



Title	On the first main theorem of holomorphic mappings from C^2 into $Q_{(n-1)}(C)$
Author(s)	Suyama, Yoshihiko
Citation	大阪大学, 1975, 博士論文
Version Type	VoR
URL	https://hdl.handle.net/11094/25977
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

ON THE FIRST MAIN THEOREM OF HOLOMORPHIC MAPPINGS FROM C^2 INTO $Q_{n-1}(C)$

YOSHIHIKO SUYAMA

(Received December 10, 1973)

0. Introduction

Let f be a holomorphic mapping of a complex line C into a complex projective space $P_n(C)$ and suppose that the image $f(C)$ is not contained in any hyperplane of $P_n(C)$. Put $V[t] = \{z \in C : \log|z| < t\}$, and for a hyperplane ξ in $P_n(C)$ let $n(t, \xi)$ be the number of points in $V[t] \cap f^{-1}(\xi)$. Let Ω be the closed form of degree 2 associated with the Fubini-Study metric on $P_n(C)$ and normalized as $\int_{P_n} \Omega^n = 1$. The counting function $N(r, \xi)$ and the order function $T(r)$ being defined by

$$(0.1) \quad N(r, \xi) = \int_0^r n(t, \xi) dt,$$

$$(0.2) \quad T(r) = \int_0^r dt \int_{V[t]} f^* \Omega$$

respectively, the following equation is known as the First Main Theorem:

$$(0.3) \quad N(r, \xi) + (m(r, \xi) - m(0, \xi)) = T(r),$$

where $m(r, \xi)$ is a non-negative function defined for $r \in \mathbf{R}^+$ and hyperplanes ξ in $P_n(C)$. The term $(m(r, \xi) - m(0, \xi))$ is called the compensating term. It follows from the equation (0.3) that the image $f(C)$ intersects with almost all hyperplanes in $P_n(C)$. Furthermore it is known that the number of hyperplanes in general position not intersecting with $f(C)$ is at most $n+1$. These results are originally due to Ahlfors, and treated also by H. Wu [6] and S. S. Chern [1] in a modernized form.

Let f be a holomorphic mapping of C^2 into a complex quadratic $Q_{n-1}(C)$ ($n \geq 3$) satisfying certain non-degenerate conditions [§2]. We consider $Q_{n-1}(C)$ as a fixed hypersurface in $P_n(C)$. We consider a special family of $(n-2)$ -dimensional projective spaces $P_{n-2}(C)$ in $P_n(C)$ parametrized by a Grassmann manifold $G(\mathbf{R})$ of 2-dimensional linear spaces in \mathbf{R}^{n+1} [§1]. This family determines a family of $(n-3)$ -dimensional complex quadratic ξ_α ($\alpha \in G(\mathbf{R})$) in $Q_{n-1}(C)$, each of whose elements is a Poincaré dual of the form Ω^2 in $Q_{n-1}(C)$.

In this paper, we shall consider a value distribution problem in two complex variables with respect to the holomorphic mapping f and the family $\{\xi_\alpha\}$. The complex quadratic $Q_{n-1}(\mathbf{C})$ being a double covering space of $G(\mathbf{R})$, we may take $Q_{n-1}(\mathbf{C})$ as a parametrizing space of the family $\{\xi_\alpha\}$ in place of $G(\mathbf{R})$. Thus we have a setting similar to the case of holomorphic curves (holomorphic mappings of \mathbf{C} into $P_n(\mathbf{C})$). Furthermore Ω is an invariant form on $Q_{n-1}(\mathbf{C})$ by a certain transformation group [§5]. This fact also plays an important role as in the case of holomorphic curves [§6].

Our main results are as follows: (1) First Main Theorem [§4], (2) the Crofton formula [§6] and (3) the Distribution theorem [§7]. In more detail, put

$$\Delta(r) = \{(z_1, z_2) \in \mathbf{C}^2 : \log |z_i| < r (i = 1, 2)\}$$

and define

$$n(\Delta(r), \alpha) = \sum_{p_i \in \Delta(r), f(p_i) \in \xi_\alpha} n(p_i, \alpha),$$

where $n(p_i, \alpha)$ is a certain real number [§3] such that $n(p_i, \alpha) = 1$ if $f(\mathbf{C}^2)$ intersects transversely with ξ_α at $f(p_i)$. We also define the following functions:

$$(0.4) \quad N(r, \alpha) = \int_0^r n(\Delta(t), \alpha) dt \quad (\text{counting function})$$

$$(0.5) \quad T(r) = \int_0^r dt \int_{\Delta(t)} f^* \Omega^2 \quad (\text{order function}).$$

Then our First Main Theorem states:

$$(0.6) \quad N(r, \alpha) + m(r, \alpha) - m(0, \alpha) = T(r),$$

where $m(r, \alpha)$ is a non-negative function defined for $r \in \mathbf{R}^+$ and submainifold ξ_α ($\alpha \in G(\mathbf{R})$) [§4]. The Crofton formula is as follows:

$$(0.7) \quad \int_{Q_{n-1}} n(\Delta(t), \alpha) \Omega^{n-1}(\alpha) = 2 \int_{\Delta(t)} f^* \Omega^2.$$

Finally the distribution theorem says: The image $f(\mathbf{C}^2)$ intersects with almost all submanifolds in $\{\xi_\alpha\}$ ($\alpha \in G(\mathbf{R})$) i.e., we have $\int_W \Omega^{n-1} = 0$ for $W = \{\alpha \in Q_{n-1}(\mathbf{C}) : f(\mathbf{C}^2) \cap \xi_\alpha = \emptyset\}$.

We note that W. Stoll [4], P. Griffiths and J. King [2] also developed the First Main Theorem in several complex variables. But our setting is different from theirs.

The author expresses his hearty thanks to Professor S. Murakami and Professor T. Ochiai for their kind encouragement and guidance.

1. Preliminaries

We shall recall several basic facts about the complex projective space $P_n(\mathbf{C})$

and the complex quadratic $Q_{n-1}(\mathbf{C})$ (c.f. [3]), and moreover we shall define a special family of submanifolds in $Q_{n-1}(\mathbf{C})$. Let \mathbf{C}^{n+1} (resp. \mathbf{R}^{n+1}) be the complex (resp. real) vector space of $(n+1)$ tuples of complex numbers (z^0, \dots, z^n) (resp. real numbers (x^0, \dots, x^n)). We define a symmetric bilinear form $(,)$ on \mathbf{C}^{n+1} by

$$(1.1) \quad (Z, W) = z^0 w^0 + \dots + z^n w^n$$

for $Z = (z^0, \dots, z^n)$ and $W = (w^0, \dots, w^n)$. For $Z = (z^0, \dots, z^n)$ we put $\bar{Z} = (\bar{z}^0, \dots, \bar{z}^n)$, where the bar denotes the complex conjugation. A vector $Z \in \mathbf{C}^{n+1} - \{0\}$ is called real if $\bar{Z} = Z$. We define a hermitian inner product \langle , \rangle on \mathbf{C}^{n+1} by

$$(1.2) \quad \langle Z, W \rangle = (Z, \bar{W})$$

for $Z, W \in \mathbf{C}^{n+1}$. We put $\|Z\| = \langle Z, Z \rangle^{1/2}$. For the complex projective space $P_n(\mathbf{C})$ of dimension n , we have the natural holomorphic fibring (called the Hopf fibring)

$$(1.3) \quad \Pi: \mathbf{C}^{n+1} - \{0\} \rightarrow P_n(\mathbf{C}),$$

where $\Pi(Z)$ is the line passing through the origin and Z . We remark that the natural conjugation $Z \mapsto \bar{Z}$ in $\mathbf{C}^{n+1} - \{0\}$ induces a diffeomorphism $z \in P_n(\mathbf{C}) \rightarrow \bar{z} \in P_n(\mathbf{C})$. Let $\tilde{\Omega}$ be the 2-form of type $(1, 1)$ on $\mathbf{C}^{n+1} - \{0\}$ given by

$$(1.4) \quad \tilde{\Omega} = \frac{i}{2\pi} \frac{1}{\|Z\|^4} \left\{ \left(\sum_j |z^j|^2 \right) \left(\sum_j dz^j \wedge d\bar{z}^j \right) - \left(\sum_j \bar{z}^j dz^j \right) \wedge \left(\sum_j z^j d\bar{z}^j \right) \right\}.$$

It is well-known that there exists a unique 2-form Ω of type $(1, 1)$ on $P_n(\mathbf{C})$ such that $\Pi^* \Omega = \tilde{\Omega}$. Then Ω is the Kähler form associated with the Fubini-Study metric on $P_n(\mathbf{C})$ and we have

$$(1.5) \quad \int_{P_n(\mathbf{C})} \Omega^n = 1.$$

We consider a family of subspaces H of \mathbf{C}^{n+1} such that H is of $(n-1)$ -dimension and $\bar{Z} \in H$ whenever $Z \in H$. With such an H , we can associate uniquely a real subspace of \mathbf{R}^{n+1} of dimension 2 by

$$(1.6) \quad \{X \in \mathbf{R}^{n+1}: \langle X, H \rangle = 0\}.$$

We see that this gives a one to one correspondence, and hence the above family of H 's is parametrized by the Grassmann manifold $G(\mathbf{R})$ of 2 planes in \mathbf{R}^{n+1} . Especially we note that $[H] = \Pi(H - \{0\})$ is an $(n-2)$ -dimensional projective space in $P_n(\mathbf{C})$.

On $P_n(\mathbf{C})$ with homogeneous coordinate z^0, \dots, z^n the complex quadratic $Q_{n-1}(\mathbf{C})$ is a complex hypersurface defined by the equation

$$(1.7) \quad (z^0)^2 + \dots + (z^n)^2 = 0.$$

Now the unit sphere $S^{2n+1} = \{Z \in \mathbf{C}^{n+1}: \|Z\|=1\}$ is a principal fibre bundle over

$P_n(\mathbf{C})$ with structure group S^1 . For a point $q \in Q_{n-1}(\mathbf{C})$, take a point $Z \in S^{2n+1}$ such that $\Pi(Z) = q$. We can write Z uniquely in the form $Z = (X + iY)/\sqrt{2}$, where X and Y are orthonormal real vectors in \mathbf{C}^{n+1} . Conversely if $Z = (X + iY)/\sqrt{2} \in S^{2n+1}$ for orthonormal real vectors X and Y , then we have $\Pi(Z) \in Q_{n-1}(\mathbf{C})$. Therefore we have

$$(1.8) \quad S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbf{C})) = \{Z = (X + iY)/\sqrt{2} : X \text{ and } Y \text{ are orthonormal real vectors}\}.$$

The group $SO(n+1)$, considered as a subgroup of $U(n+1)$, acts on S^{2n+1} and leaves the submanifold $S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbf{C}))$ invariant. Moreover $SO(n+1)$ acts transitively on $S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbf{C}))$. The isotropy subgroup of $SO(n+1)$ at $Z_0 = (1/\sqrt{2}, i/\sqrt{2}, 0, \dots, 0)$ coincides with the subgroup $SO(n-1)$ of $SO(n+1)$. We denote an element g of $SO(n+1)$ by

$$g = (X_0, X_1, \dots, X_n),$$

where each X_i is a column vector. Then, in the space $SO(n+1)/SO(n-1)$, the coset including $g = (X_0, X_1, \dots, X_n)$ can be represented by the first two vectors (X_0, X_1) . Under this identification, we have a diffeomorphism $i: SO(n+1)/SO(n-1) \rightarrow S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbf{C}))$ defined by

$$(1.9) \quad i((X_0, X_1)) = \frac{1}{\sqrt{2}}(X_0 + iX_1).$$

From now on we also identify $SO(n+1)/SO(n-1)$ with $S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbf{C}))$ by the above diffeomorphism. We denote by Π_1 the projection: $SO(n+1)/SO(n-1) \rightarrow Q_{n-1}(\mathbf{C})$ defined by

$$(1.10) \quad \Pi_1((X_0, X_1)) = \Pi((X_0 + iX_1)/\sqrt{2})$$

for $(X_0, X_1) \in SO(n+1)/SO(n-1)$. Note that the space $Q_{n-1}(\mathbf{C})$ also can be identified canonically with $SO(n+1)/SO(2) \times SO(n-1)$.

