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# ON THE FIRST MAIN THEOREM OF HOLOMORPHIC MAPPINGS FROM $\mathbb{C}^2$ INTO $Q_{n-1}(\mathbb{C})$

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## 0. Introduction

Let  $f$  be a holomorphic mapping of a complex line  $\mathbb{C}$  into a complex projective space  $P_n(\mathbb{C})$  and suppose that the image  $f(\mathbb{C})$  is not contained in any hyperplane of  $P_n(\mathbb{C})$ . Put  $V[t] = \{z \in \mathbb{C} : \log|z| < t\}$ , and for a hyperplane  $\xi$  in  $P_n(\mathbb{C})$  let  $n(t, \xi)$  be the number of points in  $V[t] \cap f^{-1}(\xi)$ . Let  $\Omega$  be the colsed form of degree 2 associated with the Fubini-Study metric on  $P_n(\mathbb{C})$  and normalized as  $\int_{P_n} \Omega^n = 1$ . The counting function  $N(r, \xi)$  and the order function  $T(r)$  being defined by

$$(0.1) \quad N(r, \xi) = \int_0^r n(t, \xi) dt,$$

$$(0.2) \quad T(r) = \int_0^r dt \int_{V[t]} f^* \Omega$$

respectively, the following equation is known as the First Main Theorem:

$$(0.3) \quad N(r, \xi) + (m(r, \xi) - m(0, \xi)) = T(r),$$

where  $m(r, \xi)$  is a non-negative function defined for  $r \in \mathbb{R}^+$  and hyperplanes  $\xi$  in  $P_n(\mathbb{C})$ . The term  $(m(r, \xi) - m(0, \xi))$  is called the compensating term. It follows from the equation (0.3) that the image  $f(\mathbb{C})$  intersects with almost all hyperplanes in  $P_n(\mathbb{C})$ . Furthermore it is known that the number of hyperplanes in general position not intersecting with  $f(\mathbb{C})$  is at most  $n+1$ . These results are originally due to Ahlfors, and treated also by H. Wu [6] and S. S. Chern [1] in a modernized form.

Let  $f$  be a holomorphic mapping of  $\mathbb{C}^2$  into a complex quadratic  $Q_{n-1}(\mathbb{C})$  ( $n \geq 3$ ) satisfying certain non-degenerate conditions [§2]. We consider  $Q_{n-1}(\mathbb{C})$  as a fixed hypersurface in  $P_n(\mathbb{C})$ . We consider a special family of  $(n-2)$ -dimensional projective spaces  $P_{n-2}(\mathbb{C})$  in  $P_n(\mathbb{C})$  parametrized by a Grassmann manifold  $G(\mathbb{R})$  of 2-dimensional linear spaces in  $\mathbb{R}^{n+1}$  [§1]. This family determines a family of  $(n-3)$ -dimensional complex quadratic  $\xi_\alpha (\alpha \in G(\mathbb{R}))$  in  $Q_{n-1}(\mathbb{C})$ , each of whose elements is a Poincaré dual of the form  $\Omega^2$  in  $Q_{n-1}(\mathbb{C})$ .

In this paper, we shall consider a value distribution problem in two complex variables with respect to the holomorphic mapping  $f$  and the family  $\{\xi_\alpha\}$ . The complex quadratic  $Q_{n-1}(\mathbf{C})$  being a double covering space of  $G(\mathbf{R})$ , we may take  $Q_{n-1}(\mathbf{C})$  as a parametrizing space of the family  $\{\xi_\alpha\}$  in place of  $G(\mathbf{R})$ . Thus we have a setting similar to the case of holomorphic curves (holomorphic mappings of  $\mathbf{C}$  into  $P_n(\mathbf{C})$ ). Furthermore  $\Omega$  is an invariant form on  $Q_{n-1}(\mathbf{C})$  by a certain transformation group [§5]. This fact also plays an important role as in the case of holomorphic curves [§6].

Our main results are as follows: (1) First Main Theorem [§4], (2) the Crofton formula [§6] and (3) the Distribution theorem [§7]. In more detail, put

$$\Delta(r) = \{(z_1, z_2) \in \mathbf{C}^2 : \log |z_i| < r (i = 1, 2)\}$$

and define

$$n(\Delta(r), \alpha) = \sum_{p_i \in \Delta(r), f(p_i) \in \xi_\alpha} n(p_i, \alpha),$$

where  $n(p_i, \alpha)$  is a certain real number [§3] such that  $n(p_i, \alpha) = 1$  if  $f(\mathbf{C}^2)$  intersects transversely with  $\xi_\alpha$  at  $f(p_i)$ . We also define the following functions:

$$(0.4) \quad N(r, \alpha) = \int_0^r n(\Delta(t), \alpha) dt \quad (\text{counting function})$$

$$(0.5) \quad T(r) = \int_0^r dt \int_{\Delta(t)} f^* \Omega^2 \quad (\text{order function}).$$

Then our First Main Theorem states:

$$(0.6) \quad N(r, \alpha) + m(r, \alpha) - m(0, \alpha) = T(r),$$

where  $m(r, \alpha)$  is a non-negative function defined for  $r \in \mathbf{R}^+$  and submanifold  $\xi_\alpha$  ( $\alpha \in G(\mathbf{R})$ ) [§4]. The Crofton formula is as follows:

$$(0.7) \quad \int_{Q_{n-1}} n(\Delta(t), \alpha) \Omega^{n-1}(\alpha) = 2 \int_{\Delta(t)} f^* \Omega^2.$$

Finally the distribution theorem says: The image  $f(\mathbf{C}^2)$  intersects with almost all submanifolds in  $\{\xi_\alpha\}$  ( $\alpha \in G(\mathbf{R})$ ) i.e., we have  $\int_W \Omega^{n-1} = 0$  for  $W = \{\alpha \in Q_{n-1}(\mathbf{C}) : f(\mathbf{C}^2) \cap \xi_\alpha = \emptyset\}$ .

We note that W. Stoll [4], P. Griffiths and J. King [2] also developed the First Main Theorem in several complex variables. But our setting is different from theirs.

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## 1. Preliminaries

We shall recall several basic facts about the complex projective space  $P_n(\mathbf{C})$

and the complex quadratic  $Q_{n-1}(\mathbf{C})$  (c.f. [3]), and moreover we shall define a special family of submanifolds in  $Q_{n-1}(\mathbf{C})$ . Let  $\mathbf{C}^{n+1}$  (resp.  $\mathbf{R}^{n+1}$ ) be the complex (resp. real) vector space of  $(n+1)$  tuples of complex numbers  $(z^0, \dots, z^n)$  (resp. real numbers  $(x^0, \dots, x^n)$ ). We define a symmetric bilinear form  $(,)$  on  $\mathbf{C}^{n+1}$  by

$$(1.1) \quad (Z, W) = z^0 w^0 + \dots + z^n w^n$$

for  $Z = (z^0, \dots, z^n)$  and  $W = (w^0, \dots, w^n)$ . For  $Z = (z^0, \dots, z^n)$  we put  $\bar{Z} = (\bar{z}^0, \dots, \bar{z}^n)$ , where the bar denotes the complex conjugation. A vector  $Z \in \mathbf{C}^{n+1} - \{0\}$  is called real if  $\bar{Z} = Z$ . We define a hermitian inner product  $\langle, \rangle$  on  $\mathbf{C}^{n+1}$  by

$$(1.2) \quad \langle Z, W \rangle = (Z, \bar{W})$$

for  $Z, W \in \mathbf{C}^{n+1}$ . We put  $\|Z\| = \langle Z, Z \rangle^{1/2}$ . For the complex projective space  $P_n(\mathbf{C})$  of dimension  $n$ , we have the natural holomorphic fibring (called the Hopf fibring)

$$(1.3) \quad \Pi: \mathbf{C}^{n+1} - \{0\} \rightarrow P_n(\mathbf{C}),$$

where  $\Pi(Z)$  is the line passing through the origin and  $Z$ . We remark that the natural conjugation  $Z \mapsto \bar{Z}$  in  $\mathbf{C}^{n+1} - \{0\}$  induces a diffeomorphism  $z \in P_n(\mathbf{C}) \rightarrow \bar{z} \in P_n(\mathbf{C})$ . Let  $\tilde{\Omega}$  be the 2-form of type  $(1, 1)$  on  $\mathbf{C}^{n+1} - \{0\}$  given by

$$(1.4) \quad \tilde{\Omega} = \frac{i}{2\pi} \frac{1}{\|Z\|^4} \{ (\sum_j |z^j|^2) (\sum_j dz^j \wedge d\bar{z}^j) - (\sum_j z^j d\bar{z}^j) \wedge (\sum_j \bar{z}^j dz^j) \}.$$

It is well-known that there exists a unique 2-form  $\Omega$  of type  $(1, 1)$  on  $P_n(\mathbf{C})$  such that  $\Pi^* \Omega = \tilde{\Omega}$ . Then  $\Omega$  is the Kähler form associated with the Fubini-Study metric on  $P_n(\mathbf{C})$  and we have

$$(1.5) \quad \int_{P_n(\mathbf{C})} \Omega^n = 1.$$

We consider a family of subspaces  $H$  of  $\mathbf{C}^{n+1}$  such that  $H$  is of  $(n-1)$ -dimension and  $\bar{Z} \in H$  whenever  $Z \in H$ . With such an  $H$ , we can associate uniquely a real subspace of  $\mathbf{R}^{n+1}$  of dimension 2 by

$$(1.6) \quad \{X \in \mathbf{R}^{n+1}: \langle X, H \rangle = 0\}.$$

We see that this gives a one to one correspondence, and hence the above family of  $H$ 's is parametrized by the Grassmann manifold  $G(\mathbf{R})$  of 2 planes in  $\mathbf{R}^{n+1}$ . Especially we note that  $[H] = \Pi(H - \{0\})$  is an  $(n-2)$ -dimensional projective space in  $P_n(\mathbf{C})$ .

On  $P_n(\mathbf{C})$  with homogeneous coordinate  $z^0, \dots, z^n$  the complex quadratic  $Q_{n-1}(\mathbf{C})$  is a complex hypersurface defined by the equation

$$(1.7) \quad (z^0)^2 + \dots + (z^n)^2 = 0.$$

Now the unit sphere  $S^{2n+1} = \{Z \in \mathbf{C}^{n+1}: \|Z\| = 1\}$  is a principal fibre bundle over

$P_n(\mathbf{C})$  with structure group  $S^1$ . For a point  $q \in Q_{n-1}(\mathbf{C})$ , take a point  $Z \in S^{2n+1}$  such that  $\Pi(Z) = q$ . We can write  $Z$  uniquely in the form  $Z = (X + iY)/\sqrt{2}$ , where  $X$  and  $Y$  are orthonormal real vectors in  $\mathbf{C}^{n+1}$ . Conversely if  $Z = (X + iY)/\sqrt{2} \in S^{2n+1}$  for orthonormal real vectors  $X$  and  $Y$ , then we have  $\Pi(Z) \in Q_{n-1}(\mathbf{C})$ . Therefore we have

$$(1.8) \quad S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbf{C})) = \{Z = (X + iY)/\sqrt{2} : X \text{ and } Y \text{ are orthonormal real vectors}\}.$$

The group  $SO(n+1)$ , considered as a subgroup of  $U(n+1)$ , acts on  $S^{2n+1}$  and leaves the submanifold  $S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbf{C}))$  invariant. Moreover  $SO(n+1)$  acts transitively on  $S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbf{C}))$ . The isotropy subgroup of  $SO(n+1)$  at  $Z_0 = (1/\sqrt{2}, i/\sqrt{2}, 0, \dots, 0)$  coincides with the subgroup  $SO(n-1)$  of  $SO(n+1)$ . We denote an element  $g$  of  $SO(n+1)$  by

$$g = (X_0, X_1, \dots, X_n),$$

where each  $X_i$  is a column vector. Then, in the space  $SO(n+1)/SO(n-1)$ , the coset including  $g = (X_0, X_1, \dots, X_n)$  can be represented by the first two vectors  $(X_0, X_1)$ . Under this identification, we have a diffeomorphism  $i: SO(n+1)/SO(n-1) \rightarrow S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbf{C}))$  defined by

$$(1.9) \quad i((X_0, X_1)) = \frac{1}{\sqrt{2}}(X_0 + iX_1).$$

From now on we also identify  $SO(n+1)/SO(n-1)$  with  $S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbf{C}))$  by the above diffeomorphism. We denote by  $\Pi_1$  the projection:  $SO(n+1)/SO(n-1) \rightarrow Q_{n-1}(\mathbf{C})$  defined by

$$(1.10) \quad \Pi_1((X_0, X_1)) = \Pi((X_0 + iX_1)/\sqrt{2})$$

for  $(X_0, X_1) \in SO(n+1)/SO(n-1)$ . Note that the space  $Q_{n-1}(\mathbf{C})$  also can be identified canonically with  $SO(n+1)/SO(2) \times SO(n-1)$ .

