mathcal{C}_v(H)$.

We continue with a uniqueness result for divisor theories. Its consequences for class groups will be discussed afterwards. We proceed as in the commutative case ([29, Section 2.4]). Recently, W.A. Schmid gave a more explicit approach valid in case of torsion class groups ([60, Section 3]).

Proposition 4.15 (Uniqueness of divisor theories). *Let H be a monoid.*

1. *Let* $\varphi\colon H \to F = \mathcal{F}(P)$ *be a divisor theory. Then the maps* $\varphi^*\colon F \to \mathcal{I}_v^*(H)$ *and* $\bar\varphi\colon \mathcal{C}(\varphi) \to \mathcal{C}_v(H)$, *defined by*

$$\varphi^*(a) = \varphi^{-1}(aF)_v \quad \text{and} \quad \overline{\varphi}([a]_\varphi) = [\varphi^{-1}(aF)_v] \quad \text{for all} \quad a \in F,$$

are isomorphisms.

2. *If* $\varphi_1\colon H \to F_1$ *and* $\varphi_2\colon H \to F_2$ *are divisor theories, then there is a unique isomorphism* $\Phi\colon F_1 \to F_2$ *such that* $\Phi \circ \varphi_1 = \varphi_2$. *It induces an isomorphism* $\overline{\Phi}\colon \mathcal{C}(\varphi_1) \to \mathcal{C}(\varphi_2)$, *given by* $\overline{\Phi}([a]_{\varphi_1}) = [\Phi(a)]_{\varphi_2}$ *for all* $a \in F_1$.

Proof. 1. Note that H is a normalizing Krull monoid by Theorem 4.13. We start with the following assertion.

A. $\{\gcd(\varphi(X)) \mid \emptyset \ne X \subset H\} = F$.

Proof of A. Since $\varphi\colon H \to \mathcal{F}(P)$ is a divisor theory, it follows that $P \subset \{\gcd(\varphi(X)) \mid \emptyset \ne X \subset H\}$. Since $\gcd(\varphi(X_1X_2)) = \gcd(\varphi(X_1))\gcd(\varphi(X_2))$ for all non-empty subsets $X_1, X_2 \subset H$, it follows that $\mathcal{F}(P) \subset \{\gcd(\varphi(X)) \mid \emptyset \ne X \subset H\} \subset \mathcal{F}(P)$.

Let $a \in F$. By A, we have $a = \gcd(\varphi(X))$ for some non-empty subset $X \subset H$, and hence $\emptyset \ne X \subset \varphi^{-1}(aF)$. This implies that $\varphi^{-1}(aF)_v \in \mathcal{I}_v(H) \setminus \{\emptyset\} = \mathcal{I}_v^*(H)$. By definition, we have $aF \cap \varphi(H) = \varphi(\varphi^{-1}(aF))$, and using Lemma 4.11 4. it follows that

$$a = \gcd(aF \cap \varphi(H)) = \gcd(\varphi(\varphi^{-1}(aF))) = \gcd(\varphi(\varphi^{-1}(aF)_v)) = \gcd(\varphi(\varphi^*(a))),$$

which shows that φ^* is injective.

In order to show that φ^* is surjective, let $\mathfrak{a} \in \mathcal{I}_v^*(H)$ be given, and set $a = \gcd(\varphi(\mathfrak{a}))$. Then $\varphi^*(a) = \varphi^{-1}(aF)_v = \mathfrak{a}$ by Lemma 4.11 3., and thus φ^* is surjective.

Next we show that φ^* is a homomorphism. Let $a, b \in F$. Then Lemma 3.6 5. implies that

$$\varphi^*(a) \cdot_v \varphi^*(b) = (\varphi^{-1}(aF)_v\varphi^{-1}(bF)_v)_v = (\varphi^{-1}(aF)\varphi^{-1}(bF))_v \subset \varphi^{-1}(abF)_v = \varphi^*(ab).$$

To prove the reverse inclusion, we set $c = \gcd(\varphi(\varphi^*(a) \cdot_v \varphi^*(b))) \in F$, and note that $\varphi^*(a) \cdot_v \varphi^*(b) \supset \varphi^{-1}(aF)\varphi^{-1}(bF)$. This implies that

$$c \mid \gcd(\varphi(\varphi^{-1}(aF)\varphi^{-1}(bF))) = \gcd(aF \cap \varphi(H))\gcd(bF \cap \varphi(H)) = ab,$$

hence $abF \subset cF$, and thus $\varphi^*(ab) \subset \varphi^{-1}(cF)_v = (\varphi^*(a) \cdot_v \varphi^*(b))_v = \varphi^*(a) \cdot_v \varphi^*(b)$, where the penultimate equation follows from Lemma 4.11 3.

It remains to verify that $\bar\varphi$ is an isomorphism. Note that for every $x \in H$, we have $\varphi^* \circ \varphi(x) = \varphi^{-1}(\varphi(x)F)_v = \mathsf{q}(\varphi)^{-1}(\mathsf{q}(\varphi)(x)F)_v = xH$ by Lemma 4.11 3. Obviously, φ^* induces an epimorphism $\varphi'\colon F \to \mathcal{C}_v(H)$, where $\varphi'(a) = [\varphi^*(a)] \in \mathcal{C}_v(H)$. If $a, b \in F$ with $[a]_\varphi = [b]_\varphi$, then there exist $x, y \in H$ such that $\varphi(x)a = \varphi(y)b$. Since

$[\varphi^*(a)] = [x\varphi^*(a)] = [\varphi^*(\varphi(x)a)] = [\varphi^*(\varphi(y)b)] = [y\varphi^*(b)] = [\varphi^*(b)]$, it follows that φ' induces an epimorphism $\bar{\varphi}\colon \mathcal{C}(\varphi) \to \mathcal{C}_v(H)$. To show that $\bar{\varphi}$ is injective, let $a, b \in F$ with $[\varphi^*(a)] = [\varphi^*(b)] \in \mathcal{C}_v(H)$. Then there are $x, y \in H$ such that $x\varphi^*(a) = y\varphi^*(b)$, hence $\varphi^*(\varphi(x)a) = \varphi^*(\varphi(y)b)$, thus $\varphi(x)a = \varphi(y)b$, and therefore we get $[a]_\varphi = [b]_\varphi$.

