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A canonical cellular decomposition of the Teichmüller space of compact surfaces with boundary

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A canonical cellular decomposition of the Teichmüller space of compact surfaces with boundary

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Abstract

Using the Euclidean decomposition of the hyperbolic surface, R. C. Penner gave a canonical cellular decomposition of the decorated Teichmüller space of punctured surfaces, which is invariant by the action of the mapping class group. Adapting his method, we give a canonical cellular decomposition of the Teichmüller space of compact orientable surfaces with non-empty boundary.

1 Introduction

R. C. Penner introduced in [Pe] a method for dividing the decorated Teichmüller space of punctured surfaces by "natural" cells. Here, natural means that the decomposition is invariant by the action of the mapping class group. In his method, the Euclidean decomposition of punctured surfaces introduced in [EP] plays an important role. Since then, it has been tried to extend his construction to the Teichmüller space of other kinds of surfaces. M. Näätänen obtained a cellular decomposition of the Teichmüller space of closed surfaces with a distinguished point in [Nä]. In her study, the decomposition of such surfaces introduced in [NP] plays a role of Euclidean decomposition in Penner's work. M. Kälin tried to construct a cellular decomposition of the Teichmüller space of surfaces with boundary in [Kä], but this study was not completely finished. We note that S. Kojima introduced in [Ko] a canonical method to decompose compact hyperbolic manifolds with non-empty geodesic boundary into truncated polyhedra. In this paper, using this decomposition and Penner's method, we give a canonical cellular decomposition of the Teichmüller space of compact orientable surfaces with non-empty boundary (see Theorem 6.6).

This paper is organized as follows. We recall in Section 2 the basic facts about Minkowski three-space and hyperbolic geometry. We develop in Section 3 most of our technical machinery on the geometry of the hyperboloid of one sheet in Minkowski three-space. In Section 4, we recall the definition of the Teichmüller space and give two parameterizations of it. One is called the *s-length* (see Theorem 4.1), and the other is called the *h-length* (see Proposition 4.4). Section 5 is devoted to a brief review of the decomposition obtained from the "convex hull construction" in [Ko]. Finally, using this decomposition together with the h-length parameterization, we give in Section 6 a canonical cellular decomposition of the Teichmüller space in question (see Theorem 6.6).

2 Minkowski space and hyperbolic geometry

The *Minkowski three-space* $\mathbf{E}^{1,2}$ is the real vector space $\mathbf{R}^3$ of dimension three with the Lorentz metric $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2$. The *Lorentzian norm* of a vector $\boldsymbol{x}$ in $\mathbf{E}^{1,2}$ is defined to be the complex number $\sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle}$. If the Lorentzian norm of $\boldsymbol{x}$ is zero (resp. positive, imaginary), then $\boldsymbol{x}$ is said to be *light-like* (resp. *space-like*, *time-like*). The set $H_T^+ = \{ \boldsymbol{x} \in \mathbf{E}^{1,2} \,|\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 > 0 \}$ forms the *hyperboloid model* of the two-dimensional hyperbolic space $\mathbf{H}^2$. Let $L = \{ \boldsymbol{x} \in \mathbf{E}^{1,2} \,|\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0 \}$. A ray from the origin $\boldsymbol{o} = (0,0,0)$ of $\mathbf{E}^{1,2}$ in the positive light cone $L^+ = \{ \boldsymbol{x} \in L \,|\, x_0 > 0 \}$ corresponds to a point on the ideal boundary of $\mathbf{H}^2$. The set of such rays forms the circle at infinity and we denote it by S^1_∞.

The group of linear isomorphisms of $\mathbf{E}^{1,2}$ preserving the Lorentz metric, the orientation on $\mathbf{E}^{1,2}$, and the sheet H_T^+ is denoted by $\mathrm{SO}^+(1,2)$. Then elements of $\mathrm{SO}^+(1,2)$ are classified into three types. A *hyperbolic* element of $\mathrm{SO}^+(1,2)$ has a pair of real positive eigenvalues λ (> 1) and λ^{-1} of which the eigenvectors lie in L^+ and a third eigenvector outside L^+ with eigenvalue 1. A hyperbolic element acts on $\mathbf{H}^2 \cup S^1_\infty$ with precisely two fixed points on S^1_∞. A *parabolic* transformation has a unique eigenvector on L^+ with eigenvalue 1 and no eigenvectors inside L^+. An *elliptic* transformation has all their eigenvalues on the unit circle and one eigenvector inside L^+. So it has a fixed point in $\mathbf{H}^2$.

Let $\mathcal{P}$ be the radial projection from $\mathbf{E}^{1,2} - \{x_0 = 0\}$ to the plane $\{x_0 = 1\}$, that is, for arbitrary point $\boldsymbol{x} = (x_0, x_1, x_2)$ with $x_0 \neq 0$, the coordinate of $\mathcal{P}(\boldsymbol{x})$ is defined by $\left(1, \frac{x_1}{x_0}, \frac{x_2}{x_0}\right)$. Then $\mathcal{P}\left(H_T^+\right)$ becomes the open unit disk $\mathbf{P}$ in $\{x_0 = 1\}$ with center $(1,0,0)$, and $\mathcal{P}$ gives an isometry of the hyperboloid model to the *projective (disk) model*. If $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{H}^2$ and d denotes the hyperbolic distance between the projections of $\boldsymbol{x}$ and $\boldsymbol{y}$ to $\mathbf{P}$, then

$$\cosh d = -\langle \boldsymbol{x}, \boldsymbol{y} \rangle .$$

The *hyperboloid of one sheet* $H_S \subset \mathbf{E}^{1,2}$ is defined to be $H_S = \{ \boldsymbol{x} \in \mathbf{E}^{1,2} \,|\, \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1 \}$. We note that $\mathcal{P}\left(H_T^+ \cup L^+ \cup H_S\right)$ covers $\{x_0 = 1\}$. For any linearly independent two points $\boldsymbol{u}_i$ and $\boldsymbol{u}_j$ in H_S with $\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle < -1$, we define their *s-length* s_{ij} as follows:

$$s_{ij} = \sqrt{-\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle + 1} .$$

We call an affine plane P in $\mathbf{E}^{1,2}$ a *plane*, and a plane through the origin *linear*. If $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ are distinct points in $\mathbf{E}^{1,2}$, then we denote by $\pi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ a plane through $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$. Suppose that a plane P is not linear, then we say that P is *elliptic* (*parabolic*, *hyperbolic*) if the conic section $P \cap L$ has the corresponding attribute. If $P = \{ \boldsymbol{x} \in \mathbf{E}^{1,2} \,|\, \langle \boldsymbol{x}, \boldsymbol{p} \rangle = \xi \}$ for some $0 \neq \boldsymbol{p} \in \mathbf{E}^{1,2}$ and $\xi \in \mathbf{R}$, then these cases correspond to $\langle \boldsymbol{p}, \boldsymbol{p} \rangle < 0$ (elliptic), $\langle \boldsymbol{p}, \boldsymbol{p} \rangle = 0$ (parabolic), and $\langle \boldsymbol{p}, \boldsymbol{p} \rangle > 0$ (hyperbolic), respectively. A linear plane P is said to be *time-like* if and only if P have a time-like vector, *space-like* if and only if every nonzero vector in P is space-like, or *light-like* otherwise. Suppose P is a time-like linear plane, and let R be a half space in $\mathbf{E}^{1,2}$ bounded by P. Then we can associate a unique unit vector $\boldsymbol{w} \in H_S$ so that $\langle \boldsymbol{w}, \boldsymbol{q} \rangle \leq 0$ for arbitrary $\boldsymbol{q} \in R$. This establishes a duality between half spaces in $\mathbf{E}^{1,2}$ and points on H_S. Now, for an arbitrary $\boldsymbol{v} \in H_S$, we denote by $P_{\boldsymbol{v}}$ (resp. $R_{\boldsymbol{v}}$) the linear plane (resp. the half space) defined as above, that is,

$$P_{\boldsymbol{v}} = \{ \boldsymbol{x} \in \mathbf{E}^{1,2} \,|\, \langle \boldsymbol{x}, \boldsymbol{v} \rangle = 0 \} ,$$

$$R_{\boldsymbol{v}} = \{ \boldsymbol{x} \in \mathbf{E}^{1,2} \,|\, \langle \boldsymbol{x}, \boldsymbol{v} \rangle \leq 0 \} .$$

Finally we define two subsets of $\mathbf{R}$ as follows:

$$\begin{aligned} \mathbf{R}_+ &= \{t \in \mathbf{R} \mid t > 0\}\,, \\ \mathbf{R}_s &= \left\{t \in \mathbf{R} \,\middle|\, t > \sqrt{2}\right\}. \end{aligned}$$

We note that $\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle < -1$, where $\boldsymbol{u}_i, \boldsymbol{u}_j \in H_S$, if and only if $s_{ij} \in \mathbf{R}_s$.

3 The geometry of the hyperboloid of one sheet

The first lemma is obtained from [Ra, Theorem 3.2.7, 3.2.8].

Lemma 3.1 *Suppose that $\boldsymbol{u}_1, \boldsymbol{u}_2 \in H_S$ are vectors with $\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle < -1$. Then $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are linearly independent, the linear plane $\pi(\boldsymbol{o}, \boldsymbol{u}_1, \boldsymbol{u}_2)$ is time-like, the geodesics $P_{\boldsymbol{u}_1} \cap \mathbf{P}$ and $P_{\boldsymbol{u}_2} \cap \mathbf{P}$ are disjoint, and the shortest path between them are uniquely obtained from $\pi(\boldsymbol{o}, \boldsymbol{u}_1, \boldsymbol{u}_2) \cap \{x_0 = 1\}$. Moreover, the distance d between $P_{\boldsymbol{u}_1} \cap \mathbf{P}$ and $P_{\boldsymbol{u}_2} \cap \mathbf{P}$ are obtained as follows:*

$$\cosh d = -\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle\,.$$

Furthermore, we have the following relation between d and s_{12}:

$$s_{12} = \sqrt{2}\cosh\frac{d}{2}\,.$$

□

Suppose $\{\boldsymbol{u}_i\}_{i=1}^3 \subset H_S$ are vectors with $\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle < -1$ for $\{i, j\} \subset \{1, 2, 3\}$ and $i \neq j$. Then $\{\boldsymbol{u}_i\}_{i=1}^3$ are linearly independent and, by Lemma 3.1, six points $p_{12} = P_{\boldsymbol{u}_1} \cap \pi(\boldsymbol{o}, \boldsymbol{u}_1, \boldsymbol{u}_2) \cap \mathbf{P}$, $p_{21} = P_{\boldsymbol{u}_2} \cap \pi(\boldsymbol{o}, \boldsymbol{u}_1, \boldsymbol{u}_2) \cap \mathbf{P}$, $p_{23} = P_{\boldsymbol{u}_2} \cap \pi(\boldsymbol{o}, \boldsymbol{u}_2, \boldsymbol{u}_3) \cap \mathbf{P}$, $p_{32} = P_{\boldsymbol{u}_3} \cap \pi(\boldsymbol{o}, \boldsymbol{u}_2, \boldsymbol{u}_3) \cap \mathbf{P}$, $p_{31} = P_{\boldsymbol{u}_3} \cap \pi(\boldsymbol{o}, \boldsymbol{u}_1, \boldsymbol{u}_3) \cap \mathbf{P}$, $p_{13} = P_{\boldsymbol{u}_1} \cap \pi(\boldsymbol{o}, \boldsymbol{u}_1, \boldsymbol{u}_3) \cap \mathbf{P}$ form a right-angled hyperbolic convex hexagon in $\mathbf{P} \cap R_{\boldsymbol{u}_1} \cap R_{\boldsymbol{u}_2} \cap R_{\boldsymbol{u}_3}$ (see Figure 1).