To each point  $\alpha = \Pi_1((X_0, X_1))$  in  $Q_{n-1}(\mathbf{C})$ , we assign the 2-dimensional linear space spanned by  $\{X_0, X_1\}$  in  $\mathbf{R}^{n+1}$ . Through this assignment,  $Q_{n-1}(\mathbf{C})$  is a double covering space of  $G(\mathbf{R})$ . We see that the function  $|\langle Z, W \rangle|^2$  on  $S^{2n+1} \times S^{2n+1}$  induces a function  $|\Pi(Z), \Pi(W)|^2$  on  $P_n(\mathbf{C}) \times P_n(\mathbf{C})$ . For each  $\alpha \in Q_{n-1}(\mathbf{C})$ , we consider a complex submanifold  $\xi_\alpha$  of  $Q_{n-1}(\mathbf{C})$ , defined by

$$(1.11) \quad \xi_\alpha = \{\beta \in Q_{n-1}(\mathbf{C}) : |\beta, \alpha|^2 + |\beta, \bar{\alpha}|^2 = 0\}.$$

Let  $(X_0, X_1) \in SO(n+1)/SO(n-1)$  and set  $\Pi_1((X_0, X_1)) = \alpha$ . Consider the complex subspace  $H$  of  $\mathbf{C}^{n+1}$  orthogonal to the vectors  $X_0, X_1$ . We have  $\xi_\alpha = Q_{n-1}(\mathbf{C}) \cap [H]$ .  $[H]$  is a Poincaré dual of the form  $\Omega^2$  in  $P_n(\mathbf{C})$ , and hence  $\xi_\alpha$  is also, in  $Q_{n-1}(\mathbf{C})$ , a Poincaré dual of the form  $\Omega^2$  restricted to  $Q_{n-1}(\mathbf{C})$ . Finally we remark that each  $\xi_\alpha$  is a complex quadratic  $Q_{n-3}(\mathbf{C})$  and  $\xi_\alpha = \xi_{\bar{\alpha}}$ .

## 2. Holomorphic mapping

Let  $f$  be a holomorphic mapping of  $\mathbf{C}^2$  into  $Q_{n-1}(\mathbf{C})$  ( $n \geq 3$ ). We consider the following two conditions on  $f$ .

Condition (A):  $f$  is an immersion.

Condition (B): For each  $\alpha \in Q_{n-1}(\mathbf{C})$ , the set  $\{p \in \mathbf{C}^2: f(p) \in \xi_\alpha\}$  is discrete.

For each point  $p \in \mathbf{C}^2$ , we can take a small neighborhood  $U(p)$  of  $p$  such that there exists a holomorphic lift  $F = (f^0, \dots, f^n)$  of  $f$  on  $U(p)$  into  $\mathbf{C}^{n+1} - \{0\}$  i.e.,  $\Pi F = f$ .

**Proposition 2.1.** Condition (A) is equivalent to the following: for each point  $p$  of  $\mathbf{C}^2$ , choose a holomorphic lift  $F = (f^0, \dots, f^n)$  of  $f$  on a neighborhood  $U$  of  $p$ , then we have

$$(2.1) \quad \text{rank} \begin{pmatrix} f^0, \dots, f^n \\ \frac{\partial f^0}{\partial w_1}, \dots, \frac{\partial f^n}{\partial w_1} \\ \frac{\partial f^0}{\partial w_2}, \dots, \frac{\partial f^n}{\partial w_2} \end{pmatrix} (p) = 3,$$

where  $(w_1, w_2)$  is a coordinate system on the neighborhood  $U$ .

**Proof.** We identify the real tangent space  $T_Z(\mathbf{C}^{n+1})$  at a point  $Z$  in  $\mathbf{C}^{n+1}$  with  $\mathbf{C}^{n+1}$  in the usual way. For  $p$ , we take  $(X_0, X_1, \dots, X_n) \in SO(n+1)$  such that  $(X_0 + iX_1)/\sqrt{2} = (F/\|F\|)(p)$ . Then the tangent space  $T_{(X_0 + iX_1)/\sqrt{2}}(S^{2n+1})$  has a basis  $i(X_0 + iX_1)$ ,  $X_0 - iX_1$ ,  $i(X_0 - iX_1)$ ,  $X_2, \dots, X_n, iX_2, \dots, iX_n$ . Let  $T_{f(p)}$  be the subspace spanned by  $X_2, \dots, X_n, iX_2, \dots, iX_n$ . The projection  $\tilde{\Pi} = \Pi|_{S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(\mathbf{C}))}$  induces a linear isomorphism  $\tilde{\Pi}_*: T_{f(p)}(Q_{n-1}(\mathbf{C})) \rightarrow T_{f(p)}(\mathbf{C}^{n+1})$  (c.f. [3] p.p. 279). Hence,  $T_{f(p)}(Q_{n-1}(\mathbf{C}))$  is identified with the subspace of  $\mathbf{C}^{n+1}$  orthogonal to the vectors  $(F/\|F\|)(p)$  and  $(\bar{F}/\|F\|)(p)$  with respect to  $\langle, \rangle$ . Since we have  $\langle F, \bar{F} \rangle = 0$  on  $U$ , we see  $\langle dF, \bar{F} \rangle = 0$ . We have

$$(2.2) \quad d\left(\frac{F}{\|F\|}\right) = \frac{1}{\|F\|} \sum_{j=1}^2 \left( \frac{\partial F}{\partial w_j} - \left\langle \frac{\partial F}{\partial w_j}, \frac{F}{\|F\|} \right\rangle \frac{F}{\|F\|} \right) dw_j \\ + \sum_{j=1}^2 iF \frac{\partial}{\partial x^j} \left( \frac{1}{\|F\|} \right) dx^j - \sum_{j=1}^2 iF \frac{\partial}{\partial y^j} \left( \frac{1}{\|F\|} \right) dy^j,$$

where  $w_j = x^j + iy^j$ . Therefore we get

$$(2.3) \quad df = \sum_{j=1}^2 \tilde{\Pi}_* \left[ \frac{1}{\|F\|} \left( \frac{\partial F}{\partial w_j} - \left\langle \frac{\partial F}{\partial w_j}, \frac{F}{\|F\|} \right\rangle \frac{F}{\|F\|} \right) \right] dw_j.$$

This shows Proposition 2.1.

Q.E.D.

We define

$$(2.4) \quad Q_{n-3}(f(p)^\perp) = \{\alpha \in Q_{n-1}(C) : |f(p), \alpha|^2 + |f(p), \bar{\alpha}|^2 = 0\},$$

that is,

$$Q_{n-3}(f(p)^\perp) = \{\alpha \in Q_{n-1}(C) : f(p) \in \xi_\alpha\}.$$

Then  $Q_{n-3}(f(p)^\perp)$  can be identified with  $SO(n-1)/SO(2) \times SO(n-3)$  as follows: Choose an element  $(X_0, X_1, \dots, X_n) \in SO(n+1)$  such that  $(X_0 + iX_1)/\sqrt{2} = (F/\|F\|)(p)$ . Let  $(A_2, A_3) \in SO(n-1)/SO(n-3)$  where  $A_i = (a_{i2}, \dots, a_{in})^t$  ( $i=2, 3$ ). Consider the mapping

$$(2.5) \quad (A_2, A_3) \rightarrow (\sum_{i=2}^n a_{2i} X_i, \sum_{i=2}^n a_{3i} X_i).$$

We see easily that this gives an identification of  $SO(n-1)/SO(2) \times SO(n-3)$  with  $Q_{n-3}(f(p)^\perp)$ , which is independent of the choice of lift  $F$ .

For  $\alpha \in Q_{n-3}(f(p)^\perp)$  we take  $(X_0, X_1) \in SO(n+1)/SO(n-1)$  such that  $\Pi_1((X_0, X_1)) = \alpha$ . Then the following condition is independent of the choice of  $(X_0, X_1)$ ,

$$(2.6) \quad \left| \begin{array}{l} \langle (\partial F/\partial w_1)(p), (X_0 + iX_1)/\sqrt{2} \rangle, \langle (\partial F/\partial w_2)(p), (X_0 + iX_1)/\sqrt{2} \rangle \\ \langle (\partial F/\partial w_1)(p), (X_0 - iX_1)/\sqrt{2} \rangle, \langle (\partial F/\partial w_2)(p), (X_0 - iX_1)/\sqrt{2} \rangle \end{array} \right| \neq 0.$$

**Proposition 2.2.** *The condition (2.6) holds if and only if  $f$  intersects transversely with  $\xi_\alpha$  at  $f(p)$ .*

**Proof.** Put  $(F/\|F\|)(p) = (X_2 + iX_3)/\sqrt{2}$ . Then we take an element  $(X_0, X_1, X_2, X_3, \dots, X_n) \in SO(n+1)$ . As in the proof of Proposition 2.1, we see that the tangent space  $T_{f(p)}(Q_{n-1}(C))$  is spanned by the vectors  $X_0, iX_0, X_1, iX_1, X_2, iX_2, \dots, X_n, iX_n$  and the tangent space  $T_{f(p)}(\xi_\alpha)$  is spanned by  $X_2, iX_2, \dots, X_n, iX_n$  through the identification by  $\tilde{\Pi}_*: T_{(X_2+iX_3)/\sqrt{2}}(S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(C))) \rightarrow T_{f(p)}(Q_{n-1}(C))$ . Therefore by (2.3) (or (2.2)) it is sufficient to show that the condition (2.6) is equivalent to  $\text{rank}_R((\partial F/\partial w_1)(p), i(\partial F/\partial w_1)(p), (\partial F/\partial w_2)(p), i(\partial F/\partial w_2)(p), X_2, iX_2, \dots, X_n, iX_n) = 2(n+1)$ . Now this can be seen easily.

Q.E.D.

Now we consider the following condition for  $\alpha = \Pi_1((X_0, X_1)) \in Q_{n-3}(f(p)^\perp)$

$$(2.7) \quad \left| \begin{array}{l} \langle (\partial F/\partial w_1)(p), (X_0 + iX_1)/\sqrt{2} \rangle, \langle (\partial F/\partial w_2)(p), (X_0 + iX_1)/\sqrt{2} \rangle \\ \langle (\partial F/\partial w_1)(p), (X_0 - iX_1)/\sqrt{2} \rangle, \langle (\partial F/\partial w_2)(p), (X_0 - iX_1)/\sqrt{2} \rangle \end{array} \right| = 0$$

Since the vectors  $(\partial F/\partial w_1)(p)$  and  $(\partial F/\partial w_2)(p)$  are linearly independent, the set of elements  $\alpha \in Q_{n-3}(f(p)^\perp)$  satisfying the condition (2.7) has measure zero in  $Q_{n-3}(f(p)^\perp)$ .

**REMARK 1.** We shall remark here a certain sufficient condition for Condition (B). For  $w \in C$  we put  $C_w^1 = \{(z, w) : z \in C\}$  and  $C_w^2 = \{(w, z) : z \in C\}$ .

Assume the following condition (C): *none of  $f(C_w^i)$  ( $i=1, 2, w \in C$ ) is contained in a hyperplane in  $P_n(C)$ .* Let  $f(p) \in \xi_\alpha$  and set  $\prod_1((X_0, X_1)) = \alpha$ . We put  $g_1(w_1, w_2) = \langle F, (X_0 + iX_1)/\sqrt{2} \rangle(w_1, w_2)$  and  $g_2(w_1, w_2) = \langle F, (X_0 - iX_1)/\sqrt{2} \rangle(w_1, w_2)$  on  $U(p)$ , where  $(w_1, w_2)$  is a coordinate system on  $U(p)$  such that  $w_i(p) = 0$  ( $i=1, 2$ ). Using the Weierstrass' preparation theorem we have the following representations

$$(2.8) \quad \begin{aligned} g_1(w_1, w_2) &= (a_0(w_1) + a_1(w_1)w_2 + \cdots + a_{l_1}(w_1)w_2^{l_1})h_1(w_1, w_2) \\ g_2(w_1, w_2) &= (b_0(w_1) + b_1(w_1)w_2 + \cdots + b_{l_2}(w_1)w_2^{l_2})h_2(w_1, w_2), \end{aligned}$$

where  $a_i(w_1)$ ,  $b_i(w_1)$  and  $h_i(w_1, w_2)$  are holomorphic such that  $a_i(0) = 0$  for  $0 \leq i < l_1$ ,  $a_{l_1}(0) \neq 0$ ,  $b_i(0) = 0$  for  $0 \leq i < l_2$ ,  $b_{l_2}(0) \neq 0$  and  $h_i(w_1, w_2) \neq 0$  ( $i=1, 2$ ). We denote by  $R(w_1)$  the resultant of  $(a_0(w_1) + \cdots + a_{l_1}(w_1)w_2^{l_1})$  and  $(b_0(w_1) + \cdots + b_{l_2}(w_1)w_2^{l_2})$ . Since the function  $R(w_1)$  is holomorphic, we have that  $R(w_1) \equiv 0$  or the following (D): *the set  $\{w_1: R(w_1) = 0\}$  is discrete.* If, under the assumption of (C),  $f$  satisfies (D) for each  $p \in C^2$  and  $\alpha \in Q_{n-1}(C)$  such that  $f(p) \in \xi_\alpha$ , then Condition (B) holds.