To each point $\alpha = \Pi_1((X_0, X_1))$ in $Q_{n-1}(\mathbf{C})$, we assign the 2-dimensional linear space spanned by $\{X_0, X_1\}$ in \mathbf{C}^{n+1} . Through this assignment, $Q_{n-1}(\mathbf{C})$ is a double covering space of $G(\mathbf{R})$. We see that the function $|\langle Z, W \rangle|^2$ on $S^{2n+1} \times S^{2n+1}$ induces a function $|\Pi(Z), \Pi(W)|^2$ on $P_n(\mathbf{C}) \times P_n(\mathbf{C})$. For each $\alpha \in Q_{n-1}(\mathbf{C})$, we consider a complex submanifold ξ_α of $Q_{n-1}(\mathbf{C})$, defined by

$$(1.11) \quad \xi_\alpha = \{\beta \in Q_{n-1}(\mathbf{C}) : |\beta, \alpha|^2 + |\beta, \bar{\alpha}|^2 = 0\}.$$

Let $(X_0, X_1) \in SO(n+1)/SO(n-1)$ and set $\Pi_1((X_0, X_1)) = \alpha$. Consider the complex subspace H of \mathbf{C}^{n+1} orthogonal to the vectors X_0, X_1 . We have $\xi_\alpha = Q_{n-1}(\mathbf{C}) \cap [H]$. $[H]$ is a Poincaré dual of the form Ω^2 in $P_n(\mathbf{C})$, and hence ξ_α is also, in $Q_{n-1}(\mathbf{C})$, a Poincaré dual of the form Ω^2 restricted to $Q_{n-1}(\mathbf{C})$. Finally we remark that each ξ_α is a complex quadratic $Q_{n-3}(\mathbf{C})$ and $\xi_\alpha = \xi_{\bar{\alpha}}$.

2. Holomorphic mapping

Let f be a holomorphic mapping of \mathbf{C}^2 into $Q_{n-1}(\mathbf{C})$ ($n \geq 3$). We consider the following two conditions on f .

Condition (A): f is an immersion.

Condition (B): For each $\alpha \in Q_{n-1}(\mathbf{C})$, the set $\{p \in \mathbf{C}^2 : f(p) \in \xi_\alpha\}$ is discrete.

For each point $p \in \mathbf{C}^2$, we can take a small neighborhood $U(p)$ of p such that there exists a holomorphic lift $F = (f^0, \dots, f^n)$ of f on $U(p)$ into $\mathbf{C}^{n+1} - \{0\}$ i.e., $\Pi F = f$.

Proposition 2.1. Condition (A) is equivalent to the following: for each point p of \mathbf{C}^2 , choose a holomorphic lift $F = (f^0, \dots, f^n)$ of f on a neighborhood U of p , then we have

$$(2.1) \quad \text{rank} \begin{pmatrix} f^0, & \dots, & f^n \\ \frac{\partial f^0}{\partial w_1}, & \dots, & \frac{\partial f^n}{\partial w_1} \\ \frac{\partial f^0}{\partial w_2}, & \dots, & \frac{\partial f^n}{\partial w_2} \end{pmatrix} (p) = 3,$$

where (w_1, w_2) is a coordinate system on the neighborhood U .

Proof. We identify the real tangent space $T_Z(\mathbf{C}^{n+1})$ at a point Z in \mathbf{C}^{n+1} with \mathbf{C}^{n+1} in the usual way. For p , we take $(X_0, X_1, \dots, X_n) \in SO(n+1)$ such that $(X_0 + iX_1)/\sqrt{2} = (F/\|F\|)(p)$. Then the tangent space $T_{(X_0+iX_1)/\sqrt{2}}(S^{2n+1})$ has a basis $i(X_0+iX_1), X_0-iX_1, i(X_0-iX_1), X_2, \dots, X_n, iX_2, \dots, iX_n$. Let $T_{f(p)}$ be the subspace spanned by $X_2, \dots, X_n, iX_2, \dots, iX_n$. The projection $\tilde{\Pi} = \Pi|_{S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbf{C}))}$ induces a linear isomorphism $\tilde{\Pi}_* : T_{f(p)} \rightarrow T_{f(p)}(Q_{n-1}(\mathbf{C}))$ (c.f. [3] p.p. 279). Hence, $T_{f(p)}(Q_{n-1}(\mathbf{C}))$ is identified with the subspace of \mathbf{C}^{n+1} orthogonal to the vectors $(F/\|F\|)(p)$ and $(\bar{F}/\|F\|)(p)$ with respect to $\langle \cdot, \cdot \rangle$. Since we have $\langle F, \bar{F} \rangle = 0$ on U , we see $\langle dF, \bar{F} \rangle = 0$. We have

$$(2.2) \quad \begin{aligned} d\left(\frac{F}{\|F\|}\right) &= \frac{1}{\|F\|} \sum_{j=1}^2 \left(\frac{\partial F}{\partial w_j} - \left\langle \frac{\partial F}{\partial w_j}, \frac{F}{\|F\|} \right\rangle \frac{F}{\|F\|} \right) dw_j, \\ &+ \sum_{j=1}^2 iF \frac{\partial}{\partial y^j} \left(\frac{1}{\|F\|} \right) dx^j - \sum_{j=1}^2 iF \frac{\partial}{\partial x^j} \left(\frac{1}{\|F\|} \right) dy^j, \end{aligned}$$

where $w_j = x^j + iy^j$. Therefore we get

$$(2.3) \quad df = \sum_{j=1}^2 \tilde{\Pi}_* \left[\frac{1}{\|F\|} \left(\frac{\partial F}{\partial w_j} - \left\langle \frac{\partial F}{\partial w_j}, \frac{F}{\|F\|} \right\rangle \frac{F}{\|F\|} \right) \right] dw_j.$$

This shows Proposition 2.1. Q.E.D.

We define

$$(2.4) \quad Q_{n-3}(f(p)^\perp) = \{\alpha \in Q_{n-1}(\mathbf{C}): |f(p), \alpha|^2 + |f(p), \bar{\alpha}|^2 = 0\},$$

that is,

$$Q_{n-3}(f(p)^\perp) = \{\alpha \in Q_{n-1}(\mathbf{C}): f(p) \in \xi_\alpha\}.$$

Then $Q_{n-3}(f(p)^\perp)$ can be identified with $SO(n-1)/SO(2) \times SO(n-3)$ as follows: Choose an element $(X_0, X_1, \dots, X_n) \in SO(n+1)$ such that $(X_0 + iX_1)/\sqrt{2} = (F/||F||)(p)$. Let $(A_2, A_3) \in SO(n-1)/SO(n-3)$ where $A_i = (a_{i2}, \dots, a_{in})^t$ ($i = 2, 3$). Consider the mapping

$$(2.5) \quad (A_2, A_3) \rightarrow (\sum_{i=2}^n a_{2i} X_i, \sum_{i=2}^n a_{3i} X_i).$$

We see easily that this gives an identification of $SO(n-1)/SO(2) \times SO(n-3)$ with $Q_{n-3}(f(p)^\perp)$, which is independent of the choice of lift F .

For $\alpha \in Q_{n-3}(f(p)^\perp)$ we take $(X_0, X_1) \in SO(n+1)/SO(n-1)$ such that $\Pi_1((X_0, X_1)) = \alpha$. Then the following condition is independent of the choice of (X_0, X_1) ,

$$(2.6) \quad \begin{vmatrix} \langle (\partial F / \partial w_1)(p), (X_0 + iX_1) / \sqrt{2} \rangle, \langle (\partial F / \partial w_2)(p), (X_0 + iX_1) / \sqrt{2} \rangle \\ \langle (\partial F / \partial w_1)(p), (X_0 - iX_1) / \sqrt{2} \rangle, \langle (\partial F / \partial w_2)(p), (X_0 - iX_1) / \sqrt{2} \rangle \end{vmatrix} \neq 0.$$

Proposition 2.2. *The condition (2.6) holds if and only if f intersects transversely with ξ_α at $f(p)$.*

Proof. Put $(F/||F||)(p) = (X_2 + iX_3)/\sqrt{2}$. Then we take an element $(X_0, X_1, X_2, X_3, \dots, X_n) \in SO(n+1)$. As in the proof of Proposition 2.1, we see that the tangent space $T_{f(p)}(Q_{n-1}(\mathbf{C}))$ is spanned by the vectors $X_0, iX_0, X_1, iX_1, X_4, iX_4, \dots, X_n, iX_n$ and the tangent space $T_{f(p)}(\xi_\alpha)$ is spanned by $X_4, iX_4, \dots, X_n, iX_n$ through the identification by $\tilde{\Pi}_*: T_{(X_2+iX_3)/\sqrt{2}}(S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbf{C}))) \rightarrow T_{f(p)}(Q_{n-1}(\mathbf{C}))$. Therefore by (2.3) (or (2.2)) it is sufficient to show that the condition (2.6) is equivalent to $\text{rank}_R((\partial F / \partial w_1)(p), i(\partial F / \partial w_1)(p), (\partial F / \partial w_2)(p), i(\partial F / \partial w_2)(p), X_2, iX_2, \dots, X_n, iX_n) = 2(n+1)$. Now this can be seen easily.

Q.E.D.

Now we consider the following condition for $\alpha = \Pi_1((X_0, X_1)) \in Q_{n-3}(f(p)^\perp)$

$$(2.7) \quad \begin{vmatrix} \langle (\partial F / \partial w_1)(p), (X_0 + iX_1) / \sqrt{2} \rangle, \langle (\partial F / \partial w_2)(p), (X_0 + iX_1) / \sqrt{2} \rangle \\ \langle (\partial F / \partial w_1)(p), (X_0 - iX_1) / \sqrt{2} \rangle, \langle (\partial F / \partial w_2)(p), (X_0 - iX_1) / \sqrt{2} \rangle \end{vmatrix} = 0$$

Since the vectors $(\partial F / \partial w_1)(p)$ and $(\partial F / \partial w_2)(p)$ are linearly independent, the set of elements $\alpha \in Q_{n-3}(f(p)^\perp)$ satisfying the condition (2.7) has measure zero in $Q_{n-3}(f(p)^\perp)$.

REMARK 1. We shall remark here a certain sufficient condition for Condition (B). For $w \in \mathbf{C}$ we put $\mathbf{C}_w^1 = \{(z, w): z \in \mathbf{C}\}$ and $\mathbf{C}_w^2 = \{(w, z): z \in \mathbf{C}\}$.

Assume the following condition (C): *none of $f(C_w^i)$ ($i=1, 2, w \in C$) is contained in a hyperplane in $P_n(C)$.* Let $f(p) \in \xi_\alpha$ and set $\Pi_1((X_0, X_1)) = \alpha$. We put $g_1(w_1, w_2) = \langle F, (X_0 + iX_1)/\sqrt{2} \rangle(w_1, w_2)$ and $g_2(w_1, w_2) = \langle F, (X_0 - iX_1)/\sqrt{2} \rangle(w_1, w_2)$ on $U(p)$, where (w_1, w_2) is a coordinate system on $U(p)$ such that $w_i(p) = 0$ ($i=1, 2$). Using the Weierstrass' preparation theorem we have the following representations

$$(2.8) \quad \begin{aligned} g_1(w_1, w_2) &= (a_0(w_1) + a_1(w_1)w_2 + \cdots + a_{l_1}(w_1)w_2^{l_1})h_1(w_1, w_2) \\ g_2(w_1, w_2) &= (b_0(w_1) + b_1(w_1)w_2 + \cdots + b_{l_2}(w_1)w_2^{l_2})h_2(w_1, w_2), \end{aligned}$$

where $a_i(w_1), b_i(w_1)$ and $h_i(w_1, w_2)$ are holomorphic such that $a_i(0) = 0$ for $0 \leq i < l_1$, $a_{l_1}(0) \neq 0$, $b_i(0) = 0$ for $0 \leq i < l_2$, $b_{l_2}(0) \neq 0$ and $h_i(w_1, w_2) \neq 0$ ($i=1, 2$). We denote by $R(w_1)$ the resultant of $(a_0(w_1) + \cdots + a_{l_1}(w_1)w_2^{l_1})$ and $(b_0(w_1) + \cdots + b_{l_2}(w_1)w_2^{l_2})$. Since the function $R(w_1)$ is holomorphic, we have that $R(w_1) \equiv 0$ or the following (D): *the set $\{w_1 : R(w_1) = 0\}$ is discrete.* If, under the assumption of (C), f satisfies (D) for each $p \in C^2$ and $\alpha \in Q_{n-1}(C)$ such that $f(p) \in \xi_\alpha$, then Condition (B) holds.

3. Certain forms on $Q_{n-1}(C) - \xi_\alpha$

We define one 2-form Ω_α on $Q_{n-1}(C) - \xi_\alpha$ by

$$(3.1) \quad \Omega_\alpha(\beta) = dd^c \log \{|\beta, \alpha|^2 + |\beta, \bar{\alpha}|^2\},$$

where $d^c = \frac{1}{4\pi i}(\partial - \bar{\partial})$. We choose a unit vector Z_α such that $\Pi(Z_\alpha) = \alpha$, and define a mapping P_α of $Q_{n-1}(C) - \xi_\alpha$ into $P_1(C)$ by

$$(3.2) \quad P_\alpha(\beta) = \hat{\Pi} \left[\frac{1}{(|\beta, \alpha|^2 + |\beta, \bar{\alpha}|^2)^{1/2}} (\langle Z_\beta, Z_\alpha \rangle, \langle Z_\beta, \bar{Z}_\alpha \rangle) \right],$$

where $Z_\beta \in S^{2n+1}$ such that $\Pi(Z_\beta) = \beta$, and $\hat{\Pi}$ is the Hopf fibring $S^3 \rightarrow P_1(C)$. P_α is well-defined and holomorphic. Let ω be the Kähler 2-form associated with the Fubini-Study metric on $P_1(C)$ and normalized as $\int_{P_1(C)} \omega = 1$. Then $P_\alpha^* \omega$ is independent of the choice of Z_α . From now on we also denote by Ω the restriction of the form Ω to $Q_{n-1}(C)$.