Lemma 3.2 *Under the condition as above, the distance δ between p_{12} and p_{13} are obtained as follows:*

$$\cosh\delta = \frac{s_{23}^2 - s_{12}^2 - s_{13}^2 + s_{12}^2 s_{13}^2}{s_{12}\, s_{13}\sqrt{s_{12}^2 - 1}\sqrt{s_{13}^2 - 1}}\,.$$

Proof of Lemma 3.2. We denote by $d(\cdot,\cdot)$ the hyperbolic distance between two points in $\mathbf{P}$. Using Lemma 3.1, we have the following relations:

$$\cosh d(p_{ij}, p_{ji}) = -\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle = s_{ij}^2 - 1\,,$$

where $i, j \in \{1, 2, 3\}$ and $i < j$. By the law of cosines for right-angled hyperbolic hexagons (see [Ra, Theorem 3.5.13]), we have the following relation:

$$\cosh\delta = \cosh d(p_{12}, p_{13}) = \frac{\cosh d(p_{12}, p_{21})\cosh d(p_{13}, p_{31}) + \cosh d(p_{23}, p_{32})}{\sinh d(p_{12}, p_{21})\sinh d(p_{13}, p_{31})}\,.$$

Substitute the three relations above for this relation, and we obtain the desired equation.
□

We next show that if $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in H_S$, then the classification of $\pi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ can be expressed in terms of some linear conditions on their s-lengths.

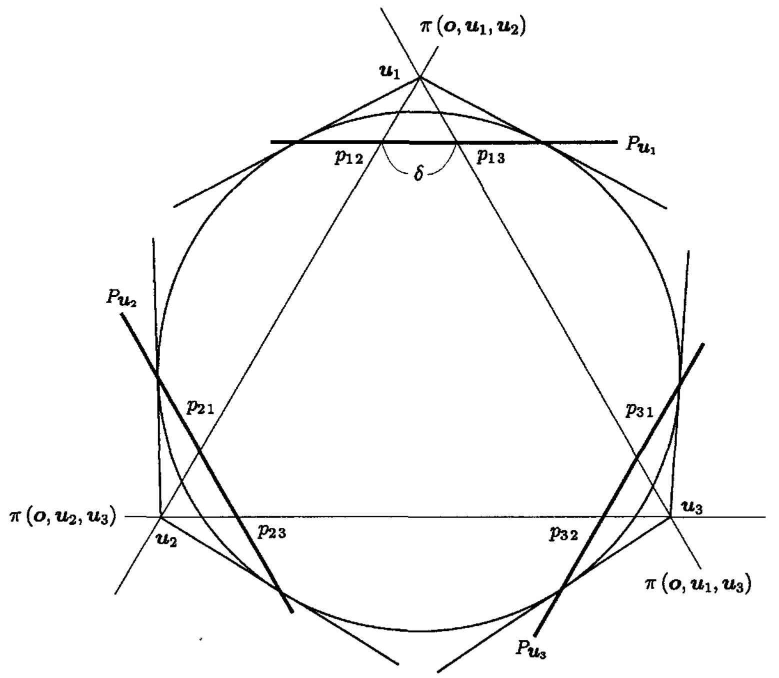


Figure 1: A hyperbolic convex hexagon in $\mathbf{P} \subset \{x_0 = 0\}$

Lemma 3.3 *Let $\{\boldsymbol{u}_i\}_{i=1}^{3} \subset H_S$ be given so that $\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle < -1$ for $i \neq j$. Then the plane $P = \pi(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)$ is elliptic if and only if the three strict triangle inequalities hold among s_{12}, s_{13}, s_{23}, P is parabolic if and only if $s_{ij} = s_{jk} + s_{ik}$ for some $\{i, j, k\} = \{1, 2, 3\}$, and P is hyperbolic if and only if some non-strict triangle inequality fails among s_{12}, s_{13}, s_{23}.*

Proof of Lemma 3.3. The tangent space to P is spanned by $\boldsymbol{v}_1 = \boldsymbol{u}_1 - \boldsymbol{u}_3$ and $\boldsymbol{v}_2 = \boldsymbol{u}_2 - \boldsymbol{u}_3$. Furthermore,

$$\begin{aligned} \langle \boldsymbol{v}_i, \boldsymbol{v}_i \rangle &= \langle \boldsymbol{u}_i - \boldsymbol{u}_3, \boldsymbol{u}_i - \boldsymbol{u}_3 \rangle \\ &= \langle \boldsymbol{u}_i, \boldsymbol{u}_i \rangle - 2 \langle \boldsymbol{u}_i, \boldsymbol{u}_3 \rangle + \langle \boldsymbol{u}_3, \boldsymbol{u}_3 \rangle \\ &= 2 - 2\left(1 - s_{i3}^2\right) \\ &= 2 s_{i3}^2 \quad \text{for} \quad i = 1, 2, \end{aligned}$$

$$\begin{aligned} \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle &= \langle \boldsymbol{u}_1 - \boldsymbol{u}_3, \boldsymbol{u}_2 - \boldsymbol{u}_3 \rangle \\ &= \langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle - \langle \boldsymbol{u}_1, \boldsymbol{u}_3 \rangle - \langle \boldsymbol{u}_2, \boldsymbol{u}_3 \rangle + \langle \boldsymbol{u}_3, \boldsymbol{u}_3 \rangle \\ &= \left(1 - s_{12}^2\right) - \left(1 - s_{13}^2\right) - \left(1 - s_{23}^2\right) + 1 \\ &= s_{13}^2 + s_{23}^2 - s_{12}^2 . \end{aligned}$$

The determinant of this form is

$$\begin{aligned} \begin{vmatrix} \langle \boldsymbol{v}_1, \boldsymbol{v}_1 \rangle & \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle \\ \langle \boldsymbol{v}_2, \boldsymbol{v}_1 \rangle & \langle \boldsymbol{v}_2, \boldsymbol{v}_2 \rangle \end{vmatrix} &= \begin{vmatrix} 2 s_{13}^2 & s_{13}^2 + s_{23}^2 - s_{12}^2 \\ s_{13}^2 + s_{23}^2 - s_{12}^2 & 2 s_{23}^2 \end{vmatrix} \\ &= -s_{12}^4 - s_{13}^4 - s_{23}^4 + 2 s_{12}^2 s_{13}^2 + 2 s_{12}^2 s_{23}^2 + 2 s_{13}^2 s_{23}^2 \end{aligned}$$

$$
\begin{aligned}
&= (s_{12}+s_{13}-s_{23})\,(s_{12}+s_{23}-s_{13})\,(s_{13}+s_{23}-s_{12}) \\
&\quad \times\,(s_{12}+s_{13}+s_{23})\,.
\end{aligned}
$$

At most one of these factors is not strictly positive, and the lemma follows. □

The next lemma provides the inductive step for our basic parameterization theorem. For an arbitrary linear subspace W in $\mathbf{E}^{1,2}$, we denote by $W^{\perp}$ its orthogonal complement with respect to $\langle\,\cdot,\cdot\,\rangle$.

Lemma 3.4 *If $\boldsymbol{u}_1,\boldsymbol{u}_2\in H_S$ are vectors with $\langle\,\boldsymbol{u}_1,\boldsymbol{u}_2\,\rangle<-1$ and $s_1,s_2\in\mathbf{R}_s$ are given, then there exists $\boldsymbol{v}_1,\boldsymbol{v}_2\in H_S$ with the following conditions:*

(1) They are separated by the linear plane $\pi\,(\boldsymbol{o},\boldsymbol{u}_1,\boldsymbol{u}_2)$.

(2) $-\langle\,\boldsymbol{u}_1,\boldsymbol{v}_i\,\rangle+1=s_2^2$ and $-\langle\,\boldsymbol{u}_2,\boldsymbol{v}_i\,\rangle+1=s_1^2$ for $i=1,2$.

Proof of Lemma 3.4. The linear plane $\pi\,(\boldsymbol{o},\boldsymbol{u}_1,\boldsymbol{u}_2)$ is time-like, so $\pi\,(\boldsymbol{o},\boldsymbol{u}_1,\boldsymbol{u}_2)^{\perp}$ intersects H_S. Let $\boldsymbol{e}$ be a vector in $\pi\,(\boldsymbol{o},\boldsymbol{u}_1,\boldsymbol{u}_2)^{\perp}$ with $\langle\,\boldsymbol{e},\boldsymbol{e}\,\rangle=1$. We solve for

$$
\boldsymbol{v}_i=\alpha_1\,\boldsymbol{u}_1+\alpha_2\,\boldsymbol{u}_2+\beta\,\boldsymbol{e}\quad\text{for}\quad i=1,2\,,
$$

where α_1,α_2 and β are unknowns. We find that

$$
\begin{aligned}
1-s_1^2 &= \langle\,\boldsymbol{u}_2,\boldsymbol{v}_i\,\rangle = \alpha_1\,\langle\,\boldsymbol{u}_1,\boldsymbol{u}_2\,\rangle+\alpha_2\,\langle\,\boldsymbol{u}_2,\boldsymbol{u}_2\,\rangle+\beta\,\langle\,\boldsymbol{u}_2,\boldsymbol{e}\,\rangle \\
&= \alpha_1\left(1-s_{12}^2\right)+\alpha_2\,,
\end{aligned}
$$

$$
\begin{aligned}
1-s_2^2 &= \langle\,\boldsymbol{u}_1,\boldsymbol{v}_i\,\rangle = \alpha_1\,\langle\,\boldsymbol{u}_1,\boldsymbol{u}_1\,\rangle+\alpha_2\,\langle\,\boldsymbol{u}_1,\boldsymbol{u}_2\,\rangle+\beta\,\langle\boldsymbol{u}_1,\boldsymbol{e}\rangle \\
&= \alpha_1+\alpha_2\left(1-s_{12}^2\right)\,.
\end{aligned}
$$

Since $s_{12}>\sqrt{2}$, we have

$$
\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}=\frac{1}{s_{12}^2\,(s_{12}^2-2)}\begin{pmatrix}s_1^2+s_2^2-s_{12}^2+s_1^2\,(s_{12}^2-2)\\ s_1^2+s_2^2-s_{12}^2+s_2^2\,(s_{12}^2-2)\end{pmatrix}.
$$

Furthermore, the condition $\langle\,\boldsymbol{v}_i,\boldsymbol{v}_i\,\rangle=1$ gives

$$
\begin{aligned}
1 = \langle\,\boldsymbol{v}_i,\boldsymbol{v}_i\,\rangle &= \langle\,\alpha_1\,\boldsymbol{u}_1+\alpha_2\,\boldsymbol{u}_2+\beta\,\boldsymbol{e}\,,\,\alpha_1\,\boldsymbol{u}_1+\alpha_2\,\boldsymbol{u}_2+\beta\,\boldsymbol{e}\,\rangle \\
&= \alpha_1^2+\alpha_2^2+\beta^2+2\,\alpha_1\,\alpha_2\left(1-s_{12}^2\right)\,.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\beta^2 &= 1-\alpha_1^2-\alpha_2^2-2\,\alpha_1\,\alpha_2\left(1-s_{12}^2\right) \\
&= \frac{\left(s_1^2+s_2^2-s_{12}^2\right)^2+2\,s_1^2\,s_2^2\,(s_{12}^2-2)}{s_{12}^2\,(s_{12}^2-2)}\,.
\end{aligned}
$$

Solving this, we have

$$
\beta=\pm\,\frac{\sqrt{\left(s_1^2+s_2^2-s_{12}^2\right)^2+2\,s_1^2\,s_2^2\,(s_{12}^2-2)}}{s_{12}\sqrt{s_{12}^2-2}}\,.
$$

Finally, the sign of β determines which side of $\pi\,(\boldsymbol{o},\boldsymbol{u}_1,\boldsymbol{u}_2)$ the vector $\boldsymbol{v}_i$ lies on. □

Suppose that P is a plane in $\mathbf{E}^{1,2}$ which does not contain the origin $\boldsymbol{o}$, so that $P=\{\,\boldsymbol{x}\in\mathbf{E}^{1,2}\,|\,\langle\,\boldsymbol{x},\boldsymbol{p}\,\rangle=-1\,\}$ for some $\boldsymbol{o}\neq\boldsymbol{p}\in\mathbf{E}^{1,2}$. We say that $\boldsymbol{y}\in\mathbf{E}^{1,2}$ lies *above* P if P separates $\boldsymbol{y}$ from $\boldsymbol{o}$ (i.e., $\langle\,\boldsymbol{y},\boldsymbol{p}\,\rangle<-1$).