### 3. Certain forms on $Q_{n-1}(C) - \xi_\alpha$

We define one 2-form  $\Omega_\alpha$  on  $Q_{n-1}(C) - \xi_\alpha$  by

$$(3.1) \quad \Omega_\alpha(\beta) = dd^c \log \{ |\beta, \alpha|^2 + |\beta, \bar{\alpha}|^2 \},$$

where  $d^c = \frac{1}{4\pi i}(\partial - \bar{\partial})$ . We choose a unit vector  $Z_\alpha$  such that  $\prod(Z_\alpha) = \alpha$ , and define a mapping  $P_\alpha$  of  $Q_{n-1}(C) - \xi_\alpha$  into  $P_1(C)$  by

$$(3.2) \quad P_\alpha(\beta) = \hat{\Pi} \left[ \frac{1}{(|\beta, \alpha|^2 + |\beta, \bar{\alpha}|^2)^{1/2}} (\langle Z_\beta, Z_\alpha \rangle, \langle Z_\beta, \bar{Z}_\alpha \rangle) \right],$$

where  $Z_\beta \in S^{2n+1}$  such that  $\prod(Z_\beta) = \beta$ , and  $\hat{\Pi}$  is the Hopf fibring  $S^3 \rightarrow P_1(C)$ .  $P_\alpha$  is well-defined and holomorphic. Let  $\omega$  be the Kähler 2-form associated with the Fubini-Study metric on  $P_1(C)$  and normalized as  $\int_{P_1(C)} \omega = 1$ . Then  $P_\alpha^* \omega$  is independent of the choice of  $Z_\alpha$ . From now on we also denote by  $\Omega$  the restriction of the form  $\Omega$  to  $Q_{n-1}(C)$ .

**Lemma 3.1.** *We have*

$$(3.3) \quad \Omega_\alpha = P_\alpha^* \omega - \Omega \quad \text{on } Q_{n-1}(C) - \xi_\alpha.$$

**Proof.** Let  $\sigma$  be a local holomorphic cross-section of the Hopf fibring  $\Pi: C^{n+1} - \{0\} \rightarrow P_n(C)$  defined on an open set  $U$  in  $Q_{n-1}(C) - \xi_\alpha$ . Then we have

$$\begin{aligned} \Omega_\alpha &= dd^c \log \left\{ \left| \left\langle \frac{\sigma}{\|\sigma\|}, Z_\alpha \right\rangle \right|^2 + \left| \left\langle \frac{\sigma}{\|\sigma\|}, \bar{Z}_\alpha \right\rangle \right|^2 \right\} \\ &= dd^c \log \{ |\langle \sigma, Z_\alpha \rangle|^2 + |\langle \sigma, \bar{Z}_\alpha \rangle|^2 \} - dd^c \log \|\sigma\|^2 \\ &= P_\alpha^* \omega - \Omega. \end{aligned} \quad \text{Q.E.D.}$$



We define another 2-form  $\Omega'_\alpha$  on  $Q_{n-1}(\mathcal{C}) - \xi_\alpha$  by

$$(3.4) \quad \Omega'_\alpha = \Omega + P_\alpha^* \omega \quad \text{on } Q_{n-1}(\mathcal{C}) - \xi_\alpha.$$

Put

$$(3.5) \quad \Omega''_\alpha = -\Omega_\alpha \wedge \Omega'_\alpha \quad \text{on } Q_{n-1}(\mathcal{C}) - \xi_\alpha.$$

By (3.3) and (3.4), we have

$$(3.5)' \quad \begin{aligned} \Omega''_\alpha &= (\Omega - P_\alpha^* \omega) \wedge (\Omega + P_\alpha^* \omega) \\ &= \Omega^2 - P_\alpha^* (\omega \wedge \omega) = \Omega^2 \quad \text{on } Q_{n-1}(\mathcal{C}) - \xi_\alpha. \end{aligned}$$

Let  $f: \mathcal{C}^2 \rightarrow Q_{n-1}(\mathcal{C})$  ( $n \geq 3$ ) be a holomorphic mapping satisfying Conditions (A) and (B) in §2. For a point  $p$  in  $\mathcal{C}^2$ , we take a small neighborhood  $U(p)$  of  $p$  and a coordinate system  $(w_1, w_2)$  on it satisfying  $w_i(p) = 0$  ( $i = 1, 2$ ). Let  $F$  be a holomorphic lift of  $f$  on  $U(p)$  into  $\mathcal{C}^{n+1} - \{0\}$ . Set  $f(p) \in \xi_\alpha$ . Then we define a real number  $n(p, \alpha)$  by

$$(3.6) \quad n(p, \alpha) = \lim_{\varepsilon \downarrow 0} \int_{\partial U_\varepsilon(p)} d^c \cdot \log \{ |\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2 \} \wedge f^* P_\alpha^* \omega,$$

where  $U_\varepsilon(p) = \{(w_1, w_2) \in U(p) : |w_1|^2 + |w_2|^2 < \varepsilon^2\}$  and  $\Pi(Z_\alpha) = \alpha$ .

**Lemma 3.2.**  $n(p, \alpha)$  is well-defined and finite. Especially if  $f$  intersects transversely with  $\xi_\alpha$  at  $f(p)$ , then we have  $n(p, \alpha) = 1$ .

*Proof.* First we choose a local lift  $F$  and a local coordinate system  $(w_1, w_2)$  such that  $w_i(p) = 0$ . Take two positive real numbers  $\varepsilon_1$  and  $\varepsilon_2$  such that  $U(p) \supset U_{\varepsilon_1}(p) \supset U_{\varepsilon_2}(p)$ . Then we have

$$(3.7) \quad \begin{aligned} 0 &= \int_{U_{\varepsilon_1}(p) - U_{\varepsilon_2}(p)} f^* P_\alpha^* (\omega \wedge \omega) \\ &= \int_{\partial U_{\varepsilon_1}(p) - \partial U_{\varepsilon_2}(p)} d^c \log \{ |\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2 \} \wedge f^* P_\alpha^* \omega. \end{aligned}$$

Therefore we obtain

$$(3.8) \quad \begin{aligned} &\int_{\partial U_{\varepsilon_1}(p)} d^c \log \{ |\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2 \} \wedge f^* P_\alpha^* \omega \\ &= \lim_{\varepsilon \downarrow 0} \int_{\partial U_\varepsilon(p)} d^c \log \{ |\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2 \} \wedge f^* P_\alpha^* \omega. \end{aligned}$$

The left hand-side of the equation (3.8) is finite and hence so is the right side. In the same way, we see that  $n(p, \alpha)$  is independent of the choice of a local coordinate system. Now we shall show that  $n(p, \alpha)$  is independent of the choice of  $F$ . Take two holomorphic lift  $F_1$  and  $F_2$  of  $f$ . Then there exists a holomorphic function  $g$  such that  $F_1 = gF_2$  and  $g(q) \neq 0$  at any  $q \in U(p)$ . We have

$$\begin{aligned}
 (3.9) \quad & d^c \log \{ |\langle F_1, Z_\alpha \rangle|^2 + |\langle F_1, \bar{Z}_\alpha \rangle|^2 \} \\
 &= d^c \log |g|^2 + d^c \log \{ |\langle F_2, Z_\alpha \rangle|^2 + |\langle F_2, \bar{Z}_\alpha \rangle|^2 \} \\
 &= \frac{1}{4\pi i} [d \log g - d \log \bar{g}] + d^c \log \{ |\langle F_2, Z_\alpha \rangle|^2 + |\langle F_2, \bar{Z}_\alpha \rangle|^2 \}.
 \end{aligned}$$

Since the form  $f^*P_\alpha^*\omega$  is closed on  $\partial U_\varepsilon(p)$ ,  $n(p, \alpha)$  is independent of the choice of  $F$ .

Next suppose that  $f$  intersects transversely with  $\xi_\alpha$  at  $f(p)$ . Then

$$\begin{vmatrix} \langle \partial F / \partial w_1, Z_\alpha \rangle, \langle \partial F / \partial w_2, Z_\alpha \rangle \\ \langle \partial F / \partial w_1, \bar{Z}_\alpha \rangle, \langle \partial F / \partial w_2, \bar{Z}_\alpha \rangle \end{vmatrix} (p) \neq 0,$$

and hence we can choose  $(w_1, w_2) = (\langle F, Z_\alpha \rangle, \langle F, \bar{Z}_\alpha \rangle)$  as a coordinate system on  $U(p)$ . We have

$$n(p, \alpha) = \lim_{\varepsilon \downarrow 0} \int_{|w_1|^2 + |w_2|^2 = \varepsilon^2} d^c \log (|w_1|^2 + |w_2|^2) \wedge f^*P_\alpha^*\omega.$$

Putting  $w_1 = r_1 e^{i\theta_1}$ ,  $w_2 = r_2 e^{i\theta_2}$ ,  $r_1 = r \cos t$  and  $r_2 = r \sin t$  ( $0 \leq \theta_i \leq 2\pi$ ,  $0 \leq t \leq \pi/2$ ), we have

$$d^c \log (r_1^2 + r_2^2) = \frac{1}{2\pi} \frac{1}{r_1^2 + r_2^2} (r_1^2 d\theta_1 + r_2^2 d\theta_2),$$

and

$$\begin{aligned}
 f^*P_\alpha^*\omega &= \frac{1}{\pi} \frac{1}{(r_1^2 + r_2^2)} (r_1 r_2^2 dr_1 \wedge d\theta_1 + r_1^2 r_2 dr_2 \wedge d\theta_2 \\
 &\quad - r_1 r_2^2 dr_1 \wedge d\theta_2 - r_1^2 r_2 dr_2 \wedge d\theta_1).
 \end{aligned}$$

Thus we see

$$d^c \log (r_1^2 + r_2^2) \wedge f^*P_\alpha^*\omega = \frac{1}{2\pi^2} \sin t \cos t d\theta_1 \wedge dt \wedge d\theta_2$$

on  $r = \text{constant}$ .

On the sphere  $\{(w_1, w_2) \in U(p) : |w_1|^2 + |w_2|^2 = r^2\}$ ,  $d\theta_1 \wedge dt \wedge d\theta_2$  is a positive form. Therefore we have  $n(p, \alpha) = 1$ . Q.E.D.

We denote by  $(z_1, z_2)$  the standard coordinate system on  $\mathbb{C}^2$ . Put  $\Delta(r) = \{(z_1, z_2) \in \mathbb{C}^2 : \log |z_i| < r (i=1, 2)\}$ .

**Theorem 1.** Let  $f: \mathbb{C}^2 \rightarrow Q_{n-1}(\mathbb{C})$  ( $n \geq 3$ ) be a holomorphic mapping satisfying (A) and (B). Suppose  $f(\partial \Delta(r)) \cap \xi_\alpha = \emptyset$ . Then we have

$$(3.10) \quad \int_{\Delta(r)} f^* \Omega^2 = n(\Delta(r), \alpha) + \int_{\partial \Delta(r)} d^c [-\log (|f, \alpha|^2 + |f, \bar{\alpha}|^2) f^*(\Omega + P_\alpha^*\omega)],$$

where  $n(\Delta(r), \alpha) = \sum_{f(p_i) \in \xi_\alpha, p_i \in \Delta(r)} n(p_i, \alpha)$ .

Proof. By (3.1), Lemma 3.1, (3.5) and (3.5)', we have

$$\begin{aligned}
 (3.11) \quad \int_{\Delta(r)} f^* \Omega^2 &= \lim_{\varepsilon \downarrow 0} \int_{\Delta(r) - \sum_i U_{\mathfrak{g}}(p_i)} f^* \Omega^2 \\
 &= \lim_{\varepsilon \downarrow 0} \int_{\Delta(r) - \sum_i U_{\mathfrak{g}}(p_i)} -dd^c \cdot \log(|f, \alpha|^2 + |f, \bar{\alpha}|^2) \wedge f^*(\Omega + P_{\alpha}^* \omega) \\
 &= \lim_{\varepsilon \downarrow 0} \int_{\Delta(r) - \sum_i U_{\mathfrak{g}}(p_i)} dd^c [-\log(|f, \alpha|^2 + |f, \bar{\alpha}|^2) f^*(\Omega + P_{\alpha}^* \omega)],
 \end{aligned}$$

where  $U_{\mathfrak{g}}(p_i)$  is such a neighborhood of  $p_i$  as given in the definition  $n(p_i, \alpha)$ . Applying Stokes Theorem to the equation (3.11), we have

$$\begin{aligned}
 (3.12) \quad \int_{\Delta(r)} f^* \Omega^2 &= \int_{\partial \Delta(r)} d^c [-\log(|f, \alpha|^2 + |f, \bar{\alpha}|^2) f^*(\Omega + P_{\alpha}^* \omega)] \\
 &\quad - \lim_{\varepsilon \downarrow 0} \sum_i \int_{\partial U_{\mathfrak{g}}(p_i)} d^c [\log \|F_i\|^2 f^*(\Omega + P_{\alpha}^* \omega)] \\
 &\quad + \lim_{\varepsilon \downarrow 0} \sum_i \int_{\partial U_{\mathfrak{g}}(p_i)} d^c [\log \{ |\langle F_i, Z_{\alpha} \rangle|^2 + |\langle F_i, \bar{Z}_{\alpha} \rangle|^2 \} f^* \Omega] \\
 &\quad + \sum_i n(p_i, \alpha),
 \end{aligned}$$

where  $F_i$  is a holomorphic lift of  $f$  on  $U(p_i)$ . We have

$$(3.13) \quad \lim_{\varepsilon \downarrow 0} \int_{\partial U_{\mathfrak{g}}(p_i)} d^c [\log \|F_i\|^2 \cdot f^* \Omega] = \lim_{\varepsilon \downarrow 0} \int_{U_{\mathfrak{g}}(p_i)} f^* \Omega^2 = 0.$$

