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FORMALITY AND HARD LEFSCHETZ PROPERTY OF ASPHERICAL MANIFOLDS

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Abstract

For a Lie group $G = \mathbb{R}^n \ltimes_{\phi} \mathbb{R}^m$ with the semi-simple action $\phi: \mathbb{R}^n \rightarrow \text{Aut}(\mathbb{R}^m)$, we show that if Γ is a finite extension of a lattice of G then $K(\Gamma, 1)$ is formal. Moreover we show that a compact symplectic aspherical manifold with the fundamental group Γ satisfies the hard Lefschetz property. By those results we give many examples of formal solvmanifolds satisfying the hard Lefschetz property but not admitting Kähler structures.

1. Introduction

Formal spaces (see Definition 5.3) in the sense of Sullivan are important in de Rham homotopy theory. Well-known examples of formal spaces are compact Kähler manifolds (see [9]). Suppose Γ is a torsion-free finitely generated nilpotent group. Then $K(\Gamma, 1)$ is formal if and only if Γ is abelian by Hasegawa's theorem in [11]. But in case Γ is a virtually polycyclic (see Definition 2.1) group, the formality of $K(\Gamma, 1)$ is more complicated. One of the purposes of this paper is to apply the way of the algebraic hull of Γ to study the formality of $K(\Gamma, 1)$. For a torsion-free virtually polycyclic group Γ , we have a unique algebraic group \mathbf{H}_{Γ} with an injective homomorphism $\psi: \Gamma \rightarrow \mathbf{H}_{\Gamma}$ so that:

- (1) $\psi(\Gamma)$ is Zariski-dense in \mathbf{H}_{Γ} .
- (2) The centralizer $Z_{\mathbf{H}_{\Gamma}}(\mathbf{U}(\mathbf{H}_{\Gamma}))$ of $\mathbf{U}(\mathbf{H}_{\Gamma})$ is contained in $\mathbf{U}(\mathbf{H}_{\Gamma})$.
- (3) $\dim \mathbf{U}(\mathbf{H}_{\Gamma}) = \text{rank } \Gamma$.

Such \mathbf{H}_{Γ} is called the algebraic hull of Γ . We call the unipotent radical of \mathbf{H}_{Γ} the unipotent hull of Γ and denote it by \mathbf{U}_{Γ} . In [3], Baues constructed a compact aspherical manifold M_{Γ} with the fundamental group Γ which is called the standard Γ -manifold by the algebraic hull of Γ . And he gave the way of computation of the de Rham cohomology of M_{Γ} . By using these results, we prove:

Proposition 1.1. *If the unipotent hull \mathbf{U}_{Γ} of Γ is abelian, $K(\Gamma, 1)$ is formal.*

So we would like to know criteria for \mathbf{U}_{Γ} to be abelian. We prove the following theorem.

Theorem 1.2. *Let Γ be a torsion-free virtually polycyclic group. Then the following two conditions are equivalent:*

- (1) U_Γ is abelian.
- (2) Γ is a finite extension group of a lattice of a Lie group $G = \mathbb{R}^n \rtimes_\phi \mathbb{R}^m$ such that the action $\phi: \mathbb{R}^n \rightarrow \text{Aut}(\mathbb{R}^m)$ is semi-simple.

Therefore we have:

Corollary 1.3. *If Γ satisfies the condition (2) in Theorem 1.2, then $K(\Gamma, 1)$ is formal.*

REMARK 1. A lattice Γ of $G = \mathbb{R}^n \rtimes_\phi \mathbb{R}^m$ is the form $\Gamma' \rtimes_\phi \Gamma''$ such that Γ' and Γ'' are lattices of \mathbb{R}^n and \mathbb{R}^m respectively and the action ϕ of Γ' preserves Γ'' .

As well as formality the hard Lefschetz property (see Definition 5.5) is an important property of a compact Kähler manifold. We have the following proposition.

Proposition 1.4. *Let M be a compact symplectic aspherical manifold with the torsion-free virtually polycyclic fundamental group Γ . If the unipotent hull U_Γ is abelian, then M satisfies the hard Lefschetz property.*

Hence we have:

Corollary 1.5. *If Γ satisfies the condition (2) in Theorem 1.2, then a compact symplectic aspherical manifold with the fundamental group Γ satisfies the hard Lefschetz property.*

In [5], Benson and Gordon showed that a compact symplectic aspherical manifold with the torsion-free nilpotent fundamental group Γ satisfies the hard Lefschetz property if and only if Γ is abelian.

As we see in [11] and [5], formality and the hard Lefschetz property are strong criteria for aspherical manifolds to admit Kähler structures. But by the results of this paper, we can obtain many non-Kähler formal aspherical manifolds satisfying the hard Lefschetz property.

Let M be a compact aspherical manifold with the virtually polycyclic fundamental group. In [4], Baues and Cortés showed that if M admits a Kähler structure then the fundamental group of M is virtually abelian (this result is an extension of the result in [1] and [12]). Let G be a simply connected solvable Lie group. We say that G is of type (I) if for any $g \in G$ all eigenvalues of the adjoint operator Ad_g have absolute value 1. In [2] it was proved that a lattice of a simply connected solvable Lie group G is virtually nilpotent if and only if G is type (I). Hence we have:

Corollary 1.6. *Let Γ be a finite extension group of a lattice of a Lie group $G = \mathbb{R}^n \rtimes_{\phi} \mathbb{R}^m$ such that the action $\phi: \mathbb{R}^n \rightarrow \text{Aut}(\mathbb{R}^m)$ is semi-simple and G is not of type (I). Then a compact aspherical manifold M with the fundamental group Γ is formal but admits no Kähler structure. If M admits a symplectic structure, then M satisfies the hard Lefschetz property.*

REMARK 2. In [12], Hasegawa showed that a simply connected solvable Lie group G with a virtually abelian lattice such that G/Γ admits Kähler structure can be written as $G = \mathbb{R}^{2k} \rtimes_{\phi} \mathbb{C}^l$ such that

$$\phi(t_j)((z_1, \dots, z_l)) = (e^{\sqrt{-1}\theta_1^j t_j} z_1, \dots, e^{\sqrt{-1}\theta_l^j t_j} z_l),$$

where each $e^{\sqrt{-1}\theta_i^j}$ is a root of unity.