Proposition 3.5 *Suppose that $\{\boldsymbol{u}_i\}_{i=1}^4 \subset H_S$ are so that any three are linearly independent, $\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle < -1$ for $i < j$, and that two points $\boldsymbol{u}_1, \boldsymbol{u}_4$ are separated by the linear plane $\pi(\boldsymbol{o}, \boldsymbol{u}_2, \boldsymbol{u}_3)$.*

(1) We have the inequality

$$s_{14}\, s_{23} \geq s_{12}\, s_{34} + s_{13}\, s_{24}\,.$$

(2) The point $\boldsymbol{u}_4$ lies above the plane $\pi(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)$

$$\iff \quad s_{24}\, s_{34}\left(s_{12}^2 + s_{13}^2 - s_{23}^2\right) + s_{12}\, s_{13}\left(s_{24}^2 + s_{34}^2 - s_{23}^2\right) > 0\,.$$

(3) The points $\{\boldsymbol{u}_i\}_{i=1}^4$ are coplanar

$$\begin{aligned} &\iff \quad s_{24}\, s_{34}\left(s_{12}^2 + s_{13}^2 - s_{23}^2\right) + s_{12}\, s_{13}\left(s_{24}^2 + s_{34}^2 - s_{23}^2\right) = 0 \\ &\iff \quad s_{14}\, s_{23} = s_{12}\, s_{34} + s_{13}\, s_{24}\,. \end{aligned}$$

Proof of Proposition 3.5. As before, let $\boldsymbol{e}$ be a vector in $\pi(\boldsymbol{o}, \boldsymbol{u}_2, \boldsymbol{u}_3)^\perp$ with $\langle \boldsymbol{e}, \boldsymbol{e} \rangle = 1$. We write $\boldsymbol{u}_1 = \beta\, \boldsymbol{e} + \alpha_2\, \boldsymbol{u}_2 + \alpha_3\, \boldsymbol{u}_3$, and three conditions $\langle \boldsymbol{u}_1, \boldsymbol{u}_1 \rangle = 1$, $\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle = 1 - s_{12}^2$ and $\langle \boldsymbol{u}_1, \boldsymbol{u}_3 \rangle = 1 - s_{13}^2$ gives

$$\begin{aligned} \alpha_2 &= \frac{A + s_{13}^2\, C}{s_{23}^2\, C}\,, \\ \alpha_3 &= \frac{A + s_{12}^2\, C}{s_{23}^2\, C}\,, \\ \beta &= \pm \frac{\sqrt{A^2 + 2\, C\, s_{12}^2\, s_{13}^2}}{s_{23}\sqrt{C}}\,, \end{aligned}$$

where $A = s_{12}^2 + s_{13}^2 - s_{23}^2$ and $C = s_{23}^2 - 2$. Similarly we have $\boldsymbol{u}_4 = \beta'\, \boldsymbol{e} + \alpha_2'\, \boldsymbol{u}_2 + \alpha_3'\, \boldsymbol{u}_3$, where

$$\begin{aligned} \alpha_2' &= \frac{B + s_{34}^2\, C}{s_{23}^2\, C}\,, \\ \alpha_3' &= \frac{B + s_{24}^2\, C}{s_{23}^2\, C}\,, \\ \beta' &= \mp \frac{\sqrt{B^2 + 2\, C\, s_{24}^2\, s_{34}^2}}{s_{23}\sqrt{C}}\,, \\ B &= s_{24}^2 + s_{34}^2 - s_{23}^2\,. \end{aligned}$$

Notice that $\beta\, \beta' < 0$, since $\boldsymbol{u}_1$ and $\boldsymbol{u}_4$ lie on different sides of the linear plane $\pi(\boldsymbol{o}, \boldsymbol{u}_2, \boldsymbol{u}_3)$. Thus

$$\beta\, \beta' = -\frac{\sqrt{A^2 + 2\, C\, s_{12}^2\, s_{13}^2}\sqrt{B^2 + 2\, C\, s_{24}^2\, s_{34}^2}}{s_{23}^2\, C}\,.$$

Now, compute

$$\begin{aligned} 1 - s_{14}^2 &= \langle \boldsymbol{u}_1, \boldsymbol{u}_4 \rangle \\ &= \beta\,\beta' + \alpha_2\,\alpha_2' + \alpha_2\,\alpha_3' \left(1 - s_{23}^2\right) + \alpha_2'\,\alpha_3 \left(1 - s_{23}^2\right) + \alpha_3\,\alpha_3' , \end{aligned}$$

and

$$\begin{aligned} -\beta\,\beta' &= (\alpha_2\,\alpha_2' + \alpha_3\,\alpha_3') + \left(1 - s_{23}^2\right)\,(\alpha_2\,\alpha_3' + \alpha_2'\,\alpha_3) + s_{14}^2 - 1 \\ &= \frac{-AB - C\,(s_{12}^2\,s_{34}^2 + s_{13}^2\,s_{24}^2 - s_{14}^2\,s_{23}^2)}{s_{23}^2\,C} , \end{aligned}$$

so

$$\begin{aligned} s_{14}^2\,s_{23}^2 &= s_{12}^2\,s_{34}^2 + s_{13}^2\,s_{24}^2 + \frac{AB + \sqrt{A^2 + 2C\,s_{12}^2\,s_{13}^2}\,\sqrt{B^2 + 2C\,s_{24}^2\,s_{34}^2}}{C} \\ &= s_{12}^2\,s_{34}^2 + s_{13}^2\,s_{24}^2 \\ &\quad + \frac{AB + \sqrt{(2C\,s_{12}\,s_{13}\,s_{24}\,s_{34} - AB)^2 + 2C\,(A\,s_{24}\,s_{34} + B\,s_{12}\,s_{13})^2}}{C} \\ &\geq s_{12}^2\,s_{34}^2 + s_{13}^2\,s_{24}^2 + \frac{AB + \sqrt{(2C\,s_{12}\,s_{13}\,s_{24}\,s_{34} - AB)^2}}{C} \\ &\geq s_{12}^2\,s_{34}^2 + s_{13}^2\,s_{24}^2 + \frac{AB + 2C\,s_{12}\,s_{13}\,s_{24}\,s_{34} - AB}{C} \\ &= (s_{12}\,s_{34} + s_{13}\,s_{24})^2 , \end{aligned}$$

that is,

$$s_{14}\,s_{23} \geq s_{12}\,s_{34} + s_{13}\,s_{24} ,$$

proving part (1).

For part (2), we may write

$$\pi\,(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3) = \{\boldsymbol{x} \in \mathbf{E}^{1,2} \mid \langle \boldsymbol{x}, \boldsymbol{p} \rangle = -1\}$$

for some $\boldsymbol{o} \neq \boldsymbol{p} \in \mathbf{E}^{1,2}$, since $\{\boldsymbol{u}_i\}_{i=1}^3$ are linearly independent. We write

$$\boldsymbol{p} = a\,\boldsymbol{e} + b\,\boldsymbol{u}_2 + c\,\boldsymbol{u}_3 ,$$

so that

$$\begin{aligned} -1 &= \langle \boldsymbol{p}, \boldsymbol{u}_2 \rangle &= b + c\,\langle \boldsymbol{u}_2, \boldsymbol{u}_3 \rangle , \\ -1 &= \langle \boldsymbol{p}, \boldsymbol{u}_3 \rangle &= b\,\langle \boldsymbol{u}_2, \boldsymbol{u}_3 \rangle + c , \end{aligned}$$

thus

$$b = c = \frac{-1}{1 + \langle \boldsymbol{u}_2, \boldsymbol{u}_3 \rangle} .$$

Furthermore,

$$\begin{aligned} -1 &= \langle \boldsymbol{p}, \boldsymbol{u}_1 \rangle = a\,\langle \boldsymbol{e}, \boldsymbol{u}_1 \rangle + b\,\langle \boldsymbol{u}_1, \boldsymbol{u}_2 \rangle + c\,\langle \boldsymbol{u}_1, \boldsymbol{u}_3 \rangle \\ &= a\,\beta - (\alpha_2 + \alpha_3) , \end{aligned}$$

so

$$a = \frac{\alpha_2 + \alpha_3 - 1}{\beta} = \frac{A}{\beta C}.$$

The condition

$$-1 > \langle \boldsymbol{p}, \boldsymbol{u}_4 \rangle = \frac{\beta' A}{\beta C} - \frac{s_{24}^2 + s_{34}^2 - 2}{C}$$

that $\boldsymbol{u}_4$ lies above $\pi(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)$ becomes

$$\begin{aligned} & B\sqrt{A^2 + 2C\, s_{12}^2\, s_{13}^2} > -A\sqrt{B^2 + 2C\, s_{24}^2\, s_{34}^2} \\ \Longleftrightarrow\ & A\, s_{24}\, s_{34} + B\, s_{12}\, s_{13} > 0, \end{aligned}$$

thus we proved part (2).

For part (3), the calculation above immediately gives the following equivalences:

$$\begin{aligned} & \text{The points } \{\boldsymbol{u}_i\}_{i=1}^4 \text{ are coplanar} \\ \Longleftrightarrow\ & B\sqrt{A^2 + 2C\, s_{12}^2\, s_{13}^2} = -A\sqrt{B^2 + 2C\, s_{24}^2\, s_{34}^2} \\ \Longleftrightarrow\ & A\, s_{24}\, s_{34} + B\, s_{12}\, s_{13} = 0 \\ \Longleftrightarrow\ & s_{24}\, s_{34}\left(s_{12}^2 + s_{13}^2 - s_{23}^2\right) + s_{12}\, s_{13}\left(s_{24}^2 + s_{34}^2 - s_{23}^2\right) = 0. \end{aligned}$$

Furthermore, using the calculation in the proof of part (1), we have the following equivalence:

$$\begin{aligned} & s_{24}\, s_{34}\left(s_{12}^2 + s_{13}^2 - s_{23}^2\right) + s_{12}\, s_{13}\left(s_{24}^2 + s_{34}^2 - s_{23}^2\right) = 0 \\ \Longleftrightarrow\ & s_{14}\, s_{23} = s_{12}\, s_{34} + s_{13}\, s_{24}, \end{aligned}$$

and we proved part (3). □

The next fact is technical and is used to give coordinates on the putative cells of our complex.

Proposition 3.6 *Suppose that $\{\boldsymbol{u}_i\}_{i=1}^n \subset H_S$ $(n \geq 4)$ satisfy the following conditions for $k = 1, 2, \ldots, n-3$:*

(1) For any two points of $\{\boldsymbol{u}_{i+k}\}_{i=0}^3$, their Lorentz metric is less than -1.

(2) Two points $\boldsymbol{u}_k$ and $\boldsymbol{u}_{k+3}$ are separated by the linear plane $\pi(\boldsymbol{o}, \boldsymbol{u}_{k+1}, \boldsymbol{u}_{k+2})$.

(3) The point $\boldsymbol{u}_{k+3}$ lies above the plane $\pi(\boldsymbol{u}_k, \boldsymbol{u}_{k+1}, \boldsymbol{u}_{k+2})$.

Then the point $\boldsymbol{u}_n$ lies above the plane $\pi(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)$.

Proof of Proposition 3.6. We proceed by induction on n, the case $n = 4$ being trivial. For the inductive step, we simply remove $\boldsymbol{u}_{n-1}$ from the sequence and must show that $\boldsymbol{u}_n$ lies above the plane $\pi(\boldsymbol{u}_{n-4}, \boldsymbol{u}_{n-3}, \boldsymbol{u}_{n-2})$, the other conditions being trivially satisfied.