Set  $r^2 = |w_i^1|^2 + |w_i^2|^2$ , where  $(w_i^1, w_i^2)$  denotes a coordinate system on  $U(p_i)$ , we see

$$(3.14) \quad d^c \log \{ |\langle F_i, Z_{\alpha} \rangle|^2 + |\langle F_i, \bar{Z}_{\alpha} \rangle|^2 \} = 0 \left( \frac{1}{r} \right) (dw_i^1 + d\bar{w}_i^1 + dw_i^2 + d\bar{w}_i^2)$$

and

$$\begin{aligned}
 (3.15) \quad dd^c \log \{ |\langle F_i, Z_{\alpha} \rangle|^2 + |\langle F_i, \bar{Z}_{\alpha} \rangle|^2 \} &= 0 \left( \frac{1}{r^2} \right) (dw_i^1 \wedge d\bar{w}_i^1 + dw_i^1 \wedge d\bar{w}_i^2 \\
 &\quad + dw_i^2 \wedge d\bar{w}_i^2 + dw_i^2 \wedge d\bar{w}_i^1).
 \end{aligned}$$

Since  $\|F_i\|$  is positive on  $U(p_i)$ , we have

$$(3.16) \quad d^c \log \|F_i\|^2 = 0(1)(dw_i^1 + d\bar{w}_i^1 + dw_i^2 + d\bar{w}_i^2)$$

and

$$(3.17) \quad f^* \Omega = 0(1)(dw_i^1 \wedge d\bar{w}_i^1 + dw_i^1 \wedge d\bar{w}_i^2 + dw_i^2 \wedge d\bar{w}_i^2 + dw_i^2 \wedge d\bar{w}_i^1).$$

Since the both sides of the equation (3.8) are finite, comparing (3.14) and (3.15) with (3.16) and (3.17), we have

$$(3.18) \quad \lim_{\varepsilon \downarrow 0} \int_{\partial U_{\mathfrak{g}}(p_i)} d^c [\log \|F_i\|^2 \cdot f^* P_{\alpha}^* \omega] = 0$$

$$(3.19) \quad \lim_{\varepsilon \downarrow 0} \int_{\partial U_{\varepsilon}(p_i)} d^c [\log \{ |\langle F_i, Z_{\alpha} \rangle|^2 + |\langle F_i, \bar{Z}_{\alpha} \rangle|^2 \} f^* \Omega] = 0.$$

Q.E.D.

#### 4. First Main Theorem

Let  $f : C^2 \rightarrow Q_{n-1}(C)$  ( $n \geq 3$ ) be a holomorphic mapping satisfying (A) and (B). For a point  $\alpha$  in  $Q_{n-1}(C)$ , we choose two real numbers  $r_1$  and  $r_2$  such that  $r_1 > r_2$  and the image  $f((r(\Delta_1) \setminus \Delta(r_2)))$  does not intersect with  $\xi_{\alpha}$ .

We see easily  $|\beta, \alpha|^2 + |\beta, \bar{\alpha}|^2 \leq 1$  for  $\beta \in Q_{n-1}(C)$ . Hence  $\psi_{\alpha} = -\log(|f, \alpha|^2 + |f, \bar{\alpha}|^2) f^*(\Omega + P_{\alpha}^* \omega)$  is a positive form (non-negative form, precisely) on  $\Delta(r_1) \setminus \Delta(r_2)$ . Putting  $z_j = e^{s_j + i\theta_j}$  ( $j=1, 2$ ), we can write  $\psi_{\alpha}$  on  $\Delta(r_1) \setminus (\Delta(r_2) \cup \{(z, 0) \in C^2\} \cup \{0, z\} \in C^2\})$  as follows:

$$(4.1) \quad \begin{aligned} \psi_{\alpha} &= -\log(|f, \alpha|^2 + |f, \bar{\alpha}|^2) f^*(\Omega + P_{\alpha}^* \omega) \\ &= \psi_1 ds_1 \wedge d\theta_1 + \psi_2 ds_1 \wedge d\theta_2 + \psi_3 ds_2 \wedge d\theta_1 \\ &\quad + \psi_4 ds_2 \wedge d\theta_2 + \psi_5 d\theta_1 \wedge d\theta_2 + \psi_6 ds_1 \wedge ds_2. \end{aligned}$$

REMARK 2. If we write  $\psi_{\alpha}$  with the standard coordinate system  $(z_1, z_2)$  on  $C^2$ , we see  $\psi_1(z_1, z_2) = \tilde{\psi}_1(z_1, z_2)e^{2s_1}$ ,  $\psi_4(z_1, z_2) = \tilde{\psi}_4(z_1, z_2)e^{2s_2}$  and  $\psi_j(z_1, z_2) = e^{s_1} \cdot e^{s_2} \tilde{\psi}_j(z_1, z_2)$  ( $j=2, 3, 5, 6$ ) for certain functions  $\tilde{\psi}_i$  ( $i=1, 2, \dots, 6$ ).

**Lemma 4.1.** *We have*

$$(4.2) \quad \psi_1 \geq 0, \psi_4 \geq 0 \text{ and } \psi_2 = \psi_3.$$

Proof. Choosing a holomorphic lift  $F$  on a sufficiently small open set  $U$  in  $\Delta(r_1) \setminus \Delta(r_2)$ , we have

$$(4.3) \quad f^*(\Omega + P_{\alpha}^* \omega) = dd^c [\log \|F\|^2 + \log(|\langle F, Z_{\alpha} \rangle|^2 + |\langle F, \bar{Z}_{\alpha} \rangle|^2)],$$

where  $\Pi(Z_{\alpha}) = \alpha$ . Now we obtain

$$(4.4) \quad \begin{aligned} d^c &= \frac{1}{4\pi} \sum_{j=1}^2 \left[ \frac{\partial}{\partial s_j} d\theta_j - \frac{\partial}{\partial \theta_j} ds_j \right] \\ d &= \sum_{j=1}^2 \left[ \frac{\partial}{\partial \theta_j} d\theta_j + \frac{\partial}{\partial s_j} ds_j \right] \end{aligned} \quad \text{on } U \setminus (\{(0, z) \in C^2\} \cup \{(z, 0) \in C^2\}),$$

where  $(e^{s_1 + i\theta_1}, e^{s_2 + i\theta_2})$  is the restriction to  $U$  of the standard coordinate system in  $C^2$ . Putting  $g = \log(|\langle F, Z_{\alpha} \rangle|^2 + |\langle F, \bar{Z}_{\alpha} \rangle|^2) + \log \|F\|^2$ , we have

$$(4.5) \quad \begin{aligned} dd^c g &= \frac{1}{4\pi} \left[ \left( \frac{\partial^2 g}{(\partial \theta_1)^2} + \frac{\partial^2 g}{(\partial s_1)^2} \right) ds_1 \wedge d\theta_1 + \left( \frac{\partial^2 g}{\partial \theta_2 \partial \theta_1} + \frac{\partial^2 g}{\partial s_1 \partial s_2} \right) ds_1 \wedge d\theta_2 \right. \\ &\quad \left. + \left( \frac{\partial^2 g}{\partial \theta_1 \partial \theta_2} + \frac{\partial^2 g}{\partial s_2 \partial s_1} \right) ds_2 \wedge d\theta_1 + \left( \frac{\partial^2 g}{(\partial \theta_2)^2} + \frac{\partial^2 g}{(\partial s_2)^2} \right) ds_2 \wedge d\theta_2 + \dots \right]. \end{aligned}$$

Comparing (4.1) with (4.5), we have  $\psi_2 = \psi_3$ .

We shall show  $\psi_1 \geq 0$  and  $\psi_4 \geq 0$ .

$$\begin{aligned}
 (4.6) \quad dd^c \log(\sum_j f^j \bar{f}^j) &= \frac{i}{2\pi} \partial \bar{\partial} \cdot \log(\sum_j f^j \bar{f}^j) \\
 &= \frac{i}{2\pi} \frac{1}{\|F\|^4} [ \|F\|^2 (\sum_j df^j \wedge d\bar{f}^j) - (\sum_k df^k \bar{f}^k) \wedge (\sum_j f^j d\bar{f}^j) ] \\
 &= \frac{i}{2\pi} \frac{1}{\|F\|^4} \left[ \left( \|F\|^2 \left\| \frac{\partial F}{\partial z_1} \right\|^2 - \left| \left( \frac{\partial F}{\partial z_1}, F \right) \right|^2 \right) dz_1 \wedge d\bar{z}_1 \right. \\
 &\quad \left. + \left( \|F\|^2 \left\| \frac{\partial F}{\partial z_2} \right\|^2 - \left| \left( \frac{\partial F}{\partial z_2}, F \right) \right|^2 \right) dz_2 \wedge d\bar{z}_2 + \dots \right],
 \end{aligned}$$

where  $F = (f^0, f^1, \dots, f^n)$ . By the Schwartz inequality and the linear independence of vectors  $F$  and  $\partial F / \partial z_j$  ( $j=1, 2$ ), we have

$$\|F\|^2 \left\| \frac{\partial F}{\partial z_j} \right\|^2 > \left| \left( \frac{\partial F}{\partial z_j}, F \right) \right|^2, \text{ and } dz_j \wedge d\bar{z}_j = e^{2s_j} (-2ids_j \wedge d\theta_j)$$

( $j=1, 2$ ). Thus we have

$$\frac{1}{\pi} \frac{1}{\|F\|^4} \left[ \|F\|^2 \left\| \frac{\partial F}{\partial z_j} \right\|^2 - \left| \left( \frac{\partial F}{\partial z_j}, F \right) \right|^2 \right] e^{2s_j} > 0 \quad (j=1, 2)$$

or

$$(4.7) \quad \frac{1}{\pi} \frac{1}{(\sum_k f^k \bar{f}^k)^2} \left[ (\sum_k f^k \bar{f}^k) \left( \sum_k \frac{\partial f^k}{\partial z_j} \overline{\frac{\partial f^k}{\partial z_j}} \right) - \left| \left( \sum_k \frac{\partial f^k}{\partial z_j} \bar{f}^k \right) \right|^2 \right] e^{2s_j} > 0 \quad (j=1, 2).$$

As for  $dd^c[\log(|\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2)]$ , putting  $f^0 = \langle F, Z_\alpha \rangle$ ,  $f^1 = \langle F, \bar{Z}_\alpha \rangle$  and  $f^j = 0$  ( $j=2, \dots, n$ ) in the equation (4.6), we have also the inequality (4.7) (in this case we replace  $>$  by  $\geq 0$ ) with respect to the coefficient of  $ds_j \wedge d\theta_j$  ( $j=1, 2$ ).

Q.E.D.

Let  $r$  be in  $[r_2, r_1]$ . We divide  $\partial\Delta(r)$  into  $\partial\Delta_1(r)$  and  $\partial\Delta_2(r)$ , where

$$(4.8) \quad \partial\Delta_i(r) = \{(z_1, z_2) \in \partial\Delta(r) : \log|z_i| = r\} \quad (i=1, 2).$$

**Lemma 4.2.** *We have*

$$\begin{aligned}
 (4.9) \quad \int_{\partial\Delta(r)} d\psi_\alpha &= \frac{1}{4\pi} \left[ - \int_{S^1 \times S^1} \psi_\alpha(e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_2 \wedge d\theta_1 \right. \\
 &\quad \left. - \int_{S^1 \times S^1} \psi_1(e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_1 \wedge d\theta_2 \right] \\
 &\quad + \frac{1}{4\pi} \frac{\partial}{\partial r} \left[ \int_{\partial\Delta_1(r)} \psi_\alpha \wedge d\theta_1 + \int_{\partial\Delta_2(r)} \psi_\alpha \wedge d\theta_2 \right].
 \end{aligned}$$

**Proof.** First we remark that  $d\theta_1 \wedge ds_2 \wedge d\theta_2$  and  $d\theta_2 \wedge ds_1 \wedge d\theta_1$  are positive forms on  $\partial\Delta_1(r)$  and  $\partial\Delta_2(r)$  respectively.