Lemma 3.1. *We have*

$$(3.3) \quad \Omega_\alpha = P_\alpha^* \omega - \Omega \quad \text{on } Q_{n-1}(C) - \xi_\alpha.$$

Proof. Let σ be a local holomorphic cross-section of the Hopf fibring $\Pi: C^{n+1} - \{0\} \rightarrow P_n(C)$ defined on an open set U in $Q_{n-1}(C) - \xi_\alpha$. Then we have

$$\begin{aligned} \Omega_\alpha &= dd^c \log \left\{ \left| \left\langle \frac{\sigma}{\|\sigma\|}, Z_\alpha \right\rangle \right|^2 + \left| \left\langle \frac{\sigma}{\|\sigma\|}, \bar{Z}_\alpha \right\rangle \right|^2 \right\} \\ &= dd^c \log \{|\langle \sigma, Z_\alpha \rangle|^2 + |\langle \sigma, \bar{Z}_\alpha \rangle|^2\} - dd^c \log \|\sigma\|^2 \\ &= P_\alpha^* \omega - \Omega. \end{aligned} \quad \text{Q.E.D.}$$

We define another 2-form Ω'_α on $Q_{n-1}(\mathbf{C}) - \xi_\alpha$ by

$$(3.4) \quad \Omega'_\alpha = \Omega + P_\alpha^* \omega \quad \text{on } Q_{n-1}(\mathbf{C}) - \xi_\alpha.$$

Put

$$(3.5) \quad \Omega''_\alpha = -\Omega_\alpha \wedge \Omega'_\alpha \quad \text{on } Q_{n-1}(\mathbf{C}) - \xi_\alpha.$$

By (3.3) and (3.4), we have

$$(3.5)' \quad \begin{aligned} \Omega''_\alpha &= (\Omega - P_\alpha^* \omega) \wedge (\Omega + P_\alpha^* \omega) \\ &= \Omega^2 - P_\alpha^* (\omega \wedge \omega) = \Omega^2 \quad \text{on } Q_{n-1}(\mathbf{C}) - \xi_\alpha. \end{aligned}$$

Let $f: \mathbf{C}^2 \rightarrow Q_{n-1}(\mathbf{C})$ ($n \geq 3$) be a holomorphic mapping satisfying Conditions (A) and (B) in §2. For a point p in \mathbf{C}^2 , we take a small neighborhood $U(p)$ of p and a coordinate system (w_1, w_2) on it satisfying $w_i(p) = 0$ ($i = 1, 2$). Let F be a holomorphic lift of f on $U(p)$ into $\mathbf{C}^{n+1} - \{0\}$. Set $f(p) \in \xi_\alpha$. Then we define a real number $n(p, \alpha)$ by

$$(3.6) \quad n(p, \alpha) = \lim_{\varepsilon \downarrow 0} \int_{\partial U_\varepsilon(p)} d^c \cdot \log \{ |\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2 \} \wedge f^* P_\alpha^* \omega,$$

where $U_\varepsilon(p) = \{(w_1, w_2) \in U(p) : |w_1|^2 + |w_2|^2 < \varepsilon^2\}$ and $\Pi(Z_\alpha) = \alpha$.

Lemma 3.2. *$n(p, \alpha)$ is well-defined and finite. Especially if f intersects transversely with ξ_α at $f(p)$, then we have $n(p, \alpha) = 1$.*

Proof. First we choose a local lift F and a local coordinate system (w_1, w_2) such that $w_i(p) = 0$. Take two positive real numbers ε_1 and ε_2 such that $U(p) \supset U_{\varepsilon_1}(p) \supset U_{\varepsilon_2}(p)$. Then we have

$$(3.7) \quad \begin{aligned} 0 &= \int_{U_{\varepsilon_1}(p) - U_{\varepsilon_2}(p)} f^* P_\alpha^* (\omega \wedge \omega) \\ &= \int_{\partial U_{\varepsilon_1}(p) - \partial U_{\varepsilon_2}(p)} d^c \log \{ |\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2 \} \wedge f^* P_\alpha^* \omega. \end{aligned}$$

Therefore we obtain

$$(3.8) \quad \begin{aligned} &\int_{\partial U_{\varepsilon_1}(p)} d^c \log \{ |\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2 \} \wedge f^* P_\alpha^* \omega \\ &= \lim_{\varepsilon \downarrow 0} \int_{\partial U_\varepsilon(p)} d^c \log \{ |\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2 \} \wedge f^* P_\alpha^* \omega. \end{aligned}$$

The left hand-side of the equation (3.8) is finite and hence so is the right side. In the same way, we see that $n(p, \alpha)$ is independent of the choice of a local coordinate system. Now we shall show that $n(p, \alpha)$ is independent of the choice of F . Take two holomorphic lift F_1 and F_2 of f . Then there exists a holomorphic function g such that $F_1 = gF_2$ and $g(q) \neq 0$ at any $q \in U(p)$. We have

$$\begin{aligned}
(3.9) \quad & d^c \log \{ |\langle F_1, Z_\alpha \rangle|^2 + |\langle F_1, \bar{Z}_\alpha \rangle|^2 \} \\
& = d^c \log |g|^2 + d^c \log \{ |\langle F_2, Z_\alpha \rangle|^2 + |\langle F_2, \bar{Z}_\alpha \rangle|^2 \} \\
& = \frac{1}{4\pi i} [d \log g - d \log \bar{g}] + d^c \log \{ |\langle F_2, Z_\alpha \rangle|^2 + |\langle F_2, \bar{Z}_\alpha \rangle|^2 \}.
\end{aligned}$$

Since the form $f^*P_\alpha^*\omega$ is closed on $\partial U_\alpha(p)$, $n(p, \alpha)$ is independent of the choice of F .

Next suppose that f intersects transversely with ξ_α at $f(p)$. Then

$$\begin{vmatrix}
\langle \partial F / \partial w_1, Z_\alpha \rangle, \langle \partial F / \partial w_2, Z_\alpha \rangle \\
\langle \partial F / \partial w_1, \bar{Z}_\alpha \rangle, \langle \partial F / \partial w_2, \bar{Z}_\alpha \rangle
\end{vmatrix} (p) \neq 0,$$

and hence we can choose $(w_1, w_2) = (\langle F, Z_\alpha \rangle, \langle F, \bar{Z}_\alpha \rangle)$ as a coordinate system on $U(p)$. We have

$$n(p, \alpha) = \lim_{\varepsilon \downarrow 0} \int_{|w_1|^2 + |w_2|^2 = \varepsilon^2} d^c \log (|w_1|^2 + |w_2|^2) \wedge f^*P_\alpha^*\omega.$$

Putting $w_1 = r_1 e^{i\theta_1}$, $w_2 = r_2 e^{i\theta_2}$, $r_1 = r \cos t$ and $r_2 = r \sin t$ ($0 \leq \theta_i \leq 2\pi$, $0 \leq t \leq \pi/2$), we have

$$d^c \log (r_1^2 + r_2^2) = \frac{1}{2\pi} \frac{1}{r_1^2 + r_2^2} (r_1^2 d\theta_1 + r_2^2 d\theta_2),$$

and

$$\begin{aligned}
f^*P_\alpha^*\omega = & \frac{1}{\pi} \frac{1}{(r_1^2 + r_2^2)} (r_1 r_2^2 dr_1 \wedge d\theta_1 + r_1^2 r_2 dr_2 \wedge d\theta_2 \\
& - r_1 r_2^2 dr_1 \wedge d\theta_2 - r_1^2 r_2 dr_2 \wedge d\theta_1).
\end{aligned}$$

Thus we see

$$d^c \log (r_1^2 + r_2^2) \wedge f^*P_\alpha^*\omega = \frac{1}{2\pi^2} \sin t \cos t d\theta_1 \wedge dt \wedge d\theta_2$$

on $r = \text{constant}$.

On the sphere $\{(w_1, w_2) \in U(p) : |w_1|^2 + |w_2|^2 = r^2\}$, $d\theta_1 \wedge dt \wedge d\theta_2$ is a positive form. Therefore we have $n(p, \alpha) = 1$. Q.E.D.

We denote by (z_1, z_2) the standard coordinate system on \mathbf{C}^2 . Put $\Delta(r) = \{(z_1, z_2) \in \mathbf{C}^2 : \log |z_i| < r (i=1, 2)\}$.

Theorem 1. *Let $f: \mathbf{C}^2 \rightarrow Q_{n-1}(\mathbf{C})$ ($n \geq 3$) be a holomorphic mapping satisfying (A) and (B). Suppose $f(\partial \Delta(r)) \cap \xi_\alpha = \phi$. Then we have*

$$(3.10) \quad \int_{\Delta(r)} f^* \Omega^2 = n(\Delta(r), \alpha) + \int_{\partial \Delta(r)} d^c [-\log (|f, \alpha|^2 + |f, \bar{\alpha}|^2) f^*(\Omega + P_\alpha^*\omega)],$$

where $n(\Delta(r), \alpha) = \sum_{f(p_i) \in \xi_\alpha, p_i \in \Delta(r)} n(p_i, \alpha)$.

Proof. By (3.1), Lemma 3.1, (3.5) and (3.5)', we have

$$\begin{aligned}
 (3.11) \quad & \int_{\Delta(r)} f^* \Omega^2 = \lim_{\varepsilon \downarrow 0} \int_{\Delta(r) - \sum_i U_{\varepsilon}(p_i)} f^* \Omega^2 \\
 & = \lim_{\varepsilon \downarrow 0} \int_{\Delta(r) - \sum_i U_{\varepsilon}(p_i)} -dd^c \cdot \log(|f, \alpha|^2 + |f, \bar{\alpha}|^2) \wedge f^*(\Omega + P_\alpha^* \omega) \\
 & = \lim_{\varepsilon \downarrow 0} \int_{\Delta(r) - \sum_i U_{\varepsilon}(p_i)} dd^c [-\log(|f, \alpha|^2 + |f, \bar{\alpha}|^2) f^*(\Omega + P_\alpha^* \omega)] ,
 \end{aligned}$$

where $U_\varepsilon(p_i)$ is such a neighborhood of p_i as given in the definition $n(p_i, \alpha)$. Applying Stokes Theorem to the equation (3.11), we have

$$\begin{aligned}
 (3.12) \quad & \int_{\Delta(r)} f^* \Omega^2 = \int_{\partial \Delta(r)} d^c [-\log(|f, \alpha|^2 + |f, \bar{\alpha}|^2) f^*(\Omega + P_\alpha^* \omega)] \\
 & - \lim_{\varepsilon \downarrow 0} \sum_i \int_{\partial U_{\varepsilon}(p_i)} d^c [\log \|F_i\|^2 f^*(\Omega + P_\alpha^* \omega)] \\
 & + \lim_{\varepsilon \downarrow 0} \sum_i \int_{\partial U_{\varepsilon}(p_i)} d^c [\log \{|\langle F_i, Z_\alpha \rangle|^2 + |\langle F_i, \bar{Z}_\alpha \rangle|^2\} f^* \Omega] \\
 & + \sum_i n(p_i, \alpha) ,
 \end{aligned}$$

where F_i is a holomorphic lift of f on $U(p_i)$. We have

$$(3.13) \quad \lim_{\varepsilon \downarrow 0} \int_{\partial U_{\varepsilon}(p_i)} d^c [\log \|F_i\|^2 \cdot f^* \Omega] = \lim_{\varepsilon \downarrow 0} \int_{U_{\varepsilon}(p_i)} f^* \Omega^2 = 0 .$$

Set $r^2 = |w_i^1|^2 + |w_i^2|^2$, where (w_i^1, w_i^2) denotes a coordinate system on $U(p_i)$, we see

$$(3.14) \quad d^c \log \{|\langle F_i, Z_\alpha \rangle|^2 + |\langle F_i, \bar{Z}_\alpha \rangle|^2\} = 0 \left(\frac{1}{r} \right) (dw_i^1 + d\bar{w}_i^1 + dw_i^2 + d\bar{w}_i^2)$$

and

$$\begin{aligned}
 (3.15) \quad & dd^c \log \{|\langle F_i, Z_\alpha \rangle|^2 + |\langle F_i, \bar{Z}_\alpha \rangle|^2\} = 0 \left(\frac{1}{r^2} \right) (dw_i^1 \wedge d\bar{w}_i^1 + dw_i^1 \wedge d\bar{w}_i^2 \\
 & + dw_i^2 \wedge d\bar{w}_i^2 + dw_i^2 \wedge d\bar{w}_i^1) .
 \end{aligned}$$

Since $\|F_i\|$ is positive on $U(p_i)$, we have

$$(3.16) \quad d^c \log \|F_i\|^2 = 0(1) (dw_i^1 + d\bar{w}_i^1 + dw_i^2 + d\bar{w}_i^2)$$

and

$$(3.17) \quad f^* \Omega = 0(1) (dw_i^1 \wedge d\bar{w}_i^1 + dw_i^1 \wedge d\bar{w}_i^2 + dw_i^2 \wedge d\bar{w}_i^2 + dw_i^2 \wedge d\bar{w}_i^1) .$$

Since the both sides of the equation (3.8) are finite, comparing (3.14) and (3.15) with (3.16) and (3.17), we have

$$(3.18) \quad \lim_{\varepsilon \downarrow 0} \int_{\partial U_{\varepsilon}(p_i)} d^c [\log \|F_i\|^2 \cdot f^* P_\alpha^* \omega] = 0$$

$$(3.19) \quad \lim_{\varepsilon \downarrow 0} \int_{\partial U_{\varepsilon}(p_i)} d^c [\log \{ |\langle F_i, Z_\alpha \rangle|^2 + |\langle F_i, \bar{Z}_\alpha \rangle|^2 \} f^* \Omega] = 0.$$

Q.E.D.