Adopt the notation of Figure 2, where a symbol next to an edge indicates the corresponding inner product corresponding s-length. By Proposition 3.5, we have

(i) $$ab\left(c^2 + d^2 - e^2\right) + cd\left(a^2 + b^2 - e^2\right) > 0,$$

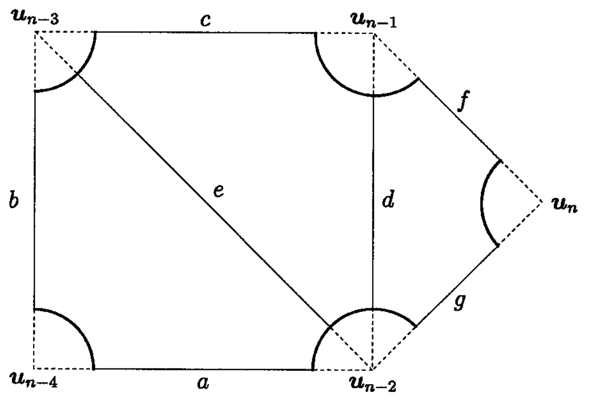


Figure 2: The s-lengths of edges

that is,

(ii) $$(a\,c + b\,d)\ (a\,d + b\,c) > e^2\ (a\,b + c\,d)\ ,$$

and

(iii) $$c\,e\ \left(f^2 + g^2 - d^2\right) + f\,g\ \left(c^2 + e^2 - d^2\right) > 0\,.$$

By Proposition 3.5 (2), we will prove

$$a\,b\ \left(s^2 + g^2 - e^2\right) + s\,g\ \left(a^2 + b^2 - e^2\right) > 0\,,$$

where $s = \sqrt{-\langle\, \boldsymbol{u}_{n-3}, \boldsymbol{u}_n \,\rangle + 1}$. We multiply both sides of the inequality above by d^2. So it is sufficient to show

$$a\,b\ \left(d^2\,s^2 + d^2\,g^2 - d^2\,e^2\right) + d\,s\,g\ \left(a^2\,d + b^2\,d - d\,e^2\right) > 0\,.$$

By Proposition 3.5 (1) and (3), we have $d\,s > c\,g + e\,f$. So we substitute it for the left side of the inequality above, and thus it is sufficient to show

$$\begin{aligned} &g^2\ (a\,c + b\,d)\ (a\,d + b\,c) + 2\,a\,b\,c\,e\,f\,g - c\,d\,e^2\,g^2 \\ &\quad + e^2\,a\,b\ \left(f^2 - d^2\right) + d\,e\,f\,g\ \left(a^2 + b^2 - e^2\right) \geq 0\,. \end{aligned}$$

The inequality (ii) gives a lower bound on the first term, so it is sufficient to show that

$$a\,b\,e^2\ \left(f^2 + g^2 - d^2\right) + d\,e\,f\,g\ \left(a^2 + b^2 - e^2\right) + 2\,a\,b\,c\,e\,f\,g \geq 0\,.$$

The inequality (iii) then gives a lower bound on the new first term, so it remains to show that

$$a\,b\ \left(c^2 + d^2 - e^2\right) + c\,d\ \left(a^2 + b^2 - e^2\right) \geq 0\,,$$

which follows from (i). $\square$

Remark

(1) Using the same method as above, we can omit the condition (iv) of Proposition 2.7 in [Pe].

(2) We can extend the statement of Lemma 3.3 into the following one:

Lemma. *Suppose that* $\{\boldsymbol{u}_i\}_{i=1}^3$ *are three linearly independent (non-zero) vectors in* $\mathbf{E}^{1,2}$*, and set*

$$s_{ij} = \sqrt{-\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle + \frac{\langle \boldsymbol{u}_i, \boldsymbol{u}_i \rangle + \langle \boldsymbol{u}_j, \boldsymbol{u}_j \rangle}{2}}.$$

Then the plane $P = \pi(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3)$ *is elliptic if and only if the three strict triangle inequalities hold among* s_{12}*,* s_{13}*,* s_{23}*,* P *is parabolic if and only if* $s_{ij} = s_{jk} + s_{ik}$ *for some* $\{i, j, k\} = \{1, 2, 3\}$*, and* P *is hyperbolic if and only if some non-strict triangle inequality fails among* s_{12}*,* s_{13}*,* s_{23}*.*

The proof of this lemma is the same as that of Lemma 3.3. We note that this lemma contains Lemma 2.2 in [Pe] and Lemma 3.1 in [Nä].

4 Coordinates on the Teichmüller space

Consider a closed orientable surface F_g of genus g with a subset $\{D_1, D_2, \ldots, D_r\}$ of disjoint closed disks on F_g, where $2g - 2 + r > 0$, and let $F_{g,r}$ be the closure of $F_g - \{D_1, D_2, \ldots, D_r\}$. We denote by $\mathcal{T}_{g,r}$ the Teichmüller space of $F_{g,r}$, that is the space of marked hyperbolic structures on $F_{g,r}$, where each boundary component $b_i = \partial D_i$, $i = 1, 2, \ldots, r$, becomes totally geodesic (see [Th, Section 4.6]). We restrict attention to the case where $r \geq 1$.

A point of $\mathcal{T}_{g,r}$ gives rise to an isomorphism $\pi_1(F_{g,r}) \longrightarrow \Gamma < \mathrm{SO}^+(1,2)$, where Γ is a marked discrete group defined up to conjugacy in $\mathrm{SO}^+(1,2)$. We will denote a marking on Γ by Γ_m. Then $\mathbf{H}^2/\Gamma_m$ is a marked complete hyperbolic surface of infinite volume, and its Nielsen kernel (see [Ab]) is homeomorphic to $F_{g,r}$. Thus each point of $\mathcal{T}_{g,r}$ gives the hyperbolic structure on $F_{g,r}$. Furthermore, it also gives the complete hyperbolic structure on $\hat{F}_{g,r}$, where $\hat{F}_{g,r}$ denotes a surface homeomorphic to $\mathbf{H}^2/\Gamma$. So, by the correspondence described above, we can naturally identify $\mathcal{T}_{g,r}$ with the Teichmüller space of $\hat{F}_{g,r}$, and we also denote it by $\mathcal{T}_{g,r}$ for convenience. We will also consider the corresponding group action on the projective disk $\mathbf{P}$. Let $\pi\colon \mathbf{P} \longrightarrow \hat{F}_{g,r}$ denote the universal cover with group Γ. Then $\pi^{-1}(F_{g,r})$ is a simply connected region bounded by geodesics. The hyperbolic metric on $\mathbf{P}$ projects to a metric on $F_{g,r}$ (resp. $\hat{F}_{g,r}$) which we refer to as the "Γ-metric" on $F_{g,r}$ (resp. $\hat{F}_{g,r}$). We refer to geodesics for the Γ-metric as "Γ-geodesics", etc.

We choose a hyperbolic element γ_i in Γ_m corresponding to a boundary component b_i. Let $\boldsymbol{z}_i$ be the point in H_S which is fixed by γ_i and induces a half space $R_{\boldsymbol{z}_i}$ including $\pi^{-1}(F_{g,r})$, and set $V_i = \Gamma \boldsymbol{z}_i$. Each point of V_i has a stabilizer in Γ which is generated by a hyperbolic element; different stabilizers for different points of V_i are conjugate in Γ. Now we fix a point Γ_m in $\mathcal{T}_{g,r}$. Let c be a homotopy class of path in $F_{g,r}$, not necessarily simple, running from b_i to b_j, where we may have $i = j$, and straighten c to a Γ-geodesic C. From now on, we assume that C is not entirely contained in a boundary component. Then it naturally induces a geodesic $\hat{C}$ in $\hat{F}_{g,r}$ having infinite length. We denote by $\hat{c}$ a homotopy class of $\hat{C}$. We call such a homotopy class in $F_{g,r}$ (resp. $\hat{F}_{g,r}$) a *seam* (resp. an *extended seam*). Now $\pi^{-1}(\hat{C})$ consists of infinitely many geodesics in $\mathbf{P}$, and we choose one of them. Then there is a line $\check{C}$ in $\{x_0 = 1\}$ containing the geodesic above, and it intersects each of $\mathcal{P}(V_i)$ and $\mathcal{P}(V_j)$ at one point, say $\mathcal{P}(\boldsymbol{u}_i)$ and $\mathcal{P}(\boldsymbol{u}_j)$ respectively, where $\boldsymbol{u}_i \in V_i$ and $\boldsymbol{u}_j \in V_j$. We note that π maps $\check{C} \cap R\boldsymbol{u}_i \cap R\boldsymbol{u}_j$ to C. Now we define the *s-length* of c or $\hat{c}$ (relative to Γ_m) by

$$s(c; \Gamma_m) = s(\hat{c}; \Gamma_m) := s_{ij} = \sqrt{-\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle + 1}\,.$$

This value depends only on the choice of c or $\hat{c}$, and naturally gives a continuous positive real-valued function defined on $\mathcal{T}_{g,r}$. We note that this s-length is an analogue to the "λ-length" in [Pe] and "L-length" in [NP, Nä, NN]. We next fix an appropriate finite number of seams $c_1, c_2, \ldots, c_q$ to obtain a map from $\mathcal{T}_{g,r}$ to $\mathbf{R}^q_s$, which will be shown to be a surjective homeomorphism.

Let $\triangle$ be a set of seams in $F_{g,r}$ with the following conditions: each arc is a disjointly embedded simple arc, and the closure of each complementary region of $\{c_i \mid c_i \in \triangle\}$ in $F_{g,r}$ is a hexagon. We call it a *truncated triangle*, and $\triangle$ a *truncated triangulation*. Euler characteristic considerations show that there are $q = 6g - 6 + 3r$ seams in $\triangle$. For a truncated triangle, its side obtained from a seam in $\triangle$ (resp. a boundary of $F_{g,r}$) is called an *edge* (resp. a *boundary*). A truncated triangle in $F_{g,r}$ induces a simply connected region in $\hat{F}_{g,r}$ bounded by three extended seams. We call this region an *infinite triangle*. We define a map $S_\triangle$ from $\mathcal{T}_{g,r}$ to $\mathbf{R}^q_s$, depending on a given truncated triangulation $\triangle$, as follows: for each point Γ_m in $\mathcal{T}_{g,r}$, the image $S_\triangle(\Gamma_m)$ is defined by $(s(c_1;\Gamma_m), s(c_2;\Gamma_m), \ldots, s(c_q;\Gamma_m))$. Now we show the following theorem.

Theorem 4.1 *If* $\triangle = (c_1, c_2, \ldots, c_q)$*, where* $q = 6g-6+3r$*, is a truncated triangulation of* $F_{g,r}$*, then*

$$S_\triangle : \mathcal{T}_{g,r} \longrightarrow \mathbf{R}^q_s$$

is a homeomorphism.

Proof of Theorem 4.1. We will find the inverse map to $\mathcal{T}_{g,r}$. So suppose that we are given $(s_1, s_2, \ldots, s_q) \in \mathbf{R}^q_s$ and wish to construct a surface. A universal cover of $\hat{F}_{g,r}$ is homeomorphic to the projective disk model $\mathbf{P}$, which is tesselated by infinite triangles with sides arising from the lifts of the extended seams $(\hat{c}_1, \hat{c}_2, \ldots, \hat{c}_q)$. Fix attention on one of these infinite triangles, and suppose that its sides correspond clockwise to extended seams $(\hat{c}_{\eta_1}, \hat{c}_{\eta_2}, \hat{c}_{\eta_3})$ (not necessarily distinct). By the existence theorem for right-angled hyperbolic hexagons (see [Ra, Theorem 3.5.14]), there exists a right-angled hyperbolic hexagon η in $\mathbf{P}$ with clockwise alternate sides of lengths $\cosh^{-1}(s^2_{\eta_1} - 1)$, $\cosh^{-1}(s^2_{\eta_2} - 1)$, $\cosh^{-1}(s^2_{\eta_3} - 1)$ respectively. Let π_i be the linear plane in $\mathbf{E}^{1,2}$ containing the side of η with length $\cosh^{-1}(s^2_{\eta_i} - 1)$, where $i \in \{1,2,3\}$, and $\boldsymbol{z}_i$ a point in $\pi_j \cap \pi_k \cap H_S$ with $\eta \subset R\boldsymbol{z}_i$, where $\{i,j,k\} = \{1,2,3\}$. We denote by ζ the Euclidean triangle in $\mathbf{E}^{1,2}$ with vertices $\boldsymbol{z}_1$, $\boldsymbol{z}_2$, $\boldsymbol{z}_3$. Then $\mathcal{P}(\zeta) \cap \mathbf{P}$ is an infinite triangle containing η.