By (4.1) and the preceeding remark 2, we have

$$\begin{aligned} \int_{\partial\Delta_1(r)} d^c\psi_\alpha &= \int_{\partial\Delta_1(r) \setminus \{(e^{r+i\theta_1}, 0)\}} d^c\psi_\alpha \\ &= \frac{1}{4\pi} \int_{\partial\Delta_1(r) \setminus \{(e^{r+i\theta_1}, 0)\}} \left[ -\frac{\partial\psi_3}{\partial s_2} + \frac{\partial\psi_4}{\partial s_1} + \frac{\partial\psi_5}{\partial\theta_2} \right] d\theta_1 \wedge ds_2 \wedge d\theta_2 \\ &= \frac{1}{4\pi} \int_{\partial\Delta_1(r)} \left[ -\frac{\partial\psi_3}{\partial s_2} + \frac{\partial\psi_4}{\partial s_1} + \frac{\partial\psi_5}{\partial\theta_2} \right] d\theta_1 \wedge ds_2 \wedge d\theta_2. \end{aligned}$$

Clearly we have

$$\int_{\partial\Delta_1(r)} \frac{\partial\psi_5}{\partial\theta_2} d\theta_1 \wedge ds_2 \wedge d\theta_2 = 0.$$

Therefore we obtain

$$(4.10) \quad \int_{\partial\Delta_1(r)} d^c\psi_\alpha = \frac{1}{4\pi} \int_{\partial\Delta_1(r)} \left[ -\frac{\partial\psi_3}{\partial s_2} + \frac{\partial\psi_4}{\partial s_1} \right] d\theta_1 \wedge ds_2 \wedge d\theta_2.$$

Similarly we obtain

$$(4.11) \quad \int_{\partial\Delta_2(r)} d^c\psi_\alpha = \frac{1}{4\pi} \int_{\partial\Delta_2(r)} \left[ \frac{\partial\psi_1}{\partial s_2} - \frac{\partial\psi_2}{\partial s_1} \right] d\theta_2 \wedge ds_1 \wedge d\theta_1.$$

Now we shall consider the equation (4.10). We have

$$\begin{aligned} (4.12) \quad & \frac{1}{4\pi} \int_{\partial\Delta_1(r)} \frac{\partial\psi_3}{\partial s_2} d\theta_1 \wedge ds_2 \wedge d\theta_2 \\ &= \frac{1}{4\pi} \int_{\partial\Delta_1(r)} d(\psi_3 d\theta_2 \wedge d\theta_1) \\ &= \frac{1}{4\pi} \int_{\partial\Delta_1(r) \cap \partial\Delta_2(r)} \psi_3 d\theta_2 \wedge d\theta_1 \\ &= \frac{1}{4\pi} \int_{S^1 \times S^1} \psi_3(e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_2 \wedge d\theta_1. \end{aligned}$$

Since we have

$$\begin{aligned} & \int_{\partial\Delta_1(r)} \psi_4 d\theta_1 \wedge ds_2 \wedge d\theta_2 \\ &= \int_{\partial\Delta_1(r)} d \left\{ \left( \int_{-\infty}^{s_2} \psi_4(e^{r+i\theta_1}, e^{t+i\theta_2}) dt \right) d\theta_2 \wedge d\theta_1 \right\} \\ &= \int_{S^1 \times S^1} \left( \int_{-\infty}^r \psi_4(e^{r+i\theta_1}, e^{t+i\theta_2}) dt \right) d\theta_2 \wedge d\theta_1, \end{aligned}$$

we obtain

$$\begin{aligned}
(4.13) \quad & \frac{\partial}{\partial r} \int_{\partial \Delta_1(r)} \psi_4 d\theta_1 \wedge ds_2 \wedge d\theta_2 \\
&= \int_{S^1 \times S^1} \psi_4(e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_2 \wedge d\theta_1 \\
&+ \int_{S^1 \times S^1} \left( \int_{-\infty}^r \frac{\partial \psi_4}{\partial r}(e^{r+i\theta_1}, e^{t+i\theta_2}) dt \right) d\theta_2 \wedge d\theta_1.
\end{aligned}$$

By (4.10), (4.12) and (4.13), we obtain

$$\begin{aligned}
(4.14) \quad & \int_{\partial \Delta_1(r)} d^c \psi_\alpha = \frac{1}{4\pi} \int_{S^1 \times S^1} [-\psi_3 - \psi_4](e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_2 \wedge d\theta_1 \\
&+ \frac{1}{4\pi} \frac{\partial}{\partial r} \int_{\partial \Delta_1(r)} \psi_4 d\theta_1 \wedge ds_2 \wedge d\theta_2.
\end{aligned}$$

By the similar argument as we derived (4.14) from (4.10), we derive the following from (4.11)

$$\begin{aligned}
(4.15) \quad & \frac{1}{4\pi} \int_{\partial \Delta_2(r)} d^c \psi_\alpha = \frac{1}{4\pi} \int_{S^1 \times S^1} [-\psi_2 - \psi_1](e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_1 \wedge d\theta_2 \\
&+ \frac{1}{4\pi} \frac{\partial}{\partial r} \int_{\partial \Delta_2(r)} \psi_1 d\theta_2 \wedge ds_1 \wedge d\theta_1.
\end{aligned}$$

By (4.14), (4.15) and the definition of  $\psi_\alpha$  we obtain (4.9).

Q.E.D.

**Lemma 4.3.** *We have*

$$(4.16) \quad \int_{\Delta(r)} f^* \Omega^2 = \frac{1}{4\pi} \frac{\partial}{\partial r} \left[ \int_{\partial \Delta_1(r)} \psi_\alpha \wedge d\theta_1 + \int_{\partial \Delta_2(r)} \psi_\alpha \wedge d\theta_2 \right] + n(\Delta(r), \alpha).$$

*Proof.* By Theorem 1 and Lemma 4.2, we have only to prove that

$$\frac{1}{4\pi} \int_{S^1 \times S^1} [\psi_4 - \psi_1](e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_2 \wedge d\theta_1 = 0.$$

We define a mapping  $h: C^2 \rightarrow C^2$  by  $h((z_1, z_2)) = (z_2, z_1)$ . Then  $(f \circ h)$  satisfies Conditions (A) and (B), and we have

$$(|f \circ h, \alpha|^2 + |f \circ h, \bar{\alpha}|^2)(z_1, z_2) = (|f, \alpha|^2 + |f, \bar{\alpha}|^2)(z_2, z_1)$$

and

$$\begin{aligned}
n_f((z_1, z_2), \alpha) &= \lim_{\varepsilon \downarrow 0} \int_{\partial U_\varepsilon((z_1, z_2))} d^c \log[|\langle F, Z_\alpha \rangle|^2 + |\langle F, \bar{Z}_\alpha \rangle|^2] \wedge f^* P_\alpha^* \omega \\
&= \lim_{\varepsilon \downarrow 0} \int_{\partial U_\varepsilon((z_2, z_1))} d^c \log[|\langle F \circ h, Z_\alpha \rangle|^2 + |\langle F \circ h, \bar{Z}_\alpha \rangle|^2] \wedge (fh)^* P_\alpha^* \omega \\
&= n_{f \circ h}((z_2, z_1), \alpha).
\end{aligned}$$

On the other hand, we have from (4.1)

$$(4.17) \quad (h^*\psi_\omega) = \psi_1 \circ h \, ds_2 \wedge d\theta_2 + \psi_2 \circ h \, ds_2 \wedge d\theta_1 + \psi_3 \circ h \, ds_1 \wedge d\theta_2 \\ + \psi_4 \circ h \, ds_1 \wedge d\theta_1 + \psi_5 \circ h \, d\theta_2 \wedge d\theta_1 + \psi_6 \circ h \, ds_2 \wedge ds_1.$$

By Theorem 1, (4.14) and (4.15) in Lemma 4.2, comparing (4.1) with (4.17) we have

$$(4.18) \quad \int_{\Delta(r)} f^* \Omega^2 = \int_{\Delta(r)} h^* f^* \Omega^2 = n(\Delta(r), \alpha) \\ + \frac{1}{4\pi} \left[ - \int_{S^1 \times S^1} \psi_1 \circ h (e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_2 \wedge d\theta_1 - \int_{S^1 \times S^1} \psi_4 \circ h (e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_1 \wedge d\theta_2 \right] \\ + \frac{1}{4\pi} \frac{\partial}{\partial r} \left[ \int_{\partial\Delta_1(r)} \psi_1 \circ h \, d\theta_1 \wedge ds_2 \wedge d\theta_2 + \int_{\partial\Delta_2(r)} \psi_4 \circ h \, d\theta_2 \wedge ds_1 \wedge d\theta_1 \right].$$

We see easily

$$\int_{\partial\Delta_1(r)} \psi_1 \circ h \, d\theta_1 \wedge ds_2 \wedge d\theta_2 = \int_{\partial\Delta_2(r)} \psi_1 d\theta_2 \wedge ds_1 \wedge d\theta_1 \\ = \int_{\partial\Delta_2(r)} \psi_\omega \wedge d\theta_2$$

and

$$\int_{\partial\Delta_2(r)} \psi_4 \circ h \, d\theta_2 \wedge ds_1 \wedge d\theta_1 = \int_{\partial\Delta_1(r)} \psi_4 d\theta_1 \wedge ds_2 \wedge d\theta_2 \\ = \int_{\partial\Delta_1(r)} \psi_\omega \wedge d\theta_1.$$

Therefore we have only to prove

$$\int_{S^1 \times S^1} ((\psi_i \circ h) - \psi_i) (e^{r+i\theta_1}, e^{r+i\theta_2}) d\theta_1 \wedge d\theta_2 = 0 \quad (i = 1, 4).$$

For any  $\alpha, \beta \in [0, 2\pi]$ , we have

$$((\psi_i \circ h) - \psi_i) (e^{r+i\alpha}, e^{r+i\beta}) = \psi_i(e^{r+i\beta}, e^{r+i\alpha}) - \psi_i(e^{r+i\alpha}, e^{r+i\beta}) \\ ((\psi_i \circ h) - \psi_i) (e^{r+i\beta}, e^{r+i\alpha}) = \psi_i(e^{r+i\alpha}, e^{r+i\beta}) - \psi_i(e^{r+i\beta}, e^{r+i\alpha})$$

Thus we obtain

$$((\psi_i \circ h) - \psi_i) (e^{r+i\alpha}, e^{r+i\beta}) = -((\psi_i \circ h) - \psi_i) (e^{r+i\beta}, e^{r+i\alpha}).$$

Q.E.D.

For the holomorphic mapping  $f: \mathbf{C}^2 \rightarrow Q_{n-1}(\mathbf{C})$  ( $n \geq 3$ ) satisfying Conditions (A) and (B), we put

$$T(r) = \int_0^r dt \int_{\Delta(t)} f^* \Omega^2 \quad (\text{order function})$$



$$(4.19) \quad N(r, \alpha) = \int_0^r n(\Delta(t), \alpha) dt \quad (\text{counting function})$$

$$m(r, \alpha) = \frac{1}{4\pi} \left[ \int_{\partial \Delta_1(r)} \psi_\alpha \wedge d\theta_1 + \int_{\partial \Delta_2(r)} \psi_\alpha \wedge d\theta_2 \right].$$

We need the following lemma, which can be proved in a similar way as ([5] p.p. 502).

**Lemma 4.4.** *For any  $\alpha$ ,  $m(r, \alpha)$  is continuous with respect to  $r \in [0, \infty)$ .*

**Theorem 2.** *We have*

$$(4.20) \quad T(r) = m(r, \alpha) - m(0, \alpha) + N(r, \alpha) \quad \text{for any } r \geq 0,$$

and  $m(r, \alpha)$  is non-negative.

*Proof.* Integrating the equation in Lemma 4.3 with respect to  $r \in [r_2, r_1]$ , we have

$$\int_{r_2}^{r_1} dr \int_{\Delta(r)} f^* \Omega^2 = \int_{r_2}^{r_1} n(\Delta(r), \alpha) dr + m(r_1, \alpha) - m(r_2, \alpha).$$

By Lemma 4.4 we obtain the equation (4.20). It follows from Lemma 4.1 and Lemma 4.4 that the function  $m(r, \alpha)$  is non-negative. Q.E.D.

**Lemma 4.5.** *For any  $r$ ,  $m(r, \alpha)$  is continuous with respect to  $\alpha \in Q_{n-1}(C)$ .*

We also omit this proof by the same reason as in Lemma 4.4. (c.f. [5] p.p. 504).

**Theorem 3.** *There exists a positive constant  $C$  satisfying*

$$(4.21) \quad T(r) + C > N(r, \alpha) \quad \text{whenever } r \geq 0 \text{ and } \alpha \in Q_{n-1}(C).$$

*Proof.* By Theorem 2 we have

$$T(r) + m(0, \alpha) \geq N(r, \alpha) \quad \text{for any } r \geq 0.$$

Therefore by Lemma 4.5 we have the equation (4.21). Q.E.D.

## 5. Induced form by $f$

We denote by  $(X_0, X_1, \dots, X_n)$  an element of  $SO(n+1)$ , where  $X_i$ 's ( $0 \leq i \leq n$ ) are column vectors, and we put  $X_i = (x_{i0}, \dots, x_{in})^t$ . The left invariant forms  $\theta_{ij}$  ( $0 \leq i, j \leq n$ ) on  $SO(n+1)$  are defined by the following equation:

$$(5.1) \quad - \begin{pmatrix} dX_0^t \\ dX_1^t \\ \vdots \\ dX_n^t \end{pmatrix} (X_0, \dots, X_n) = \begin{pmatrix} X_0^t \\ X_1^t \\ \vdots \\ X_n^t \end{pmatrix} (dX_0, \dots, dX_n) = \begin{pmatrix} 0 & \theta_{10} & \dots & \theta_{n0} \\ \theta_{01} & 0 & \dots & \theta_{n1} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{0n} & \theta_{1n} & \dots & 0 \end{pmatrix},$$

where  $\theta_{ij} = -\theta_{ji}$ .