Solvmanifolds are homogeneous spaces of connected solvable Lie groups. These are examples of aspherical manifolds with the polycyclic fundamental groups. In particular for a simply connected solvable Lie group G with a lattice Γ , the solvmanifold G/Γ is a compact aspherical manifold with the fundamental group Γ . As generalizations of solvmanifolds we define infra-solvmanifolds. Let G be a simply connected solvable Lie group. Consider the group $\text{Aut}(G) \rtimes G$ of affine transformations of G and the projection $p: \text{Aut}(G) \rtimes G \rightarrow \text{Aut}(G)$. An infra-solvmanifold is a manifold of the form G/Δ for a torsion-free subgroup Δ of $\text{Aut}(G) \rtimes G$ such that $p(\Delta)$ is contained in a compact subgroup of $\text{Aut}(G)$. In [3] Baues showed that every compact infra-solvmanifold is diffeomorphic to a standard Γ -manifold and for any torsion-free virtually polycyclic group Γ the standard Γ -manifold is diffeomorphic to an infra-solvmanifold G/Γ such that $\Gamma \subset \text{Aut}(G) \rtimes G$ is a discrete subgroup and $p(\Gamma)$ is finite. Thus for any Γ satisfying the condition (2) in Theorem 1.2 we have a compact infra-solvmanifold G/Γ for some $G = \mathbb{R}^n \rtimes_{\phi} \mathbb{R}^m$ such that the action $\phi: \mathbb{R}^n \rightarrow \text{Aut}(\mathbb{R}^m)$ is semi-simple.

Notations and terminology: Let k be a subfield of \mathbb{C} . A group \mathbf{G} is called k -algebraic group if \mathbf{G} is a Zariski-closed subgroup of $GL_n(\mathbb{C})$ which is defined by polynomials with coefficients in k . Let $\mathbf{G}(k)$ denote the set of k -points of \mathbf{G} and $\mathbf{U}(\mathbf{G})$ the maximal Zariski-closed unipotent normal k -subgroup of \mathbf{G} called the unipotent radical of \mathbf{G} . A general reference is [7]. In this paper, algebraic groups are always written in the bold face.

2. Algebraic hulls

In this section we explain the algebraic hulls of polycyclic groups or simply connected solvable Lie groups.

DEFINITION 2.1. A group Γ is *polycyclic* if it admits a sequence

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \dots \supset \Gamma_k = \{e\}$$

of subgroups such that each Γ_i is normal in Γ_{i-1} and Γ_{i-1}/Γ_i is cyclic. We set $\text{rank } \Gamma = \sum_{i=1}^{i=k} \text{rank } \Gamma_{i-1}/\Gamma_i$ which is independent of the choice of a sequence Γ_i .

There are close relations between polycyclic groups and solvable Lie groups.

Theorem 2.2 ([21, Proposition 3.7, Theorem 4.28]). *Let G be a simply connected solvable Lie group and Γ a lattice in G . Then Γ is torsion-free polycyclic and $\dim G = \text{rank } \Gamma$. Conversely every polycyclic group admits a finite index normal subgroup which is isomorphic to a lattice in a simply connected solvable Lie group.*

Let Γ be a virtually polycyclic group and Γ' be a finite index polycyclic subgroup. We set $\text{rank } \Gamma = \text{rank } \Gamma'$.

DEFINITION 2.3. Let k be a subfield \mathbb{C} . Let Γ be a torsion-free virtually polycyclic group (resp. simply connected solvable Lie group). Then a k -algebraic group \mathbf{H}_Γ is a k -algebraic hull of Γ if there exists an injective homomorphism $\psi : \Gamma \rightarrow \mathbf{H}_\Gamma(k)$ and \mathbf{H}_Γ satisfies the following conditions:

- (1) $\psi(\Gamma)$ is Zariski-dense in \mathbf{H}_Γ .
- (2) $Z_{\mathbf{H}_\Gamma}(\mathbf{U}(\mathbf{H}_\Gamma)) \subset \mathbf{U}(\mathbf{H}_\Gamma)$.
- (3) $\dim \mathbf{U}(\mathbf{H}_\Gamma) = \text{rank } \Gamma$ (resp. $\dim \Gamma$).

Theorem 2.4 ([3, Theorem A.1, Corollary A.3], [21, Proposition 4.40, Lemma 4.41]). *Let Γ be a torsion-free virtually polycyclic group (resp. simply connected solvable Lie group). Then there exists a \mathbb{Q} -algebraic (resp. \mathbb{R} -algebraic) hull of Γ and for any subfield $k \subset \mathbb{C}$ which contains \mathbb{Q} (resp. \mathbb{R}) a k -algebraic hull of Γ is unique up to k -algebraic group isomorphism.*

We call the unipotent radical of \mathbf{H}_Γ the unipotent hull of Γ and denote it by \mathbf{U}_Γ .

Lemma 2.5. *Let Γ be a torsion-free virtually polycyclic group and Δ a finite index subgroup of Γ . Let $\psi : \Gamma \rightarrow \mathbf{H}_\Gamma$ be the k -algebraic hull of Γ and \mathbf{G} the Zariski-closure of $\psi(\Delta)$ in \mathbf{H}_Γ . Then the algebraic group \mathbf{G} is the k -algebraic hull of Δ and we have $\mathbf{U}_\Delta = \mathbf{U}_\Gamma$.*

Proof. Let \mathbf{H}_Γ^0 be the identity component of \mathbf{H}_Γ . Since \mathbf{G} is a closed finite index subgroup of \mathbf{H}_Γ , we have $\mathbf{H}_\Gamma^0 \subset \mathbf{G}$. Since Γ is virtually polycyclic, \mathbf{H}_Γ^0 is solvable. Hence we have $\mathbf{U}(\mathbf{H}_\Gamma) = (\mathbf{H}_\Gamma^0)_{\text{unip}} = \mathbf{U}(\mathbf{G})$. Since $\text{rank } \Gamma = \text{rank } \Delta$, we have

$$\dim \mathbf{U}(\mathbf{G}) = \text{rank } \Delta,$$

and we have

$$Z_{\mathbf{G}}(\mathbf{U}(\mathbf{G})) \subset Z_{\mathbf{H}_\Gamma}(\mathbf{U}(\mathbf{H}_\Gamma)) \subset \mathbf{U}(\mathbf{H}_\Gamma) = \mathbf{U}(\mathbf{G}).$$

Hence the lemma follows. □

Lemma 2.6 ([21, Proof of Theorem 4.34]). *Let G be a simply connected solvable Lie group with a lattice Γ . Let $\psi: G \rightarrow \mathbf{H}_G$ be the \mathbb{R} -algebraic hull of G and \mathbf{H}' the Zariski-closure of $\psi(\Gamma)$ in \mathbf{H}_G . Then \mathbf{H}' is the \mathbb{R} -algebraic hull of Γ and we have $\mathbf{U}_G = \mathbf{U}_\Gamma$.*

3. Cohomology computations of aspherical manifolds with virtually torsion-free polycyclic fundamental groups

Let Γ be a torsion-free virtually polycyclic group and \mathbf{H}_Γ the \mathbb{Q} -algebraic hull of Γ . Denote $H_\Gamma = \mathbf{H}_\Gamma(\mathbb{R})$. Let U_Γ be the unipotent radical of H_Γ and let T be a maximal reductive subgroup. Then H_Γ decomposes as a semi-direct product $H_\Gamma = T \ltimes U_\Gamma$. Let \mathfrak{u} be the Lie algebra of U_Γ . Since the exponential map $\exp: \mathfrak{u} \rightarrow U_\Gamma$ is a diffeomorphism, U_Γ is diffeomorphic to \mathbb{R}^n such that $n = \text{rank } \Gamma$. The splitting $H_\Gamma = T \ltimes U_\Gamma$ gives rise to the affine action $\alpha: H_\Gamma \rightarrow \text{Aut}(U_\Gamma) \ltimes U_\Gamma$ such that α is an injective homomorphism.