We now inductively map further infinite triangles in $\mathbf{P}$ into Euclidean triangles in $\mathbf{E}^{1,2}$. Each infinite triangle in $\mathbf{P}$ has one side already mapped in, say with vertices $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in H_S. Then, using Lemma 3.4, we obtain the third vertex $\boldsymbol{u}_3$ of a new Euclidean triangle. The lemma gives two choices of points with the required Lorentz metric. Since one side of $\pi(\boldsymbol{o}, \boldsymbol{u}_1, \boldsymbol{u}_2)$ already contains points of the lifted triangles by induction, $\boldsymbol{u}_3$ must lie on the other side of $\pi(\boldsymbol{o}, \boldsymbol{u}_1, \boldsymbol{u}_2)$. In other words, $\boldsymbol{u}_3$ is determined by the condition that

$R\boldsymbol{u}_3$ contains $\pi(\boldsymbol{o}, \boldsymbol{u}_1, \boldsymbol{u}_2)$. This determines $\boldsymbol{u}_3$ uniquely, and we have a new Euclidean triangle in $\mathbf{E}^{1,2}$. By these inductive steps, we obtain Euclidean triangles $\mathcal{T}$ in $\mathbf{E}^{1,2}$. We denote by V ($\subset H_S$) the vertices of $\mathcal{T}$.

Each element $\beta \in \pi_1(F_{g,r}) \cong \pi_1(\hat{F}_{g,r})$ acts on the tesselation of the universal cover. Let T be one infinite triangle in $\pi^{-1}(\hat{F}_{g,r}) \approx \mathbf{P}$, and τ_1 (resp. τ_2) the Euclidean triangle in $\mathcal{T}$ corresponding to T (resp. βT). Then there is a unique $g(\beta) \in \mathrm{SO}^+(1,2)$ taking τ_1 to τ_2 mapping vertices correctly. From the inductive construction of $\mathcal{T}$, we see that the definition of $g(\beta)$ is independent of the choice of T. The same reason shows that

$$g\colon \pi_1(F_{g,r}) \cong \pi_1(\hat{F}_{g,r}) \longrightarrow \mathrm{SO}^+(1,2)$$

is a homomorphism.

To see that g is injective with a discrete image, note that the inductive construction above guarantees that $\mathcal{T}$ is mapped injectively to a tesselation of $\mathbf{P}$ by $\mathcal{P}$. If the image $\Gamma_m := g\big(\pi_1(F_{g,r})\big) \cong g\big(\pi_1(\hat{F}_{g,r})\big)$ were not discrete, there would be a non-trivial element arbitrary near the identity, and then triangles in $\mathbf{P}$ would overlap. Thus we have the injectivity of g.

To complete the discussion of the tesselation and group Γ_m, we claim that the image tesselation $\mathbf{T} = \mathcal{P}(\mathcal{T}) \cap \mathbf{P}$ actually covers all of $\mathbf{P}$. To this end, note first that the inductive construction of $\mathcal{T}$ guarantees that $\mathbf{T}$ is open in $\mathbf{P}$. We show also that $\mathbf{T}$ is close in $\mathbf{P}$. For each infinite triangle $\tau \in \mathbf{T}$, by Lemma 3.2, there is some $\varepsilon > 0$ so that each edge of τ has distance at least ε from other edges in $\mathbf{T}$. It follows easily that $\mathbf{T}$ is closed. Furthermore, from the inductive construction of $\mathcal{T}$, the tesselation $\mathbf{T}$ is connected. So connectivity of $\mathbf{P}$ guarantees that $\mathbf{T} = \mathbf{P}$. Thus we obtain the tesselation $\mathcal{P}|_{H_T^+}{}^{-1}(\mathbf{T})$ in $\mathbf{H}^2$.

The quotient of this tesselation by Γ_m is a marked complete hyperbolic surface of infinite volume, and

$$\left(\mathcal{P}|_{H_T^+}{}^{-1}(\mathbf{T}) \cap \bigcap_{\boldsymbol{b} \in V} R_{\boldsymbol{b}}\right) \Big/ \Gamma_m$$

is a marked compact hyperbolic surface with totally geodesic boundary. This gives our map from $\mathbf{R}_s^q$ to $\mathcal{T}_{g,r}$, and it is clearly inverse to $S_\triangle$. Thus the theorem is proved. □

The (full) mapping class group $\mathrm{MC}_{g,r}$ of isotopy class of (orientation preserving) homeomorphisms (which may permute the boundaries) acts on $\mathcal{T}_{g,r}$ in the natural way by change of marking. For an arbitrary element φ in $\mathrm{MC}_{g,r}$, we denote by φ_* the corresponding homeomorphism on $\mathcal{T}_{g,r}$. We note that, by Lemma 3.1, the s-length is a quantity with respect to hyperbolic metric. Thus we have the following theorem and corollary:

Theorem 4.2 *s-lengths are natural for the action of $\mathrm{MC}_{g,r}$ in the sense that if $\varphi \in \mathrm{MC}_{g,r}$, $\Gamma_m \in \mathcal{T}_{g,r}$, and c is a seam in $F_{g,r}$, then*

$$s(c; \Gamma_m) = s(\varphi\, c; \varphi_*\, \Gamma_m)\,.$$

□

Corollary 4.3 *Suppose that $\triangle$ is a truncated triangulation of $F_{g,r}$ and Σ is an assignment of numbers in $\mathbf{R}_s$ to the seams of $\triangle$ so that $(\triangle, \Sigma)$ determines the point $\Gamma_m \in \mathcal{T}_{g,r}$. If $\varphi \in \mathrm{MC}_{g,r}$, then φ induces a one-to-one correspondence between components of $\triangle$ and of $\varphi^{-1}\triangle$. If Σ' denotes the assignment of numbers to components of $\varphi^{-1}\triangle$ induced from Σ by φ, then $(\varphi^{-1}\triangle, \Sigma')$ determines the point $\varphi_*\,\Gamma_m \in \mathcal{T}_{g,r}$.* □

We close this section with yet another parameterization of $\mathcal{T}_{g,r}$. Fix a truncated triangulation $\triangle$ of $F_{g,r}$. Suppose that a truncated triangle T on $F_{g,r}$ has edges $\{a, b, e\} \subset \triangle$. The orientation on $T \subset F_{g,r}$ induces a cyclic ordering (a, b, e) on $\{a, b, e\}$ as in Figure 3. Then the boundary E is said to be *opposite* the edge e, and, in Figure 3 (1), the edge a is said to *abut* on the boundaries B and E.

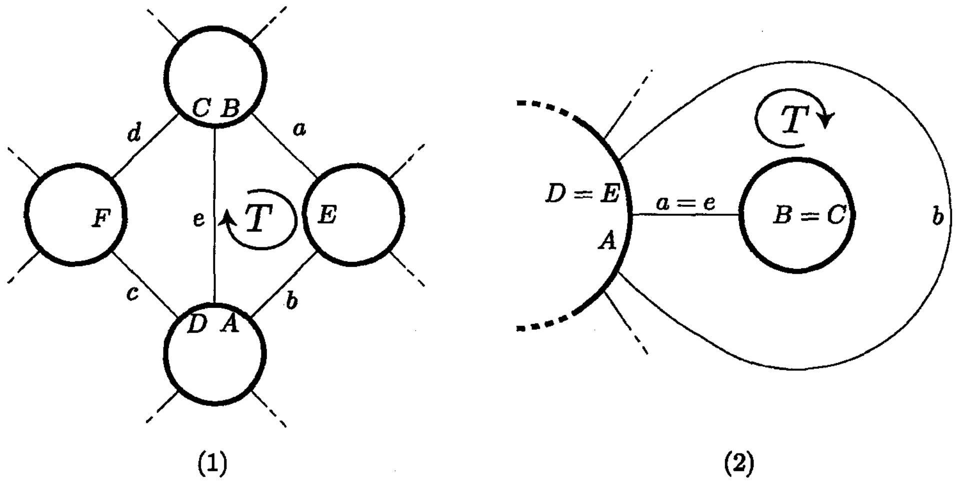


Figure 3: Sides of truncated triangles on $F_{g,r}$

Let $\mathcal{B}_\triangle$ be the set of boundaries of truncated triangles obtained from $\triangle$. Now we define a map

$$I_\triangle \colon \mathcal{T}_{g,r} \approx \mathbf{R}^q_s \longrightarrow \mathbf{R}^{2q}_+ = \mathbf{R}^{\mathcal{B}_\triangle}_+$$

and develop the corresponding parameterization of $\mathcal{T}_{g,r}$. To compute the coordinate entries in the target, suppose that T is a truncated triangle in $F_{g,r}$ with edges (a, b, e) and that the boundary $E \in \mathcal{B}_\triangle$ of T is opposite e. For arbitrary assignment Σ of s-lengths on the edges of $\triangle$, let Γ_m be a point of $\mathcal{T}_{g,r}$ corresponding to $(\triangle, \Sigma)$. Then the *h-length* of E for $(\triangle, \Sigma)$ is defined as follows:

$$h(E, \Gamma_m) = \frac{\Sigma(e)}{\Sigma(a)\,\Sigma(b)},$$

where $\Sigma(a)$ (resp. $\Sigma(b)$, $\Sigma(e)$) means the s-length of the seam a (resp. b, e) assigned by Σ. This defines the map $I_\triangle$.

We observe that

$$\Sigma(e)^{-2} = h(A, \Gamma_m)\, h(B, \Gamma_m),$$

so $I_\triangle$ is an embedding. We call the right side of this equation a *coupling* of e. Since $\Sigma(e)$ is greater than $\sqrt{2}$, the equation above induces the following inequality:

$$(0 <)\, h(A, \Gamma_m)\, h(B, \Gamma_m) < \frac{1}{2}.$$

We call this inequality a *coupling inequality* of e. Moreover, suppose e is a seam of $\triangle$, which abuts on four boundaries $A, B, C, D \in \mathcal{B}_\triangle$ as in Figure 3. Then the condition

$$h(A, \Gamma_m)\, h(B, \Gamma_m) = h(C, \Gamma_m)\, h(D, \Gamma_m)$$

is called the *coupling equation* of e. Now we summarize the consideration above.

Proposition 4.4 *The map $I_\triangle\colon \mathcal{T}_{g,r} \longrightarrow \mathbf{R}_+^{\mathcal{B}_\triangle}$ is an embedding of $\mathcal{T}_{g,r}$ into an intersection of homogeneous quadrics. Explicitly, $I_\triangle(\mathcal{T}_{g,r}) \subset \mathbf{R}_+^{\mathcal{B}_\triangle}$ is characterized by the coupling equations and the coupling inequalities.* $\square$

5 The convex hull construction

In this section, we give a brief review of the decomposition of a compact hyperbolic surface with non-empty totally geodesic boundary introduced by S. Kojima (see [Ko]).