Therefore we have  $-\langle dX_i, X_j \rangle = \theta_{ji}$  i.e.,

$$(5.2) \quad dX_i = \sum_j \theta_{ij} X_j.$$

Taking its exterior derivative, we see

$$(5.3) \quad d\theta_{01} = \sum_k \theta_{0k} \wedge \theta_{k1} = -\sum_k \theta_{0k} \wedge \theta_{1k}.$$

We remark that  $d\theta_{01}$  is a 2-form on  $SO(n+1)/SO(n-1)$ . Furthermore it is a lift of a 2-form on  $Q_{n-1}(C)$  by  $\Pi_1$ . In fact, let  $U$  be an open neighborhood of  $Q_{n-1}(C)$ , and  $(X_0, X_1)$  be a local cross-section of  $U$  into  $SO(n+1)/SO(n-1)$ :  $\Pi_1((X_0, X_1)) = \text{identity on } U$ . We have

$$(5.4) \quad \Pi_1^{-1}(\Pi_1(X_0, X_1)) = \{(X_0, X_1) \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} : 0 \leq \theta < 2\pi\}.$$

Then we have on  $\Pi_1^{-1}(U)$ ,

$$(5.5) \quad \begin{aligned} d\theta_{01} &= d\langle d(\cos \theta \cdot X_0 + \sin \theta \cdot X_1), (-\sin \theta \cdot X_1 + \cos \theta \cdot X_1) \rangle \\ &= d(d\theta + \langle dX_0, X_1 \rangle) = d\langle dX_0, X_1 \rangle. \end{aligned}$$

Let  $\sigma$  be a local holomorphic cross-section on  $U$  into  $C^{n+1} - \{0\}$  with respect to the Hopf fibring:  $\Pi\sigma = \text{identity on } U$ . We can write  $\sigma$  in the form  $\sigma = X + iY$  for orthogonal real vectors  $X$  and  $Y$  at each point of  $U$ . Then we see

$$(5.6) \quad \Omega = dd^c \log \|\sigma\|^2 = -\frac{1}{2\pi} d\langle d(X/\|X\|), Y/\|Y\| \rangle.$$

Thus,  $d\theta_{01}$  is the lift of  $-2\pi\Omega$  by  $\Pi_1^*$  i.e.,

$$(5.7) \quad \Pi_1^* \Omega = -\frac{1}{2\pi} d\theta_{01}.$$

In the equation (5.1) we defined  $\theta_{0j}$ 's and  $\theta_{1j}$ 's ( $0 \leq j \leq n$ ) as 1-forms on  $SO(n+1)$ . They are also regarded as 1-forms on  $SO(n+1)/SO(n-1)$ . To prove this fact we shall identify  $SO(n+1)/SO(n-1)$  with  $S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(C))$ . We take a local coordinate  $x = (x^1, \dots, x^{2n-1})$  on a small open set  $U$  in  $S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(C))$  and write a point  $Z(x)$  of  $U$  in the form  $(X_0(x) + iX_1(x))/\sqrt{2}$ , where  $\langle X_0, X_0 \rangle(x) = \langle X_1, X_1 \rangle(x) = 1$  and  $\langle X_0, X_1 \rangle(x) = 0$ . For each  $x$ , extending  $X_0(x)$  and  $X_1(x)$ , we take a real orthonormal basis  $X_0(x), \dots, X_n(x)$  in  $C^{n+1}$  such that  $(X_0, \dots, X_n)(x) \in SO(n+1)$ . Then the tangent space  $T_{Z(x)}(S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(C)))$  has a basis  $(iX_0 - X_1)(x), X_2(x), \dots, X_n(x), iX_2(x), \dots, X_n(x)$  (c.f. [3] p.p. 279).

In the equation  $dZ = \sum_{i=1}^{2n-1} \frac{\partial Z}{\partial x^i} dx^i$ , we see  $\frac{\partial Z}{\partial x^i} = Z_* \left( \frac{\partial}{\partial x^i} \right)$  ( $1 \leq i \leq 2n-1$ ) and hence  $\frac{\partial Z}{\partial x^i}$ 's are tangent vectors of  $T_{Z(x)}(S^{2n+1} \cap \Pi^{-1}(Q_{n-1}(C)))$ . Thus there exists 1-

forms  $\theta_j$ 's ( $1 \leq j \leq n$ ) and  $\tilde{\theta}_j$ 's ( $2 \leq j \leq n$ ) on  $U$  such that  $dZ = \theta_1(iX_0 - X_1) + \sum_{j=2}^n (\theta_j + i\tilde{\theta}_j)X_j$ . Comparing this form with (5.2), we have  $\theta_1 = \theta_{10}/\sqrt{2}$ ,  $\theta_j = \theta_{0j}/\sqrt{2}$  ( $2 \leq j \leq n$ ) and  $\tilde{\theta}_j = \theta_{1j}/\sqrt{2}$  ( $2 \leq j \leq n$ ). Thus we have from (5.2), (5.3) and (5.7)

$$(5.8) \quad (\Pi^*\Omega)_{(X_0, X_1)} = \frac{1}{2\pi} \sum_{j=2}^n \langle dX_0, X_j \rangle \wedge \langle dX_1, X_j \rangle,$$

where  $(X_0, X_1, \dots, X_n) \in SO(n+1)$ . For the volume form  $\Omega^{n-1}$  on  $Q_{n-1}(C)$ , we have

$$(5.9) \quad (\Pi^*\Omega^{n-1})_{(X_0, X_1)} = \left(\frac{1}{2\pi}\right)^{n-1} (n-1)! \langle dX_0, X_2 \rangle \wedge \langle dX_1, X_2 \rangle \wedge \dots \wedge \langle dX_0, X_n \rangle \wedge \langle dX_1, X_n \rangle.$$

We shall obtain a formula for  $f^*\Omega^2$  on  $C^2$ . Let  $F$  be a holomorphic lift of  $f$  on a neighborhood  $U$  in  $C^2$  by  $\Pi$ . Set  $(X_0 + iX_1)/\sqrt{2} = F/||F||$ , where  $X_i$  ( $i=0, 1$ ) are the orthonormal real vectors. With the coordinate system  $(x_1 + iy_1, x_2 + iy_2)$  on  $C^2$ , we can write:

$$(5.10) \quad \begin{aligned} dX_0 &= \omega_1 X_1 + \lambda_2 \tilde{B}_2 dx_1 - \lambda_3 \tilde{B}_3 dy_1 + \lambda_4 \tilde{B}_4 dx_2 - \lambda_5 \tilde{B}_5 dy_2, \\ dX_1 &= \omega_2 X_0 + \lambda_3 \tilde{B}_3 dx_1 + \lambda_2 \tilde{B}_2 dy_1 + \lambda_5 \tilde{B}_5 dx_2 + \lambda_4 \tilde{B}_4 dy_2, \end{aligned}$$

where  $\tilde{B}_i$ 's ( $2 \leq i \leq 5$ ) are differentiable vectors satisfying  $\langle \tilde{B}_i, \tilde{B}_i \rangle = 1$ ,  $\lambda_i$ 's ( $2 \leq i \leq 5$ ) are differentiable functions and  $\omega_i$ 's ( $1 \leq i \leq 2$ ) are 1-forms on  $U$ . Then we take differentiable orthonormal vectors  $B_i$  ( $2 \leq i \leq 5$ ) such that  $\tilde{B}_2 = B_2$ ,  $\tilde{B}_3 = \alpha_2 B_2 + \alpha_3 B_3$ ,  $\tilde{B}_4 = \beta_2 B_2 + \beta_3 B_3 + \beta_4 B_4$  and  $\tilde{B}_5 = \gamma_2 B_2 + \gamma_3 B_3 + \gamma_4 B_4 + \gamma_5 B_5$ , where  $\alpha_i$ ,  $\beta_i$  and  $\gamma_i$  are differentiable functions satisfying  $\sum \alpha_i^2 = 1$ ,  $\sum \beta_i^2 = 1$  and  $\sum \gamma_i^2 = 1$ . We choose differentiable vectors  $B_6, \dots, B_n$  on  $U$  such that  $(X_0, X_1, B_2, \dots, B_n) \in SO(n+1)$  at each point of  $U$ . By (5.8) we have

$$(5.11) \quad \begin{aligned} f^*\Omega &= \frac{1}{2\pi} \sum_{j=2}^n \langle dX_0, B_j \rangle \wedge \langle dX_1, B_j \rangle \\ &= \frac{1}{2\pi} \{ [\lambda_2 \lambda_5 \gamma_2 - \lambda_3 \lambda_4 \alpha_2 \beta_2 - \lambda_3 \lambda_4 \beta_3 \alpha_3] (dx_1 \wedge dx_2 + dy_1 \wedge dy_2) \\ &\quad + [\lambda_2^2 + \lambda_3^2] dx_1 \wedge dy_1 + [\lambda_4^2 + \lambda_5^2] dx_2 \wedge dy_2 \\ &\quad + [\lambda_2 \lambda_4 \beta_2 + \lambda_3 \lambda_5 \alpha_2 \gamma_2 + \lambda_3 \lambda_5 \alpha_3 \gamma_3] (dx_1 \wedge dy_2 - dy_1 \wedge dx_2) \}. \end{aligned}$$

Furthermore we obtain

$$(5.12) \quad \begin{aligned} f^*\Omega^2 &= \left(\frac{1}{2\pi}\right)^2 \times 2 \times \{ [\lambda_2^2 + \lambda_3^2] [\lambda_4^2 + \lambda_5^2] \\ &\quad - [\lambda_2 \lambda_4 \beta_2 + \lambda_3 \lambda_5 \alpha_2 \gamma_2 + \lambda_3 \lambda_5 \alpha_3 \gamma_3]^2 \\ &\quad - [\lambda_2 \lambda_5 \gamma_2 - \lambda_3 \lambda_4 \alpha_2 \beta_2 - \lambda_3 \lambda_4 \alpha_3 \beta_3]^2 \} dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2. \end{aligned}$$

## 6. Crofton formula

In §3 we have defined  $n(\Delta(r), \alpha)$  for a holomorphic mapping  $f: C^2 \rightarrow Q_{n-1}(C)$  ( $n \geq 3$ ) satisfying Conditions (A) and (B). Then we have:

**Theorem 4** (Crofton formula). *Let  $D$  be an open set in  $C^2$  with compact closure. Then we have*

$$(6.1) \quad \int_{Q_{n-1}(C)} n(D, \xi) d\xi = 2 \int_D f^* \Omega^2,$$

where  $d\xi = d\xi_\alpha = d\alpha = \Omega^{n-1}$ .

*Proof.* First we assume that  $D$  is so small that there exists a differentiable lift  $\sigma = (X_0, X_1)$  of  $f$  on  $D$ :  $\Pi_1 \sigma = f$ . Let  $q$  be a point in  $D$  and set  $f(q) \in \xi_\alpha$ . For any real orthonormal vectors  $Y_0, Y_1$  such that  $\Pi_1((Y_0, Y_1)) = \alpha$ , we have

$$(6.2) \quad \langle X_0(q), Y_0 \rangle = \langle X_0(q), Y_1 \rangle = \langle X_1(q), Y_0 \rangle = \langle X_1(q), Y_1 \rangle = 0.$$

We set

$$(6.3) \quad \begin{aligned} Q_{n-3}(f(q)^\perp) &= \{\alpha \in Q_{n-1}(C): f(q) \in \xi_\alpha\} \\ f(D)^\perp &= \{\alpha \in Q_{n-1}(C): f(D) \cap \xi_\alpha \neq \emptyset\}. \end{aligned}$$

and

$$(6.4) \quad \begin{aligned} D' &= \Pi_1^{-1}(f(D)^\perp) \\ D'' &= \{(q, a): q \in D, a = (A_2, A_3, \dots, A_n) \in SO(n-1)\}. \end{aligned}$$

For  $a = (A_2, A_3, \dots, A_n) \in SO(n-1)$  we write its column vector  $A_i$  as  $A_i = (a_{i2}, \dots, a_{in})^t$ . Then we define a mapping  $t: D'' \rightarrow SO(n+1)$  by

$$(6.5) \quad t((q, a)) = (B_2, B_3, X_0, X_1, B_4, \dots, B_n)(q)$$

$$\times \begin{pmatrix} a_{22} & a_{32} & 0 & 0 & a_{42} & \dots & a_{n2} \\ a_{23} & a_{33} & 0 & 0 & a_{43} & \dots & a_{n3} \\ 0 & 0 & 1 & 0 & 0 & & 0 \\ 0 & 0 & 0 & 1 & 0 & & 0 \\ a_{24} & a_{34} & 0 & 0 & a_{44} & \dots & a_{n4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{2n} & a_{3n} & 0 & 0 & a_{4n} & \dots & a_{nn} \end{pmatrix},$$

where  $(X_0, X_1, B_2, \dots, B_n)(q)$  is the one given in §5. Let  $\Pi'$  be the projection  $D \times (SO(n-1)/SO(n-3)) \rightarrow D \times Q_{n-3}(C)$  defined by  $\Pi'((q, (A_2, A_3))) = (q, \Pi''((A_2, A_3)))$ , where  $\Pi''$  is the projection with respect to the Hopf fibring  $SO(n-1)/SO(n-3) \rightarrow Q_{n-3}(C)$ . We consider the following diagram;

$$(6.6) \quad \begin{array}{ccc} D \times (SO(n-1)/SO(n-3)) & \xrightarrow{t'} & D' \subset SO(n+1)/SO(n-1) \\ \downarrow \Pi' & & \downarrow \Pi_1 \\ D \times Q_{n-3}(C) & \xrightarrow{t''} & f(D)^\perp \subset Q_{n-1}(C), \end{array}$$

where  $t'((q, (A_2, A_3))) = (\sum_{i=2}^n a_{2i} B_i(q), \sum_{i=2}^n a_{3i} B_i(q))$  and  $t''$  is defined by  $\Pi_1 \circ t' = t'' \circ \Pi'$ . Then, in the above diagram, we remark that  $t''((q, Q_{n-3}(C))) = Q_{n-3}(f(q)^\perp)$  for each  $q \in D$ . Putting  $t((q, a)) = (X'_0, X'_1, \dots, X'_n)$ , we obtain