2. For $i \in \{1,2\}$, let $\varphi_i^*\colon F_i \to \mathcal{I}_v^*(H)$ and $\bar{\varphi}_i\colon \mathcal{C}(\varphi_i) \to \mathcal{C}_v(H)$ be the isomorphisms as defined in 1. Then $\Phi = \varphi_2^{*-1} \circ \varphi_1^*\colon F_1 \to F_2$ and $\overline{\Phi} = \bar{\varphi}_2^{-1} \circ \bar{\varphi}_1\colon \mathcal{C}(\varphi_1) \to \mathcal{C}(\varphi_2)$ are isomorphisms as asserted.

Let $\psi\colon F_1 \to F_2$ be an arbitrary isomorphism with the property that $\psi \circ \varphi_1 = \varphi_2$. Then for every $a \in F_1$ we have

$$\psi(a) = \psi(\gcd(\varphi_1(\varphi_1^{-1}(aF_1)))) = \gcd(\psi \circ \varphi_1(\varphi_1^{-1}(aF_1))) = \gcd(\varphi_2(\varphi_1^{-1}(aF_1))),$$

which shows that ψ is uniquely determined. □

Let H be a Krull monoid and $\iota\colon \mathcal{P}^{\mathsf{n}}(H) \hookrightarrow \mathcal{I}_v^*(H)$ be the inclusion map which is a divisor homomorphism by Lemma 4.7 2. Then

$$\mathcal{C}^{\mathsf{n}}(H) = \mathcal{C}(\iota) \tag{4.1}$$

is called the *normalizing class group* of H (as studied by Jespers and Wauters, see [38, p. 332]). The monomorphism $f\colon \mathcal{I}_v^*(\mathsf{N}(H)) \to \mathcal{I}_v^*(H)$, discussed in Corollary 4.14, induces a monomorphism

$$\bar{f}\colon \mathcal{C}_v(\mathsf{N}(H)) = \mathcal{I}_v^*(\mathsf{N}(H))/\mathcal{P}(\mathsf{N}(H)) \to \mathcal{C}^{\mathsf{n}}(H).$$

In particular, if H is normalizing and $\varphi\colon H \to D$ is a divisor theory, then Proposition 4.15 shows that

$$\mathcal{C}(\varphi) \cong \mathcal{C}_v(H) = \mathcal{C}^{\mathsf{n}}(H), \tag{4.2}$$

and thus all concepts of class groups coincide.

5. Examples of Krull monoids

In this section we provide a rough overview on the different places where Krull monoids show up. We start with ring theory.

Let R be a commutative integral domain. Then R is a Krull domain if and only if its multiplicative monoid of non-zero elements is a Krull monoid. This was first proved independently by Wauters ([63, Corollary 3.6]) and Krause ([43]). A thorough treatment of this relationship and various generalizations can be found in [36, Chapters 22 and 23] and [29, Chapter 2]). If R is a Marot ring (this is a commutative ring having not too many zero-divisors), then R is a Krull ring if and only if the monoid of regular elements is a Krull monoid ([35]).

Next we consider the non-commutative setting. A large number of concepts of non-commutative Krull rings has been introduced (see [9, 49, 50, 57, 58, 51, 12, 54, 10, 64, 41, 42, 21], and in particular the survey article [38]). Our definition of a Krull ring (given below) follows Jespers and Okniński ([40, p. 56]). The following proposition summarizes the relationship between the ideal theory of rings and the ideal theory of the associated monoids of regular elements. This relationship was first observed by Wauters in [63]. More detailed references to the literature will be given after the proposition. For clarity reasons, we carefully fix our setting for rings, and then the proof of the proposition will be straightforward.

Let R be a prime Goldie ring, and let Q denote its classical quotient ring (we follow the terminology of [53] and [32]; in particular, by a Goldie ring, we mean a left and right Goldie ring, and then the quotient ring is a left and right quotient ring; an ideal is always a two-sided ideal). Then Q is simple artinian, and every regular element of Q is invertible. Since R is prime, every non-zero ideal $\mathfrak a \subset R$ is essential, and hence it is generated as a left R-module (and also as a right R-module) by its regular elements (see [53, Corollary 3.3.7]). By a fractional ideal $\mathfrak a$ of R we mean a left and right R-submodule of Q for which there exist $a, b \in Q^\times$ such that $a\mathfrak a \subset R$ and $\mathfrak a b \subset R$. Clearly, every non-zero fractional ideal is generated by regular elements. Let $\mathfrak a$ be a fractional ideal. If $(R :_l (R :_r \mathfrak a)) = (R :_r (R :_l \mathfrak a))$, then we set $\mathfrak a_v = (R :_l (R :_r \mathfrak a))$, and we say that $\mathfrak a$ is divisorial if $\mathfrak a = \mathfrak a_v$. We denote by $\mathcal F_v(R)$ the set of divisorial fractional ideals (fractional v-ideals), by $\mathcal I_v(R)$ the set of divisorial ideals of R, and by v-spec(R) the set of divisorial prime ideals of R. We say that R is completely integrally closed if $(\mathfrak a :_l \mathfrak a) = (\mathfrak a :_r \mathfrak a) = R$ for all non-zero ideals $\mathfrak a$ of R. Suppose that R is completely integrally closed. Then left and right quotients coincide, and for $\mathfrak a, \mathfrak b \in \mathcal F_v(R)$, we define v-multiplication as $\mathfrak a \cdot_v \mathfrak b = (\mathfrak a \mathfrak b)_v$. Equipped with v-multiplication, $\mathcal F_v(R)$ is a semigroup, and $\mathcal I_v(R)$ is a subsemigroup. A prime Goldie ring is said to be a *Krull ring* if it is completely integrally closed and satisfies the ascending chain condition on divisorial ideals.