In [3] Baues constructed a compact aspherical manifold $M_\Gamma = \alpha(\Gamma) \backslash U_\Gamma$ with $\pi_1(M_\Gamma) = \Gamma$. We call M_Γ a standard Γ -manifold.

Theorem 3.1 ([3, Theorem 1.2]). *Standard Γ -manifold is unique up to diffeomorphism.*

Let $A^*(M_\Gamma)$ be the de Rham complex of M_Γ . Then $A^*(M_\Gamma)$ is the set of the Γ -invariant differential forms $A^*(U_\Gamma)^\Gamma$ on U_Γ . Let $(\bigwedge \mathfrak{u}^*)^T$ be the left-invariant forms on U_Γ which are fixed by T . Since $\Gamma \subset H_\Gamma = T \ltimes U_\Gamma$, we have the inclusion

$$\left(\bigwedge \mathfrak{u}^*\right)^T = A^*(U_\Gamma)^{H_\Gamma} \subset A^*(U_\Gamma)^\Gamma = A^*(M_\Gamma).$$

Theorem 3.2 ([3, Theorem 1.8]). *This inclusion induces a cohomology isomorphism.*

4. Proof of Theorem 1.2

4.1. The embeddings of solvable Lie algebras in splittable Lie algebras. The idea of this subsection is based on [22]. Let \mathfrak{g} be a solvable Lie algebra and $\mathfrak{n} = \{X \in \mathfrak{g} \mid \text{ad}_X \text{ is nilpotent}\}$. Then \mathfrak{n} is the maximal nilpotent ideal of \mathfrak{g} and called the nilradical of \mathfrak{g} .

Lemma 4.1 ([18, p. 58]). *We have $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}$.*

Let $D(\mathfrak{g})$ denote the space of the derivations of \mathfrak{g} . By the Jordan decomposition, we have the decomposition $\text{ad}_X = d_X + n_X$ such that d_X is a semi-simple operator and n_X is a nilpotent operator.

Lemma 4.2 ([22, Proposition 3]). *We have $d_X, n_X \in D(\mathfrak{g})$.*

Then we have the homomorphism $f: \mathfrak{g} \rightarrow D(\mathfrak{g})$ such that $f(X) = d_X$ for $X \in \mathfrak{g}$. Since $\ker f = \mathfrak{n}$, we have $\text{Im} f \cong \mathfrak{g}/\mathfrak{n}$.

Let $\bar{\mathfrak{g}} = \text{Im} f \ltimes \mathfrak{g}$ and $\bar{\mathfrak{n}} = \{X - d_X \in \bar{\mathfrak{g}} \mid X \in \mathfrak{g}\}$. Since $\text{ad}_{X-d_X} = \text{ad}_X - d_X$ on \mathfrak{g} , ad_{X-d_X} is a nilpotent operator. So $\bar{\mathfrak{n}}$ consists of nilpotent elements.

Proposition 4.3. *We have $d_X(\bar{\mathfrak{n}}) \subset \mathfrak{n}$ for any $X \in \mathfrak{g}$, $\bar{\mathfrak{n}}$ is a nilpotent ideal of $\bar{\mathfrak{g}}$ and $\bar{\mathfrak{g}} = \text{Im} f \ltimes \bar{\mathfrak{n}}$.*

Proof. By Lie’s theorem, we have a basis X_1, \dots, X_l of $\mathfrak{n} \otimes \mathbb{C}$ such that $\text{ad}_{\mathfrak{g}}$ on \mathfrak{n} are represented by upper triangular matrices. Then for any $X \in \mathfrak{g}$, we have

$$\begin{aligned} \text{ad}_X(X_1) &= a_{X,1}X_1, \\ \text{ad}_X(X_2) &= a_{X,2}X_2 + b_{X,12}X_1, \\ &\dots \\ \text{ad}_X(X_l) &= a_{X,l}X_l + b_{X,l-1l}X_{l-1} + \dots + b_{X,1l}X_1. \end{aligned}$$

We take X_{l+1}, \dots, X_{l+m} such that $X_1, \dots, X_l, X_{l+1}, \dots, X_{l+m}$ is a basis of $\mathfrak{g} \otimes \mathbb{C}$. By Lemma 4.1, we have $\text{ad}_X(X_i) \in \mathfrak{n}$. Hence we have

$$\begin{aligned} \text{ad}_X(X_{l+1}) &= b_{X,l+1l}X_l + \dots + b_{X,l+11}X_1, \\ &\dots \\ \text{ad}_X(X_{l+m}) &= b_{X,l+m}X_l + \dots + b_{X,l+m1}X_1. \end{aligned}$$

Then we have

$$\begin{aligned} d_X(X_i) &= a_{X,i}X_i, \quad 1 \leq i \leq l, \\ d_X(X_i) &= 0, \quad l + 1 \leq i \leq l + m. \end{aligned}$$

Hence we have $d_X(\mathfrak{g}) \subset \mathfrak{n}$ and $d_X(\bar{\mathfrak{n}}) \subset \mathfrak{n}$. This implies $[\bar{\mathfrak{g}}, \bar{\mathfrak{g}}] \subset \mathfrak{n}$. In particular, $\bar{\mathfrak{n}}$ is an ideal of $\bar{\mathfrak{g}}$. Since $\bar{\mathfrak{n}}$ consists of nilpotent elements, $\bar{\mathfrak{n}}$ is a nilpotent ideal. By $\bar{\mathfrak{g}} = \{d_X + Y - d_Y \mid X, Y \in \mathfrak{g}\}$, we have $\bar{\mathfrak{g}} = \text{Im} f \ltimes \bar{\mathfrak{n}}$. □

By this proposition, we have the inclusion $i: \mathfrak{g} \rightarrow D(\bar{\mathfrak{n}}) \ltimes \bar{\mathfrak{n}}$ given by $i(X) = d_X + X - d_X$ for $X \in \mathfrak{g}$.

4.2. Constructions of algebraic hulls of simply connected solvable Lie groups.

Let G be a simply connected solvable Lie group and \mathfrak{g} the Lie algebra of G . Let N be the maximal normal nilpotent subgroup of G which corresponds to the nilradical \mathfrak{n} of \mathfrak{g} . Consider the injection $i: \mathfrak{g} \rightarrow \text{Im} f \ltimes \bar{\mathfrak{n}} \subset D(\bar{\mathfrak{n}}) \ltimes \bar{\mathfrak{n}}$ constructed in the last

subsection. Let \bar{N} be the simply connected Lie group which corresponds to $\bar{\mathfrak{n}}$. Since the Lie algebra of $\text{Aut}(\bar{N}) \times \bar{N}$ is $D(\bar{\mathfrak{n}}) \times \bar{\mathfrak{n}}$, we have the Lie group homomorphism $I: G \rightarrow \text{Aut}(\bar{N}) \times \bar{N}$ induced by the injective homomorphism $i: \mathfrak{g} \rightarrow D(\bar{\mathfrak{n}}) \times \bar{\mathfrak{n}}$.