We recall that $F_{g,r}$ is a compact orientable surface of genus g with r boundary components, where $r \geq 1$ and $2g-2+r>0$, and that $\mathcal{T}_{g,r}$ is its Teichmüller space. Then, for each point Γ_m in $\mathcal{T}_{g,r}$, the surface $F_{g,r}$ has a hyperbolic structure induced by Γ_m. Now each boundary component b_i of $F_{g,r}$ becomes totally geodesic. As in Section 4, let V_i be the set of points in H_S corresponding to $\pi^{-1}(b_i)$, and set $V = V_1 \cup V_2 \cup \cdots \cup V_r$. We note that $\pi^{-1}(F_{g,r})$ is identified with $\mathcal{P}\left(H_T^+ \cap (\bigcap_{\boldsymbol{b}\in V} R_{\boldsymbol{b}})\right)$. We call a point on a closed set X in $\mathbf{E}^{1,2}$ *visible* if the segment between the point and the origin $\boldsymbol{o}$ contains no other points in X. Let $\mathcal{H}_V$ be a closed (Euclidean) convex hull of V in $\mathbf{E}^{1,2}$. Then $\mathcal{H}_V$ does not contain the origin $\boldsymbol{o}$ (see [Ko, Lemma 4.2]), and the projection $\mathcal{P}(\mathcal{H}_V)$ contains $\mathbf{P}$ (see [Ko, Lemma 4.3]). Thus each ray must reach to a visible point on $\mathcal{H}_V$ if it passes through H_T^+. Furthermore, any visible point on $\mathcal{H}_V$ lies on a two-dimensional compact convex visible face of $\mathcal{H}_V$, and such a face lies in an elliptic plane (see [Ko, Proposition 4.6]).

Now, let $\mathcal{V}$ be the set of visible points on $\mathcal{H}_V$. Then it is shown in [Ko, Theorem 4.8] that the intersection of $\mathcal{P}(\mathcal{V})$ with $\pi^{-1}(F_{g,r})$ defines a Γ-invariant polygonal decomposition on $\pi^{-1}(F_{g,r})$. In particular, it induces a cellular decomposition of $F_{g,r}$. We denote by $\triangle(\Gamma_m)$ the collection of geodesics on $F_{g,r}$ arising from the edges of $\mathcal{V}$ as above. Then $\triangle(\Gamma_m)$ consists of a finite collection of simple geodesic arcs disjointly embedded in $F_{g,r}$ connecting boundaries. We also call these arcs *seams*. We call the isotopy class of a decomposition obtained as above a *truncated cellular decomposition* of $F_{g,r}$, and we also denote it by $\triangle(\Gamma_m)$ for convenience.

6 A cellular decomposition of the Teichmüller space

In this section, imitating the method in Section 5 of [Pe], we construct a cellular decomposition of $\mathcal{T}_{g,r}$. For a point Γ_m in $\mathcal{T}_{g,r}$, the convex hull construction of Section 5 determines a canonical truncated cellular decomposition $\triangle(\Gamma_m)$ of $F_{g,r}$. Conversely, if $\triangle$ is a fixed truncated cellular decomposition of $F_{g,r}$, then we define subsets $\mathring{\mathcal{C}}(\triangle)$ and $\mathcal{C}(\triangle)$ of $\mathcal{T}_{g,r}$ as follows:

$$\begin{aligned}\mathring{\mathcal{C}}(\triangle) &= \{\Gamma_m \in \mathcal{T}_{g,r} \mid \triangle(\Gamma_m) = \triangle\}\,,\\ \mathcal{C}(\triangle) &= \{\Gamma_m \in \mathcal{T}_{g,r} \mid \triangle(\Gamma_m) \subseteq \triangle\}\,.\end{aligned}$$

Our immediate goal is to characterize $\mathcal{C}(\triangle)$ and $\mathring{\mathcal{C}}(\triangle)$ in terms of s-lengths on $\triangle$ in the special case that $\triangle$ is a truncated triangulation of $F_{g,r}$ (see Theorem 6.1).

To establish notation, fix a seam e in a truncated triangulation $\triangle$, and consider a lift $\widetilde{e}$ of e to $\mathbf{P}$. The lift $\widetilde{e}$ separates two truncated triangles $\widetilde{Q}$ and $\widetilde{T}$ given by the lift $\widetilde{\triangle}$ of $\triangle$ to $\mathbf{P}$, and we adopt the notation of Figure 4 for their edges. It may be that $\pi(\widetilde{Q}) = \pi(\widetilde{T})$, where we recall that π is the canonical projection from $\mathbf{P}$ to $F_{g,r}$, and

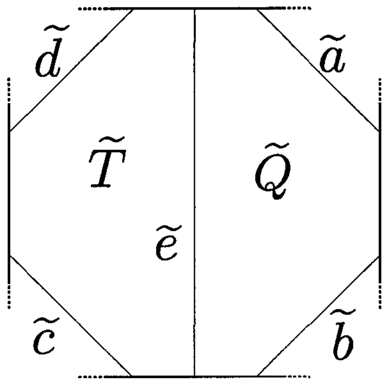


Figure 4: The notation of a lift of truncated triangles

$a = \pi(\tilde{a}), b = \pi(\tilde{b}), c = \pi(\tilde{c}), d = \pi(\tilde{d})$ need not be distinct; see Figure 5, where we enumerate the various cases. In any case, if $\Sigma \in \mathbf{R}_s^{\triangle} = \{\Sigma : \triangle \longrightarrow \mathbf{R}_s\}$, then we say that Σ satisfies the *strict face condition* on a seam e in $\triangle$ if the following inequality holds:

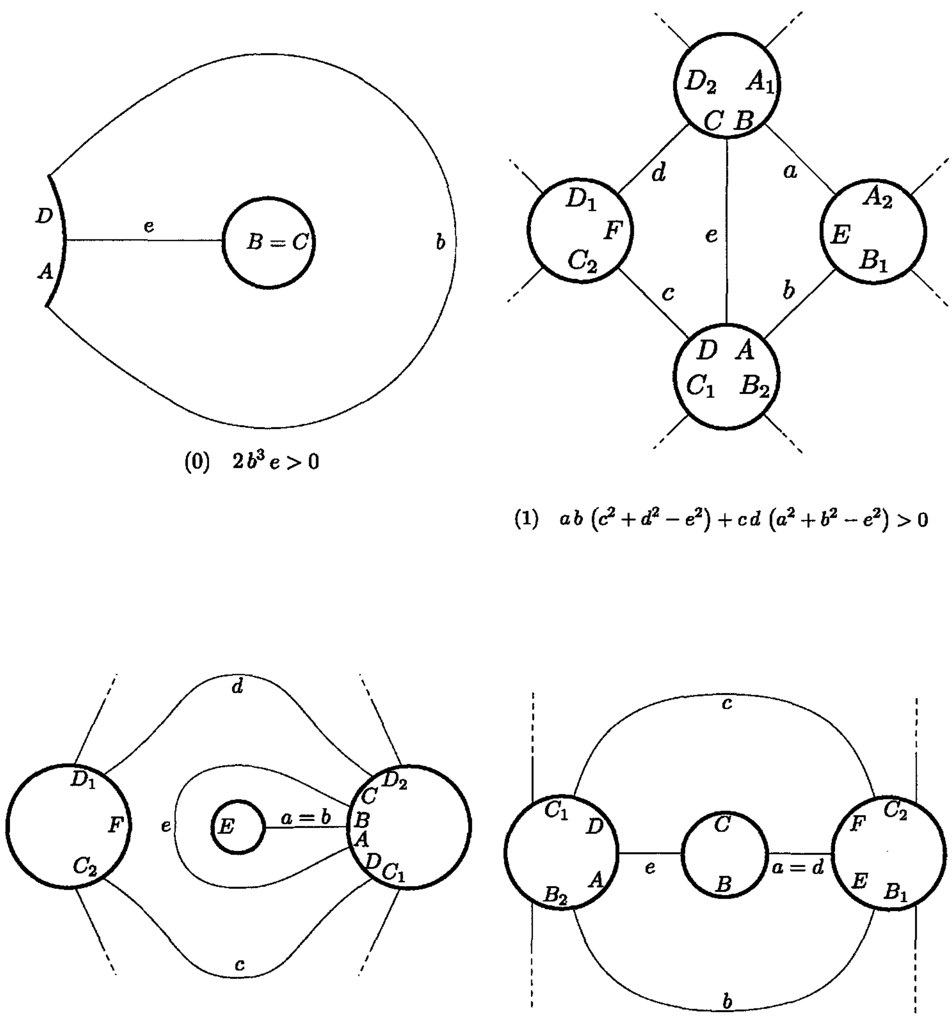
$$\Sigma(a)\,\Sigma(b)\,\left\{\Sigma(c)^2 + \Sigma(d)^2 - \Sigma(e)^2\right\} + \Sigma(c)\,\Sigma(d)\,\left\{\Sigma(a)^2 + \Sigma(b)^2 - \Sigma(e)^2\right\} > 0\,.$$

The strict face condition on e is indicated in Figure 5 in the various cases (where we identify a symbol of an edge with its Σ-value for convenience). We will also call the *face equality* when we replace $>$ in the above by $=$, and the *weak face condition* when replace by $\geq$. Furthermore, if $\triangle'$ is a subset of $\triangle$, not necessarily a truncated cellular decomposition, then we say that $\Sigma \in \mathbf{R}_s^{\triangle}$ satisfies the *face relations on $\triangle$ relative to $\triangle'$* if the strict face conditions hold for Σ on each $e \in \triangle' \subseteq \triangle$, and the face equalities hold on each $e \in \triangle - \triangle'$. In particular, we say that the *strict face relations* hold for $\triangle$ if Σ satisfies the face relations on $\triangle$ relative to $\triangle$, that is, if the strict face conditions hold on all $e \in \triangle$.

Theorem 6.1 *Suppose that $\triangle$ is a truncated triangulation of $F_{g,r}$, and let $\mathbf{R}_s^{\triangle} \ni \Sigma = \Gamma_m \in \mathcal{T}_{g,r}$. Then a necessary and sufficient condition for $\Gamma_m \in \mathring{\mathcal{C}}(\triangle)$ is that Σ satisfies the strict face relations on $\triangle$. Furthermore, if a subset $\triangle'$ of $\triangle$ is a truncated cellular decomposition, then a necessary and sufficient condition for $\Gamma_m \in \mathring{\mathcal{C}}(\triangle') \subset \mathcal{C}(\triangle)$ is that Σ satisfies the face relations on $\triangle$ relative to $\triangle'$.*

Proof of necessity in Theorem 6.1. Fix a point Γ_m in $\mathcal{T}_{g,r}$. We recall that the construction of $\triangle(\Gamma_m)$ from Γ_m in Section 5, and the definition of the discrete subset $V \subset H_S$ corresponding to the boundaries of $F_{g,r}$ in Section 4. We may assume that the truncated triangulation $\triangle$ lifts to a collection of Euclidean geodesics in $\mathbf{E}^{1,2}$ connecting points of V. If $\tilde{e}$ is such a lift of $e \in \triangle$ separating Euclidean triangles $\tilde{Q}$ and $\tilde{T}$ in the lift, then $\tilde{e}$ is external in the hull of V and so in particular in the hull of $\tilde{Q} \cup \tilde{T}$. Comparison of the strict face condition with Proposition 3.5 (2) thus guarantees necessity. The proof of necessity in the second assertion is analogous. □

Before we undertake a proof of sufficiency, we develop some generalities. Fix a truncated triangulation $\triangle$ of $F_{g,r}$. Suppose that $(T_j)_{j=1}^n$ is a *cycle* of truncated triangles in the sense that $T_j \cap T_{j+1} = e_j$, for all j, where we henceforth regard the index j as cycle,



(0) $2b^3 e > 0$

(1) $ab\,(c^2+d^2-e^2)+cd\,(a^2+b^2-e^2)>0$

(2) $a=b \;\Rightarrow$ $a^2\,(c^2+d^2-e^2)+cd\,(2a^2-e^2)>0$

(3) $a=d \;\Rightarrow$ $ab\,(a^2+c^2-e^2)+ac\,(a^2+b^2-e^2)>0$

Figure 5: The various cases of the truncated triangulation (first part)

so for instance, $T_{n+1} = T_1$. We note that e_j is an edge of both T_j and T_{j+1}. If the edges of T_j are $\{e_{j-1}, e_j, b_j\}$, $j = 1, 2, \dots n$, then the collection $(b_j)_{j=1}^n \subset \triangle$ is called the *boundary of the cycle* $(T_j)_{j=1}^n$.