For a subset $I \subset Q$, we denote by $I^\bullet = I \cap Q^\times$ the set of regular elements of I. Then the set of all regular elements $H = R^\bullet$ of R is a monoid, and $\mathsf q(H) = Q^\times$ is a quotient group of H. Let $\mathfrak a$, $\mathfrak b$, $\mathfrak c$ be fractional ideals of R. Since $\mathfrak c$ is generated (as a left R-module and also as a right R-module) by the regular elements, we have $\mathfrak c = {}_R\langle \mathfrak c^\bullet \rangle = \langle \mathfrak c^\bullet \rangle_R$, and thus also

$$(\mathfrak b :_l \mathfrak a)^\bullet = (\mathfrak b^\bullet :_l \mathfrak a^\bullet) \quad \text{and} \quad (\mathfrak b :_r \mathfrak a)^\bullet = (\mathfrak b^\bullet :_r \mathfrak a^\bullet).$$

Proposition 5.1. *Let R be a prime Goldie ring, and let H be the monoid of regular elements of R.*

1. *R is completely integrally closed if and only if H is completely integrally closed.*

2. *The maps*

$$\iota^\bullet\colon \begin{cases} \mathcal{F}_v(R) \to \mathcal{F}_v(H), \\ \quad \mathfrak{a} \ \ \mapsto \ \ \mathfrak{a}^\bullet, \end{cases} \quad \textit{and} \quad \iota^\circ\colon \begin{cases} \mathcal{F}_v(H) \to \mathcal{F}_v(R), \\ \quad \mathfrak{a} \ \ \mapsto \ \ \langle \mathfrak{a} \rangle_R, \end{cases}$$

are inclusion preserving isomorphisms which are inverse to each other. Furthermore,

(a) $\iota^\bullet \mid \mathcal{I}_v(R)\colon \mathcal{I}_v(R) \to \mathcal{I}_v(H)$ *and* $\iota^\bullet \mid v\text{-spec}(R)\colon\ v\text{-spec}(R) \to v\text{-spec}(H)$ *are bijections.*

(b) *R satisfies the ascending chain condition on divisorial ideals of R if and only if H satisfies the ascending chain condition on divisorial ideals of H.*

3. *R is a Krull ring if and only if H is a Krull monoid, and if this holds, then $\mathsf{N}(H)$ is a normalizing Krull monoid.*

Proof. 1. Suppose that H is completely integrally closed, and let $\mathfrak{a} \subset R$ be a non-zero ideal. Then $\mathfrak{a}^\bullet \subset H$ is an ideal, $(\mathfrak{a}^\bullet :_l \mathfrak{a}^\bullet) = H$ by Lemma 3.10 and hence

$$(\mathfrak{a} :_l \mathfrak{a}) = {}_R\langle (\mathfrak{a} :_l \mathfrak{a})^\bullet \rangle = {}_R\langle (\mathfrak{a}^\bullet :_l \mathfrak{a}^\bullet) \rangle = {}_R\langle H \rangle = R.$$

Similarly, we get $(\mathfrak{a} :_r \mathfrak{a}) = R$.

Conversely, suppose that R is completely integrally closed, and let $\mathfrak{a} \subset H$ be a non-empty ideal. If $A \subset R$ denotes the ideal generated by $\mathfrak{a}$, then

$$H \subset (\mathfrak{a} :_l \mathfrak{a}) \subset (A :_l A)^\bullet = R^\bullet = H.$$

Similarly, we get $(\mathfrak{a} :_r \mathfrak{a}) = H$.

2. Clearly, $\iota^\bullet$ and ι° are inclusion preserving and map fractional ideals to fractional ideals. If $\mathfrak{a} \in \mathcal{F}_v(R)$, then

$$\begin{aligned}(H :_l (H :_r \mathfrak{a}^\bullet)) &= (R^\bullet :_l (R :_r \mathfrak{a})^\bullet) = (R :_l (R :_r \mathfrak{a}))^\bullet = \mathfrak{a}^\bullet \\ &= (R :_r (R :_l \mathfrak{a}))^\bullet = (H :_r (H :_l \mathfrak{a}^\bullet)),\end{aligned}$$

and hence $\mathfrak{a}^\bullet$ is a divisorial fractional ideal of H. Similarly, we obtain that $\iota^\circ(\mathcal{F}_v(H)) \subset \mathcal{F}_v(R)$. If $\mathfrak{a} \in \mathcal{F}_v(R)$, then

$$\iota^\circ \circ \iota^\bullet(\mathfrak{a}) = \langle \mathfrak{a} \cap Q^\times \rangle_R = \mathfrak{a},$$

and, if $\mathfrak{a} \in \mathcal{F}_v(H)$, then

$$\iota^\bullet \circ \iota^\circ(\mathfrak{a}) = \langle \mathfrak{a} \rangle_R \cap Q^\times = \mathfrak{a}.$$

Thus $\iota^\bullet$ and ι° are inverse to each other, and it remains to show that $\iota^\bullet$ is a homomorphism.

Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{F}_v(R)$. In the next few calculations, we write—for clarity reasons—$\mathfrak{a} \cdot_R \mathfrak{b}$ for the ring theoretical product, $\mathfrak{a} \cdot_S \mathfrak{b}$ for the semigroup theoretical product, v_R

for the v-operation on R and v_H for the v-operation on H. If $C \subset \mathfrak{c}^\bullet \cap H$ is an ideal of H such that $\langle C \rangle_R = \mathfrak{c}$, then $(R :_r \mathfrak{c})^\bullet = (H :_r C)$, and hence

$$\mathfrak{c}_{v_R} \cap Q^\times = (R :_l (R :_r \langle C \rangle))^\bullet = (R^\bullet :_l (R :_r \langle C \rangle)^\bullet) = (H :_l (H :_r C)) = C_{v_H} .$$

Applying this relationship to $C = (\mathfrak{a} \cap Q^\times) \cdot_S (\mathfrak{b} \cap Q^\times)$ we obtain that

$$\begin{aligned} \iota^\bullet(\mathfrak{a} \cdot_{v_R} \mathfrak{b}) &= (\mathfrak{a} \cdot_R \mathfrak{b})_{v_R} \cap Q^\times = ((\langle \mathfrak{a} \cdot_S \mathfrak{b} \rangle_R)_{v_R} \cap Q^\times \\ &= ((\langle (\mathfrak{a} \cap Q^\times) \cdot_S (\mathfrak{b} \cap Q^\times) \rangle_R)_{v_R} \cap Q^\times \\ &= ((\mathfrak{a} \cap Q^\times) \cdot_S (\mathfrak{b} \cap Q^\times))_{v_H} = \iota^\bullet(\mathfrak{a}) \cdot_{v_H} \iota^\bullet(\mathfrak{b}). \end{aligned}$$

2. (a) It is clear that the restriction $\iota^\bullet \mid \mathcal{I}_v(R) \colon \mathcal{I}_v(R) \to \mathcal{I}_v(H)$ is bijective. We verify that $\iota^\bullet \mid v\text{-spec}(R) \colon v\text{-spec}(R) \to v\text{-spec}(H)$ is bijective. Indeed, if $\mathfrak{p} \in v\text{-spec}(R)$ and $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_s(H)$ such that $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}^\bullet$, then $\langle \mathfrak{a} \rangle_R \langle \mathfrak{b} \rangle_R = \langle \mathfrak{a}\mathfrak{b} \rangle_R \subset \mathfrak{p}$, whence $\langle \mathfrak{a} \rangle_R \subset \mathfrak{p}$ or $\langle \mathfrak{b} \rangle_R \subset \mathfrak{p}$ and thus $\mathfrak{a}^\bullet \subset \mathfrak{p}^\bullet$ or $\mathfrak{b}^\bullet \subset \mathfrak{p}^\bullet$. Therefore $\mathfrak{p}^\bullet$ is a prime ideal by Lemma 3.7 (a), and hence $\mathfrak{p}^\bullet \in s\text{-spec}(H) \cap \mathcal{I}_v(H) = v\text{-spec}(H)$. Conversely, suppose that $\mathfrak{p} \in \mathcal{I}_v(R)$ such that $\mathfrak{p}^\bullet \in v\text{-spec}(H)$. In order to show that $\mathfrak{p} \subset R$ is a prime ideal, let $\mathfrak{a}, \mathfrak{b} \subset R$ be ideals such that $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$. Then $\mathfrak{a}^\bullet \mathfrak{b}^\bullet \subset (\mathfrak{a}\mathfrak{b})^\bullet \subset \mathfrak{p}^\bullet$, and thus $\mathfrak{a}^\bullet \subset \mathfrak{p}^\bullet$ or $\mathfrak{b}^\bullet \subset \mathfrak{p}^\bullet$, which implies that $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$.

2. (b) Since the restriction of $\iota^\bullet$ to $\mathcal{I}_v(R)$ and the restriction of ι° to $\mathcal{I}_v(H)$ are both inclusion preserving and bijective, this follows immediately.

3. The equivalence follows immediately from 1. and 2. (b). Moreover, if H is a Krull monoid, then $\mathsf{N}(H)$ is a normalizing Krull monoid by Corollary 4.14. $\square$

Suppose that R is a prime P.I.-ring. Then R is a Krull ring if and only if R is a Chamarie–Krull ring ([63, Proposition 3.5]), and moreover the notions of Ω-Krull rings, central Ω-Krull rings, Krull rings in the sense of Marubayashi, in the sense of Chamarie and others coincide ([38, Theorem 2.4]). Classical orders in central simple algebras over Dedekind domains are Asano prime rings ([53, Theorem 5.3.16]), and if R is an Asano prime ring (in other words, an Asano order), then R is a Krull ring ([53, Proposition 5.2.6]). Moreover, if R is a maximal order in a central simple algebra over a Dedekind domain with finite class group, then the central class group and hence the normalizing class group of R are finite (for more general results see [59, Corollary 37.32], [38, Proposition 8.1], [55, Chapter E, Proposition 2.3]). Krull rings, in which every element is normalizing, are discussed in [11, 64]. Further results and examples of non-commutative Dedekind and Krull rings may be found in [1, 64].

If a monoid H is normalizing, then every non-unit $a \in H$ is contained in the divisorial ideal $aH \neq H$. But this does not hold in general. We provide the announced example of a Krull monoid H having an element $a \in H \setminus H^\times$ which is not contained in a divisorial ideal distinct from H (we thank Daniel Smertnig for his assistance).

EXAMPLE 5.2. Let R be a commutative principal ideal domain with quotient field K and $n \in \mathbb{N}$. Then $M_n(R)$ is a classical order in the central simple algebra $M_n(K)$ and hence an Asano prime ring. By Proposition 5.1, $H = M_n(R)^\bullet = M_n(R) \cap \mathrm{GL}_n(K)$ is a Krull monoid with quotient group $\mathrm{GL}_n(K)$. Since every ideal of $M_n(R)$ is divisorial ([53, Proposition 5.2.6]), we get

$$\mathcal{I}_v(R) = \{M_n(aR) \mid a \in R\}.$$

Again by Proposition 5.1, this implies that

$$\mathcal{I}_v(H) = \{M_n(aR)^\bullet \mid a \in R\},$$

where

$$M_n(aR)^\bullet = \{C = (c_{i,j})_{1 \le i,j \le n} \mid c_{i,j} \in aR \text{ for all } i, j \in [1, n] \text{ and } \det(C) \neq 0\}.$$

Thus, if $C \in M_n(R)$ with $\mathrm{GCD}(\{c_{i,j} \mid i, j \in [1, n]\}) = R^\times$ and $\det(C) \neq 0$, then $(HCH)_v = H$.