Lemma 4.4. *The homomorphism $I: G \rightarrow \text{Aut}(\bar{N}) \times \bar{N}$ is injective.*

Proof. Since the restriction of $i: \mathfrak{g} \rightarrow D(\bar{\mathfrak{n}}) \times \bar{\mathfrak{n}}$ on \mathfrak{n} is injective, the restriction $I: G \rightarrow \text{Aut}(\bar{N}) \times \bar{N}$ on N is also injective. Let T_f be the subgroup of $\text{Aut}(\bar{N})$ which corresponds to $\text{Im} f$. We have $I: G \rightarrow T_f \times \bar{N}$. By Proposition 4.3, $\bar{\mathfrak{g}}/\mathfrak{n} = \text{Im} f \oplus \bar{\mathfrak{n}}/\mathfrak{n}$. So we have the induced map $I: G/N \rightarrow T_f \times \bar{N}/N$ and it is sufficient to show that this map is injective. Let $j: \text{Im} f \oplus \bar{\mathfrak{n}}/\mathfrak{n} \rightarrow \bar{\mathfrak{n}}/\mathfrak{n}$ be the projection and $J: T_f \times \bar{N}/N \rightarrow \bar{N}/N$ be the homomorphism which corresponds to j . Since the composition

$$j \circ i(X \text{ mod } \mathfrak{n}) = X - d_X \text{ mod } \mathfrak{n}$$

is surjective, $j \circ i: \mathfrak{g}/\mathfrak{n} \rightarrow \bar{\mathfrak{n}}/\mathfrak{n}$ is an isomorphism. Since G/N and \bar{N}/N are simply connected abelian groups, $J \circ I: G/N \rightarrow \bar{N}/N$ is also an isomorphism. Hence $I: G/N \rightarrow T_f \times \bar{N}/N$ is injective. \square

A simply connected nilpotent Lie group is considered as the real points of a unipotent \mathbb{R} -algebraic group (see [19, p.43]) by the exponential map. We have the unipotent \mathbb{R} -algebraic group $\bar{\mathbf{N}}$ with $\bar{\mathbf{N}}(\mathbb{R}) = \bar{N}$. We identify the group $\text{Aut}_a(\bar{\mathbf{N}})$ of automorphisms of algebraic groups with $\text{Aut}(\mathfrak{n}_{\mathbb{C}})$ and $\text{Aut}_a(\bar{\mathbf{N}})$ has the \mathbb{R} -algebraic group structure with $\text{Aut}_a(\bar{\mathbf{N}})(\mathbb{R}) = \text{Aut}(N)$. So we have the \mathbb{R} -algebraic group $\text{Aut}_a(\bar{\mathbf{N}}) \times \bar{N}$. By the above lemma, we have the injection $I: G \rightarrow \text{Aut}(N) \times N = \text{Aut}_a(\bar{\mathbf{N}}) \times \bar{N}(\mathbb{R})$. Let \mathbf{G} be the Zariski-closure of $I(G)$ in $\text{Aut}_a(\bar{\mathbf{N}}) \times \bar{N}$.

Lemma 4.5. *We have $\mathbf{U}(\mathbf{G}) = \bar{N}$.*

Proof. Let \mathbf{T} be the Zariski-closure of T_f in $\text{Aut}_a(\bar{\mathbf{N}})$. Then $\mathbf{G} \subset \mathbf{T} \times \bar{N}$. Since \mathbf{G} is connected solvable and \mathbf{T} consists of semi-simple automorphisms, we have $\mathbf{U}(\mathbf{G}) = \mathbf{G} \cap \bar{N}$. By this, it is sufficient to show $\dim \mathbf{U}(\mathbf{G}) = \dim \bar{N}$. Let \mathbf{N} be the Zariski-closure of $I(N)$. By $I(N) \subset \bar{N}$, we have $\mathbf{U}(\mathbf{G})/\mathbf{N} = \mathbf{U}(\mathbf{G}/\mathbf{N})$. Thus it is sufficient to show $\mathbf{U}(\mathbf{G}/\mathbf{N}) = G/N$. Consider the induced map $I: G/N \rightarrow T_f \times \bar{N}/N$ as the proof of Lemma 4.4. The Zariski-closure of $I(G/N)$ in $\mathbf{T} \times \bar{N}/N$ is \mathbf{G}/\mathbf{N} . Since $\mathbf{T} \times \bar{N}/N$ is commutative, the projection $\mathbf{T} \times \bar{N}/N \rightarrow \bar{N}/N$ is an \mathbb{R} -algebraic group homomorphism. Since we showed that $J \circ I: G/N \rightarrow \bar{N}/N$ is isomorphism In the proof of Lemma 4.4, the image $J \circ I(G/N)$ is Zariski-dense in \bar{N}/N . This implies $\bar{N}/N = \mathbf{U}(\mathbf{G}/\mathbf{N})$. Hence the lemma follows. \square

By this lemma we have the following proposition.

Proposition 4.6. *\mathbf{G} is the algebraic hull of G and the Lie algebra of the unipotent hull \mathbf{U}_G is $\bar{\mathfrak{n}}_{\mathbb{C}}$.*

Proof. We show that \mathbf{G} satisfies the properties of the algebraic hull of G . We have $\dim \mathbf{U}(\mathbf{G}) = \dim \bar{\mathbf{N}} = \dim G$. Let $(t, x) \in Z_G(\mathbf{U}(\mathbf{G})) \subset \text{Aut}_a \bar{\mathbf{N}} \times \bar{\mathbf{N}}$. Since $\mathbf{U}(\mathbf{G}) = \mathbf{N}$ and t is a semi-simple automorphism, we have $t(y) = y$ for any $y \in \bar{\mathbf{N}}$. So we have $t = \text{id}_{\bar{\mathbf{N}}}$. We have $Z_G(\mathbf{U}(\mathbf{G})) \subset \mathbf{U}(\mathbf{G})$. Hence the proposition follows. \square

4.3. Proof of Theorem 1.2. We first prove:

Theorem 4.7. *Let G be a simply connected solvable Lie group. Then \mathbf{U}_G is abelian if and only if $G = \mathbb{R}^n \rtimes_{\phi} \mathbb{R}^m$ such that the action $\phi: \mathbb{R}^n \rightarrow \text{Aut}(\mathbb{R}^m)$ is semi-simple.*