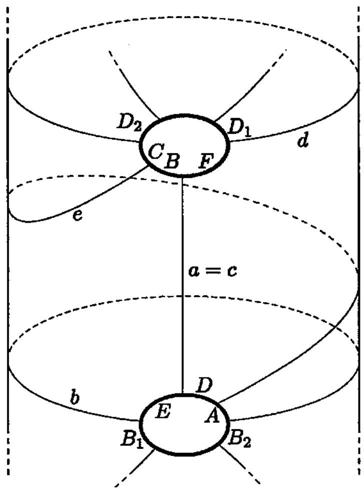
Lemma 6.2 *Suppose the weak face conditions hold for* $\mathbf{R}_s^\triangle \ni \Sigma = \Gamma_m \in \mathcal{T}_{g,r}$ *on each seam* $e \in \triangle$. *Then all three strict triangle inequalities on* $\{\Sigma(c), \Sigma(d), \Sigma(e)\}$ *hold whenever there is a truncated triangle obtained from* $\triangle$ *with edges* c, d, e.

Proof of Lemma 6.2. To get a contradiction, we suppose for instance that $\Sigma(e) \geq \Sigma(c) + \Sigma(d)$, and adopt the usual notation for the edges adjacent to e (see Figure 5). Thus,

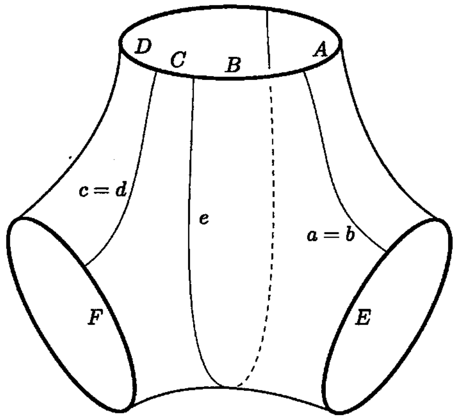
$$\Sigma(c)^2 + \Sigma(d)^2 - \Sigma(e)^2 \leq -\,2\,\Sigma(c)\,\Sigma(d)\,,$$

so the face condition on e gives

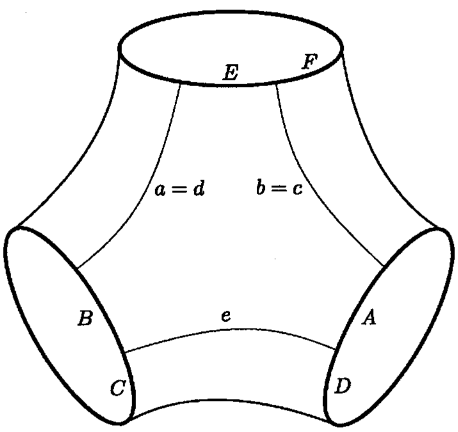
$$0 \leq \Sigma(c)\,\Sigma(d)\,\left\{(\Sigma(a) - \Sigma(b))^2 - \Sigma(e)^2\right\}\,,$$



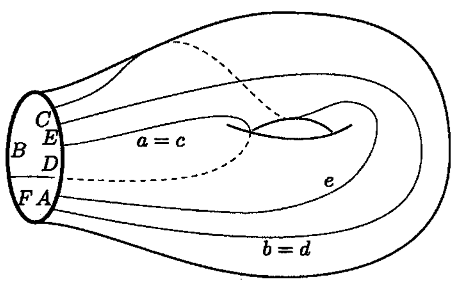
(4) $a=c \Rightarrow$
$ab\,(a^2+d^2-e^2)+ad\,(a^2+b^2-e^2)>0$



(5) $a=b$ & $c=d \Rightarrow$
$a^2\,(2c^2-e^2)+c^2\,(2a^2-e^2)>0$



(6) $a=d$ & $b=c \Rightarrow$
$2ab\,(a^2+b^2-e^2)>0$



(7) $a=c$ & $b=d \Rightarrow$
$2ab\,(a^2+b^2-e^2)>0$

Figure 5: The various cases of the truncated triangulation (second part)

and we find a second edge-triangle pair so that the triangle inequality fails. It follows that there is a cycle $(T_j)_{j=1}^n$ of truncated triangles obtained from $\triangle$ so that a strict triangle inequality fails at the edge-triangle pair (T_j, e_j), for all j. As before, let $(b_j)_{j=1}^n$ denote the seams of $\triangle$ corresponding to the boundary of the cycle. We have

$$\Sigma(e_{j+1}) \geq \Sigma(b_j) + \Sigma(e_j)\,, \quad j = 1, 2, \ldots, n\,.$$

Summing these inequalities and canceling $\Sigma(e_j)$, we obtain

$$0 \geq \Sigma(b_1) + \Sigma(b_2) + \cdots + \Sigma(b_n)\,,$$

which is absurd for $\Gamma_m \in \mathcal{T}_{g,r}$. $\square$

Proof of Sufficiency in Theorem 6.1. To prove sufficiency in the first claim, we suppose that $\Sigma \in \mathbf{R}_s^{\triangle}$ satisfies the strict face relations on $\triangle$ (and hence the "triangle inequality" condition of Lemma 6.2) and prove that $\Gamma_m = (\triangle, \Sigma) \in \overset{\circ}{\mathcal{C}}(\triangle)$. To this end, adopt the notation in the proof of necessity, so that $V \subset H_S$ arises from $\Sigma \in \mathbf{R}_s^{\triangle}$. By Lemma 3.3, the triangle inequality condition is equivalent to ellipticity of the affine planes spanned by triples in V arising as the vertices of a lift of an infinite triangle corresponding to a truncated triangle obtained from $\triangle$. Furthermore, we saw above that the face condition is equivalent to "local extremality." Finally, from the inductive definition of $V \subset H_S$ in Theorem 4.1, it follows by induction and an appeal to Proposition 3.6 that $\Gamma_m \in \overset{\circ}{\mathcal{C}}(\triangle)$. The proof of sufficiency in the second assertion is analogous. □

Recall the h-length parameterization of $\mathcal{T}_{g,r}$ given in Proposition 4.4. A pleasant algebraic fact relating h-lengths and the face condition is the observation that the face condition is linear in h-length coordinates. Indeed, suppose first that a seam e in a truncated triangulation $\triangle$ separates two truncated triangles $Q \neq T$ obtained from $\triangle$ with edges $(a, b, e), (c, d, e)$ respectively, where $\#\{a, b, c, d\} = 4$, and let $(\alpha, \beta, \varepsilon)$ ($(\gamma, \delta, \varphi)$ respectively) denote the h-lengths of the boundaries of Q opposite (a, b, e) (of T opposite (c, d, e) respectively); see Figure 6 (1). We see that the strict face condition on e is equivalent to

$$\alpha + \beta + \gamma + \delta > \varepsilon + \varphi\,,$$

by dividing the former by $\Sigma(a)\,\Sigma(b)\,\Sigma(c)\,\Sigma(d)\,\Sigma(e)$. Since the various cases (indicated in Figure 5) give rise to linear quotients, the claim follows.

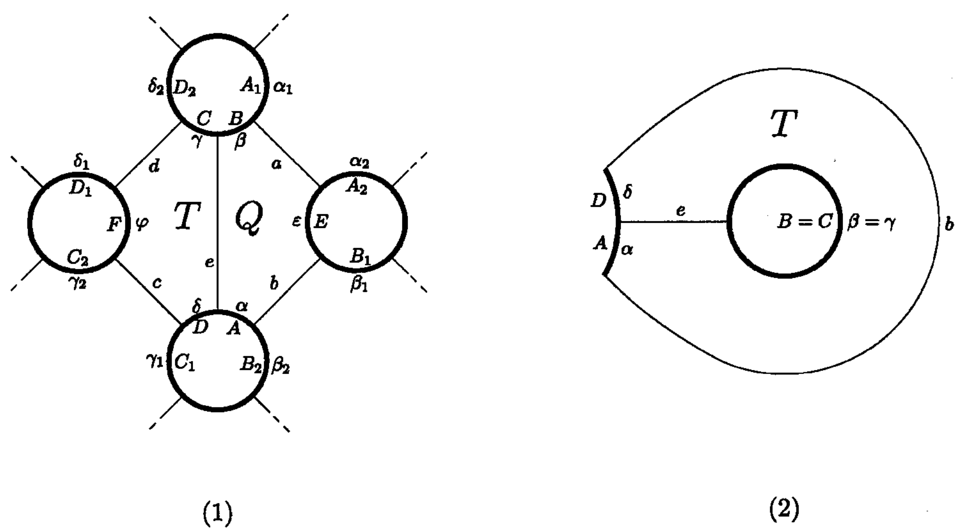


Figure 6: The h-lengths of triangulations

Thus, for a subset $\triangle'$ of a truncated triangulation $\triangle$, a point z in $\mathbf{R}^{\mathcal{B}_\triangle}$ satisfies the *face relations on $\triangle$ relative to $\triangle'$* if the strict face conditions hold for z on each $e \in \triangle' \subseteq \triangle$, and the face equalities hold on each $e \in \triangle - \triangle'$. Here, we recall that $\mathcal{B}_\triangle$ is the set of boundaries of truncated triangles given by $\triangle$ (see Section 4). In particular, we say that z satisfies the *strict face relations on* $\triangle$ if it satisfies the face relations on $\triangle$ relative to $\triangle$, that is, if the strict face conditions hold on all $e \in \triangle$. We note that these terms are used even when z is not in $I_\triangle \circ S_\triangle\,(\mathcal{T}_{g,r})$.

For each seam e in a truncated triangulation Δ, we next define a pair of vectors $B_e, C_e \in \mathbf{R}^{\mathcal{B}_\Delta}$. Adopt the notation of Figure 6 for the boundaries $A, B, C, D \in \mathcal{B}_\Delta$ on which e abuts. The vectors B_e and C_e each lie in the coordinate subspace of $\mathbf{R}^{\mathcal{B}_\Delta}$ corresponding to A, B, C, D (in this order), and B_e (C_e respectively) has entries $(1,1,1,1)$ ($(1,1,-1,-1)$ respectively); the boundaries A, B, C, D need not be distinct. See Figure 7 (1).