We end this section with some more examples of Krull monoids. Apart from their appearance as monoids of regular elements in Krull rings, they occur in various other circumstances. We offer a brief overview:

- Regular congruence monoids in Krull domains are Krull monoids ([29, Proposition 2.11.6]).
- Module Theory: Let R be a ring and $\mathcal{C}$ a class of right (or left) R-modules—closed under finite direct sums, direct summands and isomorphisms—such that $\mathcal{C}$ has a set $V(\mathcal{C})$ of representatives (that is, every module $M \in \mathcal{C}$ is isomorphic to a unique $[M] \in V(\mathcal{C})$). Then $V(\mathcal{C})$ becomes a commutative semigroup under the operation $[M]+[N] = [M \oplus N]$, which carries detailed information about the direct-sum behavior of modules in $\mathcal{C}$. If every R-module $M \in \mathcal{C}$ has a semilocal endomorphism ring, then $\mathcal{V}(C)$ is a Krull monoid (see [22], and [23] for a survey).
- Diophantine monoids: A Diophantine monoid is a monoid which consists of the set of solutions in nonnegative integers to a system of linear Diophantine equations (see [15, Proposition 4.3] and [29, Theorem 2.7.14]).
- Monoids of zero-sum sequences over abelian groups.

Since monoids of zero-sum sequences will be needed in the next section, we discuss them in greater detail. Let G be an additively written abelian group and $G_0 \subset G$ a subset. The elements of the free abelian monoid $\mathcal{F}(G_0)$ over G_0 are called *sequences over* G_0. Thus a sequence $S \in \mathcal{F}(G_0)$ will be written in the form

$$S = g_1 \cdot \dots \cdot g_l = \prod_{g \in G_0} g^{\mathsf{v}_g(S)},$$

and we use all notions (such as the length) as in general free abelian monoids (see Section 2). Furthermore, we denote by $\sigma(S) = g_1 + \cdots + g_l$ the *sum* of S, and

$$\mathcal{B}(G_0) = \{S \in \mathcal{F}(G_0) \mid \sigma(S) = 0\}$$

is called the *monoid of zero-sum sequences* over G_0. Clearly, $\mathcal{B}(G_0) \subset \mathcal{F}(G_0)$ is a saturated submonoid, and hence it is a Krull monoid by Theorem 4.13 (b). In Theorem 6.5 we will outline the relationship between a general Krull monoid and an associated monoid of zero-sum sequences. An element $S = g_1 \cdots g_l$ is an atom in $\mathcal{B}(G_0)$ if and only if it is a minimal zero-sum sequence (that is, $\sigma(S) = 0$ but $\sum_{i \in I} g_i \neq 0$ for all $\emptyset \neq I \subsetneq [1, l]$). The *Davenport constant*

$$\mathsf{D}(G_0) = \sup\{|U| \mid U \in \mathcal{A}(\mathcal{B}(G_0))\} \in \mathbb{N}_0 \cup \{\infty\},$$

of G_0 is a central invariant in zero-sum theory (see [24]), and for its relevance in factorization theory we refer to [25]. For a finite set G_0 we have $\mathsf{D}(G_0) < \infty$ (see [29, Theorem 3.4.2]).

6. Arithmetic of Krull monoids

The theory of non-unique factorizations (in commutative monoids and domains) has its origin in algebraic number theory, and in the last two decades it emerged as an independent branch of algebra and number theory (see [2, 14, 13, 28, 29] for some recent surveys and conference proceedings). Its main objective is to describe the non-uniqueness of factorizations by arithmetical invariants (such as sets of lengths, defined below), and to study the relationship between these arithmetical parameters and classical algebraic parameters (such as class groups) of the rings under investigation. Transfer homomorphisms play a crucial role in this theory. They allow to shift problems from the original objects of interest to auxiliary monoids, which are easier to handle; then one has to settle the problems in the auxiliary monoids and shift the answer back to the initial monoids or domains. This machinery is best established—but not restricted to—in the case of commutative Krull monoids, and it allows to employ methods from additive and combinatorial number theory ([25]).

In this section, we first show that the concept of a transfer homomorphism carries over to the non-commutative setting in perfect analogy. Then we give a criterion for a Krull monoid to be a bounded factorization monoid, and show that, if a Krull monoid admits a divisor homomorphism with finite Davenport constant, then all the arithmetical invariants under consideration are finite too (Theorem 6.5). In order to do so we need all the ideal and divisor theoretic tools developed in Sections 3 and 4.

Let H be a monoid. If $a \in H$ and $a = u_1 \cdots u_k$, where $k \in \mathbb{N}$ and $u_1, \ldots, u_k \in \mathcal{A}(H)$, then we say that k is the *length* of the factorization. For $a \in H \setminus H^\times$, we call

$$\mathsf{L}_H(a) = \mathsf{L}(a) = \{k \in \mathbb{N} \mid a \text{ has a factorization of length } k\} \subset \mathbb{N}$$

the *set of lengths* of a. For convenience, we set $\mathsf{L}(a) = \{0\}$ for all $a \in H^\times$. By definition, H is atomic if and only if $\mathsf{L}(a) \neq \emptyset$ for all $a \in H$. We say that H is a BF-*monoid* (or a bounded factorization monoid) if $\mathsf{L}(a)$ is finite and non-empty for all $a \in H$. We call

$$\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$$

the *system of sets of lengths* of H. So if H is a BF-monoid, then $\mathcal{L}(H)$ is a set of finite non-empty subsets of the non-negative integers.

We recall some invariants describing the arithmetic of BF-monoids. Let H be a BF-monoid. If $L = \{l_1, \ldots, l_t\} \subset \mathbb{N}$, where $t \in \mathbb{N}$ and $l_1 < \cdots < l_t$, is a finite non-empty subset of the positive integers, then

- $\rho(L) = \max L / \min L \in \mathbb{Q}_{\geq 1}$ is called the *elasticity* of L, and
- $\Delta(L) = \{l_i - l_{i-1} \mid i \in [2, t]\}$ is called the *set of distances* of L.