4. First Main Theorem

Let $f : C^2 \rightarrow Q_{n-1}(C)$ ($n \geq 3$) be a holomorphic mapping satisfying (A) and (B). For a point α in $Q_{n-1}(C)$, we choose two real numbers r_1 and r_2 such that $r_1 > r_2$ and the image $f((r(\Delta_1) \setminus \Delta(r_2)))$ does not intersect with ξ_α .

We see easily $|\beta, \alpha|^2 + |\beta, \bar{\alpha}|^2 \leq 1$ for $\beta \in Q_{n-1}(C)$. Hence $\psi_\alpha = -\log (|f, \alpha|^2 + |f, \bar{\alpha}|^2) f^*(\Omega + P_\alpha^* \omega)$ is a positive form (non-negative form, precisely) on $\Delta(r_1) \setminus \Delta(r_2)$. Putting $z_j = e^{s_j + i\theta_j}$ ($j=1, 2$), we can write ψ_α on $\Delta(r_1) \setminus (\Delta(r_2) \cup \{(z, 0) \in C^2\} \cup \{(0, z) \in C^2\})$ as follows:

$$(4.1) \quad \begin{aligned} \psi_\alpha &= -\log (|f, \alpha|^2 + |f, \bar{\alpha}|^2) f^*(\Omega + P_\alpha^* \omega) \\ &= \psi_1 ds_1 \wedge d\theta_1 + \psi_2 ds_1 \wedge d\theta_2 + \psi_3 ds_2 \wedge d\theta_1 \\ &\quad + \psi_4 ds_2 \wedge d\theta_2 + \psi_5 d\theta_1 \wedge d\theta_2 + \psi_6 ds_1 \wedge ds_2. \end{aligned}$$

REMARK 2. If we write ψ_α with the standard coordinate system (z_1, z_2) on C^2 , we see $\psi_1(z_1, z_2) = \tilde{\psi}_1(z_1, z_2) e^{2s_1}$, $\psi_4(z_1, z_2) = \tilde{\psi}_4(z_1, z_2) e^{2s_2}$ and $\psi_j(z_1, z_2) = e^{s_1} \cdot e^{s_2} \tilde{\psi}_j(z_1, z_2)$ ($j=2, 3, 5, 6$) for certain functions $\tilde{\psi}_i$ ($i=1, 2, \dots, 6$).

Lemma 4.1. *We have*

$$(4.2) \quad \psi_1 \geq 0, \psi_4 \geq 0 \text{ and } \psi_2 = \psi_3.$$

Proof. Choosing a holomorphic lift F on a sufficiently small open set U in $\Delta(r_1) \setminus \Delta(r_2)$, we have

$$(4.3) \quad f^*(\Omega + P_\alpha^* \omega) = dd^c [\log ||F||^2 + \log (|\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2)],$$

where $\Pi(Z_\alpha) = \alpha$. Now we obtain

$$(4.4) \quad \begin{aligned} d^c &= \frac{1}{4\pi} \sum_{j=1}^2 \left[\frac{\partial}{\partial s_j} d\theta_j - \frac{\partial}{\partial \theta_j} ds_j \right] \quad \text{on } U \setminus (\{(0, z) \in C^2\} \cup \{(z, 0) \in C^2\}), \\ d &= \sum_{j=1}^2 \left[\frac{\partial}{\partial \theta_j} d\theta_j + \frac{\partial}{\partial s_j} ds_j \right] \end{aligned}$$

where $(e^{s_1+i\theta_1}, e^{s_2+i\theta_2})$ is the restriction to U of the standard coordinate system in C^2 . Putting $g = \log (|\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2) + \log ||F||^2$, we have

$$(4.5) \quad \begin{aligned} dd^c g &= \frac{1}{4\pi} \left[\left(\frac{\partial^2 g}{(\partial \theta_1)^2} + \frac{\partial^2 g}{(\partial s_1)^2} \right) ds_1 \wedge d\theta_1 + \left(\frac{\partial^2 g}{\partial \theta_2 \partial \theta_1} + \frac{\partial^2 g}{\partial s_1 \partial s_2} \right) ds_1 \wedge d\theta_2 \right. \\ &\quad \left. + \left(\frac{\partial^2 g}{\partial \theta_1 \partial \theta_2} + \frac{\partial^2 g}{\partial s_2 \partial s_1} \right) ds_2 \wedge d\theta_1 + \left(\frac{\partial^2 g}{(\partial \theta_2)^2} + \frac{\partial^2 g}{(\partial s_2)^2} \right) ds_2 \wedge d\theta_2 + \dots \right]. \end{aligned}$$

Comparing (4.1) with (4.5), we have $\psi_2 = \psi_3$.

We shall show $\psi_1 \geqq 0$ and $\psi_4 \geqq 0$.

$$\begin{aligned}
 (4.6) \quad dd^c \log(\sum_j f^j \bar{f}^j) &= \frac{i}{2\pi} \partial \bar{\partial} \cdot \log(\sum_j f^j \bar{f}^j) \\
 &= \frac{i}{2\pi} \frac{1}{\|F\|^4} [\|F\|^2 (\sum_j df^j \wedge d\bar{f}^j) - (\sum_k df^k \bar{f}^k) \wedge (\sum_j f^j d\bar{f}^j)] \\
 &= \frac{i}{2\pi} \frac{1}{\|F\|^4} \left[\left(\|F\|^2 \left\| \frac{\partial F}{\partial z_1} \right\|^2 - \left| \left(\frac{\partial F}{\partial z_1}, F \right) \right|^2 \right) dz_1 \wedge d\bar{z}_1 \right. \\
 &\quad \left. + \left(\|F\|^2 \left\| \frac{\partial F}{\partial z_2} \right\|^2 - \left| \left(\frac{\partial F}{\partial z_2}, F \right) \right|^2 \right) dz_2 \wedge d\bar{z}_2 + \dots \right],
 \end{aligned}$$

where $F = (f^0, f^1, \dots, f^n)$. By the Schwartz inequality and the linear independence of vectors F and $\partial F / \partial z_j$ ($j=1, 2$), we have

$$\left\| F \right\|^2 \left\| \frac{\partial F}{\partial z_j} \right\|^2 > \left| \left(\frac{\partial F}{\partial z_j}, F \right) \right|^2, \text{ and } dz_j \wedge d\bar{z}_j = e^{2s_j} (-2i ds_j \wedge d\theta_j)$$

($j=1, 2$). Thus we have

$$\frac{1}{\pi} \frac{1}{\|F\|^4} \left[\left\| F \right\|^2 \left\| \frac{\partial F}{\partial z_j} \right\|^2 - \left| \left(\frac{\partial F}{\partial z_j}, F \right) \right|^2 \right] e^{2s_j} > 0 \quad (j=1, 2)$$

or

$$(4.7) \quad \frac{1}{\pi} \frac{1}{(\sum_k f^k \bar{f}^k)^2} \left[(\sum_k f^k \bar{f}^k) \left(\sum_k \frac{\partial f^k}{\partial z_j} \frac{\partial \bar{f}^k}{\partial z_j} \right) - \left| \left(\sum_k \frac{\partial f^k}{\partial z_j} \bar{f}^k \right) \right|^2 \right] e^{2s_j} > 0 \quad (j=1, 2).$$

As for $dd^c [\log(|\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2)]$, putting $f^0 = \langle F, Z_\alpha \rangle$, $f^1 = \langle F, \bar{F}_\alpha \rangle$ and $f^j = 0$ ($j=2, \dots, n$) in the equation (4.6), we have also the inequality (4.7) (in this case we replace $>$ by $\geqq 0$) with respect to the coefficient of $ds_j \wedge d\theta_j$ ($j=1, 2$).

Q.E.D.

Let r be in $[r_2, r_1]$. We devide $\partial\Delta(r)$ into $\partial\Delta_1(r)$ and $\partial\Delta_2(r)$, where

$$(4.8) \quad \partial\Delta_i(r) = \{(z_1, z_2) \in \partial\Delta(r) : \log|z_i| = r\} \quad (i=1, 2).$$

Lemma 4.2. *We have*

$$\begin{aligned}
 (4.9) \quad \int_{\partial\Delta(r)} d^c \psi_\alpha &= \frac{1}{4\pi} \left[- \int_{S^1 \times S^1} \psi_4(e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_2 \wedge d\theta_1 \right. \\
 &\quad \left. - \int_{S^1 \times S^1} \psi_1(e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_1 \wedge d\theta_2 \right] \\
 &\quad + \frac{1}{4\pi} \frac{\partial}{\partial r} \left[\int_{\partial\Delta_1(r)} \psi_\alpha \wedge d\theta_1 + \int_{\partial\Delta_2(r)} \psi_\alpha \wedge d\theta_2 \right].
 \end{aligned}$$

Proof. First we remark that $d\theta_1 \wedge ds_2 \wedge d\theta_2$ and $d\theta_2 \wedge ds_1 \wedge d\theta_1$ are positive forms on $\partial\Delta_1(r)$ and $\partial\Delta_2(r)$ respectively.

By (4.1) and the preceding remark 2, we have

$$\begin{aligned} \int_{\partial\Delta_1(r)} d^c \psi_\alpha &= \int_{\partial\Delta_1(r) \setminus \{(e^{r+i\theta_1}, 0)\}} d^c \psi_\alpha \\ &= \frac{1}{4\pi} \int_{\partial\Delta_1(r) \setminus \{(e^{r+i\theta_1}, 0)\}} \left[-\frac{\partial \psi_3}{\partial s_2} + \frac{\partial \psi_4}{\partial s_1} + \frac{\partial \psi_5}{\partial \theta_2} \right] d\theta_1 \wedge ds_2 \wedge d\theta_2 \\ &= \frac{1}{4\pi} \int_{\partial\Delta_1(r)} \left[-\frac{\partial \psi_3}{\partial s_2} + \frac{\partial \psi_4}{\partial s_1} + \frac{\partial \psi_5}{\partial \theta_2} \right] d\theta_1 \wedge ds_2 \wedge d\theta_2. \end{aligned}$$

Clearly we have

$$\int_{\partial\Delta_1(r)} \frac{\partial \psi_5}{\partial \theta_2} d\theta_1 \wedge ds_2 \wedge d\theta_2 = 0.$$

Therefore we obtain

$$(4.10) \quad \int_{\partial\Delta_1(r)} d^c \psi_\alpha = \frac{1}{4\pi} \int_{\partial\Delta_1(r)} \left[-\frac{\partial \psi_3}{\partial s_2} + \frac{\partial \psi_4}{\partial s_1} \right] d\theta_1 \wedge ds_2 \wedge d\theta_2.$$

Similarly we obtain

$$(4.11) \quad \int_{\partial\Delta_2(r)} d^c \psi_\alpha = \frac{1}{4\pi} \int_{\partial\Delta_2(r)} \left[\frac{\partial \psi_1}{\partial s_2} - \frac{\partial \psi_2}{\partial s_1} \right] d\theta_2 \wedge ds_1 \wedge d\theta_1.$$