$$(6.7) \quad \begin{aligned} & (\Pi')^*(t'')^*\Omega^{n-1} = (t')^*(\Pi_1)^*\Omega^{n-1} \\ &= \left(\frac{1}{2\pi}\right)^{n-1} (n-1)! \langle dX'_0, X'_2 \rangle \wedge \langle dX'_1, X'_2 \rangle \wedge \dots \wedge \langle dX'_0, X'_n \rangle \wedge \langle dX'_1, X'_n \rangle \\ &= \left(\frac{1}{2\pi}\right)^{n-1} (n-1)! \times \frac{1}{16} \times \langle d(X_0 + iX_1), X'_0 + iX'_1 \rangle \wedge \langle d(X_0 - iX_1), X'_0 - iX'_1 \rangle \\ &\quad \wedge \langle d(X_0 + iX_1), X'_0 - iX'_1 \rangle \wedge \langle d(X_0 - iX_1), X'_0 + iX'_1 \rangle \wedge \langle dA_2, A_4 \rangle \\ &\quad \wedge \langle dA_3, A_4 \rangle \wedge \dots \wedge \langle dA_2, A_n \rangle \wedge \langle dA_3, A_n \rangle \\ &= -\frac{1}{4} \left(\frac{1}{2\pi}\right)^2 (n-1)(n-2) \left| \begin{array}{cc} \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, X'_0 + iX'_1 \rangle, & \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, \\ & X'_0 + iX'_1 \rangle \\ \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, X'_0 - iX'_1 \rangle, & \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, \\ & X'_0 - iX'_1 \rangle \end{array} \right|^2 \\ &\quad \times dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \left(\frac{1}{2\pi}\right)^{n-3} (n-3)! \langle dA_2, A_4 \rangle \wedge \langle dA_3, A_4 \rangle \wedge \dots \\ &\quad \wedge \langle dA_2, A_n \rangle \wedge \langle dA_3, A_n \rangle. \end{aligned}$$

We put  $C = \{\beta \in f(D)^\perp : \text{there exists } \beta' \in (t'')^{-1}(\beta) \text{ such that } (dt'')(\beta') \text{ is singular}\}$ . From Sard's Theorem the set  $C$  has measure zero. If we take  $\alpha \in (f(D)^\perp \setminus C)$ , the set  $(t'')^{-1}(\alpha)$  consists of finite elements because of the compactness of  $\bar{D}$  and Condition (B). We denote by  $n_\alpha$  the number of elements  $(t'')^{-1}(\alpha)$ . Then, for each  $\alpha \in (f(D)^\perp \setminus C)$  there exists a connected neighborhood  $V$  of  $\alpha$  in  $(f(D)^\perp \setminus C)$  such that  $(t'')^{-1}(V)$  has  $n_\alpha$  connected components and  $t''$  maps each component onto  $V$  diffeomorphically. Let  $\{V_i\}$  be a locally finite covering of  $f(D)^\perp \setminus C$  by such open sets and  $\{\phi_i\}$  be a partition of unity subordinated to  $\{V_i\}$ . Now we have

$$(6.8) \quad \begin{aligned} \int_{f(D)^\perp} n_\alpha d\alpha &= \int_{f(D)^\perp - C} n_\alpha d\alpha = \sum_i \int_{f(D)^\perp - C} \phi_i(\alpha) n_\alpha d\alpha \\ &= \sum_i \int_{V_i} n_\alpha (\phi_i(\alpha) d\alpha) = \sum_i \int_{(t'')^{-1}(V_i)} -(t'')^*(\phi_i(\alpha) d\alpha) \\ &= \sum_i \int_{(t'')^{-1}(V_i)} -((t'')^* \phi_i(\alpha)) ((t'')^* d\alpha) \\ &= \int_{D \times Q_{n-2} - C'} -(t'')^* d\alpha = \int_{D \times Q_{n-3}} -(t'')^* d\alpha, \end{aligned}$$

where  $C'$  is the set of critical points of  $t''$ . If

$$t''((q, \alpha_j)) = \alpha \text{ and } \begin{vmatrix} \langle \partial F / \partial z_1, Z_\alpha \rangle, \langle \partial F / \partial z_2, Z_\alpha \rangle \\ \langle \partial F / \partial z_1, \bar{Z}_\alpha \rangle, \langle \partial F / \partial z_2, \bar{Z}_\alpha \rangle \end{vmatrix} (q) \\ \left( \text{which is equal to } \frac{\|F\|}{2} \begin{vmatrix} \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, Z_\alpha \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, Z_\alpha \rangle \\ \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, \bar{Z}_\alpha \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, \bar{Z}_\alpha \rangle \end{vmatrix} (q) \right) = 0$$

for  $\Pi(Z_\alpha) = \alpha$ , then  $dt''((q, \alpha_j))$  is singular because of (6.7). By Lemma 3.2 we have  $n(D, \alpha) = n_\alpha$  on  $f(D)^\perp \setminus C$ . Therefore we have

$$(6.9) \quad \int_{Q_{n-1}} n(D, \alpha) d\alpha = \frac{1}{4} \left( \frac{1}{2\pi} \right)^2 (n-1)(n-2) \int_D dx^1 \wedge dy^1 \wedge dx^2 \wedge dy^2 \\ \times \int_{Q_{n-3}(f(q)^\perp)} \left\| \begin{vmatrix} \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, X'_0 + iX'_1 \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, X'_0 + iX'_1 \rangle \\ \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, X'_0 - iX'_1 \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, X'_0 - iX'_1 \rangle \end{vmatrix} \right\|^2 \Omega^{n-3}.$$

Next we have the following equation:

$$(6.10) \quad \int_{Q_{n-3}(f(q)^\perp)} \left\| \begin{vmatrix} \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, X'_0 + iX'_1 \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, X'_0 + iX'_1 \rangle \\ \langle \lambda_2 \tilde{B}_2 + i\lambda_3 \tilde{B}_3, X'_0 - iX'_1 \rangle, \langle \lambda_4 \tilde{B}_4 + i\lambda_5 \tilde{B}_5, X'_0 - iX'_1 \rangle \end{vmatrix} \right\|^2 \Omega^{n-3} \\ = [(\lambda_2 \lambda_4 \beta_3 - \lambda_3 \lambda_5 \alpha_2 \gamma_3 + \lambda_3 \lambda_5 \alpha_3 \gamma_2)^2 + (\lambda_3 \lambda_4 \alpha_2 \beta_3 + \lambda_2 \lambda_5 \gamma_3 - \lambda_3 \lambda_4 \alpha_3 \beta_2)^2] (q) \\ \times \int_{Q_{n-3}(f(q)^\perp)} \left\| \begin{vmatrix} \langle B_2, X'_0 + iX'_1 \rangle, \langle B_3, X'_0 + iX'_1 \rangle \\ \langle B_2, X'_0 - iX'_1 \rangle, \langle B_3, X'_0 - iX'_1 \rangle \end{vmatrix} \right\|^2 \Omega^{n-3} \\ + (\lambda_2^2 + \lambda_3^2 \alpha_2^2) (\lambda_4^2 \beta_4^2 + \lambda_5^2 \gamma_4^2) (q) \int_{Q_{n-3}(f(q)^\perp)} \left\| \begin{vmatrix} \langle B_2, X'_0 + iX'_1 \rangle, \\ \langle B_2, X'_0 - iX'_1 \rangle, \\ \langle B_4, X'_0 + iX'_1 \rangle \\ \langle B_4, X'_0 - iX'_1 \rangle \end{vmatrix} \right\|^2 \Omega^{n-3} \\ + (\lambda_2^2 + \lambda_3^2 \alpha_2^2) (\lambda_5^2 \gamma_5^2) (q) \int_{Q_{n-3}(f(q)^\perp)} \left\| \begin{vmatrix} \langle B_2, X'_0 + iX'_1 \rangle, \langle B_5, X'_0 + iX'_1 \rangle \\ \langle B_2, X'_0 - iX'_1 \rangle, \langle B_5, X'_0 - iX'_1 \rangle \end{vmatrix} \right\|^2 \Omega^{n-3} \\ + (\lambda_3^2 \alpha_3^2) (\lambda_4^2 \beta_4^2 + \lambda_5^2 \gamma_4^2) (q) \int_{Q_{n-3}(f(q)^\perp)} \left\| \begin{vmatrix} \langle B_3, X'_0 + iX'_1 \rangle, \langle B_4, X'_0 + iX'_1 \rangle \\ \langle B_3, X'_0 - iX'_1 \rangle, \langle B_4, X'_0 - iX'_1 \rangle \end{vmatrix} \right\|^2 \Omega^{n-3} \\ + (\lambda_3^2 \alpha_3^2 \lambda_5^2 \gamma_5^2) (q) \int_{Q_{n-3}(f(q)^\perp)} \left\| \begin{vmatrix} \langle B_3, X'_0 + iX'_1 \rangle, \langle B_5, X'_0 + iX'_1 \rangle \\ \langle B_3, X'_0 - iX'_1 \rangle, \langle B_5, X'_0 - iX'_1 \rangle \end{vmatrix} \right\|^2 \Omega^{n-3}.$$

In fact, the integral of the other terms which appear at the right hand side of (6.10) turns out to be zero. For example we consider the following integral:

$$l = \int_{Q_{n-3}(\mathcal{F}(\mathcal{Q})^\perp)} \left| \begin{array}{cc} \langle B_2(q), X'_0 + iX'_1 \rangle, \langle B_3(q), X'_0 + iX'_1 \rangle \\ \langle B_2(q), X'_0 - iX'_1 \rangle, \langle B_3(q), X'_0 - iX'_1 \rangle \end{array} \right| \frac{\overline{\langle B_2(q), X'_0 + iX'_1 \rangle}, \overline{\langle B_2(q), X'_0 - iX'_1 \rangle}}{\langle B_4(q), X'_0 + iX'_1 \rangle, \langle B_4(q), X'_0 - iX'_1 \rangle} \Omega^{n-3}.$$

We have

$$l = \int_{SO(n-1)/SO(n-3)} \left| \begin{array}{cc} (a_{22} - ia_{32}), (a_{23} - ia_{33}) \\ (a_{22} + ia_{32}), (a_{23} + ia_{33}) \end{array} \right| \frac{\overline{(a_{22} - ia_{32}), (a_{24} - ia_{34})}}{(a_{22} + ia_{32}), (a_{24} + ia_{34})} \times \left( \frac{1}{2\pi} \right)^{n-2} (n-3)! d\theta \wedge \langle dA_2, A_4 \rangle \wedge \langle dA_3, A_4 \rangle \wedge \cdots \wedge \langle dA_2, A_n \rangle \wedge \langle dA_3, A_n \rangle,$$

where  $0 \leq \theta \leq 2\pi$ . For each vector  $A_i = (a_{i2}, a_{i3}, a_{i4}, \dots, a_{in})^t$  we set  $\tilde{A}_i$  by  $\tilde{A}_i = (a_{i2}, -a_{i3}, a_{i4}, \dots, a_{in})^t$ . This induces a diffeomorphism  $k; SO(n-1) \rightarrow SO(n-1)$  by  $k((A_2, A_3, A_4, A_5, \dots, A_n)) = (\tilde{A}_2, \tilde{A}_3, \tilde{A}_5, \tilde{A}_4, \dots, \tilde{A}_n)$ . Then we have

$$l = \int_{SO(n-1)/SO(n-3)} \left| \begin{array}{cc} (a_{22} - ia_{32}), (a_{23} - ia_{33}) \\ (a_{22} + ia_{32}), (a_{23} + ia_{33}) \end{array} \right| \frac{\overline{(a_{22} - ia_{32}), (a_{24} - ia_{34})}}{(a_{22} + ia_{32}), (a_{24} + ia_{34})} \times \left( \frac{1}{2\pi} \right)^{n-2} (n-3)! d\theta \wedge \langle d\tilde{A}_2, d\tilde{A}_5 \rangle \wedge \langle d\tilde{A}_3, \tilde{A}_5 \rangle \wedge \langle d\tilde{A}_2, \tilde{A}_4 \rangle \wedge \langle d\tilde{A}_3, \tilde{A}_4 \rangle \wedge \langle d\tilde{A}_2, \tilde{A}_6 \rangle \wedge \langle d\tilde{A}_3, \tilde{A}_6 \rangle \wedge \cdots \wedge \langle d\tilde{A}_2, \tilde{A}_n \rangle \wedge \langle d\tilde{A}_3, \tilde{A}_n \rangle.$$