For convenience, we set $\rho(\{0\}) = 1$ and $\Delta(\{0\}) = \emptyset$. We call

- $\rho(H) = \sup\{\rho(L) \mid L \in \mathcal{L}(H)\} \in \mathbb{R}_{\geq 1} \cup \{\infty\}$ the *elasticity* of H, and
- $\Delta(H) = \bigcup_{L \in \mathcal{L}(H)} \Delta(L) \subset \mathbb{N}$ the *set of distances* of H.

Clearly, we have $\rho(H) = 1$ if and only if $\Delta(H) = \emptyset$. Suppose that $\Delta(H) \neq \emptyset$, in other words that there is some $L \in \mathcal{L}(H)$ such that $|L| \geq 2$. Then there exists some $a \in H$ such that $a = u_1 \cdots u_k = v_1 \cdots v_l$ where $1 < k < l$ and $u_1, \ldots, u_k, v_1, \ldots, v_l \in \mathcal{A}(H)$. Then for every $n \in \mathbb{N}$, we have

$$a^n = (u_1 \cdots u_k)^\nu (v_1 \cdots v_l)^{n-\nu} \quad \text{for all} \quad \nu \in [0, n]$$

and hence $\{ln - \nu(l-k) \mid \nu \subset [0, n]\} \subset \mathsf{L}(a^n)$. Therefore sets of lengths get arbitrarily large. We will see that—under suitable algebraic finiteness conditions—sets of lengths are well-structured. In order to describe their structure we need the notion of almost arithmetical progressions.

Let $d \in \mathbb{N}$, $M \in \mathbb{N}_0$ and $\{0, d\} \subset \mathcal{D} \subset [0, d]$. A subset $L \subset \mathbb{Z}$ is called an *almost arithmetical multiprogression* (AAMP for short) with *difference* d, *period* $\mathcal{D}$, and *bound* M, if

$$L = y + (L' \cup L^* \cup L'') \subset y + \mathcal{D} + d\mathbb{Z},$$

where $y \in \mathbb{Z}$ is a shift parameter,

- L^* is finite nonempty with $\min L^* = 0$ and $L^* = (\mathcal{D} + d\mathbb{Z}) \cap [0, \max L^*]$ and
- $L' \subset [-M, -1]$ and $L'' \subset \max L^* + [1, M]$.

We say that *the Structure Theorem for Sets of Lengths* holds for the monoid H if H is atomic and there exist some $M^* \in \mathbb{N}_0$ and a finite nonempty set $\Delta^* \subset \mathbb{N}$ such that every $L \in \mathcal{L}(H)$ is an AAMP with some difference $d \in \Delta^*$ and bound M^* (in this case we say more precisely, that the Structure Theorem holds with parameters M^* and Δ^*).

We start with a characterization of BF-monoids, and for that we need the notion of length functions. A function $\lambda\colon H \to \mathbb{N}_0$ is called a *length function* if $\lambda(a) < \lambda(b)$

for all $b \in (aH \cup Ha) \setminus (aH^\times \cup H^\times a)$.

Lemma 6.1. *Let H be a monoid and $\mathfrak{m} = H \setminus H^\times$. Then the following statements are equivalent*:
(a) *H is a* BF-*monoid.*
(b) $\bigcap_{n \ge 0} \mathfrak{m}^n = \emptyset$.
(c) *There exists a length function $\lambda \colon H \to \mathbb{N}_0$.*

Proof. (a) $\Rightarrow$ (b) Let $a \in \mathfrak{m}^k$ for some $k \in \mathbb{N}$. Then there exist $a_1, \ldots, a_k \in \mathfrak{m}$ such that $a = a_1 \cdots a_k$ and hence $\max \mathsf{L}(a) \ge k$. Since $\mathsf{L}(a)$ is finite, there exists some $l \in \mathbb{N}$ such that $a \notin \mathfrak{m}^l \supset \bigcap_{n \ge 0} \mathfrak{m}^n$.

(b) $\Rightarrow$ (c) We define a map $\lambda \colon H \to \mathbb{N}_0$ by setting $\lambda(a) = \max\{n \in \mathbb{N}_0 \mid a \in \mathfrak{m}^n\}$, and assert that λ is a length function. Let $a \in H$ and $b \in (aH \cup Ha) \setminus (aH^\times \cup H^\times a)$, say $b \in aH$. Then $b = ac$ for some $c \in \mathfrak{m}$. If $\lambda(a) = k$, then $a \in \mathfrak{m}^k$, $b = ac \in \mathfrak{m}^{k+1}$, and thus $\lambda(b) \ge k + 1 > \lambda(a)$.

(c) $\Rightarrow$ (a) Let $\lambda \colon H \to \mathbb{N}_0$ be a length function. Note that, if $b \in H^\times$ and $c \in H \setminus H^\times$, then $c \in bH = H$ implies that $\lambda(c) > \lambda(b) \ge 0$. We assert that every $a \in H \setminus H^\times$ can be written as a product of atoms, and that $\sup \mathsf{L}(a) \le \lambda(a)$. If $a \in \mathcal{A}(H)$, then $\mathsf{L}(a) = \{1\}$, and the assertion holds. Suppose that $a \in H$ is neither an atom nor a unit. Then a has a product decomposition of the form

$$(*) \qquad a = u_1 \cdots u_k \quad \text{where} \quad k \ge 2 \quad \text{and} \quad u_1, \ldots, u_k \in H \setminus H^\times.$$

For $i \in [0,k]$, we set $a_i = u_1 \cdots u_i$, and then $a_{i+1} \in a_i H \setminus a_i H^\times$ for all $i \in [0, k-1]$. This implies that $\lambda(a) = \lambda(a_k) > \lambda(a_{k-1}) > \cdots > \lambda(a_1) > 0$ and thus $\lambda(a) \ge k$. Therefore there exists a $k \in \mathbb{N}$ maximal such that $a = u_1 \cdots u_k$ where $u_1, \ldots, u_k \in H \setminus H^\times$, aand this implies that $u_1, \ldots, u_k \in \mathcal{A}(H)$ and $k = \max \mathsf{L}(a) \le \lambda(a)$. □