Proof. Consider the inclusion $i: \mathfrak{g} \rightarrow \text{Im} f \times \bar{\mathfrak{n}}$. By the above argument, the Lie algebra of \mathbf{U}_G is $\bar{\mathfrak{n}}_{\mathbb{C}}$. Suppose $G = \mathbb{R}^n \rtimes_{\phi} \mathbb{R}^m$ such that the action $\phi: \mathbb{R}^n \rightarrow \text{Aut} \mathbb{R}^m$ is semi-simple. It is sufficient to show $\bar{\mathfrak{n}} = \{X - d_X \mid X \in \mathfrak{g}\} \subset \text{Im} f \times \mathfrak{g}$ is an abelian Lie algebra. Let $X, Y \in \mathfrak{g}$ and $X = X_1 + X_2, Y = Y_1 + Y_2$ be the decompositions induced by the semi-direct product $\mathfrak{g} = \mathbb{R}^n \rtimes_{\phi_*} \mathbb{R}^m$. Then we have $d_{X_2} = 0, d_{Y_2} = 0, [X_1, Y_1] = 0$ and $[X_2, Y_2] = 0$ by the assumption. Hence we have

$$[X - d_X, Y - d_Y] = [X_1, Y_2] + [X_2, Y_1] - d_{X_1}(Y_2) + d_{Y_1}(X_2).$$

Since the action ϕ_* is semi-simple, we have $d_{X_1}(Y_2) = [X_1, Y_2]$ and $d_{Y_1}(X_2) = [Y_1, X_2]$. Therefore we have $[X - d_X, Y - d_Y] = 0$. This implies $\bar{\mathfrak{n}}$ is abelian.

Conversely we assume \mathbf{U}_G is abelian. By Proposition 4.6, $\bar{\mathfrak{n}}$ is abelian. By $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}, \mathfrak{g}$ is two-step solvable. By [8, Lemma 4.1], we have the decomposition $\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{g}^{\infty}$ for some nilpotent subalgebra \mathfrak{a} of \mathfrak{g} where $\mathfrak{g}^{\infty} = \bigcap \mathfrak{g}^i$ for the lower central series $\mathfrak{g} = \mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \dots$ of \mathfrak{g} . Since $\bar{\mathfrak{n}}$ is abelian, the subspace $\{X - d_X \mid X \in \mathfrak{a}\}$ is a abelian subalgebra of $\bar{\mathfrak{n}}$. Since \mathfrak{a} is nilpotent, the Lie algebra $\{X - d_X \mid X \in \mathfrak{a}\}$ is identified with \mathfrak{a} . Hence \mathfrak{a} is abelian. Finally we show that the action of \mathfrak{a} on \mathfrak{g}^{∞} is semi-simple. We suppose that ad_X on \mathfrak{g}^{∞} is not semi-simple for some $X \in \mathfrak{a}$. Then the action of $\text{ad}_X - d_X$ on \mathfrak{g}^{∞} is non-trivial. Since we have $\bar{\mathfrak{n}} = \{X - d_X \mid X \in \mathfrak{g}\} \subset \text{Im} f \times \bar{\mathfrak{n}}$, we have $[\bar{\mathfrak{n}}, \mathfrak{a}] \neq \{0\}$. This contradicts $\bar{\mathfrak{n}}$ is abelian. Hence the action of \mathfrak{a} on \mathfrak{g}^{∞} is semi-simple and we have the theorem. \square

Proof of Theorem 1.2. By Theorem 2.2, we have a finite index subgroup of Γ which is isomorphic to a lattice of some simply connected solvable Lie group G . By Lemma 2.5 and 2.6, we have $\mathbf{U}_{\Gamma} = \mathbf{U}_G$. Hence by Theorem 4.7 we have the theorem. \square

REMARK 3. A virtually polycyclic group Γ has the maximal nilpotent normal subgroup called the nilradical of Γ . Since the nilradical of Γ is contained in \mathbf{U}_{Γ} (see

[3, Proposition A.7]), if U_Γ is abelian then the nilradical of Γ is also abelian. But the converse is not true. Consider $G = \mathbb{R} \ltimes_\phi \mathbb{R}^4$ with

$$\phi(t) = \begin{pmatrix} e^{rt} & 0 & 0 & 0 \\ 0 & e^{-rt} & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then for some $r \neq 0$ G has a lattice $\mathbb{Z} \ltimes_\phi \Gamma''$ for a lattice Γ'' of \mathbb{R}^4 . We have $U_\Gamma = U_G = \mathbb{C}^2 \times U_3(\mathbb{C})$ and it is not abelian. On the other hand the nilradical of Γ (resp. G) is isomorphic to \mathbb{Z}^4 (resp. \mathbb{R}^4).

5. Formality and hard Lefschetz properties of aspherical manifolds

5.1. Formality. We review the definition of formality and prove Proposition 1.1.

DEFINITION 5.1. A differential graded algebra (called DGA) is a graded \mathbb{R} -algebra A^* with the following properties:

- (1) A^* is graded commutative, i.e.

$$y \wedge x = (-1)^{p_q} x \wedge y, \quad x \in A^p, y \in A^q.$$

- (2) There is a differential operator $d: A \rightarrow A$ of degree one such that $d \circ d = 0$ and

$$d(x \wedge y) = dx \wedge y + (-1)^p x \wedge dy, \quad x \in A^p.$$

Let A and B be DGAs. If a morphism of graded algebra $\varphi: A \rightarrow B$ satisfies $d \circ \varphi = \varphi \circ d$, we call φ a morphism of DGAs. If a morphism of DGAs induces a cohomology isomorphism, we call it a quasi-isomorphism.

DEFINITION 5.2. A and B are weakly equivalent if there is a finite diagram of DGAs

$$A \leftarrow C_1 \rightarrow C_2 \leftarrow \cdots \leftarrow C_n \rightarrow B$$

such that all the morphisms are quasi-isomorphisms.

Let M be a smooth manifold. The de Rham complex $A^*(M)$ of M is a DGA. The cohomology algebra $H^*(M, \mathbb{R})$ is a DGA with $d = 0$.

DEFINITION 5.3. A smooth manifold M is formal if $A^*(M)$ and $H^*(M, \mathbb{R})$ are weakly equivalent.

Proposition 5.4. *Let Γ be a torsion-free virtually polycyclic group. If the unipotent hull U_Γ is abelian, the standard Γ -manifold M_Γ is formal.*

Proof. We use same notations as in Section 3. If the k -unipotent hull of Γ is abelian, $(\bigwedge u^*, d) = (\bigwedge u^*, 0)$. By Theorem 3.2, we have the diagram of DGAs

$$A^*(M_\Gamma) \leftarrow \left(\left(\bigwedge u^* \right)^T \right) = H^*(M_\Gamma)$$

such that the map $A^*(M_\Gamma) \leftarrow \left(\left(\bigwedge u^* \right)^T \right)$ is a quasi-isomorphism. Hence the proposition follows. □

Hence we have Proposition 1.1.