Now we shall consider the equation (4.10). We have

$$\begin{aligned} (4.12) \quad & \frac{1}{4\pi} \int_{\partial\Delta_1(r)} \frac{\partial \psi_3}{\partial s_2} d\theta_1 \wedge ds_2 \wedge d\theta_2 \\ &= \frac{1}{4\pi} \int_{\partial\Delta_1(r)} d(\psi_3 d\theta_2 \wedge d\theta_1) \\ &= \frac{1}{4\pi} \int_{\partial\Delta_1(r) \cap \partial\Delta_2(r)} \psi_3 d\theta_2 \wedge d\theta_1 \\ &= \frac{1}{4\pi} \int_{S^1 \times S^1} \psi_3(e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_2 \wedge d\theta_1. \end{aligned}$$

Since we have

$$\begin{aligned} & \int_{\partial\Delta_1(r)} \psi_4 d\theta_1 \wedge ds_2 \wedge d\theta_2 \\ &= \int_{\partial\Delta_1(r)} d \left\{ \left(\int_{-\infty}^{s_2} \psi_4(e^{r+i\theta_1}, e^{t+i\theta_2}) dt \right) d\theta_2 \wedge d\theta_1 \right\} \\ &= \int_{S^1 \times S^1} \left(\int_{-\infty}^r \psi_4(e^{r+i\theta_1}, e^{t+i\theta_2}) dt \right) d\theta_2 \wedge d\theta_1, \end{aligned}$$

we obtain

$$\begin{aligned}
(4.13) \quad & \frac{\partial}{\partial r} \int_{\partial\Delta_1(r)} \psi_4 d\theta_1 \wedge ds_2 \wedge d\theta_2 \\
&= \int_{S^1 \times S^1} \psi_4(e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_2 \wedge d\theta_1 \\
&+ \int_{S^1 \times S^1} \left(\int_{-\infty}^r \frac{\partial \psi_4}{\partial r}(e^{r+i\theta_1}, e^{t+i\theta_2}) dt \right) d\theta_2 \wedge d\theta_1.
\end{aligned}$$

By (4.10), (4.12) and (4.13), we obtain

$$\begin{aligned}
(4.14) \quad \int_{\partial\Delta_1(r)} d^c \psi_\alpha &= \frac{1}{4\pi} \int_{S^1 \times S^1} [-\psi_3 - \psi_4](e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_2 \wedge d\theta_1 \\
&+ \frac{1}{4\pi} \frac{\partial}{\partial r} \int_{\partial\Delta_1(r)} \psi_4 d\theta_1 \wedge ds_2 \wedge d\theta_2.
\end{aligned}$$

By the similar argument as we derived (4.14) from (4.10), we derive the following from (4.11)

$$\begin{aligned}
(4.15) \quad \frac{1}{4\pi} \int_{\partial\Delta_2(r)} d^c \psi_\alpha &= \frac{1}{4\pi} \int_{S^1 \times S^1} [-\psi_2 - \psi_1](e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_1 \wedge d\theta_2 \\
&+ \frac{1}{4\pi} \frac{\partial}{\partial r} \int_{\partial\Delta_2(r)} \psi_1 d\theta_2 \wedge ds_1 \wedge d\theta_1.
\end{aligned}$$

By (4.14), (4.15) and the definition of ψ_α we obtain (4.9). Q.E.D.

Lemma 4.3. *We have*

$$(4.16) \quad \int_{\Delta(r)} f^* \Omega^2 = \frac{1}{4\pi} \frac{\partial}{\partial r} \left[\int_{\partial\Delta_1(r)} \psi_\alpha \wedge d\theta_1 + \int_{\partial\Delta_2(r)} \psi_\alpha \wedge d\theta_2 \right] + n(\Delta(r), \alpha).$$

Proof. By Theorem 1 and Lemma 4.2, we have only to prove that

$$\frac{1}{4\pi} \int_{S^1 \times S^1} [\psi_4 - \psi_1](e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_2 \wedge d\theta_1 = 0.$$

We define a mapping $h: \mathbf{C}^2 \rightarrow \mathbf{C}^2$ by $h((z_1, z_2)) = (z_2, z_1)$. Then $(f \circ h)$ satisfies Conditions (A) and (B), and we have

$$(|f \circ h, \alpha|^2 + |f \circ h, \bar{\alpha}|^2)(z_1, z_2) = (|f, \alpha|^2 + |f, \bar{\alpha}|^2)(z_2, z_1)$$

and

$$\begin{aligned}
n_f((z_1, z_2), \alpha) &= \lim_{\varepsilon \downarrow 0} \int_{\partial U_\varepsilon((z_1, z_2))} d^c \log [|\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2] \wedge f^* P_\alpha^* \omega \\
&= \lim_{\varepsilon \downarrow 0} \int_{\partial U_\varepsilon((z_2, z_1))} d^c \log [|\langle F \circ h, Z_\alpha \rangle|^2 + |\langle F \circ h, \bar{Z}_\alpha \rangle|^2] \wedge (fh)^* P_\alpha^* \omega \\
&= n_{f \circ h}((z_2, z_1), \alpha).
\end{aligned}$$

On the other hand, we have from (4.1)

$$(4.17) \quad \begin{aligned} (h^* \psi_\alpha) &= \psi_1 \circ h \, ds_2 \wedge d\theta_2 + \psi_2 \circ h \, ds_2 \wedge d\theta_1 + \psi_3 \circ h \, ds_1 \wedge d\theta_2 \\ &\quad + \psi_4 \circ h \, ds_1 \wedge d\theta_1 + \psi_5 \circ h \, d\theta_2 \wedge d\theta_1 + \psi_6 \circ h \, ds_2 \wedge ds_1. \end{aligned}$$

By Theorem 1, (4.14) and (4.15) in Lemma 4.2, comparing (4.1) with (4.17) we have

$$(4.18) \quad \begin{aligned} \int_{\Delta(r)} f^* \Omega^2 &= \int_{\Delta(r)} h^* f^* \Omega^2 = n(\Delta(r), \alpha) \\ &\quad + \frac{1}{4\pi} \left[- \int_{S^1 \times S^1} \psi_1 \circ h(e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_2 \wedge d\theta_1 - \int_{S^1 \times S^1} \psi_4 \circ h(e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_1 \wedge d\theta_2 \right] \\ &\quad + \frac{1}{4\pi} \frac{\partial}{\partial r} \left[\int_{\partial \Delta_1(r)} \psi_1 \circ h \, d\theta_1 \wedge ds_2 \wedge d\theta_2 + \int_{\partial \Delta_2(r)} \psi_4 \circ h \, d\theta_2 \wedge ds_1 \wedge d\theta_1 \right]. \end{aligned}$$

We see easily

$$\begin{aligned} \int_{\partial \Delta_1(r)} \psi_1 \circ h \, d\theta_1 \wedge ds_2 \wedge d\theta_2 &= \int_{\partial \Delta_2(r)} \psi_1 \, d\theta_2 \wedge ds_1 \wedge d\theta_1 \\ &= \int_{\partial \Delta_2(r)} \psi_\alpha \wedge d\theta_2 \end{aligned}$$

and

$$\begin{aligned} \int_{\partial \Delta_2(r)} \psi_4 \circ h \, d\theta_2 \wedge ds_1 \wedge d\theta_1 &= \int_{\partial \Delta_1(r)} \psi_4 \, d\theta_1 \wedge ds_2 \wedge d\theta_2 \\ &= \int_{\partial \Delta_1(r)} \psi_\alpha \wedge d\theta_1. \end{aligned}$$

Therefore we have only to prove

$$\int_{S^1 \times S^1} ((\psi_i \circ h) - \psi_i)(e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_1 \wedge d\theta_2 = 0 \quad (i = 1, 4).$$

For any $\alpha, \beta \in [0, 2\pi]$, we have

$$\begin{aligned} ((\psi_i \circ h) - \psi_i)(e^{r+i\alpha}, e^{r+i\beta}) &= \psi_i(e^{r+i\beta}, e^{r+i\alpha}) - \psi_i(e^{r+i\alpha}, e^{r+i\beta}) \\ ((\psi_i \circ h) - \psi_i)(e^{r+i\beta}, e^{r+i\alpha}) &= \psi_i(e^{r+i\alpha}, e^{r+i\beta}) - \psi_i(e^{r+i\beta}, e^{r+i\alpha}) \end{aligned}$$

Thus we obtain

$$((\psi_i \circ h) - \psi_i)(e^{r+i\alpha}, e^{r+i\beta}) = -((\psi_i \circ h) - \psi_i)(e^{r+i\beta}, e^{r+i\alpha}).$$

Q.E.D.

For the holomorphic mapping $f: \mathbf{C}^2 \rightarrow Q_{n-1}(\mathbf{C})$ ($n \geq 3$) satisfying Conditions (A) and (B), we put

$$T(r) = \int_0^r dt \int_{\Delta(t)} f^* \Omega^2 \quad (\text{order function})$$

$$(4.19) \quad \begin{aligned} N(r, \alpha) &= \int_0^r n(\Delta(t), \alpha) dt \quad (\text{counting function}) \\ m(r, \alpha) &= \frac{1}{4\pi} \left[\int_{\partial\Delta_1(r)} \psi_\alpha \wedge d\theta_1 + \int_{\partial\Delta_2(r)} \psi_\alpha \wedge d\theta_2 \right]. \end{aligned}$$

We need the following lemma, which can be proved in a similar way as ([5] p.p. 502).

Lemma 4.4. *For any α , $m(r, \alpha)$ is continuous with respect to $r \in [0, \infty)$.*

Theorem 2. *We have*

$$(4.20) \quad T(r) = m(r, \alpha) - m(0, \alpha) + N(r, \alpha) \quad \text{for any } r \geq 0,$$

and $m(r, \alpha)$ is non-negative.

Proof. Integrating the equation in Lemma 4.3 with respect to $r \in [r_2, r_1]$, we have

$$\int_{r_2}^{r_1} dr \int_{\Delta(r)} f^* \Omega^2 = \int_{r_2}^{r_1} n(\Delta(r), \alpha) dr + m(r_1, \alpha) - m(r_2, \alpha).$$

By Lemma 4.4 we obtain the equation (4.20). It follows from Lemma 4.1 and Lemma 4.4 that the function $m(r, \alpha)$ is non-negative. Q.E.D.

Lemma 4.5. *For any r , $m(r, \alpha)$ is continuous with respect to $\alpha \in Q_{n-1}(C)$.*

We also omit this proof by the same reason as in Lemma 4.4. (c.f. [5] p.p. 504).

Theorem 3. *There exists a positive constant C satisfying*

$$(4.21) \quad T(r) + C > N(r, \alpha) \quad \text{whenever} \quad r \geq 0 \quad \text{and} \quad \alpha \in Q_{n-1}(C).$$

Proof. By Theorem 2 we have

$$T(r) + m(0, \alpha) \geq N(r, \alpha) \quad \text{for any } r \geq 0.$$

Therefore by Lemma 4.5 we have the equation (4.21). Q.E.D.

5. Induced form by f

We denote by (X_0, X_1, \dots, X_n) an element of $SO(n+1)$, where X_i 's ($0 \leq i \leq n$) are column vectors, and we put $X_i = (x_{i0}, \dots, x_{in})^t$. The left invariant forms θ_{ij} ($0 \leq i, j \leq n$) on $SO(n+1)$ are defined by the following equation:

$$(5.1) \quad - \begin{pmatrix} dX_0^t \\ dX_1^t \\ \vdots \\ dX_n^t \end{pmatrix} (X_0, \dots, X_n) = \begin{pmatrix} X_0^t \\ X_1^t \\ \vdots \\ X_n^t \end{pmatrix} (dX_0, \dots, dX_n) = \begin{pmatrix} 0, & \theta_{10}, & \dots, & \theta_{n0} \\ \theta_{01}, & 0, & \dots, & \theta_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{0n}, & \theta_{1n}, & \dots, & 0 \end{pmatrix},$$

where $\theta_{ij} = -\theta_{ji}$.