Lemma 6.2. *Let H be a monoid and Ω a set of prime ideals of H such that*

$$\bigcap_{n \in \mathbb{N}} \mathfrak{p}^n = \emptyset \quad \textit{for all} \quad \mathfrak{p} \in \Omega.$$

If for every $a \in H \setminus H^\times$ the set $\Omega_a = \{\mathfrak{p} \in \Omega \mid a \in \mathfrak{p}\}$ is finite and non-empty, then H is a BF-*monoid.*

Proof. By Lemma 6.1, it suffices to show that H has a length function. If $a \in H$ and $\Omega_a = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$, we define

$$\lambda(a) = \sup\{n_1 + \cdots + n_k \mid n_1, \ldots, n_k \in \mathbb{N}_0,\ a \in \mathfrak{p}_1^{n_1} \cap \cdots \cap \mathfrak{p}_k^{n_k}\}.$$

By assumption, there exists some $n \in \mathbb{N}$ such that $a \notin \mathfrak{p}_i^n$ for all $i \in [1, k]$, whence $\lambda(a) \le kn$. We assert that $\lambda \colon H \to \mathbb{N}_0$ is a length function. Let $a \in H$ and $b \in (aH \cup$

$Ha) \setminus (aH^\times \cup H^\times a)$, say $b = ac$ for some $c \in H \setminus H^\times$. Since $\Omega_c \neq \emptyset$, there is a $\mathfrak{q} \in \Omega$ with $c \in \mathfrak{q}$. We assume that $\Omega_a = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$, $a \in \mathfrak{p}_1^{n_1} \cap \cdots \cap \mathfrak{p}_k^{n_k}$ and $\lambda(a) = n_1 + \cdots + n_k$. If $\mathfrak{q} \in \Omega_a$, say $\mathfrak{q} = \mathfrak{p}_k$, then $b = ac \in (\mathfrak{p}_1^{n_1} \cap \mathfrak{p}_2^{n_2} \cap \cdots \cap \mathfrak{p}_k^{n_k})\mathfrak{p}_k \subset \mathfrak{p}_1^{n_1} \cap \mathfrak{p}_2^{n_2} \cap \cdots \cap \mathfrak{p}_k^{n_k+1}$ and therefore $\lambda(b) \geq n_1 + \cdots + (n_k + 1) > \lambda(a)$. If $\mathfrak{q} \notin \Omega_a$, then $b = ac \in (\mathfrak{p}_1^{n_1} \cap \cdots \cap \mathfrak{p}_k^{n_k})\mathfrak{q} \subset \mathfrak{p}_1^{n_1} \cap \cdots \cap \mathfrak{p}_k^{n_k} \cap \mathfrak{q}$ and thus again $\lambda(b) \geq n_1 + \cdots + n_k + 1 > \lambda(a)$. □

DEFINITION 6.3. A monoid homomorphism $\theta\colon H \to B$ from a monoid H onto a reduced monoid B is called a *transfer homomorphism* if it has the following properties:
(T1) $B = \theta(H)$ and $\theta^{-1}(1) = H^\times$.
(T2) If $a \in H$, $b_1, b_2 \in B$ and $\theta(a) = b_1 b_2$, then there exist $a_1, a_2 \in H$ such that $a = a_1 a_2$, $\theta(a_1) = b_1$ and $\theta(a_2) = b_2$.

Transfer homomorphisms in a non-commutative setting were first used by Baeth, Ponomarenko et al. in [7].

Proposition 6.4. *Let H and B be monoids, $\theta\colon H \to B$ a transfer homomorphism and $a \in H$.*
1. *If $k \in \mathbb{N}$, $b_1, \ldots, b_k \in B$ and $\theta(a) = b_1 \cdots b_k$, then there exist $a_1, \ldots, a_k \in H$ such that $a = a_1 \cdots a_k$ and $\theta(a_\nu) = b_\nu$ for all $\nu \in [1, k]$.*
2. *a is an atom of H if and only if $\theta(a)$ is an atom of B.*
3. $\mathsf{L}_H(a) = \mathsf{L}_B(\theta(a))$.
4. *H is atomic* (*a* BF-*monoid resp.*) *if and only if B is atomic* (*a* BF-*monoid resp.*).
5. *Suppose that H is a* BF-*monoid. Then $\rho(H) = \rho(B)$, $\Delta(H) = \Delta(B)$, and the Structure Theorem for Sets of Lengths holds for H if and only if it holds for B* (*with the same parameters*).

Proof. 1. This follows by induction on k.

2. Let $a \in H$ be an atom, and suppose that $\theta(a) = b_1 b_2$ with $b_1, b_2 \in B$. By (T2), there exist $a_1, a_2 \in H$ with $a = a_1 a_2$ and $\theta(a_i) = b_i$ for $i \in [1, 2]$. Since a is an atom, we infer that $a_1 \in H^\times$ or $a_2 \in H^\times$, and thus $b_1 = 1$ or $b_2 = 1$. Conversely, suppose that $\theta(a)$ is an atom of B. If $a = a_1 a_2$, then $\theta(a) = \theta(a_1)\theta(a_2)$. Thus $\theta(a_1) = 1$ or $\theta(a_2) = 1$, and therefore $a_1 \in H^\times$ or $a_2 \in H^\times$.