Since we have  $\langle dA_i, A_j \rangle = \langle d\tilde{A}_i, \tilde{A}_j \rangle$  ( $2 \leq i \leq 3, 4 \leq j \leq n$ ), we obtain  $l=0$ . In the equation (6.10), the integrals

$$\begin{aligned} & \int_{Q_{n-3}(\mathcal{F}(\mathcal{Q})^\perp)} \left| \begin{array}{cc} \langle B_2, X'_0 + iX'_1 \rangle, \langle B_3, X'_0 + iX'_1 \rangle \\ \langle B_2, X'_0 - iX'_1 \rangle, \langle B_3, X'_0 - iX'_1 \rangle \end{array} \right| {}^2\Omega^{n-3}, \\ & \int_{Q_{n-3}(\mathcal{F}(\mathcal{Q})^\perp)} \left| \begin{array}{cc} \langle B_2, X'_0 + iX'_1 \rangle, \langle B_4, X'_0 + iX'_1 \rangle \\ \langle B_2, X'_0 - iX'_1 \rangle, \langle B_4, X'_0 - iX'_1 \rangle \end{array} \right| {}^2\Omega^{n-3}, \\ & \int_{Q_{n-3}(\mathcal{F}(\mathcal{Q})^\perp)} \left| \begin{array}{cc} \langle B_2, X'_0 + iX'_1 \rangle, \langle B_5, X'_0 + iX'_1 \rangle \\ \langle B_2, X'_0 - iX'_1 \rangle, \langle B_5, X'_0 - iX'_1 \rangle \end{array} \right| {}^2\Omega^{n-3}, \\ & \int_{Q_{n-3}(\mathcal{F}(\mathcal{Q})^\perp)} \left| \begin{array}{cc} \langle B_3, X'_0 + iX'_1 \rangle, \langle B_4, X'_0 + iX'_1 \rangle \\ \langle B_3, X'_0 - iX'_1 \rangle, \langle B_4, X'_0 - iX'_1 \rangle \end{array} \right| {}^2\Omega^{n-3} \end{aligned}$$

and

$$\int_{Q_{n-3}(\mathcal{F}(\mathcal{Q})^\perp)} \left| \begin{array}{cc} \langle B_3, X'_0 + iX'_1 \rangle, \langle B_5, X'_0 + iX'_1 \rangle \\ \langle B_3, X'_0 - iX'_1 \rangle, \langle B_5, X'_0 - iX'_1 \rangle \end{array} \right| {}^2\Omega^{n-3}$$

are all equal and furthermore its value is independent of  $q$ . We denote by  $C_0$  its common value. Then by (5.12), (6.9) and (6.10) we have

$$(6.11) \quad \int_{Q_{n-1}(C)} n(D, \alpha) d\alpha = \frac{1}{8} (n-1)(n-2) C_0 \int_D f^* \Omega^2.$$

We shall calculate the value  $C_0$ . Let  $SO(n-1)/SO(n-3) \rightarrow Q_{n-3}(C)$  be the Hopf fibring. For arbitrary fixed pair  $(C_2, C_3)$  of  $SO(n-1)/SO(n-3)$  we have

$$(6.12) \quad C_0 = \int_{Q_{n-3}(C)} \left| \begin{array}{cc} \langle C_2, A_2 + iA_3 \rangle, \langle C_3, A_2 + iA_3 \rangle \\ \langle C_2, A_2 - iA_3 \rangle, \langle C_3, A_2 - iA_3 \rangle \end{array} \right|^2 \Omega^{n-3}.$$

We take an orthonormal pair  $(D_4, D_5)$  of  $SO(n-1)/SO(n-3)$  such that  $\langle C_i, D_j \rangle = 0$  ( $2 \leq i \leq 3, 4 \leq j \leq 5$ ) and set real orthonormal vectors  $A_2, A_3, A_4$  and  $A_5$  by

$$(6.13) \quad \begin{aligned} A_2 &= \sin\varphi(\sin\theta \cdot C_2 - \cos\theta \cdot C_3) + \cos\varphi(\sin\alpha \cdot D_4 - \cos\alpha \cdot D_5) \\ A_3 &= \sin\eta(\cos\theta \cdot C_2 + \sin\theta \cdot C_3) + \cos\eta(\cos\alpha \cdot D_4 + \sin\alpha \cdot D_5) \\ A_4 &= -\cos\varphi(\sin\theta \cdot C_2 - \cos\theta \cdot C_3) + \sin\varphi(\sin\alpha \cdot D_4 - \cos\alpha \cdot D_5) \\ A_5 &= -\cos\eta(\cos\theta \cdot C_2 + \sin\theta \cdot C_3) + \sin\eta(\cos\alpha \cdot D_4 + \sin\alpha \cdot D_5), \end{aligned}$$

where  $0 < \theta, \alpha < \pi, -\pi/2 < \varphi, \eta < \pi/2$ . By extending  $A_2, A_3, A_4$  and  $A_5$  to an ordered real orthonormal basis  $A_2, A_3, \dots, A_n$  in  $\mathbb{C}^{n-1}$  we get  $(A_2, A_3, \dots, A_n) \in SO(n-1)$ . Take an open set  $U \subset Q_{n-5}(C)$ , where  $Q_{n-5}(C)$  is a set  $\{\beta \in Q_{n-3}(C) : |\beta, \Pi'''((C_2, C_3))|^2 + |\beta, \Pi'''((C_2, -C_3))|^2 = 0\}$  in  $Q_{n-3}(C)$ , and a local cross-section  $\sigma = (D_4, D_5)$  of  $U$  into  $SO(n-3)/SO(n-5)$  with respect to the Hopf fibring:  $SO(n-3)/SO(n-5) \rightarrow Q_{n-5}(C)$ . Then we see easily the set  $\{(A_2, A_3) \in SO(n-1)/SO(n-3) : (A_2, A_3) \text{ is defined at (6.13) for } \sigma = (D_4, D_5)\}$  is a double covering of an open set in  $Q_{n-3}(C)$ . We have

$$(6.14) \quad \begin{aligned} \langle dA_2, A_4 \rangle &= -d\varphi, \langle dA_3, A_5 \rangle = -d\eta, \\ \langle dA_2, A_5 \rangle &= -\sin\varphi\cos\eta d\theta + \sin\eta\cos\varphi d\alpha + \cos\varphi\sin\eta \langle dD_4, D_5 \rangle, \\ \langle dA_3, A_4 \rangle &= \sin\eta\cos\varphi d\theta - \sin\varphi\cos\eta d\alpha - \cos\eta\sin\varphi \langle dD_4, D_5 \rangle, \\ \langle dA_2, A_i \rangle &= \cos\varphi(\sin\alpha \langle dD_4, A_i \rangle - \cos\alpha \langle dD_5, A_i \rangle) \\ \langle dA_3, A_i \rangle &= \cos\eta(\cos\alpha \langle dD_4, A_i \rangle + \sin\alpha \langle dD_5, A_i \rangle) \quad (i \geq 6). \end{aligned}$$

By (6.14) we get

$$(6.15) \quad \begin{aligned} &\langle dA_2, A_4 \rangle \wedge \langle dA_3, A_4 \rangle \wedge \dots \wedge \langle dA_2, A_n \rangle \wedge \langle dA_3, A_n \rangle \\ &= (\sin^2\eta\cos^2\varphi - \sin^2\varphi\cos^2\eta)(\cos\varphi\cos\eta)^{n-5} \\ &\quad \times d\varphi \wedge d\theta \wedge d\alpha \wedge d\eta \wedge \prod_{i \geq 6} \langle dD_4, A_i \rangle \wedge \langle dD_5, A_i \rangle, \end{aligned}$$

and

$$(6.16) \quad \left| \begin{array}{cc} \langle C_2, A_2 + iA_3 \rangle, \langle C_3, A_2 + iA_3 \rangle \\ \langle C_2, A_2 - iA_3 \rangle, \langle C_3, A_2 - iA_3 \rangle \end{array} \right|^2 = 4|\sin\varphi\sin\eta|^2$$

Thus we obtain



$$\begin{aligned}
(6.12)' \quad C_0 &= (n-3)(n-4) \int |\sin\varphi \sin\eta|^2 |\sin^2\eta \cos^2\varphi - \sin^2\varphi \cos^2\eta| \\
&\quad |\cos\varphi \cos\eta|^{n-5} d\varphi d\eta \times \int_{Q_{n-5}(C)} \Omega^{n-5} \\
&= 2(n-3)(n-4) \int |\sin\varphi \sin\eta|^2 |\sin^2\eta \cos^2\varphi - \sin^2\varphi \cos^2\eta| \\
&\quad \times |\cos\varphi \cos\eta|^{n-5} d\varphi d\eta \\
&= \frac{16}{(n-1)(n-2)},
\end{aligned}$$

because of  $\int_{Q_i(C)} \Omega^i = 2$  and  $\int_E (\sin\varphi \sin\eta)^2 (\sin^2\varphi \cos^2\eta - \sin^2\eta \cos^2\varphi)$

$$\times (\cos\varphi \cos\eta)^{n-5} d\varphi d\eta = \frac{2}{(n-1)(n-2)(n-3)(n-4)}, \text{ where}$$

$E = \{(\eta, \varphi) : 0 \leq \varphi \leq \pi/2 \text{ and } 0 \leq \eta \leq \varphi\}$ . Thus we have proved the equation (6.1) for a sufficiently small  $D$ . Now let  $D$  be an arbitrary open set in  $C^2$  with compact closure. We take a finite covering  $\{D_s\}_{s=1}^l$  of  $D$  such that each  $D_s$  has a differentiable local cross-section of  $f$  into  $SO(n+1)/SO(n-1)$ . Let  $\{g_s\}$  be a partition of unity subordinated to  $\{D_s\}$ . Taking a mapping  $P_s : D_s \times Q_{n-3}(C) \rightarrow D_s$  defined by  $P_s((q, \alpha)) = q$  for  $(q, \alpha) \in D_s \times Q_{n-3}(C)$ , we put  $n'(D_s, \alpha) = \sum_k n(p_k, \alpha) g_s(p_k)$ . Then we obtain

$$\begin{aligned}
(6.17) \quad \int_{Q_{n-1}} n(D, \alpha) d\alpha &= \sum_{s=1}^l \int_{Q_{n-1}} n'(D_s, \alpha) d\alpha \\
&= \sum_s \int_{D_s \times Q_{n-3}} -g_s(P_s(\alpha')) (t_s'')^* d\alpha \\
&= 2 \sum_s \int_{D_s} g_s f^* \Omega^2 \\
&= 2 \int_D f^* \Omega^2,
\end{aligned}$$

where  $t_s''$  is a mapping of  $D_s \times Q_{n-3}(C)$  onto  $f(D_s)^\perp$  defined by (6.6). Q.E.D.

## 7. Equidistribution theorem

We define the defect  $\delta(\alpha)$  of  $\xi_\alpha$  by

$$(7.1) \quad \delta(\alpha) = \liminf_{r \rightarrow \infty} \frac{m(r, \alpha)}{T(r)}.$$

Since  $m(r, \alpha)$  is non-negative,  $\delta(\alpha)$  is non-negative for any  $\alpha \in Q_{n-1}(C)$ . We see clearly that  $\delta(\alpha) = \delta(\bar{\alpha})$  for any  $\alpha \in Q_{n-1}(C)$ . By Theorem 2, Lemma 4.5 and the fact that  $T(r) \rightarrow \infty$  if  $r \rightarrow \infty$ , we have

$$(7.2) \quad \delta(\alpha) = \liminf_{r \rightarrow \infty} \left( 1 - \frac{N(r, \alpha)}{T(r)} \right).$$

Then we have the following equidistribution theorem.

**Theorem 5.**  $\delta(\alpha)$  is equal to zero for almost all  $\alpha \in Q_{n-1}(C)$  with respect to the volume  $\Omega^{n-1}$ .

Proof. By the Fatou's preparation theorem we have

$$\begin{aligned} 0 &\leq \int_{Q_{n-1}} \delta(\alpha) d\alpha \leq \int_{Q_{n-1}} \left\{ \liminf_{r \rightarrow \infty} \left( 1 - \frac{N(r, \alpha)}{T(r)} \right) \right\} d\alpha \\ &\leq \liminf_{r \rightarrow \infty} \int_{Q_{n-1}} \left( 1 - \frac{N(r, \alpha)}{T(r)} \right) d\alpha = \liminf_{r \rightarrow \infty} \left( 2 - \frac{1}{T(r)} \int_{Q_{n-1}} N(r, \alpha) d\alpha \right) \\ &= \liminf_{r \rightarrow \infty} \left( 2 - \frac{1}{T(r)} \int_{Q_{n-1}} \left\{ \int_0^r n(\Delta(t), \alpha) dt \right\} d\alpha \right) \\ &= \liminf_{r \rightarrow \infty} \left( 2 - \frac{1}{T(r)} \int_0^r dt \int_{Q_{n-1}} n(\Delta(t), \alpha) d\alpha \right) \\ &= \liminf_{r \rightarrow \infty} (2 - 2) = 0 \quad (\text{by Theorem 4}). \end{aligned}$$

Thus we obtain  $\delta(\alpha) = 0$  for almost all  $\alpha \in Q_{n-1}(C)$ .

Q.E.D.

If the image  $f(C^2)$  does not intersect with  $\xi_a$ , we have  $\delta(\alpha) = 1$ . So we have

**Corollary.** Let  $f$  be a holomorphic mapping of  $C^2$  into  $Q_{n-1}(C)$  ( $n \geq 3$ ) satisfying Conditions (A) and (B). We put  $W = \{\alpha \in Q_{n-1}(C) : f(C^2) \cap \xi_a = \emptyset\}$ . Then the set  $W$  has measure zero with respect to volume  $\Omega^{n-1}$ .

REMARK 3. In the case of holomorphic curves ( $f: C \rightarrow P_n(C)$  holomorphic mapping), it is known that  $0 \leq \delta(\xi) \leq 1$  for each hyperplane  $\xi$  (c.f. [1], [5] and [6]). But in our case we can not prove that  $\delta(\alpha) \leq 1$ .

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### References

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