Therefore we have $-\langle dX_i, X_j \rangle = \theta_{ji}$ i.e.,

$$(5.2) \quad dX_i = \sum_j \theta_{ij} X_j.$$

Taking its exterior derivative, we see

$$(5.3) \quad d\theta_{01} = \sum_k \theta_{0k} \wedge \theta_{k1} = -\sum_k \theta_{0k} \wedge \theta_{1k}.$$

We remark that $d\theta_{01}$ is a 2-form on $SO(n+1)/SO(n-1)$. Furthermore it is a lift of a 2-form on $Q_{n-1}(C)$ by Π_1 . In fact, let U be an open neighborhood of $Q_{n-1}(C)$, and (X_0, X_1) be a local cross-section of U into $SO(n+1)/SO(n-1)$: $\Pi_1((X_0, X_1)) = \text{identity on } U$. We have

$$(5.4) \quad \Pi_1^{-1}(\Pi_1(X_0, X_1)) = \{(X_0, X_1) \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} : 0 \leq \theta < 2\pi\}.$$

Then we have on $\Pi_1^{-1}(U)$,

$$(5.5) \quad \begin{aligned} d\theta_{01} &= d\langle d(\cos \theta \cdot X_0 + \sin \theta \cdot X_1), (-\sin \theta \cdot X_1 + \cos \theta \cdot X_0) \rangle \\ &= d(d\theta + \langle dX_0, X_1 \rangle) = d\langle dX_0, X_1 \rangle. \end{aligned}$$

Let σ be a local holomorphic cross-section on U into $C^{n+1} - \{0\}$ with respect to the Hopf fibring: $\Pi \sigma = \text{identity on } U$. We can write σ in the form $\sigma = X + iY$ for orthogonal real vectors X and Y at each point of U . Then we see

$$(5.6) \quad \Omega = dd^c \log \|\sigma\|^2 = -\frac{1}{2\pi} d\langle d(X/\|X\|), Y/\|Y\| \rangle.$$

Thus, $d\theta_{01}$ is the lift of $-2\pi\Omega$ by Π_1^* i.e.,

$$(5.7) \quad \Pi_1^* \Omega = -\frac{1}{2\pi} d\theta_{01}.$$

In the equation (5.1) we defined θ_{0j} 's and θ_{1j} 's ($0 \leq j \leq n$) as 1-forms on $SO(n+1)$. They are also regarded as 1-forms on $SO(n+1)/SO(n-1)$. To prove this fact we shall identify $SO(n+1)/SO(n-1)$ with $S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(C))$. We take a local coordinate $x = (x^1, \dots, x^{2n-1})$ on a small open set U in $S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(C))$ and write a point $Z(x)$ of U in the form $(X_0(x) + iX_1(x))/\sqrt{2}$, where $\langle X_0, X_0 \rangle(x) = \langle X_1, X_1 \rangle(x) = 1$ and $\langle X_0, X_1 \rangle(x) = 0$. For each x , extending $X_0(x)$ and $X_1(x)$, we take a real orthonormal basis $X_0(x), \dots, X_n(x)$ in C^{n+1} such that $(X_0, \dots, X_n)(x) \in SO(n+1)$. Then the tangent space $T_{Z(x)}(S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(C)))$ has a basis $(iX_0 - X_1)(x), X_2(x), \dots, X_n(x), iX_2(x), \dots, X_n(x)$ (c.f. [3] p.p. 279). In the equation $dZ = \sum_{i=1}^{2n-1} \frac{\partial Z}{\partial x^i} dx^i$, we see $\frac{\partial Z}{\partial x^i} = Z_* \left(\frac{\partial}{\partial x^i} \right)$ ($1 \leq i \leq 2n-1$) and hence $\frac{\partial Z}{\partial x^i}$'s are tangent vectors of $T_{Z(x)}(S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(C)))$. Thus there exists 1-

forms θ_j 's ($1 \leq j \leq n$) and $\tilde{\theta}_j$'s ($2 \leq j \leq n$) on U such that $dZ = \theta_1(iX_0 - X_1) + \sum_{j=2}^n (\theta_j + i\tilde{\theta}_j)X_j$. Comparing this form with (5.2), we have $\theta_1 = \theta_{10}/\sqrt{2}$, $\theta_j = \theta_{0j}/\sqrt{2}$ ($2 \leq j \leq n$) and $\tilde{\theta}_j = \theta_{1j}/\sqrt{2}$ ($2 \leq j \leq n$). Thus we have from (5.2), (5.3) and (5.7)

$$(5.8) \quad (\Pi_i^* \Omega)_{(X_0, X_1)} = \frac{1}{2\pi} \sum_{j=2}^n \langle dX_0, X_j \rangle \wedge \langle dX_1, X_j \rangle,$$

where $(X_0, X_1, \dots, X_n) \in SO(n+1)$. For the volume form Ω^{n-1} on $Q_{n-1}(\mathbf{C})$, we have

$$(5.9) \quad (\Pi_i^* \Omega^{n-1})_{(X_0, X_1)} = \left(\frac{1}{2\pi} \right)^{n-1} (n-1)! \langle dX_0, X_2 \rangle \wedge \langle dX_1, X_2 \rangle \wedge \dots \wedge \langle dX_0, X_n \rangle \wedge \langle dX_1, X_n \rangle.$$

We shall obtain a formula for $f^* \Omega^2$ on \mathbf{C}^2 . Let F be a holomorphic lift of f on a neighborhood U in \mathbf{C}^2 by Π . Set $(X_0 + iX_1)/\sqrt{2} = F/\|F\|$, where X_i ($i=0, 1$) are the orthonormal real vectors. With the coordinate system $(x_1 + iy_1, x_2 + iy_2)$ on \mathbf{C}^2 , we can write:

$$(5.10) \quad \begin{aligned} dX_0 &= \omega_1 X_1 + \lambda_2 \tilde{B}_2 dx_1 - \lambda_3 \tilde{B}_3 dy_1 + \lambda_4 \tilde{B}_4 dx_2 - \lambda_5 \tilde{B}_5 dy_2, \\ dX_1 &= \omega_2 X_0 + \lambda_3 \tilde{B}_3 dx_1 + \lambda_2 \tilde{B}_2 dy_1 + \lambda_5 \tilde{B}_5 dx_2 + \lambda_4 \tilde{B}_4 dy_2, \end{aligned}$$

where \tilde{B}_i 's ($2 \leq i \leq 5$) are differentiable vectors satisfying $\langle \tilde{B}_i, \tilde{B}_i \rangle = 1$, λ_i 's ($2 \leq i \leq 5$) are differentiable functions and ω_i 's ($1 \leq i \leq 2$) are 1-forms on U . Then we take differentiable orthonormal vectors B_i ($2 \leq i \leq 5$) such that $\tilde{B}_2 = B_2$, $\tilde{B}_3 = \alpha_2 B_2 + \alpha_3 B_3$, $\tilde{B}_4 = \beta_2 B_2 + \beta_3 B_3 + \beta_4 B_4$ and $\tilde{B}_5 = \gamma_2 B_2 + \gamma_3 B_3 + \gamma_4 B_4 + \gamma_5 B_5$, where α_i , β_i and γ_i are differentiable functions satisfying $\sum \alpha_i^2 = 1$, $\sum \beta_i^2 = 1$ and $\sum \gamma_i^2 = 1$. We choose differentiable vectors B_6, \dots, B_n on U such that $(X_0, X_1, B_2, \dots, B_n) \in SO(n+1)$ at each point of U . By (5.8) we have

$$(5.11) \quad \begin{aligned} f^* \Omega &= \frac{1}{2\pi} \sum_{j=2}^n \langle dX_0, B_j \rangle \wedge \langle dX_1, B_j \rangle \\ &= \frac{1}{2\pi} \{ [\lambda_2 \lambda_5 \gamma_2 - \lambda_3 \lambda_4 \alpha_2 \beta_2 - \lambda_3 \lambda_4 \beta_3 \alpha_3] (dx_1 \wedge dx_2 + dy_1 \wedge dy_2) \\ &\quad + [\lambda_2^2 + \lambda_3^2] dx_1 \wedge dy_1 + [\lambda_4^2 + \lambda_5^2] dx_2 \wedge dy_2 \\ &\quad + [\lambda_2 \lambda_4 \beta_2 + \lambda_3 \lambda_5 \alpha_2 \gamma_2 + \lambda_3 \lambda_5 \alpha_3 \gamma_3] (dx_1 \wedge dy_2 - dy_1 \wedge dx_2) \}. \end{aligned}$$

Furthermore we obtain

$$(5.12) \quad \begin{aligned} f^* \Omega^2 &= \left(\frac{1}{2\pi} \right)^2 \times 2 \times \{ [\lambda_2^2 + \lambda_3^2] [\lambda_4^2 + \lambda_5^2] \\ &\quad - [\lambda_2 \lambda_4 \beta_2 + \lambda_3 \lambda_5 \alpha_2 \gamma_2 + \lambda_3 \lambda_5 \alpha_3 \gamma_3]^2 \\ &\quad - [\lambda_2 \lambda_5 \gamma_2 - \lambda_3 \lambda_4 \alpha_2 \beta_2 - \lambda_3 \lambda_4 \alpha_3 \beta_3]^2 \} dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2. \end{aligned}$$

6. Crofton formula

In §3 we have defined $n(\Delta(r), \alpha)$ for a holomorphic mapping $f: C^2 \rightarrow Q_{n-1}(C)$ ($n \geq 3$) satisfying Conditions (A) and (B). Then we have:

Theorem 4 (Crofton formula). *Let D be an open set in C^2 with compact closure. Then we have*

$$(6.1) \quad \int_{Q_{n-1}(C)} n(D, \xi) d\xi = 2 \int_D f^* \Omega^2,$$

where $d\xi = d\xi_\alpha = d\alpha = \Omega^{n-1}$.

Proof. First we assume that D is so small that there exists a differentiable lift $\sigma = (X_0, X_1)$ of f on D : $\Pi_1 \sigma = f$. Let q be a point in D and set $f(q) \in \xi_\alpha$. For any real orthonormal vectors Y_0, Y_1 such that $\Pi_1((Y_0, Y_1)) = \alpha$, we have

$$(6.2) \quad \langle X_0(q), Y_0 \rangle = \langle X_0(q), Y_1 \rangle = \langle X_1(q), Y_0 \rangle = \langle X_1(q), Y_1 \rangle = 0.$$

We set

$$(6.3) \quad \begin{aligned} Q_{n-3}(f(q)^\perp) &= \{\alpha \in Q_{n-1}(C) : f(q) \in \xi_\alpha\} \\ f(D)^\perp &= \{\alpha \in Q_{n-1}(C) : f(D) \cap \xi_\alpha \neq \emptyset\}. \end{aligned}$$

and

$$(6.4) \quad \begin{aligned} D' &= \Pi_1^{-1}(f(D)^\perp) \\ D'' &= \{(q, a) : q \in D, a = (A_2, A_3, \dots, A_n) \in SO(n-1)\}. \end{aligned}$$

For $a = (A_2, A_3, \dots, A_n) \in SO(n-1)$ we write its column vector A_i as $A_i = (a_{i2}, \dots, a_{in})^t$. Then we define a mapping $t: D'' \rightarrow SO(n+1)$ by

$$(6.5) \quad t((q, a)) = (B_2, B_3, X_0, X_1, B_4, \dots, B_n)(q)$$

$$\times \begin{pmatrix} a_{22} & a_{32} & 0 & 0 & a_{42} & \dots & a_{n2} \\ a_{23} & a_{33} & 0 & 0 & a_{43} & \dots & a_{n3} \\ 0 & 0 & 1 & 0 & 0 & & 0 \\ 0 & 0 & 0 & 1 & 0 & & 0 \\ a_{24} & a_{34} & 0 & 0 & a_{44} & \dots & a_{n4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{2n} & a_{3n} & 0 & 0 & a_{4n} & \dots & a_{nn} \end{pmatrix},$$

where $(X_0, X_1, B_2, \dots, B_n)(q)$ is the one given in §5. Let Π' be the projection $D \times (SO(n-1)/SO(n-3)) \rightarrow D \times Q_{n-3}(C)$ defined by $\Pi'((q, (A_2, A_3))) = (q, \Pi''((A_2, A_3)))$, where Π'' is the projection with respect to the Hopf fibring $SO(n-1)/SO(n-3) \rightarrow Q_{n-3}(C)$. We consider the following diagram;

$$(6.6) \quad \begin{array}{ccc} D \times (SO(n-1)/SO(n-3)) & \xrightarrow{t'} & D' \subset SO(n+1)/SO(n-1) \\ \downarrow \Pi' & & \downarrow \Pi_1 \\ D \times Q_{n-3}(\mathbf{C}) & \xrightarrow{t''} & f(D)^\perp \subset Q_{n-1}(\mathbf{C}), \end{array}$$

where $t'((q, (A_2, A_3))) = (\sum_{i=2}^n a_{2i} B_i(q), \sum_{i=2}^n a_{3i} B_i(q))$ and t'' is defined by $\Pi_1 \circ t' = t'' \circ \Pi'$. Then, in the above diagram, we remark that $t''((q, Q_{n-3}(\mathbf{C}))) = Q_{n-3}(f(q)^\perp)$ for each $q \in D$. Putting $t((q, a)) = (X_0', X_1', \dots, X_n')$, we obtain

$$(6.7) \quad \begin{aligned} & (\Pi')^*(t'')^* \Omega^{n-1} = (t')^*(\Pi_1)^* \Omega^{n-1} \\ &= \left(\frac{1}{2\pi}\right)^{n-1} (n-1)! \langle dX_0', X_2' \rangle \wedge \langle dX_1', X_2' \rangle \wedge \dots \wedge \langle dX_0', X_n' \rangle \wedge \langle dX_1', X_n' \rangle \\ &= \left(\frac{1}{2\pi}\right)^{n-1} (n-1)! \times \frac{1}{16} \times \langle d(X_0 + iX_1), X_0' + iX_1' \rangle \wedge \langle d(X_0 - iX_1), X_0' - iX_1' \rangle \\ & \quad \wedge \langle d(X_0 + iX_1), X_0' - iX_1' \rangle \wedge \langle d(X_0 - iX_1), X_0' + iX_1' \rangle \wedge \langle dA_2, A_4 \rangle \\ & \quad \wedge \langle dA_3, A_4 \rangle \wedge \dots \wedge \langle dA_2, A_n \rangle \wedge \langle dA_3, A_n \rangle \\ &= -\frac{1}{4} \left(\frac{1}{2\pi}\right)^2 (n-1) (n-2) \left| \begin{array}{c} \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, X_0' + iX_1' \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, X_0' + iX_1' \rangle \\ \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, X_0' - iX_1' \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, X_0' - iX_1' \rangle \end{array} \right|^2 \\ & \quad \times dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \left(\frac{1}{2\pi}\right)^{n-3} (n-3)! \langle dA_2, A_4 \rangle \wedge \langle dA_3, A_4 \rangle \wedge \dots \\ & \quad \wedge \langle dA_2, A_n \rangle \wedge \langle dA_3, A_n \rangle. \end{aligned}$$