5.2. The hard Lefschetz property. We review the definition of the hard Lefschetz property and prove Proposition 1.4.

DEFINITION 5.5. Let (M, ω) be a $2n$ -dimensional symplectic manifold. We say that (M, ω) satisfies the *hard Lefschetz property* if the linear map

$$[\omega^{n-i}] \wedge : H^i(M, \mathbb{R}) \rightarrow H^{2n-i}(M, \mathbb{R})$$

is an isomorphism for any $0 \leq i \leq n$.

Proof of Proposition 1.4. As in the proof of Proposition 1.1, we have an isomorphism $(\bigwedge u^*)^T \cong H^*(M, \mathbb{R})$. Consider the cohomology class of a symplectic form ω on M . We have $\omega_0 \in (\bigwedge^2 u^*)^T$ which represents the cohomology class $[\omega] \in H^2(M, \mathbb{R})$. Since $\omega_0^n \neq 0$ for $2n = \dim u = \dim M$, ω_0 is a symplectic form on the vector space u . Since the linear map

$$\omega_0^{n-i} \wedge : \bigwedge u^i \rightarrow \bigwedge u^{2n-i}$$

is injective for any $0 \leq i \leq n$ by the hard Lefschetz property of a torus, the restriction

$$\omega_0^{n-i} \wedge : \left(\bigwedge u^i \right)^T \rightarrow \left(\bigwedge u^{2n-i} \right)^T$$

is also injective and so

$$[\omega^{n-i}] \wedge : H^i(M, \mathbb{R}) \rightarrow H^{2n-i}(M, \mathbb{R})$$

is injective and thus it is an isomorphism by the Poincaré duality. Hence we have the proposition. □

6. Examples

EXAMPLE 1. Let $G = \mathbb{R} \ltimes_{\phi} \mathbb{R}^2$ with $\phi(t) = \begin{pmatrix} e^{rt} & 0 \\ 0 & e^{-rt} \end{pmatrix}$. Then for some $r \neq 0$, $\phi(1)$ is conjugate to an element of $SL_2(\mathbb{Z})$. Hence we have a lattice $\Gamma = \mathbb{Z} \ltimes \mathbb{Z}^2$. $G \times \mathbb{R}$ has a left-invariant symplectic form. In [10] (see also [20]) by direct computations Fernandez and Gray showed that $G/\Gamma \times S^1$ is formal and satisfies the hard Lefschetz property and admits no Complex structure. This is also a simple example for the result of this paper.

EXAMPLE 2. Let $G = \mathbb{C} \ltimes_{\phi} \mathbb{C}^2$ with $\phi(x) = \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix}$. Then the cochain complex $(\wedge \mathfrak{g}^*, d)$ of the Lie algebra of G is given by:

$$\begin{aligned} \mathfrak{g}^* &= \langle x_1, x_2, y_1, y_2, z_1, z_2 \rangle, \\ dx_1 &= dx_2 = 0, \\ dy_1 &= -x_1 \wedge y_1 + x_2 \wedge y_2, \quad dy_2 = -x_2 \wedge y_1 - x_1 \wedge y_2, \\ dz_1 &= x_1 \wedge z_1 - x_2 \wedge z_2, \quad dz_2 = x_1 \wedge z_2 + x_2 \wedge z_1. \end{aligned}$$

We have an invariant symplectic form $\omega = x_1 \wedge x_2 + z_1 \wedge y_1 + y_2 \wedge z_2$. For some $p, q \in \mathbb{R}$ $\phi(p\mathbb{Z} + \sqrt{-1}q\mathbb{Z})$ is conjugate to a subgroup of $SL_4(\mathbb{Z})$ and hence we have a lattice $\Gamma = (p\mathbb{Z} + \sqrt{-1}q\mathbb{Z}) \ltimes \Gamma''$ for a lattice Γ'' of \mathbb{C}^2 (see [15] and [13]). For any lattice Γ , G/Γ is complex, symplectic with the hard Lefschetz property and formal but not Kähler.

REMARK 4. For a Lie group G in Example 2, the de Rham cohomology of G/Γ depends on a choice of a lattice Γ . Under some conditions, the de Rham cohomology of a solvmanifold G/Γ is isomorphic to the cohomology of Lie algebra \mathfrak{g} of G (see [14], [21, Section 7]). But for a general solvmanifold G/Γ it is difficult to compute the de Rham cohomology of G/Γ . By the results of this paper, for a Lie group $G = \mathbb{R}^n \ltimes_{\phi} \mathbb{R}^m$ with the semi-simple action ϕ , we can say that G/Γ is formal and hard Lefschetz for any lattice Γ even if an isomorphism $H^*(G/\Gamma, \mathbb{R}) \cong H^*(\mathfrak{g})$ fails to hold.

EXAMPLE 3. Let $G = \mathbb{R} \ltimes_{\phi} \mathbb{R}^4$ with

$$\phi(t) = \begin{pmatrix} e^{pt} \cos(qt) & -e^{pt} \sin(qt) & 0 & 0 \\ e^{pt} \sin(qt) & e^{pt} \cos(qt) & 0 & 0 \\ 0 & 0 & e^{-pt} \cos(-qt) & -e^{-pt} \sin(-qt) \\ 0 & 0 & e^{-pt} \sin(-qt) & e^{-pt} \cos(-qt) \end{pmatrix}.$$

Then for p, q as Example 2, $\phi(1)$ is conjugate to an element of $SL_4(\mathbb{Z})$ and hence G has a lattice $\Gamma = \mathbb{Z} \ltimes \Gamma''$ for a lattice Γ'' of \mathbb{R}^4 . The cochain complex $(\wedge(\mathfrak{g} \oplus \mathbb{R})^*, d)$

of the Lie algebra of $G \times \mathbb{R}$ is given by:

$$\begin{aligned}
 (\mathfrak{g} \oplus \mathbb{R})^* &= \langle w, x_1, x_2, x_3, x_4, y \rangle, \\
 dx_1 &= -pw \wedge x_1 + qw \wedge x_2, \quad dx_2 = -qw \wedge x_1 - pw \wedge x_2, \\
 dx_3 &= pw \wedge x_3 - qw \wedge x_4, \quad dx_4 = qw \wedge x_3 + pw \wedge x_4.
 \end{aligned}$$

We have a left-invariant symplectic form $\omega = w \wedge y + x_1 \wedge x_3 + x_4 \wedge x_2$. We regard $w + \sqrt{-1}y, x_1 + \sqrt{-1}x_2, x_3 + \sqrt{-1}x_4$ as $(1,0)$ -forms, we obtain a left-invariant complex structure. By the result of this paper, for any lattice $\Gamma, G/\Gamma \times S^1$ is formal and any symplectic form on $G/\Gamma \times S^1$ satisfies the hard Lefschetz property.