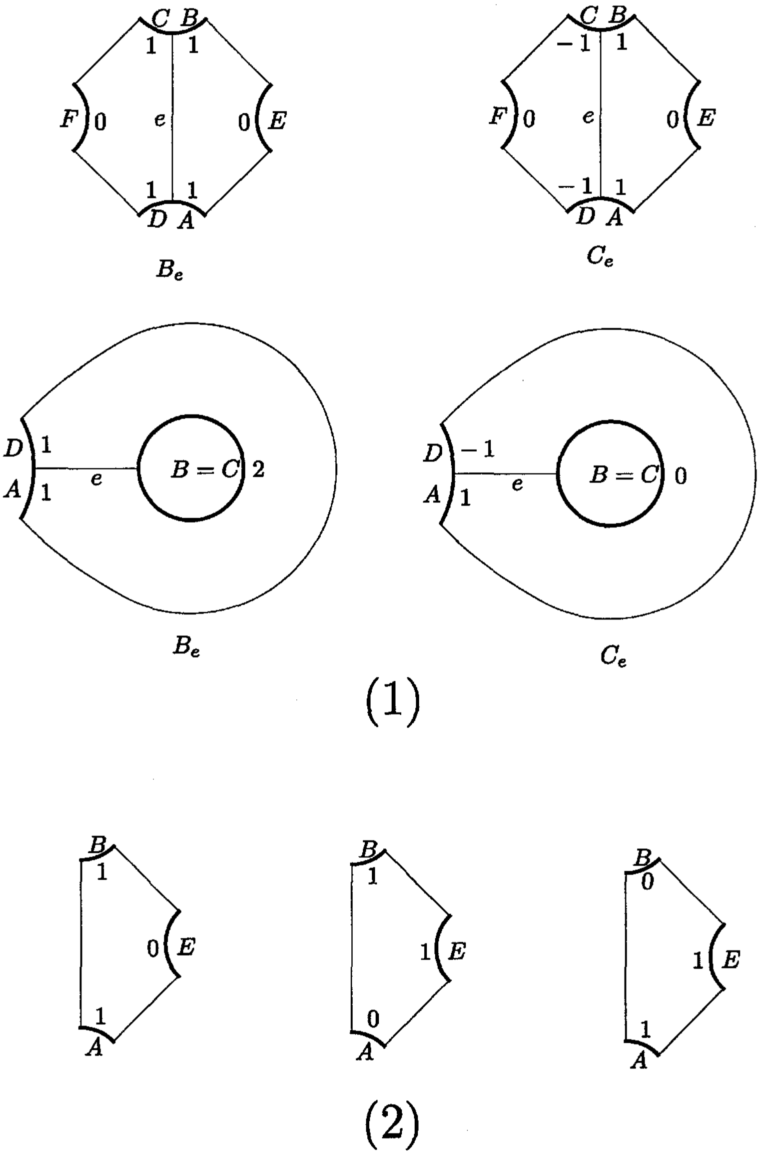


Figure 7: A basis of $\mathbf{R}^{\mathcal{B}_\Delta}$

Lemma 6.3 *Fix a truncated triangulation Δ of $F_{g,r}$. Then the set $\left\{ B_e, C_e \in \mathbf{R}^{\mathcal{B}_\Delta} \,\middle|\, e \in \Delta \right\}$ is a basis for $\mathbf{R}^{\mathcal{B}_\Delta}$. Furthermore, suppose*

$$z = x + y = \sum_{e \in \Delta} x_e \, B_e + \sum_{e \in \Delta} y_e \, C_e \,,$$

where $x_e, y_e \in \mathbf{R}$. Then z satisfies the face relations on Δ relative to Δ' if and only if $x_e > 0$ for $e \in \Delta'$ and $x_e = 0$ for $e \in \Delta - \Delta'$.

Proof of Lemma 6.3. The span of $\left\{ B_e, C_e \in \mathbf{R}^{\mathcal{B}_\triangle} \,\middle|\, e \in \triangle \right\}$ is clearly identical with the span of the vectors $\left\{ B'_e = \frac{B_e+C_e}{2}, C'_e = \frac{B_e-C_e}{2} \in \mathbf{R}^{\mathcal{B}_\triangle} \,\middle|\, e \in \triangle \right\}$. Let us fix a truncated triangle T obtained from $\triangle$, say with boundaries (A, B, E). There are exactly three vectors among $\left\{ B'_e, C'_e \in \mathbf{R}^{\mathcal{B}_\triangle} \,\middle|\, e \in \triangle \right\}$ with a non-zero projection into the subspace of $\mathbf{R}^{\mathcal{B}_\triangle}$ corresponding to (A, B, E); namely, $(1,1,0)$, $(0,1,1)$ and $(1,0,1)$. See Figure 7 (2). Insofar as these projections are linearly independent, $\left\{ B'_e, C'_e \in \mathbf{R}^{\mathcal{B}_\triangle} \,\middle|\, e \in \triangle \right\}$, and hence $\left\{ B_e, C_e \in \mathbf{R}^{\mathcal{B}_\triangle} \,\middle|\, e \in \triangle \right\}$ forms a linearly independent set, proving the first part.

Since the face condition is linear, the second part follows at once from the fact that equality $\alpha + \beta + \gamma + \delta = \varepsilon + \varphi$ holds on every edge for any C_e, $e \in \triangle$. □

Fix a truncated triangulation $\triangle$ of $F_{g,r}$. Now we define subspaces of $\mathbf{R}^{\mathcal{B}_\triangle}$ as follows:

$$\begin{aligned}
X &= \left\{ z \in \mathbf{R}^{\mathcal{B}_\triangle} \,\middle|\, z = \sum_{e\in\triangle} x_e B_e \text{ for } x_e \in \mathbf{R} \right\}, \\
Y &= \left\{ z \in \mathbf{R}^{\mathcal{B}_\triangle} \,\middle|\, z = \sum_{e\in\triangle} y_e C_e \text{ for } y_e \in \mathbf{R} \right\}, \\
\overline{X} &= \left\{ \sum_{e\in\triangle} x_e B_e \in X \,\middle|\, x_e \geq 0 \right\}, \\
\mathring{X} &= \left\{ \sum_{e\in\triangle} x_e B_e \in X \,\middle|\, x_e > 0 \right\}.
\end{aligned}$$

The (open) faces of $\overline{X}$ correspond to subsets $\triangle'$ of $\triangle$, where the face relations hold on $\triangle$ relative to $\triangle'$. A face F of $\overline{X}$ is said to be *finite* if the corresponding subset $\triangle' = \{ e \in \triangle \,|\, x_e \neq 0 \}$ of $\triangle$ is a truncated cellular decomposition, and we define

$$X^+ = \mathring{X} \sqcup \left\{ \text{faces } F \text{ of } \overline{X} \,\middle|\, F \text{ is finite} \right\} \subset \overline{X}.$$

We denote by $\Pi_\triangle$ the projection of $\mathbf{R}^{\mathcal{B}_\triangle}$ along Y onto X. We next define further subspaces of $\mathbf{R}_+^{\mathcal{B}_\triangle}$. When we fix a subset $\triangle'$ of $\triangle$, we define $\mathring{\mathcal{D}}_\triangle(\triangle')$ to be the set of points satisfying the coupling equations and the face relations on $\triangle$ relative to $\triangle'$, and $\mathring{\mathcal{G}}_\triangle(\triangle')$ to be the set of points in $\mathring{\mathcal{D}}_\triangle(\triangle')$ satisfying the coupling inequalities. An immediate consequence of Theorem 4.1, Proposition 4.4 and Theorem 6.1 is that $I_\triangle \circ S_\triangle(\mathring{\mathcal{C}}(\triangle')) = \mathring{\mathcal{G}}_\triangle(\triangle')$ if $\triangle' \subseteq \triangle$ is a truncated cellular decomposition. Now we define $\mathcal{D}(\triangle)$ as follows:

$$\mathcal{D}(\triangle) = \bigsqcup \left\{ \mathring{\mathcal{D}}_\triangle(\triangle') \,\middle|\, \triangle' \subseteq \triangle \text{ is a truncated cellular decomposition} \right\}.$$

We then have the following Theorem 6.4. The proof of this theorem is just a literal translation of that of Theorem 5.4 in [Pe].

Theorem 6.4 *For each truncated triangulation $\triangle$ of $F_{g,r}$, the projection $\Pi_\triangle$ induces a homeomorphism*

$$\Pi_\triangle \colon \mathcal{D}(\triangle) \longrightarrow X^+$$

which maps $\mathring{\mathcal{D}}_\triangle(\triangle)$ to $\mathring{X}$. If a proper subset $\triangle'$ of $\triangle$ is a truncated cellular decomposition, then $\Pi_\triangle$ maps $\mathring{\mathcal{D}}_\triangle(\triangle')$ to the corresponding (open) finite face of X^+. □

Using this theorem, we obtain the following fact about $\mathcal{C}(\triangle)$.

Theorem 6.5 *For a truncated cellular decomposition $\triangle$ of $F_{g,r}$, $\mathring{\mathcal{C}}(\triangle)$ is homeomorphic to an open ball of dimension $\#\triangle$.*

Proof of Theorem 6.5. We first suppose that $\triangle$ is a truncated triangulation. Since $I_\triangle \circ S_\triangle(\mathring{\mathcal{C}}(\triangle)) = \mathring{\mathcal{G}}_\triangle(\triangle)$, all we have to show is that $\mathring{\mathcal{G}}_\triangle(\triangle)$ is homeomorphic to an open ball of dimension $\#\triangle$.

For an arbitrary point $\boldsymbol{r}$ in $\mathring{\mathcal{D}}_\triangle(\triangle)$, we denote by R the ray in $\mathbf{R}_+^{\mathcal{B}_\triangle}$ with direction $\boldsymbol{r}$, that is, $R = \left\{ h\boldsymbol{r} \in \mathbf{R}_+^{\mathcal{B}_\triangle} \,\middle|\, h > 0 \right\}$. Then, since the couplings and the face conditions for h-lengths are homogeneous, R is entirely contained in $\mathring{\mathcal{D}}_\triangle(\triangle)$. So $\mathring{\mathcal{D}}_\triangle(\triangle)$ is homeomorphic to an (open) cone in $\mathbf{R}_+^{\mathcal{B}_\triangle}$. Furthermore, the same reasoning guarantees that there exists a unique number $k(\boldsymbol{r}) > 0$ such that the maximal value of the couplings at the point $k(\boldsymbol{r})\,\boldsymbol{r}$ is equal to $\frac{1}{2}$, and then $R \cap \mathring{\mathcal{G}}_\triangle(\triangle) = \{h\boldsymbol{r} \in R \,|\, 0 < h < k(\boldsymbol{r})\}$ and $\{h\boldsymbol{r} \in R \,|\, k(\boldsymbol{r}) \le h\} \cap \mathring{\mathcal{G}}_\triangle(\triangle) = \emptyset$. Moreover, $k(\boldsymbol{r})$ is a continuous function of $\boldsymbol{r}$. Hence $\mathring{\mathcal{G}}_\triangle(\triangle)$ is homeomorphic to the intersection of $\mathring{\mathcal{D}}_\triangle(\triangle)$ and the open unit ball in $\mathbf{R}^{\mathcal{B}_\triangle}$ centered at the origin.

Thus, if $\triangle$ is a truncated triangulation, $\mathring{\mathcal{G}}_\triangle(\triangle)$ is homeomorphic to an open ball of dimension $\#\triangle$, and so is $\mathring{\mathcal{C}}(\triangle)$. The proof when $\triangle$ is not a truncated triangulation but a truncated cellular decomposition is analogous. $\square$

Theorem 6.5 also shows that, for an arbitrary truncated cellular decomposition $\triangle$ of $F_{g,r}$, $\mathring{\mathcal{C}}(\triangle)$ is non-empty. Furthermore, by the definition of $\mathcal{C}(\,\cdot\,)$, $\mathcal{C}(\triangle_1) \cap \mathcal{C}(\triangle_2) \neq \emptyset$ if and only if $\triangle_1 \cap \triangle_2$ is also a truncated cellular decomposition of $F_{g,r}$, and in this case, $\mathcal{C}(\triangle_1) \cap \mathcal{C}(\triangle_2) = \mathcal{C}(\triangle_1 \cap \triangle_2)$. Thus, as an immediate consequence of Theorem 6.5 and Corollary 4.3, we have the following Theorem 6.6. This is an analogue of Theorem 5.5 in [Pe] and is the main theorem of this paper.

Theorem 6.6 *If $\triangle$ is a truncated cellular decomposition of $F_{g,r}$, $\mathring{\mathcal{C}}(\triangle)$ is an open cell of dimension $\#\triangle$. The set $\left\{ \mathring{\mathcal{C}}(\triangle) \,\middle|\, \triangle \text{ is a truncated cellular decomposition of } F_{g,r} \right\}$ is a $\mathrm{MC}_{g,r}$-invariant cellular decomposition of $\mathcal{T}_{g,r}$ itself. Furthermore, the isotropy group of $\mathcal{C}(\triangle)$ in $\mathrm{MC}_{g,r}$ is isomorphic to the (finite) group of mapping classes of $F_{g,r}$ leaving $\triangle$ invariant.* $\square$

Remark

Strictly speaking, $\mathring{\mathcal{C}}(\triangle)$ is not a cell in the usual sense (see, for example, [Ma, p. 226]) since the closure of $\mathring{\mathcal{C}}(\triangle)$ in $\mathcal{T}_{g,r}$ is not compact.

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