We put $C = \{\beta \in f(D)^\perp : \text{there exists } \beta' \in (t'')^{-1}(\beta) \text{ such that } (dt'')(\beta') \text{ is singular}\}$. From Sard's Theorem the set C has measure zero. If we take $\alpha \in (f(D)^\perp \setminus C)$, the set $(t'')^{-1}(\alpha)$ consists of finite elements because of the compactness of \bar{D} and Condition (B). We denote by n_α the number of elements $(t'')^{-1}(\alpha)$. Then, for each $\alpha \in (f(D)^\perp \setminus C)$ there exists a connected neighborhood V of α in $(f(D)^\perp \setminus C)$ such that $(t'')^{-1}(V)$ has n_α connected components and t'' maps each component onto V diffeomorphically. Let $\{V_i\}$ be a locally finite covering of $f(D)^\perp \setminus C$ by such open sets and $\{\phi_i\}$ be a partition of unity subordinated to $\{V_i\}$. Now we have

$$(6.8) \quad \begin{aligned} \int_{f(D)^\perp} n_\alpha d\alpha &= \int_{f(D)^\perp \setminus C} n_\alpha d\alpha = \sum_i \int_{f(D)^\perp \setminus C} \phi_i(\alpha) n_\alpha d\alpha \\ &= \sum_i \int_{V_i} n_\alpha(\phi_i(\alpha) d\alpha) = \sum_i \int_{(t'')^{-1}(V_i)} -(t'')^*(\phi_i(\alpha) d\alpha) \\ &= \sum_i \int_{(t'')^{-1}(V_i)} -((t'')^* \phi_i(\alpha)) ((t'')^* d\alpha) \\ &= \int_{D \times Q_{n-2} \setminus C'} -(t'')^* d\alpha = \int_{D \times Q_{n-3}} -(t'')^* d\alpha, \end{aligned}$$

where C' is the set of critical points of t'' . If

$$t''((q, \alpha_j)) = \alpha \text{ and } \begin{vmatrix} \langle \partial F / \partial z_1, Z_\alpha \rangle, \langle \partial F / \partial z_2, Z_\alpha \rangle \\ \langle \partial F / \partial z_1, \bar{Z}_\alpha \rangle, \langle \partial F / \partial z_2, \bar{Z}_\alpha \rangle \end{vmatrix}(q) \\ \left(\text{which is equal to } \frac{\|F\|}{2} \begin{vmatrix} \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, Z_\alpha \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, Z_\alpha \rangle \\ \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, \bar{Z}_\alpha \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, \bar{Z}_\alpha \rangle \end{vmatrix}(q) \right) = 0$$

for $\Pi(Z_\alpha) = \alpha$, then $dt''((q, \alpha_j))$ is singular because of (6.7). By Lemma 3.2 we have $n(D, \alpha) = n_\alpha$ on $f(D)^\perp \setminus C$. Therefore we have

$$(6.9) \quad \int_{Q_{n-1}} n(D, \alpha) d\alpha = \frac{1}{4} \left(\frac{1}{2\pi} \right)^2 (n-1)(n-2) \int_D dx^1 \wedge dy^1 \wedge dx^2 \wedge dy^2 \\ \times \int_{Q_{n-3}(f(q)^\perp)} \left| \begin{vmatrix} \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, X'_0 + iX_1 \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, X'_0 + iX_1 \rangle \\ \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, X'_0 - iX_1 \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, X'_0 - iX_1 \rangle \end{vmatrix} \right|^2 \Omega^{n-3}.$$

Next we have the following equation:

$$(6.10) \quad \int_{Q_{n-3}(f(q)^\perp)} \left| \begin{vmatrix} \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, X'_0 + iX_1 \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, X'_0 + iX_1 \rangle \\ \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, X'_0 - iX_1 \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, X'_0 - iX_1 \rangle \end{vmatrix} \right|^2 \Omega^{n-3} \\ = [(\lambda_2 \lambda_4 \beta_3 - \lambda_3 \lambda_5 \alpha_2 \gamma_3 + \lambda_3 \lambda_5 \alpha_3 \gamma_2)^2 + (\lambda_3 \lambda_4 \alpha_2 \beta_3 + \lambda_2 \lambda_5 \gamma_3 - \lambda_3 \lambda_4 \alpha_3 \beta_2)^2](q) \\ \times \int_{Q_{n-3}(f(q)^\perp)} \left| \begin{vmatrix} \langle B_2, X'_0 + iX_1 \rangle, \langle B_3, X'_0 + iX_1 \rangle \\ \langle B_2, X'_0 - iX_1 \rangle, \langle B_3, X'_0 - iX_1 \rangle \end{vmatrix} \right|^2 \Omega^{n-3} \\ + (\lambda_2^2 + \lambda_3^2 \alpha_2^2)(\lambda_4^2 \beta_4^2 + \lambda_5^2 \gamma_4^2)(q) \int_{Q_{n-3}(f(q)^\perp)} \left| \begin{vmatrix} \langle B_2, X'_0 + iX_1 \rangle, \\ \langle B_2, X'_0 - iX_1 \rangle, \langle B_4, X'_0 + iX_1 \rangle \\ \langle B_4, X'_0 - iX_1 \rangle \end{vmatrix} \right|^2 \Omega^{n-3} \\ + (\lambda_2^2 + \lambda_3^2 \alpha_2^2)(\lambda_5^2 \gamma_5^2)(q) \int_{Q_{n-3}(f(q)^\perp)} \left| \begin{vmatrix} \langle B_2, X'_0 + iX_1 \rangle, \langle B_5, X'_0 + iX_1 \rangle \\ \langle B_2, X'_0 - iX_1 \rangle, \langle B_5, X'_0 - iX_1 \rangle \end{vmatrix} \right|^2 \Omega^{n-3} \\ + (\lambda_3^2 \alpha_3^2)(\lambda_4^2 \beta_4^2 + \lambda_5^2 \gamma_4^2)(q) \int_{Q_{n-3}(f(q)^\perp)} \left| \begin{vmatrix} \langle B_3, X'_0 + iX_1 \rangle, \langle B_4, X'_0 + iX_1 \rangle \\ \langle B_3, X'_0 - iX_1 \rangle, \langle B_4, X'_0 - iX_1 \rangle \end{vmatrix} \right|^2 \Omega^{n-3} \\ + (\lambda_3^2 \alpha_3^2 \lambda_5^2 \gamma_5^2)(q) \int_{Q_{n-3}(f(q)^\perp)} \left| \begin{vmatrix} \langle B_3, X'_0 + iX_1 \rangle, \langle B_5, X'_0 + iX_1 \rangle \\ \langle B_3, X'_0 - iX_1 \rangle, \langle B_5, X'_0 - iX_1 \rangle \end{vmatrix} \right|^2 \Omega^{n-3}.$$

In fact, the integral of the other terms which appear at the right hand side of (6.10) turns out to be zero. For example we consider the following integral:

$$l = \int_{Q_{n-3}(f(q)^\perp)} \left| \begin{array}{l} \langle B_2(q), X_0' + iX_1' \rangle, \langle B_3(q), X_0' + iX_1' \rangle \\ \langle B_2(q), X_0' - iX_1' \rangle, \langle B_3(q), X_0' - iX_1' \rangle \end{array} \right| \overline{\left| \begin{array}{l} \langle B_2(q), X_0' + iX_1' \rangle, \\ \langle B_4(q), X_0' + iX_1' \rangle \end{array} \right|} \Omega^{n-3} \cdot \\ \left| \begin{array}{l} \langle B_4(q), X_0' + iX_1' \rangle \\ \langle B_4(q), X_0' - iX_1' \rangle \end{array} \right|$$

We have

$$l = \int_{SO(n-1)/SO(n-3)} \left| \begin{array}{l} (a_{22} - ia_{32}), (a_{23} - ia_{33}) \\ (a_{22} + ia_{32}), (a_{23} + ia_{33}) \end{array} \right| \overline{\left| \begin{array}{l} (a_{22} - ia_{32}), (a_{24} - ia_{34}) \\ (a_{22} + ia_{32}), (a_{24} + ia_{34}) \end{array} \right|} \\ \times \left(\frac{1}{2\pi} \right)^{n-2} (n-3)! d\theta \wedge \langle dA_2, A_4 \rangle \wedge \langle dA_3, A_4 \rangle \wedge \cdots \wedge \langle dA_2, A_n \rangle \wedge \langle dA_3, A_n \rangle,$$

where $0 \leq \theta \leq 2\pi$. For each vector $A_i = (a_{i2}, a_{i3}, a_{i4}, \dots, a_{in})^t$ we set \tilde{A}_i by $\tilde{A}_i = (a_{i2}, -a_{i3}, a_{i4}, \dots, a_{in})^t$. This induces a diffeomorphism $k: SO(n-1) \rightarrow SO(n-1)$ by $k((A_2, A_3, A_4, A_5, \dots, A_n)) = (\tilde{A}_2, \tilde{A}_3, \tilde{A}_5, \tilde{A}_4, \dots, \tilde{A}_n)$. Then we have

$$l = \int_{SO(n-1)/SO(n-3)} \left| \begin{array}{l} (a_{22} - ia_{32}), (a_{23} - ia_{33}) \\ (a_{22} + ia_{32}), (a_{23} + ia_{33}) \end{array} \right| \overline{\left| \begin{array}{l} (a_{22} - ia_{32}), (a_{24} - ia_{34}) \\ (a_{22} + ia_{32}), (a_{24} + ia_{34}) \end{array} \right|} \\ \times \left(\frac{1}{2\pi} \right)^{n-2} (n-3)! d\theta \wedge \langle d\tilde{A}_2, d\tilde{A}_5 \rangle \wedge \langle d\tilde{A}_3, \tilde{A}_5 \rangle \wedge \langle d\tilde{A}_2, \tilde{A}_4 \rangle \wedge \langle d\tilde{A}_3, \tilde{A}_4 \rangle \\ \wedge \langle d\tilde{A}_2, \tilde{A}_6 \rangle \wedge \langle d\tilde{A}_3, \tilde{A}_6 \rangle \wedge \cdots \wedge \langle d\tilde{A}_2, \tilde{A}_n \rangle \langle d\tilde{A}_3, \tilde{A}_n \rangle.$$

Since we have $\langle dA_i, A_j \rangle = \langle d\tilde{A}_i, \tilde{A}_j \rangle$ ($2 \leq i \leq 3, 4 \leq j \leq n$), we obtain $l = 0$. In the equation (6.10), the integrals

$$\int_{Q_{n-3}(f(q)^\perp)} \left| \begin{array}{l} \langle B_2, X_0' + iX_1' \rangle, \langle B_3, X_0' + iX_1' \rangle \\ \langle B_2, X_0' - iX_1' \rangle, \langle B_3, X_0' - iX_1' \rangle \end{array} \right| \Omega^{n-3}, \\ \int_{Q_{n-3}(f(q)^\perp)} \left| \begin{array}{l} \langle B_2, X_0' + iX_1' \rangle, \langle B_4, X_0' + iX_1' \rangle \\ \langle B_2, X_0' - iX_1' \rangle, \langle B_4, X_0' - iX_1' \rangle \end{array} \right| \Omega^{n-3}, \\ \int_{Q_{n-3}(f(q)^\perp)} \left| \begin{array}{l} \langle B_2, X_0' + iX_1' \rangle, \langle B_5, X_0' + iX_1' \rangle \\ \langle B_2, X_0' - iX_1' \rangle, \langle B_5, X_0' - iX_1' \rangle \end{array} \right| \Omega^{n-3}, \\ \int_{Q_{n-3}(f(q)^\perp)} \left| \begin{array}{l} \langle B_3, X_0' + iX_1' \rangle, \langle B_4, X_0' + iX_1' \rangle \\ \langle B_3, X_0' - iX_1' \rangle, \langle B_4, X_0' - iX_1' \rangle \end{array} \right| \Omega^{n-3}$$

and

$$\int_{Q_{n-3}(f(q)^\perp)} \left| \begin{array}{l} \langle B_3, X_0' + iX_1' \rangle, \langle B_5, X_0' + iX_1' \rangle \\ \langle B_3, X_0' - iX_1' \rangle, \langle B_5, X_0' - iX_1' \rangle \end{array} \right| \Omega^{n-3}$$

are all equal and furthermore its value is independent of q . We denote by C_0 its common value. Then by (5.12), (6.9) and (6.10) we have

$$(6.11) \quad \int_{Q_{n-1}(C)} n(D, \alpha) d\alpha = \frac{1}{8} (n-1)(n-2) C_0 \int_D f^* \Omega^2.$$