REMARK 5. In [6], Bock studies formality and the hard Lefschetz property of solvmanifolds of dimension ≤ 6 by direct computations. The cohomology of G/Γ may vary for a choice of Γ and Bock does not decide whether $G/\Gamma \times S^1$ is formal and satisfies the hard Lefschetz property.

By combining the above examples we obtain:

EXAMPLE 4. Let $G = \mathbb{R}^2 \rtimes_{\phi} \mathbb{R}^{2k+4(l+m+n)}$ such that

$$\begin{aligned}
 \phi(t_1, t_2) &= \bigoplus_{i=1}^k \begin{pmatrix} \cos a_i t_1 & -\sin a_i t_1 \\ \sin a_i t_1 & \cos a_i t_1 \end{pmatrix} \\
 &\quad \bigoplus_{i=1}^l \begin{pmatrix} e^{b_i t_1} & 0 & 0 & 0 \\ 0 & e^{-b_i t_1} & 0 & 0 \\ 0 & 0 & e^{b_i t_1} & 0 \\ 0 & 0 & 0 & e^{-b_i t_1} \end{pmatrix} \\
 &\quad \bigoplus_{i=1}^m \begin{pmatrix} e^{c_i t_1} \cos(d_i t_2) & -e^{c_i t_1} \sin(d_i t_2) & 0 & 0 \\ e^{c_i t_1} \sin(d_i t_2) & e^{c_i t_1} \cos(d_i t_2) & 0 & 0 \\ 0 & 0 & e^{-c_i t_1} \cos(-d_i t_2) & -e^{-c_i t_1} \sin(-d_i t_2) \\ 0 & 0 & e^{-c_i t_1} \sin(-d_i t_2) & e^{-c_i t_1} \cos(-d_i t_2) \end{pmatrix} \\
 &\quad \bigoplus_{i=1}^n \begin{pmatrix} e^{e_i t_1} \cos(f_i t_1) & -e^{e_i t_1} \sin(f_i t_1) & 0 & 0 \\ e^{e_i t_1} \sin(f_i t_1) & e^{e_i t_1} \cos(f_i t_1) & 0 & 0 \\ 0 & 0 & e^{-e_i t_1} \cos(-f_i t_1) & -e^{-e_i t_1} \sin(-f_i t_1) \\ 0 & 0 & e^{-e_i t_1} \sin(-f_i t_1) & e^{-e_i t_1} \cos(-f_i t_1) \end{pmatrix}.
 \end{aligned}$$

We write $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ for matrices A, B .

We suppose $a_i = 2\pi/K_i$ for $K_i = 2, 3, 4,$ or $6, b_i = rL_i$ for r as Example 1 and $L_i \in \mathbb{Z}, c_i = pM_i, d_i = qM'_i, e_i = pN_i$ and $f_i = qN'_i$ for p, q as Example 2 and $M_i, M'_i, N_i, N'_i \in \mathbb{Z}$. Then each component of $\phi(\mathbb{Z}^2)$ for the direct product is conjugate to a subgroup of $SL_2(\mathbb{Z})$ or $SL_4(\mathbb{Z})$ and hence we have a lattice $\Gamma = \mathbb{Z}^2 \rtimes \Gamma''$ for a

lattice Γ'' of $\mathbb{R}^{2k+4(l+m+n)}$. The cochain complex $(\wedge \mathfrak{g}^*, d)$ of the Lie algebra of G is given by:

$$\begin{aligned} \mathfrak{g}^* &= \langle u_1, u_2, w_1, \dots, w_{2k}, x_1, \dots, x_{4l}, y_1, \dots, y_{4m}, z_1, \dots, z_{4n} \rangle, \\ du_1 &= du_2 = 0, \\ dw_{2i-1} &= a_i u_1 \wedge w_{2i}, \quad dw_{2i} = -a_i u_1 \wedge w_{2i-1} \quad (1 \leq i \leq k), \\ dx_{2i-1} &= -b_i u_1 \wedge x_{2i-1}, \quad dx_{2i} = b_i u_1 \wedge x_{2i} \quad (1 \leq i \leq 2l), \\ dy_{4i-3} &= -c_i u_1 \wedge y_{4i-3} + d_i u_2 \wedge y_{4i-2}, \quad dy_{4i-2} = -d_i u_2 \wedge y_{4i-3} - c_i u_1 \wedge y_{4i-2}, \\ dy_{4i-1} &= c_i u_1 \wedge y_{4i-1} - d_i u_2 \wedge y_{4i}, \quad dy_{4i} = d_i u_2 \wedge y_{4i-1} + c_i u_1 \wedge y_{4i} \quad (1 \leq i \leq m), \\ dz_{4i-3} &= -e_i u_1 \wedge z_{4i-3} + f_i u_1 \wedge z_{4i-2}, \quad dz_{4i-2} = -f_i u_1 \wedge z_{4i-3} - e_i u_1 \wedge z_{4i-2}, \\ dz_{4i-1} &= e_i u_1 \wedge z_{4i-1} - f_i u_1 \wedge z_{4i}, \quad dz_{4i} = f_i u_1 \wedge z_{4i-1} + e_i u_1 \wedge z_{4i} \quad (1 \leq i \leq n). \end{aligned}$$

G has a left-invariant symplectic form

$$\begin{aligned} \omega &= u_1 \wedge u_2 + \sum_{i=1}^k w_{2i-1} \wedge w_{2i} + \sum_{i=1}^{2l} x_{2i-1} \wedge x_{2i} \\ &+ \sum_{i=1}^m (y_{4i-3} \wedge y_{4i-1} + y_{4i} \wedge y_{4i-2}) + \sum_{i=1}^n (z_{4i-3} \wedge z_{4i-1} + z_{4i} \wedge z_{4i-2}). \end{aligned}$$

Regarding

$$\begin{aligned} &u_1 + \sqrt{-1}u_2, \\ &w_{2i-1} + \sqrt{-1}w_{2i} \quad (1 \leq i \leq k), \\ &x_{4i-3} + \sqrt{-1}x_{4i-1}, \quad x_{4i-2} + \sqrt{-1}x_{4i} \quad (1 \leq i \leq l), \\ &y_{2i-1} + \sqrt{-1}y_{2i} \quad (1 \leq i \leq 2m), \\ &z_{2i-1} + \sqrt{-1}z_{2i} \quad (1 \leq i \leq 2n) \end{aligned}$$

as $(1, 0)$ -forms, we have a left-invariant complex structure on G . By the results of this paper, for any lattice Γ of G , G/Γ is formal and satisfies the hard Lefschetz property but admits no Kähler structure.