3. By (T1), it follows that $a \in H^\times$ if and only if $\theta(a) = 1$. Suppose that $a \notin H^\times$, and choose $k \in \mathbb{N}$. If $k \in \mathsf{L}_H(a)$, then there exist $u_1, \ldots, u_k \in \mathcal{A}(H)$ such that $a = u_1 \cdots u_k$. Then $\theta(a) = \theta(u_1) \cdots \theta(u_k)$. Since $\theta(u_1), \ldots, \theta(u_k) \in \mathcal{A}(B)$ by 2., it follows that $k \in \mathsf{L}_B(\theta(a))$. Conversely, suppose that $k \in \mathsf{L}_B(\theta(a))$. Then there are $b_1, \ldots, b_k \in \mathcal{A}(B)$ such that $\theta(a) = b_1 \cdots b_k$. Now 1. and 2. imply that $k \in \mathsf{L}_H(a)$.

4. A monoid S is atomic (a BF-monoid resp.) if and only if for all $s \in S$, we have $\mathsf{L}(s) \neq \emptyset$ ($\mathsf{L}(s)$ is finite and non-empty resp.). Thus the assertion follows from 3.

5. This follows immediately from 3. and 4. □

Theorem 6.5 (Arithmetic of Krull monoids)**.** *Let H be a Krull monoid.*
1. *If every $a \in H \setminus H^\times$ lies in a divisorial ideal distinct from H, then H is a* BF-*monoid.*
2. *Let $\varphi\colon H \to D = \mathcal{F}(P)$ be a divisor homomorphism, $G = \mathcal{C}(\varphi)$ its class group and $G_P \subset G$ the set of classes containing prime divisors.*
   (a) *Let $\tilde{\boldsymbol{\beta}}\colon \mathcal{F}(P) \to \mathcal{F}(G_P)$ denote the unique homomorphism satisfying $\tilde{\boldsymbol{\beta}}(p) = [p]$ for all $p \in P$. Then, for all $\alpha \in D$, we have $\tilde{\boldsymbol{\beta}}(\alpha) \in \mathcal{B}(G_P)$ if and only if $\alpha \in \varphi(H)$, and the map $\boldsymbol{\beta} = \tilde{\boldsymbol{\beta}} \circ \varphi\colon H \to \mathcal{B}(G_P)$ is a transfer homomorphism.*
   (b) *If $\mathsf{D}(G_P) < \infty$, then $\rho(H) < \infty$, $\Delta(H)$ is finite, and there exists some $M^* \in \mathbb{N}_0$ such that the Structure Theorem for Sets of Lengths holds for H with parameters M^* and $\Delta(H)$.*

Proof. 1. We show that $\Omega = v\text{-spec}(H) \setminus \{\emptyset\}$ satisfies the assumptions of Lemma 6.2. Then H is a BF-monoid.

Let $a \in H \setminus H^\times$. By assumption, the set $\Omega'_a = \{\mathfrak{a} \in \mathcal{I}_v(H) \mid a \in \mathfrak{a} \text{ with } \mathfrak{a} \cap \{1\} = \emptyset\}$ is non-empty, and since H is v-noetherian, Ω'_a has a maximal element $\mathfrak{p}$ by Lemma 3.13, which is prime by Lemma 3.8 1. Therefore the set $\Omega_a = \{\mathfrak{p} \in v\text{-spec}(H) \mid a \in \mathfrak{p}\}$ is finite and non-empty. Let $\mathfrak{p} \in v\text{-spec}(H)$. If the intersection of all powers of $\mathfrak{p}$ would be non-empty, it would be a non-empty v-ideal and hence divisible by arbitrary powers of $\mathfrak{p}$, a contradiction to the fact that $\mathcal{I}_v^*(H)$ is free abelian by Theorem 3.14.

2. (a) If $\alpha \in D$, then $\alpha = p_1 \cdots p_l$, where $l \in \mathbb{N}_0$ and $p_1, \ldots, p_l \in P$, $\tilde{\boldsymbol{\beta}}(\alpha) = [p_1] \cdots [p_l]$ and $\sigma(\tilde{\boldsymbol{\beta}}(\alpha)) = [p_1] + \cdots + [p_l] = [\alpha]$. Thus we have $[\alpha] = 0$ if and only if $\alpha \in \varphi(H)$. Therefore we obtain that $\boldsymbol{\beta} = \tilde{\boldsymbol{\beta}} \circ \varphi\colon H \to \mathcal{B}(G_P)$ is a monoid epimorphism onto a reduced monoid with $\boldsymbol{\beta}^{-1}(1) = H^\times$. To verify (T2), let $a \in H$ with $\varphi(a) = p_1 \cdots p_l \in D$, where $l \in \mathbb{N}_0$ and $p_1, \ldots, p_l \in P$, and $\boldsymbol{\beta}(a) = [p_1] \cdots [p_l] = b_1 b_2$ with $b_1, b_2 \in \mathcal{B}(G_P)$. After renumbering if necessary there is some $k \in [0, l]$ such that $b_1 = [p_1] \cdots [p_k]$ and $b_2 = [p_{k+1}] \cdots [p_l]$. Setting $\alpha_1 = p_1 \cdots p_k$, $\alpha_2 = p_{k+1} \cdots p_l$ we infer that $\alpha_1, \alpha_2 \in \varphi(H)$, say $\alpha_i = \varphi(a_i)$ with $a_i \in H$, and $\tilde{\boldsymbol{\beta}}(\alpha_i) = b_i$ for $i \in [1, 2]$. Then $\varphi(a) = \varphi(a_1)\varphi(a_2)$, and hence by Lemma 4.6 2., we get $aH^\times = a_1 a_2 H^\times$. Thus there is an $\varepsilon \in H^\times$ such that $a = (\varepsilon a_1) a_2$, $\boldsymbol{\beta}(\varepsilon a_1) = \boldsymbol{\beta}(a_1) = b_1$ and $\boldsymbol{\beta}(a_2) = b_2$.

2. (b) Suppose that $\mathsf{D}(G_P) < \infty$. By Proposition 6.4 5., it suffices to prove all assertions for the monoid $\mathcal{B}(G_P)$. Thus the finiteness of the elasticity and of the set of distances follows from [29, Theorem 3.4.11], and the validity of the Structure Theorem follows from [30, Theorem 5.1] or from [26, Theorem 4.4]. □

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