We shall calculate the value C_0 . Let $SO(n-1)/SO(n-3) \rightarrow Q_{n-3}(C)$ be the Hopf fibring. For arbitrary fixed pair (C_2, C_3) of $SO(n-1)/SO(n-3)$ we have

$$(6.12) \quad C_0 = \int_{Q_{n-3}(C)} \left| \begin{array}{l} \langle C_2, A_2 + iA_3 \rangle, \langle C_3, A_2 + iA_3 \rangle \\ \langle C_2, A_2 - iA_3 \rangle, \langle C_3, A_2 - iA_3 \rangle \end{array} \right|^2 \Omega^{n-3}.$$

We take an orthonormal pair (D_4, D_5) of $SO(n-1)/SO(n-3)$ such that $\langle C_i, D_j \rangle = 0$ ($2 \leq i \leq 3, 4 \leq j \leq 5$) and set real orthonormal vectors A_2, A_3, A_4 and A_5 by

$$(6.13) \quad \begin{aligned} A_2 &= \sin\varphi(\sin\theta \cdot C_2 - \cos\theta \cdot C_3) + \cos\varphi(\sin\alpha \cdot D_4 - \cos\alpha \cdot D_5) \\ A_3 &= \sin\eta(\cos\theta \cdot C_2 + \sin\theta \cdot C_3) + \cos\eta(\cos\alpha \cdot D_4 + \sin\alpha \cdot D_5) \\ A_4 &= -\cos\varphi(\sin\theta \cdot C_2 - \cos\theta \cdot C_3) + \sin\varphi(\sin\alpha \cdot D_4 - \cos\alpha \cdot D_5) \\ A_5 &= -\cos\eta(\cos\theta \cdot C_2 + \sin\theta \cdot C_3) + \sin\eta(\cos\alpha \cdot D_4 + \sin\alpha \cdot D_5), \end{aligned}$$

where $0 < \theta, \alpha < \pi, -\pi/2 < \varphi, \eta < \pi/2$. By extending A_2, A_3, A_4 and A_5 to an ordered real orthonormal basis A_2, A_3, \dots, A_n in C^{n-1} we get $(A_2, A_3, \dots, A_n) \in SO(n-1)$. Take an open set $U \subset Q_{n-5}(C)$, where $Q_{n-5}(C)$ is a set $\{\beta \in Q_{n-3}(C) : |\beta, \Pi''((C_2, C_3))|^2 + |\beta, \Pi''((C_2, -C_3))|^2 = 0\}$ in $Q_{n-3}(C)$, and a local cross-section $\sigma = (D_4, D_5)$ of U into $SO(n-3)/SO(n-5)$ with respect to the Hopf fibring: $SO(n-3)/SO(n-5) \rightarrow Q_{n-5}(C)$. Then we see easily the set $\{(A_2, A_3) \in SO(n-1)/SO(n-3) : (A_2, A_3) \text{ is defined at (6.13) for } \sigma = (D_4, D_5)\}$ is a double covering of an open set in $Q_{n-3}(C)$. We have

$$(6.14) \quad \begin{aligned} \langle dA_2, A_4 \rangle &= -d\varphi, \langle dA_3, A_5 \rangle = -d\eta, \\ \langle dA_2, A_5 \rangle &= -\sin\varphi\cos\eta d\theta + \sin\eta\cos\varphi d\alpha + \cos\varphi\sin\eta \langle dD_4, D_5 \rangle, \\ \langle dA_3, A_4 \rangle &= \sin\eta\cos\varphi d\theta - \sin\varphi\cos\eta d\alpha - \cos\eta\sin\varphi \langle dD_4, D_5 \rangle, \\ \langle dA_2, A_i \rangle &= \cos\varphi(\sin\alpha \langle dD_4, A_i \rangle - \cos\alpha \langle dD_5, A_i \rangle) \quad (i \geq 6), \\ \langle dA_3, A_i \rangle &= \cos\eta(\cos\alpha \langle dD_4, A_i \rangle + \sin\alpha \langle dD_5, A_i \rangle) \end{aligned}$$

By (6.14) we get

$$(6.15) \quad \begin{aligned} \langle dA_2, A_4 \rangle \wedge \langle dA_3, A_5 \rangle \wedge \dots \wedge \langle dA_2, A_n \rangle \wedge \langle dA_3, A_n \rangle \\ = (\sin^2\eta\cos^2\varphi - \sin^2\varphi\cos^2\eta)(\cos\varphi\cos\eta)^{n-5} \\ \times d\varphi \wedge d\theta \wedge d\alpha \wedge d\eta \wedge \prod_{i \geq 6} \langle dD_4, A_i \rangle \wedge \langle dD_5, A_i \rangle, \end{aligned}$$

and

$$(6.16) \quad \begin{aligned} \left| \begin{array}{l} \langle C_2, A_2 + iA_3 \rangle, \langle C_3, A_2 + iA_3 \rangle \\ \langle C_2, A_2 - iA_3 \rangle, \langle C_3, A_2 - iA_3 \rangle \end{array} \right|^2 = 4|\sin\varphi\sin\eta|^2 \end{aligned}$$

Thus we obtain

$$\begin{aligned}
(6.12)' \quad C_0 &= (n-3)(n-4) \int |\sin\varphi\sin\eta|^2 |\sin^2\eta\cos^2\varphi - \sin^2\varphi\cos^2\eta| \\
&\quad |\cos\varphi\cos\eta|^{n-5} d\varphi d\eta \times \int_{Q_{n-5}(C)} \Omega^{n-5} \\
&= 2(n-3)(n-4) \int |\sin\varphi\sin\eta|^2 |\sin^2\eta\cos^2\varphi - \sin^2\varphi\cos^2\eta| \\
&\quad \times |\cos\varphi\cos\eta|^{n-5} d\varphi d\eta \\
&= \frac{16}{(n-1)(n-2)},
\end{aligned}$$

because of $\int_{Q_i(C)} \Omega^i = 2$ and $\int_E (\sin\varphi\sin\eta)^2 (\sin^2\varphi\cos^2\eta - \sin^2\eta\cos^2\varphi)$

$$\times (\cos\varphi\cos\eta)^{n-5} d\varphi d\eta = \frac{2}{(n-1)(n-2)(n-3)(n-4)}, \text{ where}$$

$E = \{(\eta, \varphi) : 0 \leq \varphi \leq \pi/2 \text{ and } 0 \leq \eta \leq \varphi\}$. Thus we have proved the equation (6.1) for a sufficiently small D . Now let D be an arbitrary open set in C^2 with compact closure. We take a finite covering $\{D_s\}_{s=1}^l$ of D such that each D_s has a differentiable local cross-section of f into $SO(n+1)/SO(n-1)$. Let $\{g_s\}$ be a partition of unity subordinated to $\{D_s\}$. Taking a mapping $P_s : D_s \times Q_{n-3}(C) \rightarrow D_s$ defined by $P_s((q, \alpha)) = q$ for $(q, \alpha) \in D_s \times Q_{n-3}(C)$, we put $n'(D_s, \alpha) = \sum_k n(p_k, \alpha) g_s(p_k)$. Then we obtain

$$\begin{aligned}
(6.17) \quad \int_{Q_{n-1}} n(D, \alpha) d\alpha &= \sum_{s=1}^l \int_{Q_{n-1}} n'(D_s, \alpha) d\alpha \\
&= \sum_s \int_{D_s \times Q_{n-3}} -g_s(P_s(\alpha')) (t_s'')^* d\alpha \\
&= 2 \sum_s \int_{D_s} g_s f^* \Omega^2 \\
&= 2 \int_D f^* \Omega^2,
\end{aligned}$$

where t_s'' is a mapping of $D_s \times Q_{n-3}(C)$ onto $f(D_s)^\perp$ defined by (6.6). Q.E.D.

7. Equidistribution theorem

We define the defect $\delta(\alpha)$ of ξ_α by

$$(7.1) \quad \delta(\alpha) = \liminf_{r \rightarrow \infty} \frac{m(r, \alpha)}{T(r)}.$$

Since $m(r, \alpha)$ is non-negative, $\delta(\alpha)$ is non-negative for any $\alpha \in Q_{n-1}(C)$. We see clearly that $\delta(\alpha) = \delta(\bar{\alpha})$ for any $\alpha \in Q_{n-1}(C)$. By Theorem 2, Lemma 4.5 and the fact that $T(r) \rightarrow \infty$ if $r \rightarrow \infty$, we have

$$(7.2) \quad \delta(\alpha) = \liminf_{r \rightarrow \infty} \left(1 - \frac{N(r, \alpha)}{T(r)} \right).$$

Then we have the following equidistribution theorem.

Theorem 5. $\delta(\alpha)$ is equal to zero for almost all $\alpha \in Q_{n-1}(\mathbf{C})$ with respect to the volume Ω^{n-1} .

Proof. By the Fatou's preparation theorem we have

$$\begin{aligned}
 0 &\leq \int_{Q_{n-1}} \delta(\alpha) d\alpha \leq \int_{Q_{n-1}} \left\{ \liminf_{r \rightarrow \infty} \left(1 - \frac{N(r, \alpha)}{T(r)} \right) \right\} d\alpha \\
 &\leq \liminf_{r \rightarrow \infty} \int_{Q_{n-1}} \left(1 - \frac{N(r, \alpha)}{T(r)} \right) d\alpha = \liminf_{r \rightarrow \infty} \left(2 - \frac{1}{T(r)} \int_{Q_{n-1}} N(r, \alpha) d\alpha \right) \\
 &= \liminf_{r \rightarrow \infty} \left(2 - \frac{1}{T(r)} \int_{Q_{n-1}} \left\{ \int_0^r n(\Delta(t), \alpha) dt \right\} d\alpha \right) \\
 &= \liminf_{r \rightarrow \infty} \left(2 - \frac{1}{T(r)} \int_0^r dt \int_{Q_{n-1}} n(\Delta(t), \alpha) d\alpha \right) \\
 &= \liminf_{r \rightarrow \infty} (2 - 2) = 0 \quad (\text{by Theorem 4}).
 \end{aligned}$$

Thus we obtain $\delta(\alpha) = 0$ for almost all $\alpha \in Q_{n-1}(\mathbf{C})$. Q.E.D.

If the image $f(\mathbf{C}^2)$ does not intersect with ξ_α , we have $\delta(\alpha) = 1$. So we have

Corollary. Let f be a holomorphic mapping of \mathbf{C}^2 into $Q_{n-1}(\mathbf{C})$ ($n \geq 3$) satisfying Conditions (A) and (B). We put $W = \{\alpha \in Q_{n-1}(\mathbf{C}) : f(\mathbf{C}^2) \cap \xi_\alpha = \emptyset\}$. Then the set W has measure zero with respect to volume Ω^{n-1} .

REMARK 3. In the case of holomorphic curves ($f: \mathbf{C} \rightarrow P_n(\mathbf{C})$ holomorphic mapping), it is known that $0 \leq \delta(\xi) \leq 1$ for each hyperplane ξ (c.f. [1], [5] and [6]). But in our case we can not prove that $\delta(\alpha) \leq 1$.

OSAKA UNIVERSITY

References

- [1] S.S. Chern: *Holomorphic curves in the plane*, Differential Geometry in honor of K. Yano, Kinokuniya, Tokyo, 1972, 73–94.
- [2] P. Griffiths and J. King: *Nevanlinna theory and holomorphic mappings between algebraic varieties*, Acta. Math. **130** (1973), 145–220.
- [3] S. Kobayashi and K. Nomizu: Foundations of Differential Geometry II. Interscience Pub. 1969.
- [4] W. Stoll: Value Distribution of Holomorphic Maps into Compact Complex Manifold, Lect. Notes in Math. 135, Springer-Verlag, Berlin-Heidelberg-New York, 1970.
- [5] H. Wu: *Mappings of Riemannian surfaces (Nevanlinna theory)*, Proc. Sympos. Pure Math. XI “Entire functions and related topics of analysis”. Amer. Math. Soc., 1968, 480–532.
- [6] H. Wu: Equidistribution Theory of Holomorphic Curves, Annals of Math. Studies, 64, Princeton, 1970.