EXAMPLE 5 (Oeljeklaus–Toma manifolds). We apply the result of this paper to non-Kähler complex manifolds constructed by Oeljeklaus and Toma in [17]. Let K be a finite extension field of \mathbb{Q} with the degree $s + 2t$ for positive integers s, t . Suppose K admits embeddings $\sigma_1, \dots, \sigma_s, \sigma_{s+1}, \dots, \sigma_{s+2t}$ into \mathbb{C} such that $\sigma_1, \dots, \sigma_s$ are real embeddings and $\sigma_{s+1}, \dots, \sigma_{s+2t}$ are complex ones satisfying $\sigma_{s+i} = \bar{\sigma}_{s+i+t}$ for $1 \leq i \leq t$. We can choose K admitting such embeddings (see [17]). Denote \mathcal{O}_K the ring of

$H^2(X(K, U), \mathbb{R})$ is spanned by $\{[dt_i \wedge dt_j]\}_{1 \leq i < j \leq s}$ and hence simple type OT-manifolds admit no symplectic structure.

EXAMPLE 6. Infra-solvmanifolds appear in study of geometries of 3-manifolds. See [23] for the general theory of geometries of 3-manifolds. A compact aspherical 3-manifold M with the virtually solvable fundamental group admits a one of the three geometries E^3, Nil, Sol i.e. M is diffeomorphic to G/Γ such that G is $\mathbb{R}^3, \mathbf{U}_3(\mathbb{R})$ or $\mathbb{R} \rtimes_{\phi} \mathbb{R}^2$ as an Example 1 with a left-invariant metric and $\Gamma \subset C \times G$ is a lattice for the group C of isometric automorphisms of G . In the E^3 case, Γ is virtually abelian by Bieberbach’s first theorem. In the Sol case, C is finite (see [23]). Hence a compact 3-manifold M admitting the geometry E^3 or Sol is formal.

7. Remarks

In this section we give an example of a formal standard Γ -manifold with the hard Lefschetz property such that U_{Γ} is not abelian. In addition this is also an example of formal manifold satisfying the hard Lefschetz property such that it is finitely covered by a non-formal manifold not satisfying the hard Lefschetz property. We notice that compact manifolds finitely covered by non-Kähler manifolds are not Kähler.

Let $\Gamma = \mathbb{Z} \rtimes_{\phi} \mathbb{Z}^2$ such that for $t \in \mathbb{Z}$

$$\phi(t) = \begin{pmatrix} (-1)^t & (-1)^t t \\ 0 & (-1)^t \end{pmatrix}.$$

Lemma 7.1. The algebraic hull of Γ is given by $\mathbf{H}_{\Gamma} = \{\pm 1\} \times \mathbf{U}_3(\mathbb{C})$ such that

$$(-1) \cdot \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & (-1)z \\ 0 & 1 & (-1)y \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. We have the inclusion

$$\Gamma \cong \left((-1)^x, \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right) \subset \{\pm 1\} \times \mathbf{U}_3(\mathbb{C}).$$

Then Γ is Zariski-dense in $\{\pm 1\} \times \mathbf{U}_3(\mathbb{C})$ and $\text{rank } \Gamma = 3 = \dim \mathbf{U}_3(\mathbb{C})$. Since the action of $\{\pm 1\}$ on $\mathbf{U}_3(\mathbb{C})$ is faithful, the centralizer of $\mathbf{U}_3(\mathbb{C})$ is contained in $\mathbf{U}_3(\mathbb{C})$. Hence the lemma follows. □

We have $\mathbf{H}_{\Gamma}(\mathbb{R}) = \{\pm 1\} \times U_{\Gamma}$ such that $U_{\Gamma} = \mathbf{U}_3(\mathbb{R})$. Let \mathfrak{u} be the Lie algebra of U_{Γ} . We have $\mathfrak{u} = \langle X_1, X_2, X_3 \rangle$ such that the bracket is given by

$$[X_1, X_2] = -[X_2, X_1] = X_3.$$

The $\{\pm 1\}$ -action on u is given by

$$(-1) \cdot X_1 = X_1, \quad (-1) \cdot X_i = -X_i, \quad i = 2, 3.$$

Let x_1, x_2, x_3 be the basis of u^* which is dual to X_1, X_2, X_3 . Then the DGA $(\wedge u^*)^{(\pm 1)}$ is the subalgebra of $\wedge u^*$ generated by $\{x_1, x_2 \wedge x_3\}$ and the derivation on $(\wedge u^*)^{(\pm 1)}$ is trivial. Let M_Γ be the standard Γ -manifold. Then by Theorem 3.2, we have the quasi-isomorphism $(\wedge u^*)^{(\pm 1)} \rightarrow A^*(M_\Gamma)$. Since the derivation on $(\wedge u^*)^{(\pm 1)}$ is trivial, we have the isomorphism $(\wedge u^*)^{(\pm 1)} \cong H^*(M)$. Hence we have:

Proposition 7.2. M_Γ is formal.

REMARK 8. Since U_Γ is not abelian, the converse of Proposition 5.4 is not true.

REMARK 9. We have the finite index subgroup $2\mathbb{Z} \times \mathbb{Z}^2$ which is nilpotent. So Γ is virtually nilpotent but not virtually abelian. By the result of [11], $K(2\mathbb{Z} \times \mathbb{Z}^2, 1)$ is not formal. But for the finite extension group Γ , $K(\Gamma, 1)$ is formal.

REMARK 10. Since $\{\pm 1\}$ acts isometrically on U_Γ with the invariant metric, M_Γ admits the *Nil* geometry. So we have a formal 3-dimensional compact manifold admitting the *Nil* geometry.

Let $\Delta = \Gamma \times \mathbb{Z}$. Then we have $H_\Delta = H_\Gamma \times \mathbb{R}$ and $U_\Delta = U_\Gamma \times \mathbb{R}$. As above we have the quasi-isomorphism inclusion $(\wedge u^*)^{(\pm 1)} \otimes \wedge(y) \subset A^*(M_\Delta)$. Let $\omega = x_1 \wedge y + x_2 \wedge x_3$. Then ω is a symplectic form on M_Δ . Since $H^1(M_\Delta, \mathbb{R}) \cong \langle x_1, y \rangle$ and $H^3(M_\Delta, \mathbb{R}) \cong \langle x_1 \wedge x_2 \wedge x_3, x_2 \wedge x_3 \wedge y \rangle$, the linear map $[\omega] \wedge : H^1(M_\Delta, \mathbb{R}) \rightarrow H^3(M_\Delta, \mathbb{R})$ is an isomorphism and hence we have the following proposition.

Proposition 7.3. $M_\Gamma \times S^1$ satisfies the hard Lefschetz property.

REMARK 11. Δ is a finite extension group of the non-abelian nilpotent group $2\mathbb{Z} \times \mathbb{Z}^2 \times \mathbb{Z}$ as Remark 9. By the result of [5], a compact $K(2\mathbb{Z} \times \mathbb{Z}^2 \times \mathbb{Z}, 1)$ -manifold is not a Lefschetz 4-manifold. Thus M_Δ is a example of a Lefschetz 4-manifold with non-Lefschetz finite covering space. In [16, Example 3.4], Lin showed the existence of Lefschetz 4-manifolds with non-Lefschetz finite covering space. M_Δ is a simpler and more constructive example.

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